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Spatiotemporal dynamics in epidemic models with Lévy flights: A fractional diffusion approach

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A B S T R A C T

Recent field and experimental studies show that mobility patterns for humans exhibit scale-free nonlocal dynamics with heavy-tailed distributions characterized by Lévy flights. To study the long-range geographical spread of infectious diseases, in this paper we propose a susceptible-infectious-susceptible epidemic model with Lévy flights in which the dispersal of susceptible and infectious individuals follows a heavy-tailed jump distribution. Owing to the fractional diffusion described by a spectral fractional Neumann Laplacian, the nonlocal diffusion model can be used to address the spatiotemporal dynamics driven by the nonlocal dispersal. The primary focuses are on the existence and stability of disease-free and endemic equilibria and the impact of dispersal rates and fractional powers on the spatial profiles of these equilibria. A variational characterization of the basic reproduction number $\mathscr{R}_0$ is obtained and its dependence on dispersal rates and fractional powers is also examined. Then $\mathscr{R}_0$ is utilized to investigate the effects of spatial heterogeneity on the transmission dynamics. It is shown that $\mathscr{R}_0$ serves as a threshold for determining the existence and nonexistence of an epidemic equilibrium as well as the stability of the disease-free and endemic equilibria. In particular, in low-risk regions both dispersal rates and fractional powers play a critical role and are capable of altering the threshold value. Numerical simulations were performed to illustrate the theoretical results.

R É S U M É

Des études récentes sur le terrain et expérimentales montrent que les modèles de mobilité pour les humains présentent une dynamique non locale sans échelle avec des distributions à queue lourde caractérisées par des vols de Lévy. Étudier la propagation géographique à longue distance des maladies infectieuses, dans cet article nous proposons un modàle épidémique sensible-infectieux-sensible avec des vols de Lévy dans lequel la dispersion des individus sensibles et infectieux suit une distribution de sauts à queue lourde. En raison de la diffusion fractionnaire décrite par un Laplacien de Neumann fractionnaire spectral, le modèle de diffusion non locale peut être utilisé pour traiter la dynamique spatio-temporelle entraînée par la dispersion non locale. Les principaux objectifs sont l'existence et la stabilité d'équilibres sans maladie et endémiques et l'impact des taux de dispersion

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et des puissances fractionnaires sur les profils spatiaux de ces équilibres. Une caractérisation variationnelle du nombre de reproduction de base $\mathscr{R}_0$ est obtenue et sa dépendance aux taux de dispersion et aux puissances fractionnaires est également examinée. Ensuite $\mathscr{R}_0$ est utilisé pour étudier les effets de l'hétérogénéité spatiale sur la dynamique de transmission. On montre que $\mathscr{R}_0$ sert de seuil pour déterminer l'existence et l'inexistence d'un équilibre épidémique ainsi que la stabilité des équilibres sans maladie et endémique. En particulier, dans les régions à faible risque, les taux de dispersion et les puissances fractionnaires jouent un rôle critique et sont capables de modifier la valeur seuil. Des simulations numériques ont été réalisées pour illustrer les résultats théoriques.

1. Introduction

In studying the spatiotemporal properties of biological phenomena, such as the population dynamics of biological species and transmission dynamics of infectious diseases, classical reaction-diffusion equations are usually used to simulate the geographic diffusion and interaction of individuals (Murray [32], Okubo and Levin [35], Ruan and Wu [38]). In such models, Laplace operators are employed based on the assumption that the diffusion process can be described by Brownian motion, namely a group of particles spreads as a whole according to the irregular motion of each particle, and the fact that the probability density function of the continuous random walk (a Gaussian distribution) is a solution of the Fickian diffusion equation (Murray [32], Okubo and Levin [35]).

Note that Fickian diffusion applies to a diffusion process that corresponds to the random walk only when the step size and time size are small compared with the spatial variable and time, respectively. Consequently, classical reaction-diffusion equations only characterize spatial properties of biological systems locally. However, recent studies have shown that mobility patterns for humans exhibit scale-free dynamics with heavier tails distribution, a characteristic of Lévy flights (Mandellbrot [30], Zaburdaev et al. [54]). Lévy-flight patterns have been observed in human traveling by analyzing the circulation of banknotes (Brockmann et al. [7]), mobile phone data (González et al. [21]), hunter–gathers data when foraging for a wide variety of food items (Raichlen et al. [36]), as well as in the dispersal patterns of many biological species (Viswanathan et al. [45], Zaburdaev et al. [54]).

It has been shown that the density function representing the population with Lévy flight diffusion is the solution of a fractional-order diffusion equation and fractional-order derivatives are nonlocal integro-differential operators which can be used to characterize memory effects and long-distance diffusion processes (Chaves [14], Metzler and Klafter [31]). Such diffusion processes are often mathematically described by the spectral fractional Laplacians $(-\Delta_N)^s$ and $(-\Delta_D)^s$, where $0 < s < 1$, (the precise definition of $(-\Delta_N)^s$ will be given soon), which are the fractional counterparts of $-\Delta$ depending on the boundary condition under consideration (Neumann and Dirichlet). We refer to some recent studies by Caffarelli and Stinga [11], Grubb [22], Stinga [41], Stinga and Torrea [42], and Zhao [51], a survey of Vázquez [44], and the monographs of Bucur and Valdinoci [8] and Dipierro [16] on the fundamental theories of fractional diffusion equations. For studies on specific equations, we refer to Gui and Zhao [23] and Ma et al. [29] on the Allen-Cahn equation with a fractional Laplacian, Cabré and Roquejoffre [9], Caffarelli et al. [10], Felmer and Yangari [19], and Stan and Vázquez [40] on fractional Fisher-KPP equation, Estrada-Rodriguez et al. [18], Salem [39], and Stinga and Volzone [43] on Keller-Segel model with fractional diffusion, Bendahmane et al. [5] and Dannemann et al. [17] on Lotka-Volterra systems with Lévy flight. It is worth noting that, as shown in [11], $(-\Delta_N)^s$ can be represented by a nonlocal diffusion operator with a singular integral kernel, which highlights the nonlocal nature of $(-\Delta_N)^s$ and is yet contrast with the bounded nonlocal diffusion operators studied in Bates et al. [3], Bates and Zhao [4], Andreu-Vaillo et al. [2], Xu et al. [48], Yang et al. [49], Zhao and Ruan [52], and references therein.

The geographical spread of infectious disease through a population consists of two processes, the short-range local transmission and the long-range travel of infectious individuals, with the latter introducing the infectious diseases to new locations and causing potentially global outbreaks (Murray [32], Ruan [37]). Hence, it is the tail of the probability distribution of diffusion that has a significant impact on the spatial transmission dynamics of infectious diseases. It is believed that utilizing a heavy-tailed human movement process such as the Lévy flight can serve as a starting point for developing a new class of epidemic models for the spread of human infectious diseases (Brockmann et al. [7]). Indeed, epidemic models with fractional-diffusion have been developed to simulate the spatial spread of epidemics driven by long-range displacements in the infectious and susceptible populations, see Hanert et al. [24] and the references cited therein.

In this paper, we propose a susceptible-infectious-susceptible (SIS) endemic model with fractional diffusion of the following form

$$\begin{cases} u_t + (-d_u\Delta_N)^{s_1}u = a(x)u - b(x)u^2 - \dfrac{p(x)uv}{u+v} + q(x)v, & (t,x) \in \mathbb{R}^+ \times \Omega; \\ v_t + (-d_v\Delta_N)^{s_2}v = \dfrac{p(x)uv}{u+v} - q(x)v, & (t,x) \in \mathbb{R}^+ \times \Omega, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), \end{cases} \tag{1.1}$$

where $u(x,t)$ and $v(x,t)$ are the densities of a susceptible population and an infectious population at location $x \in \Omega$ and time t, respectively, $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with smooth boundary, $0 < s_i < 1$ ($i = 1,2$) are the fractional powers of the diffusion operator $-\Delta_N$, which is the Neumann realization of $-\Delta$ in a suitable function space. For any given $d > 0$ and $0 < s < 1$, the spectral fractional Neumann Laplace operator $(-d\Delta_N)^s$ is defined by

$$(-d\Delta_N)^s u = \sum_{k=1}^{\infty} d^s \lambda_k^s u_k \varphi_k = \frac{d^s}{\Gamma(-s)} \int_0^{\infty} \frac{e^{\Delta_N t}u - u}{t^{1+s}} dt,$$

where $e^{\Delta_N t}$ is the semigroup generated by Δ_N, $(\lambda_k, \varphi_k)_{k=1}^{\infty}$ are eigen-pairs of $-\Delta_N$, and $u(x) = \sum_{k=1}^{\infty} u_k\varphi_k(x)$ in which $u_k = \langle u, \varphi_k\rangle$, and $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\Omega)$. d_u and d_v are two positive constants that stand for the dispersal rates of u and v. We assume that, in the absence of the disease, the population has a density-dependent demographic structure (Gao and Hethcote [20]); that is, its growth is governed by a logistic term $a(x)u - b(x)u^2$, where $a(x)$ is the intrinsic growth rate and $b(x)/a(x)$ is the carrying capacity of the environment. $p(x)$ and $q(x)$ denote the transmission rate and recovery rate of the infectious individuals, respectively. It is assumed that $a(x), b(x), p(x), q(x)$ are Hölder continuous in $\overline{\Omega}$. In addition, $b(x), p(x), q(x)$ are non-negative. In (1.1), the local susceptible population is subject to the logistic growth, hence the size of total population is not constant. It is also assumed that the movements of $u(x,t)$ and $v(x,t)$ are both described by a power law probability distribution and are modeled by two spectral fractional Laplacians. Note that when $s_1 = s_2 = 1$, system (1.1) along with other diffusive epidemic models have been intensively investigated, see Allen et al. [1], Huang et al. [25], Li et al. [26], Li et al. [27], Murray [32], Webb [46], and the references cited therein.

The present paper is the second piece of a two-part series of studies on fractional diffusion equations, with the first part dealing with singularly perturbed fractional diffusion eigenvalue problems (Zhao and Ruan [53]). Our interest in (1.1) arises from a desire to understand its spatiotemporal transmission dynamics driven by nonlocal diffusion. Since the susceptible and infectious populations rarely display the exactly same dispersal behavior, it seems reasonable to assume that the movements of these two groups follow jump distributions at different microscopic scales, so $s_1 \neq s_2$ in general. Similar to Allen et al. [1], to gain a better understanding of the effects of spatial heterogeneity on the occurrence of an endemic, a threshold, being referred to as the basic reproduction number $\mathscr{R}_0$ and characterized by a variational formula, is introduced in the present work. Much like the conventional diffusive models considered in [1], in high-risk regions

($\bar{q} \le \bar{p}$) it always holds that $\mathscr{R}_0 > 1$. Given that $\mathscr{R}_0 > 1$, our analysis shows that (1.1) possesses an endemic equilibrium under the condition that $\bar{q} \le \bar{p} \le \bar{a}$, where $\bar{q}, \bar{p}$, and $\bar{a}$ are the averages of $q(x), p(x)$, and $a(x)$ over Ω, respectively. Hence in a high-risk region where population growth is sustainable, an endemic is bounded to occur. On the other hand, (1.1) has no endemic equilibria if $\mathscr{R}_0 \le 1$. For low-risk regions ($\bar{q} > \bar{p}$) and $\inf_{x\in\Omega}[q(x) - p(x)] < 0$, then for each s there exists a threshold value of d_0, such that $\mathscr{R}_0 \le 1$ if $d \ge d_0$ whereas $\mathscr{R}_0 > 1$ if $d < d_0$. Furthermore, if $d_0 > \frac{1}{\lambda_2}$, where λ_2 is the second least eigenvalue of $-\Delta_N$ over Ω, then the aforementioned s is also a threshold value in the sense that $\mathscr{R}_0 < 1$ when $d \ge d_0$ and $\theta > s$, and $\mathscr{R}_0 > 1$ provided that $d = d_0$ and $0 < \theta < s$, where θ stands for a different fractional power. In particular, as long as $d > \frac{1}{\lambda_2}$, then $\mathscr{R}_0$ is decreasing in s. Thus, from the perspective of disease prevention and control, in such regions, increasing s is likely to reduce potential risks of infection. This also underscores the importance of the underlying geometry of Ω.

The paper is organized as follows: Section 2 collects a series of properties concerning the basic reproduction number $\mathscr{R}_0$ and presents sufficient conditions that ensure the existence of an endemic equilibrium. Emphasis is placed on the dependence of $\mathscr{R}_0$ on dispersal rates and fractional powers. In this section, we also examine possible influences of dispersal rates d_u and d_v on the spatial profiles of the aforementioned equilibrium. Section 3 focuses on the stabilities of the disease-free and endemic equilibria. More specifically, we will show that the disease-free equilibrium is globally asymptotically stable when $\mathscr{R}_0 \le 1$. On the other hand, if $\mathscr{R}_0 > 1$ and the endemic equilibrium is independent of x, then it is globally asymptotically stable. In particular, a universal bound is also obtained for solutions of (1.1) regardless of their initial data. Finally, Section 4 presents numerical simulations that simulate the global asymptotic stabilities of disease-free and endemic equilibria under the condition that all coefficients are constants.

For future reference, we adopt all notations used in [53]. Given any two functions u and v, $u \gneqq v$ means that $u \ge v$ and u and v are not identical. Also, $u \lneqq v$ indicates $-u \gneqq -v$, and $u \vee v = \max\{u, v\}$, $u \wedge v = \min\{u, v\}$, $u_+ = u \vee 0$, $u_- = u \wedge 0$. Given any $w \in L^1(\Omega)$, the average of w over Ω is defined by $\bar{w} = \fint_\Omega w dx$. Given $u, v \in L^2(\Omega)$, the inner product of u, v is defined by $\langle u, v\rangle = \int_\Omega uv dx$.

Let $\mu \in \mathbb{R}^+ \setminus \mathbb{N}$ and $s = \mu - [\mu]$, where $[\mu]$ denotes the integer part of μ. Let $H^\mu(\Omega)$ be the Sobolev-Slobodeckii space defined by

$$H^\mu(\Omega) = \left\{ \partial^{|\alpha|} w \in L^2(\Omega),\ 0 \le |\alpha| \le [\mu] \middle| \ [w]^2_{H^\mu} := \sum_{|\alpha|=0}^{[\mu]} \int_\Omega \int_\Omega \frac{|\partial^{|\alpha|} w(x) - \partial^{|\alpha|} w(y)|^2}{|x-y|^{N+2s}} dy dx < \infty \right\}.$$

If $\mu > \frac{3}{2}$, set $H^\mu_N(\Omega) = \left\{ w \in H^\mu(\Omega) \mid \frac{\partial w}{\partial n} |_{\partial\Omega} = 0 \right\}$, where n is the outward unit normal on $\partial\Omega$. Throughout this paper, we let $\langle (-\Delta_N)^s \cdot \mid \cdot \rangle : H^s(\Omega) \times H^s(\Omega) \to \mathbb{R}$ be the bilinear form associated with $(-\Delta_N)^s$, which is defined by

$$\langle (-\Delta_N)^s u \mid v \rangle = \int_\Omega \int_\Omega K_{s,N}(x, y)[u(x) - u(y)][v(x) - v(y)] dy dx \qquad u, v \in H^s(\Omega), \tag{1.2}$$

where $K_{s,N}(x, y)$ is given by

$$K_{s,N}(x, y) = \frac{1}{2|\Gamma(-s)|} \int_0^\infty \frac{G_N(t, x, y)}{t^{1+s}} dt, \tag{1.3}$$

and $G_N(t, x, y)$ is the heat kernel of $e^{\Delta_N t} |_{t>0}$. As shown in Caffarelli and Stinga [11], $K_{s,N}$ is symmetric and enjoys two-sided Gaussian estimates, and there exist two positive constants $c_*(s, \Omega)$ and $c^*(s, \Omega)$ such that

$$\frac{c_*(s, \Omega)}{|x-y|^{N+2s}} \le K_{s,N}(x, y) \le \frac{c^*(s, \Omega)}{|x-y|^{N+2s}}, \quad x \ne y. \tag{1.4}$$

2. Existence of endemic equilibria

In this section, we consider the existence of endemic equilibria of (1.1). That is, the existence of component-wise positive solutions to

$$\begin{cases} (-d_u\Delta_N)^{s_1}u = a(x)u - b(x)u^2 - \dfrac{p(x)uv}{u+v} + q(x)v, & x \in \Omega, \\ (-d_v\Delta_N)^{s_2}v = \dfrac{p(x)uv}{u+v} - q(x)v, & x \in \Omega. \end{cases} \tag{2.1}$$

The following are the assumptions that will be used throughout the rest of the paper.

(H1) $a, b, p, q \in C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$; $b > 0$, $q > 0$, and $p \gneq 0$ for all $x \in \overline{\Omega}$.
(H2) $\overline{a} \geq 0$ if a is a non-constant function, or $a > 0$ if a is a constant.

2.1. Basic reproduction number

From now on, we set

$$\mathscr{R}_0 = \sup_{w\in H^s(\Omega)} \left\{ \frac{\langle p(x)w, w\rangle}{\langle(-d\Delta_N)^s w \mid w\rangle + \langle q(x)w, w\rangle} \right\}. \tag{2.2}$$

$\mathscr{R}_0$ is referred to as the *basic reproduction number*, which is defined in the manner similar to that of Allen et al. [1], where $\langle(-d\Delta_N)^s w \mid w\rangle = d^s\langle(-\Delta_N)^s w \mid w\rangle$, and $\langle(-\Delta_N)^s w \mid w\rangle$ is defined in (1.2). It can be shown that $\mathscr{R}_0$ is well defined, which is confirmed in the next proposition.

Proposition 2.1. *Let $\mathscr{R}_0$ be defined in (2.2). Then $\frac{\bar{p}}{\bar{q}} \leq \mathscr{R}_0 \leq \sup_{x\in\Omega} \frac{p(x)}{q(x)}$. In addition, let $\lambda_{p,q} = \inf_{w\in H^s(\Omega), \|w\|_{L^2(\Omega)=1}} \langle(-d\Delta_N)^s w \mid w\rangle + \langle[q(x) - p(x)]w, w\rangle$, that is, $\lambda_{p,q}$ is the principal eigenvalue of $(-d\Delta_N)^s + [q(x) - p(x)]I$. Then*

$$\mathrm{sign}\lambda_{p,q} = \mathrm{sign}\left(\frac{1}{\mathscr{R}_0} - 1\right). \tag{2.3}$$

Proof. We start to show the first part with an observation similar to Yang et al. [49] that

$$\begin{aligned} \frac{\langle p(x)w, w\rangle}{\langle(-d\Delta_N)^s w \mid w\rangle + \langle q(x)w, w\rangle} &\leq \sup_{x\in\Omega} \frac{p(x)}{q(x)} \frac{\langle q(x)w, w\rangle}{\langle(-d\Delta_N)^s w \mid w\rangle + \langle q(x)w, w\rangle} \\ &\leq \sup_{x\in\Omega} \frac{p(x)}{q(x)}. \end{aligned}$$

Meanwhile, by choosing $w = 1$ in (2.2), we find that $\mathscr{R}_0 \geq \frac{\bar{p}}{\bar{q}}$. Thus, $\frac{\bar{p}}{\bar{q}} \leq \mathscr{R}_0 \leq \sup_{x\in\Omega} \frac{p}{q}$. Let $I : H^s(\Omega) \to \mathbb{R}$ be given by $I(w) = \langle(-d\Delta_N)^s w \mid w\rangle + \langle q(x)w, w\rangle,\ w \in H^s(\Omega)$. Thanks to the fact that I is weakly lower semi-continuous, by the standard arguments (see the proof of Theorem 2.4 of [12]), it can be shown that $\mathscr{R}_0$ is attained by a maximizer $w^* \in H^s(\Omega)$. In particular, it is not difficult to see that w^* does not change sign. Indeed, given any $w \in H^s(\Omega)$, we can see that

$$\begin{aligned} \langle(-\Delta_N)^s|w| \mid |w|\rangle &= \iint_{\Omega\times\Omega} K_{N,s}(x,y)(w_+(x) - w_+(y) + w_-(x) - w_-(y))^2 dydx \\ &= \iint_{\Omega\times\Omega} K_{N,s}(x,y)(w_+(x) - w_+(y))^2 dydx \end{aligned}$$

$$+\iint_{\Omega\times\Omega} K_{N,s}(x,y)(w_-(x)-w_-(y))^2dydx$$
$$-2\iint_{\Omega\times\Omega} K_{N,s}(x,y)w_+(x)w_-(y)dydx$$
$$\leq \iint_{\Omega\times\Omega} K_{N,s}(x,y)(w(x)-w(y))^2dydx,$$

where $K_{N,s}(x,y)$ is given in (1.2). Note that $\langle(-\Delta_N)^s|w| \mid |w|\rangle < \langle(-\Delta_N)^s w \mid w\rangle$ if both $\{x \mid w > 0\}$ and $\{x \mid w < 0\}$ have positive measures, it then follows that w^* does not change sign. Now, to show (2.3), let ϕ be a positive eigenfunction corresponding to $\lambda_{p,q}$, and w^* be a positive maximizer associated with $\mathscr{R}_0$, respectively. Due to the Hölder continuity of p and q, Theorem 1.4 of Caffarelli and Stinga [11] and Lemma A.4 imply that $w^* > 0$ and $\phi > 0$ on $\overline{\Omega}$. Then, similar to [1], we have

$$\langle(-d\Delta_N)^s\phi \mid w^*\rangle + \langle[q(x)-p(x)]\phi, w^*\rangle = \lambda_{p,q}\langle\phi, w^*\rangle$$

and

$$\langle(-d\Delta_N)^s w^* \mid \phi\rangle + \langle[q(x)-p(x)]w^*, \phi\rangle = \left(\frac{1}{\mathscr{R}_0}-1\right)\langle p(x)w^*, \phi\rangle.$$

Subtracting these two equations yields that

$$\lambda_{p,q}\int_\Omega \phi w^* dx = \left(\frac{1}{\mathscr{R}_0}-1\right)\int_\Omega p(x)w^*\phi dx.$$

Namely,

$$\text{sign}\lambda_{p,q} = \text{sign}\left(\frac{1}{\mathscr{R}_0}-1\right).$$

The proof is completed. □

Proposition 2.1 together with Proposition A.6 immediately yield the following.

Proposition 2.2. *Assume that (H1) is satisfied, then*

(i) $\mathscr{R}_0 > 1$ *if* $\bar{q} \leq \bar{p}$ *and* $q - p$ *is a non-constant function, or* $\bar{q} < \bar{p}$ *and* $q-p$ *is a constant;*
(ii) $\mathscr{R}_0 = 1$ *if* $q = p$;
(iii) $\mathscr{R}_0 < 1$ *if* $q \gneq p$;
(iv) *If* $\bar{q} > \bar{p}$ *and* $\inf_{x\in\Omega}[q(x)-p(x)] < 0$, *then there exists* $d_0 > 0$ *such that* $\mathscr{R}_0 > 1$ *when* $d < d_0$ *whereas* $\mathscr{R}_0 \leq 1$ *if* $d \geq d_0$.

The next proposition deals with the dependence of $\mathscr{R}_0$ on p, q, d, and s.

Proposition 2.3. *Assume that (H1) is satisfied, then*

(i) $\lim_{d\to 0^+}\left|\mathscr{R}_0 - \sup_{x\in\Omega}\frac{p(x)}{q(x)}\right| = 0$. *Moreover,* $\mathscr{R}_0 \to \sup_{x\in\Omega}\frac{p(x)}{q(x)}$ *uniformly in* s *for* $s \in [\eta, 1)$ *as* $d \to 0^+$, *where* $0 < \eta < 1$;
(ii) $\lim_{d\to\infty}\left|\mathscr{R}_0 - \frac{\bar{p}}{\bar{q}}\right| = 0$;

(iii) *If* $d\lambda_2 \geq 1$, *then* $\mathscr{R}_0$ *is non-increasing in* s, *where* λ_2 *is the second least eigenvalue of* $-\Delta_N$ *in* Ω.

Proof. Given any $w \in H^s(\Omega)$ with $w \neq 0$, it follows from the definition of $\mathscr{R}_0$ that

$$\int_\Omega [p(x) - \mathscr{R}_0 q(x)] w^2 dx \leq \mathscr{R}_0 \langle (-d\Delta_N)^s w \mid w \rangle.$$

Let $m = \sup_{x\in\Omega} \frac{p(x)}{q(x)}$ and $x^* \in \overline{\Omega}$ be such that $\frac{p(x^*)}{q(x^*)} = \sup_{x\in\Omega} \frac{p(x)}{q(x)}$. Also, let x_r be defined as that in Proposition A.6. That is, $x_r = x^*$ if $x^* \in \Omega$, whereas $x_r = x^* + rn^+(x^*)$ if $x^* \in \partial\Omega$, where $n^+(x^*)$ is the inward unit normal to Ω at $x^* \in \partial\Omega$, and $r > 0$ is chosen such that $\overline{B}_r(x_r) \subset \overline{\Omega}$. Let again $\varphi_r(x) = (r^2 - |x - x_r|^2)_+^s$. As shown in Proposition 2.1 that $\mathscr{R}_0 \leq m$, we then have

$$\begin{aligned} m\langle (-d\Delta_N)^s \varphi_r \mid \varphi_r \rangle &\geq \int_{B_r(x_r)} \left[(m - \mathscr{R}_0) q(x) + \left(\frac{p(x)}{q(x)} - m \right) q(x) \right] \varphi_r^2 dx \\ &\geq (m - \mathscr{R}_0) q(x^*) \int_{B_r(x_r)} \varphi_r^2 dx + \int_{B_r(x_r)} (m - \mathscr{R}_0)[q(x) - q(x^*)] \varphi_r^2 dx \\ &\quad + \int_{B_r(x_r)} \left[\left(\frac{p(x)}{q(x)} - m \right) q(x) \right] \varphi_r^2 dx. \end{aligned}$$

Note that $\langle (-d\Delta_N)^s \varphi_r \mid \varphi_r \rangle / \langle \varphi_r, \varphi_r \rangle \leq C^N d^s r^{-2s}$, where $C^N > 0$ is the constant given in the proof of Proposition A.6. Dividing both sides of the above inequality by $q(x^*)\langle \varphi_r, \varphi_r \rangle$ gives

$$m - \mathscr{R}_0 \leq \frac{1}{q(x^*)} \left\{ mC^N \left(\frac{d}{r^2} \right)^s + 2m \sup_{x \in B_r(x_r)} |q(x) - q(x^*)| + \sup_{x\in\Omega} q(x) \sup_{x \in B_r(x_r)} |(p/q)(x) - m| \right\}.$$

Fix $0 < \eta < 1$. Thanks to the continuity of q and p/q, given $\epsilon > 0$, r and d can be chosen sufficiently small such that $m - \mathscr{R}_0 \leq \epsilon$ for all $s \in [\eta, 1)$, this together the fact that $\mathscr{R}_0 \leq m$ confirm (i).

(ii) Let w_d be the maximizer of (2.2) with $\|w_d\|_{L^2(\Omega)} = 1$. Then

$$(-\Delta_N)^s w_d = \frac{q(x) w_d}{d^s} + \frac{p(x) w_d}{d^s \mathscr{R}_0}.$$

By arguing along the same lines as those in Proposition A.6, we reach the conclusion that $\mathscr{R}_0 \to \frac{\bar{p}}{\bar{q}}$ as $d \to \infty$.

(iii) Instead of $\mathscr{R}_0$, we temporarily denote the basic reproduction number by $\mathscr{R}_0(s)$ to emphasize its dependence on s. Given that $s_1 < s_2$, let $w_{s_2} \in H^{s_2}(\Omega)$ be a maximizer of (2.2) for $s = s_2$. Clearly, $w_{s_2} \in H^{s_1}(\Omega)$ as $s_2 > s_1$. Then, from the condition that $d\lambda_2 \geq 1$, it follows that

$$\langle (-d\Delta_N)^{s_1} w_{s_2} \mid w_{s_2} \rangle = \sum_{i=2}^{\infty} (d\lambda_i)^{s_1} |w_{s_2,i}|^2 \leq \sum_{i=2}^{\infty} (d\lambda_i)^{s_2} |w_{s_2,i}|^2 = \langle (-d\Delta_N)^{s_2} w_{s_2} \mid w_{s_2} \rangle,$$

where $w_{s_2,i} = \langle w_{s_2}, \varphi_i \rangle$ and $(\lambda_i, \varphi_i)_{i=1}^{\infty}$ are eigen-pairs of $-\Delta_N$. The inequality is strict if $d\lambda_2 > 1$. Thus,

$$\mathscr{R}_0(s_1) \geq \frac{\langle p(x) w_{s_2}, w_{s_2} \rangle}{\langle (-d\Delta_N)^{s_1} w_{s_2} \mid w_{s_2} \rangle + \langle q(x) w_{s_2}, w_{s_2} \rangle} \geq \mathscr{R}_0(s_2).$$

This completes the proof. □

2.2. Existence of endemic equilibria

Now we study the existence of endemic equilibria.

Proposition 2.4. *Let $(u,v) \in H^{s_1}(\Omega) \times H^{s_2}(\Omega)$ be any non-negative solution of (2.1). Then*

$$\int_\Omega u^2 dx \le \frac{4}{\inf_{x\in\Omega} b(x)} \int_\Omega \frac{a^2(x)}{b(x)} dx, \quad \int_\Omega v^2 dx \le \left(\frac{\sup_{x\in\Omega} p(x)}{\inf_{x\in\Omega} q(x)}\right)^2 \int_\Omega u^2 dx.$$

For any $\theta \ge 1$,

$$\|u\|_{L^\theta(\Omega)} + \|v\|_{L^\theta(\Omega)} \le C,$$

where $C > 0$ is a constant depending on $\theta, a, b, p,$ and q only.

Proof. Note that

$$\int_\Omega a(x) u dx - \int_\Omega b(x) u^2 dx = \langle (-d_u \Delta_N)^{s_1} u \mid 1\rangle + \langle (-d_v \Delta_N)^{s_2} v \mid 1\rangle = 0.$$

It immediately follows from the Hölder inequality and the Young inequality that

$$\int_\Omega b(x) u^2 dx \le \frac{1}{2} \int_\Omega b(x) u^2 dx + 2 \int_\Omega \frac{a^2(x)}{b(x)} dx.$$

This confirms the first inequality.

Also, in conjunction with the facts that $|\frac{u}{u+v}|_\infty \le 1$ and $|\frac{v}{u+v}|_\infty \le 1$, using the Hölder inequality and Young inequality again gives rise to

$$\int_\Omega q(x) v^2 dx + \langle (-d\Delta_N)^s v \mid v\rangle \le \int_\Omega p(x) uv dx \le \frac{1}{2} \int_\Omega q(x) v^2 dx + 2 \int_\Omega \frac{p^2(x)}{q(x)} u^2 dx.$$

Hence, the second inequality follows.

Given any $k \ge 1$, in light of Lemma A.2 of Brasco and Parini [6], we have

$$\frac{4 d_u^{s_1} c_* k}{(k+1)^2} [u^{\frac{k+1}{2}}]^2_{H^{s_1}} \le \langle (-d_u \Delta_N)^{s_1} u \mid u^k\rangle, \quad \frac{4 d_v^{s_2} c_* k}{(k+1)^2} [v^{\frac{k+1}{2}}]^2_{H^{s_2}} \le \langle (-d_v \Delta_N)^{s_1} v \mid v^k\rangle,$$

where $c_* > 0$ is the constant given in (1.4). Then, as in Li et al. [26], multiplying the first and second equations of (2.1) by u^k and v^k, respectively, yields that

$$\begin{aligned} \int_\Omega b(x) u^{k+2} dx &\le \int_\Omega |a(x)| u^{k+1} dx + \int_\Omega q(x) v u^k dx \\ &\le \int_\Omega \left(|a(x)| + \frac{k}{k+1}\right) u^{k+1} dx + \frac{1}{k+1} \int_\Omega q^{k+1}(x) v^{k+1} dx \end{aligned}$$

and

$$\int_\Omega q(x)v^{k+1}dx \le \frac{1}{2}\int_\Omega q(x)v^{k+1}dx + \sqrt[k]{\frac{2k}{k+1}}\int_\Omega \frac{p^{k+1}(x)}{q^k(x)}u^k dx.$$

Therefore,

$$\int_\Omega u^{k+1}dx \le C(a,b,q,k)\Big[\int_\Omega u^k dx + \int_\Omega v^k dx\Big], \quad \int_\Omega v^{k+1}dx \le C(p,q,k)\int_\Omega u^k dx.$$

Given any $\theta > 2$, by virtue of the inequalities given above, the implementation of finite iterations yields $\|u\|_{L^\theta(\Omega)} + \|v\|_{L^\theta(\Omega)} \le C$ for $C > 0$ depending on a, b, p, q and θ only. The proof is completed. □

In the following we always let $\mathscr{R}_0$ be defined in (2.2) with $s = s_2$

Proposition 2.5. *Assume that (H1) is fulfilled and $\mathscr{R}_0 \le 1$. Then (2.1) has no positive endemic equilibria. Suppose further that (H2) is satisfied, then (2.1) possesses only one semi-positive solution $(u_1, 0)$, which is the disease-free equilibrium, where u_1 is the unique positive solution to*

$$(-d_u\Delta_N)^{s_1}w = (a(x) - b(x)w)w. \tag{2.4}$$

Proof. Again, let λ_v be the principal eigenvalue of $(-d_v\Delta_N)^{s_2} + [q(x) - p(x)]I$. Since $\mathscr{R}_0 \le 1$, Proposition 2.1 shows that $\lambda_v \ge 0$. Assume to the contrary that (2.1) has a positive endemic equilibrium (u, v). Then Propositions A.3, A.2 and Lemma A.4 imply that $u > 0$ and $v > 0$ on $\overline{\Omega}$. Let ψ be a positive eigenfunction corresponding to λ_v, Lemma A.4 again implies that $\psi > 0$ on $\overline{\Omega}$. Observe that

$$\langle(-d_v\Delta_N)^{s_2}v \mid \psi\rangle + \int_\Omega \Big[q(x) - \frac{p(x)u}{u+v}\Big]v\psi dx = 0$$

and

$$\langle(-d_v\Delta_N)^{s_2}\psi \mid v\rangle + \int_\Omega [q(x) - p(x)]v\psi dx = \lambda_v \int_\Omega v\psi dx.$$

By subtracting the second equation from the first one, we find that

$$\int_\Omega p(x)\Big[1 - \frac{u}{u+v}\Big]v\psi dx \le 0.$$

Note that $\sup_{x\in\Omega} \frac{u}{u+v} < 1$ as $u > 0$ and $v > 0$ on $\overline{\Omega}$. Hence, we have reached a contradiction. The contradiction implies that (2.1) has no positive endemic equilibria. Additionally, with (H2), the existence of the disease-free equilibrium is an immediate consequence of Lemma A.8. The proof is completed. □

The next theorem is the main result of this section that gives sufficient conditions for the existence of an endemic equilibrium.

Theorem 2.6. *Suppose that (H1) is satisfied and $\mathscr{R}_0 > 1$. Assume that $\lambda_{d_u,a-p} < 0$, where $\lambda_{d_u,a-p}$ is the principal eigenvalue of $(-d_u\Delta_N)^{s_1} - [a(x) - p(x)]I$. Then (2.1) has at least a positive endemic steady state.*

Proof. We will obtain the existence of a positive endemic steady state by the homotopy invariance of Leray-Schauder degree. To this end, the following auxiliary system is considered:

$$\begin{cases}(-d_u\Delta_N)^{s_1}u = [a(x)-p(x)-b(x)u]u,\\ (-d_v\Delta_N)^{s_2}v = \left[\dfrac{p(x)u}{u+v}-q(x)\right]v.\end{cases} \tag{2.5}$$

Notice that (2.5) is a weakly coupled system, it is relatively easier to establish the existence of positive endemic steady states of (2.5). Since $\lambda_{d_u,a-p}<0$, it follows from Lemma A.8 that the first equation of (2.5) has a unique positive solution $u_l>0$ on $\overline{\Omega}$. Set $l_0=-\frac{\inf_{x\in\Omega}u_l}{2}$. Now consider the existence of positive solutions to

$$(-d_v\Delta_N)^{s_2}v = vh(x,u_l,v),\quad h(x,u,v)=\frac{p(x)u(x)}{u(x)+v}-q(x)$$

Note that $h(x,u_l,\cdot)$ is differentiable in (l_0,∞) and $h(x,u_l,0)=p(x)-q(x)$. Since $\mathscr{R}_0>1$, (2.3) implies that $\lambda_{d_v,p-q}<0$, where $\lambda_{d_v,p-q}$ is the principal eigenvalue of $(-d_v\Delta_N)^{s_2}-[p(x)-q(x)]I$. Again, Lemma A.8 shows that the equation has a unique positive solution v_l. Namely, (2.5) has a unique positive endemic steady state (u_l,v_l). Clearly, $u_l>0, v_l>0$ on $\overline{\Omega}$.

Set $X=C(\overline{\Omega})\times C(\overline{\Omega})$, we next consider the spectrum of the linearization of (2.5) at (u_l,v_l), which is the linear operator $\mathscr{L}:\mathrm{dom}(\mathscr{L})\subset X\to X$ given by

$$\mathscr{L}(u,v)=\begin{pmatrix}(-d_u\Delta_N)^{s_1}u+[2b(x)u_l-a(x)-p(x)]u & 0\\ -\frac{p(x)v_l^2}{(u_l+v_l)^2}u & (-d_v\Delta_N)^{s_2}v+[q(x)-\frac{p(x)u_l^2}{(u_l+v_l)^2}]v\end{pmatrix}.$$

Since $(-d\Delta_N)^s$ has compact resolvent, $\mathscr{L}$ is a Fredholm operator of index zero. In particular, it is not difficult to see that

$$\sigma(\mathscr{L})\setminus\{+\infty\}=\sigma_p(\mathscr{L})=\sigma_p((-d_u\Delta_N)^{s_1}+c_{11}(x)I)\bigcup\sigma_p((-d_v\Delta_N)^{s_2}+c_{22}(x)I),$$

where $\sigma_p(\cdot)$ denotes the point spectrum of an operator, and $c_{11}(x)=2b(x)u_l-a(x)-p(x)$, $c_{22}(x)=q(x)-\frac{p(x)u_l^2}{(u_l+v_l)^2}$. Let λ_1 be the principal eigenvalue of $(-d_u\Delta_N)^{s_1}+c_{11}(x)I$ and λ_2 be the principal eigenvalue of $(-d_v\Delta_N)^{s_2}+c_{22}(x)I$. Note that 0 is the principal eigenvalue of $(-d_v\Delta_N)^{s_2}+[q(x)-\frac{p(x)u_l}{u_l+v_l}]I$ as $(-d_v\Delta_N)^{s_2}v_l+[q(x)-\frac{p(x)u_l}{u_l+v_l}]v_l=0$. Since $c_{22}(x)\gneqq[q(x)-\frac{p(x)u_l}{u_l+v_l}]$, it follows that $\lambda_2>0$. For similar reasons, $\lambda_1>0$. Thus,

$$\mathfrak{s}(\mathscr{L}):=\inf\{\mathrm{Re}\lambda\mid\lambda\in\sigma(\mathscr{L})\}>0.$$

Next write

$$f^l(x,u)=[a(x)-p(x)-b(x)u]u,\quad f^r(x,u,v)=[a(x)-b(x)u]u+q(x)v$$

and

$$f^c(x,u,v)=\left[a(x)-b(x)u-\frac{p(x)v}{u+v}\right]u+q(x)v,\quad g(x,u,v)=\left[\frac{p(x)u}{u+v}-q(x)\right]v.$$

Obviously, for all $x\in\overline{\Omega}$,

$$f^l(x,u) \lneq f^c(x,u,v) \lneq f^r(x,u,v)$$

provided that $u > 0$ and $v > 0$ on $\overline{\Omega}$.

We now select sub- and super-solutions to form the domain for the needed Leray-Schauder degree. Chose $0 < \epsilon < 1$ and set $\tilde{u}_l = \epsilon u_l$. Then we have $(-d_u\Delta_N)^{s_1}\tilde{u}_l \lneq f^l(x,\tilde{u}_l)$. Accordingly, let $\tilde{v}_l$ be the positive solution to $(-d_v\Delta_N)^{s_2}v = g(x,\tilde{u}_l,v)$. As shown above, the existence and uniqueness are a consequence of Lemma A.8. That is, $(-d_v\Delta_N)^{s_2}\tilde{v}_l = g(x,\tilde{u}_l,\tilde{v}_l)$. On the one hand, since $u_l > \tilde{u}_l$ on $\overline{\Omega}$, and $\partial_u g(x,u,v_l) \gneq 0$, we have $(-d_v\Delta_N)^{s_2}v_l = g(x,u_l,v_l) \gneq g(x,\tilde{u}_l,v_l)$. On the other hand, since $g(x,\tilde{u}_l,v) = vh(x,\tilde{u}_l,v)$ and $\partial_v h(x,\tilde{u}_l,v) \lneq 0$, from Lemma A.4 and Proposition A.7, it follows that $v_l > \tilde{v}_l$ on $\overline{\Omega}$. Later, $(\tilde{u}_l,\tilde{v}_l)$ serves as a lower bound. To obtain an upper bound, let $C > 0$ be such a number that $\sup_{x\in\Omega} p(x) \le C \inf_{x\in\Omega} q(x)$. Also, let $m > 1$ and set $u_r = m, v_r = Cm$. Simple calculation shows that $g(x,u_r,v_r) = Cm[\frac{p(x)}{C+1} - q(x)] < 0$ on $\overline{\Omega}$. In addition, let m be so chosen that $f_r(x,u_r,v_r) = a(x)m - b(x)m^2 + q(x)Cm \lneq 0$ on $\overline{\Omega}$. Apparently, u_r and v_r are independent of d_u, d_v, and s_i $(i = 1,2)$, and $\tilde{u}_l < u_l < u_r$, $\tilde{v}_l < v_l < v_r$ on $\overline{\Omega}$. Now it is straightforward to verify that

$$\begin{cases}(-d_u\Delta_N)^{s_1}\tilde{u}_l \lneq f^l(x,\tilde{u}_l) \lneq f^c(x,\tilde{u}_l,v), & (x,v) \in \overline{\Omega}\times[\tilde{v}_l,v_r],\\ (-d_v\Delta_N)^{s_2}\tilde{v}_l \le g(x,u,\tilde{v}_l), & (x,u) \in \overline{\Omega}\times[\tilde{u}_l,u_r],\end{cases}$$

and

$$\begin{cases}(-d_u\Delta_N)^{s_1}u_r \gneq f^r(x,u_r,v) \gneq f^c(x,u_r,v), & (x,v) \in \overline{\Omega}\times[\tilde{v}_l,v_r],\\ (-d_v\Delta_N)^{s_2}v_r > g(x,u,v_r), & (x,u) \in \overline{\Omega}\times[\tilde{u}_l,u_r].\end{cases}$$

Next set

$$\mathcal{O} = \{(u,v) \in C(\overline{\Omega}) \times C(\overline{\Omega}) \mid \tilde{u}_l < u < u_r,\ \tilde{v}_l < v < v_r,\ x \in \overline{\Omega}\},$$

and let $\mathscr{F}_t : [0,1] \times \mathcal{O} \to C(\overline{\Omega}) \times C(\overline{\Omega})$ be defined by

$$\mathscr{F}_t(u,v) = \begin{pmatrix}(-d_u\Delta_N)^{s_1} + I)^{-1}[u + (1-t)f^l(x,u) + tf^c(x,u,v)]\\ (-d_v\Delta_N)^{s_2} + I)^{-1}[v + g(x,u,v)]\end{pmatrix}.$$

Clearly, $\mathscr{F}_t$ is compact for $t \in [0,1]$. As indicated by the above calculation, $(u,v) \ne \mathscr{F}_t(u,v)$ on $\partial\mathcal{O}$ for any $t \in [0,1]$. Thus, $\deg(I - \mathscr{F}_t, \mathcal{O}, 0)$ is well defined and is independent of t. Since $[I - \mathscr{F}_0](u_l,v_l) = 0$ and $\mathfrak{s}(\mathscr{L}) > 0$, it follows from Nirenberg [34] that $\deg(I - \mathscr{F}_0, \mathcal{O}, 0) = 1$. Thus, $\deg(I - \mathscr{F}_1, \mathcal{O}, 0) = 1$ and $\mathscr{F}_1$ has a fixed point in $\mathcal{O}$, which is a positive endemic steady state of (2.1). Thanks to (H1), Proposition A.2 shows that $u \in C^{2s_1+2\alpha}(\overline{\Omega})$ with $s_1 < \frac{1}{2}$ and $2s_1 + 2\alpha < 1$ or $u \in C_N^{2s_1+2\alpha}(\overline{\Omega})$ with $s_1 \ge \frac{1}{2}$ and $2s_1 + 2\alpha < 2$. Likewise, $v \in C^{2s_2+2\alpha}(\overline{\Omega})$ with $s_2 < \frac{1}{2}$ and $2s_2 + 2\alpha < 1$ or $v \in C_N^{2s_2+2\alpha}(\overline{\Omega})$ with $s_2 \ge \frac{1}{2}$ and $2s_2 + 2\alpha < 2$. The proof is completed. □

Corollary 2.7. *Assume that $\overline{q} \le \overline{p} \le \overline{a}$. Suppose further that $\overline{q} < \overline{p}$ if $q - p$ is a constant, and $\overline{p} < \overline{a}$ if $p - a$ is a constant. Then, for any $0 < s_i < 1$ $(i = 1,2)$, $d_u > 0$, and $d_v > 0$, (2.1) has at least a positive endemic steady state.*

Proof. The assumptions and Proposition A.6 imply that $\lambda_{d_u,a-p} < 0$ for any $0 < s_1 < 1$ and $d_u > 0$. In addition, Proposition 2.2 shows that $\mathscr{R}_0 > 1$ for any $d_v > 0$ and $0 < s_2 < 1$. Thus, the desired conclusion follows from Theorem 2.6. □

2.3. Influence of diffusion rates

We next analyze the impacts of d_u and d_v on the spatial profiles of the positive endemic steady state established in Theorem 2.6. We begin with the case $d_u \to 0$.

Theorem 2.8. *Suppose that (H1) is satisfied, $\mathscr{R}_0 > 1$, and $\inf_{x\in\Omega}[a(x)-p(x)] > 0$. Then (2.1) has a positive endemic steady state (u,v) for any $0 < s_1 < 1$ and $d_u > 0$. Moreover,*

$$\lim_{d_u\to 0^+} \|u-u_0\|_{C(\overline{\Omega})} = 0, \quad \lim_{d_u\to 0^+} \|v-v_0\|_{C(\overline{\Omega})} = 0$$

for some $(u_0,v_0) \in C(\overline{\Omega}) \times C(\overline{\Omega})$, where $u_0 > 0$, $v_0 > 0$ on $\overline{\Omega}$, and (u_0,v_0) solves

$$\begin{cases} [a(x)-b(x)u_0]u_0 - \dfrac{p(x)u_0v_0}{u_0+v_0} + q(x)v_0 = 0, \\ (-d_v\Delta_N)^{s_2} v_0 = \left[\dfrac{p(x)u_0}{u_0+v_0} - q(x)\right] v_0. \end{cases} \tag{2.6}$$

Proof. Under the condition that $\inf_{x\in\Omega}[a(x)-p(x)] > 0$, Proposition A.6 shows that $\lambda_{d_u,a-p} < 0$ for any $0 < s_1 < 1$ and $d_u > 0$. Hence, it follows from Theorem 2.6 that (2.1) has a positive endemic steady state for any $0 < s_1 < 1$ and $d_u > 0$.

Now let u_l be the unique positive solution of $(-d_u\Delta_N)^{s_1}w = (a(x)-p(x)-b(x)w)w$ and let $x_0 \in \overline{\Omega}$ be such that $u_l(x_0) = \inf_{x\in\Omega} u_l$. Then Proposition A.5 implies that

$$[a(x_0)-p(x_0)-b(x_0)u_l(x_0)]u_l(x_0) \le 0.$$

Since $u_l(x_0) > 0$, $u_l(x_0) \ge \frac{a(x_0)-p(x_0)}{b(x_0)} \ge \frac{\inf_{x\in\Omega}(a-p)(x)}{\sup_{x\in\Omega} b(x)} := l_0 > 0$. Subsequently, it follows from Proposition A.7 that $u \ge u_l \ge l_0$ on $\overline{\Omega}$. Next let v_l be the unique positive solution of

$$(-d_v\Delta_N)^{s_2} w = \left[\frac{p(x)l_0}{l_0+w} - q(x)\right] w.$$

Then, Lemma A.4 implies that $\inf_{x\in\Omega} v_l \ge C \sup_{x\in\Omega} v_l$, where $C > 0$ is a constant depending on $p, q, d_v > 0$, and Ω only. Hence, $\inf_{x\in\Omega} v_l := k_0 > 0$. Since $u \ge l_0$, we have

$$(-d_v\Delta_N)^{s_2} v \ge \left[\frac{p(x)l_0}{l_0+v} - q(x)\right] v.$$

Hence, it follows again from Proposition A.7 that $v \ge v_l \ge k_0$ on $\overline{\Omega}$. On the other hand, as a consequence of Propositions A.3, 2.4 and Theorem 1.5 of Caffarelli and Stinga [11] (or see Grubb [22]), there exist constants $0 < \gamma < 1$ and $C > 0$ such that $\|v\|_{H^{s_2}(\Omega)} + \|v\|_{C^\gamma(\overline{\Omega})} \le C$. Here $C > 0$ depends on d_v, s_2, p, and q, and $\|v\|_{L^2(\Omega)}$. Thus, upon the extraction of a subsequence of d_u, we have

$$\lim_{d_u\to 0} \|v-v_0\|_{C^\mu(\overline{\Omega})} = 0, \quad v \xrightarrow{\text{w}} v_0 \text{ in } H^{s_2}(\Omega)$$

for any $0 < \mu < \gamma$, $v_0 \in C^\mu(\overline{\Omega}) \cap H^{s_2}(\Omega)$.

We then turn to

$$(-d_u\Delta_N)^{s_1} u = \left[a(x) - b(x)u - \frac{p(x)u}{u+v} + q(x)\frac{v}{u}\right] u.$$

Set

$$g(x,v,u)=u,\quad h(x,v,u)=a(x)-b(x)u-\frac{p(x)u}{u+v}+q(x)\frac{v}{u}.$$

To complete the proof, we just need to show that $h(x,\cdot,\cdot)$ satisfies the assumptions of Corollary A.9. Fix $x\in\overline{\Omega}$ and consider

$$h(x,v_0(x),\tau)=a(x)-b(x)\tau-\frac{p(x)\tau}{\tau+v_0(x)}+q(x)\frac{v_0(x)}{\tau}.$$

Clearly, for each fixed $x\in\overline{\Omega}$, as a function of τ, h is defined in $(0,\infty)$. In particular, $h\to\infty$ as $\tau\to 0^+$, and $h\to-\infty$ as $\tau\to\infty$. Thus, $h(x,v_0(x),\tau)=0$ has at least one root $\tau(x)>0$ in $(0,\infty)$. Simple calculation shows that

$$h_\tau(x,v_0(x),\tau)=-b(x)-p(x)-\frac{p(x)v_0(x)}{(v_0(x)+\tau)^2}-\frac{q(x)v_0(x)}{\tau}<0,\quad \tau\in(0,\infty).$$

Hence, $\tau(x)$ is the only root in $(0,\infty)$. The implicit function theorem implies that $\tau(x)\in C(\overline{\Omega})$.

We next show that $\tau(x)$ is bounded from below by a positive number. Note that

$$h(x,v(x),\tau)\geq a(x)-b(x)\tau-p(x)+\frac{q(x)v_0(x)}{\tau}=\frac{[a(x)-p(x)]\tau-b(x)\tau^2+q(x)v_0(x)}{\tau},$$

which implies that

$$\frac{a(x)-p(x)+\sqrt{[a(x)-p(x)]^2+4b(x)q(x)v_0(x)}}{2b(x)}:=r_2(x)\leq\tau(x).$$

The assumption that $\inf_{x\in\Omega}[a(x)-p(x)]>0$ implies that $\inf_{x\in\Omega}r_2(x):=r_*>0$. Set $\eta_1=0$ and $\eta_3=0$. Obviously, $h(x,\cdot,\cdot)\in C^{\alpha,1}(\overline{\Omega}\times(\eta_1,\infty)\times(\eta_3,\infty))$ and $h_\tau(x,v_0(x),\tau)<0$ for all $x\in\overline{\Omega}$. Rename $\tau(x)$ by $u_0(x)$. That is, (u_0,v_0) solves the first equation of (2.6). On the other hand, set $\eta_2=-1$. Since $g(x,v,u)=u$, $g\in C^{\alpha,1}(\overline{\Omega}\times(\eta_1,\infty)\times(\eta_2,\infty))$. Also, $\partial_3 g=1>0$. Thus, Corollary A.9 yields that

$$\lim_{d_u\to0^+}\|u-u_0\|_{C(\overline{\Omega})}=0.$$

Returning to v, since $u\geq l_0>0$ and $v\geq k_0>0$ on $\overline{\Omega}$, $\frac{u}{u+v}\to\frac{u_0}{u_0+v_0}$ in $C(\overline{\Omega})$ as $d_u\to0^+$. Thus,

$$\lim_{d_u\to0^+}\int_\Omega\left|v+\frac{p(x)uv}{u+v}-q(x)v-v_0-\frac{p(x)u_0v_0}{u_0+v_0}+q(x)v_0\right|^2dx=0.$$

Then, in view of Proposition A.2, the continuity of $(-d_v\Delta_N)^{s_2}+I)^{-1}$ yields

$$\lim_{d_u\to0^+}\left\|v-(-d_v\Delta_N)^{s_2}+I)^{-1}\left[v_0+\frac{p(x)u_0v_0}{u_0+v_0}-q(x)v_0\right]\right\|_{H^{s_2}(\Omega)}=0,$$

and it follows from the definition of $(-d_v\Delta_N)^{s_2}+I)^{-1}$ that

$$\langle(-d_v\Delta_N)^{s_2}v_0\mid\psi\rangle=\int_\Omega\left[\frac{p(x)u_0}{u_0+v_0}-q(x)\right]v_0\psi dx$$

for any $\psi \in H^{s_2}(\Omega)$. It is not difficult to see that u_0 is Hölder continuous. Hence, Theorem 1.4 of [11] shows that $v_0 \in C^{2s_2+2\theta}(\overline{\Omega})$ with $s_2 < \frac{1}{2}$ and $2s_2 + 2\theta < 1$ or $v_0 \in C_N^{2s_2+2\theta}(\overline{\Omega})$ with $s_2 \geq \frac{1}{2}$ and $2s_2 + 2\theta < 2$ for some $0 < \theta < 1$, and (u_0, v_0) solves the second equation of (2.6). The proof is completed. □

The next theorem is concerned with the spatial profiles of the positive endemic steady state (u, v) of (2.1) as $d_v \to 0$.

Theorem 2.9. *Suppose that* $\sup_{x\in\Omega}[p(x) - q(x)] > 0$ *and* $\lambda_{d_u,a-p} < 0$. *Assume that (H1) is satisfied. Then there exists* $d^* > 0$ *such that (2.1) has a positive endemic steady state* (u, v) *for any* $0 < d_v \leq d^*$. *Moreover,*

$$\lim_{d_v\to 0^+} \|u - u_0\|_{C(\overline{\Omega})} = 0, \quad \lim_{d_v\to 0^+} \|v - v_0^+\|_{C(\overline{\Omega})} = 0$$

for some $(u_0, v_0) \in C(\overline{\Omega}) \times C(\overline{\Omega})$, *where* $u_0 > 0$ *on* $\overline{\Omega}$, $v_0^+ = v_0 \vee 0 := \max\{v_0(x), 0\}$, *and* (u_0, v_0) *solves*

$$\begin{cases} (-d_u\Delta_N)^{s_1} u_0 = [a(x) - b(x)u_0]u_0 - \dfrac{p(x)u_0v_0}{u_0 + v_0} + q(x)v_0, \\ [p(x) - q(x)]u_0(x) = q(x)v_0(x). \end{cases} \tag{2.7}$$

Proof. Under the condition that $\sup_{x\in\Omega}[p(x) - q(x)] > 0$, Proposition 2.2 implies that there exists $d_0 > 0$ such that $\mathscr{R}_0 > 1$ for all $0 < d < d_0$, where d_0 may depend on s_2. Fix $d^* < d_0$, then $\mathscr{R}_0 > 1$ for any $0 < d \leq d^*$. Thus, in terms of Theorem 2.6, (2.1) has a positive endemic steady state (u, v) for any $0 < d_v \leq d^*$ as long as d_u and s_i $(i = 1, 2)$ are fixed.

Let u_l be the positive solution given in the proof of Theorem 2.8. Since $d_u > 0$ and s_1 are fixed, Lemma A.4 shows that $\inf_{x\in\Omega} u_l \geq C \sup_{x\in\Omega} u_l$, where $C > 0$ depends on d_u, s_1, a, and p only. Hence, $l_0 = \inf_{x\in\Omega} u_l > 0$. It then follows from Proposition A.7 that $u \geq u_l \geq l_0 > 0$. Now chose $\theta > \frac{N}{2s_2}$, then in view of Proposition 2.4, we see that $\|v\|_{L^\theta(\Omega)} \leq C_\theta$ for some $C_\theta > 0$ depending on a, b, p, q and θ only. Also note that

$$\langle (-d_u\Delta_N)^{s_1} u \mid \psi \rangle \leq \langle a(x)u, \psi \rangle + \langle q(x)v, \psi \rangle$$

for any $\psi \in H^{s_1}(\Omega)$ satisfying $\psi \geq 0$. Consequently, Proposition A.3 and Theorem 1.5 of Caffarelli and Stinga [11] imply that $\|u\|_{H^{s_1}(\Omega)} + \|u\|_{C^\gamma(\overline{\Omega})} \leq C$ for some $0 < \gamma < 1$ and $C > 0$ depending on d_u, s_1, $|a|_\infty, |q|_\infty$, and $\|v\|_{L^\theta(\Omega)}$. By extracting a subsequence of d_v if necessary, we obtain

$$\lim_{d_v\to 0^+} \|u - u_0\|_{C^\mu(\overline{\Omega})} = 0, \quad u \xrightarrow{\text{w}} u_0 \ \text{ in } \ H^{s_1}(\Omega)$$

for any $0 < \mu < \gamma$, $u_0 \in H^{s_1}(\Omega) \cap C^\mu(\overline{\Omega})$. We next turn to v. Note that

$$(-d_v\Delta_N)^{s_2} v = \left(\frac{v}{u + v}\right)\left[p(x)u - q(x)u - q(x)v\right]$$

and set

$$g(x, u, v) = \frac{v}{u + v}, \quad h(x, u, v) = [p(x) - q(x)]u - q(x)v.$$

As done in Theorem 2.8, we proceed to show that g and h satisfy the assumptions of Corollary A.9. Clearly, $h(x, u_0(x), v) = 0$ has a unique root $v_0(x) = \frac{[p(x)-q(x)]u_0(x)}{q(x)}$ and $v_0 \in C^\beta(\overline{\Omega})$ for some $0 < \beta < 1$. Let $\eta_1 = \frac{l_0}{2}$, $\eta_2 = -\frac{l_0}{4}$, and $\eta_3 = -2|v_0|_\infty$. Then

$$g \in C^{\alpha,1}(\overline{\Omega} \times (\eta_1, \infty) \times (\eta_2, \infty)), \quad h \in C^{\alpha,1}(\overline{\Omega} \times (\eta_1, \infty) \times (\eta_3, \infty))$$

and

$$g(x, \cdot, 0) = 0, \quad \theta g(x, \cdot, \tau) \geq g(x, \cdot, \theta\tau), \quad \partial_\tau g(x, u, \tau) = \frac{u}{(u+\tau)^2} > 0, \quad \partial_\tau h(x, u, \tau) = -q(x) < 0.$$

Here $\theta \geq 1$ and $\tau \geq 0$ are arbitrary. Note that $v_0^+ \gneq 0$ as $\sup_{x\in\Omega}[p(x) - q(x)] > 0$ and $u_0 \geq l_0$. Thus, Corollary A.9 implies that

$$\lim_{d_v \to 0^+} \|v - v_0^+\|_{C(\overline{\Omega})} = 0.$$

Now, by employing the same arguments used in the proof of Theorem 2.8, we infer that (u_0, v_0) solves the first equation of (2.7). This completes the proof. □

We next study the spatial profiles of the positive endemic steady state of (2.1) as either $d_u \to \infty$ or $d_v \to \infty$.

Theorem 2.10. *Suppose that (H1) is satisfied and $\mathscr{R}_0 > 1$. Assume that $\overline{a} > \overline{p}$. Then (2.1) has a positive endemic steady state (u, v) for any $d_u > 0$. Moreover,*

$$\lim_{d_u \to \infty} \|u - u_\infty\|_{C(\overline{\Omega})} = 0, \quad \lim_{d_u \to \infty} \|v - v_\infty\|_{C(\overline{\Omega})} = 0$$

for some $(u_\infty, v_\infty) \in C(\overline{\Omega}) \times C(\overline{\Omega})$, where $u_\infty > 0$ and $v_\infty > 0$ on $\overline{\Omega}$, u_∞ is a constant, and (u_∞, v_∞) solves

$$\begin{cases} \displaystyle\int_\Omega [a(x) - b(x)u_\infty]u_\infty dx = \int_\Omega \left[\frac{p(x)u_\infty}{u_\infty + v_\infty} - q(x)\right] v_\infty dx, \\ (-d_v\Delta_N)^{s_2} v_\infty = \left[\dfrac{p(x)u_\infty}{u_\infty + v_\infty} - q(x)\right] v_\infty. \end{cases} \tag{2.8}$$

Proof. In terms of the condition that $\overline{a} > \overline{p}$, Proposition 2.2 implies that $\lambda_{d_u, a-p} < 0$ for any $d_u > 0$. Thus, the existence of an endemic steady state is obtained via Theorem 2.6 for any $d_u > 0$. As $d_v > 0$ and s_2 are fixed, with the same reasoning shown in the proof of Theorem 2.8, we obtain

$$\lim_{d_u \to \infty} \|v - v_\infty\|_{C^\mu(\overline{\Omega})} = 0, \quad v \xrightarrow{\text{w}} v_\infty \text{ in } H^{s_2}(\Omega)$$

for some $0 < \mu < 1$, $v_\infty \in C^\mu(\overline{\Omega}) \cap H^{s_2}(\Omega)$. Again, notice that

$$\langle(-\Delta_N)^{s_1} u \mid \psi\rangle \leq d_u^{-s_1}\langle a(x)u, \psi\rangle + d_u^{-s_1}\langle q(x)v, \psi\rangle$$

for any $\psi \in H^{s_1}(\Omega)$ with $\psi \geq 0$. Hence, it follows from Proposition A.3 and Theorem 1.4 of Caffarelli and Stinga [11] that

$$\|u\|_{C^{2s_1+2\theta}(\overline{\Omega})} \leq C,$$

where $0 < \theta < 1$ is such that $2s_1 + 2\theta < 1$ if $0 < s_1 < \frac{1}{2}$, and $2s_1 + 2\theta < 2$ if $s_1 \geq \frac{1}{2}$, $C > 0$ is independent of d_u as long as $d_u \geq 1$. Then, as in the proof of Proposition A.6, we have $u \to u_\infty$ in $\overline{\Omega}$ as $d_u \to \infty$, where $u_\infty \geq 0$ is a constant. In particular, by passing the limits in

$$0 = \langle (-d_u\Delta_N)^{s_1} u \mid 1 \rangle = \int_\Omega \left[a(x)u - b(x)u^2 - \frac{p(x)uv}{u+v} + q(x)v \right] dx,$$

we infer that (u_∞, v_∞) solves the first equation of (2.8).

Next we show $u_\infty > 0$ and $v_\infty > 0$ on $\overline{\Omega}$. Let again u_l be the positive solution of $(-d_u\Delta_N)^{s_1} w = [a(x) - p(x) - b(x)w]w$. In view of the fact that $\langle (-d_u\Delta_N)^{s_1} u_l \mid \frac{1}{u_l} \rangle \le 0$ (see the discussion at the end of Section 3 of [53]), we have

$$⨍_\Omega b(x)u_l dx \ge ⨍_\Omega [a(x) - p(x)] dx.$$

The inequality is strict unless u_l is a constant. Since $u \ge u_l$ on $\overline{\Omega}$ and $\lim_{d_u\to\infty} \|u - u_\infty\|_{C(\overline{\Omega})} = 0$, we have $u_\infty \ge \frac{\bar{a}-\bar{p}}{\bar{b}} := l_0 > 0$. In addition, it is clear that $u \ge \frac{l_0}{2}$ on $\overline{\Omega}$ if d_u is sufficiently large. Now let v_l be the positive solution of

$$(-d_v\Delta_N)^{s_2} w = \frac{p(x)l_0 w}{l_0 + 2w} - q(x)w.$$

Since d_v and s_2 remain unchanged, as shown before, $\inf_{x\in\Omega} v_l := k_0 > 0$. Because $u \ge \frac{l_0}{2}$ once d_u is large enough, Proposition A.7 shows that $v \ge v_l \ge k_0$ on $\overline{\Omega}$ if d_u is sufficiently large. As a result, $v_\infty \ge k_0$ on $\overline{\Omega}$. Finally, the same arguments used in Theorem 2.8 imply that u_∞ and v_∞ satisfy the second equation of (2.8). The proof is completed. □

Theorem 2.11. *Suppose that (H1) is satisfied and $\lambda_{d_u,a-p} < 0$. Assume that $\overline{p} > \overline{q}$. Then (2.1) has a positive endemic steady state (u, v) for any $d_v > 0$. Moreover,*

$$\lim_{d_v\to\infty} \|u - u_\infty\|_{C(\overline{\Omega})} = 0, \quad \lim_{d_v\to\infty} \|v - v_\infty\|_{C(\overline{\Omega})} = 0$$

for some $(u_\infty, v_\infty) \in C(\overline{\Omega}) \times C(\overline{\Omega})$, where $u_\infty > 0$ and $v_\infty > 0$ on $\overline{\Omega}$, v_∞ is a constant, and (u_∞, v_∞) solves

$$\begin{cases} (-d_u\Delta_N)^{s_1} u_\infty = [a(x) - b(x)u_\infty]u_\infty - \dfrac{p(x)u_\infty v_\infty}{u_\infty + v_\infty} + q(x)v_\infty, \\ \displaystyle\int_\Omega \left[\frac{p(x)u_\infty}{u_\infty + v_\infty} - q(x) \right] v_\infty dx = 0. \end{cases} \tag{2.9}$$

Proof. The proof is mainly the same as that of Theorem 2.10. We present a sketch with a few key details. First, as in the proof of Theorem 2.10, we have $u \to u_\infty$ and $v \to v_\infty$ in $\overline{\Omega}$ for some $(u_\infty, v_\infty) \in C(\overline{\Omega}) \times C(\overline{\Omega})$, where $v_\infty \ge 0$ is a constant. As d_u is fixed, we still have $u \ge u_l \ge l_0 := \inf_{x\in\Omega} u_l > 0$, where u_l is the positive solution given in the proof of Theorem 2.10. Hence $u_\infty \ge l_0$ on $\overline{\Omega}$.

It remains to show that $v_\infty > 0$. Similar to Ni [33], we consider $w = \frac{v}{\|v\|_{L^2(\Omega)}}$. Then

$$(-d_v\Delta_N)^{s_2} w = \left[\frac{p(x)u}{u+v} - q(x) \right] w dx.$$

By using Proposition A.3 and Theorem 1.4 of [11] again, we obtain that $w \in C^{2s_2+2\theta}(\overline{\Omega})$ for some $0 < \theta < 1$. Then, the same arguments show that $w \to w_\infty$ in $\overline{\Omega}$ for some positive constant w_∞ as $d_v \to \infty$. In particular,

$$\int_\Omega \left[\frac{p(x)u_\infty}{u_\infty + v_\infty} - q(x) \right] dx = 0.$$

Note that $v_\infty > 0$, otherwise, we would have $\bar{p} = \bar{q}$, which contradicts the assumption. By passing the limits in (2.1), we find that (u_∞, v_∞) solves (2.9). The proof is completed. $\square$

Corollary 2.12. *Suppose that (H1) is satisfied and $\bar{q} < \bar{p} < \bar{a}$. Then (2.1) has a positive solution (u, v) for any $d_u > 0$, $d_v > 0$, and $0 < s_i < 1$ $(i = 1, 2)$. Moreover,*

$$u \to \frac{\bar{a}}{\bar{b}}, \quad v \to \frac{\bar{a}}{\bar{b}}(\mathscr{R}_0 - 1)$$

uniformly in $\overline{\Omega}$ as $d_u \to \infty$ and $d_v \to \infty$.

Proof. The existence of a positive solution (u, v) of (2.1) is a special case of Corollary 2.7. The convergence of u and v and the positivity of the corresponding limits are established via the same arguments employed in the proofs of Theorems 2.10 and 2.11. Let u_∞ and v_∞ be the limits of u and v, respectively. Clearly, u_∞ and v_∞ are two positive constants. Then

$$\begin{cases} \int_\Omega [a(x) - b(x)u_\infty]u_\infty dx = \int_\Omega \left[\frac{p(x)u_\infty}{u_\infty + v_\infty} - q(x)\right] v_\infty dx, \\ \int_\Omega \left[\frac{p(x)u_\infty}{u_\infty + v_\infty} - q(x)\right] v_\infty dx = 0. \end{cases}$$

Therefore $u_\infty = \frac{\bar{a}}{\bar{b}}$ and $v_\infty = \frac{\bar{a}}{\bar{b}}(\frac{\bar{p}}{\bar{q}} - 1)$. Meanwhile, by Proposition 2.3, we have $\mathscr{R}_0 \to \frac{\bar{p}}{\bar{q}}$ as $d_v \to \infty$. Thus, the desired conclusion follows. $\square$

3. Stability of disease-free and endemic equilibria

This section focuses on the stability of the disease-free and endemic equilibria that were obtained in Section 3. We first consider time-dependent positive solutions to

$$\begin{cases} u_t + (-d_u\Delta_N)^{s_1}u = a(x)u - b(x)u^2 - \frac{p(x)uv}{u+v} + q(x)v, & (t, x) \in \mathbb{R}^+ \times \Omega; \\ v_t + (-d_v\Delta_N)^{s_2}v = \frac{p(x)uv}{u+v} - q(x)v, & (t, x) \in \mathbb{R}^+ \times \Omega, \\ u(0, x) = u_0(x), \ v(0, x) = v_0(x), \end{cases} \tag{3.1}$$

where $(u_0, v_0) \in X = C(\overline{\Omega}) \times C(\overline{\Omega})$. According to Yosida [50], $-(-d\Delta_N)^s$ is the infinitesimal generator of an analytic semigroup given by

$$e^{-(-d\Delta_N)^s t}w = \int_0^\infty T_{s,t}(\tau)e^{d\Delta_N \tau}w d\tau,$$

where $T_{s,t} \in L^1(\mathbb{R}^+)$ are a family of non-negative functions satisfying

$$\int_0^\infty T_{s,t}(\tau)d\tau = 1, \ t > 0; \quad T_{s,t} * T_{s,\eta} = T_{s,t+\eta}, \ t, \eta > 0, \quad T_{s,t}(\tau) = t^{-\frac{1}{s}}T_{s,1}(t^{-\frac{1}{s}}\tau), \ t > 0.$$

The analytic semigroup generated by $-(-d\Delta_D)^s$ on $L^2(\Omega)$ and $C_0(\overline{\Omega})$ can be defined in a similar way. The following lemma summarizes a few basic estimates of $(-d\Delta_N)^\sigma e^{-(-d\Delta_N)^s t}$ and $(-d\Delta_D)^\sigma e^{-(-d\Delta_D)^s t}$ in either $L^2(\Omega)$ or $C(\overline{\Omega})$ ($C_0(\overline{\Omega})$ for $e^{-(-d\Delta_D)^s t}$) as $t \to \infty$, where $0 < \sigma \leq s$

Lemma 3.1. *Let $e^{-(-d\Delta_N)^s t}$ and $e^{-(-d\Delta_D)^s t}$ be the analytic semigroups generated by $-(-d\Delta_N)^s$ and $-(-d\Delta_D)^s$ in either $L^2(\Omega)$ or $C(\overline{\Omega})$ ($C_0(\overline{\Omega})$ for $e^{-(-d\Delta_D)^s t}$), respectively. Then*

$$e^{-(-d\Delta_N)^s t} w = \int_0^\infty T_{s,t}(\tau) e^{d\Delta_N \tau} w d\tau, \quad e^{-(-d\Delta_D)^s t} w = \int_0^\infty T_{s,t}(\tau) e^{d\Delta_D \tau} w d\tau.$$

Let $0 < \sigma, s < 1$. Then, for any $t > 0$,

(i) $\|e^{-(-d\Delta_N)^s t} w\|_{L^2(\Omega)} \le \|w\|_{L^2(\Omega)}$, $\|(-d\Delta_N)^\sigma e^{-(-d\Delta_N)^s t} w\|_{L^2(\Omega)} \le C e^{-\frac{(d\lambda_2)^s t}{2}} t^{-\frac{\sigma}{s}} \|w\|_{L^2(\Omega)}$;
(ii) $\|e^{-(-d\Delta_N)^s t} w\|_{C(\overline{\Omega})} \le C_\Omega \|w\|_{C(\overline{\Omega})}$, $\|(-d\Delta_N)^\sigma e^{-(-d\Delta_N)^s t} w\|_{C(\overline{\Omega})} \le C t^{-\frac{\sigma}{s}} \|w\|_{C(\overline{\Omega})}$, *where $C > 0$ are constants depending on σ, s, and Ω, $C_\Omega > 0$ depends on Ω;*
(iii) $\|e^{-(-d\Delta_D)^s t} w\|_{L^2(\Omega)} \le e^{-(d\mu_1)^s t} \|w\|_{L^2(\Omega)}$, $\|(-d\Delta_D)^\sigma e^{-(-d\Delta_D)^s t} w\|_{L^2(\Omega)} \le C e^{-\frac{(d\mu_1)^s t}{2}} t^{-\frac{\sigma}{s}} \|w\|_{L^2(\Omega)}$;
(iv) $\|e^{-(-d\Delta_D)^s t} w\|_{C(\overline{\Omega})} \le C_\Omega e^{\frac{-(d\mu_1)^s t}{2}} \|w\|_{C(\overline{\Omega})}$, $\|(-d\Delta_D)^\sigma e^{-(-d\Delta_D)^s t} w\|_{C(\overline{\Omega})} \le C t^{-\frac{\sigma}{s}} \|w\|_{C(\overline{\Omega})}$, *where $\mu_1 > 0$ is the principal eigenvalue of $-\Delta_D$ in Ω, $C > 0$ are constants depending on σ, s, and Ω, $C_\Omega > 0$ depends on Ω.*

Proof. We only prove (i) and (ii) as the proofs of (iii) and (iv) are parallel. To show (i), notice that $e^{d\Delta_N \tau} w = \sum_{k=1}^\infty e^{-d\lambda_k \tau} w_k \varphi_k$, where $(\lambda_k, \varphi_k)_{k=1}^\infty$ are the eigen-pairs of $-\Delta_N$ in $L^2(\Omega)$, and $w_k = \langle w, \varphi_k \rangle$. From the Laplace transforms of $T_{s,t} |_{t>0}$ (see Sec. IX, 11 of Yosida [50]), it follows that

$$e^{-(-d\Delta_N)^s t} w = \sum_{k=1}^\infty \int_0^\infty T_{s,t}(\tau) e^{-d\lambda_k \tau} d\tau w_k \varphi_k = \sum_{k=1}^\infty e^{-(d\lambda_k)^s t} w_k \varphi_k. \tag{3.2}$$

Note that $e^{-(-d\Delta_N)^s t} 1 = 1$. Moreover, the Parseval's identity implies that

$$\|e^{-(-d\Delta_N)^s t} w\|_{L^2(\Omega)} \le \|w\|_{L^2(\Omega)}.$$

To prove the second estimate of (i), we use the fact that $(-d\Delta_N)^\sigma \varphi_k = (d\lambda_k)^\sigma \varphi_k$. For the sake of simplicity, we assume without loss of generality that $d = 1$. Then, it follows from (3.2) that

$$\begin{aligned} (-\Delta_N)^\sigma e^{-(-\Delta_N)^s t} w &= \sum_{k=2}^\infty \lambda_k^\sigma e^{-\lambda_k^s t} w_k \varphi_k \\ &= \sum_{k=2}^\infty \lambda_k^\sigma e^{-\frac{\lambda_k^s}{2} t} e^{-\frac{\lambda_k^s}{2} t} w_k \varphi_k \\ &= t^{-\frac{\sigma}{s}} \sum_{k=2}^\infty (2\tau_k)^{\frac{\sigma}{s}} e^{-\tau_k} e^{-\frac{\lambda_k^s}{2} t} w_k \varphi_k, \end{aligned}$$

where $\tau_k = \frac{\lambda_k^s t}{2}$. Therefore, the Parseval's identity implies that

$$\|(-\Delta_N)^\sigma e^{-(-\Delta_N)^s t} w\|_{L^2(\Omega)}^2 \le \left(t^{-\frac{\sigma}{s}} e^{-\frac{\lambda_2^s t}{2}}\right)^2 \sum_{k=2}^\infty (2\tau_k)^{\frac{2\sigma}{s}} e^{-2\tau_k} |w_k|^2.$$

Observe that $\sup_{\tau \ge 0} (2\tau)^{\frac{2\sigma}{s}} e^{-2\tau} = (\frac{2\sigma}{s})^{\frac{2\sigma}{s}} e^{-\frac{2\sigma}{s}} < \infty$. Hence,

$$\|(-\Delta_N)^\sigma e^{-(-\Delta_N)^s t} w\|_{L^2(\Omega)} \le C t^{-\frac{\sigma}{s}} e^{-\frac{\lambda_2^s t}{2}} \|w\|_{L^2(\Omega)}$$

for some $C > 0$ depending on s, σ, and Ω.

To prove (ii), assume that $w \in C(\overline{\Omega})$. Write $w = \bar{w} + (w - \bar{w})$. By results in Lunardi [28],

$$e^{\Delta_N \tau} w = \bar{w} + e^{\Delta_N \tau}(w - \bar{w}), \quad \|e^{\Delta_N \tau}(w - \bar{w})\|_{C(\overline{\Omega})} \le C' e^{-(\lambda_2 - \delta)\tau} \|w - \bar{w}\|_{C(\overline{\Omega})},$$

where $0 < \delta < \lambda_2$ and $C' > 0$ depends on Ω. Thus, by the definition of $e^{-(-\Delta_N)^s t}$,

$$\begin{aligned} \|e^{-(-\Delta_N)^s t} w\|_{C(\overline{\Omega})} &\le \int_0^\infty T_{s,t}(\tau)\big(\|\bar{w}\|_{C(\overline{\Omega})} + \|e^{\Delta_N \tau}(w - \bar{w})\|_{C(\overline{\Omega})}\big) d\tau \\ &\le |\bar{w}| + C' e^{-(\lambda_2 - \delta)^s t} \|w - \bar{w}\|_{C(\overline{\Omega})} \\ &\le C_\Omega \|w\|_{C(\overline{\Omega})} \end{aligned}$$

for some constant $C_\Omega > 0$. Furthermore, by [28] again, it holds that

$$\|(-\Delta_N)^\sigma e^{\Delta_N \tau} w\|_{C(\overline{\Omega})} \le C_\sigma \tau^{-\sigma} \|w\|_{C(\overline{\Omega})}, \quad \tau > 0,$$

where $C_\sigma > 0$ depends on σ and Ω. Consequently,

$$\begin{aligned} \|(-\Delta_N)^\sigma e^{-(-\Delta_N)^s t} w\|_{C(\overline{\Omega})} &\le \int_0^\infty T_{s,t}(\tau) \|(-\Delta_N)^\sigma e^{\Delta_N \tau} w\|_{C(\overline{\Omega})} d\tau \\ &\le C_\sigma \|w\|_{C(\overline{\Omega})} \int_0^\infty \frac{t^{-\frac{1}{s}} T_{s,1}(t^{-\frac{1}{s}}\tau)}{\tau^\sigma} d\tau \\ &= C_\sigma t^{-\frac{\sigma}{s}} \|w\|_{C(\overline{\Omega})} \int_0^\infty \theta^{-\sigma} T_{s,1}(\theta) d\theta \\ &\le C_\sigma t^{-\frac{\sigma}{s}} \|w\|_{C(\overline{\Omega})} \left[\int_0^1 \theta^{-\sigma} T_{s,1}(\theta) d\theta + \int_0^\infty T_{s,1}(\theta) d\theta \right] \\ &\le C t^{-\frac{\sigma}{s}} \|w\|_{C(\overline{\Omega})} \end{aligned}$$

for a positive constant C depending in σ, s and Ω. Here we used the fact that $T_{s,1}(\cdot)$ is bounded (see page 262 of Yosida [50]). Let Y denote either $L^2(\Omega)$ or $C(\overline{\Omega})$, when $\sigma = s$, it is worth mentioning that the estimate of $\|(-\Delta_N)^s e^{-(-\Delta_N)^s t} w\|_Y$ is already established in [50] where it was shown that

$$\left\| \frac{d}{dt} e^{-(-\Delta_N)^s t} w \right\|_Y \le C t^{-1} \|w\|_Y,$$

where $C > 0$ is a constant. The proof is completed. □

3.1. Global existence and point dissipativity

We first establish the global existence of solutions and point dissipativity of the semiflow generated by problem (3.1).

Proposition 3.2. *Suppose that $(u_0, v_0) \in X$ with $u_0 \geq 0$ and $v_0 \geq 0$. Then there exists a unique global positive solution $(u(t, u_0, v_0), v(t, u_0, v_0))$ to (3.1) such that $u(0, u_0, v_0)(x) = u_0(x)$ and $v(0, u_0, v_0)(x) = v_0(x)$. Moreover, $u(t, u_0, v_0) > 0$ and $v(t, u_0, v_0) > 0$ on $\overline{\Omega}$ for each $t > 0$ if $u_0 \gneq 0$ and $v_0 \gneq 0$. In particular,*

$$\limsup_{t\to\infty} |u(t, u_0, v_0)|_\infty \leq C, \quad \limsup_{t\to\infty} |v(t, u_0, v_0)|_\infty \leq C,$$

where $C > 0$ is a constant depending on d_u, d_v, s_i $(i = 1, 2), N,$ and a, p, q.

Proof. We shall first obtain the local existence and uniqueness of a positive solution with initial data (u_0, v_0). For the sake of clearance, write again

$$f^l(x, u) = [a(x) - p(x) - b(x)u]u, \quad f^r(x, u, v) = [a(x) - b(x)u]u + q(x)v$$

and

$$f^c(x, u, v) = \left[a(x) - b(x)u - \frac{p(x)v}{u + v}\right]u + q(x)v, \quad g(x, u, v) = \left[\frac{p(x)u}{u + v} - q(x)\right]v.$$

Notice that the (mild) solution of (3.1) with initial data (u_0, v_0) is given by

$$\begin{cases} u(t, x) = e^{-(-d_u\Delta_N)^{s_1}t}u_0(x) + \int_0^t e^{-(-d_u\Delta_N)^{s_1}(t-\tau)}f^c(x, u, v)d\tau, \\ v(t, x) = e^{-(-d_v\Delta_N)^{s_2}t}v_0(x) + \int_0^t e^{-(-d_v\Delta_N)^{s_2}(t-\tau)}g(x, u, v)d\tau. \end{cases}$$

Thus, the local existence and uniqueness of $(u(t, u_0, v_0), v(t, u_0, v_0))$ follow from the contraction mapping theorem.

We next prove the positivity of $(u(t, u_0, v_0), v(t, u_0, v_0))$ given that $u_0 \gneq 0$ and $v_0 \gneq 0$. To this end, let $(u_l(t, u_0, v_0), v_l(t, u_0, v_0))$ and $(u_r(t, u_0, v_0), v_r(t, u_0, v_0))$ be the solutions of

$$(L) \quad \begin{cases} u_t + (-d_u\Delta_N)^{s_1}u = f^l(x, u), \\ v_t + (-d_v\Delta_N)^{s_2}v = g(x, u, v), \\ u(t, x) = u_0(x), \ v(t, x) = v_0(x); \end{cases} \qquad (R) \quad \begin{cases} u_t + (-d_u\Delta_N)^{s_1}u = f^r(x, u, v), \\ v_t + (-d_v\Delta_N)^{s_2}v = g(x, u, v); \\ u(t, x) = u_0(x), \ v(t, x) = v_0(x), \end{cases}$$

respectively. Both systems are cooperative as $\partial_v f^l = 0$, $\partial_v f^r > 0$, and $\partial_u g \geq 0$ in X^+, where $X^+ := \{(u, v) \in X \mid u \geq 0, v \geq 0\}$. The local existence and uniqueness of these two solutions are again established by the contraction mapping theorem. In addition, thanks to the strong positivity of $e^{-(-d\Delta_N)^s t}\ |_{t>0}$ and the comparison principle (see Lemma 2.1 of Chapter 8 of Wu [47]), we have

$$(0, 0) < (u_l(t, u_0, v_0), v_l(t, u_0, v_0)) \leq (u(t, u_0, v_0), v(t, u_0, v_0)) \leq (u_r(t, u_0, v_0), v_r(t, u_0, v_0)) \tag{3.3}$$

for $x \in \overline{\Omega}$ over the maximal interval of existence shared by all three solutions. It is not difficult to see that $(u(t, u_0, v_0), v(t, u_0, v_0))$ is a global solution. In fact, set $\tilde{u}_r = m$ and $\tilde{v}_r = mC$, where m and C are two positive constants satisfying $\sup_{x\in\Omega} p(x) \leq C \inf_{x\in\Omega} q(x)$ and $m > 1$ as in the proof of Theorem 2.6. It is a simple matter to verify that

$$\partial_t\tilde{u}_r + (-d_u\Delta_N)^{s_1}\tilde{u}_r \geq f^r(x, \tilde{u}_r, \tilde{v}_r), \quad \partial_t\tilde{v}_r + (-d_v\Delta_N)^{s_2}\tilde{v}_r \geq g(x, \tilde{u}_r, \tilde{v}_r), \quad (t, x) \in \mathbb{R}^+ \times \overline{\Omega}.$$

Let m be so chosen that $|u_0|_\infty \le \tilde{u}_r$ and $|v_0|_\infty \le \tilde{v}_r$. Then, the comparison principle implies that

$$(u_l(t,u_0,v_0), v_l(t,u_0,v_0)) \le (u(t,u_0,v_0), v(t,u_0,v_0)) \le (u_r(t,u_0,v_0), v_r(t,u_0,v_0)) \le (\tilde{u}_r, \tilde{v}_r).$$

Thus, it follows from the continuation of solution that $(u(t,u_0,v_0), v(t,u_0,v_0))$ is a global solution.

We next show the second part. As the arguments are standard, only a sketch is given. We start to show that there exists $0 < \theta < \alpha < 1$ such that

$$u(t,u_0,v_0) \in C^{0,2s_1+2\theta}([\varepsilon,\infty)\times\overline{\Omega}), \quad u_t(t,u_0,v_0) \in C([\varepsilon,\infty)\times\overline{\Omega})$$

for any $\varepsilon > 0$, where α is given in (H1). If $s_1 \ge \frac{1}{2}$, then $u(t) \in C_N^{2s_1+2\theta}(\overline{\Omega})$ for each $t \ge \varepsilon$. Likewise,

$$v(t,u_0,v_0) \in C^{0,2s_2+2\theta}([\varepsilon,\infty)\times\overline{\Omega}), \quad v_t(t,u_0,v_0) \in C([\varepsilon,\infty)\times\overline{\Omega}).$$

To this end, select $0 < \sigma < s_1$ such that $\sigma + \sigma \wedge \frac{\alpha}{4} \wedge s_2 > s_1$. By using the first equation of (3.1) and Lemma 3.1, we find that, for any $t > 0$,

$$\begin{aligned}\|(-d_u\Delta_N)^\sigma u\|_{C(\overline{\Omega})} &\le C\Big[t^{-\frac{\sigma}{s_1}}\|u_0\|_{C(\overline{\Omega})} + \|f^c\|_{C(\overline{\Omega})}\int_0^t (t-\tau)^{-\frac{\sigma}{s_1}}d\tau\Big] \\ &\le C(t^{-\frac{\sigma}{s_1}} + t^{1-\frac{\sigma}{s_1}}),\end{aligned}$$

where $C > 0$ depends on s_1, σ, d_u, $|u|_\infty, |v|_\infty$, and $|f^c|_\infty$. In view of Proposition 2.2.15 of Lunardi [28], $u \in C^{2\sigma}(\overline{\Omega})$. Assume without loss of generality that $s_i \ge \frac{\alpha}{4}$, (H1) and Proposition A.1 show that $f^c \in \mathrm{dom}(-d_u\Delta_N)^{\frac{\alpha}{4}}$. Employing Lemma 3.1 again gives that

$$\begin{aligned}\|(-d_u\Delta_N)^{\sigma+\frac{\alpha}{4}} u\|_{C(\overline{\Omega})} &\le C\Big[t^{-(\frac{\sigma}{s_1}+\frac{\alpha}{4s_1})}\|u_0\|_{C(\overline{\Omega})} + \|(-d_u\Delta_N)^{\frac{\alpha}{4}} f^c\|_{C(\overline{\Omega})}\int_0^t (t-\tau)^{-\frac{\sigma}{s_1}}d\tau\Big] \\ &\le C(t^{-(\frac{\sigma}{s_1}+\frac{\alpha}{4s_1})} + t^{1-\frac{\sigma}{s_1}})\end{aligned}$$

for some $C > 0$. Let $\beta = \sigma + \frac{\alpha}{4} - s_1$. Then $u(t) \in C^{2s_1+2\beta}(\overline{\Omega})$. In particular, $u(t) \in C_N^{2s_1+2\beta}(\overline{\Omega})$ if $s_1 \ge \frac{1}{2}$. Proposition A.1 implies that $(-d_u\Delta_N)^{s_1}u(t) \in C(\overline{\Omega})$. Then the interpolation theory gives rise to the desired estimates since $u_t = -(-d_u\Delta_N)^{s_1}u + f^c(x,u,v)$. The regularity of v and v_t follows from the same arguments.

We now are ready to prove the last part. Given any $k \ge 1$, note that

$$\frac{1}{k+1}\frac{d}{dt}\int_\Omega v^{k+1}dx + \frac{4d_v^{s_2}c_*k}{(k+1)^2}\|v^{\frac{k+1}{2}}\|_{H^{s_2}(\Omega)}^2 \le \int_\Omega \big[p(x) + 4d_v^{s_2}c_*\big]v^{k+1}dx,$$

where $c_* > 0$ is the constant given in (1.4). Write $w = v^{\frac{k+1}{2}}$. Then the Sobolev embedding theorem implies that

$$\frac{d}{dt}\int_\Omega w^2dx + \frac{kr}{k+1}\int_\Omega w^{2s_2^*}dx \le (k+1)\int_\Omega \big[p(x) + 4d_v^{s_2}c_*\big]w^2dx,$$

where $r > 0$ is a constant depending on $s_2, c_*, d_v^{s_2}, N$ only, and $2s_2^* = \frac{2N}{N-2s_2}$. Similar to the proof of Lemma 6.1.1 of Cholewa and Dlotko [15], it follows from the Bernoulli inequality that

$$\limsup_{t\to\infty} |v(t,u_0,v_0)|_\infty \le C,$$

where $C > 0$ depends on $s_2, c_*, d_v^{s_2}, N$, and p only. Likewise, we have

$$\frac{1}{k+1}\frac{d}{dt}\int_\Omega u^{k+1}dx + \frac{4d_u^{s_1}c_*k}{(k+1)^2}\|u^{\frac{k+1}{2}}\|^2_{H^{s_1}(\Omega)} \le \int_\Omega [a(x) + 4d_u^{s_1}c_*]u^{k+1}dx + \int_\Omega q(x)vu^k dx.$$

Hence, the same reasoning shows that

$$\limsup_{t\to\infty} |u(t, u_0, v_0)|_\infty \le C,$$

where $C > 0$ depends on $s_1, c_*, d_u^{s_1}, N$, and a, q only. The proof is completed. □

3.2. Stability of the disease-free equilibrium

Proposition 3.2 implies that the semiflow $(u(t, u_0, v_0), v(t, u_0, v_0))$ generated by (3.1) is point dissipative in X^+, where $X^+ := \{(u, v) \in X \mid u \ge 0, v \ge 0\}$, and $X = C(\overline{\Omega})\times C(\overline{\Omega})$. This along with the compactness of the trajectory of $(u(t, u_0, v_0), v(t, u_0, v_0))$ suggest the existence of a global attractor. The next two theorems (Theorems 3.3 and 3.5) specify what the global attractor is under certain conditions.

Theorem 3.3. *Suppose that all assumptions of Proposition 2.5 are satisfied. Assume that $\mathscr{R}_0 < 1$. Then the disease-free equilibrium $(u_1, 0)$ is globally asymptotic stable, where u_1 is the unique positive solution of (2.4).*

Proof. As the first step, we show that $\lim_{t\to\infty} \|v(t, u_0, v_0)\|_{C(\overline{\Omega})} = 0$. Thanks to (3.3), $\frac{u}{u+v} \le 1$. Thus,

$$v_t + (-d_v\Delta_N)^{s_2}v \le [p(x) - q(x)]v.$$

Let ψ be the positive eigenfunction corresponding to $\lambda_{d_v,p-q}$ with $\inf_{x\in\Omega}\psi \ge 1$. Let C be the constant obtained in Proposition 3.2 and set $\xi = C_v e^{-\lambda_{d_v,p-q}t}\psi$, where $C_v = C \vee |v_0|_\infty$. Then

$$\xi_t + (-d_v\Delta_N)^{s_2}\xi = [p(x) - q(x)]\xi.$$

Subsequently, it follows from the comparison principle that

$$0 < v(t, u_0, v_0) \le C_v e^{-\lambda_{d_v,p-q}t}\psi, \quad x \in \overline{\Omega}.$$

Since $\mathscr{R}_0 < 1$, $\lambda_{d_v,p-q} > 0$. This confirms that $\lim_{t\to\infty} \|v(t, u_0, v_0)\|_{C(\overline{\Omega})} = 0$.

Now it remains to show that $u(t, u_0, v_0)$ converges to u_1 as $t \to \infty$. Since $u_1 > 0$ on $\overline{\Omega}$ and $v(t, u_0, v_0) \to 0$ as $t \to \infty$, given any $0 < \delta < 1$, there exists $T_\delta > 0$ such that

$$[a(x) - b(x)u]u - \delta b(x)u_1^2 \le u_t + (-d_u\Delta_N)^{s_1}u \le [a(x) - b(x)u]u + \delta b(x)u_1^2, \quad t \ge T_\delta.$$

Notice that there exists a unique positive solution $w_{\delta\pm}$ to each of the following equations

$$(-d_u\Delta_N)^{s_1}w = [a(x) - b(x)w]w \pm \delta b(x)u_1^2, \tag{3.4}$$

respectively, as long as $0 < \delta < \frac{1}{4}$. As a matter of fact, set $\epsilon^- = \frac{1-\sqrt{1-4\delta}}{2}$, $w_{\delta-} = (1 - \epsilon^-)u_1$; and $\epsilon^+ = \frac{\sqrt{1+4\delta}-1}{2}$, $w_{\delta+} = (1 + \epsilon^+)u_1$. Direct calculation shows that

$$(-d_u\Delta_N)^{s_1}w_{\delta-} = [a(x) - b(x)u_1]w_{\delta-} = [a(x) - b(x)w_{\delta-}]w_{\delta-} - \epsilon^-(1 - \epsilon^-)b(x)u_1^2$$

and

$$(-d_u\Delta_N)^{s_1}w_{\delta+} = [a(x) - b(x)u_1]w_{\delta+} = [a(x) - b(x)w_{\delta+}]w_{\delta+} + \epsilon^+(1+\epsilon^+)b(x)u_1^2.$$

Namely, $w_{\delta\pm}$ are positive solutions of (3.4)$_\pm$.

We invoke the implicit function theorem to prove the uniqueness of $w_{\delta\pm}$. To do so, let $m = 2\sup_{x\in\Omega}\frac{a(x)}{b(x)}$ and define

$$j(x,\tau) = \begin{cases} a(x)\tau, & \tau \le 0, \\ a(x)\tau - b(x)\tau^2, & 0 \le \tau \le m; \\ b(x)m^2 + a(x)\tau - 2b(x)m\tau, & \tau \ge m. \end{cases}$$

Hence, $j \in C^{\alpha,1}(\overline{\Omega}\times\mathbb{R})$. Also let $\mathscr{F} : H^s(\Omega)\times\mathbb{R} \to H^{-s}(\Omega)$ be defined by

$$\langle \mathscr{F}(w,\delta) \mid v\rangle = \langle(-d_w\Delta_N)^{s_1}w \mid v\rangle - \langle j(x,w) + \delta b(x)u_1^2, v\rangle, \quad w, v \in H^s(\Omega).$$

Note that $\mathscr{F} \in C^1(H^s(\Omega)\times\mathbb{R})$. In addition,

$$\mathscr{F}(u_1, 0) = 0, \quad \partial_w\mathscr{F}(u_1,0) = (-d_u\Delta_N)^{s_1} - [a(x) - 2b(x)u_1]I.$$

Since $\lambda_{d_u,a-2bu_1} > 0$, $\langle\partial_w\mathscr{F}(u_1,0)w \mid w\rangle \ge \lambda_{d_u,a-2bu_1}\|w\|^2_{L^2(\Omega)}$, it follows that $\partial_u\mathscr{F}(u_1,0)$ is invertible. Therefore, the implicit function theorem implies that $\mathscr{F}(w,\delta) = 0$ has a unique solution for $\delta \in (-\varepsilon,\varepsilon)$ provided that $\varepsilon > 0$ is sufficiently small. Obviously, $w_{\delta\pm}$ are two solutions of $\mathscr{F}(w,\delta) = 0$ when δ is sufficiently small as $j(x,\tau) = (a(x) - b(x)\tau)\tau$ if $0 \le \tau \le m$. So the uniqueness of $w_{\delta\pm}$ follows.

Finally consider

$$\begin{cases} w_t + (-d_u\Delta_N)^{s_1}w = [a(x) - b(x)w]w - \delta b(x)u_1^2, & (t,x) \in (t_0,\infty)\times\overline{\Omega}, \\ w(t_0,\cdot) = u(T_\delta, u_0, v_0). \end{cases}$$

In terms of the theory of monotone dynamical systems, both $w_{\delta\pm}$ are globally asymptotic stable. Hence, there exists $T_{\epsilon^-} > 0$ for which $w(t,t_0) \ge w_{\delta^-} - \epsilon^-|u_1|_\infty$ if $t \ge T_{\epsilon^-} + T_\delta$. Meanwhile, the comparison principle shows that $u(t,u_0,v_0) \ge w(t,t_0)$ for $t \ge T_\delta$. Thus, if $t \ge T_{\epsilon^-} + T_\delta$, then $u_1 - 2\epsilon^-|u_1|_\infty \le u(t,u_0,v_0)$. For the same reason, $u(t,u_0,v_0) \le u_1 + 2\epsilon^+|u_1|_\infty$. In view of the setups of $\epsilon^\pm$, given any $\varepsilon > 0$, we have $|u(t,u_0,v_0) - u_1|_\infty \le \varepsilon$ if t is sufficiently large. This ends the proof. □

The following theorem accounts for the stability of $(u_1, 0)$ when $\mathscr{R}_0 = 1$. It shows that $(u_1, 0)$ is still stable as long as $\langle v_0, \phi_1\rangle$ is sufficiently small, where ϕ_1 is a positive eigenfunction associated with $\lambda_{d_v,p-q}$.

Theorem 3.4. *Suppose that all assumptions of Proposition 2.5 are satisfied. Assume that $\mathscr{R}_0 = 1$. Then, given any $\varepsilon > 0$, there exist $\delta > 0$ and $T > 0$ depending on ε such that*

$$\|u(t,u_0,v_0) - u_1\|_{C(\overline{\Omega})} + \|v(t,u_0,v_0)\|_{C(\overline{\Omega})} \le \varepsilon$$

whenever $\langle v_0,\phi_1\rangle \le \delta$ and $t \ge T$, where ϕ_1 is the positive eigenfunction corresponding to $\lambda_{d_v,p-q}$ satisfying $\|\phi_1\|_{L^2(\Omega)} = 1$.

Proof. Given any $\delta > 0$ such that $\langle v_0, \psi_1\rangle \le \delta$, we first show that $|v(t,u_0,v_0)|_\infty \le C_1\delta$ when t is sufficiently large, where $C_1 > 0$ is a constant depending on ϕ_1. Since $\mathscr{R}_0 = 1$, (2.3) says that $\lambda_{d_v,p-q} = 0$. Denote $(-d_v\Delta_N)^{s_2} - (p-q)I$ by A and consider A as a linear operator on $L^2(\Omega)$ with $\mathrm{dom}(A) \subset L^2(\Omega)$. In view of the proof of Lemma A.8, $\lambda I - A$ is invertible if $\lambda < 0$ and $|\lambda|$ is sufficiently large. In particular, $(\lambda I - A)^{-1}$ is compact. Namely, A has compact resolvent. Hence, the self-adjointness of A implies that $\sigma(A) \setminus \{\infty\} = \sigma_p(A) = \{\nu_1\}_{k=1}^{\infty}$, where $\nu_k \in \mathbb{R}^+$, $\nu_k \le \nu_{k+1}$, and $\nu_1 = \lambda_{d_v,p-q} = 0$. Moreover, ν_k are characterized by the min-max formula

$$\nu_1 = \inf_{w \in H^s(\Omega), \|w\|_{L^2(\Omega)}=1} \langle (-d_v\Delta_N)^{s_2} w \mid w\rangle + \langle (q-p)w, w\rangle,$$

$$\nu_k = \inf_{w \in X_{k-1}^{\perp}, \|w\|_{L^2(\Omega)}=1} \langle (-d_v\Delta_N)^{s_2} w \mid w\rangle + \langle (q-p)w, w\rangle, \quad k \ge 2,$$

where $X_k = \bigoplus_{i=1}^{k} \ker(A - \nu_i I)$ and each ν_k has finite geometric multiplicity. Now we view A as a linear operator on $C(\overline{\Omega})$ with $\mathrm{dom}(A) \subset C(\overline{\Omega})$. The spectrum $\sigma_p(A)$ remains the same. In particular, $\nu_1 = \lambda_{d_v,p-q}$ is a simple pole of $(\nu I - A)^{-1}$. Since $\nu_1 = 0$, in terms of Proposition A.2.2 of [28], we have $C(\overline{\Omega}) = \ker(-A) \oplus \mathrm{range}(-A)$. Note that $\mathrm{range}(-A) = \{w \in C(\overline{\Omega}) \mid \langle w, \phi_1\rangle = 0\}$. According to Proposition 5.3.2 of Carracedo and Alix [13], A is sectorial. Let P be the spectral projection associated with $\{0\}$, then Corollary 2.3.5 of [28] shows that

$$e^{-At}w = \langle w, \phi_1\rangle\phi_1 + e^{-At}(I-P)w, \quad \|e^{-At}(I-P)w\|_{C(\overline{\Omega})} \le C_\epsilon e^{-(\nu_2-\epsilon)t}\|w\|_{C(\overline{\Omega})},$$

where $w \in C(\overline{\Omega})$, $0 < \epsilon < \nu_2$ and $C_\epsilon > 0$ are two constants. Now let $\zeta = e^{-At}v_0$. Clearly, ξ solves $\zeta_t + (-d_v\Delta_N)^{s_2}\zeta = [p(x) - q(x)]\zeta$. By the comparison principle again, we have $v(t, u_0, v_0) \le \zeta$. In view of the estimate given above, it is easy to see that $|v(t, u_0, v_0)|_\infty \le C_1\delta$ for some positive constant C_1 as long as t is sufficiently large. Due to the arbitrariness of δ, given any $\varepsilon > 0$, by employing the arguments used in Theorem 3.3, we can reach the desired conclusion. The proof is completed. □

3.3. Stability of the endemic equilibrium

Our final result is about the stability of a homogeneous endemic equilibrium in system (3.1). We thereafter assume that a, b, p, and q are constants, and $q < p < a$. Under these assumptions, one can construct a Lyapunov function as done in Li et al. [26].

Theorem 3.5. *Suppose that a, b, p, and q are constants, and $q < p < a$. Then system (3.1) has a unique endemic equilibrium $(u^*, v^*) = \left(\frac{a}{b}, \frac{a(p-q)}{bq}\right)$ which is globally asymptotic stable.*

Proof. As in [26], let $V(u(t), v(t))$ be given by

$$V(u(t), v(t)) = \int_\Omega [u - u^* \log u + v - v^* \log v] dx.$$

In terms of the fact that $\langle (-d\Delta_N)^s w \mid 1\rangle = 0$, a straightforward calculation shows that

$$\begin{aligned}\frac{dV}{dt} &= u^*\langle (-d_u\Delta_N)^{s_1} u \mid u^{-1}\rangle + v^*\langle (-d_v\Delta_N)^{s_2} v \mid v^{-1}\rangle \\ &\quad + \int_\Omega \left[(u-u^*)(a-bu) - v\left(\frac{pu}{u+v}\right)\left(\frac{v^*}{v} - \frac{u^*}{u}\right)\right] dx \\ &\le -\int_\Omega \left[b(u-u^*)^2 + \frac{p(uv^* - u^*v)^2}{u(u+v)(u^*+v^*)}\right] dx \le 0.\end{aligned}$$

Here we used the fact that $\langle(-d\Delta_N)^s w \mid w^{-1}\rangle \le 0$. It then follows from the theory of dynamical systems that

$$\lim_{t\to\infty}\big(\|u-u^*\|_{L^2(\Omega)}+\|v-v^*\|_{L^2(\Omega)}\big)=0.$$

We next show that

$$|u-u^*|_\infty+|v-v^*|_\infty \le C\big(\|u-u^*\|_{L^2(\Omega)}+\|v-v^*\|_{L^2(\Omega)}\big)$$

for some positive constant $C>0$. This will complete the proof. To this end, write

$$c_{11}(t,x)=\int_0^1 f_u^c(x,\tau\mathbf{w}+(1-\tau)\mathbf{w}^*)d\tau,\quad c_{12}(t,x)=\int_0^1 f_v^c(x,\tau\mathbf{w}+(1-\tau)\mathbf{w}^*)d\tau,$$

$$c_{21}(t,x)=\int_0^1 g_u(x,\tau\mathbf{w}+(1-\tau)\mathbf{w}^*)d\tau,\quad c_{22}(t,x)=\int_0^1 g_v(x,\tau\mathbf{w}+(1-\tau)\mathbf{w}^*)d\tau,$$

where $\mathbf{w}=(u,v)$ and $\mathbf{w}^*=(u^*,v^*)$. Due to Proposition 3.2, $c_{i,j}\in L^\infty(\mathbb{R}^+\times\Omega),\ 1\le i,j\le 2$. Accordingly we rewrite (3.1) as

$$\begin{cases}(u-u^*)_t+(-d_u\Delta_N)^{s_1}(u-u^*)=c_{11}(t,x)(u-u^*)+c_{12}(t,x)(v-v^*),\\ (v-v^*)_t+(-d_v\Delta_N)^{s_2}(v-v^*)=c_{21}(t,x)(u-u^*)+c_{22}(t,x)(v-v^*).\end{cases}$$

In what follows, for the sake of clearance and simplicity, we assume without loss of generality that $d_u=d_v=1$ and $c_*=1$. Let $w=u-u^*$, and $m\ge 2$ be even. Then by multiplying both sides of the first equation of the system by w^{m-1} and integrating the resulting equation over Ω, we obtain

$$\begin{aligned}\frac{1}{m}\frac{d}{dt}\int_\Omega w^m dx+\frac{4(m-1)}{m^2}\|w^{\frac{m}{2}}\|^2_{H^{s_1}(\Omega)} &\le \int_\Omega |c_{11}+1|w^m dx+\int_\Omega |c_{22}||v-v^*||w|^{m-1}dx\\ &\le C\Big[\int_\Omega w^m dx+\Big(\int_\Omega |v-v^*|^m dx\Big)^{\frac{1}{m}}\Big(\int_\Omega w^m dx\Big)^{\frac{m-1}{m}}\Big]\\ &\le C\Big[\int_\Omega w^m dx+\Big(\int_\Omega w^m dx\Big)^{\frac{m-1}{m}}\Big],\end{aligned}$$

where $C>0$ depends on c_{11},c_{12}, $|v-v^*|_\infty$, and $|\Omega|$. Let $\xi=w^{\frac{m}{2}}$, then

$$\frac{d}{dt}\int_\Omega \xi^2 dx+\frac{4(m-1)}{m}\|\xi\|^2_{H^{s_1}(\Omega)}\le Cm\big(\|\xi\|^2_{L^2(\Omega)}+\|\xi\|^{\frac{2(m-1)}{m}}_{L^2(\Omega)}\big). \tag{3.5}$$

By the Sobolev embedding theorem and interpolation inequality, we have

$$\|\xi\|_{L^2(\Omega)}\le C_{s_1}\|\xi\|^\theta_{H^{s_1}(\Omega)}\|\xi\|^{1-\theta}_{L^1(\Omega)},\quad \theta=\frac{N-2s_1}{N+2s_1}.$$

Here we already assumed that $N \geq 2$ or $N = 1 > 2s_1$. It follows from the Young inequality that

$$Cm\|\xi\|^2_{L^2(\Omega)} \leq \frac{2(m-1)}{m}\|\xi\|^2_{H^{s_1}(\Omega)} + \left[\frac{Cm^2\theta}{2(m-1)}\right]^{\frac{1-\theta}{\theta}}\|\xi\|^2_{L^1(\Omega)}.$$

Likewise,

$$Cm\|\xi\|^{\frac{2(m-1)}{m}}_{L^2(\Omega)} \leq \frac{m-1}{m}\|\xi\|^2_{H^{s_1}(\Omega)} + \left[\frac{Cm\theta}{2}\right]^{\frac{(m-1)\theta}{m(1-\theta)+\theta}}\|\xi\|^{2\theta_m}_{L^1(\Omega)},$$

where $\theta_m = \frac{m(1-\theta)-(1-\theta)}{m(1-\theta)+\theta}$. By inserting these two inequalities into (3.5), we find that

$$\frac{d}{dt}\int_\Omega \xi^2 dx + \frac{(m-1)}{m}\|\xi\|^2_{H^{s_1}(\Omega)} \leq \left\{\left[\frac{Cm^2\theta}{2(m-1)}\right]^{\frac{1-\theta}{\theta}} + \left[\frac{Cm\theta}{2}\right]^{\frac{(m-1)\theta}{m(1-\theta)+\theta}}\right\}\left(\|\xi\|^2_{L^1(\Omega)} + \|\xi\|^{2\theta_m}_{L^1(\Omega)}\right).$$

Thus, Lemma 1.2.5 of Cholewa and Dlotko [15] shows that

$$\int_\Omega \xi^2 dx \leq C_\theta m^\mu\left(\|\xi\|^2_{L^1(\Omega)} + \|\xi\|^{2\theta_m}_{L^1(\Omega)}\right),$$

where $\mu = \frac{4s_1}{N-2s_1} \vee 2$. Set $m = 2^k,\ k = 1, 2, \cdots$, then

$$\int_\Omega w^{2^k} dx \leq C_\theta 2^{k\mu}\left[\left(\int_\Omega w^{2^{k-1}} dx\right)^2 + \left(\int_\Omega w^{2^{k-1}} dx\right)^{2\theta_k}\right],$$

where $\theta_k = \frac{2^k(1-\theta)-(1-\theta)}{2^k(1-\theta)+\theta}$. Given any $t_0 > 0$ sufficiently large, let $j_k = \sup_{t\geq t_0}\|w\|_{L^{2^k}(\Omega)}$. Then, we have

$$j_{k+1} \leq (C_\theta 2^{k\mu})^{\frac{1}{2^k}}\left[j_k^{2^k} + j_k^{2^k\theta_{k+1}}\right]^{\frac{1}{2^k}}.$$

Note that $\{j_k\}_{k=1}^\infty$ is bounded and $j_k \to |w|_\infty$, $\theta_k \to 1$ as $k \to \infty$. Hence, there exists $\kappa > 1$ such that $j_k^{2^k\theta_{k+1}} \leq \kappa j_k^{2^k}$. As a result, j_{k+1} is dominated by $j'_{k+1} = [C_\theta(1+\kappa)2^{k\mu}]^{\frac{1}{2^k}} j'_k$ with $j'_1 = j_1$. As in the proof of Lemma 9.3.1 of [15], we have

$$\lim_{k\to\infty} j'_k = j_1\prod_{k=2}^\infty [C_\theta(1+\kappa)2^{k\mu}]^{\frac{1}{2^k}} := C_{1,\infty}.$$

Through the passage of the limit, we obtain

$$|u-u^*|_\infty \leq C_{1,\infty}\|u-u^*\|_{L^2(\Omega)}, \quad t \geq t_0.$$

Repeating the same argument for $v - v^*$ yields that

$$|v-v^*|_\infty \leq C_{2,\infty}\|v-v^*\|_{L^2(\Omega)}, \quad t \geq t_0$$

for a constant $C_{2,\infty} > 0$. Thus, the desired conclusion follows if $N \geq 2$ or $N = 1 > 2s_1$.

We now end the proof with a brief account for the cases that $N = 1 \leq 2s_1$ or ($N = 1 \leq 2s_2$). Under these circumstances, the conclusion can be reached via the regularity of $u - u^*$ (or $v - v^*$). Chose $0 < \sigma < s_1$, let $-\frac{\sigma}{s_1} = \theta - 1$, assume without loss of generality that $C \leq 1$, where C is given in Lemma 3.1, and write $\delta = \frac{\lambda_2^s}{4}$, then Lemma 3.1 implies that

$$\begin{aligned}
\|(-\Delta_N)^\sigma w\|_{L^2(\Omega)} &\leq e^{-\delta t} t^{-\frac{\sigma}{s_1}} \|w\|_{L^2(\Omega)} + \int_0^t e^{-\delta(t-\tau)}(t-\tau)^{-\frac{\sigma}{s_1}} \|h\|_{L^2(\Omega)} d\tau \\
&\leq e^{-\delta t} t^{-\frac{\sigma}{s_1}} \|w\|_{L^2(\Omega)} + \|h\|_{L^2(\Omega)} \int_0^\infty t^{\theta-1} e^{-\delta t} dt \\
&\leq e^{-\delta t} t^{-\frac{\sigma}{s_1}} \|w\|_{L^2(\Omega)} + C\|h\|_{L^2(\Omega)}, \quad t > 0.
\end{aligned}$$

Here $h = c_{11}(u - u^*) + c_{12}(v - v^*)$ and $C > 0$ depends on s_1, λ_2, θ. Since $s_1 \geq \frac{1}{2}$, then σ can be so chosen that $2\sigma > \frac{1}{2}$. Thus, the Sobolev embedding theorem shows that

$$|u - u^*|_\infty \leq C(\|u - u^*\|_{L^2(\Omega)} + \|v - v^*\|_{L^2(\Omega)})$$

for some positive constant C. The proof is completed. □

4. Numerical simulations

This section provides numerical simulations of solutions of (3.1) to illustrate the stabilities of both disease-free and endemic equilibria under the condition that all coefficients are constant. Regarding the asymptotic stability of the disease-free equilibrium, we assume that the spatial domain is either $\Omega = (-1, 1) \subset \mathbb{R}$ or $\Omega = (-1, 1) \times (-1, 1) \subset \mathbb{R}^2$, and the parameters are given as follows:

$$s_1 = s_2 = 0.6, \quad d_u = \sqrt[0.6]{0.8}, \quad d_v = \sqrt[0.6]{0.7}, \quad a = 2, \quad b = 1, \quad p = 0.3, \quad q = 10.$$

Then it follows from Proposition 2.2 and Theorem 3.3 that $\mathscr{R}_0 < 1$ and the disease-free equilibrium $(2, 0)$ is globally asymptotically stable (Figs. 4.1 and 4.2). The asymptotic stability of $(2, 0)$ is manifested in the solutions with initial data in $(-1, 1)$ or $(-1, 1) \times (-1, 1)$ given below:

$$\begin{aligned}
&u_0(x) = 1.001 + 0.05 \sin(5\pi x + 0.1), \quad x \in (-1, 1) \\
&v_0(x) = 0.5 + 0.02 \cos(3\pi x + 0.1) + 0.0005 e^{2x} \sin(\pi x), \quad x \in (-1, 1),
\end{aligned}$$

or

$$\begin{aligned}
&u_0(x, y) = 1.001 + 0.05 \sin(3\pi x + 0.1) \cos(5\pi y + 0.2), \quad (x, y) \in (-1, 1) \times (-1, 1) \\
&v_0(x, y) = 0.2 + 0.01 \sin(2\pi x + 0.1) \cos(4\pi y), \quad (x, y) \in (-1, 1) \times (-1, 1).
\end{aligned}$$

Concerning the asymptotic stability of the endemic equilibrium, we assume that the spatial domain is either $\Omega = (-1, 1) \subset \mathbb{R}$ or $\Omega = (-1, 1) \times (-1, 1) \subset \mathbb{R}^2$, and the parameters take the following values:

$$s_1 = s_2 = 0.6, \quad d_u = \sqrt[0.6]{0.8}, \quad d_v = \sqrt[0.6]{0.7}, \quad a = 2, \quad b = 1, \quad p = 0.3, \quad q = 0.2.$$

Clearly, (3.1) has an endemic equilibrium $(2, 1)$. In addition, Proposition 2.2 and Theorem 3.5 show that $(2, 1)$ is globally asymptotically stable (Figs. 4.3 and 4.4). The asymptotic stability of $(2, 1)$ is demonstrated by the convergence of the solutions with the same initial data in $(-1, 1)$ or $(-1, 1) \times (-1, 1)$ given above.

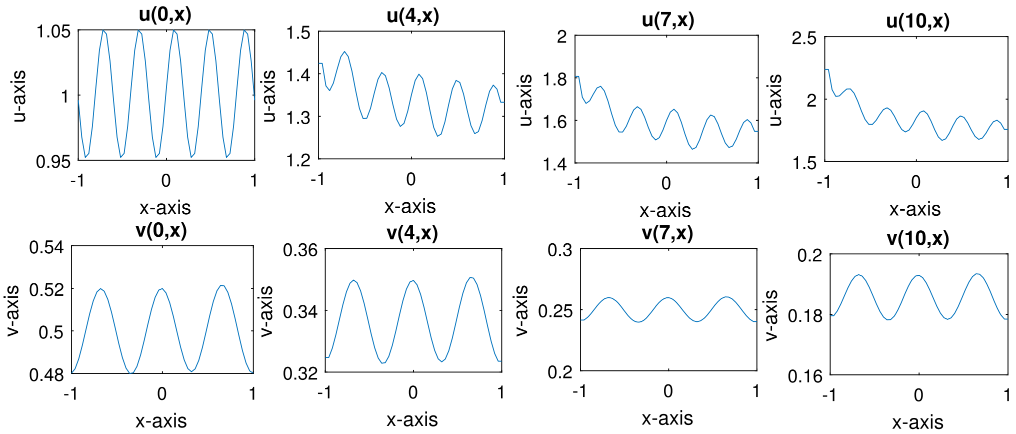

Fig. 4.1. $\mathscr{R}_0 < 1$. The snapshots of the solution $(u(t,x), v(t,x))$ of (3.1) in the spatial domain $(-1,1)$ at $t = 0, 4, 7, 10$, which converges to the disease-free equilibrium $(2,0)$.

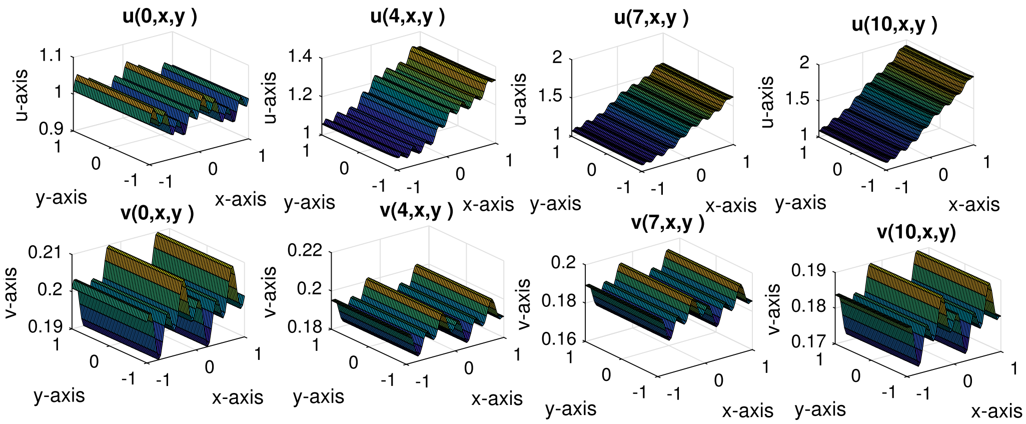

Fig. 4.2. $\mathscr{R}_0 < 1$. The snapshots of the solution $(u(t,x,y), v(t,x,y))$ of (3.1) in the spatial domain $(-1,1) \times (-1,1)$ at $t = 0, 4, 7, 10$, which converges to the disease-free equilibrium $(2,0)$.

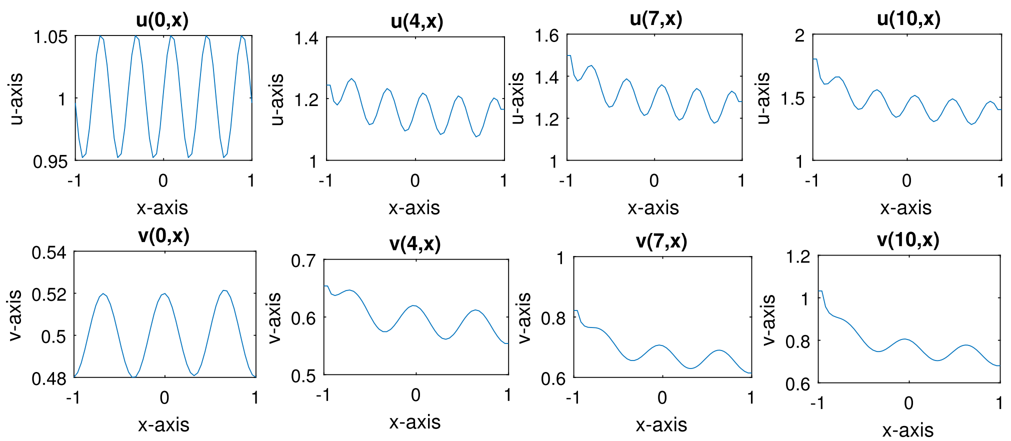

Fig. 4.3. $\mathscr{R}_0 > 1$. The snapshots of the solution $(u(t,x), v(t,x))$ of (3.1) in the spatial domain $(-1,1)$ at $t = 0, 4, 7, 10$, which converges to the endemic equilibrium $(2,1)$.

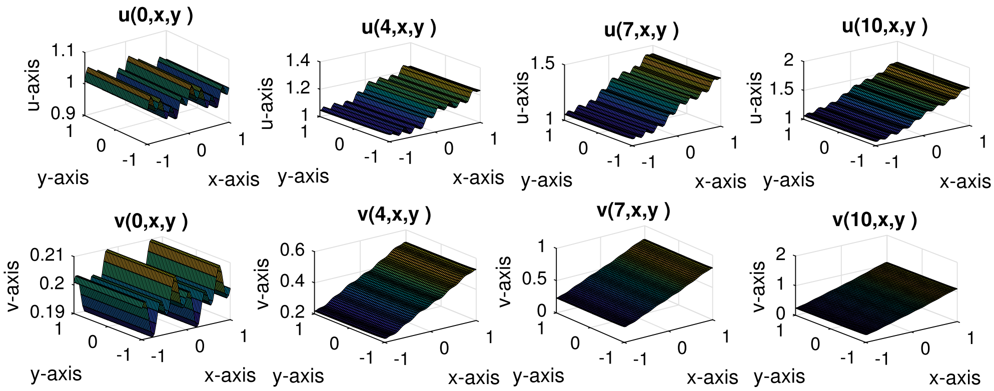

Fig. 4.4. $\mathscr{R}_0 > 1$. The snapshots of the solution $(u(t, x, y), v(t, x, y))$ of (3.1) in the spatial domain $(-1, 1) \times (-1, 1)$ at $t = 0, 4, 7, 10$, which converges to the endemic equilibrium $(2, 1)$.

5. Discussion

Spatial movement of the host is the most crucial factor for the geographic spread of infectious diseases. Classical reaction-diffusion equations have been used extensively to model the spatial spread of various infectious diseases (Murray [32], Ruan and Wu [38]) based on the fact that Laplace operators can be employed to describe the random walk of the host population. Note that the probability density of the continuous random walk is a Gaussian distribution and reaction-diffusion equations only describe the local spatial spread of infectious diseases.

In modern times, humans travel on many spatial scales ranging from a few kilometers to thousands of kilometers over short periods. In a series of recent studies, it was observed that mobility patterns for humans exhibit scale-free dynamics with heavier tails distribution, a characteristic of Lévy flights (Brockmann et al. [7], González et al. [21], Mandellbrot [30], Zaburdaev et al. [54]). Roughly speaking, around 80% of people travel in short distances (locally) and about 20% of people travel in long distances (non-locally). Interestingly, Brockmann et al. [7] commented that *"We believe that these results can serve as a starting point for developing a new class of models for the spread of human infectious diseases because universal features of human travel can now be accounted for in a quantitative way."* It seems that these observations about human travel patterns and Brockmann et al.'s message have not been well-received by the community of mathematical modelers of infectious diseases, and the ongoing COVID-19 pandemic has confirmed these observations: it is the small fraction of long-distance travelers who spread the virus from countries to countries and from continents to continents. The random Laplace diffusion certainly is not suitable to describe such long-distance geographic spread of the virus and fractional diffusion is a reasonable approach.

Since Lévy flights are drawn from a probability distribution function with heavier tails rather than a normal distribution, they are superdiffusive as they disperse particles faster than a Gaussian random walk and large displacements and long jumps are more likely. Motivated by these recent observations of human travel patterns and the comment of Brockmann et al. [7], in this paper, we proposed a susceptible-infectious-susceptible epidemic model with Lévy flights, i.e., fractional diffusion. By using our recent results on fractional diffusion equations (Zhao and Ruan [53]), we established the existence and the stabilities of disease-free and endemic equilibria and studied the impact of dispersal rates and fractional powers on spatial profiles of these equilibria. The basic reproduction number $\mathscr{R}_0$ was obtained and was used to investigate the effects of spatial heterogeneity on the transmission dynamics. It was also used to determine the existence and nonexistence of an epidemic equilibrium as well as stabilities of the disease-free and endemic equilibria. It was found that for low-risk regions both dispersal rates and fractional powers play a critical role and are capable of altering the threshold value. Numerical simulations were carried out to confirm the theoretical results.

To the best of our knowledge, this is the first piece of theoretical study on epidemic models with Lévy flight (fractional diffusion). One of our key assumptions is that the susceptible and infected individuals under consideration do not leave the region Ω. Hence, spectral fractional Neumann Laplacian was adopted to describe the underlying transport process. In case that the region outside of Ω is uninhabitable, then spectral fractional Dirichlet Laplacians with different fractional powers would be a natural choice for the diffusion operators in the model, and it is anticipated that most results obtained in this paper would still hold. Nonetheless, technical details are obviously needed in this regard. In addition, there are more interesting questions about such models that deserve further consideration, such as the existence of traveling waves, spatial and temporal patterns, calibration of geographic epidemic data, etc.

Acknowledgements

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Appendix A

This section contains a sequence of results established in Zhao and Ruan [53] that were frequently used in the present paper. For readers' convenience, these results are listed here without proofs. All proofs can be founded in [53].

Proposition A.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Suppose that $\frac{1}{2} \le s < 1$. Let $\theta > 0$ such that $1 < 2s + 2\theta < 2$. Set*

$$C_N^{2s+2\theta}(\overline{\Omega}) := \left\{ w \in C^{1,2s+2\theta-1}(\overline{\Omega}) \mid \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega \right\},$$
$$C_0^{2s+2\theta}(\overline{\Omega}) := \left\{ w \in C^{1,2s+2\theta-1}(\overline{\Omega}) \mid w = 0 \text{ on } \partial\Omega \right\}.$$

If $0 < s < \frac{1}{2}$ and $2s + 2\theta < 1$, then set

$$C_0^{2s+2\theta}(\overline{\Omega}) := \left\{ w \in C^{2s+2\theta}(\overline{\Omega}) \mid w = 0 \text{ on } \partial\Omega \right\}.$$

(i) *Assume that either $w \in C_N^{2s+2\theta}(\overline{\Omega})$ with $s \ge \frac{1}{2}$ and $2s+2\theta < 2$ or $w \in C^{2s+2\theta}(\overline{\Omega})$ with $0 < 2s+2\theta < 1$. Then, for any $0 \le \alpha < \theta$,*

$$\|(-\Delta_N)^s w\|_{C^{2\alpha}(\overline{\Omega})} \le C\|w\|_{C^{2s+2\theta}(\overline{\Omega})}$$

for some positive constant C which depends on s, α, θ, and Ω only.

(ii) *Assume that $0 < s < 1$, $2s + 2\theta < 2$, and $w \in C_0^{2s+2\theta}(\overline{\Omega})$. Then, for any $0 \le \alpha < \theta$,*

$$\|(-\Delta_D)^s w\|_{C^{2\alpha}(\overline{\Omega})} \le C\|w\|_{C^{2s+2\theta}(\overline{\Omega})}$$

for some positive constant C which depends on s, θ, and Ω only.

Proposition A.2. *Let $u = ((-\Delta_N)^s + I)^{-1} g$ stand for the unique solution of*

$$\langle (-\Delta_N)^s u \mid \psi \rangle + \langle u, \psi \rangle = \langle g, \psi \rangle$$

where $\psi \in H^s(\Omega)$ is arbitrary. Then

(i) $u \in H^s(\Omega)$ *and*

$$\|u\|_{H^s(\Omega)} \leq C\|g\|_{L^2(\Omega)},$$

where $C > 0$ depends on s and Ω only;

(ii) *Suppose that $g \in L^p(\Omega)$, where $\frac{N}{2s} < p < \frac{N}{(2s-1)_+}$. Then $u \in C^\alpha(\overline{\Omega})$ for $\alpha = 2s - \frac{N}{p}$, and*

$$[u]_{C^\alpha(\overline{\Omega})} \leq C(\|g\|_{L^p(\Omega)} + \|g\|_{L^2(\Omega)});$$

(iii) *Suppose that $s > \frac{1}{2}$ and $g \in L^p(\Omega)$, where $p > \frac{N}{2s-1}$. Then $u \in C^{1,\alpha}(\overline{\Omega})$ for $\alpha = 2s - \frac{N}{p} - 1$, and*

$$[u]_{C^{1,\alpha}(\overline{\Omega})} \leq C(\|g\|_{L^p(\Omega)} + \|g\|_{L^2(\Omega)});$$

(iv) *Suppose $0 < s < \frac{1}{2}$ and $g \in C^\alpha(\overline{\Omega})$ for some $0 < \alpha < \frac{1}{2}$ such that $0 < 2s + \alpha < 1$ Then $u \in C^{2s+\alpha}(\overline{\Omega})$ and*

$$\|u\|_{C^{2s+\alpha}(\overline{\Omega})} \leq C\|g\|_{C^\alpha(\overline{\Omega})};$$

(v) *Suppose $s \geq \frac{1}{2}$ and $g \in C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$ such that $0 < 2s + 2\alpha < 2$. Then $u \in C_N^{2s+\alpha}(\overline{\Omega})$ and*

$$\|u\|_{C^{2s+\alpha}(\overline{\Omega})} \leq C\|g\|_{C^\alpha(\overline{\Omega})}.$$

Here the positive constants C in (ii)-(v) depend on s, N, Ω, and $\|g\|_{L^p(\Omega)}$ for $p > \frac{N}{2s}$.

Proposition A.3. *Assume that either $u \in H^s(\Omega)$ is a weak solution of $(-d\Delta_N)^s u = c(x)u + f$. Namely, $\langle(-d\Delta_N)^s u \mid \psi\rangle = \langle cu + f, \psi\rangle$ for any $\psi \in H^s(\Omega)$, or $u \in H^s(\Omega)$ is a weak sub-solution of $(-d\Delta_N)^s u = c(x)u + f$ with $u \geq 0$. That is, $\langle(-d\Delta_N)^s u \mid \psi\rangle \leq \langle cu + f, \psi\rangle$ for any $\psi \in H^s(\Omega)$ with $\psi \geq 0$, where $c \in L^\infty(\Omega)$ and $f \in L^p(\Omega)$ with $p > \frac{N}{2s}$. Then*

$$|u|_\infty \leq C(\|u\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)}),$$

where $C > 0$ is a constant depending on s, N, d; $|c|_\infty, \Omega$, and $\|f\|_{L^p(\Omega)}$ only.

Lemma A.4. *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain with $\partial\Omega \in C^k$ $(k \geq 2)$. Suppose that u_d is a non-negative function satisfying*

$$u_d \in \begin{cases} H^{2s}(\Omega) \cap C(\overline{\Omega}) & \text{if } \ 0 < s < \frac{3}{4}, \\ H_N^{2s}(\Omega) \cap C(\overline{\Omega}) & \text{if } \ \frac{3}{4} < s < 1. \end{cases}$$

In case that $s = \frac{3}{4}$, assume further that $u_d \in H_N^{2s+2\alpha}(\Omega) \cap C(\overline{\Omega})$ for some $0 < \alpha < 1$ with $s + \alpha < 1$. Furthermore, assume that

$$c_0(x)u_d \leq (-d\Delta_N)^s u_d \leq c_1(x)u_d, \quad x \in \Omega, \tag{A.1}$$

where $c_0, c_1 \in L^\infty(\Omega)$ with $c_0 \leq c_1$. Let $x \in \overline{\Omega}$ and $\delta > 0$. Then

$$\sup_{B_\delta(x)\cap\Omega} u_d \leq C \inf_{B_\delta(x)\cap\Omega} u_d, \tag{A.2}$$

where $C > 0$ is a constant depending on Ω, s, d, N; $\delta, |c_0|_\infty$, and $|c_1|_\infty$ only.

Proposition A.5. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. Given that $u \in H^{2s}(\Omega) \cap C(\overline{\Omega})$ with $0 < s < \frac{3}{4}$ or $u \in H_N^{2s}(\Omega) \cap C(\overline{\Omega})$ with $\frac{3}{4} < s < 1$. If $s = \frac{3}{4}$, assume further that $u \in H_N^{2s+2\alpha}(\Omega) \cap C(\overline{\Omega})$ for some $0 < \alpha < 1$ satisfying $s + \alpha < 1$. Then the following statements hold:*

(i) *Assume that u satisfies*

$$(-d\Delta_N)^s u \le f(x), \quad x \in \Omega,$$

where $f \in C(\overline{\Omega})$. Suppose that $u(x_0) = \max_{x \in \overline{\Omega}} u$ for some $x_0 \in \overline{\Omega}$. Then $f(x_0) \ge 0$.

(ii) *Assume that u satisfies that*

$$(-d\Delta_N)^s u \ge f(x), \quad x \in \Omega,$$

where $f \in C(\overline{\Omega})$. Suppose that $u(x_0) = \min_{x \in \overline{\Omega}} u$ for some $x_0 \in \overline{\Omega}$. Then $f(x_0) \le 0$.

We now gather a number of properties pertained to the principal eigenvalue problems associated with $(-\Delta_N)^s$:

$$(-d\Delta_N)^s w + \mu c(x) w = \lambda w, \tag{A.3}$$

where $d > 0$, $c \in L^\infty(\Omega)$, and $\mu \in \mathbb{R}$. These properties will be used in Sections 3 and 4.

Proposition A.6. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $d > 0$, $c \in L^\infty(\Omega)$, and $\mu \in \mathbb{R}$. Then (A.3) possesses a unique principal eigenvalue which is simple and is the least eigenvalue of $(-d\Delta_N)^s + \mu c(x) I$. The principal eigenvalue, denoted by $\lambda(d, s, \mu c)$, is given by*

$$\begin{aligned}\lambda(d, s, \mu c) &= \inf_{u \in H^s(\Omega), \|u\|_{L^2(\Omega)}=1} \int_\Omega \int_\Omega d^s K_{s,N}(x,y)|u(x) - u(y)|^2 dy dx + \int_\Omega \mu c(x) u^2 dx \\ &= \inf_{u \in H^s(\Omega), \|u\|_{L^2(\Omega)}=1} \langle (-d\Delta_N)^s u \mid u \rangle + \langle \mu c u, u \rangle,\end{aligned}$$

where $K_{s,N}(x,y)$ is given in (1.2). Then we have the following

(i) *$\lambda(d, s, \mu c)$ is analytic with respect to d, s, and μ. In particular, let $\dot{\lambda}_d$ and $\dot{\lambda}_s$ be the partial derivative with respect to d and s, respectively, then $\dot{\lambda}_d \ge 0$, and $\dot{\lambda}_d > 0$ if c is not a constant. Moreover, $\ddot{\lambda}_{\mu\mu} \le 0$, where $\ddot{\lambda}_{\mu\mu}$ is the second derivative of λ with respect to μ. In case that $d\lambda_2 \ge 1$, then $\dot{\lambda}_s \ge 0$.*

(ii) *$\lambda(d, s, \mu c) \le \mu \bar{c}$. In addition, $\lambda(d, s, \mu c) < \mu \bar{c}$ if c is a nonconstant function.*

(iii) *$\lim_{d \to \infty} \lambda(d, s, \mu c) = \mu \bar{c}$. In particular, $\lambda(d, s, \mu c) \to \mu \bar{c}$ uniformly for s in $[\eta, 1)$ as $d \to \infty$, where $0 < \eta < 1$.*

(iv) *Assume that $c_d \in C(\overline{\Omega})$ are a family of functions such that $\lim_{d \to 0^+} \|c_d - c\|_{C(\overline{\Omega})} = 0$, where $c \in C(\overline{\Omega})$. Then $\lim_{d \to 0^+} \lambda(d, s, \mu c_d) = \inf_{x \in \Omega} \mu c$, where $\lambda(d, s, \mu c_d)$ is the principal eigenvalue of $(-d\Delta_N)^s + \mu c_d I$. Moreover, $\lambda(d, s, \mu c_d) \to \inf_{x \in \Omega} \mu c$ uniformly for s in $[\eta, 1)$ as $d \to 0^+$, where $0 < \eta < 1$.*

(v) *If $\bar{c} > 0$ and $\inf_{x \in \Omega} c < 0$, then for each $(s, \mu) \in (0,1) \times \mathbb{R}^+$, $\lambda(d, s, \mu) = 0$ has a unique root $d_0(s, \mu)$ such that $\lambda(d, s, \mu) > 0$ for any $d > d_0$, and $\lambda(d, s, \mu) < 0$ for any $d < d_0$. In case that $d_0 \lambda_2 > 1$, then $\lambda(d, \sigma, \mu) > 0$ for any $d \ge d_0$ and $\sigma > s$. Moreover, $\lambda(d_0, \sigma, \mu) < 0$ for any $0 < \sigma < s$.*

Proposition A.7. *Suppose that $u_d, w_d \in H^{2s}(\Omega) \cap C(\overline{\Omega})$ with $0 < s < \frac{3}{4}$ and $0 < 2s + 2\alpha < \frac{3}{2}$ or $u_d, w_d \in H_N^{2s}(\Omega) \cap C(\overline{\Omega})$ with $\frac{3}{4} \le s < 1$ and $1 < 2s + 2\alpha < 2$ satisfy*

$$(-d\Delta_N)^s u_d \geq f(x, u_d), \quad (-d\Delta_N)^s w_d \leq f(x, w_d), \quad x \in \Omega,$$

where $0 < \alpha < 1$, $u_d \gneq 0$ *on* $\overline{\Omega}$, *and* $f \in C^{0,1}(\overline{\Omega} \times (\eta, \infty))$ *for some* $0 < \mu < 1$ *and* $\eta < 0$. *Furthermore,* $f(x, 0) = 0$ *for all* $x \in \overline{\Omega}$ *and* $f(x, \tau\theta) \lneq \tau f(x, \theta)$ *for all* $x \in \overline{\Omega}$ *as long as* $\theta > 0$ *and* $\tau > 1$. *Then* $u_d \geq w_d$ *on* $\overline{\Omega}$.

Concerning the existence of positive solutions to $(-d\Delta_N)^s u = f(x, u)$, $x \in \Omega$. The following assumptions will be used in the rest of this section.

(A1) $f \in C^{\mu,1}(\overline{\Omega} \times (\eta, \infty))$ for some $0 < \mu < 1$, $\eta < 0$, and $f(x, 0) = 0$ for all $x \in \overline{\Omega}$.

(A2) Let $h(x, \tau) = \begin{cases} \frac{f(x,\tau)}{\tau}, & \tau \neq 0 \\ f_\tau(x, 0), & \tau = 0, \end{cases}$ $h \in C^{\mu,1}(\overline{\Omega} \times (\eta, \infty))$ and $h_\tau(x, \tau) < 0$ in $\overline{\Omega} \times (\eta, \infty)$.

(A3) There exists a constant $m > 0$ such that $f(x, m) \leq 0$ for all $x \in \overline{\Omega}$.

(A4) There exists $\zeta \in C(\overline{\Omega})$ such that $\inf_{x \in \Omega} \zeta > \eta$, $h(x, \zeta(x)) = 0$ for all $x \in \overline{\Omega}$, and $\zeta_+ \gneq 0$, where $\zeta_+(x) = \zeta \vee 0 := \max\{\zeta(x), 0\}$.

Lemma A.8. *Suppose that (A1), (A2) and (A3) are fulfilled. Assume that* $\lambda_{d,h} < 0$, *where* $\lambda_{d,h}$ *is the principal eigenvalue of* $(-d\Delta_N)^s - h(x, 0)I$. *Then there exists a unique positive solution* $u_d \in C^{2s+2\alpha}(\overline{\Omega})$ *with* $2s + 2\alpha < 1$ *if* $0 < s < \frac{1}{2}$, *or* $u_d \in C_N^{2s+2\alpha}(\overline{\Omega})$ *with* $2s + 2\alpha < 2$ *if* $s \geq \frac{1}{2}$ *satisfying*

$$(-d\Delta_N)^s u = f(x, u), \quad x \in \Omega, \tag{A.4}$$

where $0 < \alpha < 1$.

Corollary A.9. *Suppose that*

$$g(x, \cdot, \cdot) \in C^{\mu,1}\big(\overline{\Omega} \times (\eta_1, +\infty) \times (\eta_2, +\infty)\big), \quad h(x, \cdot, \cdot) \in C^{\mu,1}\big(\overline{\Omega} \times (\eta_1, +\infty) \times (\eta_3, +\infty)\big),$$

where $0 < \mu < 1$, η_i $(i = 1, 2, 3)$ *are three constants, and* $\eta_2 < 0$. *In addition,* $\partial_3 g(x, \cdot, \cdot) > 0$ *on* $\overline{\Omega} \times (\eta_1, +\infty) \times [0, +\infty)$, $g(x, \cdot, 0) = 0$. *For any* $\theta \geq 1$ *and* $\tau \geq 0$, $\theta g(x, \cdot, \tau) \geq g(x, \cdot, \theta\tau)$ *in* $\overline{\Omega} \times (\eta_1, +\infty)$. *Assume that there exist a family of functions* $v_d \in C(\overline{\Omega})$ *such that* $v_d > \eta_1$ *on* $\overline{\Omega}$, *and* $\lim_{d \to 0^+} \|v_d - v^*\|_{C(\overline{\Omega})} = 0$ *for some* $v^* \in C(\overline{\Omega})$. *Moreover,* $\partial_3 h(x, v^*(x), \cdot) < 0$ *on* $\overline{\Omega} \times (\eta_3, +\infty)$ *and there exists* $\zeta \in C(\overline{\Omega})$ *for which* $h(x, v^*(x), \zeta) = 0$, $\inf_{x \in \Omega} \zeta > \eta_3$, *and* $\zeta_+ \neq 0$. *Moreover, for each* v_d, *there exists* $u_d \in C^{2s+2\alpha}(\overline{\Omega})$ *with* $2s + 2\alpha < 1$ *if* $0 < s < \frac{1}{2}$, *or* $u_d \in C_N^{2s+2\alpha}(\overline{\Omega})$ *with* $2s + 2\alpha < 2$ *if* $s \geq \frac{1}{2}$ *satisfying* $u_d \gneq 0$, $u_d > \eta_2 \vee \eta_3$ *on* $\overline{\Omega}$, *and*

$$(-d\Delta_N)^s u = g(x, v_d(x), u) h(x, v_d(x), u), \quad x \in \Omega,$$

where $0 < \alpha < 1$. *Then*

$$\lim_{d \to 0^+} \|u_d - \zeta_+\|_{C(\overline{\Omega})} = 0.$$

References

[1] L.J.S. Allen, B.M. Bolker, Y. Lou, A.L. Nevai, Asymptotic profiles of the steady states for an SIS epidemic reaction-diffusion model, Discrete Contin. Dyn. Syst. 21 (2008) 1–20.
[2] F. Andreu-Vaillo, J.M. Mazón, J.D. Rossi, J.J. Toledo-Melero, Nonlocal Diffusion Problems, Mathematical Surveys and Monographs, vol. 165, Amer. Math. Soc., Providence, Rhode Island, 2010.
[3] P.W. Bates, J. Han, G. Zhao, On a nonlocal phase-field system, Nonlinear Anal. 64 (2006) 2251–2278.
[4] P.W. Bates, G. Zhao, Existence, uniqueness and stability of the stationary solution to a nonlocal evolution equation arising in population dispersal, J. Math. Anal. Appl. 332 (2007) 428–440.

[5] M. Bendahmane, R. Ruiz-Baier, C. Tian, Turing pattern dynamics and adaptive discretization for a super-diffusive Lotka-Volterra model, J. Math. Biol. 72 (2016) 1441–1465.
[6] L. Brasco, E. Parini, The second eigenvalue of the fractional p-Laplacian, Adv. Calc. Var. 9 (2015) 323–355.
[7] D. Brockmann, L. Hufnagel, T. Geisel, The scaling laws of human travel, Nature 439 (2006) 462–465.
[8] C. Bucur, E. Valdinoci, Nonlocal Diffusion and Applications, Lecture Notes of the Unione Matematica Italiana, vol. 20, Springer, Cham, 2016.
[9] X. Cabré, J.-M. Roquejoffre, The influence of fractional diffusion in Fisher-KPP equations, Commun. Math. Phys. 320 (2013) 679–722.
[10] L. Caffarelli, S. Dipierro, E. Valdinoci, A logistic equation with nonlocal interactions, Kinet. Relat. Models 10 (2017) 141–170.
[11] L. Caffarelli, P. Stinga, Fractional elliptic equations, Caccioppoli estimates and regularity, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 33 (2016) 767–807.
[12] R.S. Cantrell, C. Cosner, Spatial Ecology via Reaction-Diffusion Equations, John Wiley and Sons Inc., West Sussex, England, 2003.
[13] C.M. Carracedo, M.S. Alix, The Theory of Fractional Powers of Operators, Elsevier Science B. V., New York, 2001.
[14] A.S. Chaves, A fractional diffusion equation to describe Lévy flights, Phys. Lett. A 239 (1998) 13–16.
[15] J. Cholewa, T. Dlotko, Global Attractors in Abstract Parabolic Problems, Cambridge University Press, New York, 2000.
[16] S. Dipierro, Contemporary Research in Elliptic PDEs and Related Topics, Springer, Switzerland, 2019.
[17] T. Dannemann, D. Boyerd, O. Miramontes, Lévy flight movements prevent extinctions and maximize population abundances in fragile Lotka-Volterra systems, Proc. Natl. Acad. Sci. USA 115 (2018) 3794–3799.
[18] G. Estrada-Rodriguez, H. Gimperlein, K.J. Painter, Fractional Patlak–Keller–Segel equations for chemotactic superdiffusion, SIAM J. Appl. Math. 78 (2018) 1155–1173.
[19] P. Felmer, M. Yangari, Fast propagation for fractional KPP equations with slowly decaying initial conditions, SIAM J. Math. Anal. 45 (2013) 662–678.
[20] L. Gao, H.W. Hethcote, Disease transmission models with density-dependent demographics, J. Math. Biol. 30 (1992) 717–731.
[21] M.C. González, C.A. Hidalgo, A.-L. Barabási, Understanding individual human mobility patterns, Nature 453 (2008) 779–782.
[22] G. Grubb, Regularity of spectral fractional Dirichlet and Neumann problems, Math. Nachr. 289 (2016) 831–844.
[23] C. Gui, M. Zhao, Traveling wave solutions of Allen–Cahn equation with a fractional Laplacian, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 32 (2015) 785–812.
[24] E. Hanert, E. Schumacher, E. Deleersnijder, Front dynamics in fractional-order epidemic models, J. Theor. Biol. 279 (2011) 9–16.
[25] W. Huang, M. Han, K. Liu, Dynamics of an SIS reaction-diffusion epidemic model for disease transmission, Math. Biosci. Eng. 7 (2010) 51–66.
[26] B. Li, H. Li, Y. Tong, Analysis on a diffusive SIS epidemic model with logistic source, Z. Angew. Math. Phys. 68 (2017) 68–96.
[27] H. Li, R. Peng, F.-B. Wang, Varying total population enhances disease persistence: qualitative analysis on a diffusive SIS epidemic model, J. Differ. Equ. 262 (2017) 885–913.
[28] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, Basel, 1995.
[29] L. Ma, H.-T. Niu, Z.-C. Wang, Global asymptotic stability of traveling waves to the Allen-Cahn equation with a fractional Laplacian, Commun. Pure Appl. Anal. 18 (2019) 2457–2472.
[30] B.B. Mandelbrot, The Fractal Geometry of Nature, W. H. Freeman and Company, New York, 1982.
[31] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, Phys. Rep. 339 (2000) 1–77.
[32] J.D. Murray, Mathematical Biology II: Spatial Models and Biomedical Applications, Springer-Verlag, Berlin, 2003.
[33] W.-M. Ni, The Mathematics of Diffusion, Society for Industrial and Applied Mathematics, Philadelphia, 2011.
[34] L. Nirenberg, Topic in Nonlinear Functional Analysis, American Mathematical Society, Providence, RI, 1974.
[35] A. Okubo, S.A. Levin, Diffusion and Ecological Problems: Modern Perspectives, Springer-Verlag, New York, 2002.
[36] D.A. Raichlen, B. Wood, A.D. Gordon, A.Z.P. Mabulla, F.W. Marlowe, H. Pontzer, Evidence of Lévy walk foraging patterns in human hunter–gatherers, Proc. Natl. Acad. Sci. USA 111 (2014) 728–733.
[37] S. Ruan, Spatial-temporal dynamics in nonlocal epidemiological models, in: Y. Takeuchi, K. Sato, Y. Iwasa (Eds.), Mathematics for Life Science and Medicine, Springer-Verlag, Berlin, 2007, pp. 97–122.
[38] S. Ruan, J. Wu, Modeling spatial spread of communicable diseases involving animal hosts, in: R.S. Cantrell, C. Cosner, S. Ruan (Eds.), Spatial Ecology, Chapman and Hall/CRC, Boca Raton, FL, 2009, pp. 293–316.
[39] S. Salem, Propagation of chaos for fractional Keller Segel equations in diffusion dominated and fair competition cases, J. Math. Pures Appl. 132 (2019) 79–132.
[40] D. Stan, J.L. Vázquez, The Fisher-KPP equation with nonlinear fractional diffusion, SIAM J. Math. Anal. 46 (2014) 3241–3276.
[41] P.R. Stinga, User's Guide to the Fractional Laplacian and the Method of Semigroups, Handbook of Fractional Calculus with Applications, vol. 2, Walter de Gruyter, Boston, 2019.
[42] P.R. Stinga, J.L. Torrea, Extension problem and Harnack's inequality for some fractional operators, Commun. Partial Differ. Equ. 35 (2010) 2092–2122.
[43] P.R. Stinga, B. Volzone, Fractional semilinear Neumann problems arising from a fractional Keller-Segel model, Calc. Var. Partial Differ. Equ. 54 (2015) 1009–1042.
[44] J.L. Vázquez, The mathematical theories of diffusion: nonlinear and fractional diffusion, in: M. Bonforte, G. Grillo (Eds.), Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions, in: Lecture Notes in Math., vol. 2186, Springer, Cham, 2017, pp. 205–278.

[45] G.M. Viswanathan, E.P. Raposo, M.G.E. da Luz, Lévy flights and superdiffusion in the context of biological encounters and random searches, Phys. Life Rev. 5 (2008) 133–150.
[46] G.F. Webb, A reaction-diffusion model for a deterministic diffusive epidemic, J. Math. Anal. Appl. 84 (1981) 150–161.
[47] J. Wu, Theory and Applications of Partial Functional Differential Equations, Springer, New York, 1996.
[48] W.-B. Xu, W.-T. Li, S. Ruan, Spatial propagation in an epidemic model with nonlocal diffusion: the influences of initial data and dispersal, Sci. China Math. 63 (2020) 2177–2206.
[49] F.-Y. Yang, W.-T. Li, S. Ruan, Dynamics of a nonlocal dispersal SIS epidemic model with Neumann boundary conditions, J. Differ. Equ. 267 (2019) 2011–2051.
[50] K. Yosida, Functional Analysis, Springer, Berlin, 1995.
[51] G. Zhao, The principal eigenvalue problems for perturbed fractional Laplace operators, Tamkang J. Math. 52 (2021) 189–220.
[52] G. Zhao, S. Ruan, Spatial and temporal dynamics of a nonlocal viral infection model, SIAM J. Appl. Math. 78 (2018) 1954–1980.
[53] G. Zhao, S. Ruan, Singularly perturbed eigenvalue problems of spectral fractional Laplace operators, submitted for publication.
[54] V. Zaburdaev, S. Denisov, J. Klafter, Lévy walks, Rev. Mod. Phys. 87 (2015) 483–530.