Journal of Computational and Applied Mathematics 436 (2024) 115385

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Localized pointwise error estimates for hybrid finite element methods

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ARTICLE INFO

Article history:
Received 30 December 2022
Received in revised form 2 June 2023

MSC:
65N30
65N15

Keywords:
Pointwise estimates
Hybrid finite element

ABSTRACT

We present localized pointwise error estimates for a numerical method designed for accurate and efficient approximations for the flux variables. The method can be considered as a reduced version of the mixed finite element method. The method uses much less degrees of freedom compared to the mixed finite element methods while produces approximation solutions for the flux variable of the same accuracy. In this paper, we present optimal pointwise error estimates for the method, and they show local dependence of the error at a point and weaker dependence on global norms.
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1. Introduction

Recently, a new hybrid finite element method for second-order elliptic partial differential equations (PDEs) is developed [1] for accurate and efficient approximations of the flux variables. It is a two-step procedure. On coarse meshes of mesh size H, the primary variable is approximated by a standard method such as the standard Galerkin method. With this crude approximation as a problem data, a $H(\mathrm{div})$ projection of dual variables is sought as an approximation solution on a finer meshes of mesh size h. This new method can be considered as a reduced version of the mixed finite element method.

Typically, there is a restriction for the choice of fine meshsize h to achieve the optimal rate of convergence. For example, h should be taken as H^2, see [2]. The hybrid method circumvents this restriction by introducing a parameter ϵ, which can be chosen arbitrary small. To achieve optimal convergence rate in L_2 norm on the finer meshes, ϵ needs to satisfy

$$\sqrt{\epsilon}H^2 = \mathcal{O}(h^{k+1}), \tag{1.1}$$

where h^{k+1} is the optimal convergence rate for the approximation space on the finer meshes. The above equality can be interpreted as one can choose fine mesh size as small as one wishes by choosing the problem parameter ϵ to satisfy (1.1). In other words, choosing small ϵ enables one to use arbitrary small meshsize h independent of the coarse meshsize H. This freedom of choice for the approximation spaces makes the new method efficient and accurate. A possible problem of choosing small ϵ is that the resulting algebraic equation becomes nearly singular as $\epsilon \to 0$. This problem is resolved using the iterative solver developed in [3].

While optimal error estimates in L_2-norm are provided for the hybrid method in [1], there is no pointwise error estimates. In order to obtain accurate and reliable determination concerning the behavior of the approximation solutions, one needs to use pointwise error. In this paper, we provide the highly localized pointwise error estimates showing that the solutions of the hybrid method are higher order perturbation of the standard mixed finite element methods with the assumption (1.1). Demlow [4] obtained localized pointwise error estimates for the mixed finite element methods, that

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https://doi.org/10.1016/j.cam.2023.115385
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is an extension of the Schatz's results for the standard Galerkin method [5]. Similar arguments are used in [6] for the pointwise error estimates for the least-squares formulation. We also refer to [7,8] for maximum norm error estimates for the mixed finite element methods.

This paper is organized as follows. In Section 2, we present mathematical formulations. In Section 3, finite element spaces are defined and numerical methods are defined. In Section 4, some preliminaries results are presented. Finally, pointwise error estimate is presented in Section 5.

2. Problem formulation

Let Ω be a bounded domain in $\mathbf{R}^n$, $n = 2, 3$, with smooth boundary $\partial\Omega$. We use the Sobolev spaces W_p^m and $H_2^m = W_2^m$ and the associated norms and seminorms. For $m = 0$, W_p^m coincides with L_p, and $H_0^1(D)$ denotes the functions in $H_2^1(D)$ with zero trace on ∂D. Also, we use $[Y]^n$ for the set of all ordered n-tuples of Y. We define

$$\mathbf{X} = H(\mathrm{div}) = \{\mathbf{v} \in [L_2(\Omega)]^n : \nabla \cdot \mathbf{v} \in L_2\},$$

which is a Hilbert space under the norm $\|\mathbf{v}\|_{H(\mathrm{div})} = (\|\mathbf{v}\|_{L_2}^2 + \|\nabla \cdot \mathbf{v}\|_{L_2}^2)^{1/2}$. For brevity, the norm in W_2^m will be denoted by $\|\cdot\|_m$.

2.1. Mathematical equations

We consider a model second-order elliptic partial differential equations with homogeneous Dirichlet boundary condition:

$$-\nabla \cdot (\mathcal{A}\nabla u) = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{2.1}$$

where the matrix $\mathcal{A}$ is symmetric, uniformly positive definite, and bounded.

We assume that for u satisfying (2.1), there exists a positive constant C independent of f satisfying

$$\|u\|_2 \leq C\|f\|_0. \tag{2.2}$$

We transform the original model problem into a system of first-order by introducing a flux variable $\boldsymbol{\sigma} = -\mathcal{A}\nabla u \in H(\mathrm{div})$. Then, we obtain

$$\boldsymbol{\sigma} + \mathcal{A}\nabla u = 0 \quad \text{in } \Omega, \quad \nabla \cdot \boldsymbol{\sigma} = f \quad \text{in } \Omega, \tag{2.3}$$

with the boundary condition $u = 0$ on $\partial\Omega$.

Here and hereafter, we use C with or without subscripts in this paper to denote a generic positive constant, possibly different at different occurrences, that is independent of the mesh size h.

For localized estimates, the following weight-function and weighted norms are commonly used [4,5]. For each point x of $\overline{\Omega}$, a real number s and an arbitrary $y \in \Omega$,

$$\sigma_{x,h}^s(y) = \left(\frac{h}{|x-y| + h}\right)^s. \tag{2.4}$$

For $1 \leq p \leq \infty$ and fixed x, we define weighted norms

$$\|u\|_{L_p(\Omega),x,s} = \|\sigma_x^s(\cdot)\, u(\cdot)\|_{L_p(\Omega)} \tag{2.5}$$

and

$$\|u\|_{W_p^1(\Omega),x,s} = \|u\|_{L_p(\Omega),x,s} + \|\nabla u\|_{L_p(\Omega),x,s}. \tag{2.6}$$

3. Finite element approximation

Let $\mathcal{T}_h$ be a quasiuniform triangulation of Ω with triangular/tetrahedra elements with meshsize h. Boundary elements are allowed to have one curved face, see [9]. In order to approximate functions in $\mathbf{X} = H(\mathrm{div})$, we use the Raviart–Thomas elements (RT) or Brezzi–Douglas–Marini (BDM) space of index k, see [9–12]. We present our results based on RT elements for simplicity of presentation. Similar results holds for BDM family of spaces with obvious modifications.

In order to place some assumptions on the approximation spaces, we first define

$$Q_h^k = \{q \in L_2(\Omega) : q|_T \in P^k(T), \text{ for each } T \in \mathcal{T}_h\}, \tag{3.1}$$

where $P^k(T)$ is the space of polynomials of degree k on the triangle T.

Let $P_h : L_2(\Omega) \to Q_h^k$ be the local L_2 projection satisfying

$$(v - P_h v, q_h) = 0, \text{ for all } q_h \in Q_h^k. \tag{3.2}$$

It is well known that the local L_2 projection P_h satisfies the following approximation property: For $D \subset \Omega$,

$$\|v - P_h v\|_{L_p(D)} \le Ch^s |v|_{W_p^s(D')}, \text{ where } D' = \bigcup_{\{T: T \bigcap D \neq \emptyset\}} T, \tag{3.3}$$

for all $v \in W_p^s(D')$, all $1 \le s \le k$ and $1 \le p \le \infty$. We also recall that P_h is stable in W_p^0 norm for any $1 \le p \le \infty$, that is,

$$\|P_h v\|_{W_p^0(\Omega)} \le C\|v\|_{W_p^0(\Omega)}, \tag{3.4}$$

for $1 \le p \le \infty$.

Let $\mathbf{X}_h \subset \mathbf{X}$ be the RT spaces of order k. Let $\Pi_h : \mathbf{X} \to \mathbf{X}_h$ be the Fortin projection satisfying the following commuting diagram property:

$$\nabla \cdot \Pi_h = P_h \nabla \cdot : \mathbf{X} \to Q_h^k. \tag{3.5}$$

It satisfies the following approximation property:

$$\|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{[L_p(D)]^n} \le Ch^s |\boldsymbol{\sigma}|_{[W_p^s(D')]^n}, \tag{3.6}$$

for $1 \le s \le k$ and $1 \le p \le \infty$.

For the approximation spaces for the primary variable on a coarse mesh $\mathcal{T}_H$, we use the standard continuous piecewise linear polynomial space S_H, i.e.

$$S_H = \{v \in H_0^1(\Omega) : v_{|_T} \in P^1(T), \text{ for each } T \in \mathcal{T}_H\}.$$

Remark 3.1. For the remaining of this paper, k is the index of the RT space.

3.1. Discrete δ-function

For any given $x_0 \in \Omega$, let $\boldsymbol{\delta}_i^0 \in \mathbf{X}_h$ be a function such that

$$(\mathcal{A}^{-1}\mathbf{p}_h, \boldsymbol{\delta}_i^0) = [\mathbf{p}_h(x_0)]_i \qquad \text{for all } \mathbf{p}_h \in \mathbf{X}_h, \tag{3.7}$$

where $[\mathbf{p}_h(x_0)]_i$ is the ith component of the vector $\mathbf{p}_h(x_0)$. The following inequalities concerning $\boldsymbol{\delta}_i^0$ is obtained in [4]: For any $y \in \Omega$,

$$|\boldsymbol{\delta}_i^0(y)| + h|\nabla \cdot \boldsymbol{\delta}_i^0(y)| \le Ch^{-n} e^{-c|y-x_0|/h}, \tag{3.8}$$

and

$$\|\boldsymbol{\delta}_i^0\|_{[L_p(\Omega)]^n} + h\|\nabla \cdot \boldsymbol{\delta}_i^0\|_{[L_p(\Omega)]} \le Ch^{n(1/p-1)}. \tag{3.9}$$

3.2. Mixed finite element methods

The mixed finite element method corresponding to (2.3) is as follows: Find a pair $(u_h^m, \boldsymbol{\sigma}_h^m) \in Q_h^k \times \mathbf{X}_h$ such that

$$\begin{aligned} (\mathcal{A}^{-1}\boldsymbol{\sigma}_h^m, \boldsymbol{\tau}_h) - (\nabla \cdot \boldsymbol{\tau}_h, u_h^m) &= 0, \\ (\nabla \cdot \boldsymbol{\sigma}_h^m, v_h) &= (f, v_h) \end{aligned} \tag{3.10}$$

for all $(v_h, \boldsymbol{\tau}_h) \in Q_h^k \times \mathbf{X}_h$. Then, the pair $(u - u_h^m, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m)$ satisfy the following error equations

$$(\mathcal{A}^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m), \boldsymbol{\tau}_h) - (\nabla \cdot \boldsymbol{\tau}_h, u - u_h^m) = 0, \tag{3.11}$$

$$(\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m), v_h) = 0, \tag{3.12}$$

for all $(v_h, \boldsymbol{\tau}_h) \in Q_h^k \times \mathbf{X}_h$.

Remark 3.2. Using the commuting diagram property and (3.2), we have

$$\nabla \cdot (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m) = 0. \tag{3.13}$$

It is well-known that the mixed method has the basic L_2 norm error estimates, see [13]:

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \le Ch^{k+1}\|\boldsymbol{\sigma}\|_{k+1}.$$

The following lemma can be also obtained by taking advantage of the orthogonality property in (3.12).

Lemma 3.1. *Let $f \in Q_h^k$ and let $(u_h^m, \boldsymbol{\sigma}_h^m)$ satisfies (3.10). Then,*

$$\nabla \cdot \boldsymbol{\sigma} = \nabla \cdot \boldsymbol{\sigma}_h^m = f.$$

The following superconvergence results are obtained in [14, Theorem 4.2].

Lemma 3.2. *Let P_h be the local L_2 projection satisfying (3.11) and (3.12). Then,*

$$\|P_h u - u_h^m\|_0 \le Ch\Big(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_m\|_0 + \|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m)\|_0 + \|u - u_h^G\|_1\Big). \tag{3.14}$$

In particular, if $f \in Q_h^k$, then,

$$\|P_h u - u_h^m\|_0 \le Ch^2\|u\|_2. \tag{3.15}$$

The following weighted-norm error estimate for the mixed method is obtain in [4, Theorem 1.1]. We recall the Kronecker delta δ_{ij}, where $\delta_{ij} = 0$ if $i \ne j$ and $\delta_{ij} = 1$ if $i = j$.

Lemma 3.3. *Let the assumptions concerning Π_h, P_h, and the mesh be satisfied. Then there exists a constant C independent of u, $\boldsymbol{\sigma}$ and h such that for any $x_0 \in \Omega$, $0 \le s \le k+1$, and $0 \le t \le k$,*

$$|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m)(x_0)| \le C(\ln\frac{1}{h})^{\delta_{s(k+1)}}\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{L_\infty(\Omega),x,s}$$
$$+ Ch\Big(\|u - P_h u\|_{L_\infty(\Omega),x,t} + (\ln\frac{1}{h})^{\delta_{tk}}\|\nabla\cdot\boldsymbol{\sigma} - P_h\nabla\cdot\boldsymbol{\sigma}\|_{L_\infty(\Omega),x,t}\Big).$$

3.3. Hybrid finite element methods

An efficient flux approximation scheme, hybrid finite element method, is developed in [1]. The methods can be consider as a reduced mixed finite element method due to its close tie to the mixed finite element methods. It is imperative to note that the new method reduces the degrees of freedom (DOFs) significantly compared to the mixed methods while maintaining the accuracy of the approximations, see [1, Section 6]. The method is defined as follows:

Step 1 (Coarse-grid solution) On a coarse mesh $\mathcal{T}_H$, obtain a crude approximation, e.g. standard Galerkin method defined: Let S_H be the continuous piecewise linear polynomial spaces on $\mathcal{T}_H$. Define the Galerkin approximation solution $u_H^G \in S_H$ as

$$(\mathcal{A}\nabla u_H^G\,,\ \nabla v_h) = (f, \nabla v_h),\ \text{ for all } v_h \in S_H.$$

Step 2 (Fine-grid solution) On a finer mesh $\mathcal{T}_h$, find the $H(\text{div})$ projection $\boldsymbol{\sigma}_h \in \mathbf{X}_h$ for the given data $f + \epsilon u_H^G$, *i.e.*

$$(\nabla\cdot\boldsymbol{\sigma}_h, \nabla\cdot\boldsymbol{\tau}_h) + \epsilon(\mathcal{A}^{-1}\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) = (f + \epsilon u_H^G, \nabla\cdot\boldsymbol{\tau}_h) \quad \forall\,\boldsymbol{\tau}_h \in \mathbf{X}_h. \tag{3.16}$$

Remark 3.3. By taking $v_h = \nabla\cdot\boldsymbol{\tau}_h$ and multiplying ϵ to the first equation in the mixed formulation (3.10), we have

$$(\nabla\cdot\boldsymbol{\sigma}_h^m, \nabla\cdot\boldsymbol{\tau}_h) + \epsilon(\mathcal{A}^{-1}\boldsymbol{\sigma}_h^m, \boldsymbol{\tau}_h) = (f + \epsilon u_h^m, \nabla\cdot\boldsymbol{\tau}_h) \quad \forall\,\boldsymbol{\tau}_h \in \mathbf{X}_h. \tag{3.17}$$

This is similar to (3.16) except the problem data $f + \epsilon u_h^m$ replacing $f + \epsilon u_H^G$. Note that the hybrid method first obtains a crude approximation u_H^G and obtain $\boldsymbol{\sigma}_h$ separately. Thus, the problem size is smaller.

Remark 3.4. Generally, there is a restriction concerning the meshsize h and H. The reduced mixed method overcomes this restriction by introducing parameter ϵ. With a proper choice of ϵ, one can choose arbitrary small meshsize h regardless of coarse meshsize H. For optimal convergence rate, ϵ needs to be chosen to satisfy

$$\sqrt{\epsilon}H^2 = \mathcal{O}(h^{k+1}).$$

Remark 3.5. The Galerkin solution satisfies the following orthogonality property:

$$(\mathcal{A}\nabla(u - u_H^G)\,,\ \nabla v_H) = 0, \quad \text{ for all } v_H \in S_H, \tag{3.18}$$

and it has the following approximation property, [5]:

$$\|u - u_H^G\|_{W_\infty^1(\Omega)} \le CH\|u\|_{W_\infty^2(\Omega)}. \tag{3.19}$$

Note that the true solution $(u, \boldsymbol{\sigma})$ satisfies

$$(\nabla\cdot\boldsymbol{\sigma}, \nabla\cdot\boldsymbol{\tau}_h) + \epsilon(\mathcal{A}^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau}_h) = (f + \epsilon u, \nabla\cdot\boldsymbol{\tau}_h) \quad \forall\,\boldsymbol{\tau}_h \in \mathbf{X}_h.$$

Subtracting (3.16) from the above, we obtain the following quasi-orthogonality property:

$$(\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\,,\ \nabla\cdot\boldsymbol{\tau}_h) + \epsilon(\mathcal{A}^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\,,\ \boldsymbol{\tau}_h) = \epsilon(u - u_H^G, \nabla\cdot\boldsymbol{\tau}_h) \tag{3.20}$$

The following basic error estimate is obtained in [1, Theorem 4.4]:

Theorem 3.4. *Assume that $\boldsymbol{\sigma} \in H^{k+1}(\Omega)^n$. Then, and $\mathbf{X}_h$ is the RT space of order k. Then,*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \le Ch^{k+1}\|\boldsymbol{\sigma}\|_{k+1} + C\sqrt{\epsilon}H^2\|u\|_2, \tag{3.21}$$

and

$$\|\nabla \cdot (\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^m)\|_0 \le C\epsilon\|P_h(u_h^m - u_H^G)\|_0 \le C\epsilon H^2(\|u\|_2 + \|f\|_1). \tag{3.22}$$

4. Some preliminaries

We shall use the Green's function for our problem (2.1), i.e. let $G(x,y)$ be a function satisfying

$$u(x) = \int_\Omega \mathcal{A}\nabla u(y)\nabla G(x,y)dy = \int_\Omega f(y)G(x,y)dy,$$

and $G(x,y) = 0$ for $y \in \partial\Omega$. Then, we have the following estimates, see [15]

Lemma 4.1. *There exists a constant C such that for x and y in Ω,*

$$|D_x^\alpha D_y^\beta G(x,y)| \le C|x-y|^{2-n-|\alpha+\beta|} \text{ for } |\alpha+\beta| > 0 \tag{4.1}$$

and

$$|G(x,y)| \le C|x-y|^{1-n}, \tag{4.2}$$

where D_x^α is a differential operator with respect to x defined in [16, Chapter 1.2].

The above lemma is used to obtain an a priori estimate for the following auxiliary problem (4.3). The estimate plays an crucial role in terms of obtaining pointwise estimate for the hybrid finite element methods.

We shall employ the following problem involving the discrete $\boldsymbol{\delta}_i^0$ function defined in [4].

$$\begin{aligned} -\mathrm{div}\mathcal{A}\nabla z &= \nabla\cdot\boldsymbol{\delta}_i^0 \quad \text{in} \quad \Omega, \\ z &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{4.3}$$

By a priori estimates, we have

$$\|z\|_1 \le C\|\nabla\cdot\boldsymbol{\delta}_i^0\|_{-1} \le C\|\boldsymbol{\delta}_i^0\|_0. \tag{4.4}$$

For the remainder of the paper, z will denote the solution of the above problem. Our main concern is obtaining W_1^1 error estimate for z. In order to present our result, we first define the following: For $d > 0$ and any fixed $x \in \overline{\Omega} \subset \mathbf{R}^n$, $B_d(x)$ is defined as follows;

$$B_d(x) = \{y \in \Omega;\ |y-x| < d\}. \tag{4.5}$$

Without loss of generality we assume that $\mathrm{diam}(\Omega) \le 1$. Let

$$d_j = 2^{-j} \qquad \text{for } j = 0, 1, 2, \ldots,$$

and for fixed x, set

$$\begin{aligned} \Omega_j &= \{y \in \Omega : d_{j+1} < |y-x_0| < d_j\}, \\ \Omega_j' &= \{y \in \Omega : d_{j+2} < |y-x_0| < d_{j-1}\}, \\ \Omega_j'' &= \{y \in \Omega : d_{j+3} < |y-x_0| < d_{j-2}\}. \end{aligned}$$

Let $M \gg 2$ be a constant which will be chosen later on to be sufficiently large. For convenience we shall choose M to begin with so that for some integer J

$$Mh = 2^{-J}.$$

Notice that since $M > 2$

$$J = \frac{1}{\ln 2}(\ln\frac{1}{M} + \ln\frac{1}{h}) \le C\ \ln\frac{1}{h}. \tag{4.6}$$

Lemma 4.2. *Let z satisfy (4.3). Then,*

$$\|z\|_{W_1^1(\Omega)} \le C\ln\frac{1}{h}. \tag{4.7}$$

Proof. Let ω be a smooth cutoff function which is 1 on Ω_i'', 0 on $\Omega \setminus \Omega_i'''$, satisfies $0 \le \omega \le 1$, and has a bounded first derivative, i.e. $\|\nabla\omega\|_{L_\infty(\Omega)} \le \frac{C}{d_i}$. We let z_1 and z_2 satisfy

$$\begin{aligned} -\mathrm{div}\mathcal{A}\nabla z_1 &= \nabla\cdot(\omega\boldsymbol{\delta}_i^0) \quad \text{in} \quad \Omega, \\ z_1 &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} -\mathrm{div}\mathcal{A}\nabla z_2 &= \nabla\cdot((1-\omega)\boldsymbol{\delta}_i^0) \quad \text{in} \quad \Omega, \\ z_2 &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{4.9}$$

Note that

$$z = z_1 + z_2. \tag{4.10}$$

Also, using the a priori estimate in (3.9) and (4.4), we have

$$\|z_1\|_{W_2^1(\Omega)} \le C\|\omega\boldsymbol{\delta}_i^0\|_0 \le C\|\boldsymbol{\delta}_i^0\|_0 \le Ch^{-n/2}, \tag{4.11}$$

and similarly

$$\|z_2\|_{W_2^1(\Omega)} \le C\|(1-\omega)\boldsymbol{\delta}_i^0\|_0 \le C\|\boldsymbol{\delta}_i^0\|_0 \le Ch^{-n/2}. \tag{4.12}$$

Using the a priori estimate in (4.4), (3.8), Hölder's inequality, and $\|\boldsymbol{\delta}_i^0\|_{L_2(\Omega_j'')} \le Cd_j^{n/2}\|\boldsymbol{\delta}_i^0\|_{L_\infty(\Omega_j'')}$, we have the following local estimates:

$$\|z_1\|_{W_2^1(\Omega_j)} \le \|\omega\boldsymbol{\delta}_i^0\|_{L_2(\Omega_j'')} \le Ch^{-n}e^{-cd_j/h}d_j^{n/2}. \tag{4.13}$$

First, we obtain the estimate for $\|z_1\|_{W_1^1(\Omega)}$. Using the measure of the subdomain, (4.11), and (4.13), we have

$$\begin{aligned} \|z_1\|_{W_1^1(\Omega)} &\le \|z_1\|_{W_1^1(B_{Mh}(x))} + \sum_{j=0}^{J}\|z_1\|_{W_1^1(\Omega_j)} \\ &\le Ch^{n/2}\|z_1\|_{W_2^1(B_{Mh}(x_0))} + C\sum_{i=0}^{J} d_j^{n/2}\|z_1\|_{W_2^1(\Omega_j)} \\ &\le Ch^{n/2}\|z_1\|_{W_2^1(\Omega)} + C\sum_{j=0}^{J} d_j^{n/2}\|z_1\|_{W_2^1(\Omega_j)} \\ &\le C + C\sum_{j=0}^{J} d_j^{n/2}h^{-n}e^{-cd_j/h}d_j^{n/2} \\ &\le C + C\sum_{j=0}^{J}(\frac{d_j}{h})^n e^{-cd_j/h}. \end{aligned} \tag{4.14}$$

Note that for any $l \ge 0$,

$$\sum_{j=0}^{J}\left(\frac{d_j}{h}\right)^l e^{-cd_j/h} \le C\int_0^\infty x^l e^{-cx}dx \le C. \tag{4.15}$$

Thus, using (4.15) into (4.14), we have

$$\|z_1\|_{W_1^1(\Omega)} \le C. \tag{4.16}$$

In order to obtain the estimate for $\|z_2\|_{W_1^1(\Omega)}$, using the measure of the subdomain, (4.12), we have

$$\begin{aligned} \|z_2\|_{W_1^1(\Omega)} &\le \|z_2\|_{W_1^1(B_{Mh}(x_0))} + \sum_{j=0}^{J}\|z_2\|_{W_1^1(\Omega_j)} \\ &\le Ch^{n/2}\|z_2\|_{W_2^1(B_{Mh}(x_0))} + C\sum_{j=0}^{J} d_j^n\|z_2\|_{W_\infty^1(\Omega_j)}, \end{aligned} \tag{4.17}$$

where we used $\|z_2\|_{W_1^1(\Omega_j)} \le Cd_j^n\|z_2\|_{W_\infty^1(\Omega_j)}$.

We observe that z_2 is smooth on Ω_i'' since $\nabla\cdot((1-\omega)\boldsymbol{\delta}_i^0) = 0$ on Ω_i''. Thus, for $x \in \Omega_i''$ and $|\alpha| \le 1$, using the cutoff function ω and (4.1),

$$\begin{aligned} D_x^\alpha z_2(x) &= D_x^\alpha \int_{\Omega\setminus\Omega_i''} G(x,y)(\nabla\cdot(1-\omega)\boldsymbol{\delta}_i^0)dy \\ &= -D_x^\alpha \int_{\Omega\setminus\Omega_i''} \nabla_y G(x,y)(1-\omega)\boldsymbol{\delta}_i^0 dy \end{aligned}$$

$$= -\int_{\Omega\setminus\Omega_i''} D_x^\alpha \nabla_y G(x,y)(1-\omega)\delta_i^0 dy$$
$$\le d_j^{2-n-2}\|\delta_i^0\|_{L_1(\Omega)} \le C d_j^{-n}. \tag{4.18}$$

Thus, using (4.12) and (4.18) into (4.17), and using (4.6), we have

$$\|z_2\|_{W_1^1(\Omega)} \le C + C\cdot J \le C\ln\frac{1}{h}. \tag{4.19}$$

Finally, using the equalities $z = z_1 + z_2$ combining the inequalities (4.16) and (4.19), we have

$$\|z\|_{W_1^1(\Omega)} \le \|z_1\|_{W_1^1(\Omega)} + \|z_2\|_{W_1^1(\Omega)} \le C\ln\frac{1}{h}.$$

This completes the proof.

As a consequence of Lemma 4.2, we have the following a priori estimate using Sobolev embedding Theorem.

$$\|z\|_{W_2^0(\Omega)} \le C\|z\|_{W_1^1(\Omega)} \le C\ln\frac{1}{h} \quad \text{in } \mathbf{R}^2, \text{ and}$$
$$\|z\|_{W_{3/2}^0(\Omega)} \le C\|z\|_{W_1^1(\Omega)} \le C\ln\frac{1}{h} \quad \text{in } \mathbf{R}^3 \tag{4.20}$$

We shall use the following auxiliary problem involving z. Consider

$$-\nabla\cdot\mathcal{A}\nabla v = P_h z \text{ in } \Omega, \text{ and } v = 0 \text{ on } \partial\Omega. \tag{4.21}$$

Then, we have the following a priori estimates.

Corollary 4.3. *Let v satisfy (4.21). Then,*

$$\|v\|_2 \le Ch^{1-n/2}\ln\frac{1}{h}, \quad \textit{and } \|v\|_1 \le C\ln\frac{1}{h}, \quad \textit{for } n = 2, 3, \tag{4.22}$$

and for $n = 3$, we have

$$\|v\|_{W_{3/2}^2} \le C\ln\frac{1}{h}. \tag{4.23}$$

Proof. Using the a priori estimate, (3.4) and (4.20),

$$\|v\|_{W_2^2} \le C\|P_h z\|_{W_2^0} \le C\|z\|_{W_2^0} \le C\ln\frac{1}{h}, \quad \text{for } n = 2.$$

and using the a priori estimate, inverse inequality, (3.4) and (4.20),

$$\|v\|_{W_2^2} \le C\|P_h z\|_{W_2^0} \le Ch^{-1/2}\|P_h z\|_{W_{3/2}^0}$$
$$\le Ch^{-1/2}\|z\|_{W_{3/2}^0} \le Ch^{-1/2}\ln\frac{1}{h}, \quad \text{for } n = 3.$$

This proves the first inequality in (4.22). For the second inequality, first note that

$$\|v\|_1 \le \|v\|_2 \le C\ln\frac{1}{h}, \quad \text{for } n = 2.$$

For $n = 3$, using a priori estimate, Sobolev embedding Theorem, (3.4), and (4.20), we have

$$\|v\|_{W_2^1} \le C\|P_h z\|_{W_2^{-1}} \le C\|P_h z\|_{W_{3/2}^0} \le \|z\|_{W_{3/2}^0} \le \|z\|_{W_1^1} \le C\ln\frac{1}{h}. \tag{4.24}$$

This completes the proof for (4.22).

For (4.23), using the similar argument for (4.24)

$$\|v\|_{W_{3/2}^2} \le C\|P_h z\|_{W_{3/2}^0} \le \|z\|_{W_{3/2}^0} \le C\ln\frac{1}{h}. \tag{4.25}$$

This completes the proof.

We need the following bound for $\|v - v_I\|_{W_1^1}$, where v_I is an interpolation [17].

Lemma 4.4. *Let v be the solution of (4.21) and $v_I \in S_H$ be the interpolation. Then,*

$$\|v - v_I\|_{W_1^1} \le C\,H\ln\frac{1}{h}.$$

Proof. We present the proof for $n = 2$ and $n = 3$ separately.

Case 1, $n = 2$.

Using the measure of the subdomain, Cauchy–Schwarz inequality, approximation property of the interpolation and (4.22), we have

$$\begin{aligned}
\|v - v_I\|_{W_1^1(\Omega)} &= \|v - v_I\|_{W_1^1(B_{MH}(x_0))} + \sum_{j=1}^{J} \|v - v_I\|_{W_1^1(\Omega_j)} \\
&\le CH\|v - v_I\|_{W_2^1(B_{MH}(x_0))} + C\sum_{j=1}^{J} d_j \|v - v_I\|_{W_2^1(\Omega_j)} \\
&\le CH^2\|v\|_{W_2^2(B_{MH}(x_0))} + C\sum_{j=1}^{J} d_j H\|v\|_{W_2^2(\Omega_j)} \\
&\le C\Big(H^2\|v\|_{W_2^2(\Omega)} + CH\Big[\sum_{j=1}^{J} d_j^2\Big]^{1/2}\Big[\sum_{j=1}^{J}\|v\|^2_{W_2^2(\Omega_j)}\Big]^{1/2}\Big) \\
&\le C(H^2\|v\|_{W_2^2(\Omega)} + H\|v\|_{W_2^2(\Omega)}) \\
&\le CH \ln\frac{1}{h}.
\end{aligned} \tag{4.26}$$

Case 2, $n = 3$.

Using the measure of the subdomain, Hölder's inequality with $p = 3$ and $q = 3/2$, approximation property of the interpolation and (4.23), we have

$$\begin{aligned}
\|v - v_I\|_{W_1^1(\Omega)} &= \|v - v_I\|_{W_1^1(B_{MH}(x_0))} + \sum_{j=1}^{J} \|v - v_I\|_{W_1^1(\Omega_j)} \\
&\le CH\|v - v_I\|_{W_{3/2}^1(B_{MH}(x_0))} + C\sum_{j=1}^{J} d_j \|v - v_I\|_{W_{3/2}^1(\Omega_j)} \\
&\le CH^2\|v\|_{W_{3/2}^2(B_{MH}(x_0))} + C\sum_{j=1}^{J} d_j H\|v\|_{W_{3/2}^2(\Omega_j)} \\
&\le C\Big(H^2\|v\|_{W_{3/2}^2(\Omega)} + CH\Big[\sum_{j=1}^{J} d_j^3\Big]^{1/3}\Big[\sum_{j=1}^{J}\|v\|^{3/2}_{W_{3/2}^2(\Omega_j)}\Big]^{2/3}\Big) \\
&\le C(H^2\|v\|_{W_{3/2}^2(\Omega)} + H\|v\|_{W_{3/2}^2(\Omega)}) \\
&\le CH \ln\frac{1}{h}.
\end{aligned} \tag{4.27}$$

This completes the proof.

Lemma 4.5. *Let $\boldsymbol{\sigma}_h$ be the solution of (3.16). Let P_h be the L_2 projection in (3.2) and a be the solution of $-\nabla \cdot \mathcal{A}\nabla a = P_h b$ for some $b \in L_2(\Omega)$. Let $\boldsymbol{\eta} = -\mathcal{A}\nabla a$. Then,*

$$\begin{aligned}
&(\mathcal{A}^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), \boldsymbol{\eta} - \boldsymbol{\eta}_h^m) \\
&\quad \le Ch^{k+2}\Big(\|u\|_{k+2} + \|f\|_{k+1}\Big)\|a\|_2 + C\sqrt{\epsilon}H^2h^2\Big(\|u\|_2 + \|f\|_1\Big)\|a\|_2.
\end{aligned}$$

Proof. Note that

$$-\nabla \cdot \mathcal{A}\nabla a = \nabla \cdot \boldsymbol{\eta} = P_h b, \text{ and } \|P_h b\|_0 \le \|a\|_2.$$

Now using the estimate (3.15) and $P_h b \in Q_h^k$, we have

$$\|P_h a - a_h^m\|_0 \le Ch^2\|a\|_2. \tag{4.28}$$

Using $\boldsymbol{\eta} = -\mathcal{A}\nabla a$, integration by parts, (3.11), Lemma 3.1, (3.2), Cauchy–Schwarz inequality, approximation property (3.3), and (3.22), we have

$$\begin{aligned}
&(\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h), \boldsymbol{\eta}-\boldsymbol{\eta}_h^m) = (\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^m), \boldsymbol{\eta}-\boldsymbol{\eta}_h^m) + (\mathcal{A}^{-1}(\boldsymbol{\sigma}_h^m-\boldsymbol{\sigma}_h), \boldsymbol{\eta}-\boldsymbol{\eta}_h^m)\\
&\quad = (\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^m), \boldsymbol{\eta}) - (\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^m), \boldsymbol{\eta}_h^m) + (\mathcal{A}^{-1}(\boldsymbol{\sigma}_h^m-\boldsymbol{\sigma}_h), \boldsymbol{\eta}-\boldsymbol{\eta}_h^m)\\
&\quad = (\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^m), -\mathcal{A}\nabla a) - (\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^m), \boldsymbol{\eta}_h^m) + (\mathcal{A}^{-1}(\boldsymbol{\sigma}_h^m-\boldsymbol{\sigma}_h), \boldsymbol{\eta}-\boldsymbol{\eta}_h^m)\\
&\quad = (\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^m), a) - (u-u_h^m, \nabla\cdot\boldsymbol{\eta}_h^m) + (\nabla\cdot(\boldsymbol{\sigma}_h^m-\boldsymbol{\sigma}_h), a-a_h^m)\\
&\quad = (f-P_hf, a) - (u-u_h^m, \nabla\cdot\boldsymbol{\eta}_h^m) + (\nabla\cdot(\boldsymbol{\sigma}_h^m-\boldsymbol{\sigma}_h), a-a_h^m)\\
&\quad = (f-P_hf, a-P_ha) - (P_hu-u_h^m, P_hb) + (\nabla\cdot(\boldsymbol{\sigma}_h^m-\boldsymbol{\sigma}_h), P_ha-a_h^m)\\
&\quad \le \|f-P_hf\|_0\|a-P_ha\|_0 + \|P_hu-u_h^m\|_0\|P_hb\|_0 + \|\nabla\cdot(\boldsymbol{\sigma}_h^m-\boldsymbol{\sigma}_h)\|_0\|P_ha-a_h^m\|_0\\
&\quad \le Ch^{k+2}\|f\|_{k+1}\|a\|_2 + Ch^{k+2}\Big(\|u\|_{k+2}+\|f\|_{k+1}\Big)\|a\|_2\\
&\qquad + \sqrt{\epsilon}H^2h^2(\|u\|_2+\|f\|_1)\|a\|_2\\
&\quad \le Ch^{k+2}\Big(\|u\|_{k+2}+\|f\|_{k+1}\Big)\|a\|_2 + \sqrt{\epsilon}H^2h^2\Big(\|u\|_2+\|f\|_1\Big)\|a\|_2.
\end{aligned}\tag{4.29}$$

This completes the proof.

Using the above estimates, we obtain the following result.

Lemma 4.6. *Let $\boldsymbol{\sigma}_h$ be the solution of (3.16). Let P_h be the L_2 projection and z be the solution of (4.3). Then,*

$$\begin{aligned}
(\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h), P_hz) \le{}& C\epsilon h^{k+3/2}\ln\frac{1}{h}\Big(\|u\|_{k+2}+\|f\|_{k+1}\Big)\\
&+C\epsilon^{1/2}h^{k+1}\ln\frac{1}{h}\Big(\|u\|_2+\|f\|_1\Big) + C\epsilon^{1/2}h^{k+1}\ln\frac{1}{h}\|u\|_{W^2_\infty(\Omega)}.
\end{aligned}$$

Proof. Let v be the solution of (4.21). By defining $\boldsymbol{\eta} = -\mathcal{A}\nabla v$, we obtain the following system of equations:

$$\boldsymbol{\eta} + \mathcal{A}\nabla v = 0, \quad \text{and } \nabla\cdot\boldsymbol{\eta} = P_hz. \tag{4.30}$$

Let $(v_h^m, \boldsymbol{\eta}_h^m)$ be the mixed finite approximation solution for $(v, \boldsymbol{\eta})$.

Note that using (3.12), we have

$$\nabla\cdot\boldsymbol{\eta} = \nabla\cdot\boldsymbol{\eta}_h^m = P_hz, \tag{4.31}$$

Using (4.31), (3.20) and (4.31)

$$\begin{aligned}
&(\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\ P_hz) = (\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\ \nabla\cdot\boldsymbol{\eta}_h^m)\\
&\quad = -\epsilon(\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\ \boldsymbol{\eta}_h^m) + \epsilon(u-u_H^G, \nabla\cdot\boldsymbol{\eta}_h^m)\\
&\quad = -\epsilon(\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\ \boldsymbol{\eta}_h^m-\boldsymbol{\eta}) - \epsilon(\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\ \boldsymbol{\eta}) + \epsilon(u-u_H^G, \nabla\cdot\boldsymbol{\eta})\\
&\quad = I_1 + I_2 + I_3.
\end{aligned}\tag{4.32}$$

Using Lemma 4.5 with $a = v$, (1.1), and (4.22), we have

$$\begin{aligned}
I_1 &\le C\epsilon h^{k+2}\Big(\|u\|_{k+2}+\|f\|_{k+1}\Big)\|v\|_2 + \epsilon^{3/2}H^2h^2(\|u\|_2+\|f\|_1)\|v\|_2\\
&\le C\epsilon h^{k+3/2}\ln\frac{1}{h}\Big(\|u\|_{k+2}+\|f\|_{k+1}\Big) + C\epsilon h^{k+3}(\|u\|_2+\|f\|_1)h^{-1/2}\ln\frac{1}{h}\\
&\le C\epsilon h^{k+3/2}\ln\frac{1}{h}\Big(\|u\|_{k+2}+\|f\|_{k+1}\Big) + C\epsilon h^{k+5/2}\ln\frac{1}{h}(\|u\|_2+\|f\|_1).
\end{aligned}\tag{4.33}$$

Using the first equality in (4.30), integration by parts, (3.2), $\nabla\cdot\boldsymbol{\sigma} = f$, and (4.22), we have

$$\begin{aligned}
I_2 &= \epsilon(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \nabla v) = -\epsilon(\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, v))\\
&= -\epsilon(\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h), v-P_hv) - \epsilon(\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h), P_hv)\\
&= -\epsilon(f-P_hf, v-P_hv) - \epsilon(\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h), P_hv)\\
&\le \epsilon\|f-P_hf\|_0\|v-P_hv\|_0 - \epsilon(\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h), P_hv)\\
&\le C\epsilon h^{k+2}\|f\|_{k+1}\|v\|_1 - \epsilon(\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h), P_hv)\\
&\le C\epsilon h^{k+2}\ln\frac{1}{h}\|f\|_{k+1} - \epsilon(\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h), P_hv).
\end{aligned}\tag{4.34}$$

In order to obtain an upper bound for the second term in the above, let a be the solution of

$$-\nabla\cdot\mathcal{A}\nabla b = P_hv \text{ in } \Omega, \text{ and } b = 0 \text{ on } \partial\Omega. \tag{4.35}$$

Using the a priori estimate, (3.4) and (4.22),

$$\|b\|_2 \le C\|P_h v\|_0 \le C\|v\|_0 \le C \ln \frac{1}{h}. \tag{4.36}$$

Defining $\boldsymbol{\theta} = -\mathcal{A}\nabla b$, we obtain the following system of equations:

$$\boldsymbol{\theta} + \mathcal{A}\nabla b = 0, \quad \text{ and } \nabla \cdot \boldsymbol{\theta} = P_h v. \tag{4.37}$$

Note that using Lemma 3.1, we have

$$\nabla \cdot \boldsymbol{\theta} = \nabla \cdot \boldsymbol{\theta}_h^m = P_h v. \tag{4.38}$$

Now, using (3.20), (4.38)

$$\begin{aligned} -\epsilon(\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), P_h v) &= -\epsilon(\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), \nabla \cdot \boldsymbol{\theta}_h^m) \\ &= \epsilon^2(\mathcal{A}^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\,,\ \boldsymbol{\theta}_h^m) - \epsilon^2(u - u_H^G, \nabla \cdot \boldsymbol{\theta}_h^m) \\ &= \epsilon^2(\mathcal{A}^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\,,\ \boldsymbol{\theta}_h^m - \boldsymbol{\theta}) + \epsilon^2(\mathcal{A}^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\,,\ \boldsymbol{\theta}) - \epsilon^2(u - u_H^G, \nabla \cdot \boldsymbol{\theta}) \\ &= J_1 + J_2 + J_3. \end{aligned} \tag{4.39}$$

Using Lemma 4.5 and (4.36)

$$\begin{aligned} J_1 &\le C\epsilon^2\Big(h^{k+2}(\|u\|_{k+2} + \|f\|_{k+1})\|b\|_2 + +\sqrt{\epsilon}H^2h^2(\|u\|_2 + \|f\|_1)\|b\|_2\Big) \\ &\qquad \le C\epsilon^2 h^{k+2} \ln \frac{1}{h}\Big(\|u\|_{k+2} + \|f\|_{k+1}\Big). \end{aligned} \tag{4.40}$$

Using the first equality in (4.37), integration by parts, (3.12), (3.3), (3.22), (3.4), (4.36), and (1.1), we have

$$\begin{aligned} J_2 &= \epsilon^2(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\,,\ \nabla b) = -\epsilon^2(\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\,,\ b) \\ &= -\epsilon^2(\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\,,\ b - P_h b) - \epsilon^2(\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\,,\ P_h b) \\ &= -\epsilon^2(f - P_h f\,,\ b - P_h b) - \epsilon^2(\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\,,\ P_h b) \\ &= -\epsilon^2(f - P_h f\,,\ b - P_h b) - \epsilon^2(\nabla \cdot (\boldsymbol{\sigma}_h^m - \boldsymbol{\sigma}_h)\,,\ P_h b) \\ &\le \epsilon^2\|f - P_h f\|_0\|b - P_h b\|_0 + \epsilon^2\|\nabla \cdot (\boldsymbol{\sigma}_h^m - \boldsymbol{\sigma}_n)\|_0\|P_h b\|_0 \\ &\le C\epsilon^2 h^{k+2}\|f\|_{k+1}\|b\|_1 + C\epsilon^3 H^2(\|u\|_2 + \|f\|_1)\,\|b\|_0 \\ &\le C\epsilon^2 h^{k+2} \ln \frac{1}{h}\|f\|_{k+1} + C\epsilon^{5/2} h^{k+1} \ln \frac{1}{h}(\|u\|_2 + \|f\|_1). \end{aligned} \tag{4.41}$$

For J_3, using integration by parts, the first equality in (4.37), the orthogonality of the Galerkin solution u_H^G, Cauchy–Schwarz inequality, (4.36), and (1.1), we have

$$\begin{aligned} J_3 &= \epsilon^2(\mathcal{A}\nabla(u - u_H^G)\,,\ \nabla b) = \epsilon^2(\mathcal{A}\nabla(u - u_H^G)\,,\ \nabla(b - b_I)) \\ &\le C\epsilon^2 H^2\|u\|_2\|b\|_2 \le C\epsilon^{3/2} h^{k+1} \ln \frac{1}{h}\|u\|_2. \end{aligned} \tag{4.42}$$

Plugging (4.40), (4.41) and (4.42) into (4.39) and then (4.34), we have

$$I_2 \le C\epsilon^2 h^{k+2} \ln \frac{1}{h}\Big(\|u\|_{k+2} + \|f\|_{k+1}\Big) + C\epsilon^{3/2} h^{k+1} \ln \frac{1}{h}\Big(\|u\|_2 + \|f\|_1\Big). \tag{4.43}$$

Using the orthogonality, and L_∞ error estimate for the Galerkin solution and Lemma 4.4, we have

$$\begin{aligned} I_3 &= \epsilon(\mathcal{A}(u - u_H^G), \nabla v) = \epsilon(\mathcal{A}(u - u_H^G), \nabla(v - v_I)) \\ &\le \epsilon\|u - u_H^G\|_{W^1_\infty(\Omega)}\|v - v_I\|_{W^1_1(\Omega)} \le C\epsilon H^2 \ln \frac{1}{h}\|u\|_{W^2_\infty(\Omega)} \\ &\le C\epsilon^{1/2} h^{k+1} \ln \frac{1}{h}\|u\|_{W^2_\infty(\Omega)}. \end{aligned} \tag{4.44}$$

Plugging (4.33), (4.43) and (4.44) into (4.32), we have

$$\begin{aligned} (\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), P_h z) &\le C\epsilon h^{k+3/2} \ln \frac{1}{h}\Big(\|u\|_{k+2} + \|f\|_{k+1}\Big) \\ &\qquad + C\epsilon^{1/2} h^{k+1} \ln \frac{1}{h}\Big(\|u\|_2 + \|f\|_1\Big) + C\epsilon^{1/2} h^{k+1} \ln \frac{1}{h}\|u\|_{W^2_\infty(\Omega)}. \end{aligned}$$

This completes the proof.

5. Pointwise error estimates

In this section, we present localized error estimates for $\boldsymbol{\sigma}-\boldsymbol{\sigma}_h$ for all the possible choices of $0<\epsilon\le 1$ for our reduced mixed methods. The main result in this section implies that the solution of the hybrid finite element method is higher order perturbation of the standard mixed finite element methods for all $\epsilon\le 1$ satisfying (1.1), i.e. $\sqrt{\epsilon}H^2=\mathcal{O}(h^{k+1})$.

Theorem 5.1. *Let $\boldsymbol{\sigma}_h$ be the solution of (3.16). Let the assumptions concerning Π_h, P_h, and the mesh be satisfied. Then there exists a constant C independent of u, $\boldsymbol{\sigma}$ and h such that for any $x_0\in\Omega$, $0\le s\le k+1$, and $0\le t\le k$,*

$$\begin{aligned}
&|(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)(x_0)|\\
&\quad\le C(\ln\frac{1}{h})^{\delta_{s(k+1)}}\|\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}\|_{L_\infty(\Omega),x,s}+Ch\Big(\|u-P_hu\|_{L_\infty(\Omega),x,t}+(\ln\frac{1}{h})^{\delta_{tk}}\|\nabla\cdot\boldsymbol{\sigma}-P_h\nabla\cdot\boldsymbol{\sigma}\|_{L_\infty(\Omega),x,t}\Big)\\
&\quad+C\epsilon h^{k+3/2}\ln\frac{1}{h}\Big(\|u\|_{k+2}+\|f\|_{k+1}\Big)+C\sqrt{\epsilon}h^{k+1}\ln\frac{1}{h}\Big(\|u\|_2+\|f\|_1\Big)+C\sqrt{\epsilon}h^{k+1}\ln\frac{1}{h}\|u\|_{W^2_\infty(\Omega)}.
\end{aligned}$$

Remark 5.1. Note that we have

$$\sqrt{\epsilon}\,H^2\le Ch^{k+1}.$$

This indicates that the last three global terms in the above estimate are higher order. For example, if $\epsilon = h^2$ (and $k=0, H=1$), then the dependence of the pointwise estimate on the global terms is 1- order higher modulo logarithm.

Proof.
By the triangle inequality, we have

$$|(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)(x_0)|\le|(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^m)(x_0)|+|(\boldsymbol{\sigma}_h^m-\boldsymbol{\sigma}_h)(x_0)|. \tag{5.1}$$

For the first term in the above, i.e. $|(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^m)(x_0)|$, we use Lemma 3.3. For the second term in (5.1), using the discrete δ function in (3.7), we have

$$\begin{aligned}
|(\boldsymbol{\sigma}_h^m-\boldsymbol{\sigma}_h)(x_0)| &= (\mathcal{A}^{-1}(\boldsymbol{\sigma}_h^m-\boldsymbol{\sigma}_h)\,,\ \boldsymbol{\delta}_i^0)\\
&=(\mathcal{A}^{-1}(\boldsymbol{\sigma}_h^m-\boldsymbol{\sigma})\,,\boldsymbol{\delta}_i^0)+(\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\boldsymbol{\delta}_i^0)\\
&=I_1+I_2.
\end{aligned}\tag{5.2}$$

Note that I_1 contains only mixed finite element solutions $\boldsymbol{\sigma}_h^m$ and discrete delta function. An upper bound for I_1 is obtained in [4, Lemma 4.1 and Lemma 4.4].

$$\begin{aligned}
I_1 &\le C(\ln\frac{1}{h})^{\delta_{s(k+1)}}\|\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}\|_{L_\infty(\Omega),x,s}\\
&\quad+Ch\Big(\|u-P_hu\|_{L_\infty(\Omega),x,t}+(\ln\frac{1}{h})^{\delta_{tk}}\|\nabla\cdot\boldsymbol{\sigma}-P_h\nabla\cdot\boldsymbol{\sigma}\|_{L_\infty(\Omega),x,t}\Big)
\end{aligned}\tag{5.3}$$

In order to obtain an upper bound for I_2, consider the following is the first-order system corresponding to (4.3):

$$\left\{\begin{array}{rcll}\mathbf{r}+\mathcal{A}\nabla z &=& \mathbf{0} & \text{in}\quad\Omega,\\ \nabla\cdot\mathbf{r} &=& \nabla\cdot\boldsymbol{\delta}_i^0 & \text{in}\quad\Omega\end{array}\right.\tag{5.4}$$

with boundary conditions

$$z=0\quad\text{on }\partial\Omega \tag{5.5}$$

Let $(z_h^m,\mathbf{r}_h^m)$ be the approximation solution of the mixed finite element method. Then, the following estimates are obtained in [4, (4.50a), (4,21)].

$$\|\mathbf{r}-\mathbf{r}_h^m\|_{L_1,x_0,-s}\le C(\ln\frac{1}{h})^{\delta_{k,s}},\qquad\text{for }0\le s\le k+1,\tag{5.6}$$

and

$$\|z-P_hz\|_{L_1,x_0,-t}\le Ch(\ln\frac{1}{h})^{\delta_{k,t}},\qquad\text{for }0\le t\le k.\tag{5.7}$$

Using Lemma 3.1 with $f=\nabla\cdot\boldsymbol{\delta}_i^0\in Q_h^k$, we have

$$\nabla\cdot\mathbf{r}=\nabla\cdot\mathbf{r}_h^m=\nabla\cdot\boldsymbol{\delta}_i^0.\tag{5.8}$$

Using the quasi-orthogonality property (3.20), we have

$$\begin{aligned} I_2 &= (\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h), \boldsymbol{\delta}_i^0) = -\frac{1}{\epsilon}(\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\ \nabla\cdot\boldsymbol{\delta}_i^0) + (u-u_H^G, \nabla\cdot\boldsymbol{\delta}_i^0) \\ &= -\frac{1}{\epsilon}(\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\ \nabla\cdot\mathbf{r}) + (u-u_H^G, \nabla\cdot\mathbf{r}) \\ &= -\frac{1}{\epsilon}\Big((\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\ \nabla\cdot(\mathbf{r}-\mathbf{r}_h^m)) + (\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\ \mathbf{r}_h^m) - \epsilon(u-u_H^G, \nabla\cdot\mathbf{r}_h^m)\Big) \\ &\qquad +(u-u_G^H, \nabla\cdot\mathbf{r}) \\ &= (\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\ \mathbf{r}_h^m) \\ &= (\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\ \mathbf{r}_h^m-\mathbf{r}) + (\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\ \mathbf{r}) \\ &= (\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^m),\ \mathbf{r}_h^m-\mathbf{r}) + (\mathcal{A}^{-1}(\boldsymbol{\sigma}_h^m-\boldsymbol{\sigma}_h),\ \mathbf{r}_h^m-\mathbf{r}) + (\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\ \nabla z) \\ &= J_1 + J_2 + J_3. \end{aligned} \tag{5.9}$$

For J_1, using the error Eqs. (3.11), (3.12) and (3.13), we have

$$\begin{aligned} J_1 &= (\mathcal{A}^{-1}(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}),\ \mathbf{r}_h^m-\mathbf{r}) + (\mathcal{A}^{-1}(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^m),\ \mathbf{r}_h^m-\mathbf{r}) \\ &= (\mathcal{A}^{-1}(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}),\ \mathbf{r}_h^m-\mathbf{r}) + (\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^m),\ z_h^m-z) \\ &= (\mathcal{A}^{-1}(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}),\ \mathbf{r}_h^m-\mathbf{r}) \\ &\le C\|\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}\|_{L_\infty,x_0,s}\|\mathbf{r}-\mathbf{r}_h^m\|_{L_1,x_0,-s} \\ &\le C(\ln\frac{1}{h})^{\delta_{k,s}}\|\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}\|_{L_\infty,x_0,s}, \end{aligned} \tag{5.10}$$

for $0 \le s \le k+1$.

For J_2, using (3.2) and (3.11), we have

$$\begin{aligned} J_2 &= (\nabla\cdot(\boldsymbol{\sigma}_h^m-\boldsymbol{\sigma}_h),\ z_h^m-z) = (\nabla\cdot(\boldsymbol{\sigma}_h^m-\boldsymbol{\sigma}_h),\ z_h^m-P_hz) \\ &\le \|\nabla\cdot(\boldsymbol{\sigma}_h^m-\boldsymbol{\sigma}_h)\|_0\cdot\|z_h^m-P_hz\|_0. \end{aligned} \tag{5.11}$$

Using (3.14), (5.8), and (3.9) with $p=2$, we have

$$\begin{aligned} \|z_h^m-P_hz\|_0 &\le Ch(\|\mathbf{r}-\mathbf{r}_h^m\|_0 + \|\nabla\cdot\mathbf{r}-\nabla\cdot\mathbf{r}_h^m\|_0 + \|z-z_h^G\|_1) \\ &\le Ch^2(\|\mathbf{r}\|_1+\|z\|_2) \\ &\le Ch^2\|\nabla\cdot\boldsymbol{\delta}_i^0\|_0 \le C. \end{aligned} \tag{5.12}$$

Plugging (3.22) and (5.12) into (5.11) and using (1.1), we have

$$J_2 \le C\epsilon H^2(\|u\|_2+\|f\|_1) \le C\sqrt{\epsilon}\,h^{k+1}(\|u\|_2+\|f\|_1). \tag{5.13}$$

Using integration by parts, we have

$$\begin{aligned} J_3 &= -(\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\ z) \\ &= -(\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\ z-P_hz) - (\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\ P_hz) \\ &= K_1+K_2. \end{aligned} \tag{5.14}$$

Using (3.2), (3.5), and (5.7)

$$\begin{aligned} K_1 &= -(\nabla\cdot\boldsymbol{\sigma},\ z-P_hz) = -(\nabla\cdot(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}),\ z-P_hz) \\ &\le \|\nabla\cdot(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma})\|_{L_\infty,x_0,t}\|z-P_hz\|_{L_1,x_0,-t} \\ &\le Ch(\ln\frac{1}{h})^{\delta_{k,t}}\|\nabla\cdot(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma})\|_{L_\infty,x_0,t}. \end{aligned} \tag{5.15}$$

From Lemma 4.6, we have

$$\begin{aligned} K_2 &\le |(\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\ P_hz)| \\ &\le C\epsilon h^{k+3/2}\ln\frac{1}{h}\Big(\|u\|_{k+2}+\|f\|_{k+1}\Big) \\ &\qquad + C\epsilon^{1/2}h^{k+1}\ln\frac{1}{h}\Big(\|u\|_2+\|f\|_1\Big) + C\epsilon^{1/2}h^{k+1}\ln\frac{1}{h}\|u\|_{W_\infty^2(\Omega)}. \end{aligned} \tag{5.16}$$

Plugging (5.15) and (5.16) into (5.14), we obtain

$$J_3 \le Ch\|\nabla\cdot(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma})\|_{L_\infty,x_0,t} + C\epsilon h^{k+3/2}\ln\frac{1}{h}\Big(\|u\|_{k+2}+\|f\|_{k+1}\Big)$$
$$+C\sqrt{\epsilon}h^{k+1}\ln\frac{1}{h}\Big(\|u\|_2+\|f\|_1\Big)+C\sqrt{\epsilon}h^{k+1}\ln\frac{1}{h}\|u\|_{W^2_\infty(\Omega)}. \tag{5.17}$$

Collecting (5.10), (5.13) and (5.17) into (5.9), we obtain

$$I_2 \le (\ln\frac{1}{h})^{\delta_{k,s}}\|\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}\|_{L_\infty,x_0,s} + Ch(\ln\frac{1}{h})^{\delta_{k,t}}\|\nabla\cdot(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma})\|_{L_\infty,x_0,t}$$
$$+C\epsilon h^{k+3/2}\ln\frac{1}{h}\Big(\|u\|_{k+2}+\|f\|_{k+1}\Big)+C\sqrt{\epsilon}h^{k+1}\ln\frac{1}{h}\Big(\|u\|_2+\|f\|_1\Big)+C\sqrt{\epsilon}h^{k+1}\ln\frac{1}{h}\|u\|_{W^2_\infty(\Omega)}.$$

Finally, combining (5.3) and the above inequality into (5.2) then (5.1) with Lemma 3.3, we obtain the desired result. This completes the proof.

Remark 5.2. Using the above pointwise error estimate and standard argument, one can obtain so-called asymptotic error expansion inequality, [4,5].

Data availability

No data was used for the research described in the article.

Acknowledgments

This research is supported in part by National Science Foundation Grant DMS-2208289.

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