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Calculus of Variations

Long time behavior for a periodic Lotka–Volterra reaction–diffusion system with strong competition

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Abstract

This paper is concerned with the long time behavior of bounded solutions to a two-species time-periodic Lotka–Volterra reaction–diffusion system with strong competition. It is well known that solutions of the Cauchy problem of this system with front-like initial values converge to a bistable periodic traveling front. One may ask naturally how solutions of such time-periodic systems with other types of initial data evolve as time increases. In this paper, by transforming the system into a cooperative system on $[\mathbf{0}, \mathbf{1}]$, we first show that if the bounded initial value $\boldsymbol{\varphi}(x)$ has compact support and equals $\mathbf{1}$ for a sufficiently large x-level, then solutions converge to a pair of diverging periodic traveling fronts. As a by-product, we obtain a sufficient condition for solutions to spread to $\mathbf{1}$. We also prove that if the two species are initially absent from the right half-line $x > 0$ and the slower one dominates the faster one on $x < 0$, then solutions approach a propagating terrace, which means that several invasion speeds can be observed.

Mathematics Subject Classification 35K40 · 35K57 · 35B40

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1 Introduction

In this paper, we are interested in the long time behavior of bounded solutions for the following time-periodic Lotka–Volterra reaction–diffusion system with strong competition:

$$\begin{cases} u_t = d_1 u_{xx} + u[r_1(t) - a_1(t)u - b_1(t)v], \\ v_t = d_2 v_{xx} + v[r_2(t) - a_2(t)u - b_2(t)v], \end{cases} \tag{1.1}$$

where $t > 0,\ x \in \mathbb{R}, d_1, d_2 > 0$ are diffusion coefficients and the given functions $a_i(\cdot), b_i(\cdot), r_i(\cdot)$ $(i = 1, 2)$ satisfy the following basic assumptions:

(H$_1$) $r_i(t),\ a_i(t),\ b_i(t) \in C^{\frac{\theta}{2}}(\mathbb{R})$ $(i = 1, 2)$ are positive T-periodic functions for some constants $T > 0$ and $\theta \in (0, 1)$, and $\overline{r_i} := \frac{1}{T}\int_0^T r_i(t)dt > 0,\ i = 1, 2$;

(H$_2$) $\overline{r_1} < \min\limits_{t\in[0,T]}\left(\frac{b_1(t)}{b_2(t)}\right)\overline{r_2},\ \overline{r_2} < \min\limits_{t\in[0,T]}\left(\frac{a_2(t)}{a_1(t)}\right)\overline{r_1}$;

(H$_3$) $\overline{r_1} + \overline{r_2} > \max\limits_{t\in[0,T]}\left(\frac{a_2(t)}{a_1(t)}\right)\overline{r_1},\ \overline{r_1} + \overline{r_2} > \max\limits_{t\in[0,T]}\left(\frac{b_1(t)}{b_2(t)}\right)\overline{r_2}$.

The spatially homogeneous system of (1.1) takes the following form:

$$\begin{cases} u'(t) = u(t)[r_1(t) - a_1(t)u(t) - b_1(t)v(t)], \\ v'(t) = v(t)[r_2(t) - a_2(t)u(t) - b_2(t)v(t)]. \end{cases} \tag{1.2}$$

Under assumptions (H$_1$)–(H$_2$), system (1.2) has a trivial solution $(0, 0)$ and two stable semi-trivial T-periodic solutions $(p(t), 0)$ and $(0, q(t))$, where

$$p(t) = \frac{p_0 e^{\int_0^t r_1(s)ds}}{1 + p_0\int_0^t e^{\int_0^s r_1(\varrho)d\varrho} a_1(s)ds}, \quad p_0 = \frac{e^{\int_0^T r_1(s)ds} - 1}{\int_0^T e^{\int_0^s r_1(\varrho)d\varrho} a_1(s)ds} > 0,$$

$$q(t) = \frac{q_0 e^{\int_0^t r_2(s)ds}}{1 + q_0\int_0^t e^{\int_0^s r_2(\varrho)d\varrho} b_2(s)ds}, \quad q_0 = \frac{e^{\int_0^T r_2(s)ds} - 1}{\int_0^T e^{\int_0^s r_2(\varrho)d\varrho} b_2(s)ds} > 0.$$

Moreover, according to Hess [25, Theorem 35.1 and Proposition 36.3], system (1.2) admits a unique T-periodic coexistence state $(\bar{u}(t), \bar{v}(t))$, which is unstable and satisfies $0 < \bar{u}(t) < p(t)$ and $0 < \bar{v}(t) < q(t)$ for $t \in [0, T]$. Biologically, assumption (H$_2$) indicates that the interspecific competition between the two species is stronger than the intraspecific competition within each of the two species. An example is that a variety of bird species house in small islands off the coast of New Guinea, while similar species (i.e., similar in size, diet, and habitat use) often fail to coexist with each other (c.f. [6]). We also mention that (H$_3$) is a technique assumption which ensures that the periodic eigenvalue problem associated to the linearized system of (1.2) at $(p(t), 0)$ and $(0, q(t))$, respectively, admits exactly a positive eigenvalue and the corresponding periodic eigenfunction is positive, which will be used in establishing the stability of a pair of diverging periodic traveling fronts.

One of the central questions in the study of parabolic equations is how bounded solutions of their initial value problems with various initial data evolve as time increases. It is well known that traveling fronts can describe the long time behavior of solutions of the Cauchy problem of many reaction–diffusion systems with front-like initial values. In recent years, traveling fronts of competition-diffusion systems in homogeneous habitat have been studied very extensively, we refer to [8, 16, 17, 21–24, 26, 28, 31, 33]. There are also a few of significant and interesting results on periodic traveling fronts of reaction–diffusion equations/systems in periodic media (c.f. [1–5, 12–14, 19, 27, 32, 38, 40–46, 48–51]) and the long time behavior for scalar reaction–diffusion equations (see e.g., [9, 15, 20, 52]). For example, Zhao and Ruan [49] investigated the existence, uniqueness and stability of monostable periodic traveling

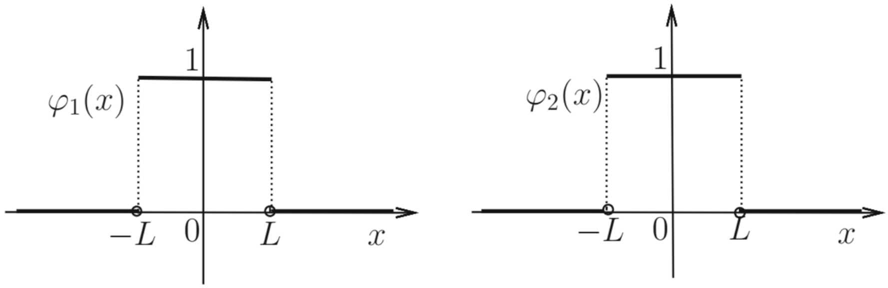

Fig. 1 Profiles of the initial value $\boldsymbol{\varphi}(x)$ with compact support

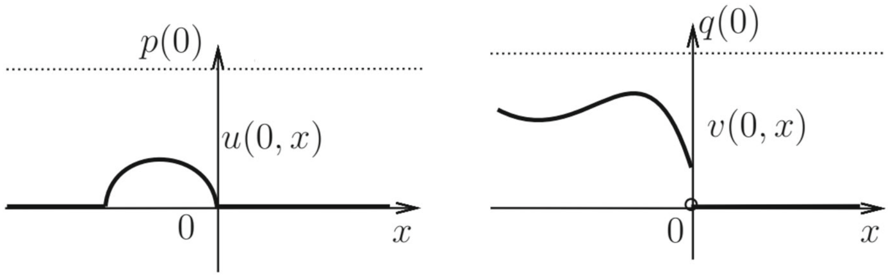

Fig. 2 Profiles of the initial value $(u(0,x), v(0,x))$ without compact support

fronts for (1.1). Bao and Wang [1] studied the existence of bistable periodic traveling fronts for (1.1) by applying the theory of monotone semiflows (c.f. [18]). They also showed that the solution of the Cauchy problem of (1.1) with front-like initial value converges to a bistable traveling front. One may ask naturally how solutions of time-periodic reaction–diffusion systems, such as (1.1), with other types of initial data evolve as time increases.

The aim of this paper is to consider the long time behavior of solutions to (1.1) with two types of initial values, one type has compact support (see (1.6) and Fig. 1) while the other is not compactly supported (see (1.7) and Fig. 2). We shall show that solutions of (1.1) with the first type of initial values evolve into a pair of diverging periodic traveling fronts. It should be mentioned that Kanel [30] and Fife and McLeod [20] considered the stability of a pair of traveling fronts for scalar reaction–diffusion equations; Roquejoffre [39] and Ma and Wang [35] investigated the same issue for autonomous and periodic parabolic equations in cylinders, respectively. In this paper, we adapt nontrivially the methods in [20] for (1.1) by constructing an appropriate subsolution (see Lemma 2.2). To the best of our knowledge, this may be the first time that the stability of a pair of diverging periodic traveling fronts for *time-periodic reaction–diffusion systems* is considered. To state this result, by making a change of variables

$$u_1(t,x) = \frac{u(t,x)}{p(t)} \quad \text{and} \quad u_2(t,x) = \frac{q(t) - v(t,x)}{q(t)}, \tag{1.3}$$

we transform the competitive system (1.1) into the following cooperative system

$$\begin{cases} (u_1)_t = d_1(u_1)_{xx} + u_1[a_1(t)p(t)(1-u_1) - b_1(t)q(t)(1-u_2)], \\ (u_2)_t = d_2(u_2)_{xx} + (1-u_2)[a_2(t)p(t)u_1 - b_2(t)q(t)u_2]. \end{cases} \tag{1.4}$$

The four periodic steady states $(0,0)$, $(0,q(t))$, $(p(t),0)$ and $(\bar{u}(t),\bar{v}(t))$ of (1.1) become $(0,1)$, $\mathbf{0}:=(0,0)$, $\mathbf{1}:=(1,1)$ and $\mathbf{u}^*(t):=(u_1^*(t),u_2^*(t))$ of (1.4), respectively. Moreover, $0<u_i^*(t)<1$, $u_i^*(t+T)=u_i^*(t), i=1,2$. Clearly, the stability of a pair of diverging periodic traveling fronts of (1.1) is equivalent to that of (1.4). For simplicity, we denote

$$\mathcal{X}=\left\{\boldsymbol{\varphi}=(\varphi_1,\varphi_2)\in L^\infty(\mathbb{R},\mathbb{R}^2)\middle|\boldsymbol{\varphi}(x)\in[\mathbf{0},\mathbf{1}] \text{ and } \boldsymbol{\varphi} \text{ has compact support}\right\}.$$

Recall that periodic traveling waves of (1.1) connecting $(0,q(t))$ and $(p(t),0)$ are bounded solutions of the special form $(u(t,x),v(t,x))=(U(t,z),V(t,z))=:W(t,z)$, $z=x-ct$ and $W(t+T,z)=W(t,z)$, which satisfy

$$\lim_{z\to-\infty}W(t,z)=(0,q(t)) \text{ and } \lim_{z\to+\infty}W(t,z)=(p(t),0) \text{ uniformly in } t\in[0,T],$$

where c and (U,V) are called the *wave speed* and *wave profile*, respectively. If $U(t,z)$ and $V(t,z)$ are monotone with respect to z, then $(U(t,z),V(t,z))$ is called *a periodic traveling front*. Obviously, the wave profile (U,V) satisfies the following periodic parabolic system:

$$\begin{cases} U_t=d_1U_{zz}+cU_z+U[r_1(t)-a_1(t)U-b_1(t)V], \\ V_t=d_2V_{zz}+cV_z+V[r_2(t)-a_2(t)U-b_2(t)V]. \end{cases} \tag{1.5}$$

Moreover, it is clear that the function

$$\boldsymbol{\Phi}(t,z):=(\Phi_1(t,z),\Phi_2(t,z))=\Big(\frac{U(t,z)}{p(t)},\frac{q(t)-V(t,z)}{q(t)}\Big),\ z=x-ct$$

is a periodic traveling front of (1.4) connecting $\mathbf{0}$ and $\mathbf{1}$. The existence and uniqueness of bistable periodic traveling fronts of (1.4) come from Bao and Wang [1].

Lemma 1.1 (Bao and Wang [1]) *Assume that* $(\mathrm{H}_1)-(\mathrm{H}_3)$ *hold. Then there exists* $c\in\mathbb{R}$ *such that* (1.4) *admits a periodic traveling front* $\boldsymbol{\Phi}(t,x-ct)$ *satisfying* $\boldsymbol{\Phi}(t,-\infty)=\mathbf{0}$ *and* $\boldsymbol{\Phi}(t,+\infty)=\mathbf{1}$ *uniformly in* $t\in[0,T]$ *and* $\boldsymbol{\Phi}_z(\cdot,\cdot)>0$. *Moreover,* $(\boldsymbol{\Phi};c)$ *is unique in the following sense: the speed* c *is unique and the profile* $\boldsymbol{\Phi}(t,\cdot)$ *is unique up to a translation.*

We now state our main results on the long time behavior of (1.4) with initial value (1.6) as follows.

Theorem 1.2 *Assume that* (H_1)–(H_3) *hold and that* $\boldsymbol{\Phi}(t,x-ct)$ *is the periodic traveling front of system* (1.4) *connecting* $\mathbf{0}$ *and* $\mathbf{1}$ *obtained in Lemma 1.1 with speed* $c<0$. *Let* $\mathbf{u}(t,x)$ *be the unique solution of system* (1.4) *with initial data* $\boldsymbol{\varphi}(x)$ *such that*

$$\boldsymbol{\varphi}(x)\in\mathcal{X} \text{ and } \boldsymbol{\varphi}(x)=\mathbf{1} \text{ for } |x|<L, \tag{1.6}$$

where $L>0$ *is a constant. Then there exists a large enough constant* $\tilde{L}>0$ *such that for any* $L\geq\tilde{L}$ *and constants* ξ_1 *and* ξ_2*,*

$$\lim_{t\to+\infty}\|\mathbf{u}(t,x)-\boldsymbol{\Phi}(t,x-ct+\xi_1)\|_{L^\infty_{\mathrm{loc}}((-\infty,0])}=0;$$
$$\lim_{t\to+\infty}\|\mathbf{u}(t,x)-\boldsymbol{\Phi}(t,-x-ct+\xi_2)\|_{L^\infty_{\mathrm{loc}}([0,+\infty))}=0.$$

Moreover, we obtain a sufficient condition for solutions to spread to $\mathbf{1}$.

Theorem 1.3 *Assume that the conditions in Theorem 1.2 hold and* $\mathbf{u}(t,x)$ *is the unique solution of* (1.4) *with initial data* $\boldsymbol{\varphi}(x)$ *satisfying* (1.6)*. Then there exists* $\tilde{L}>0$ *large enough such that for any* $L\geq\tilde{L}$, $\mathbf{u}(t,\cdot)\to\mathbf{1}$ *in* $L^\infty_{loc}(\mathbb{R},\mathbb{R}^2)$ *as* $t\to\infty$.

Remark 1.4 (i) If the sign of the wave speed in Theorem 1.2 is reversed and suitable changes in (1.6) are made, then one can obtain similar convergence results.

(ii) In the bistable competition model, the sign of the wave speed of traveling waves decides which species eventually wins the competition. Thus, it is an interesting and important issue to determine the sign of the wave speed in the bistable dynamics. Recently, Ma et al. [34] obtained some sufficient conditions on the sign of the wave speed c of the bistable periodic traveling waves for system (1.1) connecting $(0, q(t))$ and $(p(t), 0)$, see [34, Theorems 4.1 and 4.2] and [34, Theorems 4.3 and 4.4] for detailed results on $c > 0$ and $c < 0$, respectively.

Next, we consider the long time behavior of solutions to (1.1) with another type of initial values which do not have compact support. More specifically, we consider the spreading properties of solutions to (1.1) with the following initial conditions (see Fig. 2):

$$\begin{cases} u(0,x) = v(0,x) = 0 \text{ for } x \geq 0, \\ 0 \leq u(0,x) < p(0) \text{ and } u(0,x) \text{ has nontrivial compact support for } x < 0, \\ 0 < v_{0m} \leq v(0,x) \leq v_{0M} < q(0) \text{ for } x < 0, \end{cases} \tag{1.7}$$

where $v_{0m} = \inf_{x\in\mathbb{R}} v(0,x)$ and $v_{0M} = \sup_{x\in\mathbb{R}} v(0,x)$. This type of initial values means that species u and v are initially absent from the right half-line $x > 0$, and the species v dominates the species u on $x < 0$. It is interesting to consider how the two species invade the right half-line. We shall show that solutions of (1.1) with such initial values will approach a propagating terrace, which means that several invasion speeds can be observed.

Before stating the long time behavior of solutions to (1.1) with initial values (1.7), we state some known results on periodic traveling fronts and spreading speed for monostable time-periodic reaction–diffusion equations/systems (cf. [1, 32]). Let c_{UV} be the unique speed of the bistable periodic traveling front of (1.1) connecting $(0, q(t))$ and $(p(t), 0)$ and $c_U > 0$ be the minimal wave speed of the periodic traveling front $W_1(t, x - ct)$ connecting $p(t)$ and 0 of the Fisher-KPP equation:

$$w_t = d_1 w_{xx} + w[r_1(t) - a_1(t)w]. \tag{1.8}$$

Similarly, let $c_V > 0$ be the minimal wave speed of the periodic traveling front $W_2(t, x-ct)$ connecting $q(t)$ and 0 of the Fisher-KPP equation:

$$w_t = d_2 w_{xx} + w[r_2(t) - b_2(t)w]. \tag{1.9}$$

From Liang et al. [32, Theorems 4.1 and 4.2] (take time delay $\tau = 0$), $c_U = 2\sqrt{d_1\overline{r_1}}$ and $c_V = 2\sqrt{d_2\overline{r_2}}$. It is clear that c_U and c_V are the spreading speed of one species in the absence of the other species, respectively. For simplicity, we denote $f_M := \max_{t\in[0,T]} f(t)$ and $f_m := \min_{t\in[0,T]} f(t)$ for a given T-periodic and continuous function f. The results on the long time behavior of solutions to (1.1) with initial values (1.7) are stated as follows.

Theorem 1.5 *Assume that* (H$_1$)–(H$_3$) *and the following assumption hold*

(H$_4$) $r_1(t) \equiv r_1 > 0$, $r_1 < \min\{\frac{5}{4}a_{1m}p_m, b_{1m}q_m\}$ *and* $r_{2M} < a_{2m}p_m$.

Let $\mathbf{u}(t,x) = (u(t,x), v(t,x))$ *be the unique solution of system* (1.1) *with the initial values* (1.7). *Then the following spreading results hold:*

(i) *For any* $c > \max\{c_U, c_V\}$, $\lim_{t\to+\infty} \sup_{x>ct}(|u(t,x)| + |v(t,x)|) = 0$;
(ii) *For any* $c < c_{UV}$, $\lim_{t\to+\infty} \sup_{x<ct}(|u(t,x)| + |v(t,x) - q(t)|) = 0$;
(iii) *Suppose furthermore that* $c_V < c_U$, *then for any* $c_{UV} < c_1 < c_2 < c_U$,

$$\lim_{t\to+\infty} \sup_{c_1 t < x < c_2 t} (|u(t,x) - p(t)| + |v(t,x)|) = 0.$$

Obviously, Theorem 1.5 means that if the two species are initially absent from the right half-line $x > 0$ and the slower one dominates the faster one on $x < 0$, then the solution of (1.1) with such initial values approaches a propagating terrace, which connects the unstable state $(0, 0)$ to the stable state $(p(t), 0)$, and then connects the stable state $(p(t), 0)$ to the other stable state $(0, q(t))$.

Remark 1.6 (i) It should be mentioned that the assumption (H_4) is technique, which is used in the proofs of Lemmas 3.4 and 3.8. We feel that it is probably not necessary for the main results to hold. We leave it for our future research.

(ii) We note that (H_2), (H_3) and (H_4) are compatible. Let us consider the simplest positive constant coefficients $r_i, a_i, b_i (i = 1, 2)$ as an example. In this case, (H_2), (H_3) and (H_4) become

$$r_1 < \frac{b_1 r_2}{b_2}, \; r_2 < \frac{a_2 r_1}{a_1}, \tag{1.10}$$

$$r_1 + r_2 > \frac{a_2 r_1}{a_1}, \; r_1 + r_2 > \frac{b_1 r_2}{b_2}, \tag{1.11}$$

$$r_1 < \min\left\{\frac{5}{4} r_1, \frac{b_1 r_2}{b_2}\right\}, \; r_2 < \frac{a_2 r_1}{a_1}, \tag{1.12}$$

respectively. Clearly, (1.10) implies (1.12). Then, (H_2), (H_3) and (H_4) are equivalent to

$$1 + \frac{r_1}{r_2} > \frac{a_2 r_1}{a_1 r_2} > 1 \text{ and } 1 + \frac{r_2}{r_1} > \frac{b_1 r_2}{b_2 r_1} > 1,$$

which means that the competition between species u and v is not too stronger.

(iii) The results in Theorem 1.5 also hold when the conditions in (1.7) that $u(0, x)$ has nontrivial compact support in $(-\infty, 0)$ is replaced by $0 < u(0, x) \leq -Lxe^{\tilde{\mu}x}$ for some $L > 0 \, as \, x \ll -1$, where $\tilde{\mu}$ is a constant with $\tilde{\mu} \geq \sqrt{r_1/d_1}$.

We remark that propagating terraces have been widely investigated for reaction–diffusion equations and autonomous competition systems (c.f. [7, 10, 11, 20, 36, 37, 47]). For instance, in the heterogeneous case, Ding and Matano [10], Ducrot et al. [11] and Poláčik [36, 37] proved the existence and convergence of the minimal propagating terrace by using the zero-number argument; Carrère [7] and Zhang and Zhao [47] considered the propagating terrace for autonomous competition systems with local and nonlocal dispersal, respectively. Although the zero-number argument is a powerful tool, it cannot be applied for reaction–diffusion *systems*. Here, we generalize nontrivially the techniques in [7, 47] for *autonomous* diffusion systems to time-periodic Lotka–Volterra competition-diffusion systems. It should be mentioned that the presence of time-periodicity makes the problem more difficult than the autonomous case. For example, to construct appropriate super- and subsolutions, we need to prove the continuity of the bistable wave speed with respect to parameter κ (see Lemma 3.7). Such a property cannot be proved by the method in [29], and we prove it by using the uniqueness and stability of bistable traveling fronts.

The rest of this paper is organized as follows. In Sect. 2, we show that solutions of system (1.4) with initial values (1.7) develop into a pair of diverging periodic traveling fronts. In Sect. 3, we prove that solutions of (1.1) with initial conditions (1.7) will approach a propagating terrace.

2 Stability of a pair of diverging periodic fronts

In this section, we consider the stability of a pair of diverging periodic fronts for the time-periodic reaction–diffusion system (1.4) with initial values (1.6) under the assumptions (H_1)-(H_3), i.e. Theorem 1.2. For $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$ and $\mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$, we denote $\mathbf{a} \le \mathbf{b}$ if $a_i \le b_i,\ i = 1, 2$; $\mathbf{a} < \mathbf{b}$ if $\mathbf{a} \le \mathbf{b}$ but $\mathbf{a} \ne \mathbf{b}$; and $\mathbf{a} \ll \mathbf{b}$ if $a_i < b_i,\ i = 1, 2$. Let $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^2$.

We first recall some known results on the asymptotic behavior of periodic traveling fronts of (1.4) with $c = c_{uv} \ne 0$ at $\pm\infty$, which will play an important role in the proof of our main results.

Lemma 2.1 [12, Theorems 1.3 and 1.4] *Assume* (H_1)-(H_3). *Let* $(\Phi_1(t,z), \Phi_2(t,z))$ *be a periodic traveling wave of* (1.4) *connecting* $\mathbf{0}$ *and* $\mathbf{1}$ *with* $c \ne 0$. *Then*

$$\lim_{z\to+\infty} \frac{1-\Phi_1(t,z)}{k_1 e^{-\tilde{\nu}_1 z}\tilde{\phi}_1(t)} = 1, \quad \lim_{z\to+\infty} \frac{1-\Phi_2(t,z)}{k_1 e^{-\tilde{\nu}_1 z}\tilde{\phi}_2(t)} = 1 \text{ uniformly in } t \in \mathbb{R},$$

and

$$\lim_{z\to-\infty} \frac{\Phi_1(t,z)}{k_2 e^{\tilde{\nu}_2 z}\tilde{\psi}_1(t)} = 1, \quad \lim_{z\to-\infty} \frac{\Phi_2(t,z)}{k_2 e^{\tilde{\nu}_2 z}\tilde{\psi}_2(t)} = 1 \text{ uniformly in } t \in \mathbb{R},$$

where $k_i > 0, \tilde{\nu}_i > 0, i = 1, 2$ *are some constants,* $\tilde{\phi}_i(t)$ *and* $\tilde{\psi}_i(t), i = 1, 2,$ *are some positive* T*-periodic functions in* $\mathbb{R}$.

Under the assumptions of Theorem 1.2, it is easy to see that the Cauchy problem (1.1) with the initial values (1.6) has a unique solution $\mathbf{u}(t,x)$ satisfying $\mathbf{0} \le \mathbf{u}(t,x) \le \mathbf{1}$ for any $t \ge 0,\ x \in \mathbb{R}$. In the remainder of this section, we always assume that the hypotheses of Theorem 1.2 hold. For convenience, we set

$$\begin{aligned} g_1(t,u_1,u_2) &:= u_1[a_1(t)p(t)(1-u_1) - b_1(t)q(t)(1-u_2)],\\ g_2(t,u_1,u_2) &:= (1-u_2)[a_2(t)p(t)u_1 - b_2(t)q(t)u_2]. \end{aligned} \tag{2.1}$$

To construct an explicit subsolution, let us consider the following eigenvalue problem

$$\begin{cases} \phi_1'(t) - [a_1(t)p(t) - b_1(t)q(t)]\phi_1(t) = \lambda_0\phi_1(t),\\ \phi_2'(t) - a_2(t)p(t)\phi_1(t) + b_2(t)q(t)\phi_2(t) = \lambda_0\phi_2(t),\\ \phi_1(t+T) = \phi_1(t),\ \ \phi_2(t+T) = \phi_2(t). \end{cases} \tag{2.2}$$

By direct calculations and assumptions $(H_1) - (H_3)$, we know that the periodic eigenvalue problem (2.2) admits an eigenvalue $\lambda_0 = -\frac{1}{T}\int_0^T [a_1(t)p(t) - b_1(t)q(t)]dt > 0$. Moreover, the eigenfunction $(\phi_1(t), \phi_2(t))$ associated with λ_0 satisfies $(0,0) < (\phi_1(t), \phi_2(t)) \le (1,1)$ (c.f. Bao and Wang [1]). Similarly, the periodic eigenvalue problem

$$\begin{cases} \psi_1'(t) + a_1(t)p(t)\psi_1(t) - b_1(t)q(t)\psi_2(t) = \lambda_1\psi_1(t),\\ \psi_2'(t) - [b_2(t)q(t) - a_2(t)p(t)]\psi_2(t) = \lambda_1\psi_2(t),\\ \psi_1(t+T) = \psi_1(t), \psi_2(t+T) = \psi_2(t) \end{cases}$$

admits an eigenvalue pair $(\lambda_1, (\psi_1(t), \psi_2(t))$ with

$$\lambda_1 = -\frac{1}{T}\int_0^T [b_2(t)q(t) - a_2(t)p(t)]dt > 0 \quad \text{and} \quad (0,0) < (\psi_1(t), \psi_2(t)) \le (1,1).$$

Denote $\bar{p} := \min\left\{\phi_{1m}, \phi_{2m}, \psi_{1m}, \psi_{2m}\right\} > 0$. Define a function $\zeta(z) \in C^2(\mathbb{R}, \mathbb{R}^+)$ such that

$$\zeta(z) = 0 \quad \text{for} \quad z \leq -2 \quad \text{and} \quad \zeta(z) = 1 \quad \text{for} \quad z \geq 2;$$

$$0 \leq \zeta'(z) \leq 1, \; |\zeta''(z)| \leq 1 \quad \text{for} \quad z \in \mathbb{R}.$$

Take a positive vector function $\mathbf{p}(t, z) := (p_1(t, z), p_2(t, z))$, where

$$p_1(t, z) = \zeta(z)\psi_1(t) + (1 - \zeta(z))\phi_1(t) \quad \text{and} \quad p_2(t, z) = \zeta(z)\psi_2(t) + (1 - \zeta(z))\phi_2(t).$$

It is obvious that $\mathbf{0} \leq \mathbf{p}(\cdot, \cdot) \leq \mathbf{1}$.

Lemma 2.2 *There exist positive constants β_1, σ_1 and δ_1 such that for any $\delta^- \in (0, \delta_1)$, $\xi^- \in \mathbb{R}$, the function $\underline{\mathbf{u}}(t, x) = (\underline{u}_1(t, x), \underline{u}_2(t, x)) := \max\{\mathbf{0}, \mathbf{u}^-(t, x)\}$ is a subsolution of (1.4) on $t \geq 0$, where $\mathbf{u}^-(t, x) = (u_1^-(t, x), u_2^-(t, x))$ is given by*

$$\begin{aligned}\mathbf{u}^-(t, x) = &\,\mathbf{\Phi}(t, x - ct + \xi^- - \sigma_1\delta^-(1 - e^{-\beta_1 t})) + \mathbf{\Phi}(t, -x - ct + \xi^- - \sigma_1\delta^-(1 - e^{-\beta_1 t})) \\ &- \mathbf{1} - \delta^-\mathbf{p}(t, x - ct + \xi^- - \sigma_1\delta^-(1 - e^{-\beta_1 t}))e^{-\beta_1 t}, \; t \geq 0, x \in \mathbb{R}.\end{aligned}$$

Proof Let $\eta^\pm = \eta^\pm(t, x) := \pm x - ct + \xi^- - \sigma_1\delta^-(1 - e^{-\beta_1 t})$. Then

$$\mathbf{u}^-(t, x) = \mathbf{\Phi}(t, \eta^+(t, x)) + \mathbf{\Phi}(t, \eta^-(t, x)) - \mathbf{1} - \delta^-\mathbf{p}(t, \eta^+(t, x))e^{-\beta_1 t}, \; t \geq 0, x \in \mathbb{R}.$$

It is clear that $\mathbf{u}^-(\cdot, \cdot) \in [-\mathbf{2}, \mathbf{1}]$, where $-2 := (-2, -2)$. For simplicity, we denote

$$\begin{aligned}
\mathcal{L}_k[\underline{\mathbf{u}}](t, x) &:= (\underline{u}_k)_t - d_k(\underline{u}_k)_{xx} - g_k(t, \underline{u}_1, \underline{u}_2), \quad k = 1, 2, \; t \geq 0, x \in \mathbb{R}, \\
\partial_1 g_i(t, \gamma^+; \theta) &:= \partial_{u_1} g_i\big(t, \Phi_i(t, \gamma^+) + \theta(1 - \Phi_i(t, \gamma^+)), \Phi_j(t, \gamma^+) + \theta(1 - \Phi_j(t, \gamma^+))\big), \\
\partial_2 g_i(t, \gamma^+; \theta) &:= \partial_{u_2} g_i\big(t, \Phi_i(t, \gamma^+) + \theta(1 - \Phi_i(t, \gamma^+)), \Phi_j(t, \gamma^+) + \theta(1 - \Phi_j(t, \gamma^+))\big), \\
\partial_1 \tilde{g}_i(t, \gamma^+, \gamma^-; \theta, \delta) &:= \partial_{u_1} g_i\big(t, \Phi_i(t, \gamma^-) + \theta(\Phi_i(t, \gamma^+) - 1 - \delta p_i(t, \gamma^+)e^{-\beta_1 t}), \\
&\qquad \Phi_j(t, \gamma^-) + \theta(\Phi_j(t, \gamma^+) - 1 - \delta p_j(t, \gamma^+)e^{-\beta_1 t})\big), \\
\partial_2 \tilde{g}_i(t, \gamma^+, \gamma^-; \theta, \delta) &:= \partial_{u_2} g_i\big(t, \Phi_i(t, \gamma^-) + \theta(\Phi_i(t, \gamma^+) - 1 - \delta p_i(t, \gamma^+)e^{-\beta_1 t}), \\
&\qquad \Phi_j(t, \gamma^-) + \theta(\Phi_j(t, \gamma^+) - 1 - \delta p_j(t, \gamma^+)e^{-\beta_1 t})\big)
\end{aligned}$$

for $i \neq j \in \{1, 2\}$, $\gamma^\pm \in \mathbb{R}$, $\theta \in (0, 1)$ and $\delta > 0$. We further define

$$\begin{aligned}
G_0(t, \gamma^+, \gamma^-; \theta, \delta) &:= \sum_{i=1}^{2} \big[|(g_i)_{u_1}(t, 0, 0) - \partial_1 \tilde{g}_i(t, \gamma^+, \gamma^-; \theta, \delta)| \\
&\qquad + |(g_i)_{u_2}(t, 0, 0) - \partial_2 \tilde{g}_i(t, \gamma^+, \gamma^-; \theta, \delta)|\big], \\
H_0(t, \gamma^+, \gamma^-; \theta, \delta) &:= \sum_{i=1}^{2} \big[|(g_i)_{u_1}(t, 1, 1) - \partial_1 \tilde{g}_i(t, \gamma^+, \gamma^-; \theta, \delta)| \\
&\qquad + |(g_i)_{u_2}(t, 1, 1) - \partial_2 \tilde{g}_i(t, \gamma^+, \gamma^-; \theta, \delta)|\big]
\end{aligned}$$

for any $\gamma^\pm \in \mathbb{R}$, $\theta \in (0, 1)$ and $\delta > 0$.

By the definition of $\zeta(\cdot)$, we have $\zeta(z) = 1$ and $\zeta(-z) = 0$ for $z \geq 2$, and hence $p_1(t, z) = \psi_1(t)$, $p_2(t, z) = \psi_2(t)$, $p_1(t, -z) = \phi_1(t)$ and $p_2(t, -z) = \phi_2(t)$ for $z \geq 2$. Note that

$$\lim_{z\to-\infty}(\Phi_1(t,z),\Phi_2(t,z))=(0,0),\ \lim_{z\to+\infty}(\Phi_1(t,z),\Phi_2(t,z))=(1,1) \text{ uniformly in } t\in[0,T].$$

It then follows that there exist $M^+>2$ large enough and $\delta_1'\in(0,1)$ small enough such that for any $\delta^-\in(0,\delta_1')$ and $\theta\in(0,1)$,

$$\sup_{t\geq 0,\gamma^-\leq -M^+,\gamma^+\geq M^+}|G_0(t,\gamma^+,\gamma^-;\theta,\delta^-)|\leq\frac{\lambda_0\min\{\phi_{1m},\phi_{2m}\}}{2(\phi_{1M}+\phi_{2M})},\tag{2.3}$$

$$\sup_{t\geq 0,\gamma^\pm\geq M^+}|H_0(t,\gamma^+,\gamma^-;\theta,\delta^-)|\leq\frac{\lambda_1\min\{\psi_{1m},\psi_{2m}\}}{2(\psi_{1M}+\psi_{2M})}.\tag{2.4}$$

Take

$$K_0:=\sup_{t\geq 0,\gamma^\pm\in\mathbb{R},\delta^-\in(0,\delta_1'),\theta\in(0,1)}\Big\{\sum_{i=1}^2\big[|\partial_1\tilde{g}_i(t,\gamma^+,\gamma^-;\theta,\delta^-)|+|\partial_2\tilde{g}_i(t,\gamma^+,\gamma^-;\theta,\delta^-)|\big]\Big\},$$

$$K_1:=\max\Big\{\sup_{t\geq 0,z\in\mathbb{R},i=1,2}\Big|\frac{\partial}{\partial z}p_i(t,z)\Big|,\ \sup_{t\geq 0,z\in\mathbb{R},i=1,2}\Big|\frac{\partial^2}{\partial zz}p_i(t,z)\Big|\Big\},$$

$$K_2:=\max\Big\{\max_{t\geq 0,z\in\mathbb{R}}\Big|\frac{\partial}{\partial t}p_1(t,z)\Big|,\ \max_{t\geq 0,z\in\mathbb{R}}\Big|\frac{\partial}{\partial t}p_2(t,z)\Big|\Big\},$$

$$K_3:=\min\Big\{\min_{t\in[0,T],z\in[-M^+,M^+]}\frac{\partial}{\partial z}\Phi_1(t,z),\ \min_{t\in[0,T],z\in[-M^+,M^+]}\frac{\partial}{\partial z}\Phi_2(t,z)\Big\},$$

$$K_4:=\max_{t\geq 0,\gamma^\pm\in\mathbb{R},\delta^-\in(0,\delta_1'),\theta\in(0,1)}\sum_{i=1}^2\Big\{\big|\partial_i\tilde{g}_1(t,\gamma^+,\gamma^-;\theta,\delta^-)-\partial_i g_1(t,\gamma^+;\theta)\big|$$
$$+\big|\partial_i\tilde{g}_2(t,\gamma^+,\gamma^-;\theta,\delta^-)-\partial_i g_2(t,\gamma^+;\theta)\big|\Big\}.$$

By Lemma 2.1, we may assume that there exist $\nu,\upsilon>0$ and $\bar{K}>0$ such that

$$\mathbf{1}-\mathbf{\Phi}(t,\xi)\leq\bar{K}e^{-\nu\xi}\mathbf{1}\text{ for } t\geq 0 \text{ and } \xi\geq 0\tag{2.5}$$

and

$$\mathbf{\Phi}(t,\xi)\leq\bar{K}e^{\upsilon\xi}\mathbf{1}\text{ for } t\geq 0 \text{ and } \xi\leq 0.\tag{2.6}$$

Fix β_1 satisfying $0<\beta_1<\min\{\frac{\lambda_1}{4},\frac{\lambda_0}{4},-\nu c\}$ and take

$$\sigma_1>\frac{|c|K_1+\beta_1+K_2+dK_1+K_0}{\beta_1K_3},\ \delta_1:=\min\Big\{\frac{K_3}{K_1},\delta_1',\frac{1}{\sigma_1}\Big\},$$

where $d=\max\{d_1,d_2\}$. Choose $\xi^->\max\{1,\sigma_1\}$ such that

$$\delta_1\min\{\psi_{1m},\psi_{2m},\phi_{1m},\phi_{2m}\}\big(\beta_1-\frac{1}{2}\min\{\lambda_0,\lambda_1\}\big)+K_4\bar{K}e^{-\nu(\xi^--1)}\leq 0,\tag{2.7}$$

$$\delta_1\big[K_1|c|-\sigma_1\beta_1K_3+\beta_1+d_1K_1+K_2+K_0\big]+K_4\bar{K}e^{-\nu(\xi^--1)}\leq 0.\tag{2.8}$$

Given any $\delta^-\in(0,\delta_1)$. Let us take $C_i=\{(t,x)\in[0,\infty)\times\mathbb{R}|u_i^-(t,x)>0\}$ and $D_i=\{(t,x)\in[0,\infty)\times\mathbb{R}|u_i^-(t,x)\leq 0\}$ for $i=1,2$. Next, we consider the following four cases.

Case I. $(t,x)\in D_1\cap D_2$. In this case $\underline{\mathbf{u}}\equiv\mathbf{0}$ and hence $\mathcal{L}_k[\underline{\mathbf{u}}](t,x)=0$ for $k=1,2$.

Case II. $(t,x)\in C_1\cap C_2$. Then $\underline{u}_1(t,x)=u_1^-(t,x)$ and $\underline{u}_2(t,x)=u_2^-(t,x)$. Note that $(\Phi_1)_t=d_1(\Phi_1)_{zz}+c(\Phi_1)_z+g_1(t,\Phi_1,\Phi_2)$ and $g_1(t,1,1)=0$, direct computations show that

$$
\begin{aligned}
\mathcal{L}_1[\underline{\mathbf{u}}](t,x) &= (u_1^-)_t-d_1(u_1^-)_{xx}-g_1(t,u_1^-,u_2^-)\\
&=\frac{\partial}{\partial t}\Phi_1(t,\eta^+)-c\frac{\partial}{\partial z}\Phi_1(t,\eta^+)-\sigma_1\delta^-\beta_1e^{-\beta_1t}\frac{\partial}{\partial z}\Phi_1(t,\eta^+)-d_1\frac{\partial^2}{\partial zz}\Phi_1(t,\eta^+)\\
&\quad+\frac{\partial}{\partial t}\Phi_1(t,\eta^-)-c\frac{\partial}{\partial z}\Phi_1(t,\eta^-)-\sigma_1\delta^-\beta_1e^{-\beta_1t}\frac{\partial}{\partial z}\Phi_1(t,\eta^-)-d_1\frac{\partial^2}{\partial zz}\Phi_1(t,\eta^-)\\
&\quad-\delta^-e^{-\beta_1t}\Big[\frac{\partial}{\partial t}p_1(t,\eta^+)-c\frac{\partial}{\partial z}p_1(t,\eta^+)-\sigma_1\delta^-\beta_1e^{-\beta_1t}\frac{\partial}{\partial z}p_1(t,\eta^+)-d_1\frac{\partial^2}{\partial zz}p_1(t,\eta^+)\Big]\\
&\quad+\delta^-\beta_1e^{-\beta_1t}p_1(t,\eta^+)-g_1(t,u_1^-,u_2^-)\\
&=\delta^-e^{-\beta_1t}\Big[c\frac{\partial}{\partial z}p_1(t,\eta^+)+\sigma_1\delta^-\beta_1e^{-\beta_1t}\frac{\partial}{\partial z}p_1(t,\eta^+)+\beta_1p_1(t,\eta^+)\\
&\quad-\frac{\partial}{\partial t}p_1(t,\eta^+)+d_1\frac{\partial^2}{\partial zz}p_1(t,\eta^+)\Big]-\sigma_1\delta^-\beta_1e^{-\beta_1t}\big(\frac{\partial}{\partial z}\Phi_1(t,\eta^+)+\frac{\partial}{\partial z}\Phi_1(t,\eta^-)\big)\\
&\quad+g_1(t,\Phi_1(t,\eta^+),\Phi_2(t,\eta^+))-g_1(t,1,1)+g_1(t,\Phi_1(t,\eta^-),\Phi_2(t,\eta^-))-g_1(t,u_1^-,u_2^-)\\
&=\delta^-e^{-\beta_1t}\Big[c\frac{\partial}{\partial z}p_1(t,\eta^+)+\sigma_1\delta^-\beta_1e^{-\beta_1t}\frac{\partial}{\partial z}p_1(t,\eta^+)+\beta_1p_1(t,\eta^+)\\
&\quad-\frac{\partial}{\partial t}p_1(t,\eta^+)+d_1\frac{\partial^2}{\partial zz}p_1(t,\eta^+)\Big]-\sigma_1\delta^-\beta_1e^{-\beta_1t}\big(\frac{\partial}{\partial z}\Phi_1(t,\eta^+)+\frac{\partial}{\partial z}\Phi_1(t,\eta^-)\big)\\
&\quad-\big[\partial_{u_1}g_1\big(t,\Phi_1(t,\eta^+)+\theta(1-\Phi_1(t,\eta^+)),\Phi_2(t,\eta^+)+\theta(1-\Phi_2(t,\eta^+))\big)(1-\Phi_1(t,\eta^+))\\
&\quad+\partial_{u_2}g_1\big(t,\Phi_1(t,\eta^+)+\theta(1-\Phi_1(t,\eta^+)),\Phi_2(t,\eta^+)+\theta(1-\Phi_2(t,\eta^+))\big)(1-\Phi_2(t,\eta^+))\big]\\
&\quad+\partial_{u_1}g_1\big(t,\Phi_1(t,\eta^-)+\theta(\Phi_1(t,\eta^+)-1-\delta^-p_1(t,\eta^+)e^{-\beta_1t}),\Phi_2(t,\eta^-)\\
&\quad+\theta(\Phi_2(t,\eta^+)-1-\delta^-p_2(t,\eta^+)e^{-\beta_1t})\big)[1-\Phi_1(t,\eta^+)+\delta^-p_1(t,\eta^+)e^{-\beta_1t}]\\
&\quad+\partial_{u_2}g_1\big(t,\Phi_1(t,\eta^-)+\theta(\Phi_1(t,\eta^+)-1-\delta^-p_1(t,\eta^+)e^{-\beta_1t}),\Phi_2(t,\eta^-)\\
&\quad+\theta\big(\Phi_2(t,\eta^+)-1-\delta^-p_2(t,\eta^+)e^{-\beta_1t}\big)\big)[1-\Phi_2(t,\eta^+)+\delta^-p_2(t,\eta^+)e^{-\beta_1t}]\\
&=\delta^-e^{-\beta_1t}\Big[c\frac{\partial}{\partial z}p_1(t,\eta^+)+\sigma_1\delta^-\beta_1e^{-\beta_1t}\frac{\partial}{\partial z}p_1(t,\eta^+)+\beta_1p_1(t,\eta^+)-\frac{\partial}{\partial t}p_1(t,\eta^+)\\
&\quad+d_1\frac{\partial^2}{\partial zz}p_1(t,\eta^+)+\partial_1\tilde{g}_1(t,\eta^+,\eta^-;\theta,\delta^-)p_1(t,\eta^+)+\partial_2\tilde{g}_1(t,\eta^+,\eta^-;\theta,\delta^-)p_2(t,\eta^+)\Big]\\
&\quad+[\partial_1\tilde{g}_1(t,\eta^+,\eta^-;\theta,\delta^-)-\partial_1g_1(t,\eta^+;\theta)](1-\Phi_1(t,\eta^+))\\
&\quad+[\partial_2\tilde{g}_1(t,\eta^+,\eta^-;\theta,\delta^-)-\partial_2g_1(t,\eta^+;\theta)](1-\Phi_2(t,\eta^+))\\
&\quad-\sigma_1\delta^-\beta_1e^{-\beta_1t}\big(\frac{\partial}{\partial z}\Phi_1(t,\eta^+)+\frac{\partial}{\partial z}\Phi_1(t,\eta^-)\big).
\end{aligned}\tag{2.9}
$$

Next, we first show that $\mathcal{L}[\underline{\mathbf{u}}](t,x)\le 0$ for $t\ge 0$ and $x\ge 0$. We distinguish among three subcases.

Case II-1. $\eta^-(t,x)\ge M^+$. Since $\eta^+(t,x)\ge\eta^-(t,x)\ge M^+$ for $t\ge 0$ and $x\ge 0$, it follows that $p_1(t,\eta^+)=\psi_1(t)$ and $p_2(t,\eta^+)=\psi_2(t)$. Moreover, $\frac{\partial}{\partial z}p_1(t,\eta^+)=0$, $\frac{\partial^2}{\partial z^2}p_1(t,\eta^+)=0$ and

$$
\begin{aligned}
\frac{\partial}{\partial t}p_1(t,\eta^+)=\psi_1'(t)&=-a_1(t)p(t)\psi_1(t)+b_1(t)q(t)\psi_2(t)+\lambda_1\psi_1(t)\\
&=(g_1)_{u_1}(t,1,1)\psi_1(t)+(g_1)_{u_2}(t,1,1)\psi_2(t)+\lambda_1\psi_1(t).
\end{aligned}
$$

From (2.5), we have for any $t \geq 0$ and $x \geq 0$ that

$$\begin{aligned}\mathbf{1} - \mathbf{\Phi}(t, \eta^+) =& \mathbf{1} - \mathbf{\Phi}(t, x - ct + \xi^- - \sigma_1\delta^-(1 - e^{-\beta_1 t})) \\ \leq& \mathbf{1} - \mathbf{\Phi}(t, -ct + \xi^- - 1) \leq \bar{K} e^{-\nu(-ct+\xi^- -1)}\mathbf{1}.\end{aligned} \tag{2.10}$$

Hence, by (2.4), (2.7), (2.9)–(2.10) and the monotonicity of $\mathbf{\Phi}(t, z)$ in z, we deduce that for $t \geq 0$ and $x \geq 0$,

$$\begin{aligned}\mathcal{L}_1[\underline{\mathbf{u}}](t, x) \leq& \delta^- e^{-\beta_1 t}\big\{\beta_1\psi_1(t) - [(g_1)_{u_1}(t, 1, 1)\psi_1(t) + (g_1)_{u_2}(t, 1, 1)\psi_2(t) + \lambda_1\psi_1(t)] \\ &+ \partial_1\tilde{g}_1(t, \eta^+, \eta^-; \theta, \delta^-)\psi_1(t) + \partial_2\tilde{g}_1(t, \eta^+, \eta^-; \theta, \delta^-)\psi_2(t)\big\} \\ &+ [\partial_1\tilde{g}_1(t, \eta^+, \eta^-; \theta, \delta^-) - \partial_1 g_1(t, \eta^+; \theta)](1 - \Phi_1(t, \eta^+)) \\ &+ [\partial_2\tilde{g}_1(t, \eta^+, \eta^-; \theta, \delta^-) - \partial_2 g_1(t, \eta^+; \theta)](1 - \Phi_2(t, \eta^+)) \\ \leq& \delta^- e^{-\beta_1 t}\big\{\beta_1\psi_1(t) + [\partial_1\tilde{g}_1(t, \eta^+, \eta^-; \theta, \delta^-) - (g_1)_{u_1}(t, 1, 1)]\psi_1(t) \\ &+ [\partial_2\tilde{g}_1(t, \eta^+, \eta^-; \theta, \delta^-) - (g_1)_{u_2}(t, 1, 1)]\psi_2(t) - \lambda_1\psi_1(t)\big\} \\ &+ K_4\bar{K}e^{-\nu(-ct+\xi^- -1)} \\ \leq& \delta^- e^{-\beta_1 t}\min\{\psi_{1m}, \psi_{2m}\}\big(\beta_1 - \frac{1}{2}\lambda_1\big) + K_4\bar{K}e^{-\nu(-ct+\xi^- -1)} \\ \leq& e^{-\beta_1 t}\big[\delta^- \min\{\psi_{1m}, \psi_{2m}\}\big(\beta_1 - \frac{1}{2}\lambda_1\big) + K_4\bar{K}e^{-\nu(\xi^- -1)}\big] \leq 0.\end{aligned}$$

Case II-2. $|\eta^-(t, x)| \leq M^+$. By (2.8)–(2.10) and the definitions of σ_1, δ^- and K_1, we have for $t \geq 0$ and $x \geq 0$ that

$$\begin{aligned}\mathcal{L}_1[\underline{\mathbf{u}}](t, x) \leq& \ \delta^- e^{-\beta_1 t}\big(K_1|c| + \sigma_1\delta^-\beta_1 K_1 + \beta_1 + d_1 K_1 + K_2 + K_0 - 2\sigma_1\beta_1 K_3\big) \\ &+ K_4\bar{K}e^{-\nu(-ct+\xi^- -1)} \\ \leq& \ \delta^- e^{-\beta_1 t}(K_1|c| - \sigma_1\beta_1 K_3 + \beta_1 + d_1 K_1 + K_2 + K_0) + K_4\bar{K}e^{-\nu(-ct+\xi^- -1)} \\ \leq& \ e^{-\beta_1 t}\{\delta^-[K_1|c| - \sigma_1\beta_1 K_3 + \beta_1 + d_1 K_1 + K_2 + K_0] + K_4\bar{K}e^{-\nu(\xi^- -1)}\} \leq 0.\end{aligned}$$

Case II-3. $\eta^-(t, x) \leq -M^+$. That is, $-x - ct + \xi^- - \sigma_1\delta^-(1 - e^{-\beta_1 t}) \leq -M^+$, then $-x + \xi^- - \sigma_1\delta^- \leq -M^+$, which yields that $x \geq M^+ + \xi^- - \sigma_1\delta^-$. Hence, for any $t \geq 0$,

$$\eta^+(t, x) = x - ct + \xi^- - \sigma_1\delta^-(1 - e^{-\beta_1 t}) \geq M^+ + 2\xi^- - 2\sigma_1\delta^- \geq M^+.$$

Then, $p_1(t, \eta^+) = \phi_1(t)$ and $p_2(t, \eta^+) = \phi_2(t)$. Using (2.3), (2.7) and similar to **Case II-1**, one can easily prove that $\mathcal{L}_1[\underline{\mathbf{u}}](t, x) \leq 0$ for $t \geq 0$ and $x \geq 0$.

By the above discussions, we obtain $\mathcal{L}_1[\underline{\mathbf{u}}](t, x) \leq 0$ for $t \geq 0$ and $x \geq 0$. Similarly, one can easily verify that $\mathcal{L}_2[\underline{\mathbf{u}}](t, x) \leq 0$ for $t \geq 0$ and $x \geq 0$. Therefore, $\mathcal{L}[\underline{\mathbf{u}}](t, x) \leq 0$ for $t \geq 0$ and $x \geq 0$.

A similar argument shows that $\mathcal{L}[\underline{\mathbf{u}}](t, x) \leq 0$ for $t \geq 0$ and $x \leq 0$. Therefore, we have $\mathcal{L}[\underline{\mathbf{u}}](t, x) \leq 0$ for $t \geq 0$ and $x \in \mathbb{R}$.

Case III. $(t, x) \in C_1 \cap D_2$. Then $\underline{u}_1(t, x) = u_1^-(t, x) > 0$ and $\underline{u}_2(t, x) = 0$. Thus,

$$\mathcal{L}_2[\underline{\mathbf{u}}](t, x) = -g_2(t, u_1^-(t, x), 0) = -a_2(t)p(t)u_1^-(t, x) \leq 0.$$

Moreover, since $u_2^- \leq 0$, we have $g_1(t, u_1^-, u_2^-) \leq g_1(t, u_1^-, 0)$. Thus

$$\begin{aligned}\mathcal{L}_1[\underline{\mathbf{u}}](t, x) &= (u_1^-)_t - d_1(u_1^-)_{xx} - g_1(t, u_1^-, 0) \\ &= \delta^- e^{-\beta_1 t}\Big[c\frac{\partial}{\partial z}p_1(t, \eta^+) + \sigma_1\delta^-\beta_1 e^{-\beta_1 t}\frac{\partial}{\partial z}p_1(t, \eta^+) + \beta_1 p_1(t, \eta^+)\end{aligned}$$

$$
\begin{aligned}
&- \frac{\partial}{\partial t} p_1(t,\eta^+) + d_1 \frac{\partial^2}{\partial zz} p_1(t,\eta^+)\Big] - \sigma_1\delta^-\beta_1 e^{-\beta_1 t}\big(\frac{\partial}{\partial z}\Phi_1(t,\eta^+) + \frac{\partial}{\partial z}\Phi_1(t,\eta^-)\big)\\
&+ g_1(t,\Phi_1(t,\eta^+),\Phi_2(t,\eta^+)) - g_1(t,1,1) + g_1(t,\Phi_1(t,\eta^-),\Phi_2(t,\eta^-)) - g_1(t,u_1^-,0)\\
&\le \delta^- e^{-\beta_1 t}\Big[c\frac{\partial}{\partial z} p_1(t,\eta^+) + \sigma_1\delta^-\beta_1 e^{-\beta_1 t}\frac{\partial}{\partial z} p_1(t,\eta^+) + \beta_1 p_1(t,\eta^+)\\
&- \frac{\partial}{\partial t} p_1(t,\eta^+) + d_1 \frac{\partial^2}{\partial zz} p_1(t,\eta^+)\Big] - \sigma_1\delta^-\beta_1 e^{-\beta_1 t}\big(\frac{\partial}{\partial z}\Phi_1(t,\eta^+) + \frac{\partial}{\partial z}\Phi_1(t,\eta^-)\big)\\
&+ g_1(t,\Phi_1(t,\eta^+),\Phi_2(t,\eta^+)) - g_1(t,1,1) + g_1(t,\Phi_1(t,\eta^-),\Phi_2(t,\eta^-)) - g_1(t,u_1^-,u_2^-).
\end{aligned}
$$

By using the same method in **Case II**, we obtain that $\mathcal{L}_1[\underline{\mathbf{u}}](t,x) \le 0$ for $t \ge 0$ and $x \in \mathbb{R}$. Therefore, $\mathcal{L}[\underline{\mathbf{u}}](t,x) \le 0$ for $t \ge 0$ and $x \in \mathbb{R}$.

Case IV. $(t,x) \in D_1 \cap C_2$. In this case, $\underline{u}_1(t,x) = 0$ and $\underline{u}_2(t,x) = u_2^-(t,x) > 0$. Then, $\mathcal{L}_1[\underline{\mathbf{u}}](t,x) = -g_1(t,0,u_2^-) = 0$. Moreover, we know that $g_2(t,u_1^-,u_2^-) \le g_2(t,0,u_2^-)$ because of $u_1^- \le 0$ and (2.1), thus

$$
\begin{aligned}
\mathcal{L}_2[\underline{\mathbf{u}}](t,x) &= (u_2^-)_t - d_2(u_2^-)_{xx} - g_2(t,0,u_2^-)\\
&\le \delta^- e^{-\beta_1 t}\Big[c\frac{\partial}{\partial z} p_2(t,\eta^+) + \sigma_1\delta^-\beta_1 e^{-\beta_1 t}\frac{\partial}{\partial z} p_2(t,\eta^+) + \beta_1 p_2(t,\eta^+)\\
&\quad - \frac{\partial}{\partial t} p_2(t,\eta^+) + d_2 \frac{\partial^2}{\partial zz} p_2(t,\eta^+)\Big] - \sigma_1\delta^-\beta_1 e^{-\beta_1 t}\big(\frac{\partial}{\partial z}\Phi_2(t,\eta^+) + \frac{\partial}{\partial z}\Phi_2(t,\eta^-)\big)\\
&\quad + g_2(t,\Phi_1(t,\eta^+),\Phi_2(t,\eta^+)) - g_2(t,1,1) + g_2(t,\Phi_1(t,\eta^-),\Phi_2(t,\eta^-)) - g_2(t,u_1^-,u_2^-).
\end{aligned}
$$

By using an argument as in **Case II**, we have $\mathcal{L}_2[\underline{\mathbf{u}}](t,x) \le 0$ for $t \ge 0$ and $x \in \mathbb{R}$. Hence, $\mathcal{L}[\underline{\mathbf{u}}](t,x) \le 0$ for $t \ge 0$ and $x \in \mathbb{R}$.

Combining **Case I-Case IV**, we infer that $\mathcal{L}[\underline{\mathbf{u}}](t,x) \le 0$ for $t \ge 0$ and $x \in \mathbb{R}$; that is, $\underline{\mathbf{u}}(t,x)$ is a subsolution of system (1.4). The proof is completed. □

Lemma 2.3 *There exists a large enough constant $\tilde{L} > 0$ so that for any $L \ge \tilde{L}$, there exist constants ξ_1, ξ_2 and $\delta > 0$ such that*

$$
\begin{aligned}
&\mathbf{\Phi}(t,x-ct+\xi_2) + \mathbf{\Phi}(t,-x-ct+\xi_2) - \mathbf{1} - \delta e^{-\beta_1 t}\mathbf{1} \le \mathbf{u}(t,x)\\
&\quad \le \mathbf{\Phi}(t,x-ct+\xi_1) + \mathbf{\Phi}(t,-x-ct+\xi_1) - \mathbf{1} + \delta e^{-\beta_1 t}\mathbf{1},\ \forall t \ge 0, x \in \mathbb{R}. \qquad (2.11)
\end{aligned}
$$

Proof We first prove the left inequality of (2.11). Recall that $\lim_{z\to-\infty}\mathbf{\Phi}(t,z) = \mathbf{0}$ and $\lim_{z\to+\infty}\mathbf{\Phi}(t,z) = \mathbf{1}$ uniformly in $t \in [0,T]$. Thus, there exists $\tilde{L} > 0$ such that for any $L \ge \tilde{L}$, $\mathbf{u}^-(0,x) \le \mathbf{0}$ for $|x| \ge L$, and hence $\underline{\mathbf{u}}(0,x) = \mathbf{0} \le \boldsymbol{\varphi}(x)$ for $|x| \ge L$. From Lemma 2.2, we infer that

$$
\mathbf{u}^-(0,x) = \mathbf{\Phi}(0,x+\xi^-) + \mathbf{\Phi}(0,-x+\xi^-) - \mathbf{1} - \delta^-\mathbf{p}(0,x+\xi^-) \le \mathbf{1} - \delta^-\bar{p}\mathbf{1} < \mathbf{1},
$$

which yields that $\underline{\mathbf{u}}(0,x) = \max\{\mathbf{0},\mathbf{u}^-(0,x)\} < \mathbf{1} = \boldsymbol{\varphi}(x)$ for $|x| < L$. Therefore, for any $L \ge \tilde{L}$, we have $\underline{\mathbf{u}}(0,x) \le \boldsymbol{\varphi}(x)$, $\forall x \in \mathbb{R}$. Set $\xi_2 = \xi^- - \sigma_1\delta^-$. By Lemma 2.2 and comparison theorem, we have

$$
\begin{aligned}
\mathbf{u}(t,x) &\ge \underline{\mathbf{u}}(t,x)\\
&\ge \mathbf{\Phi}(t,x-ct+\xi^- - \sigma_1\delta^-(1-e^{-\beta_1 t})) + \mathbf{\Phi}(t,-x-ct+\xi^- - \sigma_1\delta^-(1-e^{-\beta_1 t}))\\
&\quad - \mathbf{1} - \delta^-\mathbf{p}(t,x-ct+\xi^- - \sigma_1\delta^-(1-e^{-\beta_1 t}))e^{-\beta_1 t}\\
&\ge \mathbf{\Phi}(t,x-ct+\xi^- - \sigma_1\delta^-) + \mathbf{\Phi}(t,-x-ct+\xi^- - \sigma_1\delta^-) - \mathbf{1} - \delta^- e^{-\beta_1 t}\mathbf{1}\\
&= \mathbf{\Phi}(t,x-ct+\xi_2) + \mathbf{\Phi}(t,-x-ct+\xi_2) - \mathbf{1} - \delta^- e^{-\beta_1 t}\mathbf{1},\ \forall t \ge 0, x \in \mathbb{R}.
\end{aligned}
\qquad (2.12)
$$

Next, we prove the right inequality of (2.11). For any $\boldsymbol{\varphi}(x) \in \mathcal{X}$, we can choose $\xi_0^+ > 0$ large enough such that $\boldsymbol{\varphi}(x) \le \boldsymbol{\Phi}(0, x + \xi_0^+) + \delta_0^+ \mathbf{p}(0, x + \xi_0^+)$ for $x \in \mathbb{R}$. Moreover, by Bao and Wang [1, Lemma 3.4], there exist $\sigma_0^+ > 0$, $\delta_0^+ > 0$ such that the function

$$\begin{aligned}\hat{\mathbf{u}}^+(t,x) = \boldsymbol{\Phi}(t, x - ct + \xi_0^+ + \sigma_0^+\delta_0^+(1 - e^{-\beta_1 t}))\\ + \delta_0^+ \mathbf{p}(t, x - ct + \xi_0^+ + \sigma_0^+\delta_0^+(1 - e^{-\beta_1 t}))e^{-\beta_1 t}\end{aligned}$$

is a supersolution of (1.4). Thus, we have

$$\begin{aligned}\mathbf{u}(t,x) &\le \boldsymbol{\Phi}(t, x - ct + \xi_0^+ + \sigma_0^+\delta_0^+(1 - e^{-\beta_1 t}))\\ &\quad + \delta_0^+ \mathbf{p}(t, x - ct + \xi_0^+ + \sigma_0^+\delta_0^+(1 - e^{-\beta_1 t}))e^{-\beta_1 t}\\ &\le \boldsymbol{\Phi}(t, x - ct + \xi_0^+ + \sigma_0^+\delta_0^+(1 - e^{-\beta_1 t})) + \delta_0^+ e^{-\beta_1 t}\mathbf{1},\ t \ge 0,\ x \in \mathbb{R}. \qquad (2.13)\end{aligned}$$

Similarly, we can conclude that

$$\begin{aligned}\mathbf{u}(t,x) &\le \boldsymbol{\Phi}(t, -x - ct + \xi_1^+ + \sigma_1^+\delta_1^+(1 - e^{-\beta_1 t}))\\ &\quad + \delta_1^+ \mathbf{p}(t, x - ct + \xi_1^+ + \sigma_1^+\delta_1^+(1 - e^{-\beta_1 t}))e^{-\beta_1 t}\\ &\le \boldsymbol{\Phi}(t, -x - ct + \xi_1^+ + \sigma_1^+\delta_1^+(1 - e^{-\beta_1 t})) + \delta_1^+ e^{-\beta_1 t}\mathbf{1},\ t \ge 0,\ x \in \mathbb{R} \qquad (2.14)\end{aligned}$$

for any $\xi_1^+ \in \mathbb{R}$ and some constants $\sigma_1^+ > 0$, $\delta_1^+ > 0$. Take $\xi^+ = \max\{\xi_0^+, \xi_1^+\}$, $\delta^+ = \max\{\delta_0^+, \delta_1^+\}$, and $\sigma^+ = \max\{\sigma_0^+, \sigma_1^+\}$. By the monotonicity of $\boldsymbol{\Phi}(t, z)$ with respect to z (see Lemma 1.1), we obtain

$$\begin{aligned}\mathbf{u}(t,x) \le \min\{&\boldsymbol{\Phi}(t, x - ct + \xi^+ + \sigma^+\delta^+(1 - e^{-\beta_1 t})),\\ &\boldsymbol{\Phi}(t, -x - ct + \xi^+ + \sigma^+\delta^+(1 - e^{-\beta_1 t}))\} + \delta^+ e^{-\beta_1 t}\mathbf{1}. \qquad (2.15)\end{aligned}$$

(**i**) For $x \ge 0$, by the monotonicity of $\boldsymbol{\Phi}(t, z)$ in z, one has

$$\boldsymbol{\Phi}(t, x - ct + \xi^+ + \sigma^+\delta^+(1 - e^{-\beta_1 t})) \ge \boldsymbol{\Phi}(t, -x - ct + \xi^+ + \sigma^+\delta^+(1 - e^{-\beta_1 t})).$$

By (2.5), there holds

$$\mathbf{1} - \boldsymbol{\Phi}(t, x - ct + \xi^+ + \sigma^+\delta^+(1 - e^{-\beta_1 t})) \le \mathbf{1} - \boldsymbol{\Phi}(t, -ct + \xi^+) \le \bar{K} e^{-\nu(\xi^+ - ct)}\mathbf{1}. \qquad (2.16)$$

Hence, combining (2.15) with (2.16), we have

$$\begin{aligned}\mathbf{u}(t,x) &\le \boldsymbol{\Phi}(t, -x - ct + \xi^+ + \sigma^+\delta^+(1 - e^{-\beta_1 t})) + \delta^+ e^{-\beta_1 t}\mathbf{1}\\ &\le \boldsymbol{\Phi}(t, -x - ct + \xi^+ + \sigma^+\delta^+(1 - e^{-\beta_1 t})) + \boldsymbol{\Phi}(t, x - ct + \xi^+ + \sigma^+\delta^+(1 - e^{-\beta_1 t}))\\ &\quad - \mathbf{1} + \delta^+ e^{-\beta_1 t}\mathbf{1} + \bar{K} e^{-\nu(\xi^+ - ct)}\mathbf{1}\\ &\le \boldsymbol{\Phi}(t, -x - ct + \xi^+ + \sigma^+\tilde{\delta}_1) + \boldsymbol{\Phi}(t, x - ct + \xi^+ + \sigma^+\tilde{\delta}_1) + \tilde{\delta}_1 e^{-\beta_1 t}\mathbf{1} - \mathbf{1} \qquad (2.17)\end{aligned}$$

by choosing $\tilde{\delta}_1 > \delta^+$ and $\xi^+ > 0$ large enough.

(**ii**) For $x < 0$, since

$$\boldsymbol{\Phi}(t, x - ct + \xi^+ + \sigma^+\delta^+(1 - e^{-\beta_1 t})) \le \boldsymbol{\Phi}(t, -x - ct + \xi^+ + \sigma^+\delta^+(1 - e^{-\beta_1 t})),$$

using (2.5) again, we obtain

$$\mathbf{1} - \boldsymbol{\Phi}(t, -x - ct + \xi^+ + \sigma^+\delta^+(1 - e^{-\beta_1 t})) \le \mathbf{1} - \boldsymbol{\Phi}(t, -ct + \xi^+) \le \bar{K} e^{-\nu(\xi^+ - ct)}\mathbf{1}.$$

It then follows from (2.15) that

$$\begin{aligned}\mathbf{u}(t,x) &\le \mathbf{\Phi}(t, x-ct+\xi^+ + \sigma^+\delta^+(1-e^{-\beta_1 t})) + \delta^+ e^{-\beta_1 t}\mathbf{1}\\ &\le \mathbf{\Phi}(t, x-ct+\xi^+ + \sigma^+\delta^+(1-e^{-\beta_1 t})) + \mathbf{\Phi}(t, -x-ct+\xi^+ + \sigma^+\delta^+(1-e^{-\beta_1 t}))\\ &\quad -\mathbf{1} + \delta^+ e^{-\beta_1 t}\mathbf{1} + \bar{K}e^{-\nu(\xi^+ - ct)}\mathbf{1}\\ &\le \mathbf{\Phi}(t, x-ct+\xi^+ + \sigma^+\tilde{\delta}_2) + \mathbf{\Phi}(t, -x-ct+\xi^+ + \sigma^+\tilde{\delta}_2) + \tilde{\delta}_2 e^{-\beta_1 t}\mathbf{1} - \mathbf{1} \end{aligned} \tag{2.18}$$

by choosing $\tilde{\delta}_2 > \delta^+$ and $\xi^+ > 0$ large enough.

Take $\hat{\delta}^+ = \max\{\tilde{\delta}_1, \tilde{\delta}_2\}$ and let $\xi_1 = \xi^+ + \sigma^+\hat{\delta}^+$, by (2.17) and (2.18), we have

$$\mathbf{u}(t,x) \le \mathbf{\Phi}(t, x-ct+\xi_1) + \mathbf{\Phi}(t, -x-ct+\xi_1) - \mathbf{1} + \hat{\delta}^+ e^{-\beta_1 t}\mathbf{1}, \forall t \ge 0, x \in \mathbb{R}. \tag{2.19}$$

Choosing $\delta = \max\{\delta^-, \hat{\delta}^+\}$, by (2.12) and (2.19), we complete the proof. □

We are now ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2 We only prove the first assertion since the second one can be discussed similarly. From Lemma 2.3, we have

$$\begin{aligned}&|\mathbf{u}(t,x) - \mathbf{\Phi}(t, x-ct+\xi_1)|\\ &\quad \le |\mathbf{\Phi}(t, -x-ct+\xi_1) - \mathbf{1} + \delta e^{-\beta_1 t}\mathbf{1}|\\ &\qquad + |\mathbf{\Phi}(t, x-ct+\xi_2) + \mathbf{\Phi}(t, -x-ct+\xi_2) - \mathbf{1} - \delta e^{-\beta_1 t}\mathbf{1} - \mathbf{\Phi}(t, x-ct+\xi_1)|\\ &\quad \le |\mathbf{\Phi}(t, -x-ct+\xi_1) - \mathbf{1} + \delta e^{-\beta_1 t}\mathbf{1}|\\ &\qquad + |\mathbf{\Phi}(t, x-ct+\xi_2) - \mathbf{\Phi}(t, x-ct+\xi_1)| + |\mathbf{\Phi}(t, -x-ct+\xi_2) - \mathbf{1} - \delta e^{-\beta_1 t}\mathbf{1}|.\end{aligned}$$

Note that

$$\lim_{t\to+\infty} |\mathbf{\Phi}(t, x-ct+\xi_2) - \mathbf{\Phi}(t, x-ct+\xi_1)| = 0 \text{ locally uniformly in } x.$$

Thus, combining this with $\beta_1 > 0$ and $c < 0$, we have

$$\lim_{t\to+\infty} \|\mathbf{u}(t,x) - \mathbf{\Phi}(t, x-ct+\xi_1)\|_{L^\infty_{\mathrm{loc}}((-\infty,0])} = 0.$$

This completes the proof of Theorem 1.2. □

3 Spreading properties

In this section, we consider the long time behavior of solutions to (1.1) with initial values (1.7). We always assume that the hypotheses of Theorem 1.5 hold. Obviously, in the following discussion $c_U = 2\sqrt{d_1 r_1}$. To establish the spreading properties of solutions to system (1.1) with initial values (1.7), we introduce the following two crucial comparison principles for competitive system (1.1). We denote $(u_1, v_1) \preceq (u_2, v_2)$ if $u_1 \ge u_2$ and $v_1 \le v_2$.

Definition 3.1 The bounded functions (u_1, v_1), (u_2, v_2) are called *a pair of super- and sub-solutions* of system (1.1) on $\mathbb{R}^+ \times \mathbb{R}$ if (u_1, v_1) and (u_2, v_2) are C^1 in t and C^2 in x, and satisfy

$$\begin{cases} (u_1)_t - d_1(u_1)_{xx} - u_1[r_1 - a_1(t)u_1 - b_1(t)v_1] \ge 0,\\ (v_1)_t - d_2(v_1)_{xx} - v_1[r_2(t) - a_2(t)u_1 - b_2(t)v_1] \le 0 \end{cases}$$

and

$$\begin{cases} (u_2)_t - d_1(u_2)_{xx} - u_2[r_1 - a_1(t)u_2 - b_1(t)v_2] \leq 0, \\ (v_2)_t - d_2(v_2)_{xx} - v_2[r_2(t) - a_2(t)u_2 - b_2(t)v_2] \geq 0, \end{cases}$$

respectively, for $(t, x) \in (0, \infty) \times \mathbb{R}$.

The proofs of the following two lemmas are similar to those of [1, Theorem 3.2] and [47, Lemma 4.7] and are omitted.

Lemma 3.2 *Let (u_1, v_1), (u_2, v_2) be a pair of super- and subsolutions of system (1.1) on $\mathbb{R}^+ \times \mathbb{R}$. Assume that $(u_1, v_1)(0, x) \preceq (u_2, v_2)(0, x)$, $\forall x \in \mathbb{R}$. Then, $(u_1, v_1)(t, x) \preceq (u_2, v_2)(t, x)$, $\forall t \geq 0$, $x \in \mathbb{R}$.*

Lemma 3.3 *Let (u_1, v_1), (u_2, v_2) be a pair of super- and subsolutions of system (1.1) on $\{(t, x)|t \geq 0, x \geq X(t)\}$, where $X(\cdot) : [0, +\infty) \to \mathbb{R}$ is a continuous function. Assume that $(u_1, v_1)(t, x) \preceq (u_2, v_2)(t, x)$, $\forall t > 0$, $x \leq X(t)$ and $(u_1, v_1)(0, x) \preceq (u_2, v_2)(0, x)$ for any $x \geq X(0)$. Then, $(u_1, v_1)(t, x) \preceq (u_2, v_2)(t, x)$, $\forall t \geq 0$, $x \geq X(t)$.*

3.1 Proof of Theorem 1.5

Under the assumptions of Theorem 1.5, it is easy to see that the Cauchy problem (1.1) with the initial values (1.7) has a unique solution $\mathbf{u}(t, x) = (u(t, x), v(t, x))$ satisfying $(0, 0) \leq \mathbf{u}(t, x) \leq (p(t), q(t))$ for any $t \geq 0$, $x \in \mathbb{R}$. To simplify notations, we take

$$\begin{aligned} N_1[u, v](t, x) &= u_t(t, x) - d_1 u_{xx}(t, x) - u(t, x)[r_1 - a_1(t)u(t, x) - b_1(t)v(t, x)], \\ N_2[u, v](t, x) &= v_t(t, x) - d_2 v_{xx}(t, x) - v(t, x)[r_2(t) - a_2(t)u(t, x) - b_2(t)v(t, x)]. \end{aligned}$$

3.1.1 Proof of Theorem 1.5 (i)

(1) Recall that W_2 is a decreasing periodic traveling front of (1.9). By the definition of $v(0, x)$ and the fact that $W_2(0, -\infty) = q(0)$, we can choose $x_0 \in \mathbb{R}$ such that $v(0, x) \leq W_2(0, x - x_0)$, $\forall x \in \mathbb{R}$. Let $(\underline{u}(t, x), \overline{v}(t, x)) = (0, W_2(t, x - c_V t - x_0))$. Then $N_1[\underline{u}, \overline{v}](t, x) = 0$, and

$$\begin{aligned} N_2[\underline{u}, \overline{v}](t, x) = {} & (W_2)_t(t, x - c_V t - x_0) - c_V (W_2)_z(t, x - c_V t - x_0) \\ & - d_2(W_2)_{zz}(t, x - c_V t - x_0) \\ & - W_2(t, x - c_V t - x_0)[r_2(t) - b_2(t)W_2(t, x - c_V t - x_0)] = 0. \end{aligned}$$

Since $(u, v)(0, x) \preceq (\underline{u}, \overline{v})(0, x)$, $\forall x \in \mathbb{R}$, by Lemma 3.2, we get $(u, v) \preceq (\underline{u}, \overline{v})$. Therefore, for every $t \geq 0$ and $x > ct$, we have

$$v(t, x) \leq \overline{v}(t, x) = W_2(t, x - c_V t - x_0) \leq W_2(t, (c - c_V)t - x_0). \tag{3.1}$$

Using $\lim_{z \to +\infty} W_2(t, z) = 0$ uniformly in $t \in [0, T]$, we get $\lim_{t \to +\infty} \sup_{x > ct} v(t, x) = 0$, $\forall c > c_V$.

(2) Similarly, we let $(\overline{u}(t, x), \underline{v}(t, x)) = (W_1(t, x - c_U t - \tilde{x}), 0)$, where $\tilde{x} \in \mathbb{R}$ such that $(\overline{u}, \underline{v})(0, x) \preceq (u, v)(0, x)$, $\forall x \in \mathbb{R}$. Note that $N_2[\overline{u}, \underline{v}](t, x) = 0$ and

$$\begin{aligned} N_1[\overline{u}, \underline{v}](t, x) = {} & (W_1)_t(t, x - c_U t - \tilde{x}) - c_U (W_1)_z(t, x - c_U t - \tilde{x}) \\ & - d_1(W_1)_{zz}(t, x - c_U t - \tilde{x}) \end{aligned}$$

$$- W_1(t, x - c_U t - \tilde{x})[r_1 - a_1(t) W_1(t, x - c_U t - \tilde{x})] = 0.$$

Combining $\lim_{z\to+\infty} W_1(t, z) = 0$ uniformly in $t \in [0, T]$ with Lemma 3.2, we obtain $\lim_{t\to+\infty} \sup_{x>ct} u(t, x) = 0$, $\forall c > c_U$. This completes the proof of the statement (i) of Theorem 1.5.

3.1.2 Proof of Theorem 1.5 (ii)

We first give the following lemma which will play an important role in proving Theorem 1.5 (ii).

Lemma 3.4 *Assume that $c_V < c_U$. Then for any c satisfying $c_V < c < c_U$, we have*

$$\lim_{t\to+\infty} [u(t, x + ct) - p(t)] = 0 \text{ uniformly on every compact subset of } \mathbb{R}. \tag{3.2}$$

Proof Let $\overline{v}(t, x) := W_2(t, x - c_V t - \bar{x})$, where $\bar{x}$ is selected such that $v \le \overline{v}$. Since $\lim_{z\to+\infty} W_2(t, z) = 0$ uniformly in $t \in [0, T]$, for any $\varepsilon > 0$, there exists some $x_* \in \mathbb{R}$ such that

$$\overline{v}(t, x) \le \varepsilon \text{ for any } (t, x) \text{ such that } x - c_V t - \bar{x} \ge x_*.$$

The rest of the proof is divided into three steps.

Step 1. Fix any $c' \in (c, c_U)$. We prove

$$\exists\, a > 0,\ \hat{\eta} > 0 \text{ and } x_1 \in \mathbb{R} \text{ s.t. } \liminf_{t\to+\infty} \inf_{x\in(-a,a)} u\Big(\frac{ct}{c'}, x + ct + x_1\Big) \ge \hat{\eta}. \tag{3.3}$$

For any given $a > 0$, consider the following eigenvalue problem:

$$\begin{cases} d_1 \psi''_{2a}(x) = \lambda_{2a} \psi_{2a}(x), \text{ in } (-2a, 2a), \\ \psi_{2a}(x) = 0, \text{ in } \mathbb{R}\backslash(-2a, 2a), \ \psi_{2a}(x) > 0, \text{ in } (-2a, 2a), \\ \|\psi_{2a}\|_\infty = 1. \end{cases} \tag{3.4}$$

The above eigenvalue problem has a principal eigenvalue λ_{2a} with a principal eigenfunction ψ_{2a}. Moreover, $\lambda_{2a} < 0$ for any $a > 0$ and $\lambda_{2a} \to 0$ as $a \to \infty$.

Take $\varepsilon > 0$ such that

$$c' < 2\sqrt{d_1(r_1 - b_{1M}\varepsilon)} < c_U.$$

Choose $a > 0$ large enough and $\eta > 0$ small enough such that

$$\frac{c'^2}{4d_1} - \lambda_{2a} - r_1 + a_{1M}\eta e^{\frac{c'(x_*+\bar{x})}{2d_1}} + b_{1M}\varepsilon \le 0. \tag{3.5}$$

Denote $x_1 = 2a + \bar{x} + x_*$ and define

$$\underline{u}(t, x) := \eta e^{-\frac{c'}{2d_1}(x - c't)} \psi_{2a}(x - c't - x_1).$$

Next, we show that $(u, v) \preceq (\underline{u}, \overline{v})$. It is clear that

$$\begin{aligned} N_2[\underline{u}, \overline{v}](t, x) =& \overline{v}_t - d_2 \overline{v}_{xx} - \overline{v}(t, x)[r_2(t) - a_2(t)\underline{u}(t, x) - b_2(t)\overline{v}(t, x)] \\ =& a_2(t)\underline{u}(t, x) W_2(t, x - c_V t - \bar{x}) \ge 0. \end{aligned}$$

On the other hand, for $-2a < x - c't - x_1 < 2a$ and $t > 0$, we have $\underline{u}(t, x) > 0$ and

$$\overline{v}(t, x) = W_2(t, x - c_V t - \bar{x}) \le W_2(t, x - c't - \bar{x}) \le W_2(t, x_1 - 2a - \bar{x})$$

$$= W_2(t, x_*) \leq \varepsilon,$$

by which and (3.5), we get for $-2a < x - c't - x_1 < 2a$ and $t > 0$ that

$$\begin{aligned}
N_1[\underline{u}, \overline{v}](t,x) &= \underline{u}_t - d_1 \underline{u}_{xx} - \underline{u}(t,x)[r_1 - a_1(t)\underline{u}(t,x) - b_1(t)\overline{v}(t,x)] \\
&\leq \eta \frac{c'^2}{2d_1} e^{-\frac{c'}{2d_1}(x-c't)} \psi_{2a}(x - c't - x_1) - c'\eta e^{-\frac{c'}{2d_1}(x-c't)} \psi'_{2a}(x - c't - x_1) \\
&\quad - d_1\eta\Big[\frac{c'^2}{4d_1^2} e^{-\frac{c'}{2d_1}(x-c't)} \psi_{2a}(x - c't - x_1) - \frac{c'}{d_1} e^{-\frac{c'}{2d_1}(x-c't)} \psi'_{2a}(x - c't - x_1) \\
&\quad + e^{-\frac{c'}{2d_1}(x-c't)} \psi''_{2a}(x - c't - x_1)\Big] - \underline{u}(t,x)[r_1 - a_1(t)\underline{u}(t,x) - b_1(t)\varepsilon] \\
&= \eta e^{-\frac{c'}{2d_1}(x-c't)} \Big[\frac{c'^2}{4d_1} \psi_{2a}(x - c't - x_1) - d_1 \psi''_{2a}(x - c't - x_1)\Big] \\
&\quad - \underline{u}(t,x)[r_1 - a_1(t)\underline{u}(t,x) - b_1(t)\varepsilon] \\
&\leq \underline{u}(t,x)\Big[\frac{c'^2}{4d_1} - \lambda_{2a} - r_1 + a_{1M}\eta e^{\frac{c'(x_*+\bar{x})}{2d_1}} + b_{1M}\varepsilon\Big] \leq 0.
\end{aligned}$$

Since $\underline{u}(t,x) = 0$ for $|x - c't - x_1| \geq 2a$, we infer that $N_1[\underline{u}, \overline{v}](t,x) \leq 0$ for $t > 0$, $x \in \mathbb{R}$.

Recalling that u is positive for any positive time and $\psi_{2a}(\cdot)$ has compact support, we can deduce that $u(1,x) \geq \underline{u}(1,x)$ for all $x \in \mathbb{R}$. Hence, by Lemma 3.2, we have $(u, v) \preceq (\underline{u}, \overline{v})$. Therefore, we get $u(t,x) \geq \underline{u}(t,x)$ for $t \geq 1$ and $x \in \mathbb{R}$, which implies that

$$u\Big(\frac{ct}{c'}, x\Big) \geq \underline{u}\Big(\frac{ct}{c'}, x\Big) = \eta e^{-\frac{c'}{2d_1}(x-ct)} \psi_{2a}(x - ct - x_1) \text{ for } t \geq \frac{c'}{c},\ x \in \mathbb{R}.$$

Thus,

$$u\Big(\frac{ct}{c'}, x + ct + x_1\Big) \geq \eta e^{-\frac{c'}{2d_1}(x+x_1)} \psi_{2a}(x) \geq \eta e^{-\frac{c'}{2d_1}(a+x_1)} \min_{x \in (-a,a)} \psi_{2a}(x) =: \hat{\eta} > 0.$$

Hence, (3.3) holds.

Step 2. We show that there exist $a > 0$, $\eta_2 > 0$ and $x_2 \in \mathbb{R}$ such that

$$\liminf_{t \to +\infty} \inf_{t' \in [\frac{ct}{c'}, t], x \in (-\frac{a}{2}, \frac{a}{2})} u(t', x + ct + x_2) \geq \eta_2. \tag{3.6}$$

Take $a > 0$ and $\varepsilon > 0$ such that $r_1 > b_{1M}\varepsilon - \tilde{\lambda}_a$, where $(\tilde{\lambda}_a, \tilde{\psi}_a)$ satisfies (3.4) with $2a$ replaced by a. Then choose $\eta' \in (0, \hat{\eta})$ such that $\eta' a_{1M} < \tilde{\lambda}_a + r_1 - b_{1M}\varepsilon$. Fix $t > 0$ and define

$$\underline{w}(t', x) := \eta' \tilde{\psi}_a(x - ct - x_1) \text{ for } t' \in \Big[\frac{ct}{c'}, t\Big], x \in \mathbb{R}.$$

Obviously, $N_2[\underline{w}, \overline{v}](t', x) = a_2(t')\underline{w}(t', x) W_2(t', x - c_V t' - x_0) \geq 0$. Moreover, for $-a < x - ct - x_1 < a$ and $t > 0$,

$$\overline{v}(t,x) = W_2(t, x - c_V t - \bar{x}) \leq W_2(t, x - ct - \bar{x}) \leq W_2(t, x_1 - a - \bar{x}) \leq W_2(t, x_*) \leq \varepsilon,$$

and hence

$$\begin{aligned}
N_1[\underline{w}, \overline{v}](t', x) =& \underline{w}_{t'}(t', x) - d_1 \underline{w}_{xx}(t', x) - \underline{w}(t', x)[r_1 - a_1(t')\underline{w}(t', x) - b_1(t')\overline{v}(t', x)] \\
&\leq - d_1 \eta' \tilde{\psi}''_a(x - ct - x_1) - \underline{w}(t', x)[r_1 - a_1(t')\underline{w}(t', x) - b_1(t')\varepsilon] \\
&= - \eta' \tilde{\lambda}_a \tilde{\psi}_a(x - ct - x_1) - \underline{w}(t', x)[r_1 - a_1(t')\underline{w}(t', x) - b_1(t')\varepsilon]
\end{aligned}$$

$$\leq -\underline{w}(t',x)[\tilde{\lambda}_a + r_1 - \eta' a_{1M} - b_{1M}\varepsilon] \leq 0.$$

Thus, $N_1[\underline{w},\overline{v}](t',x) \leq 0$ for $x \in \mathbb{R}$ and $t' \in \left[\frac{ct}{c'}, t\right]$.

From (3.3), for $t \gg 1$ and $x \in (-a,a)$, one has $u\left(\frac{ct}{c'}, x+ct+x_1\right) \geq \hat{\eta} \geq \eta'\tilde{\psi}_a(x)$. Since $\tilde{\psi}_a(x) = 0$ for $|x| \geq a$, it then follows that $u\left(\frac{ct}{c'}, x+ct+x_1\right) \geq \eta'\tilde{\psi}_a(x)$ for $t \gg 1$ and $x \in \mathbb{R}$. Noting that $\eta'\tilde{\psi}_a(x) = \underline{w}(\frac{ct}{c'}, x+ct+x_1)$, we have $u(\frac{ct}{c'}, x) \geq \underline{w}(\frac{ct}{c'}, x)$ for $t \gg 1$ and $x \in \mathbb{R}$. Hence, by Lemma 3.2, we get $(u,v) \preceq (\underline{w},\overline{v})$ for any $t' \in [\frac{ct}{c'}, t]$, $x \in \mathbb{R}$. Taking $\eta_2 = \eta' \inf_{x \in (-\frac{a}{2},\frac{a}{2})} \tilde{\psi}_a(x)$, then (3.6) holds.

Step 3. We prove (3.2). Let $\{t_n\}_{n\in\mathbb{Z}} = \{nT\}_{n\in\mathbb{Z}}$. Define

$$u_n(t,x) = u(t+t_n, x+ct_n),\ \ v_n(t,x) = v(t+t_n, x+ct_n) \tag{3.7}$$

for any $(t,x) \in [-t_n,+\infty)\times\mathbb{R}$. By the periodicity of $a_i(t)$, $b_i(t)$, $i=1,2$ and $r_2(t)$, it is easy to see that $(u_n(t,x), v_n(t,x))$ satisfies

$$\begin{cases} (u_n)_t = d_1(u_n)_{xx} + u_n(t,x)[r_1 - a_1(t)u_n(t,x) - b_1(t)v_n(t,x)], \\ (v_n)_t = d_2(v_n)_{xx} + v_n(t,x)[r_2(t) - a_2(t)u_n(t,x) - b_2(t)v_n(t,x)], \\ u_{n0}(-t_n,x) = u(0,x+ct_n),\ \ v_{n0}(-t_n,x) = v(0,x+ct_n) \end{cases}$$

for any $(t,x) \in [-t_n,+\infty)\times\mathbb{R}$.

By standard parabolic estimates and Ascoli-Arzela theorem, there exists a subsequence of $\{t_n\}$, still denoted by $\{t_n\}$, such that $(u_n(t,x), v_n(t,x))$ converges to $(u_\infty(t,x), v_\infty(t,x))$ locally uniformly in $(t,x)\in\mathbb{R}^2$ as $n\to\infty$, and $u_\infty(t,x)$ satisfies

$$\begin{aligned} &(u_\infty)_t - d_1(u_\infty)_{xx} - u_\infty(t,x)[r_1 - a_1(t)u_\infty(t,x) \\ &\quad -b_1(t)\varepsilon] \geq 0 \end{aligned}$$

for $(t,x)\in\mathbb{R}^2$. By (3.6), we get

$$\liminf_{t_n\to+\infty} \inf_{t'\in\left[\frac{ct_n}{c'},t_n\right], x\in(-\frac{a}{2},\frac{a}{2})} u(t', x+ct_n+x_2) \geq \eta_2. \tag{3.8}$$

Note that $t+t_n \in [\frac{ct_n}{c'}, t_n]$ for $t \leq 0$ with $|t|$ small enough. Since $u_n(t,x+x_2) = u(t+t_n, x+ct_n+x_2)$, from (3.8), we get for any $t \leq 0$ with $|t|$ small enough that $\inf_{x\in(-\frac{a}{2},\frac{a}{2})} u_\infty(t,x+x_2) \geq \eta_2$. Let u_ε be the solution of

$$\begin{cases} (u_\varepsilon)_t = d_1(u_\varepsilon)_{xx} + u_\varepsilon(t,x)[r_1 - a_1(t)u_\varepsilon(t,x) - b_1(t)\varepsilon], \\ u_\varepsilon(0,x) = \tilde{g}(x), \end{cases}$$

where $\tilde{g}(x) \in C(\mathbb{R},[0,\eta_2])$ is defined by

$$\tilde{g}(x) = \begin{cases} \text{nondecreasing}, & x \in \left(-\frac{a}{2}+x_2, -\frac{a}{4}+x_2\right), \\ \eta_2, & x \in \left(-\frac{a}{4}+x_2, \frac{a}{4}+x_2\right), \\ \text{nonincreasing}, & x \in \left(\frac{a}{4}+x_2, \frac{a}{2}+x_2\right), \\ 0, & x \in \mathbb{R}\backslash\left(-\frac{a}{2}+x_2, \frac{a}{2}+x_2\right). \end{cases}$$

By the classical comparison principle, we have $u_\infty(t,x) \geq u_\varepsilon(t,x)$ for any $(t,x)\in\mathbb{R}^2$. Moreover, by the result in Liang et al. [32], we know that $\lim_{t\to+\infty}[u_\varepsilon(t,x) - \tilde{p}_\varepsilon(t)] = 0$ locally uniformly in $x\in\mathbb{R}$, where $\tilde{p}_\varepsilon(t)$ be the unique positive T-periodic solution of

$$u'(t) = u(t)(r_1 - a_1(t)u(t) - b_1(t)\varepsilon).$$

Since $\lim_{\varepsilon\to 0^+}\tilde{p}_\varepsilon(t)=p(t)$, we may assume that $\tilde{p}_\varepsilon(t)>p(t)-\varepsilon$. Thus, $\lim_{t\to+\infty}[u_\infty(t,x)-\tilde{p}_\varepsilon(t)]\geq 0$ locally uniformly in $x\in\mathbb{R}$. By the definition of u_∞, we then obtain that

$$\liminf_{t\to+\infty}[u(t,x+ct)-p(t)]\geq \liminf_{t\to+\infty}[u(t,x+ct)-\tilde{p}_\varepsilon(t)-\varepsilon]>-\varepsilon$$

locally uniformly in x. By the arbitrariness of ε, $\lim_{t\to+\infty}[u(t,x+ct)-p(t)]=0$ locally uniformly on every compact subset of $\mathbb{R}$. The proof is completed. □

Now, combining the statement (i) of Theorem 1.5 with Lemma 3.4, we directly obtain the following result.

Lemma 3.5 *Assume $c_V<c_U$. Let c_1 and c_2 be two speeds such that $c_V<c_1<c_2<c_U$. Then*

$$\lim_{t\to+\infty}\sup_{c_1t<x<c_2t}(|u(t,x)-p(t)|+|v(t,x)|)=0. \tag{3.9}$$

Proof From the proof of Theorem 1.5 (i), we know that $\lim_{t\to+\infty}\sup_{x>ct}|v(t,x)|=0,\ \forall c>c_V$, and hence,

$$\lim_{t\to+\infty}\sup_{c_1t<x<c_2t}|v(t,x)|=0,\ \forall c_V<c_1<c_2<c_U.$$

Therefore, to prove (3.9), we only need to prove

$$\lim_{t\to+\infty}\sup_{c_1t<x<c_2t}|u(t,x)-p(t)|=0,\ \forall c_V<c_1<c_2<c_U. \tag{3.10}$$

Assume by contradiction that there exist two sequences $\{t_n\}$ and $\{x_n\}$ satisfying $c_1t_n<x_n<c_2t_n$ and $t_n\to+\infty$ as $n\to\infty$, such that $\limsup_{n\to\infty}[u(t_n,x_n)-p(t_n)]<0$. Let $c_n=\frac{x_n}{t_n}$, then $c_n\in(c_1,c_2)\subset(c_V,c_U)$. Thus, there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $\lim_{j\to\infty}c_{n_j}=c\in[c_1,c_2]\subset(c_V,c_U)$. By Lemma 3.4, it then follows that

$$u(t_{n_j},x_{n_j})-p(t_{n_j})=u(t_{n_j},c_{n_j}t_{n_j})-p(t_{n_j})\to 0 \text{ as } j\to\infty,$$

which contradicts $\limsup_{n\to\infty}[u(t_n,x_n)-p(t_n)]<0$. This completes the proof. □

Remark 3.6 By using an argument similar to the proof of Lemma 3.5, we have

$$\lim_{t\to+\infty}\sup_{x<ct}(|u(t,x)|+|v(t,x)-q(t)|)=0,\ \forall c<-c_U.$$

To show Theorem 1.5 (ii) for $c<c_{UV}$, we consider the following auxiliary time-periodic competition-diffusion system with a parameter $\kappa>0$:

$$\begin{cases} u_t=d_1u_{xx}+u(t,x)[r_1-a_1(t)u(t,x)-b_1(t)v(t,x)],\\ v_t=d_2v_{xx}+v(t,x)[r_2(t)-\kappa-a_2(t)u(t,x)-b_2(t)v(t,x)]. \end{cases} \tag{3.11}$$

Choose $\kappa_0>0$ such that, for $\kappa\in(0,\kappa_0)$, (H$_1$)-(H$_3$) hold with $r_1(t)$ and $r_2(t)$ replaced by r_1 and $r_2(t)-\kappa$, respectively. According to Lemma 1.1, we know that (3.11) admits a periodic traveling front $(U_\kappa(t,z),V_\kappa(t,z))$, which satisfies

$$\begin{cases} (U_\kappa)_t-c_\kappa(U_\kappa)_z=d_1(U_\kappa)_{zz}+U_\kappa[r_1-a_1(t)U_\kappa-b_1(t)V_\kappa],\\ (V_\kappa)_t-c_\kappa(V_\kappa)_z=d_2(V_\kappa)_{zz}+V_\kappa[r_2(t)-\kappa-a_2(t)U_\kappa-b_2(t)V_\kappa],\\ \lim_{z\to-\infty}(U_\kappa(t,z),V_\kappa(t,z))=(0,q_\kappa(t)) \text{ uniformly in } t\in[0,T],\\ \lim_{z\to+\infty}(U_\kappa(t,z),V_\kappa(t,z))=(p(t),0) \text{ uniformly in } t\in[0,T],\\ (U_\kappa(t+T,z),V_\kappa(t+T,z))=(U_\kappa(t,z),V_\kappa(t,z)),\ (t,z)\in\mathbb{R}^2, \end{cases} \tag{3.12}$$

where c_κ is wave speed, $z = x - c_\kappa t$, and $q_\kappa(t)$ is the unique positive T-periodic solution of the ODE:

$$v'(t) = v(t)(r_2(t) - \kappa - b_2(t)v(t)).$$

It is clear that

$$q_\kappa(t) = \frac{q_\kappa(0)e^{\int_0^t (r_2(s)-\kappa)ds}}{1 + q_\kappa(0)\int_0^t e^{\int_0^s (r_2(\varrho)-\kappa)d\varrho} b_2(s)ds}, \quad q_\kappa(0) = \frac{e^{\int_0^T (r_2(s)-\kappa)ds} - 1}{\int_0^T e^{\int_0^s (r_2(\varrho)-\kappa)d\varrho} b_2(s)ds} > 0.$$

Hence, $\lim_{\kappa\to 0^+} q_\kappa(t) = q(t)$ uniformly in $t \in [0, T]$.

Lemma 3.7 *Let c_κ be the unique speed of a bistable periodic traveling front (U_κ, V_κ) of (3.11) connecting $(0, q_\kappa(t))$ and $(p(t), 0)$. Then $\lim_{\kappa\to 0^+} c_\kappa = c_{UV}$.*

Proof The proof is divided into two steps.

Step 1. We show that c_κ is nondecreasing in $\kappa \in (0, \infty)$. Given $0 < \kappa_1 < \kappa_2$. Let $(U_{\kappa_i}(t, z), V_{\kappa_i}(t, z))$ be the periodic traveling front of (3.11) with κ being replaced by κ_i, $i = 1, 2$. For convenience, we make a change of variables $\tilde{u} = u$ and $\tilde{v} = -v$ in (3.11). Clearly, $(\tilde{u}, \tilde{v})$ satisfies

$$\begin{cases} \tilde{u}_t = d_1\tilde{u}_{xx} + \tilde{u}(t, x)[r_1 - a_1(t)\tilde{u}(t, x) + b_1(t)\tilde{v}(t, x)], \\ \tilde{v}_t = d_2\tilde{v}_{xx} + \tilde{v}(t, x)[r_2(t) - \kappa - a_2(t)\tilde{u}(t, x) + b_2(t)\tilde{v}(t, x)]. \end{cases} \tag{3.13}$$

Note that (3.13) is a cooperative system on $[0, \infty) \times (-\infty, 0]$. Clearly, $\tilde{\mathbf{W}}_{\kappa_i} := (\tilde{U}_{\kappa_i}, \tilde{V}_{\kappa_i}) = (U_{\kappa_i}, -V_{\kappa_i})$ is the periodic traveling front of (3.13) connecting $(0, -q_{\kappa_i}(t))$ and $(p(t), 0)$, $i = 1, 2$. Without loss of generality, we normalize $\tilde{U}_{\kappa_1}(0, 0) = \frac{p(0)}{2}$.

Let $Q^{\kappa_2}(t)[\mathbf{u}_0](x)$ be the solution of (3.13) with $\kappa = \kappa_2$ and the initial value $\mathbf{u}_0(\cdot) = \tilde{\mathbf{W}}_{\kappa_1}(0, \cdot) := (\tilde{U}_{\kappa_1}(0, \cdot), \tilde{V}_{\kappa_1}(0, \cdot))$. By the comparison principle (see Bao and Wang [1, Theorem 3.2]), we get

$$\tilde{\mathbf{W}}_{\kappa_1}(t, x - c_{\kappa_1}t) \geq Q^{\kappa_2}(t)[\tilde{\mathbf{W}}_{\kappa_1}(0, \cdot)](x) \quad \text{for } t \geq 0,\ x \in \mathbb{R}.$$

In addition, by Bao and Wang [1, Theorem 4.4], we can see that for any $\epsilon > 0$, there exist $T_\epsilon > 0$ and $s_0 \in \mathbb{R}$ such that

$$Q^{\kappa_2}(t)[\tilde{\mathbf{W}}_{\kappa_1}(0, \cdot)](x) \geq \tilde{\mathbf{W}}_{\kappa_2}(t, x - c_{\kappa_2}t + s_0) - \epsilon\mathbf{1} \text{ for } t \geq T_\epsilon,\ x \in \mathbb{R}.$$

Thus, we obtain

$$\tilde{\mathbf{W}}_{\kappa_1}(t, x - c_{\kappa_1}t) \geq \tilde{\mathbf{W}}_{\kappa_2}(t, x - c_{\kappa_2}t + s_0) - \epsilon\mathbf{1} \text{ for } t \geq T_\epsilon,\ x \in \mathbb{R}. \tag{3.14}$$

Taking $t = kT$ and $x = c_{\kappa_1}kT$ in (3.14), then for any positive integer k with $kT \geq T_\epsilon$, by the periodicity of $\tilde{\mathbf{W}}_{\kappa_i}(t, \cdot)$ in t, $i = 1, 2$, we have

$$\begin{aligned} \tilde{\mathbf{W}}_{\kappa_1}(0, 0) = \tilde{\mathbf{W}}_{\kappa_1}(kT, 0) \geq & \tilde{\mathbf{W}}_{\kappa_2}(kT, (c_{\kappa_1} - c_{\kappa_2})kT + s_0) - \epsilon\mathbf{1} \\ = & \tilde{\mathbf{W}}_{\kappa_2}(0, (c_{\kappa_1} - c_{\kappa_2})kT + s_0) - \epsilon\mathbf{1}. \end{aligned} \tag{3.15}$$

If $c_{\kappa_1} > c_{\kappa_2}$, taking $\epsilon \to 0^+$ and $k \to +\infty$ in (3.15), we deduce that

$$\tilde{U}_{\kappa_1}(0, 0) \geq \tilde{U}_{\kappa_2}(0, +\infty) = p(0),$$

which contradicts $\tilde{U}_{\kappa_1}(0, 0) = \frac{p(0)}{2}$. Hence, $c_{\kappa_1} \leq c_{\kappa_2}$; that is, c_κ is nondecreasing.

Step 2. We prove that $\lim_{\kappa\to 0^+} c_\kappa = c_{UV}$. Note that for any small enough $\kappa > 0$, $c_\kappa \leq c_{UV}$. Therefore, $\lim_{\kappa\to 0^+} c_\kappa$ exists. Set $\bar{c} := \lim_{\kappa\to 0^+} c_\kappa$. Clearly, $\bar{c} \leq c_{UV}$. Taking a sequence $\{\kappa_n\}$ such that $\kappa_n \to 0$ as $n \to \infty$. By standard parabolic estimates, there is a subsequence of $\{\kappa_n\}$, still denoted by $\{\kappa_n\}$ for simplicity, such that $(\tilde{U}_{\kappa_n}, \tilde{V}_{\kappa_n})$ converges uniformly to a function $(\bar{U}, \bar{V})$, which satisfies

$$\begin{cases} \bar{U}_t - \bar{c}\bar{U}_z = d_1\bar{U}_{zz} + \bar{U}[r_1 - a_1(t)\bar{U} + b_1(t)\bar{V}], \\ \bar{V}_t - \bar{c}\bar{V}_z = d_2\bar{V}_{zz} + \bar{V}[r_2(t) - a_2(t)\bar{U} + b_2(t)\bar{V}], \end{cases} \tag{3.16}$$

where $z = x - \bar{c}t$. Noting that $(\tilde{U}_{\kappa_n}(t, z), \tilde{V}_{\kappa_n}(t, z))$ is monotone increasing in z, $(\bar{U}(t, z), \bar{V}(t, z))$ is also monotone increasing in z and

$$0 \le \bar{U}(t, z) \le p(t), \ -q(t) \le \bar{V}(t, z) \le 0.$$

Hence, $(\bar{U}, \bar{V})(t, \pm\infty)$ exists and belongs to $\{(0, 0),\ (0, -q(t)),\ (p(t), 0),\ (\bar{u}(t),\ -\bar{v}(t))\}$.

Recall that $0 < \bar{u}(t) < p(t)$ and $0 < \bar{v}(t) < q(t)$ for $t \in [0, T]$. Without loss of generality, we normalize $\tilde{U}_{\kappa_n}(0, 0) = \frac{p(0)+\bar{u}(0)}{2}$. Thus $\bar{U}(0, 0) = \frac{p(0)+\bar{u}(0)}{2} > \bar{u}(0)$. By the monotonicity of $\bar{U}(t, z)$ in z, one has $\bar{U}(t, -\infty) \le \bar{U}(0, 0) \le \bar{U}(t, +\infty)$. Thus

$$\bar{U}(t, +\infty) = p(t) \text{ uniformly in } t \in [0, T].$$

From the first equation of (3.16), we obtain

$$\bar{U}(t, z) = T_1(t)[\bar{U}(0, \cdot)](z) + \int_0^t T_1(t - s)[\bar{U}(s, \cdot)(r_1 - a_1(s)\bar{U}(s, \cdot) + b_1(s)\bar{V}(s, \cdot))](z)ds,$$

where

$$T_1(t)[\phi](z) := \frac{1}{\sqrt{4\pi d_1 t}} \int_{-\infty}^{+\infty} \exp\{ - \frac{(z + \bar{c}t - y)^2}{4d_1 t} \}\phi(y)dy.$$

Taking the limits as $z \to +\infty$ in the above equation, we have

$$\begin{aligned} p(t) &= p(0) + \int_0^t T_1(t - s)[p(s)(r_1 - a_1(s)p(s) + b_1(s)\bar{V}(s, +\infty))]ds \\ &= p(0) + \int_0^t p(s)(r_1 - a_1(s)p(s) + b_1(s)\bar{V}(s, +\infty))ds, \end{aligned}$$

which yields that

$$p'(t) = p(t)(r_1 - a_1(t)p(t) + b_1(t)\bar{V}(t, +\infty)).$$

Since $b_1(\cdot) > 0$, we infer that $\bar{V}(t, +\infty) = 0$ uniformly in $t \in [0, T]$. Thus, we conclude that $(\bar{U}(t, +\infty), \bar{V}(t, +\infty)) = (p(t), 0)$ uniformly in $t \in [0, T]$.

Noticing that $(\bar{U}(t, z), \bar{V}(t, z))$ is increasing in z and $\bar{U}(0, 0) = \frac{p(0)+\bar{u}(0)}{2} < p(0)$, we deduce that

$$(\bar{U}, \bar{V})(t, -\infty) = (0, -q(t)) \quad \text{or} \quad (\bar{U}, \bar{V})(t, -\infty) = (\bar{u}(t), -\bar{v}(t)) \text{ uniformly in } t \in [0, T].$$

Suppose that $(\bar{U}, \bar{V})(t, -\infty) = (\bar{u}(t), -\bar{v}(t))$ uniformly in $t \in [0, T]$. Then it is easy to see that for any $\varepsilon > 0$, $\bar{U}(t, z) > \bar{u}(t) - \varepsilon/2$ for all $z \in \mathbb{R}$. Since $\tilde{U}_{\kappa_n}(t, z) \to \bar{U}(t, z)$ as $n \to \infty$, there exists $N \gg 1$ such that for $n > N$, $\tilde{U}_{\kappa_n}(t, z) \ge \bar{U}(t, z) - \varepsilon/2$. Hence, $\tilde{U}_{\kappa_n}(t, z) \ge \bar{u}(t) - \varepsilon$ for $n > N$ and $z \in \mathbb{R}$, which contradicts $\tilde{U}_{\kappa_n}(t, -\infty) = 0$ uniformly in $t \in [0, T]$. Therefore, $(\bar{U}, \bar{V})(t, -\infty) = (0, -q(t))$ uniformly in $t \in [0, T]$, which implies that $(\bar{U}(t, x - \bar{c}t), \bar{V}(t, x - \bar{c}t))$ is a solution of (3.16) connecting $(0, -q(t))$ and $(p(t), 0)$ with speed $\bar{c}$. From Lemma 1.1 on the uniqueness of bistable periodic traveling fronts, we conclude $\lim_{\kappa \to 0^+} c_\kappa = \bar{c} = c_{UV}$. The proof is completed. □

By Lemma 3.7, there holds $c < c_\kappa \le c_{UV}$ for $\kappa \in (0, \kappa_0)$. For convenience, in the rest of this subsection, we set $\mathcal{Y} := U_\kappa$ and $\mathcal{Z} := V_\kappa$ with $\tilde{c} := c_\kappa$ for $\kappa \in (0, \kappa_0)$. By (H$_4$), we can take $\epsilon > 0$ small enough such that $r_{2M} < a_{2m}(p_m - \epsilon)$ and

$$r_1 < \min\left\{\frac{7}{4}a_{1m}(p_m - \epsilon), \frac{5}{4}a_{1m}p_m, b_{1m}(q_m - \epsilon)\right\}.$$

Recall that $\lim_{\kappa\to 0^+} q_\kappa(t) = q(t)$ uniformly in $t \in [0, T]$. We can deduce $\kappa_0 > 0$ such that $r_1 < b_{1m}(q_{\kappa m} - \epsilon)$ for any $\kappa \in (0, \kappa_0)$. Since $\mathcal{Y}(t, \zeta)$ is strictly increasing in ζ and $\mathcal{Z}(t, \zeta)$ is strictly decreasing in ζ, there exists $\rho > 0$ such that, for any ζ satisfying either $\epsilon < \mathcal{Y}(t, \zeta) < p(t) - \epsilon$ or $\epsilon < \mathcal{Z}(t, \zeta) < q_\kappa(t) - \epsilon$, there hold $\mathcal{Y}_\zeta(t, \zeta) > \rho$ and $\mathcal{Z}_\zeta(t, \zeta) < -\rho$ uniformly in $t \in \mathbb{R}$.

Lemma 3.8 *Take $\eta(t) = \eta_1 + \eta_0 e^{-\mu t}$, $\vartheta_1(t) = \vartheta_0 e^{-\mu t}$ and $\gamma_1(t) = \gamma_0 e^{-\mu t}$, where ϑ_0, γ_0, η_0, $\mu \in \mathbb{R}^+$ and $\eta_1 \in \mathbb{R}$ satisfy*

$$\vartheta_0 < \frac{\kappa}{2a_{2M}}, \ \gamma_0 < \frac{\vartheta_0 a_{1m}}{4b_{1M}}, \ \vartheta_0 \le p_m, \tag{3.17}$$

$$b_{1M}\gamma_0 + r_1 + \mu < b_{1m}(q_{\kappa m} - \epsilon), \tag{3.18}$$

$$\gamma_0(\mu + r_{2M} + b_{2M}\gamma_0) < \frac{\kappa}{2}(q_{\kappa m} - \epsilon), \tag{3.19}$$

$$b_{2M}\gamma_0 + r_{2M} + \mu < a_{2m}(p_m - \epsilon), \tag{3.20}$$

$$\mu + r_1 < \min\left\{\frac{7}{4}a_{1m}(p_m - \epsilon), \frac{5}{4}a_{1m}p_m\right\}, \tag{3.21}$$

$$\mu + r_{2M} < a_{2m}p_m, \tag{3.22}$$

$$\rho\mu\eta_0 > \max\left\{\vartheta_0\mu + \vartheta_0(r_1 + b_{1M}\gamma_0), \ \gamma_0\mu + \gamma_0(r_{2M} + b_{2M}\gamma_0)\right\}. \tag{3.23}$$

Then, the following functions

$$\begin{cases} \bar{u}(t, x) = \min\{p(t), \mathcal{Y}(t, x - \tilde{c}t - \eta(t)) + \vartheta_1(t)\}, \\ \underline{v}(t, x) = \max\{0, \mathcal{Z}(t, x - \tilde{c}t - \eta(t)) - \gamma_1(t)\} \end{cases}$$

satisfy $N_1[\bar{u}, \underline{v}](t, x) \ge 0$ and $N_2[\bar{u}, \underline{v}](t, x) \le 0$ for any $(t, x) \in (\mathbb{R}^+, \mathbb{R})$.

Proof For any $(t, x) \in \mathbb{R}^2$, let $\zeta = x - \tilde{c}t - \eta(t)$. From (3.17), we see that

$$a_1(t)\vartheta_1(t) - b_1(t)\gamma_1(t) = [a_1(t)\vartheta_0 - b_1(t)\gamma_0]e^{-\mu t} \ge [a_{1m}\vartheta_0 - b_{1M}\gamma_0]e^{-\mu t} \ge 0 \tag{3.24}$$

and

$$a_2(t)\vartheta_1(t) - b_2(t)\gamma_1(t) - \kappa \le a_{2M}\vartheta_0 - \kappa < 0. \tag{3.25}$$

The rest of the proof is divided into four cases.

Case I. $\bar{u}(t, x) = \mathcal{Y}(t, \zeta) + \vartheta_1(t)$ and $\underline{v}(t, x) = \mathcal{Z}(t, \zeta) - \gamma_1(t)$. In this case, we have $\mathcal{Y}(t, \zeta) + \vartheta_1(t) \le p(t)$ and $\mathcal{Z}(t, \zeta) \ge \gamma_1(t)$. Direct computations show that

$$\begin{aligned} N_1[\bar{u}, \underline{v}](t, x) =& \bar{u}_t(t, x) - d_1\bar{u}_{xx}(t, x) - \bar{u}(t, x)[r_1 - a_1(t)\bar{u}(t, x) - b_1(t)\underline{v}(t, x)] \\ =& \mathcal{Y}_t(t, \zeta) - \tilde{c}\mathcal{Y}_\zeta(t, \zeta) - \eta'(t)\mathcal{Y}_\zeta(t, \zeta) + \vartheta_1'(t) - d_1\mathcal{Y}_{\zeta\zeta}(t, \zeta) - (\mathcal{Y}(t, \zeta) \\ &+ \vartheta_1(t))[r_1 - a_1(t)\mathcal{Y}(t, \zeta) - a_1(t)\vartheta_1(t) - b_1(t)\mathcal{Z}(t, \zeta) + b_1(t)\gamma_1(t)] \\ =& -\eta'(t)\mathcal{Y}_\zeta(t, \zeta) + \vartheta_1'(t) + \mathcal{Y}(t, \zeta)[a_1(t)\vartheta_1(t) - b_1(t)\gamma_1(t)] + \vartheta_1(t)[-r_1 \\ &+ a_1(t)\mathcal{Y}(t, \zeta) + a_1(t)\vartheta_1(t) + b_1(t)\mathcal{Z}(t, \zeta) - b_1(t)\gamma_1(t)] \end{aligned} \tag{3.26}$$

and

$$
\begin{aligned}
N_2[\bar{u}, \underline{v}](t,x) =& \underline{v}_t(t,x) - d_2\underline{v}_{xx}(t,x) - \underline{v}(t,x)[r_2(t) - a_2(t)\bar{u}(t,x) - b_2(t)\underline{v}(t,x)] \\
=& \mathcal{Z}_t(t,\zeta) - \tilde{c}\mathcal{Z}_\zeta(t,\zeta) - \eta'(t)\mathcal{Z}_\zeta(t,\zeta) - \gamma_1'(t) - d_2\mathcal{Z}_{\zeta\zeta}(t,\zeta) - (\mathcal{Z}(t,\zeta) \\
& - \gamma_1(t))[r_2(t) - a_2(t)\mathcal{Y}(t,\zeta) - a_2(t)\vartheta_1(t) - b_2(t)\mathcal{Z}(t,\zeta) + b_2(t)\gamma_1(t)] \\
=& -\eta'(t)\mathcal{Z}_\zeta(t,\zeta) - \gamma_1'(t) + \mathcal{Z}(t,\zeta)[a_2(t)\vartheta_1(t) - b_2(t)\gamma_1(t) - \kappa] + \gamma_1(t)[r_2(t) \\
& - a_2(t)\mathcal{Y}(t,\zeta) - a_2(t)\vartheta_1(t) - b_2(t)\mathcal{Z}(t,\zeta) + b_2(t)\gamma_1(t)]. \qquad (3.27)
\end{aligned}
$$

We distinguish among three subcases.

Case I − 1. Let $(t,x) \in \mathbb{R}^2$ be such that $\mathcal{Y}(t,\zeta) \le \epsilon$ and $\mathcal{Z}(t,\zeta) \ge q_\kappa(t) - \epsilon$. Then from (3.18), (3.24) and (3.26), we have

$$
\begin{aligned}
N_1[\bar{u}, \underline{v}](t,x) &\ge \vartheta_1'(t) + \vartheta_1(t)[-r_1 + b_1(t)\mathcal{Z}(t,\zeta) - b_1(t)\gamma_1(t)] \\
&\ge \vartheta_1'(t) + \vartheta_1(t)[-r_1 + b_1(t)(q_\kappa(t) - \epsilon) - b_1(t)\gamma_1(t)] \\
&= \vartheta_1(t)[-r_1 + b_1(t)(q_\kappa(t) - \epsilon) - \mu - b_1(t)\gamma_1(t)] \\
&\ge \vartheta_1(t)[-r_1 + b_{1m}(q_{\kappa m} - \epsilon) - \mu - b_{1M}\gamma_0] \ge 0,
\end{aligned}
$$

and from (3.17), (3.19) and (3.27), we get

$$
\begin{aligned}
N_2[\bar{u}, \underline{v}](t,x) \le& -\gamma_1'(t) + \gamma_1(t)[r_2(t) - b_2(t)\mathcal{Z}(t,\zeta) + b_2(t)\gamma_1(t)] + \mathcal{Z}(t,\zeta)[a_{2M}\vartheta_0 - \kappa] \\
\le& \gamma_1(t)[\mu + r_{2M} - b_{2m}(q_\kappa(t) - \epsilon) + b_{2M}\gamma_0] + (a_{2M}\frac{\kappa}{2a_{2M}} - \kappa)(q_\kappa(t) - \epsilon) \\
\le& \gamma_0 e^{-\mu t}\big[\mu + r_{2M} + b_{2M}\gamma_0\big] - \frac{\kappa}{2}(q_{\kappa m} - \epsilon) \le 0.
\end{aligned}
$$

Case I − 2. Let $(t,x) \in \mathbb{R}^2$ be such that $\mathcal{Y}(t,\zeta) \ge p(t) - \epsilon$ and $\mathcal{Z}(t,\zeta) \le \epsilon$. Then from (3.17), (3.21), (3.26) and $\mathcal{Z}(t,\zeta) \ge \gamma_1(t)$, we obtain

$$
\begin{aligned}
N_1[\bar{u}, \underline{v}](t,x) \ge& \vartheta_1'(t) + (p(t) - \epsilon)[a_1(t)\vartheta_0 - b_1(t)\gamma_0]e^{-\mu t} + \vartheta_1(t)[-r_1 + a_1(t)\mathcal{Y}(t,\zeta)] \\
\ge& \vartheta_1'(t) + (p(t) - \epsilon)\big[a_{1m}\vartheta_0 - b_{1M}\frac{a_{1m}\vartheta_0}{4b_{1M}}\big]e^{-\mu t} + \vartheta_1(t)[-r_1 + a_1(t)(p(t) - \epsilon)] \\
\ge& \vartheta_1'(t) + \frac{3a_{1m}}{4}(p(t) - \epsilon)\vartheta_1(t) + \vartheta_1(t)[-r_1 + a_{1m}(p(t) - \epsilon)] \\
\ge& \vartheta_1(t)\big[-\mu + \frac{7}{4}a_{1m}(p_m - \epsilon) - r_1\big] \ge 0.
\end{aligned}
$$

Further, by (3.20), (3.25) and (3.27), we have

$$
\begin{aligned}
N_2[\bar{u}, \underline{v}](t,x) &\le -\gamma_1'(t) + \gamma_1(t)[r_2(t) - a_2(t)\mathcal{Y}(t,\zeta) + b_2(t)\gamma_1(t)] \\
&\le -\gamma_1'(t) + \gamma_1(t)[r_2(t) - a_2(t)(p(t) - \epsilon) + b_2(t)\gamma_1(t)] \\
&\le \gamma_1(t)[r_{2M} - a_{2m}(p_m - \epsilon) + b_{2M}\gamma_0 + \mu] \le 0.
\end{aligned}
$$

Case I − 3. Let $(t,x) \in \mathbb{R}^2$ be such that either $\epsilon < \mathcal{Y}(t,\zeta) < p(t) - \epsilon$ or $\epsilon < \mathcal{Z}(t,\zeta) < q_\kappa(t) - \epsilon$. Thus, by (3.23), (3.24), (3.26) and the fact that $\mathcal{Y}_\zeta(t,\zeta) > \rho$, we obtain

$$
\begin{aligned}
N_1[\bar{u}, \underline{v}](t,x) &\ge -\eta'(t)\mathcal{Y}_\zeta(t,\zeta) + \vartheta_1'(t) + \vartheta_1(t)[-r_1 - b_1(t)\gamma_1(t)] \\
&\ge \rho\eta_0\mu e^{-\mu t} - \vartheta_0\mu e^{-\mu t} - \vartheta_0 e^{-\mu t}(r_1 + b_{1M}\gamma_0) \\
&= e^{-\mu t}[\rho\eta_0\mu - \vartheta_0\mu - \vartheta_0(r_1 + b_{1M}\gamma_0)] \ge 0.
\end{aligned}
$$

Furthermore, from (3.23), (3.25), (3.27) and the fact that $\mathcal{Z}_\zeta(t,\zeta) < -\rho$, we get

$$
N_2[\bar{u}, \underline{v}](t,x) \le -\eta'(t)\mathcal{Z}_\zeta(t,\zeta) - \gamma_1'(t) + \gamma_1(t)[r_2(t) + b_2(t)\gamma_1(t)]
$$

$$\begin{aligned}&\leq -\eta_0\mu\rho e^{-\mu t}+\gamma_0\mu e^{-\mu t}+\gamma_0 e^{-\mu t}(r_{2M}+b_{2M}\gamma_0)\\&=e^{-\mu t}[-\eta_0\mu\rho+\gamma_0\mu+\gamma_0(r_{2M}+b_{2M}\gamma_0)]\leq 0.\end{aligned}$$

Case II. $\bar{u}(t,x)=\mathcal{Y}(t,\zeta)+\vartheta_1(t)$ and $\underline{v}(t,x)=0$. Then, $\mathcal{Y}(t,\zeta)+\vartheta_1(t)\leq p(t)$ and $\mathcal{Z}(t,\zeta)\leq\gamma_1(t)$. It is obvious that $N_2[\bar{u},\underline{v}](t,x)=0$. Since $\lim_{\xi\to+\infty}\mathcal{Y}(t,\xi)=p(t)$ uniformly in $t\in[0,T]$, we assume that $\mathcal{Y}(t,\zeta)\geq\frac{3}{4}p(t)$ for any $t\in\mathbb{R}$. By (3.17) and (3.21), we have

$$\begin{aligned}N_1[\bar{u},\underline{v}](t,x)=&\,\mathcal{Y}_t(t,\zeta)-\tilde{c}\mathcal{Y}_\zeta(t,\zeta)-\eta'(t)\mathcal{Y}_\zeta(t,\zeta)+\vartheta_1'(t)-d_1\mathcal{Y}_{\zeta\zeta}(t,\zeta)\\&-\mathcal{Y}(t,\zeta)[r_1-a_1(t)\mathcal{Y}(t,\zeta)-b_1(t)\mathcal{Z}(t,\zeta)]+\mathcal{Y}(t,\zeta)[r_1-a_1(t)\mathcal{Y}(t,\zeta)\\&-b_1(t)\mathcal{Z}(t,\zeta)]-(\mathcal{Y}(t,\zeta)+\vartheta_1(t))[r_1-a_1(t)\mathcal{Y}(t,\zeta)-a_1(t)\vartheta_1(t)]\\=&-\eta'(t)\mathcal{Y}_\zeta(t,\zeta)+\vartheta_1'(t)+\mathcal{Y}(t,\zeta)a_1(t)\vartheta_1(t)-b_1(t)\mathcal{Y}(t,\zeta)\mathcal{Z}(t,\zeta)\\&+\vartheta_1(t)[-r_1+a_1(t)\mathcal{Y}(t,\zeta)+a_1(t)\vartheta_1(t)]\\\geq&\,\vartheta_1'(t)-b_1(t)\gamma_1(t)\mathcal{Y}(t,\zeta)+2\mathcal{Y}(t,\zeta)a_1(t)\vartheta_1(t)-r_1\vartheta_1(t)\\\geq&\,\vartheta_1'(t)-b_1(t)\gamma_1(t)(p(t)-\vartheta_1(t))+\frac{3}{2}a_1(t)\vartheta_1(t)p(t)-r_1\vartheta_1(t)\\\geq&\,\vartheta_1'(t)-b_{1M}\frac{\vartheta_0 a_{1m}}{4b_{1M}}e^{-\mu t}p(t)+\frac{3}{2}a_{1m}p(t)\vartheta_1(t)-r_1\vartheta_1(t)\\\geq&\,\vartheta_1(t)\big[-\mu+\frac{5}{4}a_{1m}p_m-r_1\big]\geq 0.\end{aligned}$$

Case III. $\bar{u}(t,x)=p(t)$ and $\underline{v}(t,x)=\mathcal{Z}(t,\zeta)-\gamma_1(t)$. In this case $\mathcal{Y}(t,\zeta)+\vartheta_1(t)\geq p(t)$ and $\mathcal{Z}(t,\zeta)\geq\gamma_1(t)$. Then, we have

$$\begin{aligned}N_1[\bar{u},\underline{v}](t,x)&=p'(t)-p(t)[p(t)-a_1(t)p(t)-b_1(t)\underline{v}(t,x)]\\&\geq p'(t)-p(t)[p(t)-a_1(t)p(t)]=0,\end{aligned}$$

and it follows from (3.22) and (3.25) that

$$\begin{aligned}N_2[\bar{u},\underline{v}](t,x)=&\,\mathcal{Z}_t(t,\zeta)-\tilde{c}\mathcal{Z}_\zeta(t,\zeta)-\eta'(t)\mathcal{Z}_\zeta(t,\zeta)-\gamma_1'(t)-d_2\mathcal{Z}_{\zeta\zeta}(t,\zeta)\\&-\mathcal{Z}(t,\zeta)[r_2(t)-\kappa-a_2(t)\mathcal{Y}(t,\zeta)-b_2(t)\mathcal{Z}(t,\zeta)]\\&+\mathcal{Z}(t,\zeta)[r_2(t)-\kappa-a_2(t)\mathcal{Y}(t,\zeta)-b_2(t)\mathcal{Z}(t,\zeta)]\\&-(\mathcal{Z}(t,\zeta)-\gamma_1(t))[r_2(t)-a_2(t)p(t)-b_2(t)\mathcal{Z}(t,\zeta)+b_2(t)\gamma_1(t)]\\=&-\eta'(t)\mathcal{Z}_\zeta(t,\zeta)-\gamma_1'(t)+\mathcal{Z}(t,\zeta)[-\kappa-a_2(t)\mathcal{Y}(t,\zeta)+a_2(t)p(t)-b_2(t)\gamma_1(t)]\\&+\gamma_1(t)[r_2(t)-a_2(t)p(t)-b_2(t)\mathcal{Z}(t,\zeta)+b_2(t)\gamma_1(t)]\\\leq&-\eta'(t)\mathcal{Z}_\zeta(t,\zeta)-\gamma_1'(t)+\mathcal{Z}(t,\zeta)[-\kappa-a_2(t)(p(t)-\vartheta_1(t))+a_2(t)p(t)\\&-b_2(t)\gamma_1(t)]+\gamma_1(t)[r_2(t)-a_2(t)p(t)-b_2(t)\mathcal{Z}(t,\zeta)+b_2(t)\gamma_1(t)]\\=&\quad-\eta'(t)\mathcal{Z}_\zeta(t,\zeta)-\gamma_1'(t)+\mathcal{Z}(t,\zeta)[a_2(t)\vartheta_1(t)-\kappa-b_2(t)\gamma_1(t)]\\&+\gamma_1(t)[r_2(t)-a_2(t)p(t)-b_2(t)\mathcal{Z}(t,\zeta)+b_2(t)\gamma_1(t)]\\\leq&-\gamma_1'(t)+\gamma_1(t)[r_2(t)-a_2(t)p(t)]\\=&\,\gamma_1(t)[r_2(t)-a_2(t)p(t)+\mu]\\\leq&\,\gamma_1(t)[r_{2M}-a_{2m}p_m+\mu]\leq 0.\end{aligned}$$

Case IV. $\bar{u}(t,x)=p(t)$ and $\underline{v}(t,x)=0$. It is easy to see that $N_1[\bar{u},\underline{v}](t,x)=0$ and $N_2[\bar{u},\underline{v}](t,x)=0$.

Combining the above four cases, the assertion of this lemma follows. □

Next, we compare $(\bar{u}, \underline{v})$ constructed in Lemma 3.8 with the solution (u, v) with initial conditions (1.7). First we have the following lemma.

Lemma 3.9 *There exists $T^* > 0$ and $c_* < c < c_{UV}$ such that for every $t > T^*$ and $x \le c_* t$,*

$$u(t, x) \le \bar{u}(t, x), \quad v(t, x) \ge \underline{v}(t, x), \tag{3.28}$$

and for every $x \ge c_* T^*$,

$$u(T^*, x) \le \bar{u}(T^*, x), \quad v(T^*, x) \ge \underline{v}(T^*, x). \tag{3.29}$$

Proof Let $c_* < c$ be such that $c_* < -c_U$. By Remark 3.6, we get $\lim_{t\to+\infty}(|u(t,x)| + |v(t,x) - q(t)|) = 0$ uniformly for $x \le c_* t$. Thus, there exists $T_2 > 0$ such that for each $t \ge T_2$ and $x \le c_* t$, $u(t, x) < p(t)$ and for any given $\varepsilon_1 > 0$, $v(t, x) > q(t) - \varepsilon_1 > q_\kappa(t)$ for sufficiently small $\kappa > 0$. Moreover, $\underline{v}(t, x) \le q_\kappa(t)$ for any $(t, x) \in \mathbb{R}^2$. Thus,

$$v(t, x) \ge \underline{v}(t, x), \ \forall t \ge T_2, x \le c_* t. \tag{3.30}$$

Note that $c_U > 0$ is the minimal wave speed of the monotonically decreasing periodic traveling wave $W_1(t, x - ct)$ of (1.8). Set $\tilde{W}_1(t, z) := \tilde{W}_1(t, x - c_U t) := W_1(t, -x + c_U t)$ for $(t, x) \in \mathbb{R}^2$. Then, we have

$$\begin{cases} (\tilde{W}_1)_t = d_1(\tilde{W}_1)_{zz} + c_U(\tilde{W}_1)_z + \tilde{W}_1(r_1 - a_1(t)\tilde{W}_1), \\ \lim_{z\to-\infty} \tilde{W}_1(t, z) = 0, \ \lim_{z\to+\infty} \tilde{W}_1(t, z) = p(t) \text{ uniformly in } t \in [0, T]. \end{cases}$$

Notice that

$$\tilde{W}_1(t, x) \sim -x e^{\sqrt{r_1/d_1}x} \text{ as } x \to -\infty.$$

Thus, there exists a constant $C' > 0$ such that

$$\tilde{W}_1(t, x) \le C'|x|e^{\sqrt{r_1/d_1}x}, \ \forall x \le 0. \tag{3.31}$$

Since $0 \le u(0, x) < p(0)$ for $x < 0$ and $u(0, x)$ has compact support in $(-\infty, 0]$, there exists $\hat{x}_0 \in \mathbb{R}$ such that $u(0, x) \le \tilde{W}_1(0, x - \hat{x}_0)$, $\forall x \in \mathbb{R}$. In view of

$$u_t \le d_1 u_{xx} + u(r_1 - a_1(t)u),$$

by the classical comparison principle for (1.8), we get $u(t, x) \le \tilde{W}_1(t, x - c_U t - \hat{x}_0)$ for $x \in \mathbb{R}$ and $t \ge 0$. It then follows from (3.31) that

$$u(t, x) \le \tilde{W}_1(t, (c_* - c_U)t - \hat{x}_0) \le C'|(c_* - c_U)t - \hat{x}_0|e^{\sqrt{r_1/d_1}((c_*-c_U)t-\hat{x}_0)} \tag{3.32}$$

for any $t > 0$, $x \le c_* t$. By reducing μ such that $\mu < \sqrt{r_1/d_1}(c_U - c_*)$, we see that there exists $T_3 > 0$ such that for all $t > T_3$,

$$C'|(c_* - c_U)t - \hat{x}_0|e^{\sqrt{r_1/d_1}((c_*-c_U)t-\hat{x}_0)} \le \vartheta_0 e^{-\mu t} = \vartheta_1(t). \tag{3.33}$$

Since $\vartheta_1(t) \le p(t)$ for $t \in [0, T]$, from the definition of $\overline{u}(t, x)$, we get $\overline{u}(t, x) \ge \vartheta_1(t)$. Taking $T^* = \max\{T_2, \ T_3\}$, from (3.32) and (3.33), we obtain

$$u(t, x) \le \overline{u}(t, x), \ \forall t > T^*, x \le c_* t. \tag{3.34}$$

Combining (3.30) and (3.34), we have (3.28) holds.

Note that $\mathcal{Y}$ is increasing and $\mathcal{Z}$ is decreasing. Then we choose $\eta_1 < 0$ with $|\eta_1|$ large enough such that

$$\bar{u}(T^*, x) = \min\{p(T^*), \ \mathcal{Y}(T^*, x - \tilde{c}T^* - \eta(T^*)) + \vartheta_1(T^*)\}$$

$$\geq \min\{p(T^*),\ \mathcal{Y}(T^*, c_*T^* - \tilde{c}T^* - \eta_1 - \eta_0 e^{-\mu T^*}) + \vartheta_1(T^*)\} = p(T^*) \geq u(T^*, x)$$

and

$$\begin{aligned}\underline{v}(T^*, x) &= \max\{0,\ \mathcal{Z}(T^*, x - \tilde{c}T^* - \eta(T^*)) - \gamma_1(T^*)\}\\ &\leq \max\{0,\ \mathcal{Z}(T^*, c_*T^* - \tilde{c}T^* - \eta_1 - \eta_0 e^{-\mu T^*}) - \gamma_1(T^*)\} = 0 \leq v(T^*, x)\end{aligned}$$

for $x \geq c_*T^*$. Therefore, (3.29) holds. The proof is completed. □

We now give the proof of Theorem 1.5 (ii).

Proof of Theorem 1.5 (ii) From Lemmas 3.3, 3.8 and 3.9, we have

$$(\bar{u}, \underline{v})(t, x) \preceq (u, v)(t, x) \text{ for } t \geq T^* \text{ and } x \geq c_*t. \tag{3.35}$$

Since $c > c_*$, (3.35) holds for $c_*t \leq x \leq ct$. By the proof of Lemma 3.9, we obtain that (3.35) holds for any $c_* < -c_U$. Letting $c_* \to -\infty$, we get that (3.35) holds for any $t \geq T^*$ and $-\infty < x \leq ct$. Hence, for $t \geq T^*$ and $x \leq ct$, we get

$$\begin{cases} u(t, x) \leq \bar{u}(t, x) \leq \mathcal{Y}(t, x - \tilde{c}t - \eta(t)) + \vartheta_1(t) \leq \mathcal{Y}(t, (c - \tilde{c})t - \eta(t)) + \vartheta_1(t), \\ v(t, x) \geq \underline{v}(t, x) \geq \mathcal{Z}(t, x - \tilde{c}t - \eta(t)) - \gamma_1(t) \geq \mathcal{Z}(t, (c - \tilde{c})t - \eta(t)) - \gamma_1(t). \end{cases} \tag{3.36}$$

Recall that $c < \tilde{c}$. Combining (3.36) with $\lim_{z\to-\infty} \mathcal{Y}(t, z) = 0$ and $\lim_{z\to-\infty} \mathcal{Z}(t, z) = q_\kappa(t)$ uniformly in $t \in [0, T]$, we have $\lim_{t\to+\infty} \inf_{x\leq ct} u(t, x) \leq 0$ and

$$\lim_{t\to+\infty} \inf_{x\leq ct} [v(t, x) - q_\kappa(t)] \geq 0 \tag{3.37}$$

for any $\kappa \in (0, \kappa_0)$. Given any sufficiently small $\epsilon > 0$. Since $\lim_{\kappa\to 0^+} q_\kappa(t) = q(t)$ uniformly in $t \in [0, T]$, we can deduce $\kappa_0 > 0$ such that $q_\kappa(t) \geq q(t) - \epsilon$ for any $\kappa \in (0, \kappa_0)$ and hence

$$\lim_{t\to+\infty} \inf_{x\leq ct} [v(t, x) - q(t)] \geq -\epsilon. \tag{3.38}$$

By the arbitrariness of $\epsilon > 0$ and the fact that $v(t, x) \leq q(t)$ for any $(t, x) \in \mathbb{R}^2$, we get

$$\lim_{t\to+\infty} \sup_{x\leq ct} (|u(t, x)| + |v(t, x) - q(t)|) = 0.$$

This completes the proof of Theorem 1.5 (ii). □

3.1.3 Proof of Theorem 1.5 (iii)

From Lemma 3.5, we obtain that if $c_V < c_U$, then for any $c_1,\ c_2$ satisfying $c_V < c_1 < c_2 < c_U$,

$$\lim_{t\to+\infty} \sup_{c_1t<x<c_2t} (|u(t, x) - p(t)| + |v(t, x)|) = 0. \tag{3.39}$$

To prove Theorem 1.5 (iii), we will use a similar idea as in the proof of Theorem 1.5 (ii) to show that (3.39) is true for all $c_{UV} < c_1 < c_2 < c_U$. Choose $\hat{\tau}_0 > 0$ such that, for $\hat{\tau} \in (0, \hat{\tau}_0)$, (H_1)-(H_3) hold with r_1 replaced by $r_1 - \hat{\tau}$. From Lemma 1.1, we know that the following auxiliary system

$$\begin{cases} u_t = d_1 u_{xx} + u(t, x)(r_1 - \hat{\tau} - a_1(t)u(t, x) - b_1(t)v(t, x)), \\ v_t = d_2 v_{xx} + v(t, x)(r_2(t) - a_2(t)u(t, x) - b_2(t)v(t, x)) \end{cases}$$

has a periodic traveling front $(\mathcal{U}_{\hat{\tau}}(t,z), \mathcal{V}_{\hat{\tau}}(t,z))$, which satisfies

$$\begin{cases} (\mathcal{U}_{\hat{\tau}})_t - c_{\hat{\tau}}(\mathcal{U}_{\hat{\tau}})_z = d_1(\mathcal{U}_{\hat{\tau}})_{zz} + \mathcal{U}_{\hat{\tau}}(r_1 - \hat{\tau} - a_1(t)\mathcal{U}_{\hat{\tau}} - b_1(t)\mathcal{V}_{\hat{\tau}}), \\ (\mathcal{V}_{\hat{\tau}})_t - c_{\hat{\tau}}(\mathcal{V}_{\hat{\tau}})_z = d_2(\mathcal{V}_{\hat{\tau}})_{zz} + \mathcal{V}_{\hat{\tau}}(r_2(t) - a_2(t)\mathcal{U}_{\hat{\tau}} - b_2(t)\mathcal{V}_{\hat{\tau}}), \\ \lim_{z\to-\infty}(\mathcal{U}_{\hat{\tau}}(t,z), \mathcal{V}_{\hat{\tau}}(t,z)) = (0, q(t)) \text{ uniformly in } t \in [0,T], \\ \lim_{z\to+\infty}(\mathcal{U}_{\hat{\tau}}(t,z), \mathcal{V}_{\hat{\tau}}(t,z)) = (p_{\hat{\tau}}(t), 0) \text{ uniformly in } t \in [0,T], \\ (\mathcal{U}_{\hat{\tau}}(t+T,z), \mathcal{V}_{\hat{\tau}}(t+T,z)) = (\mathcal{U}_{\hat{\tau}}(t,z), \mathcal{V}_{\hat{\tau}}(t,z)), \ (t,z) \in \mathbb{R}^2, \end{cases} \tag{3.40}$$

where $c_{\hat{\tau}}$ is the wave speed, $z = x - c_{\hat{\tau}}t$ and $p_{\hat{\tau}}(t)$ is the unique positive T-periodic solution of

$$u'(t) = u(t)(r_1 - \hat{\tau} - a_1(t)u(t)).$$

Similarly, by Lemma 3.7, we have $c_{UV} \leq \bar{c} := \lim_{\hat{\tau}\to 0^+} c_{\hat{\tau}}$. In the following, we set $\hat{\mathcal{Y}} := \mathcal{U}_{\hat{\tau}}$ and $\hat{\mathcal{Z}} := \mathcal{V}_{\hat{\tau}}$ with $c_{\hat{\tau}} = \check{c}$ for $\hat{\tau} \in (0, \hat{\tau}_0)$ and suppose that $c_{UV} \leq \check{c} < c_1$.

Lemma 3.10 *Define*

$$\begin{cases} \underline{u}(t,x) = \max\{0, \hat{\mathcal{Y}}(t, x - \check{c}t - \eta(t)) - \vartheta_1(t)\}, \\ \overline{v}(t,x) = \min\{q(t), \hat{\mathcal{Z}}(t, x - \check{c}t - \eta(t)) + \gamma_1(t)\}. \end{cases}$$

Then for any $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, there hold $N_1[\underline{u}, \overline{v}](t,x) \leq 0$ and $N_2[\underline{u}, \overline{v}](t,x) \geq 0$.

Proof The proof of this lemma is similar to that of Lemma 3.8. So we emit it. □

In order to compare $(\underline{u}, \overline{v})$ with the solution (u, v) of (1.1) and initial values (1.7), we introduce the following Lemma 3.11. Since its proof is similar to that of Lemma 3.9, we omit it here.

Lemma 3.11 *There exist $\hat{T} > 0$ and $\hat{c} > c_2$ such that for every $t > \hat{T}$ and $x \geq \hat{c}t$,*

$$u(t,x) \geq \underline{u}(t,x), \ v(t,x) \leq \overline{v}(t,x),$$

and for every $x \leq \hat{c}\hat{T}$, $u(\hat{T},x) \geq \underline{u}(\hat{T},x), \ v(\hat{T},x) \leq \overline{v}(\hat{T},x)$.

Lemma 3.12 *Assume that $c_V < c_U$, then for all $c_{UV} < c_1 < c_2 < c_U$,*

$$\lim_{t\to+\infty} \sup_{c_1 t < x < c_2 t} (|u(t,x) - p(t)| + |v(t,x)|) = 0.$$

Proof From Lemmas 3.3, 3.10 and 3.11, we have

$$(u,v)(t,x) \preceq (\underline{u}, \overline{v})(t,x) \text{ for } t \geq \hat{T} \text{ and } c_1 t \leq x \leq c_2 t. \tag{3.41}$$

Hence, for $t \geq \hat{T}$ and $c_1 t \leq x \leq c_2 t$, it then follows that

$$\begin{cases} u(t,x) \geq \underline{u}(t,x) \geq \hat{\mathcal{Y}}(t, x - \check{c}t - \eta(t)) - \vartheta_1(t) \geq \hat{\mathcal{Y}}(t, (c_1 - \check{c})t - \eta(t)) - \vartheta_1(t), \\ v(t,x) \leq \overline{v}(t,x) \leq \hat{\mathcal{Z}}(t, x - \check{c}t - \eta(t)) + \gamma_1(t) \leq \hat{\mathcal{Z}}(t, (c_1 - \check{c})t - \eta(t)) + \gamma_1(t). \end{cases} \tag{3.42}$$

Note that $c_1 > \check{c}$, then from (3.42) and

$$\lim_{z\to+\infty} \hat{\mathcal{Y}}(t,z) = p_{\hat{\tau}}(t), \quad \lim_{z\to+\infty} \mathcal{Z}(t,z) = 0 \quad \text{uniformly in } t \in [0,T],$$

we get $\lim_{t\to+\infty} \sup_{c_1 t \leq x \leq c_2 t} v(t,x) \leq 0$ and

$$\lim_{t\to+\infty} \sup_{c_1 t \leq x \leq c_2 t} [u(t,x) - p_{\hat{\tau}}(t)] \geq 0 \tag{3.43}$$

for any $\hat{\tau} \in (0, \hat{\tau}_0)$. Given any small enough $\epsilon > 0$. Since $\lim_{\hat{\tau}\to 0^+} p_{\hat{\tau}}(t) = p(t)$ uniformly in $t \in [0, T]$, we can deduce $\hat{\tau}_0 > 0$ such that $p_{\hat{\tau}}(t) \geq p(t) - \epsilon$ for any $\hat{\tau} \in (0, \hat{\tau}_0)$, and hence

$$\lim_{t\to+\infty} \sup_{c_1 t \leq x \leq c_2 t} [u(t, x) - p(t)] \geq -\epsilon.$$

By the arbitrariness of $\epsilon > 0$ and the fact that $u(t, x) \leq p(t)$ for any $(t, x) \in \mathbb{R}^2$, we have

$$\lim_{t\to+\infty} \sup_{c_1 t \leq x \leq c_2 t} (|u(t, x) - p(t)| + |v(t, x)|) = 0.$$

This completes the proof of Theorem 1.5 (iii). □

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