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Decomposing 4-manifolds with positive scalar curvature

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ABSTRACT

We show that every closed, oriented, topologically PSC 4-manifold can be obtained via 0 and 1-surgeries from a topologically PSC 4-orbifold with vanishing first Betti number and second Betti number at most as large as the original one.

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1. Introduction

Our starting point is the following well-known theorem of Schoen–Yau and Gromov–Lawson, which exhibits the richness of the class of manifolds that can carry Riemannian metrics with positive scalar curvature. For convenience and brevity, we will adopt the convention of writing “PSC” for “positive scalar curvature,” and will call a manifold “topologically PSC” if it can carry Riemannian metrics with positive scalar curvature.

Theorem 1.1 *([17,7]). Let M be a topologically PSC n-manifold with $n \geq 3$. Any manifold obtained from M by performing a sequence of 0-, 1-, ..., and/or $(n-3)$-surgeries is also topologically PSC.*

Theorem 1.1 naturally leads one to ask:

Question 1.2. Can all topologically PSC n-manifolds be built out of “simple” topologically PSC n-manifolds by performing codimension ≥ 3 surgeries?

Our understanding of 3-manifold topology implies a positive answer when $n = 3$:

Theorem 1.3 *(Schoen–Yau and Gromov–Lawson and Perelman). Every closed, oriented, topologically PSC 3-manifold can be obtained by performing 0-surgeries on a disjoint union of spherical space forms (i.e., $\mathbf{S}^3/\Gamma$'s, where the Γ's are finite subgroups of $SO(4)$ acting freely on $\mathbf{S}^3$).*

In this paper we prove:

Theorem 1.4. *Every closed, oriented, topologically PSC 4-manifold M can be obtained from a possibly disconnected, closed, oriented, topologically PSC 4-orbifold M' with isolated singularities such that $b_1(M') = 0$ and $b_2(M') \leq b_2(M)$ by performing 0- and 1-surgeries. All 1-surgeries are standard manifold ones, but 0-surgeries may occur at orbifold points.*

Recall that the jth Betti number $b_j(M')$ of the orbifold M' is defined to be the jth Betti number of M' viewed as a topological space; in our case, this is equivalent to the jth Betti number of the regular part $M'_{\mathrm{reg}} \subset M'$. In the connected case, $b_1(M')$ is the same as the rank of the abelianization of the orbifold fundamental group $\pi_1^{\mathrm{orb}}(M')$. Finally, a 0-surgery occurring at two orbifold points both modeled on $\mathbf{R}^4/\Gamma$ means that the corresponding connected sum operation is performed with a $\mathbf{S}^3/\Gamma$ neck.

Remark 1.5. Theorem 1.4 will continue to hold if M is itself a 4-orbifold rather than a 4-manifold, but this is somewhat outside the scope of our current paper.

Note that orbifolds are indeed sometimes necessary for surgery decompositions such as that of Theorem 1.4 in dimension 4.

Theorem 1.6. *Let $\mathbf{S}^3/\Gamma$ be a lens space with Γ a nontrivial finite cyclic subgroup of $SO(4)$. The 4-manifold $M = (\mathbf{S}^3/\Gamma) \times \mathbf{S}^1$ cannot be obtained by performing manifold 0 and 1-surgeries on a 4-manifold M' with $b_1(M') = 0$.*

Let us outline the proof of Theorem 1.4. Endow M with an arbitrary PSC metric. We "exhaust" the codimension-1 homology of M with a two-sided, stable minimal hypersurface Σ (Lemma 2.5). By a now-standard argument of Schoen–Yau, the metric induced on Σ is conformal to a PSC metric. Thus, Σ is topologically the result of 0-surgeries on spherical space forms (Theorem 1.3). We then show how to locally modify the metric on M to another PSC metric that is locally a product near Σ and induces a "model" PSC metric on Σ (Lemma 4.1, Definition 4.2). If Σ is merely the disjoint union of spherical space forms $\mathbf{S}^3/\Gamma$, with no 0-surgeries, then our model metrics are all round and simple 3-surgeries on M along the components of Σ yield a 4-orbifold whose b_2 is unchanged and b_1 is trivial (because Σ suitably exhausted the codimension-1 homology of M). If Σ does involve 0-surgeries, we first undo these using 2-surgeries on M near Σ's 0-surgery neck regions; this may decrease b_2. We have only performed 3- and 2-surgeries on M to get to the orbifold, so M can be obtained from the orbifold via 0- and 1-surgeries, respectively. Note that our construction preserves the spin condition; see Remark 4.4.

We conclude our introduction by posing the following:

Question 1.7. Let M be a closed, oriented, topologically PSC 4-manifold. Can one obtain M from a closed, oriented, topologically PSC 4-orbifold M' with isolated singularities and the property that each component has finite orbifold fundamental group $\pi_1^{\text{orb}}(M')$ by performing 0- and 1-surgeries?

Organization of the paper. We first illustrate our homological decomposition argument for ambient 3-manifolds in Section 2, where the picture is much simpler. Section 3 is a brief digression offering a homotopical refinement of the 3-dimensional decomposition. In Section 4, we give the proof of our main theorem. In Section 5 we give the proof of Theorem 1.6. Finally, our appendix contains the proofs of some technical lemmas.

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2. Warmup: homology decomposition in 3D

We first illustrate our homological decomposition argument on 3D manifolds, where it yields the following weaker analog of Theorem 1.3. (See, however, Theorem 3.1 and its subsequent discussion.)

Theorem 2.1. *Every closed, oriented, topologically PSC* 3*-manifold* M *can be obtained from a closed, oriented, topologically PSC* 3*-manifold* M' *with* $b_1(M') = 0$ *by performing* 0*-surgeries.*

Remark 2.2 *(The idea in 3D).* We want to capture $H_2(M; \mathbf{Z})$ by 2-spheres on which we can perform 2-surgeries within the PSC category. Of course, 2-surgeries in a 3-manifold do not a priori preserve the PSC condition (they are not of codimension ≥ 3), but we will see that 2-surgeries performed on stable minimal spheres do. Indeed, we show how to reduce a 2-surgery on any stable minimal sphere to the surgically trivial case of 2-surgery on a round, totally geodesic sphere. There, the PSC condition is obviously preserved under the surgery.

Lemma 2.3 is the preparation lemma underlying our PSC 2-surgeries. We defer its proof to Appendix A.

Lemma 2.3 *(Metric preparation lemma). Let* Σ *be a closed, embedded, two-sided, stable minimal surface in an oriented, PSC* 3*-manifold* (M, g)*. Then:*

(a) Each component of Σ *is an* $\mathbf{S}^2$*.*
(b) Given any auxiliary PSC metric ϱ *on* Σ*, there exists a new PSC metric* $\tilde{g}$ *on* M*, which:*

- *is isometric to a product cylinder* $(\Sigma, \varrho) \times (-2, 2)$ *in the distance-*2 *tubular neighborhood of* Σ*, and*
- *coincides with* g *outside a larger tubular neighborhood of* Σ*.*

Remark 2.4. If Σ has one-sided components, then the same is true for them, except they are $\mathbf{RP}^2$'s rather than $\mathbf{S}^2$'s, and around them each product cylinder $(\mathbf{S}^2, \varrho) \times (-2, 2)$ is to be taken modulo the $\mathbf{Z}_2$ action $(x, t) \mapsto (-x, -t)$; here, the auxiliary metric ϱ must be a lift from a metric on $\mathbf{RP}^2$ to a stable two-sided $\mathbf{S}^2$ covering.

The lemma below delivers the surface Σ on which Lemma 2.3 is applied. We state it in n-dimensional generality, because it is indeed very general, and we will it need again in Section 4. We defer its proof to Appendix A, and note that it can be viewed as a generalization of the "slicing" procedure in [3, Lemma 19].

Lemma 2.5 *(Minimal (hyper)surface preparation lemma). Let* (M, g) *be a closed, connected, oriented* n*-manifold, with* $n \leq 7$*. There exists a closed, embedded, two-sided, stable minimal hypersurface* $\Sigma \subset M$ *such that* $\breve{M} = M \setminus \Sigma$ *is connected and the map* $H_{n-1}(\partial \breve{M}; \mathbf{Z}) \twoheadrightarrow H_{n-1}(\breve{M}; \mathbf{Z})$ *is surjective.*

Proof of Theorem 2.1. Without loss of generality, M is connected. Endow M with a PSC metric g. Let Σ be as in Lemma 2.5, with components $\{\Sigma_i\}_{i=1}^k$. By Lemma 2.3 and the two-sidedness of each component Σ_i, and we can modify the metric g near Σ_i and arrange for a new PSC metric $\tilde{g}$ on M inducing a product metric isometric to $(\mathbf{S}^2, \varrho_{\mathbf{S}^2}) \times (-2, 2)$ on the distance-2 tubular neighborhood of Σ_i, where $\varrho_{\mathbf{S}^2}$ is the standard round metric.

Now excise the distance-1 tubular neighborhood U_i of each Σ_i and cap off the two newly formed boundary components with a pair of PSC 3-balls $\{(B_i^\alpha, \eta_i^\alpha)\}_{\alpha=1}^2$ whose metric has been suitably deformed near the boundary to match smoothly with a product cylinder over $(\mathbf{S}^2, \varrho_{\mathbf{S}^2})$. Then,

$$(M', g') := (M \setminus \cup_{i=1}^k U_i, \tilde{g}) \cup \big(\cup_{i=1}^k \cup_{\alpha=1,2} (B_i^\alpha, \eta_i^\alpha)\big)$$

with the obvious boundary identifications is a PSC 3-manifold obtained from M by performing a 2-surgery for each Σ_i. Thus, M can be obtained from M' by performing 0-surgeries.

It remains to verify $b_1(M') = 0$. Denote

$$\breve{M} = M \setminus \cup_{i=1}^k \Sigma_i.$$

Tracking the homology exact sequence

$$H_2(\partial \breve{M}; \mathbf{Z}) \xrightarrow{i_*} H_2(\breve{M}; \mathbf{Z}) \xrightarrow{j_*} H_2(\breve{M}, \partial \breve{M}; \mathbf{Z}) \xrightarrow{\partial} H_1(\partial \breve{M}; \mathbf{Z}),$$

we find $\ker j_* = \operatorname{img} i_* = H_2(\breve{M}; \mathbf{Z})$ (due to Lemma 2.5), so $j_* = 0$, so $H_2(\breve{M}, \partial \breve{M}; \mathbf{Z})$ injects into $H_1(\partial \breve{M}; \mathbf{Z})$. By Lemma 2.3, $\partial \breve{M}$ consists of 2-spheres, so $H_1(\partial \breve{M}; \mathbf{Z}) = 0$ and thus $H_2(\breve{M}, \partial \breve{M}; \mathbf{Z}) = 0$ by the exact sequence. By Lefschetz duality, $H^1(\breve{M}; \mathbf{Z}) = 0$. The Mayer–Vietoris sequence then implies $H^1(M'; \mathbf{Z}) = 0$, and thus $b_1(M') = 0$ by the universal coefficient theorem. □

3. A digression: homotopy decomposition in 3D

This section can be skipped at first reading. We digress to present a homotopy-based refinement of Theorem 2.1, show how it can directly imply a certain homotopy version of Theorem 1.3, and discuss why we find this of interest.

Theorem 3.1. *Every closed, oriented, topologically PSC 3-manifold can be obtained from a closed, oriented, topologically PSC 3-manifold with vanishing π_2 (on each connected component) by performing 0-surgeries.*

Instead of representing $H_2(M; \mathbf{Z})$ by stable minimal 2-spheres as in Section 2, we seek to represent $\pi_2(M)$. The key ingredient is the Meeks–Yau proof of the embedded sphere theorem using minimal surfaces, which will be used in lieu of Lemma 2.5.

Lemma 3.2 *([15, Theorem 7] as Minimal sphere preparation lemma). Let (M, g) be a closed, connected 3-manifold. There exist conformal maps $\{f_i : \mathbf{S}^2 \to M\}_{i=1}^k$ with pairwise disjoint images that generate $\pi_2(M)$ as a $\pi_1(M)$-module. Each f_i has least area in* *its homotopy class and is either a conformal embedding of an* $\mathbf{S}^2$ *or a* $2:1$ *conformal covering of an embedded* $\mathbf{RP}^2$*.*

Of course, the embeddings or immersions above are stable minimal spheres. (The statement of [15, Theorem 7] doesn't explicitly mention the pairwise disjointness, but see the comment in the proof of [15, p. 480, Assertion 1].)

Proof of Theorem 3.1. Without loss of generality, our initial manifold M is connected. Endow M with a PSC metric g. Let $\{f_i : \mathbf{S}^2 \to M\}_{i=1}^k$ be the maps given by Lemma 3.2. The proof is the same as that of Theorem 2.1, except for the fact that some images $\Sigma_i := f_i(\mathbf{S}^2)$ might be doubly covered embedded one-sided $\mathbf{RP}^2$'s. We apply Lemma 2.3 (see Remark 2.4) and modify g locally near each Σ_i, arranging for a new PSC metric $\tilde{g}$ that, in the distance-2 tubular neighborhood of Σ_i induces product metrics isometric to $(\mathbf{S}^2, \varrho_{\mathbf{S}^2}) \times (-2, 2)$ near Σ_i, where $\varrho_{\mathbf{S}^2}$ is round, and all is taken modulo the $\mathbf{Z}_2$ action $(x, t) \mapsto (-x, -t)$ when f_i is a double covering of Σ_i.

We excise $(M, \tilde{g})$ as before, except for all i corresponding with $\Sigma_i \approx \mathbf{RP}^2$, we only use a single PSC 3-ball and include an $\mathbf{RP}^3$ in the new manifold M'. This M' can be obtained from M by 2-surgeries, and thus M can be obtained from M' by 0-surgeries.

It remains to verify that $\pi_2(M') = 0$. Without loss of generality, we may assume M to be connected. Denote by $\pi : \tilde{M} \to M$ its universal covering and let $\tilde{\Sigma} := \pi^{-1}(\cup_{i=1}^k \Sigma_i)$. The boundary components of $\tilde{M} \setminus \tilde{\Sigma}$ are all 2-spheres.

Claim 1. *All components of* $\tilde{M} \setminus \tilde{\Sigma}$ *are simply connected.*

Proof. Let $\mathcal{E}_1$ be the closure of any component of $\tilde{M} \setminus \tilde{\Sigma}$. We inductively construct a monotone increasing exhaustion $\{\mathcal{E}_k\}_{k=1,2,\ldots}$ of $\tilde{M}$ by taking $\mathcal{E}_{k+1} := \mathcal{E}_k \cup \mathcal{C}_k$ where $\mathcal{C}_k$ is the closure of any component of $\tilde{M} \setminus \tilde{\Sigma}$ that shares a boundary component with $\mathcal{E}_k$. At each step $\mathcal{E}_k$ is connected by construction. Moreover, $\partial\mathcal{E}_k \cap \partial\mathcal{C}_k$ has at most one component (otherwise one could construct a closed loop intersecting a component of $\partial\mathcal{E}_k \cap \partial\mathcal{C}_k$ in exactly one point, in contradiction to $\pi_1(\tilde{M}) = 0$). As a result, by the Seifert–Van Kampen theorem and the fact that all boundary components are 2-spheres, the map $\pi_1(\mathcal{E}_i) \to \pi_1(\mathcal{E}_{i+1})$ induced by the inclusion is injective. But $\pi_1(\tilde{M}) = 0$ is the direct limit of $\pi_1(\mathcal{E}_i)$. Therefore, $\pi_1(\mathcal{E}_1) = 0$. □

Denote by $\tilde{M}'$ the manifold obtained by gluing one 3-ball per boundary 2-sphere of $\tilde{M} \setminus \tilde{\Sigma}$ and one 3-sphere per component of $\pi^{-1}(\Sigma_i)$ for any $\Sigma_i \approx \mathbf{RP}^2$. By construction:

$$\tilde{M}' \text{ covers } M'. \tag{3.1}$$

By Seifert–Van Kampen and Claim 1:

$$\pi_1(\tilde{M}') = 0. \tag{3.2}$$

Meanwhile, the argument in the proof of Theorem 2.1 gives:

$$H_2(\tilde{M}'; \mathbf{Z}) = 0. \tag{3.3}$$

At this point, it follows that

$$\pi_2(M') \cong \pi_2(\tilde{M}') = 0;$$

the isomorphism follows from (3.1), and the vanishing follows from (3.2), (3.3), and the Hurewicz theorem. This completes the proof. □

Theorem 3.1 implies a certain homotopy version of Theorem 1.3:

Corollary 3.3. *Every closed, oriented, topologically PSC* 3*-manifold can be obtained from a closed, oriented, topologically PSC* 3*-manifold with components covered by homotopy* 3*-spheres, by performing* 0*-surgeries.*

Proof. Let M' be the manifold obtained from Theorem 3.1. Without loss of generality, M' is connected. It suffices to show that $\pi_1(M')$ is finite. Suppose that $\pi_1(M')$ were infinite. Then the universal covering $\tilde{M}'$ of M' would be noncompact, so $H_3(\tilde{M}'; \mathbf{Z}) = 0$. On the other hand, $\pi_2(\tilde{M}') \cong \pi_2(M') = 0$. Then, $\pi_3(M') \cong \pi_3(\tilde{M}') \cong H_3(\tilde{M}'; \mathbf{Z}) = 0$, the second isomorphism being Hurewicz's. This strategy iterates to give $\pi_k(M') = 0$ for all $k \geq 2$, and thus M' is aspherical. This violates the non-asphericity result for 3D PSC: by Schoen–Yau and Gromov–Lawson (see [8, Theorem E], [18]), closed aspherical 3-manifolds are not topologically PSC. □

It is interesting to compare Corollary 3.3 with the Schoen–Yau and Gromov–Lawson approach to obtaining a homotopy version of Theorem 1.3:

(a) By the Kneser–Milnor prime decomposition of 3-manifolds [13], and further decomposing all prime $\mathbf{S}^2 \times \mathbf{S}^1$'s as 0-surgeries on $\mathbf{S}^3$'s, we see that any closed, oriented 3-manifold M can be obtained from:

- aspherical 3-manifolds, and/or
- manifolds covered by homotopy 3-spheres

by performing only 0-surgeries.

(b) One knows [8, Theorem E] (see also [18]) that aspherical 3-manifolds *cannot* occur as prime summands of PSC 3-manifolds. This rules out the first source of summands in (a). Thus, any closed oriented 3-manifold M can be obtained from a closed, oriented 3-manifold with components covered by homotopy 3-spheres, by performing 0-surgeries.

In this approach, the topological decomposition in step (a) prevents step (b) from deciding whether the decomposed pieces, which are all covered by homotopy 3-spheres, are themselves topologically PSC. (Of course, this follows from Perelman's resolution of the elliptization conjecture.) In our approach, step (a) was Theorem 3.1, a geometric PSC surgery result that allows us to directly guarantee that the decomposed pieces are, too, topologically PSC. Step (b) is essentially unchanged for us.

A note about higher dimensions. Step (b) of the program above was recently carried out for 4-(and 5-)manifolds by the second author together with Otis Chodosh [3] (see also [9,4]). Step (a) remains a challenge. There are no suitable topological decompositions to perfectly replace step (a) in 4D, and we instead also proceed with a homological decomposition obtained via geometric measure theory.

4. Main theorem: homology decomposition in 4D

We now extend the strategy of Section 2 to 4D. We capture $H_3(M; \mathbf{Z})$ by a stable minimal hypersurface Σ, using Lemma 2.5. In the current higher dimensional setting we need to use Lemma 4.1 (instead of the two-dimensional Lemma 2.3) to reduce to the product case. We state it below but defer its proof to Appendix A.

Lemma 4.1. *Let Σ be a two-sided, closed, embedded, stable, minimal hypersurface inside an oriented PSC 4-manifold (M, g). Then:*

(a) Σ must be topologically PSC and thus as in Theorem 1.3.
(b) Given any auxiliary PSC metric σ on Σ, there exists a new PSC metric $\tilde{g}$ on M, which:

- *is isometric to a product cylinder $(\Sigma, \sigma) \times (-2, 2)$ in the distance-2 tubular neighborhood of Σ, and*
- *coincides with g outside a larger tubular neighborhood of Σ.*

We will apply this lemma with various choices of σ, one being:

Definition 4.2 *(Model metric). Let Σ be obtained by performing 0-surgeries on spherical space forms. A PSC metric on Σ is called a model metric if the neck corresponding to each 0-surgery contains an isometric copy of $(\mathbf{S}^2, \varrho) \times (-2, 2)$ for some size round metric ϱ on $\mathbf{S}^2$.*

Proof of Theorem 1.4. Without loss of generality, M is connected. Fix a PSC metric g on M, let Σ be as in Lemma 2.5. We will stray slightly from the notation in the statement of Theorem 1.4 below: M' will not denote the ultimate decomposition, but only an intermediate one, and the ultimate decomposition will be denoted M''.

Step I. In this step, we assume that Σ involves at least one 0-surgery, otherwise we proceed to **Step II** by setting $(M', g') := (M, g)$ and $\Sigma' := \Sigma$.

By Lemma 4.1 we can modify the metric g of M locally near Σ and arrange for a new PSC metric $\tilde{g}$ on M that induces a product metric $(\Sigma, \sigma) \times (-2, 2)$ in the distance-2 tubular neighborhood of Σ, where σ is a model metric (Definition 4.2).

Now consider the neck regions $\{N_i\}_{i=1}^{\ell}$ of Σ where, due to the model metric structure, (Σ, σ) restricts on N_i to an isometric copy of $(\mathbf{S}^2, \varrho_i)$, where ϱ_i is a round metric on $\mathbf{S}^2$ of radius ε_i. By construction of $\tilde{g}$, the distance-2 tubular neighborhood of N_i in $(M, \tilde{g})$ is isometric to $(\mathbf{S}^2, \varrho_i) \times (-2, 2) \times (-2, 2)$. Let U_i be the interior of a smoothing of the domain $\mathbf{S}^2 \times [-1, 1] \times [-1, 1]$ in these coordinates, where the smoothing only takes place outside $\mathbf{S}^2 \times [-\frac{1}{2}, \frac{1}{2}] \times [-1, 1]$ and $\mathbf{S}^2 \times [-1, 1] \times [-\frac{1}{2}, \frac{1}{2}]$. Note that $U_i \approx \mathbf{S}^2 \times B^2$. For small ε_i, we can construct a PSC metric η_i on $V_i := B^3 \times \mathbf{S}^1$ which matches smoothly with U_i on their respective boundaries ($\approx \mathbf{S}^2 \times \mathbf{S}^1$). Then,

$$(M', g') := (M \setminus \cup_{i=1}^{\ell} U_i, \tilde{g}) \cup \big(\cup_{i=1}^{\ell} (V_i, \eta_i) \big)$$

with the obvious boundary identifications, is a PSC 4-manifold obtained from M by 2-surgeries. Thus, M can be obtained from M' by 1-surgeries. The surgeries above can be performed so that Σ gets replaced by a hypersurface Σ', whose components are spherical space forms. Moreover, by choosing (V_i, η_i) to be a local product on the $\mathbf{S}^1$ factor, we may assume that the metric g' is locally a product near Σ' and ensure that Σ' is again stable.

Note that the components of $M' \setminus \Sigma'$ arise from the components of $M \setminus \Sigma$ by attaching copies of $B^3 \times [-1, 1]$ along $\mathbf{S}^2 \times [-1, 1]$ to its boundary components. This shows that the surjectivity of the natural map $H_3(\partial(M' \setminus \Sigma'); \mathbf{Z}) \twoheadrightarrow H_3(M' \setminus \Sigma'; \mathbf{Z})$, initially guaranteed in Lemma 2.5 for $H_3(\partial(M \setminus \Sigma); \mathbf{Z}) \twoheadrightarrow H_3(M \setminus \Sigma; \mathbf{Z})$, is maintained.

Finally, since we're performing 2-surgeries, we have $b_2(M') \leq b_2(M)$. (See Lemma A.2, where the roles of M and M' are reversed.)

Step II. In this step we work with (M', g') and the components $\{\Sigma'_i\}_{i=1}^{k'}$ of Σ', each of which is, by construction, a spherical space form, i.e., $\Sigma'_i \approx \mathbf{S}^3/\Gamma_i$, for finite subgroups Γ_i of $SO(4)$ acting freely on $\mathbf{S}^3$.

We invoke Lemma 4.1 again, except we modify the metric g' of M' locally near each Σ'_i and arrange for a new PSC metric $\tilde{g}'$ that induces a product metric $(\Sigma'_i, \sigma'_i) \times (-2, 2)$ in the distance-2 tubular neighborhood of Σ'_i, where σ'_i is a round metric on Σ'_i. For convenience, we denote by $A \subseteq \{1, 2, \ldots, k'\}$ the set of i's for which $\Sigma'_i \approx \mathbf{S}^3$, and by $B \subseteq \{1, 2, \ldots, k'\}$ the remaining i's.

For each $i \in A$, we can excise the distance-1 tubular neighborhood U'_i of Σ'_i and smoothly replace it with two PSC 4-balls $\{(B_i^\alpha, \theta_i^\alpha)\}_{\alpha=1}^2$. Note that this is a 3-surgery, and is therefore undone with a 0-surgery.

For each $i \in B$, we can still excise the distance-1 tubular neighborhood U'_i of Σ'_i, but now we have to smoothly glue in two PSC 4-orbifold-balls $\{(\hat{B}_i^\alpha, \hat{\theta}_i^\alpha)\}_{\alpha=1}^2$ whose orbifold singularity is modeled on $\mathbf{R}^4/\Gamma_i$. Note that this is an orbifold 3-surgery, and is therefore undone by a 0-surgery at orbifold points.

The ultimate space we end up with is

$$(M'', g'') := (M' \setminus \cup_{i=1}^k U_i', \tilde{g}') \cup \left(\cup_{i\in A} \cup_{\alpha=1,2} (B_i^\alpha, \theta_i^\alpha) \right) \cup \left(\cup_{i\in B} \cup_{\alpha=1,2} (\hat{B}_i^\alpha, \hat{\theta}_i^\alpha) \right).$$

As explained, M can be obtained from M'' by performing 0-surgeries (possibly on the orbifold points) and then 1-surgeries (on the smooth part).

We now prove $b_1(M'') = 0$. Set $\breve{M} := M' \setminus \Sigma'$. As in the proof of Theorem 2.1, we track the homology exact sequence

$$H_3(\partial\breve{M}; \mathbf{Z}) \xrightarrow{i_*} H_3(\breve{M}; \mathbf{Z}) \xrightarrow{j_*} H_3(\breve{M}, \partial\breve{M}; \mathbf{Z}) \xrightarrow{\partial} H_2(\partial\breve{M}; \mathbf{Z}).$$

We have $H_2(\partial\breve{M}; \mathbf{Z}) = 0$ since the components Σ_i' of $\partial\breve{M}$ are spherical space forms. As before, i_* is guaranteed to be surjective, so $H_3(\breve{M}, \partial\breve{M}; \mathbf{Z}) = 0$, so $H^1(\breve{M}; \mathbf{Z}) = 0$ by Lefschetz duality, and $b_1(M'') = 0$ from Mayer–Vietoris.

Finally, $b_2(M'') = b_2(M') \leq b_2(M)$. The inequality follows from **Step I** and the equality from the 3-surgeries on M' to get M''. The latter can be verified applying Mayer–Vietoris twice to get $b_2(M'') = b_2(\breve{M}) = b_2(M')$: once on the open cover of M' by $\breve{M}$ and $\cup_{i=1}^k U_i'$, and once on the open cover of M'' by $\breve{M}$ and $(\cup_{i,\alpha} B_i^\alpha) \cup (\cup_{i,\alpha} \hat{B}_i^\alpha)$. □

Remark 4.3. The orbifold singularities in the decomposition are due to the spherical space forms appearing in Σ' in **Step II** above.

Remark 4.4. Our proof of Theorem 1.4 preserves the spin condition; i.e., if M is spin, then the ultimate orbifold M' is spin too, in the sense that its regular part M''_{reg} is spin. In Step I we can endow each $B^3 \times \mathbf{S}^1$ we are gluing in to replace U_i with a spin structure to induce the same spin structure on $\mathbf{S}^2 \times \mathbf{S}^1$ that ∂U_i did; one can see this, e.g., from that $\pi_1(\mathbf{S}^2 \times \mathbf{S}^1) \to \pi_1(B^3 \times \mathbf{S}^1)$ is an isomorphism and $\pi_2(B^3 \times \mathbf{S}^1) = 0$. Likewise, in Step II we can similarly endow each B_i^α or $\hat{B}_i^\alpha$ with a spin structure to induce the same spin structure on each $\mathbf{S}^3/\Gamma_i$ that Σ_i' did from either side.

5. Proof of Theorem 1.6

Suppose, for the sake of contradiction, that $M = (\mathbf{S}^3/\Gamma) \times \mathbf{S}^1$ can be obtained by manifold 0 and 1-surgeries from a 4-manifold M' with $b_1(M') = 0$. The key features of M are that

$$\pi_1(M) = \Gamma \times \mathbf{Z} \text{ with } \Gamma \leq SO(4) \text{ finite, cyclic, nontrivial,} \tag{5.1}$$

$$b_1(M) = 1,\ b_2(M) = 0. \tag{5.2}$$

Since $1 < 4/2$, all the 0 and 1-surgeries commute. So first take the connected sum of all connected components of M', and call it M''. Note that $b_1(M'') = 0$. Perform all remaining 0-surgeries, so that M'' turns into $M''\#k(\mathbf{S}^3 \times \mathbf{S}^1)$ for some integer $k \geq 0$. Then, M is obtained by performing 1-surgeries on $M''\#k(\mathbf{S}^3 \times \mathbf{S}^1)$, which has

$$b_1(M''\#k(\mathbf{S}^3 \times \mathbf{S}^1)) = k.$$

Combining Lemma A.2 with (5.2), we see that the integer k above is such that $k \geq 1$ and that exactly $(k-1)$ 1-surgeries were performed.

On the level of fundamental groups, since M is obtained from $M''\#k(\mathbf{S}^3 \times \mathbf{S}^1)$ by performing $(k-1)$ 1-surgeries, (5.1) yields

$$\Gamma \times \mathbf{Z} \cong \left\langle G * \mathbf{Z}^{*k} \middle| r_1, \cdots, r_{k-1} \right\rangle,$$

where $G = \pi_1(M'')$, $\mathbf{Z}^{*k} = \mathbf{Z} * \cdots * \mathbf{Z}$ (k times) and $r_1, \ldots, r_{k-1}$ represent the relations (possibly trivial) introduced by the 1-surgeries. We are led to a contradiction from this presentation of $\Gamma \times \mathbf{Z}$ and that the abelianization of G has rank $b_1(M'') = 0$ via the following group theoretic lemma.

Lemma 5.1. *Suppose that Γ is a nontrivial finite cyclic group, and that G is a group whose abelianization has rank 0. Then $\Gamma \times \mathbf{Z}$ cannot be expressed as*

$$\Gamma \times \mathbf{Z} \cong \left\langle G * \mathbf{Z}^{*k} \middle| r_1, \cdots, r_{k-1} \right\rangle. \tag{5.3}$$

Proof. If N is the minimal normal subgroup of $G * \mathbf{Z}^{*k}$ containing $\langle r_1, \cdots, r_{k-1} \rangle$, then, by (5.3),

$$\Gamma \times \mathbf{Z} \cong {G * \mathbf{Z}^{*k}} \big/ {N}. \tag{5.4}$$

Consider the standard embedding $i : G \to G * \mathbf{Z}^{*k}$, and consider the normal subgroup

$$G' = i^{-1}(N) \trianglelefteq G.$$

We will reduce to the case $G' = \{1\}$. If N' denotes the minimal normal subgroup in $G * \mathbf{Z}^{*k}$ containing $i(G')$, then

$$G * \mathbf{Z}^{*k} \big/ N \cong \left(G * \mathbf{Z}^{*k} / N'\right) \big/ (N / N')$$

If we denote by $\tilde{r}_i$ the image of r_i under the map $G * \mathbf{Z}^{*k} \to (G / G') * \mathbf{Z}^{*k}$, and we denote by $\tilde{N}$ the minimal normal subgroup of $(G / G') * \mathbf{Z}^{*k}$ containing $\langle \tilde{r}_1, \cdots, \tilde{r}_{k-1} \rangle$, then the rightmost group above is

$$\cong (G / G') * \mathbf{Z}^{*k} \big/ \tilde{N}.$$

Also, under the standard embedding $\tilde{i} : G / G' \to (G / G') * \mathbf{Z}^{*k}$, we have

$$(\tilde{i})^{-1}(\tilde{N}) = \{1\}.$$

Thus, by replacing G with G / G' and r_i with $\tilde{r}_i$, we may assume that we had

$$i^{-1}(N) = \{1\},$$

all along as desired. Note that this reduction preserves the property that the rank of the abelianization equals zero. Then,

$$G \xrightarrow{i} G * \mathbf{Z}^{*k} \to {G * \mathbf{Z}^{*k}} \big/ {N} \cong \Gamma \times \mathbf{Z},$$

(the last isomorphism is (5.4)) is injective. Therefore G is abelian since $\Gamma \times \mathbf{Z}$ is. The rank of the abelianization of G, and thus of G, is assumed to be 0, so G is finite. Since $\mathbf{Z}$ has no torsion, G must inject into $\Gamma \times \{0\} \cong \Gamma$, which is assumed to be finite and cyclic, so

$$G \cong \mathbf{Z}_m, \quad m \geq 1.$$

Therefore,

$$\Gamma \times \mathbf{Z} \cong \left\langle \mathbf{Z}_m * \mathbf{Z}^{*k} \middle| r_1, \cdots, r_{k-1} \right\rangle \cong \left\langle \mathbf{Z} * \mathbf{Z}^{*k} \middle| r_0, r_1, \cdots, r_{k-1} \right\rangle, \tag{5.5}$$

where r_0 is a word that gives order m to the generator of the first $\mathbf{Z}$ factor. Thus, for every $n \geq 1$,

$$\Gamma \times \mathbf{Z}_n \cong \left\langle \mathbf{Z} * \mathbf{Z}^{*k} \middle| r_0, r_1, \cdots, r_{k-1}, r_k \right\rangle, \tag{5.6}$$

with r_k being a word that reduces to $(0, n)$ in (5.5).

We show that (5.6) must fail for *some* n by comparing the Schur Multiplier (denoted by $M(\cdot)$) of both sides. Indeed, it follows from (5.6) that $\Gamma \times \mathbf{Z}_n$ is a finitely represented group with the same number of generators and relations (a group of zero deficiency), so

$$M(\Gamma \times \mathbf{Z}_n) = \{1\}$$

by [5, Lemma 1.2]. On the other hand, if we choose n to be equal to the order of Γ, so $\Gamma \cong \mathbf{Z}_n$, then, invoking [10, Theorem 2.1] yields

$$M(\Gamma \times \mathbf{Z}_n) = M(\mathbf{Z}_n \times \mathbf{Z}_n) = M(\mathbf{Z}_n) \times M(\mathbf{Z}_n) \times (\mathbf{Z}_n \otimes \mathbf{Z}_n) = \mathbf{Z}_n.$$

This is a contradiction since $n \geq 2$. □

Appendix A. Technical lemmas

We first prove Lemmas 2.3, 4.1. Our proof relies on the flexibility of two-sided stable minimal hypersurfaces in 3- and 4-manifolds due to the second and third authors [11].[1] We need one additional piece of notation and an auxiliary lemma. For any closed connected manifold Σ, set

$$\mathscr{M}(\Sigma) := \{\sigma \in \operatorname{Met}(\Sigma) : -\Delta_\sigma + \tfrac{1}{2} R_\sigma \text{ has positive principal eigenvalue}\};$$

here, R_σ denotes the scalar curvature of σ. This space is denoted $\mathscr{M}^{>0}_{1/2}(\Sigma)$ in [11].

Lemma A.1. *Suppose $(\sigma_t)_{t\in[0,1]}$ is a smooth path in $\mathscr{M}(\Sigma)$ with $\sigma_0' \equiv \sigma_1' \equiv 0$. There exists a smooth $u : \Sigma \times [0,1] \to (0,\infty)$ so that the metric $h = \sigma_t + u^2 dt^2$ on $\Sigma \times [0,1]$ has the following properties:*

(a) $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ are totally geodesic;
(b) $R_h > 0$ everywhere.

Proof. We make use of the curvature formulas from [11, Lemma A.1]. Right away we note that (a) is a consequence of $\sigma_0' \equiv \sigma_1' \equiv 0$, no matter what u is. For (b), the key observation is that if $\tilde{u} : \Sigma \times [0,1] \to (0,\infty)$ is held fixed and $A > 0$ is a constant, both to be determined, then

$$R_{\sigma_t + A^2 \tilde{u}^2\, dt^2} = 2\tilde{u}^{-1}(\Delta_{\sigma_t} \tilde{u} + \tfrac{1}{2} R_{\sigma_t} \tilde{u}) + O(A^{-2}) \text{ as } A \to \infty. \tag{A.1}$$

Take each $\tilde{u}(\cdot, t) : \Sigma \to (0,\infty)$ to be a positive principal eigenfunction of $-\Delta_{\sigma_t} + \frac{1}{2} R_{\sigma_t}$. This can be done smoothly over $\Sigma \times [0,1]$ because the principal eigenvalue is simple; see [14, Lemma A.1]. With this $\tilde{u}$, the first term on the right hand side of (A.1) is bounded below by the minimum principal eigenvalue over t, thus uniformly positive since $\sigma_t \in \mathscr{M}(\Sigma)$ for all t. Now, (b) follows with $u := A\tilde{u}$ and $A \gg 1$. □

Proof of Lemma 2.3. Conclusion (a) is a well-known consequence of [16].

To arrange conclusion (b), we first cut (M, g) along Σ. This introduces two boundary components isometric to (Σ, σ_0). We will join them together using a PSC cylinder constructed using Lemma A.1, and its reflection.

To that end, note that $\sigma_0 \in \mathcal{M}(\Sigma)$ by [11, Lemma C.6]. The auxiliary metric σ on Σ also satisfies $\sigma \in \mathcal{M}(\Sigma)$ because it has positive scalar curvature. The existence of a smooth path $(\sigma_t)_{t\in[0,1]} \subset \mathscr{M}(\Sigma)$ with $\sigma_1 = \sigma$ is guaranteed by [14, Proposition 1.1];

[1] Another proof relies on the conformal cobordism theory of Akutagawa–Botvinnik [1] and the known path-connectedness of the space of PSC metrics on topologically PSC manifolds due to Weyl [19] in 2D and the first author and Bruce Kleiner [2] in 3D. We thank Demetre Kazaras for bringing this alternative and particularly [1] to our attention.

the condition $\sigma_0' \equiv \sigma_1' \equiv 0$ is trivially arranged by a time reparametrization. Then Lemma A.1 yields a PSC metric h on $\Sigma \times [0,1]$ that induces σ_t on $\Sigma \times \{t\}$ for all $t \in [0,1]$, with both boundary components totally geodesic. We then extend h to a metric on $\Sigma \times [0,4]$ by taking it to be a product metric $(\Sigma, \sigma) \times (1,4]$.

Along each boundary component of $(M,g) \setminus \Sigma$, we glue in a copy of $(\Sigma \times [0,4], h)$, identifying the boundary Σ with $\Sigma \times \{0\}$, and the two copies of $\Sigma \times \{4\}$ with each other, suitably reversing orientations. This yields a new manifold diffeomorphic to M, with a Lipschitz metric $\hat{g}$ that satisfies all desired conclusions of the lemma, except it is only smooth away from the various copies of $\Sigma \times \{0,1\}$. Since these hypersurfaces are minimal from both sides in $(M, \hat{g})$, they can be smoothed out locally while preserving PSC. One can do this using [12], which is easily localized using cut-off functions in view of the strict positivity of scalar curvature in our case; see [11, Lemma 7.1] for details. □

Proof of Lemma 4.1. The proof of Lemma 2.3 carries through verbatim, except conclusion (a) is due to [17] and one needs to invoke [11, Proposition 3.1] rather than [14, Proposition 1.1] when proving (b). □

Proof of Lemma 2.5. We offer a generalization of the method [3, Lemma 19].

We proceed by induction. Suppose that we have constructed a sequence of pairwise disjoint, two-sided, stable hyperminimal surfaces $\Sigma_1, \ldots, \Sigma_k \subset M$ such that $\breve{M}_k := M \setminus \cup_{j=1}^k \Sigma_j$ is connected. If the map

$$i_k : H_{n-1}(\partial \breve{M}_k; \mathbf{Z}) \to H_{n-1}(\breve{M}_k; \mathbf{Z})$$

is surjective, we are done; set $\Sigma := \cup_{j=1}^k \Sigma_j$ in the statement of the lemma. So let us assume it is not. Using geometric measure theory [6] (this is the source of the dimensional restriction $n \leq 7$) we can find a closed, connected, two-sided, stable minimal surface $\Sigma_{k+1} \subset \breve{M}_k$ such that $\breve{M}_{k+1} := \breve{M}_k \setminus \Sigma_{k+1}$ is connected and such that

$$[\Sigma_{k+1}] \notin \operatorname{img} i_k. \tag{A.2}$$

To see that this process terminates, we will show that the cokernels

$$\Gamma_k := H_{n-1}(\breve{M}_k; \mathbf{Z}) / \operatorname{img} i_k$$

have strictly decreasing rank. Since $\Gamma_0 = H_{n-1}(M; \mathbf{Z})$ is finitely generated, the process will have to terminate after finitely many steps.

So, it remains to prove the ranks decrease strictly. We apply the Mayer–Vietoris sequence to the open cover of $\breve{M}_k$ consisting of $\breve{M}_{k+1}$ and a tubular neighborhood of Σ_{k+1}. Writing $H_j(\Sigma_{k+1}; \mathbf{Z})$ and $H_j(\Sigma_{k+1} \times \{\pm 1\}; \mathbf{Z})$ for the homology groups of the tubular neighborhood of Σ_{k+1} and for the intersection of both subsets, respectively:

$$H_{n-1}(\Sigma_{k+1} \times \{\pm 1\}; \mathbf{Z}) \to H_{n-1}(\breve{M}_{k+1}; \mathbf{Z}) \oplus H_{n-1}(\Sigma_{k+1}; \mathbf{Z}) \to H_{n-1}(\breve{M}_k; \mathbf{Z})$$

This implies that the kernel of the natural map

$$H_{n-1}(\breve{M}_{k+1};\mathbf{Z}) \to H_{n-1}(\breve{M}_k;\mathbf{Z}), \tag{A.3}$$

which is induced by the inclusion map $\breve{M}_{k+1} \hookrightarrow \breve{M}_k$, is contained in $\operatorname{img} i_{k+1}$. Thus the map (A.3) descends to an injection

$$\Gamma_{k+1} \hookrightarrow H_{n-1}(\breve{M}_k;\mathbf{Z})\big/(\operatorname{img} i_k + \mathbf{Z}\cdot[\Sigma_{k+1}]) = \Gamma_k/(\mathbf{Z}\cdot[[\Sigma_{k+1}]]), \tag{A.4}$$

where $[[\Sigma_{k+1}]]$ denotes the equivalence class of $[\Sigma_{k+1}]$ within Γ_k.

Note that the Γ_k are all torsion-free. Indeed, from the homology long exact sequence

$$H_{n-1}(\partial\breve{M}_k;\mathbf{Z}) \xrightarrow{i_*} H_{n-1}(\breve{M}_k;\mathbf{Z}) \to H_{n-1}(\breve{M}_k,\partial\breve{M}_k;\mathbf{Z}) \xrightarrow{\partial} H_{n-2}(\partial\breve{M};\mathbf{Z})$$

we get $\Gamma_k \cong \ker\partial$. However, $H_{n-1}(\breve{M}_k,\partial\breve{M}_k;\mathbf{Z}) \cong H^1(\breve{M}_k;\mathbf{Z})$ by Lefschetz duality. The latter is torsion-free by the universal coefficient theorem. Thus, Γ_k is torsion-free.

Since the Γ_k are torsion free, $\operatorname{rank}\Gamma_{k+1} < \operatorname{rank}\Gamma_k$ follows from (A.2) and (A.4). □

The following lemma is well-known to the experts but we include a proof for clarity:

Lemma A.2. *Suppose M, M' are closed, oriented 4-manifolds and that M is obtained from M' by a 1-surgery. Then $b_1(M) \le b_1(M')$ and $b_2(M) \ge b_2(M')$. In fact, either*

$$b_1(M) = b_1(M') - 1 \text{ and } b_2(M) = b_2(M')$$

or

$$b_1(M) = b_1(M') \text{ and } b_2(M) = b_2(M') + 2.$$

Proof. Let $\breve{M}$ be the manifold obtained by removing a $\mathbf{S}^1 \times B^3$ from M'. The Mayer–Vietoris sequence implies that

$$H_0(\mathbf{S}^1\times\mathbf{S}^2;\mathbf{Z}) \xrightarrow{((i_0)_*,(j_0)_*)} H_0(\breve{M};\mathbf{Z}) \oplus H_0(\mathbf{S}^1\times B^3;\mathbf{Z}) \to H_0(M';\mathbf{Z}) \to \{1\}.$$

For H_0, the map $((i_0)_*,(j_0)_*)$ is the diagonal map, so it is injective. Therefore, again from Mayer–Vietoris:

$$H_1(\mathbf{S}^1\times\mathbf{S}^2;\mathbf{Z}) \xrightarrow{((i_1)_*,(j_1)_*)} H_1(\breve{M};\mathbf{Z}) \oplus H_1(\mathbf{S}^1\times B^3;\mathbf{Z}) \to H_1(M';\mathbf{Z}) \to \{1\}.$$

Observe that $j_1 : \mathbf{S}^1\times\mathbf{S}^2 \to \mathbf{S}^1\times B^3$ induces an injection on H_1. Counting ranks in

$$\Big(H_1(\breve{M};\mathbf{Z}) \oplus H_1(\mathbf{S}^1\times B^3;\mathbf{Z})\Big)\Big/_{\operatorname{im}((i_1)_*,(j_1)_*)} \cong H_1(M';\mathbf{Z}),$$

we conclude that $b_1(\breve{M}) = b_1(M')$.

Similarly,

$$H_1(\mathbf{S}^2 \times \mathbf{S}^1; \mathbf{Z}) \xrightarrow{((i_1)_*,(k_1)_*))} H_1(\breve{M}; \mathbf{Z}) \oplus H_1(\mathbf{S}^2 \times B^2; \mathbf{Z}) \to H_1(M; \mathbf{Z}) \to \{1\}.$$

Counting ranks, $b_1(\breve{M}) = b_1(M) + \operatorname{rank}\operatorname{im}((i_1)_*, (k_1)_*) = b_1(M) + \operatorname{rank}\operatorname{im}(i_1)_*$. Combining these, we have that

$$b_1(M) = b_1(M') \text{ or } b_1(M) = b_1(M') - 1, \tag{A.5}$$

depending on whether $\operatorname{rank}\operatorname{im}(i_1)_* = 0$ or 1.

On the other hand, the Euler characteristics of X, Y satisfy:

$$\chi(M') = \chi(\breve{M}) + \chi(\mathbf{S}^1 \times B^3) - \chi(\mathbf{S}^1 \times \mathbf{S}^2),$$
$$\chi(M) = \chi(\breve{M}) + \chi(\mathbf{S}^2 \times B^2) - \chi(\mathbf{S}^1 \times \mathbf{S}^2).$$

Therefore, $\chi(M) = \chi(M') + 2$. The result follows in combination with (A.5). □

References

[1] K. Akutagawa, B. Botvinnik, Manifolds of positive scalar curvature and conformal cobordism theory, Math. Ann. 324 (4) (2002) 817–840, MR 1942251.

[2] R.H. Bamler, B. Kleiner, Ricci flow and contractibility of spaces of metrics, arXiv Mathematics e-prints, arXiv:1909.08710 [math], 2019.

[3] O. Chodosh, C. Li, Generalized soap bubbles and the topology of manifolds with positive scalar curvature, arXiv Mathematics e-prints, arXiv:2008.11888 [math], 2020.

[4] O. Chodosh, C. Li, Y. Liokumovich, Classifying sufficiently connected psc manifolds in 4 and 5 dimensions, arXiv Mathematics e-prints, arXiv:2105.07306 [math], 2021.

[5] D.B.A. Epstein, Finite presentations of groups and 3-manifolds, Q. J. Math. Oxford Ser. (2) 12 (1961) 205–212. MR 144321.

[6] H. Federer, Geometric Measure Theory, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag, New York Inc., New York, 1969. MR 0257325.

[7] M. Gromov, H.B. Lawson Jr., The classification of simply connected manifolds of positive scalar curvature, Ann. Math. (2) 111 (3) (1980) 423–434. MR 577131.

[8] M. Gromov, H.B. Lawson Jr., Positive scalar curvature and the Dirac operator on complete Riemannian manifolds, Publ. Math. Inst. Hautes Études Sci. (58) (1983) 83–196 (1984). MR 720933.

[9] M. Gromov, No metrics with positive scalar curvatures on aspherical 5-manifolds, arXiv Mathematics e-prints, arXiv:2009.05332 [math], 2020.

[10] S. Hatui, V. Kakkar, M.K. Yadav, The Schur multiplier of p-groups of order p^5, J. Group Theory 22 (4) (2019) 647–687. MR 3975686.

[11] C. Li, C. Mantoulidis, Metrics with $\lambda_1(-\Delta + kR) \geq 0$ and flexibility in the Riemannian Penrose inequality, arXiv Mathematics e-prints, arXiv:2106.15709 [math], 2021.

[12] P. Miao, Positive mass theorem on manifolds admitting corners along a hypersurface, Adv. Theor. Math. Phys. 6 (6) (2002) 1163–1182 (2003). MR 1982695.

[13] J. Milnor, A unique decomposition theorem for 3-manifolds, Am. J. Math. 84 (1962) 1–7. MR 142125.

[14] C. Mantoulidis, R. Schoen, On the Bartnik mass of apparent horizons, Class. Quantum Gravity 32 (20) (2015) 205002. MR 3406373.

[15] W.H. Meeks III, S.-T. Yau, Topology of three-dimensional manifolds and the embedding problems in minimal surface theory, Ann. Math. (2) 112 (3) (1980) 441–484. MR 595203.

[16] R. Schoen, S.-T. Yau, Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature, Ann. Math. (2) 110 (1) (1979) 127–142. MR 541332.
[17] R. Schoen, S.-T. Yau, On the structure of manifolds with positive scalar curvature, Manuscr. Math. 28 (1–3) (1979) 159–183. MR 535700.
[18] R. Schoen, S.-T. Yau, The structure of manifolds with positive scalar curvature, in: Directions in Partial Differential Equations, Madison, WI, 1985, in: Publ. Math. Res. Center Univ. Wisconsin, vol. 54, Academic Press, Boston, MA, 1987, pp. 235–242. MR 1013841.
[19] H. Weyl, On the determination of a closed convex surface by its line element, Usp. Mat. Nauk (N.S.) 3 (2(24)) (1948) 159–190. MR 0025756.