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Fusion rules for pastures and tracts

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ABSTRACT

Baker and Bowler defined a category of algebraic objects called tracts which generalize both partial fields and hyperfields. They also defined a notion of weak and strong matroids over a tract F , and proved that if F is perfect, meaning that F -vectors and F -covectors are orthogonal for every matroid over F , then the notions of weak and strong F -matroids coincide. We define the class of strongly fused tracts and prove that such tracts are perfect. We also show that both partial fields and stringent hyperfields are strongly fused; in this way, our criterion for perfection generalizes results of Baker–Bowler and Bowler–Pendavingh.

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1. Introduction

1.1. Overview

In [1], Baker and Bowler define a category of algebraic objects called tracts which generalize both partial fields and hyperfields (in particular, they generalize fields). Given a tract F , Baker and Bowler define the notions of weak and strong matroids over F , and they prove that if F is perfect (meaning that F -vectors and F -covectors are orthogonal for every F -matroid) then the notions of weak and strong F -matroids coincide. The authors also show that partial fields and doubly distributive hyperfields are always perfect.

The fact that doubly distributive hyperfields are perfect was generalized by Bowler–Pendavingh [2] and Bowler–Su [3], who showed that every stringent hyperfield¹ is perfect and every doubly distributive hyperfield is stringent.

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¹ In fact, Bowler and Pendavingh work in the more general context of not necessarily multiplicatively commutative skew hyperfields, but for simplicity we restrict to the commutative case in this paper.

In this paper, we define the class of strongly fused tracts and prove that such tracts are perfect. We also show that both partial fields and stringent hyperfields are strongly fused, so our criterion for perfection generalizes results from [1–3].

The proof of our main theorem (strongly fused tracts are perfect) is heavily influenced by the paper [4] of Dress and Wenzel, though the details differ in a number of places.

We now explain our results in more detail, deferring proofs of the main propositions and theorems to the later sections.

1.2. Pastures and tracts

Given an abelian group G , let $\mathbb{N}[G]$ denote the group semiring associated to G . For $\alpha \in \mathbb{N}[G]$, let $\|\alpha\|$ be the unique natural number m such that α is a sum of m elements of G (with $\|0\| = 0$). Thus $\|\alpha\| = 1$ iff $\alpha \in G$, and we have $\|\alpha \cdot \beta\| = \|\alpha\| \cdot \|\beta\|$ and $\|\alpha + \beta\| = \|\alpha\| + \|\beta\|$ for all $\alpha, \beta \in \mathbb{N}[G]$. For $m \in \mathbb{N}$, let $\mathbb{N}[G]_{\leq m}$ denote the set of all $\alpha \in \mathbb{N}[G]$ with $\|\alpha\| \leq m$.

Definition 1.1. A tract (resp. pasture) is a multiplicatively written commutative monoid F with an absorbing element 0 such that $F^\times := F \setminus \{0\}$ is a group, together with a subset N_F of $\mathbb{N}[F^\times]$ (resp. $\mathbb{N}[F^\times]_{\leq 3}$) satisfying:

- (T1) The zero element of $\mathbb{N}[F^\times]$ belongs to N_F .
- (T2) There is a unique element ϵ of F^\times with $1 + \epsilon \in N_F$.
- (T3) N_F is closed under the natural action of F^\times on $\mathbb{N}[F^\times]$.

We call N_F the null set of F .

We will write -1 instead of ϵ , $-x$ instead of ϵx , and $x - y$ instead of $x + \epsilon y$ for $x, y \in F$. (Note, however, that $x - x$ is not equal to 0 in $\mathbb{N}[F^\times]$, we merely have $x - x \in N_F$.)

Roughly speaking, tracts are the natural algebraic setting for considerations involving strong matroids and pastures are the natural setting for considerations involving weak matroids. Since we are interested in conditions such as perfection which guarantee that every weak matroid is strong, it is natural to explore the relationship between pastures and tracts.

Definition 1.2. A morphism of tracts (or pastures) is a map $\varphi : F \rightarrow F'$ such that $\varphi(0) = 0$, φ induces a group homomorphism from F^\times to $(F')^\times$, and $\varphi(N_F) \subseteq N_{F'}$.

1.3. Partial fields and hyperfields

Given a pair (G, R) consisting of a commutative ring R with 1 and a subgroup $G \leq R^\times$ containing -1 , we can associate a pasture $P = P_{G,R}$ with (G, R) by setting $P^\times = G$ and declaring that $x + y + z \in N_P$ if and only if $x + y + z = 0$ in P . Pastures of this form are called partial fields.

Roughly speaking, a hyperfield is an algebraic structure which behaves like a field except that addition is allowed to be multivalued. More precisely, a hyperfield H consists of a multiplicative monoid with an absorbing element 0 such that $H^\times = H \setminus \{0\}$ is an abelian group, an involution $x \mapsto -x$ fixing 0 , and a commutative hyperoperation which associates to each pair of elements $a, b \in H$ a non-empty subset $a \boxplus b$ of H . The multiplication and hyperaddition are required to satisfy a number of axioms including commutativity and distributivity, and we require for each $a, b \in H$ that $0 \in a \boxplus b$ if and only if $a = -b$. There is also a reversibility axiom which says that $c \in a \boxplus b$ if and only if $b \in c \boxplus (-a)$.

Definition 1.3. The tract F_H (resp. pasture P_H) associated to a hyperfield H has multiplicative group H^\times and null set defined by $\sum_{i=1}^k x_i \in N_H$ iff $0 \in \boxplus_{i=1}^k x_i$ (resp. $x + y + z \in N_H$ if and only if $0 \in x \boxplus y \boxplus z$).

If P is a pasture and we set $x \boxplus y = \{z \in P : x + y - z \in N_P\}$, the pasture P corresponds to a field if and only if \boxplus is an associative binary operation. Moreover, $P = P_H$ for some hyperfield H if and only if $x \boxplus y$ contains at least one element for all $x, y \in P$ and \boxplus is associative (in the sense of set-wise addition), and $P = P_{(G,R)}$ for some partial field (G, R) if and only if $x \boxplus y$ contains at most

one element for all $x, y \in P$ and satisfies a suitable associative law (which is a bit complicated to state, cf. [5, Section 2.2]). Pastures thus generalize (and simplify) both hyperfields and partial fields by imposing no conditions on the size of the sets $x \boxplus y$ and no associativity conditions.

Definition 1.4. A hyperfield H is stringent if $|a \boxplus b| = 1$ for all $a, b \in H$ with $a \neq -b$.

Definition 1.5. A hyperfield H is doubly-distributive if $(a \boxplus b)(c \boxplus d) = ac \boxplus bc \boxplus ad \boxplus bd$ for all $a, b, c, d \in H$.

Here are a few examples of hyperfields and their associated tracts:

Example 1.6 (Sign Hyperfield). The sign hyperfield \mathbb{S} consists of the multiplicative monoid $\{0, \pm 1\}$, together with the hyperaddition rule given by $1 \boxplus 1 = 1$, $(-1) \boxplus (-1) = -1$, and $1 \boxplus (-1) = \{-1, 0, 1\}$. As a tract, $N_{\mathbb{S}}$ consists of 0 and all formal sums $\sum x_i$ with at least one 1 and one -1 .

Example 1.7 ($\mathbb{S} \times \mathbb{S}$). Products exist in both the category of hyperfields and the category of tracts. As a multiplicative monoid, $\mathbb{S} \times \mathbb{S}$ is given by the Cartesian product of $\{0, \pm 1\}$ with itself, while $N_{\mathbb{S} \times \mathbb{S}}$ consists of 0 and all formal sums $\sum (x_i, y_i)$ such that both $\sum x_i$ and $\sum y_i$ are in $N_{\mathbb{S}}$.

Example 1.8 (Phase Hyperfield). The phase hyperfield \mathbb{P} consists of the multiplicative monoid $\{0\} \cup \mathbb{S}^1$, where \mathbb{S}^1 is the complex unit circle, together with the following hyperaddition rule. Given $x_i \in \mathbb{P}$, the hypersum $\boxplus_{i=1}^n x_i$ is the set of phases of all complex numbers in the cone $\sum_{i=1}^n c_i x_i$ with $c_i \in \mathbb{R}_{>0}$. As a tract, $\sum_{i=1}^n x_i \in N_{\mathbb{P}}$ if and only if there exist $c_1, \dots, c_n \in \mathbb{R}_{>0}$ such that $\sum_{i=1}^n c_i \cdot x_i = 0$ in \mathbb{C} .

1.4. The fusion axiom

Although one can trivially extend a pasture to a tract via the inclusion $\mathbb{N}[F^\times]_{\leq 3} \subset \mathbb{N}[F^\times]$, this way of viewing pastures as tracts is not very useful in practice. Instead, it is more useful to define the tract associated to a pasture by inductively “fusing” together additive relations of smaller degree to generate higher-degree relations. More precisely, consider the following fusion axiom:

(F) If $\alpha + z$ and $\beta - z$ are in N_F with $\alpha, \beta \in \mathbb{N}[F^\times]$ and $z \in F$, then $\alpha + \beta \in N_F$.

Given a pasture P , let \bar{P} be the tract whose multiplicative group is P^\times and whose null set is the smallest subset of $\mathbb{N}[P^\times]$ containing N_P and satisfying the fusion axiom.

The proof of the following result is left as an exercise:

Proposition 1.9. The map $P \mapsto \bar{P}$ defines a fully faithful functor from pastures to tracts. A tract F is equal to \bar{P} for some pasture P if and only if F satisfies the fusion axiom (F) and every $\gamma \in N_F \cap \mathbb{N}[F^\times]_{\geq 4}$ can be written as $\alpha + \beta$ for some $\alpha, \beta \in \mathbb{N}[F^\times]_{\geq 2}$ and $z \in F$ with $\alpha + z$ and $\beta - z$ in N_F .

In particular, there is no harm in identifying a pasture P with the corresponding tract \bar{P} .

Note that the tract \bar{P} associated to a pasture P is in fact an idyll (cf. [6, Section 1.2.2]), meaning that N_F is an ideal in the semiring $\mathbb{N}[F^\times]$; this is equivalent to requiring:

(I) If $\alpha, \beta \in N_F$ then $\alpha + \beta \in N_F$.

For hyperfields, we have the following pleasant correspondence (which was in fact our motivation for the fusion axiom):

Proposition 1.10. If H is a hyperfield and F_H (resp. P_H) is the associated tract (resp. pasture) then $F_H = \bar{P}_H$.

For later reference, we also define a functor from tracts to pastures: given a tract F , define the 3-term truncation of F to be the pasture whose multiplicative group is F^\times and whose null set is $N_F \cap \mathbb{N}[F^\times]_{\leq 3}$.

1.5. The strong fusion axiom

The present paper is motivated by the observation that many tracts of interest, such as partial fields and stringent hyperfields, satisfy a property that is stronger than (F) and which turns out to be sufficient to guarantee perfection.

More precisely, consider the following strong fusion axiom:

(SF) If $\alpha + \gamma$ and $\beta - \gamma$ are in N_F with $\alpha, \beta, \gamma \in \mathbb{N}[F^\times]$ and either $\gamma = 0$ or $\gamma \notin N_F$, then $\alpha + \beta \in N_F$.

Note that the fusion axiom is precisely the case where $\gamma \in F$, and in particular a tract satisfying (SF) (which we call a strongly fused tract) automatically satisfies (F).

The main result of this paper is:

Theorem 1.11. *Every strongly fused tract is perfect.*

In fact, we will prove a stronger version of [Theorem 1.11](#) in which we replace (SF) with the modified axiom:

(MSF) If $\alpha + \gamma$ and $\beta - \gamma$ are in N_F with $\alpha, \beta, \gamma \in \mathbb{N}[F^\times]$ and either $\gamma = 0$ or $\gamma \notin N_F$, and if $\|\alpha + \beta\| \geq 4$, then $\alpha + \beta \in N_F$.

Our proof will show more generally that a tract satisfying (MSF) is strongly perfect, a notion which will be defined in Section 3 (but it turns out to be equivalent to perfection in the usual sense; see [Theorem 3.7](#)).

As a corollary of the strengthened version of [Theorem 1.11](#), we obtain:

Corollary 1.12. *There is a rule $F \mapsto \sigma(F)$ which associates to each tract F a strongly perfect tract $\sigma(F)$ and which is the identity map on tracts satisfying (MSF).*

Proof. We can take $\sigma(F)$ to be the tract whose multiplicative group is F^\times and whose null set is defined as follows. Let $N^{(1)} = N_F$, and for $k \geq 2$ define $N^{(k)}$ to be the set of all elements of the form $\alpha + \beta$ with $\alpha, \beta \in N^{(k-1)}$ or $\alpha + \gamma$ and $\beta - \gamma$ in $N^{(k-1)}$ for some $\gamma \notin N^{(k-1)}$, and such that $\|\alpha + \beta\| \geq 4$. Set $N_{\sigma(F)} := \bigcup_{k \geq 1} N^{(k)}$. It is easy to see that $\sigma(F) = F$ if F satisfies (MSF). We claim that $\sigma(F)$ satisfies (MSF) for every tract F . Indeed, suppose $\alpha + \gamma$ and $\beta - \gamma$ are in $N_{\sigma(F)}$ with either $\gamma = 0$ or $\gamma \notin N_{\sigma(F)}$, and assume furthermore that $\|\alpha + \beta\| \geq 4$. Then by definition there exists $k \geq 1$ such that $\alpha + \gamma$ and $\beta - \gamma$ are in $N^{(k-1)}$, and if $\gamma \notin N_{\sigma(F)}$ then $\gamma \notin N^{(k-1)}$ since $N^{(k-1)} \subset N_{\sigma(F)}$. By the definition of $N^{(k)}$ we have $\alpha + \beta \in N^{(k)}$, hence $\alpha + \beta \in N_{\sigma(F)}$. \square

1.6. Stringent hyperfields and the strong fusion axiom

It is easy to see that the tract embedding of a partial field satisfies the strong fusion axiom. For hyperfields, we show:

Proposition 1.13. *If H is a hyperfield, then H satisfies the strong fusion axiom if and only if H is stringent.*

In particular, this gives a new proof of the fact, originally proved by Bowler and Pendavingh in [\[2\]](#), that stringent hyperfields are perfect.

Remark 1.14. If we removed the assumption that $\gamma \notin N_F$ in (SF), then stringent hyperfields would no longer satisfy this property. For example, in the sign hyperfield \mathbb{S} with $\alpha = 1, \beta = 1$, and $\gamma = 1 + (-1)$, we have $\alpha + \gamma, \beta - \gamma \in N_{\mathbb{S}}$ but $\alpha + \beta \notin N_{\mathbb{S}}$.

It would be useful to have a natural and easily verified sufficient condition which implies perfection, is satisfied by stringent hyperfields and partial fields, and which is stable under taking finite products (since one easily shows that the product of perfect pastures is perfect.) Unfortunately, neither (SF) nor (MSF) is stable under products, as the following shows:

Example 1.15. A counterexample which applies to both (SF) and (MSF) is $F = \mathbb{S} \times \mathbb{S}$, where \mathbb{S} is the sign hyperfield. Indeed, note that if $\gamma = (1, 1)+(-1, 1)$, $\alpha = (1, -1)+(1, -1)$, and $\beta = (1, 1)+(1, 1)$ then $\alpha + \gamma, \beta - \gamma \in N_F$ and $\gamma \notin N_F$ but $\alpha + \beta \notin N_F$.

1.7. Structure of the paper

The proofs of Propositions 1.10 and 1.13 are given in Section 2. In Section 3 we recall the definition of an F -matroid and define what it means for an F -matroid (resp. a tract) to be strongly perfect. We then prove that strong perfection and perfection coincide. The proof of (a strengthening of) Theorem 1.11 is given in Section 4. Finally, in Section 5 we compare our results to those of Dress–Wenzel.

2. Hyperfields, partial fields, and fusion axioms

In this section we prove Propositions 1.10 and 1.13.

2.1. Hyperfields, partial fields, and the fusion axiom

Our goal in this section is to prove Proposition 1.10. In order to do this, we first recall the precise definition of a hyperfield.

Definition 2.1. A commutative hypergroup is a set G together with a distinctive element 0 and a hyperaddition, which is a map

$$\boxplus : G \times G \longrightarrow \mathcal{P}(G)$$

into the power set $\mathcal{P}(G)$ of G , such that:

- (HG1) $a \boxplus b$ is not empty, (nonempty sums)
- (HG2) $\bigcup_{d \in b \boxplus c} a \boxplus d = \bigcup_{d \in a \boxplus b} d \boxplus c$, (associativity)
- (HG3) $0 \boxplus a = a \boxplus 0 = \{a\}$, (neutral element)
- (HG4) there is a unique element $-a$ in G such that $0 \in a \boxplus (-a)$, (inverses)
- (HG5) $a \boxplus b = b \boxplus a$, (commutativity)
- (HG6) $c \in a \boxplus b$ if and only if $b \in c \boxplus (-a)$ (reversibility)

for all $a, b, c \in G$.

Thanks to commutativity and associativity, it makes sense to define hypersums of several elements a_1, \dots, a_n unambiguously by the recursive formula

$$\boxplus_{i=1}^n a_i = \bigcup_{b \in \boxplus_{i=1}^{n-1} a_i} b \boxplus a_n.$$

Definition 2.2. A (commutative) hyperring is a set R together with distinctive elements 0 and 1 and with maps $\boxplus : R \times R \rightarrow \mathcal{P}(R)$ and $\cdot : R \times R \rightarrow R$ such that

- (HR1) $(R, \boxplus, 0)$ is a commutative hypergroup,
- (HR2) $(R, \cdot, 1)$ is a commutative monoid,
- (HR3) $0 \cdot a = a \cdot 0 = 0$,
- (HR4) $a \cdot (b \boxplus c) = ab \boxplus ac$

for all $a, b, c \in R$ where $a \cdot (b \boxplus c) = \{ad \mid d \in b \boxplus c\}$.

A hyperfield is a hyperring H such that $0 \neq 1$ and every nonzero element has a multiplicative inverse, i.e., $H^\times = H \setminus \{0\}$.

Lemma 2.3. If H is a hyperfield and F_H is the associated tract, then F_H satisfies the fusion axiom.

Proof. Suppose $\alpha + z$ and $\beta - z$ are in N_H , where $\alpha = \sum_{i=1}^k x_i$, $\beta = \sum_{j=1}^l y_j$, and $z \in H$. Then by definition, $0 \in (\boxplus_{i=1}^k x_i) \boxplus z$ and $0 \in (\boxplus_{j=1}^l y_j) \boxplus -z$. By the inverse axiom (HG4), $-z$ is in $\boxplus_{i=1}^k x_i$ and z is in $\boxplus_{j=1}^l y_j$. Hence, 0 is in $(\boxplus_{i=1}^k x_i) \boxplus (\boxplus_{j=1}^l y_j)$, i.e., $\alpha + \beta \in N_H$. \square

We can now prove [Proposition 1.10](#).

Proof of Proposition 1.10. First, notice that $F := F_H$, $P := P_H$, and \overline{P}_H all have the same multiplicative group, so it suffices to prove that F_H and \overline{P}_H have the same null set.

Next, note that the null set of P_H is contained in the null set of F_H , so by [Lemma 2.3](#) and the definition of \overline{P}_H , $N_{\overline{P}} \subseteq N_F$. Conversely, we will prove by induction on $\|\gamma\|$ that if $\gamma \in N_F$ then $\gamma \in N_{\overline{P}}$.

The base case $\|\gamma\| \leq 3$ is trivial. Assume, then, that $\gamma \in N_{\overline{P}}$ for every $\gamma \in N_F$ with $\|\gamma\| < k$, and let $\gamma \in N_F$ have norm $k \geq 4$. Then $\gamma = \sum_{i=1}^k x_i$ with $x_i \in H$, and since $\gamma \in N_F$ we have $0 \in \boxplus_{i=1}^k x_i$ in H . By the associativity axiom (HG2), we have $0 \in (\boxplus_{i=1}^{k-2} x_i) \boxplus (x_{k-1} \boxplus x_k)$. Since $k \geq 4$, there exists some element $z \in H$ such that $z \in \boxplus_{i=1}^{k-2} x_i$ and $-z \in x_{k-1} \boxplus x_k$. By the reversibility axiom (HG4), $0 \in -z \boxplus_{i=1}^{k-2} x_i$ and $0 \in z \boxplus x_{k-1} \boxplus x_k$, which means, setting $\alpha = \sum_{i=1}^{k-2} x_i$ and $\beta = x_{k-1} \boxplus x_k$, that $\alpha - z$ and $\beta + z$ both belong to N_F . By the inductive hypothesis, these two elements of N_F are in $N_{\overline{P}}$. Applying the fusion axiom gives $\gamma = \alpha + \beta \in N_{\overline{P}}$. \square

2.2. Stringent hyperfields and the strong fusion axiom

Our goal in this section is to prove [Proposition 1.13](#). The following is a more precise version of this result.

Proposition 2.4. *Let H be a hyperfield. Then the following are equivalent:*

- (1) H is stringent.
- (2) If $x_1, \dots, x_k \in H$ and $0 \notin \boxplus_{i=1}^k x_i$ then $|\boxplus_{i=1}^k x_i| = 1$.
- (3) The tract $F = F_H$ associated to H satisfies the strong fusion axiom (SF).

Proof. (1) \Rightarrow (2): This follows from [[2](#), Lemma 39].

(2) \Rightarrow (3): Suppose $\alpha = \sum_{i=1}^{\ell} x_i$, $\beta = \sum_{j=1}^m y_j$, and $\gamma = \sum_{k=1}^n z_k$ satisfy $\alpha + \gamma, \beta - \gamma \in N_F$ and $\gamma = 0$ or $\gamma \notin N_F$. If $\gamma = 0$, the result follows from [Lemma 2.3](#). So we may assume that $\gamma \notin N_F$. Thus $\boxplus_{k=1}^n z_k = \{z\}$ is a singleton and we have $-z \in \boxplus_{i=1}^{\ell} x_i$ and $z \in \boxplus_{j=1}^m y_j$. It follows that $0 \in (\boxplus_{i=1}^{\ell} x_i) \boxplus (\boxplus_{j=1}^m y_j)$, i.e. $\alpha + \beta \in N_F$.

(3) \Rightarrow (1): Suppose there exist $x, y \in H$ such that $x \neq -y$ and $|x \boxplus y| \geq 2$. Choosing $z \neq z' \in x \boxplus y$, we have $0 \in -z \boxplus x \boxplus y$ and $0 \in -z' \boxplus x \boxplus y$. Since $x \neq -y$ we have $0 \notin x \boxplus y$, and thus (SF) implies that $0 \in z \boxplus -z'$, which contradicts the fact that $z \neq z'$. \square

3. Matroids over tracts and strongly perfect tracts

Our goal in this section is to define what it means for a tract F to be strongly perfect, and to show that F is strongly perfect if and only if it is perfect. To do this, we need to introduce some terminology related to matroids over tracts.

Definition 3.1 (Involution). Let F be a tract. An involution of F is a homomorphism $\tau : F \rightarrow F$ such that τ^2 is the identity map. For an element $x \in F$, its involution is usually denoted by \bar{x} instead of $\tau(x)$.

Definition 3.2 (Orthogonality). Let F be a tract endowed with an involution $x \mapsto \bar{x}$, and let $E = \{1, \dots, m\}$. The inner product of $X = (x_1, \dots, x_m) \in \mathbb{N}[F^\times]^m$ and $Y = (y_1, \dots, y_m) \in \mathbb{N}[F^\times]^m$ is defined to be

$$X \cdot Y := x_1 \bar{y}_1 + \dots + x_m \bar{y}_m.$$

We say that X is orthogonal to Y if $X \cdot Y \in N_F$.

Note that our definition of orthogonality generalizes [1, Definition 3.4], since for us X and Y are in $\mathbb{N}[F^\times]^m$ instead of F^m .

If $S \subseteq \mathbb{N}[F^\times]^m$, we denote by S^\perp the set of all $X \in \mathbb{N}[F^\times]^m$ such that $X \perp Y$ for all $Y \in S$.

Let F be a tract endowed with an involution $x \mapsto \bar{x}$, and let \underline{M} be a (classical) matroid with ground set E . The following two definitions are taken directly from [1].

Definition 3.3 (*F-Signature*). A subset C of F^E is an F -signature of \underline{M} if C satisfies the following properties:

(C0) $0 \notin C$.

(C1) If $X \in C$ and $\alpha \in F^\times$ then $\alpha \cdot X \in C$.

(C2) Taking supports gives a bijection from the projectivization of C to the set of circuits of \underline{M} .

Definition 3.4 (*Dual Pair of F-Signatures*). Let $C, D \subseteq F^E$. We call (C, D) a dual pair of F -signatures of \underline{M} if:

(DP1) C is an F -signature of \underline{M} .

(DP2) D is an F -signature of the dual matroid \underline{M}^* .

(DP3) $C \perp D$, meaning that $X \perp Y$ for all $X \in C$ and $Y \in D$.

For the purposes of this paper, we define a (strong) F -matroid M^2 to be a matroid \underline{M} (called the underlying matroid of M), together with a dual pair of F -signatures of \underline{M} . The equivalence of this definition with the one given in [1] is proved in [1, Theorem 3.26].

We call C (resp. D) the set of F -circuits (resp. F -cocircuits) of M , and denote these by $C(M)$ and $C^*(M)$, respectively.

Definition 3.5. We say that $X \in \mathbb{N}[F^\times]^m$ is a generalized vector (resp. generalized covector) of M if $X \perp Y$ for every $Y \in C^*(M)$ (resp. for every $Y \in C(M)$). We denote the set of all generalized vectors (resp. covectors) by $\mathcal{V}(M)$ (resp. $\mathcal{V}^*(M)$).

Note that a vector of M , in the sense of [1], is just a generalized vector belonging to F^m rather than $\mathbb{N}[F^\times]^m$, and similarly for covectors. We denote by $V(M)$ (resp. $V^*(M)$) the set of vectors (resp. covectors) of M .

Definition 3.6. An F -matroid M is strongly perfect if $\mathcal{V}(M) \perp \mathcal{V}^*(M)$. A tract F is strongly perfect if every F -matroid is strongly perfect.

A strongly perfect tract is obviously perfect. We now show that the converse holds as well:

Theorem 3.7. *A tract F is perfect if and only if it is strongly perfect.*

For the proof of [Theorem 3.7](#), we will need the following straightforward lemma, whose proof we omit.

Lemma 3.8. *Let F be a tract, and let M be an F -matroid on E with underlying matroid \underline{M} . Let $e \in E$, let $\{e_1, e_2\}$ be a 2-element set disjoint from E , and let $E' = E \setminus \{e\} \cup \{e_1, e_2\}$.*

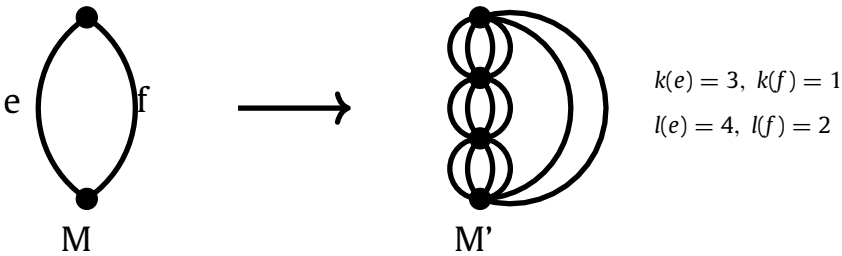
- (1) *There is an F -matroid $\sigma_e(M)$ on E' , whose underlying matroid is obtained by replacing e with two elements e_1, e_2 in series, and whose F -circuits C' are given by $C'(f) = C(f)$ for $f \in E$ and $C'(f) = C(e)$ for $f \in \{e_1, e_2\}$, where C is an F -circuit of M . The F -cocircuits D' of $\sigma_e(M)$ are given by either (i) $D'(f) = D(f)$ for $f \in E$, $D'(e_1) = 0$, and $D'(e_2) = D(e)$ or (ii) $D'(f) = D(f)$ for $f \in E$, $D'(e_1) = D(e)$, and $D'(e_2) = 0$, for D an F -cocircuit of M , or (iii) $D'(f) = 0$ for $f \in E$, $D'(e_1) = a \in F^\times$, and $D'(e_2) = -a$.*

² All F -matroids in this paper will be strong, so we sometimes omit the modifier.

(2) There is an F -matroid $\pi_e(M)$ on E' , whose underlying matroid is obtained by replacing e with two elements e_1, e_2 in parallel, and whose F -cocircuits D' are given by $D'(f) = D(f)$ for $f \in E$ and $D'(f) = D(e)$ for $f \in \{e_1, e_2\}$, where D is an F -cocircuit of M . The F -circuits C' of $\pi_e(M)$ are given by either (i) $C'(f) = C(f)$ for $f \in E$, $C'(e_1) = 0$, and $C'(e_2) = C(e)$ or (ii) $C'(f) = C(f)$ for $f \in E$, $C'(e_1) = C(e)$, and $C'(e_2) = 0$, for C an F -circuit of M , or (iii) $C'(f) = 0$ for $f \in E$, $C'(e_1) = a \in F^\times$, and $C'(e_2) = -a$.

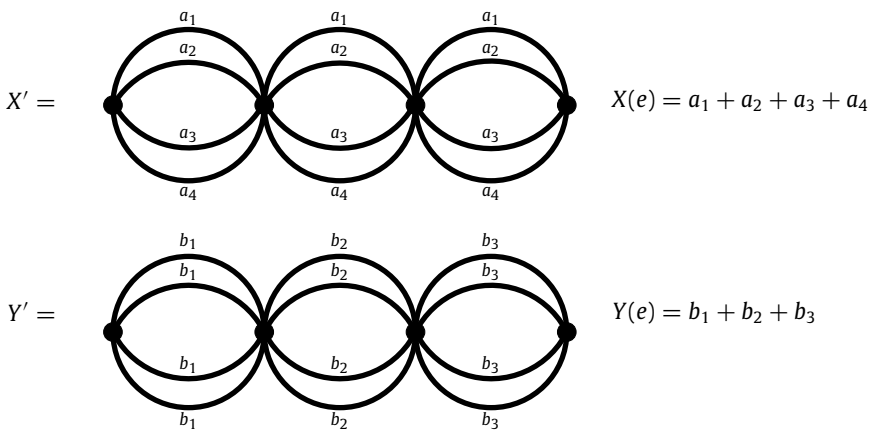
Proof of Theorem 3.7. Let F be a perfect tract, let M be an F -matroid, and let $X, Y \in \mathbb{N}[F^\times]^E$ be elements of $\mathcal{V}(M)$ and $\mathcal{V}^*(M)$, respectively. We need to show that $X \perp Y$.

To see this, for each $e \in E$ let $k(e) = \min(1, \|X(e)\|)$ and let $\ell(e) = \min(1, \|Y(e)\|)$. Let M' be the F -matroid on E' obtained from M by replacing each $e \in E$ with $k(e)$ series copies of a bundle of $\ell(e)$ parallel elements. Formally, M' is obtained from M as follows: for each $e \in E$, apply the operator σ_e $k(e) - 1$ times, thereby replacing e with $k = k(e)$ elements e_1, \dots, e_k ; now, for each $i = 1, \dots, k$ apply the operator π_{e_i} $\ell(e) - 1$ times.



For each $e \in E$, write $X(e) = a_1(e) + \dots + a_{\ell(e)}(e)$, with $a_i(e) \in F$, and similarly write $Y(e) = b_1(e) + \dots + b_{k(e)}(e)$. Define $X' \in \mathbb{N}[F^\times]^{E'}$ by setting $X'(f) = a_i(e)$ if f is the i th parallel element in any one of the $k(e)$ bundles in series for $i = 1, \dots, \ell(e)$. Similarly, define $Y' \in \mathbb{N}[F^\times]^{E'}$ by setting $Y'(f) = b_j(e)$ if f is any one of the $\ell(e)$ parallel elements in the j th series copy of the bundle of parallel elements replacing e for $j = 1, \dots, k(e)$.

Using Lemma 3.8 (which by induction provides us with an explicit description of $C^*(M')$ and $C(M')$, respectively), it is straightforward to check that $X' \in \mathcal{V}(M')$ and $Y' \in \mathcal{V}^*(M')$. Moreover, we have $X' \cdot Y' = X \cdot Y$, and since F is perfect, $X \cdot Y = 0$.



□

The following propositions concern the behavior of generalized vectors and covectors with respect to deletion and contraction. For (non-generalized) vectors and covectors, the corresponding results are proved as Propositions 4.3 and 4.4, respectively, in Laura Anderson's paper [7]. The

proofs given in [7] work *mutatis mutandis* for generalized vectors; alternatively, one can reduce the generalized case to the one treated in [7] using a trick similar to the one in the proof of Theorem 3.7.

Proposition 3.9. $\{Y \setminus e \mid Y \in \mathcal{V}^*(M), Y(e) = 0\} \subseteq \mathcal{V}^*(M/e)$ and $\{X \setminus e \mid X \in \mathcal{V}(M), X(e) = 0\} \subseteq \mathcal{V}(M \setminus e)$.

Proposition 3.10. $\{Y \setminus e \mid Y \in \mathcal{V}^*(M)\} \subseteq \mathcal{V}^*(M \setminus e)$ and $\{X \setminus e \mid X \in \mathcal{V}(M)\} \subseteq \mathcal{V}(M/e)$.

In particular, the contraction of a generalized vector is again a generalized vector, and the deletion of a generalized covector is again a generalized covector.

4. Strongly fused tracts are strongly perfect

Recall the modified strong fusion axiom:

(MSF) If $\alpha + \gamma$ and $\beta - \gamma$ are in N_F with $\alpha, \beta, \gamma \in \mathbb{N}[F^\times]$ and either $\gamma = 0$ or $\gamma \notin N_F$, and if $\|\alpha + \beta\| \geq 4$, then $\alpha + \beta \in N_F$.

Our goal in this section is to prove the following theorem, which generalizes Theorem 1.11:

Theorem 4.1. *If a tract F satisfies (MSF) then F is strongly perfect.*

The following is an example of a tract that satisfies (MSF) but not (SF).

Example 4.2. Let \mathbb{P} be the phase hyperfield and take the tract embedding $(\mathbb{P}^\times, N_{\mathbb{P}})$. Letting $N_{\mathbb{P}'} = N_{\mathbb{P}} \cup \mathbb{N}[\mathbb{P}^\times]_{\geq 4}$, it is straightforward to show that $\mathbb{P}' = (\mathbb{P}^\times, N_{\mathbb{P}'})$ satisfies the tract axiom and axiom (MSF). However, it does not satisfy the strong fusion axiom (SF). Let $\alpha = 1, \beta = 1 + 1$, and $\gamma = (-1) + (-1)$. Then, we have the following,

$$\begin{aligned} \alpha + \gamma &= 1 + (-1) + (-1) \in N_{\mathbb{P}'}. \\ \beta - \gamma &= 1 + 1 - ((-1) + (-1)) \in \mathbb{N}[\mathbb{P}^\times]_{\geq 4} \subseteq N_{\mathbb{P}'}. \\ \gamma &= (-1) + (-1) \notin N_{\mathbb{P}'}. \\ \alpha + \beta &= 1 + 1 + 1 \notin N_{\mathbb{P}'}. \end{aligned}$$

Remark 4.3. Generalizing (part of) Example 4.2, it is straightforward to check that weak hyperfields in the sense of [1, Example 2.14] satisfy (MSF).

The proof of Theorem 4.1 is fairly long and technical, so it will be broken up into a number of smaller and hopefully more digestible pieces.

Lemma 4.4. *If F is a tract satisfying the idyll property (I), then for $\alpha \in \mathbb{N}[F^\times]$ and $\beta \in N_F$ we have $\alpha\beta \in N_F$.*

Proof. If $\alpha = 0$, this is obvious. Otherwise, write $\alpha = \sum_{i=1}^k x_i$ with $x_i \in F^\times$ and inductively apply (I). \square

Lemma 4.5. *If a tract F satisfies the modified strong fusion axiom (MSF), then it also satisfies the fusion axiom (F).*

Proof. It suffices to show that (F) is satisfied when $\|\alpha + \beta\| \leq 3$.

If $z = 0$, then α and β are either zero or they belong to $\mathbb{N}[F^\times]_{\geq 2}$. Hence, either at least one of α, β is 0 or $\|\alpha + \beta\| \geq 4$.

Assume $z \in F^\times$. If $\|\alpha\| = 1$ or $\|\beta\| = 1$, the result is clear. Otherwise, both $\|\alpha\|$ and $\|\beta\|$ are at least 2. \square

Proposition 4.6. *Let F be a tract which satisfies (MSF). Suppose $\alpha, \beta, \gamma, \delta \in \mathbb{N}[F^\times]$ with $\gamma \notin N_F$, $\alpha + \beta\gamma \in N_F$, $\delta - \gamma \in N_F$, and $\|\alpha + \beta\delta\| \geq 4$. Then $\alpha + \beta\delta \in N_F$.*

Proof. We proceed by induction on $\|\beta\|$. For $\|\beta\| = 1$, the result follows immediately from (MSF).

Assuming the result holds for $\|\beta\| < k$ with $k \geq 2$, we will prove it for $\beta = y_1 + \dots + y_k \in \mathbb{N}[F^\times]$.

If $\|\gamma\| = 1$, then since $\alpha + \beta\gamma = \alpha + y_1\gamma + \dots + y_{k-1}\gamma + y_k\gamma \in N_F$ and $y_k\delta - y_k\gamma \in N_F$, it follows from the fusion axiom (F) that $\alpha + y_1\gamma + \dots + y_{k-1}\gamma + y_k\delta \in N_F$. Letting $\alpha' = \alpha + y_k\delta$ and $\beta' = y_1 + \dots + y_{k-1}$, it follows from the induction hypothesis that $\alpha + \beta\delta = \alpha' + \beta'\delta \in N_F$.

If $\|\gamma\| \geq 2$, then since $\|\beta\| = k \geq 2$ as well we have $\|\alpha + \beta\gamma\| \geq 4$ and $\|\alpha + y_1\gamma + \dots + y_{k-1}\gamma + y_k\delta\| \geq 4$. By (MSF) we have $\alpha + y_1\gamma + \dots + y_{k-1}\gamma + y_k\delta \in N_F$. As in the previous case, this implies by induction that $\alpha + \beta\delta \in N_F$. \square

Next, we introduce the “wedge product” $X \wedge_e Y$ for $X, Y \in \mathbb{N}[F^\times]^E$ and $e \in E$ which will allow us to perform an analogue of (co)circuit elimination for generalized (co)vectors of matroids over tracts satisfying (MSF).

Definition 4.7. For $X, Y \in \mathbb{N}[F^\times]^E$, and $e \in E$ we define $X \wedge_e Y \in \mathbb{N}[F^\times]^E$ by

$$X \wedge_e Y(f) = \begin{cases} 0 & \text{if } f = e, \\ Y(e)X(f) - X(e)Y(f) & \text{if } f \neq e. \end{cases}$$

The following result and its proof were inspired by [4, Lemma 2.4] (which is proved in [8, Lemma 3.2]).

Proposition 4.8. *Let F be a tract which satisfies (MSF), and let M be an F -matroid on $E = \{1, 2, \dots, m\}$. For any $X, Y \in \mathcal{V}^*(M)$ and $e \in E$ we have $X \wedge_e Y \in \mathcal{V}^*(M)$.*

Proof. It suffices to show that for any $C \in C(M)$, $C \cdot (X \wedge_e Y) \in N_F$. Note for later reference that

$$C \cdot (X \wedge_e Y) = \sum_{f \in C \setminus e} (C(f)X(e)Y(f) - C(f)Y(e)X(f)).$$

We will consider the following two cases:

Case 1: $\|C \cdot (X \wedge_e Y)\| \geq 4$.

We have the following subcases:

- If $C(e) = 0$, then $C \cdot (X \wedge_e Y) = Y(e)(C \cdot X) - X(e)(C \cdot Y) \in N_F$ by the idyll axiom (I).
- If $X(e), Y(e) \in N_F$, then $X \wedge_e Y(f) \in N_F$ for all $f \in E$ and thus $C \cdot (X \wedge_e Y) \in N_F$ by (I).
- Suppose that either $X(e) \notin N_F$ or $Y(e) \notin N_F$. By symmetry, we can assume $Y(e) \notin N_F$. Since $X \perp C$, we have $C(e)X(e) + \sum_{f \in E \setminus \{e\}} C(f)X(f) \in N_F$ and thus

$$C(e)X(e)Y(e) + \sum_{f \in E \setminus \{e\}} C(f)X(f)Y(e) \in N_F. \tag{1}$$

And since $Y \perp C$,

$$C(e)Y(e) + \sum_{f \in E \setminus \{e\}} C(f)Y(f) \in N_F. \tag{2}$$

Since $C(e)Y(e) \notin N_F$, we can apply Proposition 4.6 to (1) and (2) with $\alpha = \sum_{f \in E \setminus \{e\}} C(f)X(f)Y(e)$, $\beta = X(e)$, $\gamma = C(e)Y(e)$, and $\delta = -\sum_{f \in E \setminus \{e\}} C(f)Y(f)$. As a result, we get $C \perp (X \wedge_e Y)$.

Case 2: $\|C \cdot (X \wedge_e Y)\| < 4$.

It is easy to check that if any one of $\{C(e), X(e), Y(e)\}$ is 0, then $(X \wedge_e Y) \perp C$. Assuming none of them is 0, we have the following subcases:

- If $\|X(e)\| \geq 2$ and $\|Y(e)\| \geq 2$, then either $\underline{X} \cap \underline{C} = \{e\}$ or $\underline{Y} \cap \underline{C} = \{e\}$, since otherwise $\|C \cdot (X \wedge_e Y)\| \geq 4$. Without loss of generality, assume $\underline{X} \cap \underline{C} = \{e\}$. Then, since $X \perp C$, $X(e) \in N_F$ and therefore

$$C \cdot (X \wedge_e Y) = \sum_{f \in E \setminus \{e\}} C(f)X(e)Y(f) \in N_F.$$

- If $\|X(e)\| = 1$ or $\|Y(e)\| = 1$, then either $|\underline{X} \cap \underline{C}| = \{e, f\}$ with $\|X(f)\| = 1$ or $|\underline{Y} \cap \underline{C}| = \{e, f\}$ with $\|Y(f)\| = 1$, since otherwise $\|C \cdot (X \wedge_e Y)\| \geq 4$.
By symmetry, we may assume without loss of generality that $|\underline{X} \cap \underline{C}| = \{e, f\}$ and $\|X(f)\| = 1$. Since $X \perp C$, $X(e)C(e) + X(f)C(f) \in N_F$. And since both $X(e)C(e)$ and $X(f)C(f)$ belong to F^\times , we must have $X(e)C(e) = -X(f)C(f)$. Therefore

$$C \cdot (X \wedge_e Y) = \sum_{g \in \underline{C} \setminus e} (C(g)X(e)Y(g) - C(g)Y(e)X(g)) \tag{3}$$

$$= \left(\sum_{g \in \underline{C} \setminus e} C(g)X(e)Y(g) \right) - C(f)Y(e)X(f) \tag{4}$$

$$= \left(\sum_{g \in \underline{C} \setminus e} C(g)X(e)Y(g) \right) + C(e)Y(e)X(e) \tag{5}$$

$$= \sum_{g \in \underline{C}} C(g)X(e)Y(g) \tag{6}$$

$$= X(e)(C \cdot Y) \in N_F. \quad \square \tag{7}$$

The following result and its proof were inspired by [4, Lemma 2.6].

Proposition 4.9. *Suppose F is a tract satisfying (MSF). Let $X_1, \dots, X_n \in \mathbb{N}[F^\times]$ be such that $X_I := \sum_{i \in I} X_i \in N_F$ for every $I \subset \{1, 2, \dots, n\}$ with $n - 2 \leq |I| \leq n - 1$. Then $\sum_{i=1}^n X_i \in N_F$.*

Proof. We may assume, without loss of generality, that $X_i \notin N_F$ for all i and $X_i + X_j \notin N_F$ for all $i \neq j$, since otherwise

$$X_i + \sum_{\substack{j=1 \\ j \neq i}}^n X_j \in N_F$$

or

$$X_i + X_j + \sum_{\substack{k=1 \\ k \neq i, j}}^n X_k \in N_F.$$

Let J be a maximum non-empty proper subset of $\{1, \dots, n\}$ such that $X_J := \sum_{j \in J} X_j \notin N_F$. Since J is proper, $|J| \leq n - 3$. By symmetry, we may assume without loss of generality that $J \subseteq \{4, 5, \dots, n\}$ and that $3 \in I := J^c \setminus \{1, 2\}$.

From the maximality of J , we have

$$X_1 + X_i + X_j \in N_F, \quad X_1 + X_2 + X_j \in N_F$$

and since $X_j \notin N_F$ by assumption, (MSF) implies³

$$X_1 + X_i - X_1 - X_2 \in N_F. \tag{8}$$

³ Recall that $X_1 - X_1$ is not the same thing as zero in $\mathbb{N}[F^\times]$!

Since $X_1 + X_j \in N_F$ and $X_1 \notin N_F$, (MSF) applied to (8) yields

$$X_1 - X_I + X_2 + X_j \in N_F. \tag{9}$$

Similarly, since $X_1 + X_2 + X_j \in N_F$ and $X_j \notin N_F$, (MSF) applied to (9) yields

$$X_1 - X_I + X_2 - X_1 - X_2 \in N_F. \tag{10}$$

Finally, since $X_1 + X_2 + X_j \in N_F$ and $X_1 + X_2 \notin N_F$, (MSF) applied to (10) yields

$$\sum_{i=1}^n X_i = X_1 + X_2 + X_I + X_j \in N_F. \quad \square$$

Our next goal is to prove that for any generalized vector X and any generalized covector Y such that $X \cdot Y$ has at most three terms, we have $X \perp Y$. We first recall the following key lemma from [1]:

Lemma 4.10. *Let X be a generalized vector of M and choose $e \in E$ with $X(e) \notin N_F$. Then there is some circuit C with $e \in \underline{C} \subset \underline{X}$.*

Although [1, Lemma 3.43] is stated in the language of fuzzy rings, the same (straightforward) proof works for generalized vectors in our sense.

Proposition 4.11. *Let F be a tract and let M be an F -matroid. If $X \in \mathcal{V}(M)$ and $Y \in \mathcal{V}^*(M)$ satisfy $\|X \cdot Y\| \leq 3$, then $X \perp Y$.*

Proof. We will treat each of the four possibilities for $|\underline{X} \cap \underline{Y}|$ separately.

Case 0: $|\underline{X} \cap \underline{Y}| = 0$.

In this case, $X \cdot Y = 0 \in N_F$.

Case 1: $|\underline{X} \cap \underline{Y}| = 1$.

Writing $\underline{X} \cap \underline{Y} = \{e\}$, we claim that either $X(e)$ or $Y(e)$ is null. Indeed, suppose that neither $X(e)$ nor $Y(e)$ belongs to N_F . Then by Lemma 4.10, there is a circuit C and cocircuit D such that $e \in \underline{C} \subset \underline{X}$ and $e \in \underline{D} \subset \underline{Y}$. But this implies $\underline{C} \cap \underline{D} = \{e\}$, which is impossible.

Note that a tract that satisfies (F) is an idyll. We therefore have $X \cdot Y = X(e)Y(e) \in N_F$.

Case 2: $|\underline{X} \cap \underline{Y}| = 2$.

Writing $\underline{X} \cap \underline{Y} = \{e, f\}$, we observe that, since $X \cdot Y$ has at most 3 terms, at least three of $X(e), X(f), Y(e), Y(f)$ must lie in F^\times (and not just $\mathbb{N}[F^\times]$). Without loss of generality, we may suppose that $X(e), X(f), Y(e) \in F^\times$. In particular, these values are non-null. By Lemma 4.10, there exist a circuit C such that $e \in \underline{C} \subset \underline{X}$ and a cocircuit D such that $e \in \underline{D} \subset \underline{Y}$. Thus $e \in \underline{C} \cap \underline{D} \subset \underline{X} \cap \underline{Y} = \{e, f\}$ and $|\underline{C} \cap \underline{D}| \neq 1$, from which it follows that $\underline{C} \cap \underline{D} = \{e, f\}$. We therefore have the following relations:

$$C(e)D(e) + C(f)D(f) \in N_F \Rightarrow C(f) = -\frac{C(e)D(e)}{D(f)}.$$

$$X(e)D(e) + X(f)D(f) \in N_F \Rightarrow X(f) = -\frac{X(e)D(e)}{D(f)}.$$

Since

$$X \cdot Y = X(e)Y(e) + X(f)Y(f) = X(e) \left(Y(e) - \frac{D(e)}{D(f)} Y(f) \right)$$

and $C \perp Y$ implies that

$$C \cdot Y = C(e)Y(e) + C(f)Y(f) = C(e) \left(Y(e) - \frac{D(e)}{D(f)} Y(f) \right) \in N_F,$$

we have $Y(e) - \frac{D(e)}{D(f)}Y(f) \in N_F$ as well and thus $X \cdot Y \in N_F$ by Lemma 4.4.

Case 3: $|\underline{X} \cap \underline{Y}| = 3$.

Suppose $\underline{X} \cap \underline{Y} = \{e, f, g\}$. Let $I = E - \underline{X}, J = \underline{X} \setminus \underline{Y}$, and let $M' = M \setminus I/J$ be the corresponding minor on $E' = \{e, f, g\}$. By Propositions 3.9 and 3.10, the natural restrictions $X' = X|_{E'}$ and $Y' = Y|_{E'}$ are vectors and covectors of M' , respectively. Since $\underline{X}' = \underline{Y}' = E'$, the matroid \underline{M}' has no loops or coloops (for example, if e is a loop then $C' = \{e\}$ is a circuit with $C' \cdot Y = Y(e)C(e) \in F^\times$, which is impossible, and similarly for coloops). By [9, Table 1.1], the only matroids on 3 elements with no loops or coloops are the uniform matroids $U_{1,3}$ and $U_{2,3}$. By duality, we may assume without loss of generality that $\underline{M}' = U_{2,3}$. In this case, M' has exactly one F -circuit C , with $\underline{C} = \{e, f, g\}$, and three F -cocircuits D_e, D_f , and D_g with $\underline{D}_e = \{f, g\}, \underline{D}_f = \{e, g\}$ and $\underline{D}_g = \{e, f\}$.

The fact that $\underline{X}' = \underline{Y}' = E'$, combined with $\|X \cdot Y\| = \overline{3}$, implies that X' is a vector (not a generalized vector) of M' and Y' is a covector. By [1, Lemma 4.19], the F -circuits of M' are exactly the vectors of M' having minimal non-empty support; together with Axiom (C2) in [1, Definition 3.11], this implies in our situation that X is a scalar multiple of C . It follows that $X \perp Y$. \square

We have finally put together all the ingredients needed to prove the main theorem. The structure of the following proof is modeled on [4, Proof of Theorem 2.7].

Proof of Theorem 4.1. Assume there exists a non-perfect F -matroid M on some set E , and choose $|E|$ as small as possible. Then by definition, there exist $X \in \mathcal{V}(M)$ and $Y \in \mathcal{V}^*(M)$ such that $X \cdot Y \notin N_F$. By Proposition 4.11, $\|X \cdot Y\| \geq 4$.

Claim. $\sum_{e \in E \setminus \{e_0\}} Y(e)C(e) \in N_F$ for all $e_0 \in E$ and all $C \in C(M)$.

Assume for the sake of contradiction that there exist $e_0 \in E$ and $C \in C(M)$ such that $\sum_{e \in E'} Y(e)C(e) \notin N_F$, where $E' = E \setminus \{e_0\}$.

Let $M' = M \setminus e_0$. By Propositions 3.9 and 3.10, we have $Y|_{E'} \in \mathcal{V}^*(M')$ and $(X \wedge_{e_0} C)|_{E'} \in \mathcal{V}(M')$. By the minimality of E , M' is perfect, and therefore

$$\sum_{e \in E'} Y(e)(X \wedge_{e_0} C) \in N_F.$$

Explicitly, this means that

$$C(e_0) \sum_{e \in E'} Y(e)X(e) - X(e_0) \sum_{e \in E'} Y(e)C(e) \in N_F. \tag{11}$$

We also have

$$Y \cdot C = Y(e_0)C(e_0) + \sum_{e \in E'} Y(e)C(e) \in N_F. \tag{12}$$

Since $\|X \cdot Y\| \geq 4$, we may apply Proposition 4.6 with $\beta = X(e_0)$ and $\gamma = -\sum_{e \in E'} Y(e)C(e) \notin N_F$ to (11) and (12) to obtain

$$C(e_0)(X \cdot Y) = C(e_0) \sum_{e \in E'} Y(e)X(e) + Y(e_0)C(e_0)X(e_0) \in N_F,$$

which means that $X \perp Y$, a contradiction.

Claim. $X \perp Y$ (a contradiction which finishes the proof).

From the previous claim, $\sum_{e \in E \setminus \{e_0\}} Y(e)C(e) \in N_F$ for all $e_0 \in E$ and all $C \in C(M)$. By symmetry, we also have $\sum_{e \in E \setminus \{e_0\}} X(e)D(e) \in N_F$ for all $e_0 \in E$ and $D \in C^*(M)$. Thus $X|_{E \setminus \{e\}}$ is contained in both

$\mathcal{V}(M \setminus e)$ and $\mathcal{V}(M/e)$ for all $e \in E$, and similarly $Y|_{E \setminus \{e\}}$ is contained in both $\mathcal{V}^*(M \setminus e)$ and $\mathcal{V}^*(M/e)$ for all $e \in E$. Since both $M \setminus e$ and M/e are perfect, we have

$$\sum_{e \in E \setminus \{e_0\}} X(e)Y(e) \in N_F \quad \forall e_0 \in E. \tag{13}$$

Moreover for every $e_1, e_2 \in E$ with $e_1 \neq e_2$, we have both $X|_{E \setminus \{e_1\}} \in \mathcal{V}(M \setminus \{e_1\})$ and $Y|_{E \setminus \{e_2\}} \in \mathcal{V}^*(M/\{e_2\})$. By Propositions 3.9 and 3.10, we have $X|_{E \setminus \{e_1, e_2\}} \in \mathcal{V}(M \setminus \{e_1\}/\{e_2\})$ and $Y|_{E \setminus \{e_1, e_2\}} \in \mathcal{V}^*(M/\{e_2\} \setminus \{e_1\})$.

Since $M'' = M \setminus e_1/e_2 = M/e_2 \setminus e_1$ is perfect, we have

$$\sum_{e \in E \setminus \{e_1, e_2\}} X(e)Y(e) \in N_F \quad \forall e_1, e_2 \in E. \tag{14}$$

Applying Proposition 4.9 to (13) and (14) shows that $X \perp Y$ as claimed. \square

5. Comparison with the work of Dress–Wenzel

In this section we briefly compare our results with those in [4].

For ease of exposition, we work with Lorscheid’s “simplified fuzzy rings”. It is proved in [1, Appendix B] that every fuzzy ring in the sense of Dress–Wenzel is weakly isomorphic to a simplified fuzzy ring, and it is proved in [6, Theorem 2.21] that the category of simplified fuzzy rings can be identified with a full subcategory of the category of tracts. In particular, every simplified fuzzy ring can be identified in a natural way with a tract.⁴

A simplified fuzzy ring in the sense of Lorscheid is a tuple $(K, +, \cdot, \epsilon, K_0)$ where $(K, +, \cdot)$ is a commutative semiring equal to $\mathbb{N}[K^\times]$ and such that ϵ, K_0 satisfy the following axioms:

- (FR4) K_0 is a proper semiring ideal, i.e., $K_0 + K_0 \subseteq K_0, K \cdot K_0 \subseteq K_0, 0 \in K_0$ and $1 \notin K_0$.
- (FR5) For $\alpha \in K^*$ we have $1 + \alpha \in K_0$ if and only if $\alpha = \epsilon$.
- (FR6) If $x_1, x_2, y_1, y_2 \in K$ and $x_1 + y_1, x_2 + y_2 \in K_0$ then $x_1 \cdot x_2 + \epsilon \cdot y_1 \cdot y_2 \in K_0$.

To give a sufficient condition for perfection, Dress and Wenzel introduce the following variant of (FR6):

$$(FR6'') \text{ If } \kappa, \lambda_1, \lambda_2 \in K, \mu \in K \setminus K_0, \text{ and } \kappa + \mu \cdot \lambda_1, \mu + \lambda_2 \in K_0 \text{ then } \kappa + \epsilon \cdot \lambda_1 \cdot \lambda_2 \in K_0.$$

Proposition 5.1. *A simplified fuzzy ring satisfying (FR6''), when viewed as a tract, satisfies the strong fusion axiom (SF).*

Proof. Given a simplified fuzzy ring K , let F_K denote the tract associated to it. If $\alpha + \gamma$ and $\beta - \gamma$ are in N_{F_K} , then in terms of the fuzzy ring we have $\alpha + \gamma, \beta - \gamma \in K_0$. If $\gamma = 0$, then $\alpha + \beta \in K_0$ by (FR4). If $\gamma \notin K_0$, let $\kappa = \alpha, \mu = \epsilon \cdot \gamma, \lambda_1 = \epsilon$ and $\lambda_2 = \beta$. Then $\kappa + \mu \cdot \lambda_1, \mu + \lambda_2 \in K_0$ by (FR6''). In the language of tracts, this means precisely that $\alpha + \beta \in N_{F_K}$. \square

Combining Proposition 5.1 with Theorem 1.11, we recover the following special case⁵ of [4, Theorem 2.7]:

Theorem. *A simplified fuzzy ring which satisfies (FR6'') is perfect.*

The tract \mathbb{P}' appearing in Example 4.2 comes from a simplified fuzzy ring K . Since \mathbb{P}' does not satisfy (SF), Proposition 5.1 implies that K does not satisfy (FR6''). On the other hand, Theorem 4.1 applies to \mathbb{P}' since \mathbb{P}' does satisfy (MSF). This shows that Theorem 4.1 is strictly stronger than [4, Theorem 2.7], at least when we restrict the latter to simplified fuzzy rings.

⁴ For a discussion of which tracts come from simplified fuzzy rings, see [6, Example 2.11].

⁵ Dress and Wenzel prove in [4, Theorem 2.7] that this remains true with “simplified fuzzy ring” replaced by “weakly distributive fuzzy ring”.

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