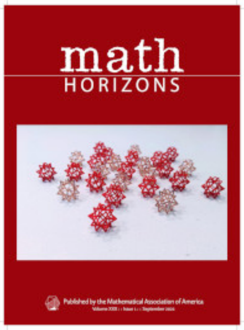

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Balancing Centrifuges with Number Theory

Matthew H. Baker

Back in 2011–2012, I spent a year as a faculty member at UC Berkeley and I became friends with some biologists there. At a barbecue one weekend, I was chatting with Iswar Hariharan, a cancer researcher. When he learned that I was a number theorist, he told me about a problem he had been thinking about on and off for more than 15 years. The problem concerns balancing centrifuges.

Figure 1. A 20-hole balanced centrifuge with 8 test tubes.

Photo from Joseph Elsbernd, used according to CC BY 2.0, https://bit.ly/3b6aufz

A Crash Course on Centrifuges

A centrifuge is a laboratory device that separates fluids based on density. The separation is achieved through centrifugal force by spinning a collection of test tubes at high speeds. As Iswar explained to me, it's very important to balance a centrifuge before operating it; running a centrifuge with an unbalanced load can cause permanent damage. A centrifuge is called *balanced* if the center of mass of the collection of test tubes coincides with the center of mass of the centrifuge itself. For example, figure 1 shows a balanced configuration of 8 test tubes in a 20-hole centrifuge. Some configurations are not as obviously balanced, such as the one with 13 test tubes in figure 2.

Figure 2. A 24-hole balanced centrifuge with 13 test tubes.

If you spend a lot of time balancing centrifuges and have a mathematically curious mind, the following question might naturally arise: For which pairs of n and k with $1 \le k \le n$ can you find a way to balance k identical test tubes in an n-hole centrifuge?

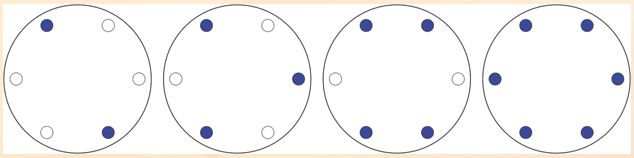
This is precisely the question that Iswar started to think about, and in 1998 he arrived at a conjectural answer. Before I tell you Iswar's conjecture (which he described to me at the barbecue), let's gather some data in a few special cases to get a feeling for the problem.

Some Special Cases

In the case that $n = 3$, you can balance the centrifuge only when $k = 3$. When $n = 4$, you can balance with $k = 2$ or $k = 4$ test tubes.

Figure 3 shows balanced centrifuges with $k = 2$, 3, 4, and 6 test tubes when $n = 6$. You

Figure 3. Balanced configurations for a 6-hole centrifuge when k is 2, 3, 4, and 6.



Title and figure 2 photos courtesy of Rishav Ray (Reddit; https://bit.ly/3zI5vvN)

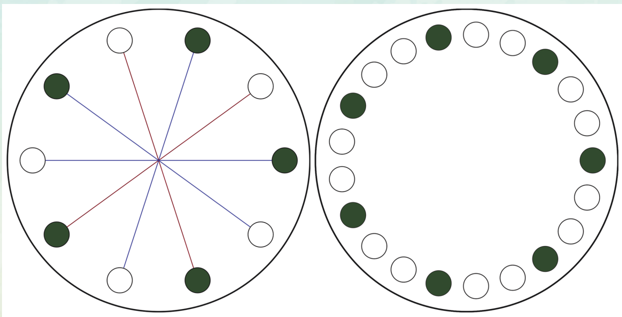


Figure 4. A balanced 10-hole centrifuge with 5 test tubes cannot accommodate 2 more (left), and a balanced 21-hole centrifuge with 7 test tubes cannot accommodate 3 more (right).

should be able to convince yourself that $k = 1$ and $k = 5$ test tubes won't balance when $n = 6$. When $n = 5$ or $n = 7$, it turns out you can only balance the centrifuge when $k = n$.

From these first few cases, we should guess that the answer might have something to do with whether n is prime… let's look at a few more examples. When $n = 8$, you can balance the centrifuge if and only if k is even, and when $n = 9$, you can balance the centrifuge when k is a multiple of 3. When $n = 11$, you can only balance if $k = 11$.

But something interesting happens when $n = 10$. In this case, you can find balanced centrifuges for $k = 2, 4, 5, 6, 8$, or 10. Notice, for example, that if you wanted to balance the centrifuge with $k = 7$, you could try to start with a balanced set of five test tubes (occupying every other slot as in figure 4) and then add two more in opposite holes, but there's a problem: one out of each pair of opposite holes is already occupied! We will call this the *overlap problem*. In some sense, the overlap problem is the key subtlety in the balanced centrifuge problem.

Let's look at one more example. When $n = 21$, you can balance if and only if k is in the set $\{3,6,7,9,12,14,15,18,21\}$. We can see yet another illustration of the overlap problem by considering the case $k = 10$. You could try to take a balanced configuration of seven test tubes (evenly spaced every three spots as in figure 4) and then add three more in the shape of an equilateral triangle, but you will find that it doesn't work.

If you stare long enough at this data, and at similar data for other values of n, you might come up with the same guess as Iswar.

Iswar's Conjecture. You can balance k identical test tubes, where $1 \le k \le n$, in an n-hole centrifuge if and only if both k and $n - k$ can be expressed as a sum of prime divisors of n.

For example, when $n = 21$ and $k = 6$, we have $6 = 3 + 3$ and $21 - 6 = 15 = 3 + 3 + 3 + 3 + 3$; moreover, the pair (21, 6) can be balanced. But when $k = 10$, although $10 = 3 + 7$, there is no way to express $n - k = 11$ as a sum of 3s and 7s. It turns out that the pair (21,10) cannot be balanced.

The conjecture predicts, for example, that if 6 divides n then every pair (n, k) with $2 \le k \le n - 2$ can be balanced (and when $n \ge 5$ this happens only if 6 divides n). This fact is presumably why the number of chambers in most commercial centrifuges is a multiple of 6.

A Solution to the Problem

After I left the barbecue, I started thinking about Iswar's conjecture. It occurred to me that a natural way to phrase the question mathematically is in terms of nth roots of unity.

By the fundamental theorem of algebra, for each positive integer n, there are precisely n solutions in the complex numbers, denoted by $\mathbb{C}$, to the equation $z^n = 1$. The roots of $z^n - 1$ are called the *nth roots of unity*, and they are equally spaced along the complex unit circle $\{z \in \mathbb{C} : |z| = 1\}$. For example, figure 5 depicts the 8th roots of unity, which are the eight solutions to the equation $z^8 - 1 = 0$.

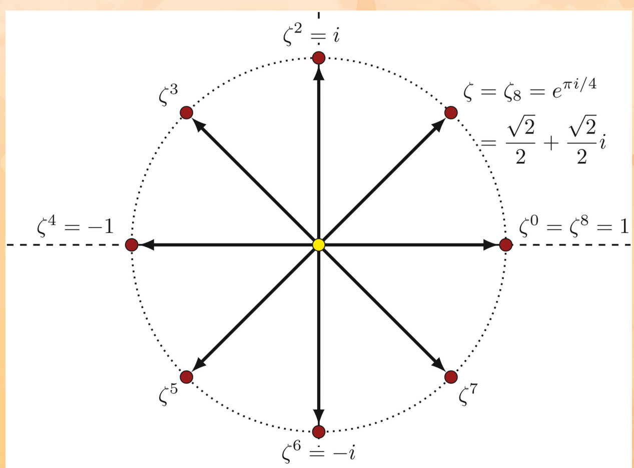


Figure 5. The 8th roots of unity equally spread around the unit circle at the angle multiples of $2\pi/8$.

By formulas attributed to Euler and de Moivre, the set of nth roots of unity can be described by

$$\{\zeta_n^k = \cos(2\pi k / n) + i\sin(2\pi k / n) : 0 \le k \le n-1\},$$

where $\zeta_n = e^{2\pi i/n}$.

Using the language of complex numbers and roots of unity, Iswar's conjecture can be restated as follows.

Iswar's Conjecture (alternate form). For any integers $n \ge 2$ and $1 \le k \le n$, one may find k distinct nth roots of unity whose sum is 0 if and only if both k and $n - k$ are expressible as linear combinations of prime factors of n with nonnegative integer coefficients.

I worked out some special cases of the conjecture from this perspective. For example, as previously noted, when $n = p$ is prime, the conjecture says that only $k = n$ should be balanced, and this follows from the fact that the pth *cyclotomic polynomial* is irreducible. But I couldn't see how to deal with the general case. However, I knew there were some papers in the literature about linear relations between roots of unity, such as Mann's theorem ("On linear relations between roots of unity," H. B. Mann, *Mathematika* **12**:2 [1965]), so I turned to Google. Lo and behold, I found that Iswar's conjecture had been proved by Gary Sivek in a 2010 paper ("On Vanishing Sums of Distinct Roots of Unity," *Integers* **10**:A31). I don't think I would have found the Sivek reference if I hadn't first translated the question into a problem about linear relations between roots of unity.

Half the Proof

The proof of the "only if"" direction of the theorem uses a result from Lam and Leung ("On Vanishing Sums of Roots of Unity," *J. Algebra*, **224** [2000]); that proof utilizes techniques that are beyond the scope of this article. We can, however, outline a proof of the "if" direction, formally stated as follows.

Theorem. If both k and $n - k$ are expressible as linear combinations of prime factors of n with nonnegative integer coefficients, then n is k-balancing, that is, there are k distinct nth roots of unity whose sum is 0.

To begin with, we will make use of a well-known theorem due to J.J. Sylvester: If the greatest common divisor of x and y is 1 (denoted by $\gcd(x,y) = 1$) and $m \ge (x-1)(y-1)$, then m can be written as a linear combination of x and y with nonnegative integer coefficients.

Next, we note that multiplying a set of roots of unity by ζ_n^b rotates the set by $2\pi b / n$ radians. If p is a prime dividing both k and n, then the pth roots of unity form a subset of the nth roots of unity, and we can obtain k / p pairwise disjoint rotations of this set by multiplying by ζ_n^b for each $0 \le b < k / p$. Each rotated set sums to zero, and so we conclude that

$$\sum_{\substack{0\le c<p\\ 0\le b<k/p}} \zeta_p^c \zeta_n^b = \sum_{0\le b<k/p} \zeta_n^b \sum_{0\le c<p} \zeta_p^c = 0.$$

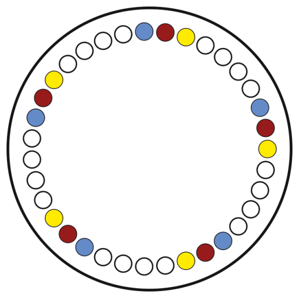


Figure 6. The 5th roots of unity (yellow) sitting inside the 35th roots of unity can be rotated up to 7 times without overlapping. Three rotations give a balanced 35-hole centrifuge with 15 test tubes.

Figure 6 illustrates this for $n = 35$ and $k = 3\cdot 5$. We restate this result as a lemma.

Lemma 1. If $\gcd(k, n) > 1$, then n is k-balancing.

As is common with number-theoretic proofs, we have several cases depending on the prime factorization of n. If $n = p^e$, then the theorem holds because $k = ap$ for some integer $a \ge 1$ and thus $\gcd(k,n) \ge p$.

Suppose $n = pq$ is a product of two distinct primes. If $k = ap + bq$ and $n - k = cp + dq$, then $pq = n = (a+c)p + (b+d)q$. Therefore, p must divide $b + d$ and q must divide $a + c$. If both $a + c$ and $b + d$ are positive, we get a contradiction by observing that $pq = xqp + ypq \ge 2pq$. Consequently either $a + c$ or $b + d$ is zero, and because all of these are nonnegative integers, we conclude that k is either a multiple of p or a multiple of q, forcing $\gcd(n,k) > 1$.

For our third case, assume that $n = pqm$ where p and q are the smallest distinct primes dividing n, $p < q$, and $m \ge p$. If $k = n / 2$, then $\gcd(n,k) = k > 1$. Otherwise, either $k > n / 2$ or $n - k > n / 2$. Because $n / 2 > (p-1)(q-1)$, we know that either k or $n - k$ is a nonnegative linear combination of p and q by Sylvester's theorem. So, without loss of generality, we can focus on values of k that are a sum of the prime factors of n satisfying $k < n / 2$.

We can further investigate two subcases: either $k < (p-1)(q-1)$ or $(p-1)(q-1) \leq k < n/2$.

Consider the case when $k < (p-1)(q-1)$. Let $\{p_1 = p, p_2 = q, p_3, \ldots, p_r\}$ be the set of prime divisors of n. Now, suppose that n is k-balanced where $k = \sum_{i=1}^{r} a_i p_i$ with $k < (p-1)(q-1)$ and suppose that there is a prime p_j that divides n satisfying $k + p_j < (p-1)(q-1)$. Because n is k-balanced, there is a set of nth roots of unity B with $|B| = k$ such that $\sum_{b \in B} \zeta_n^b = 0$. From $p_j \leq m$, we know that $kp_j < (p-1)(q-1)p_j < n$. If we consider all n possible rotations of the p_jth roots of unity, each such root will overlap each of the k roots in B once. Thus, by the pigeonhole principle, the total number of overlaps upon n rotations would be $p_j \cdot k$, which is less than n. It follows that some rotation must have no overlaps. This implies that there is an nth root of unity ζ_n^ℓ such that

$$\sum_{b \in B} \zeta_n^b + \sum_{0 \leq a \leq p_j - 1} \zeta_n^\ell \cdot \zeta_{p_j}^a = 0.$$

This technique allows us to prove, by induction on the sum $a_1 + a_2 + \cdots + a_r$, that n is k-balancing for each $k < (p-1)(q-1)$ that is a nonnegative linear combination of the primes dividing n; the base case follows from Lemma 1.

Still in the case with $n = pqm$, suppose that $(p-1)(q-1) \leq k < n/2$. By Sylvester's theorem, we know that $k = ap + bq$ where both a and b are nonnegative integers. We can assume $b < p$, and it is straightforward to prove that $a \leq qm - q$. Define

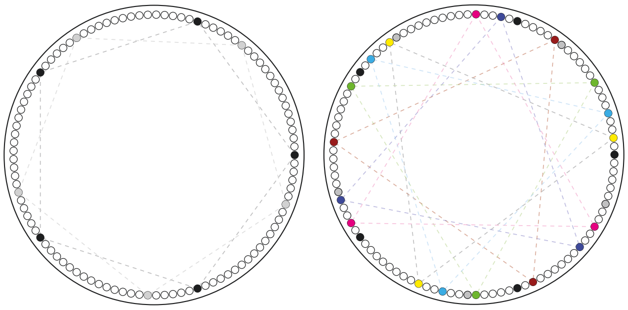
$$S = \{\zeta_{pq}^j \cdot \zeta_q^\ell : 0 \leq j \leq b-1,\ 0 \leq \ell \leq q-1\}.$$

In other words, S is a collection of b rotations of the qth roots of unity by a pqth root of unity. We again use the pigeonhole principle to deduce that there are at least $qm - q$ distinct rotations of the pth roots of unity of the form

$$T = \{\zeta \cdot \zeta_p^k : 0 \leq k \leq p-1\},$$

where ζ is an nth root of unity, such that $S \cap T = \varnothing$. Choosing a different sets of this form, $T_1, \ldots, T_a$, allows us to build the set $T_1 \cup \ldots \cup T_a \cup S$ consisting of k distinct nth roots of unity whose sum is 0. Figure 7 depicts an example of this process for $n = 3 \cdot 5 \cdot 7$ and $k = 28 = 6 \cdot 3 + 2 \cdot 5$.

Figure 7. The set S on the left consists of two copies of the 5th roots of unity; one is rotated by a 15th root of unity. The sets $T_1, T_2, \ldots, T_6$ are pictured in color on the right; they are all rotations of the set of 3rd roots of unity by a 105th root of unity. The right image shows a 28-balanced 105-centrifuge.



Denouement

I emailed Iswar telling him that the conjecture he had made back in 1998 was in fact correct. He replied:

"*I am both happy and sad to get this email. On the one hand I am happy that I 'guessed' the solution correctly (15 years ago). I am sad that I never acquired the mathematical skills to prove it myself! It is funny how our minds get stuck on some problems over multiple decades. Thanks for taking our conversation seriously.*"

Although the problem is ultimately just a stimulating curiosity, and nothing that is going to cure cancer, I still think of this episode as a nice illustration of the intellectual cross-fertilization that can take place when scientists from different disciplines get together, whether at a research conference or a barbecue.

***Matthew H. Baker** is a professor of mathematics and the Associate Dean for Faculty Development in the Georgia Tech College of Sciences. He does research in number theory, algebraic geometry, and combinatorics and is a Fellow of the American Mathematical Society. He also performs and creates original magic tricks. This article was adapted from an essay that appeared on his blog* (https://bit.ly/3QtrD32). *You can learn more from the Numberphile video based on the blog post* (youtu.be/7DHE8RnsCQ8).

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