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REGULARITY OF SOLUTIONS FOR NONLOCAL DIFFUSION EQUATIONS ON PERIODIC DISTRIBUTIONS

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This work addresses the regularity of solutions for a nonlocal diffusion equation over the space of periodic distributions. The spatial operator for the nonlocal diffusion equation is given by a nonlocal Laplace operator with a compactly supported integral kernel. We follow a unified approach based on the Fourier multipliers of the nonlocal Laplace operator, which allows the study of regular as well as distributional solutions of the nonlocal diffusion equation, with integrable as well as singular kernels, in any spatial dimension. In addition, the results extend beyond operators with singular kernels to nonlocal superdiffusion operators. We present results on the spatial and temporal regularity of solutions in terms of regularity of the initial data or the diffusion source term. Moreover, solutions of the nonlocal diffusion equation are shown to converge to the solution of the classical diffusion equation for two types of limits: as the spatial nonlocality vanishes or as the singularity of the integral kernel approaches a certain critical singularity that depends on the spatial dimension. Furthermore, we show that, for the case of integrable kernels, discontinuities in the initial data propagate and persist in the solution of the nonlocal diffusion equation. The magnitude of a jump discontinuity is shown to decay over time.

1. Introduction

In this work, we study the regularity of solutions to the nonlocal diffusion equation given by

$$\begin{cases} u_t(x,t) = L^{\delta,\beta}u(x,t) + b(x), & x \in T^n, t > 0, \\ u(x,0) = f(x), \end{cases} \tag{1}$$

over the space of periodic distributions $H^s(T^n)$, with $s \in \mathbb{R}$. Here T^n denotes the periodic torus in $\mathbb{R}^n$ and $L^{\delta,\beta}$ is a nonlocal Laplace operator defined by

$$L^{\delta,\beta}u(x) = c^{\delta,\beta} \int_{B_\delta(x)} \frac{u(y) - u(x)}{\|y - x\|^\beta}\, dy, \quad x \in \mathbb{R}^n, \tag{2}$$

where $B_\delta(x)$ denotes a ball in $\mathbb{R}^n$, $\delta > 0$ is called the horizon or the nonlocality, and the kernel exponent β satisfies $\beta < n+2$ [12; 13]. The scaling constant $c^{\delta,\beta}$ is given by

$$c^{\delta,\beta} = \frac{2(n+2-\beta)\,\Gamma\left(\frac{n}{2}+1\right)}{\pi^{\frac{n}{2}}\delta^{n+2-\beta}}.$$

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Nonlocal integral operators with compact support of the form (2) have their roots in peridynamics [24; 25] and have been introduced in nonlocal vector calculus [13]. These nonlocal operators have been used in different applied settings; see for example [4; 6; 7; 17; 19]. The work in [3] proposed a nonlocal model for transient heat transfer, which is valid when the body undergoes damage or evolving cracks. There have been many mathematical analysis studies involving nonlocal Laplace operators and peridynamic operators including the works [2; 12; 15; 21; 22; 23]. In general, exact solutions are not readily available for nonlocal models, however, different computational techniques and numerical analysis methods have been developed for solving nonlocal equations such as [1; 5; 8; 9; 10; 11; 14; 16; 20; 26].

The work in [2] introduces the Fourier multipliers for nonlocal Laplace operators, studies the asymptotic behavior of these multipliers, and then applies the asymptotic analysis in the periodic setting to prove regularity results for the nonlocal Poisson equation. In this work, we apply the Fourier multipliers approach developed in [2] to study the regularity of solutions to the nonlocal diffusion equation over the space of periodic distributions. The organization of this article and a brief description of the main contributions of this study are as follows.

- A review of the Fourier multipliers analysis for the nonlocal Laplace operator (2) is provided in Section 2.

- In Section 3, we present the regularity of solutions analysis for the nonlocal diffusion equation with initial data in $H^s(T^n)$, but without a diffusion source.
  - Theorem 3 and Proposition 5 provide the spatial and temporal regularity results, respectively, in any spatial dimension. The temporal regularity for a general periodic distribution in $H^s(T^n)$, with $s \in \mathbb{R}$, is studied in the sense of Gateaux differentiation.
  - In the case when the Fourier coefficients of the initial data $f \in H^s(T^n)$ are summable

$$\sum_{k\in\mathbb{Z}^n} |\hat{f}_k| < \infty,$$

    we have the solution of the nonlocal diffusion equation, considered as a function of the spatial variable x, is a regular $L^2(T^n)$ function and Proposition 10 of Section 3.3 provides the temporal regularity of the solution with respect to the classical derivative.
  - Theorems 13 and 14 provide convergence results for the solution of the nonlocal diffusion equation, without a diffusion source, to the solution of the corresponding classical diffusion equation with respect to two different limits: as $\delta \to 0^+$ or as $\beta \to n+2$, respectively.

- In Section 4, we present the regularity of solutions analysis for the nonlocal diffusion equation when a diffusion source $b \in H^s(T^n)$, for some $s \in \mathbb{R}$, is present.
  - Theorem 15 and Proposition 17 provide the spatial and temporal regularity results, respectively, in any spatial dimension. The temporal regularity for a general periodic distribution in $H^s(T^n)$, with $s \in \mathbb{R}$, is studied in the sense of Gateaux differentiation.
  - In the case when the Fourier coefficients of the source term $b \in H^s(T^n)$ are summable

$$\sum_{k\in\mathbb{Z}^n} |\hat{b}_k| < \infty,$$

we have the solution of the nonlocal diffusion equation, considered as a function of the spatial variable x, is a regular $L^2(T^n)$ function and Proposition 20 of Section 4.1 provides the temporal regularity of the solution with respect to the classical derivative.

– Theorems 22 and 24 provide convergence results for the solution of nonlocal diffusion equation with a nonzero diffusion source to the solution of the corresponding classical diffusion equation with respect to two kinds of limits: as $\delta \to 0^+$ or as $\beta \to n+2$, respectively.

- In Section 5, we show that, for the case of integrable kernels, that is, when $\beta < n$, discontinuities in the initial data propagate and persist in the solution of the nonlocal diffusion equation. The magnitude of a jump discontinuity is shown to decay as time increases.

2. Fourier multipliers

In this section, we give a summary of the Fourier multiplier results introduced in [2], which are relevant to the work presented in Section 3. These multipliers are defined through the Fourier transform by

$$L^{\delta,\beta}u(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} m^{\delta,\beta}\hat{u}(\nu)e^{i\nu\cdot x}\,d\nu, \tag{3}$$

where $m^{\delta,\beta}(\nu)$ is given by

$$m^{\delta,\beta}(\nu) = c^{\delta,\beta}\int_{B_\delta(0)} \frac{\cos(\nu\cdot z)-1}{\|z\|^\beta}\,dz, \tag{4}$$

for $\beta < n+2$. The following theorem gives a representation of these multipliers using the hypergeometric function ${}_2F_3$, which is defined by

$${}_2F_3(a_1, a_2; b_1, b_2, b_3, z) := \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k}{(b_1)_k(b_2)_k(b_3)_k}\frac{z^k}{k!},$$

where $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$ is the rising factorial, also known as the Pochhammer symbol.

Theorem 1. *Let $n \geq 1$, $\delta > 0$ and $\beta < n+2$. Then the Fourier multipliers can be written as*

$$m^{\delta,\beta}(\nu) = -\|\nu\|^2{}_2F_3\big(1, \tfrac{1}{2}(n+2-\beta); 2, \tfrac{1}{2}(n+2), \tfrac{1}{2}(n+4-\beta); -\tfrac{1}{4}\|\nu\|^2\delta^2\big). \tag{5}$$

The hypergeometric function ${}_2F_3$ on the right-hand side is well defined for any $\beta \neq n+4, n+6, \ldots$, hence, using (5), the definition of the multipliers is extended to the case when $\beta \geq n+2$ with $\beta \neq n+4, n+6, \ldots$. Consequently, the operator $L^{\delta,\beta}$ is extended to these larger values of β using the Fourier transform. In particular, for the case when $\beta = n+2$, and since $m^{\delta,n+2}(\nu)$ is equal to $-\|\nu\|^2$, the extended operator $L^{\delta,\beta}$ coincides with the classical Laplace operator Δ. For the case $n+2 < \beta < n+4$, the extended operator $L^{\delta,\beta}$ corresponds to a *nonlocal superdiffusion operator* [1].

The representation (5) is used to provide the asymptotic behavior of $m^{\delta,\beta}(\nu)$ for large $\|\nu\|$. This is given by the following result [2].

Theorem 2. *Let $n \geq 1$, $\delta > 0$ and $\beta \notin \{n+2, n+4, n+6, \dots\}$. Then, as $\|\nu\| \to \infty$,*

$$m^{\delta,\beta}(\nu) \sim \begin{cases} -\dfrac{2n(n+2-\beta)}{\delta^2(n-\beta)} + 2\Big(\dfrac{2}{\delta}\Big)^{n+2-\beta} \dfrac{\Gamma\big(\frac{n+4-\beta}{2}\big)\Gamma\big(\frac{n+2}{2}\big)}{(n-\beta)\Gamma\big(\frac{\beta}{2}\big)} \|\nu\|^{\beta-n}, & \text{if } \beta \neq n, \\ -\dfrac{2n}{\delta^2}\Big(2\log\|\nu\| + \log\Big(\dfrac{\delta^2}{4}\Big) + \gamma - \psi\Big(\dfrac{n}{2}\Big)\Big), & \text{if } \beta = n, \end{cases} \tag{6}$$

where γ is Euler's constant and ψ is the digamma function.

To simplify the notation, throughout this article we will denote $m^{\delta,\beta}$ simply by m. However, in places in which there is a need to emphasize the dependence of the multipliers on the parameters δ and β, such as when we take limits in those parameters, we will revert to the notation $m^{\delta,\beta}$.

3. Regularity of solutions for the peridynamic diffusion equation

In this section, we focus on the nonlocal diffusion equation with initial data and no diffusion source

$$\begin{cases} u_t(x,t) = L^{\delta,\beta}u(x,t), & x \in T^n, t > 0, \\ u(x,0) = f(x). \end{cases} \tag{7}$$

In order to study the existence, uniqueness, and regularity of solutions to (7) over the space of periodic distributions, we consider the identification $U(t) = u(\cdot,t)$, with $U : [0,\infty) \to H^s(T^n)$.

3.1. Eigenvalues on periodic domains. Let $L^{\delta,\beta}$ be defined on the periodic torus

$$T^n = \prod_{i=1}^{n} [0, r_i], \quad r_i > 0, i = 1, 2, \dots, n.$$

Define

$$\nu_k = \Big(\frac{2\pi k_1}{r_1}, \frac{2\pi k_2}{r_2}, \dots, \frac{2\pi k_n}{r_n}\Big),$$

for any $k \in \mathbb{Z}^n$. Let $\phi_k(x) = e^{i\nu_k \cdot x}$. Then,

$$L^{\delta,\beta}\phi_k(x) = m^{\delta,\beta}(\nu_k)\phi_k(x), \tag{8}$$

which shows that ϕ_k is an eigenfunction of $L^{\delta,\beta}$ with eigenvalues $m^{\delta,\beta}(\nu_k)$. To simplify the notation, we will often suppress the dependence of the multipliers on δ and β and use $m(\nu)$ to denote $m^{\delta,\beta}(\nu)$.

Consider the nonlocal diffusion equation defined in (1). For $s \in \mathbb{R}$, let $H^s(T^n)$ be the space of periodic distributions h on T^n such that

$$\|h\|^2_{H^s(T^n)} := \sum_{k \in \mathbb{Z}^n} (1 + \|k\|^2)^s |\hat{h}_k|^2 < \infty.$$

3.2. Distributional solutions for nonlocal diffusion equation. Let $f \in H^s(T^n)$ and define $U, V : [0,\infty) \to H^q(T^n)$, for some $q \in \mathbb{R}$, by

$$U(t) = \sum_k \hat{f}_k e^{m(\nu_k)t} e^{i\nu_k \cdot x}, \tag{9}$$

$$V(t) = \sum_k \hat{f}_k m(\nu_k) e^{m(\nu_k)t} e^{i\nu_k \cdot x}. \tag{10}$$

Observe that for any $t \geq 0$, $U(t)$ and $V(t)$ are well-defined periodic distributions, since $e^{m(\nu_k)t}$ and $m(\nu_k)\, e^{m(\nu_k)t}$ are both bounded functions in k.

Theorem 3. *Let $n \geq 1$, $\delta > 0$ and $\beta < n+4$. Let ϵ_1 and ϵ_2 be such that $0 < \epsilon_1 < \epsilon_2 < 1$. Assume that $f \in H^s(T^n)$ for some $s \in \mathbb{R}$. Then for any fixed $t > 0$, $U(t) \in H^p(T^n)$ and $V(t) \in H^r(T^n)$, where*

$$p = \begin{cases} s, & \text{if } \beta < n, \\ s + \dfrac{4nt}{\delta^2}(1-\epsilon_1), & \text{if } \beta = n, \\ \infty, & \text{if } \beta > n, \end{cases} \qquad r = \begin{cases} s, & \text{if } \beta < n, \\ s + \dfrac{4nt}{\delta^2}(1-\epsilon_2), & \text{if } \beta = n, \\ \infty, & \text{if } \beta > n, \end{cases} \tag{11}$$

with $H^\infty(T^n) := \bigcap_{s \in \mathbb{R}} H^s(T^n)$.

Proof. We observe that

$$\sum_{0 \neq k \in \mathbb{Z}^n} (1+\|k\|^2)^p |\hat{U}_k|^2 = \sum_{0 \neq k \in \mathbb{Z}^n} (1+\|k\|^2)^{p-s} (1+\|k\|^2)^s |\hat{f}_k e^{m(\nu_k)t}|^2.$$

Since $f \in H^s(T^n)$, the result that $U(t) \in H^p(T^n)$ follows by showing that

$$(1+\|k\|^2)^{p-s} e^{2m(\nu_k)t}$$

is bounded for $k \neq 0$. To see this, we consider three cases. For the case $\beta < n$, we have $p - s = 0$ and

$$e^{2m(\nu_k)t}(1+\|k\|^2)^{p-s} = e^{2m(\nu_k)t},$$

which is bounded since $m(\nu_k) \leq 0$.

For the case when $n < \beta < n+4$, let $q \in \mathbb{R}$ be arbitrary. Then,

$$(1+\|k\|^2)^{q-s} e^{2m(\nu_k)t} = \frac{(1+\|k\|^2)^{q-s}}{e^{2t|m(\nu_k)|}},$$

which vanishes as $\|k\| \to \infty$, and hence boundedness follows. Thus, $U(t) \in H^q(T^n)$ for all q and therefore,

$$U(t) \in \bigcap_{q \in \mathbb{R}} H^q(T^n) = H^\infty(T^n).$$

For the case when $\beta = n$, we have $p - s = ((4nt)/\delta^2)(1-\epsilon_1)$. From Theorem 2, we have

$$m(\nu_k) \sim -\frac{4n}{\delta^2} \log \|\nu_k\|,$$

which implies that

$$\lim_{\|\nu_k\| \to \infty} \frac{m(\nu_k)}{-\frac{4n}{\delta^2} \log \|\nu_k\|} = 1. \tag{12}$$

Thus, for any $\epsilon_1 > 0$, there exists $N \in \mathbb{N}$ such that

$$-\frac{4n}{\delta^2}(1+\epsilon_1) \log \|\nu_k\| \leq m(\nu_k) \leq -\frac{4n}{\delta^2}(1-\epsilon_1) \log \|\nu_k\|, \tag{13}$$

for all $\|\nu_k\| \geq N$. Therefore,

$$e^{2m(\nu_k)t} \leq e^{-\frac{8nt}{\delta^2}(1-\epsilon_1) \log \|\nu_k\|} = \|\nu_k\|^{-\frac{8nt}{\delta^2}(1-\epsilon_1)}. \tag{14}$$

Since there exists $A > 0$ such that $A\|k\| \le \|\nu_k\|$, we have

$$(1+\|k\|^2)^{p-s}e^{2m(\nu_k)t} \le \frac{(1+\|k\|^2)^{\frac{4nt}{\delta^2}(1-\epsilon_1)}}{\|\nu_k\|^{\frac{8nt}{\delta^2}(1-\epsilon_1)}} \le \frac{(1+\|k\|^2)^{\frac{4nt}{\delta^2}(1-\epsilon_1)}}{(A\|k\|)^{\frac{8nt}{\delta^2}(1-\epsilon_1)}},$$

which is bounded.

Similarly, to show that $V(t) \in H^r(T^n)$, we show that

$$(1+\|k\|^2)^{r-s}m(\nu_k)^2e^{2m(\nu_k)t}$$

is bounded for $k \ne 0$. For the case when $\beta < n$, we have $r - s = 0$ and there exists a constant $C > 0$ such that $|m(\nu_k)| \le C$. Thus,

$$(1+\|k\|^2)^{r-s}m(\nu_k)^2e^{2m(\nu_k)t} = m(\nu_k)^2e^{2m(\nu_k)t}$$

is bounded. For the case when $n < \beta < n+4$, for an arbitrary $q' \in \mathbb{R}$, we have

$$(1+\|k\|^2)^{q'-s}m(\nu_k)^2e^{2m(\nu_k)t} = \frac{m(\nu_k)^2(1+\|k\|^2)^{q'-s}}{e^{2t|m(\nu_k)|}},$$

which vanishes as $\|k\| \to \infty$, and therefore boundedness follows. Thus, $V(t) \in H^{q'}(T^n)$ for all $q' \in \mathbb{R}$ and hence,

$$V(t) \in \bigcap_{q' \in \mathbb{R}} H^{q'}(T^n) = H^\infty(T^n).$$

When $\beta = n$, we have $r - s = ((4nt)/\delta^2)(1-\epsilon_2)$. From (13),

$$|m(\nu_k)|^2 \le \left(\frac{4n}{\delta^2}(1+\epsilon_1)\right)^2 (\log\|\nu_k\|)^2. \tag{15}$$

In addition, there exists $N_2 \in \mathbb{N}$ such that

$$\log(\|\nu_k\|) \le \|\nu_k\|^{\frac{4nt}{\delta^2}(\epsilon_2-\epsilon_1)}, \tag{16}$$

for all $\|\nu_k\| > N_2$. Moreover, there exists $B > 0$ such that $\|\nu_k\| \le B\|k\|$. Hence, by using (14), (15), and (16), we obtain

$$(1+\|k\|^2)^{r-s}|m(\nu_k)|^2e^{2m(\nu_k)t} \le \frac{(1+\|k\|^2)^{\frac{4nt}{\delta^2}(1-\epsilon_2)}\left(\frac{4n}{\delta^2}(1+\epsilon_1)\right)^2(B^2\|k\|^2)^{\frac{4nt}{\delta^2}(\epsilon_2-\epsilon_1)}}{(A\|k\|)^{\frac{8nt}{\delta^2}(1-\epsilon_1)}} = M\frac{(1+\|k\|^2)^{\frac{4nt}{\delta^2}(1-\epsilon_2)}}{\|k\|^{\frac{8nt}{\delta^2}(1-\epsilon_2)}},$$

where

$$M = \frac{\left(\frac{4n}{\delta^2}(1+\epsilon_1)\right)^2 B^{\frac{8nt}{\delta^2}(\epsilon_2-\epsilon_1)}}{A^{\frac{8nt}{\delta^2}(1-\epsilon_1)}}.$$

This shows boundedness and therefore completes the proof. □

For any $J \in H^s(T^n)$, $s \in \mathbb{R}$, define

$$L^{\delta,\beta}J = \sum_{k\in\mathbb{Z}^n} m(\nu_k)\hat{J}_k e^{i\nu_k\cdot x}. \tag{17}$$

Lemma 4. *Let U and V be as defined in (9) and (10) respectively. Then $L^{\delta,\beta}U(t) = V(t)$.*

Proof. By (17), we have

$$L^{\delta,\beta}U(t)=\sum_{k\in\mathbb{Z}^n} m(\nu_k)\hat{U}_k(t)e^{i\nu_k\cdot x}=\sum_{k\in\mathbb{Z}^n} m(\nu_k)\hat{f}_k e^{m(\nu_k)t}e^{i\nu_k\cdot x}=V(t).$$ □

Proposition 5. *Let $N\in\mathbb{N}\cup\{0\}$ and define*

$$U^{(N)}(t)=\sum_k \hat{f}_k m(\nu_k)^N e^{m(\nu_k)t}e^{i\nu_k\cdot x}\quad \text{and}\quad V^{(N)}(t)=\sum_k \hat{f}_k m(\nu_k)^{N+1}e^{m(\nu_k)t}e^{i\nu_k\cdot x}.$$

Then $\frac{d}{dt}U^{(N)}(t)=V^{(N)}(t)$, for all $t\in(0,\infty)$. Equivalently,

$$\frac{d^N}{dt^N}U(t)=\sum_k \hat{f}_k m(\nu_k)^{N+1}e^{m(\nu_k)t}e^{i\nu_k\cdot x},$$

where the differentiation here is in the sense of Gateaux differentiation.

Remark 6. We note that $U^{(0)}(t)=U(t)$ and $V^{(0)}(t)=V(t)$. In addition, similar to the argument in Theorem 3, for any $t\geq 0$, both $U^{(N)}(t)$ and $V^{(N)}(t)$ are in $H^s(T^n)$ when $\beta\leq n$, and both are in $H^\infty(T^n)$ when $n<\beta<n+4$.

Proof. Let $t>0$. We show that $\frac{d}{dt}U^{(N)}(t)=V^{(N)}(t)$, where the differentiation is in the Gateaux sense, which is given by

$$\lim_{h\to 0}\left\|\frac{1}{h}[U^{(N)}(t+h)-U^{(N)}(t)]-V^{(N)}(t)\right\|^2_{H^q(T^n)}=0,$$

where $q=s$ when $\beta\leq n$ and q is arbitrary when $n<\beta<n+4$. Equivalently, we show that

$$\lim_{h\to 0}\sum_{k\in\mathbb{Z}^n}(1+\|k\|^2)^q|\hat{f}_k|^2 m(\nu_k)^{2N}\left[\frac{1}{h}(e^{m(\nu_k)h}-1)e^{m(\nu_k)t}-m(\nu_k)e^{m(\nu_k)t}\right]^2=0. \tag{18}$$

This result follows from passing the limit inside the sum, which we justify next by the dominated convergence theorem.

When $\beta<n$, we have $q=s$ and there exists a constant $C_1>0$ such that $|m(\nu_k)|<C_1$. Moreover, there exists $C_2>0$ such that

$$\left|\frac{e^{m(\nu_k)h}-1}{h}\right|<C_2, \tag{19}$$

for all k and for sufficiently small h. Combining this with the fact that $f\in H^s(T^n)$, it follows that the summand in the left-hand side of (18) is uniformly bounded.

For the cases $\beta>n$ and $\beta=n$, we first note that the summand in (18) can be written as

$$(1+\|k\|^2)^s|f_k|^2\left(\frac{e^{m(\nu_k)h}-1-m(\nu_k)h}{h}\right)^2 e^{2m(\nu_k)t}m(\nu_k)^{2N}(1+\|k\|^2)^{q-s}.$$

Since $f\in H^s(T^n)$, it is left to show that

$$\left(\frac{e^{m(\nu_k)h}-1-m(\nu_k)h}{h}\right)^2 e^{2m(\nu_k)t}m(\nu_k)^{2N}(1+\|k\|^2)^{q-s} \tag{20}$$

is uniformly bounded. By Taylor's theorem,

$$\left(\frac{e^{m(\nu_k)h}-1-m(\nu_k)h}{h}\right)\le\frac{m(\nu_k)^2}{2}, \tag{21}$$

for all $h\in(0,1)$. Therefore,

$$\left(\frac{e^{m(\nu_k)h}-1-m(\nu_k)h}{h}\right)^2 e^{2m(\nu_k)t}m(\nu_k)^{2N}(1+\|k\|^2)^{q-s}\le\tfrac{1}{4}(1+\|k\|^2)^{q-s}m(\nu_k)^{2N+4}e^{2m(\nu_k)t}.$$

For $\beta=n$, we have $q-s=0$, and since $m(\nu_k)\to-\infty$ as $\|k\|\to\infty$, we find

$$\tfrac{1}{4}m(\nu_k)^{2N+4}e^{2m(\nu_k)t}\to 0$$

as $\|k\|\to\infty$, showing that (20) is uniformly bounded.

For $\beta>n$, by using Theorem 2, we have that

$$\tfrac{1}{4}(1+\|k\|^2)^{q-s}m(\nu_k)^{2N+4}e^{2m(\nu_k)t}=\frac{(1+\|k\|^2)^{q-s}m(\nu_k)^{2N+4}}{4e^{2|m(\nu_k)|t}}\to 0$$

as $\|k\|\to\infty$, showing that (20) is uniformly bounded and therefore completing proof. □

The following theorem summarizes the results in this subsection.

Theorem 7. *Let $f\in H^s(T^n)$, $\beta<n+4$, and $s\in\mathbb{R}$. Then, there exists a unique solution $U(t)$ to the nonlocal diffusion equation*

$$\begin{cases}\dfrac{dU}{dt}=L^{\delta,\beta}U(t),\\ U(0)=f.\end{cases} \tag{22}$$

Moreover, $U\in C^\infty((0,\infty);H^p(T^n))$, where p is as defined in (11).

Remark 8. The time regularity in Theorem 7 is in the sense of Gateaux differentiation.

Proof. The existence follows from Lemma 4 and Proposition 5 by taking $N=0$. For the uniqueness, let $U_2(t)$ be another solution of (22). We define $W(t)=U(t)-U_2(t)$. Then, $W(t)$ satisfies

$$\begin{cases}\dfrac{dW}{dt}=L^{\delta,\beta}W,\\ W(0)=0.\end{cases}$$

Represent $W(t)$ by its Fourier series

$$W(t)=\sum_{k\in\mathbb{Z}^n}\hat{W}_k(t)e^{i\nu_k\cdot x}.$$

Lemma 4 implies that

$$L^{\delta,\beta}W(t)=\sum_{k\in\mathbb{Z}^n}m(\nu_k)\hat{W}_k(t)e^{i\nu_k\cdot x}\quad\text{and}\quad\frac{dW}{dt}=\sum_{k\in\mathbb{Z}^n}\frac{d\hat{W}_k}{dt}(t)e^{i\nu_k\cdot x}.$$

From (22) and the uniqueness of Fourier coefficients, we have that

$$\frac{d\hat{W}_k}{dt}(t) = m(\nu_k)\hat{W}_k(t),$$

for all k. This implies that $\hat{W}_k(t) = Ae^{m(\nu_k)t}$, where A is a constant. Since $\hat{W}_k(0) = 0$, we have $\hat{W}_k(t) = 0$, for all k, which implies that $W(t) = 0$. Therefore, $U(t) = U_2(t)$. The spatial regularity follows from Theorem 3. □

3.3. Regular functions as solutions of the nonlocal diffusion equation. In this section, we focus on functions f with absolutely summable Fourier coefficients, that is, $\sum_{k\in\mathbb{Z}^n} |\hat{f}_k| < \infty$. The following theorem gives a class of functions that satisfy this condition; see [18].

Theorem 9. *Let s be a nonnegative integer and let $0 \le \alpha < 1$. Assume that f is a function defined on T^n all of whose partial derivatives of order s lie in the space of Holder continuous functions of order α. Suppose that $s + \alpha > n/2$. Then f has an absolutely convergent Fourier series.*

Next we provide results on the temporal regularity of the nonlocal diffusion equation.

Proposition 10. *Let $f \in H^s(T^n)$ such that $\sum_{k\in\mathbb{Z}^n} |\hat{f}_k| < \infty$ and let $\beta < n + 4$. Then,*

$$u(x, \cdot) \in C^\infty((0, \infty)),$$

for all $x \in T^n$.

Proof. We use the Leibniz integral rule for the counting measure to differentiate under the summation. Let $g_k(t) = \hat{f}_k e^{m(\nu_k)t} e^{i\nu_k\cdot x}$ and consider

$$\sum_{k\in\mathbb{Z}^n} |g_k(t)| = \sum_{k\in\mathbb{Z}^n} |\hat{f}_k e^{m(\nu_k)t} e^{i\nu_k\cdot x}| = \sum_{k\in\mathbb{Z}^n} |\hat{f}_k| e^{m(\nu_k)t} \le \sum_{k\in\mathbb{Z}^n} |\hat{f}_k|.$$

Since $\sum_{k\in\mathbb{Z}^n} |\hat{f}_k| < \infty$, we find $g_k(t)$ is summable for any fixed t. Moreover,

$$\frac{dg_k}{dt} = \hat{f}_k m(\nu_k) e^{m(\nu_k)t} e^{i\nu_k\cdot x}$$

is continuous for all k. Now fix $t > 0$. Then there exists τ such that $0 < \tau < t$. We define $\theta_k := |\hat{f}_k||m(\nu_k)|e^{m(\nu_k)\tau}$. When $\beta < n$, there exists $C > 0$ such that $|m(\nu_k)| \le C$. Thus

$$\theta_k = |\hat{f}_k||m(\nu_k)|e^{m(\nu_k)\tau} \le C|\hat{f}_k|,$$

showing that θ_k is summable. When $\beta \ge n$, we have $e^{m(\nu_k)\tau} \to 0$ as $\|k\| \to \infty$. Thus $|m(\nu_k)|e^{m(\nu_k)\tau} \le 1$ for sufficiently large $\|k\|$. Hence $\theta_k \le |\hat{f}_k|$, showing that θ_k is summable. Moreover

$$\left|\frac{dg_k}{dt}\right| = |\hat{f}_k||m(\nu_k)|e^{m(\nu_k)t} \le |\hat{f}_k||m(\nu_k)|e^{m(\nu_k)\tau} = \theta_k.$$

Therefore, we can differentiate under the summation,

$$\frac{\partial u(x,t)}{\partial t} = \frac{d}{dt}\sum_{k\in\mathbb{Z}^n} g_k(t) = \sum_{k\in\mathbb{Z}^n} \frac{dg_k}{dt} = \sum_{k\in\mathbb{Z}^n} \hat{f}_k m(\nu_k) e^{m(\nu_k)t} e^{i\nu_k\cdot x}.$$

For higher derivatives, we observe that

$$\frac{d^N g_k}{dt^N} = \hat{f}_k |m(\nu_k)|^N e^{m(\nu_k)t} e^{i\nu_k\cdot x}.$$

Define $\psi_k = |\hat{f}_k||m(\nu_k)|^N e^{m(\nu_k)\tau}$. Then ψ_k is summable by following similar arguments as above. Furthermore,

$$\left|\frac{d^N g_k}{dt^N}\right| = |\hat{f}_k||m(\nu_k)|^N e^{m(\nu_k)t} \le |\hat{f}_k||m(\nu_k)|^N e^{m(\nu_k)\tau} = \psi_k.$$

This implies that $u(x,\cdot)$ is N-times continuously differentiable and

$$\frac{\partial^N u(x,t)}{\partial t^N} = \frac{d^N}{dt^N}\sum_{k\in\mathbb{Z}^n} g_k(t) = \sum_{k\in\mathbb{Z}^n}\frac{d^N g_k}{dt^N} = \sum_{k\in\mathbb{Z}^n}\hat{f}_k |m(\nu_k)|^N e^{m(\nu_k)t} e^{i\nu_k\cdot x}.$$

Since N is arbitrary, it follows that $u(x,\cdot)\in C^\infty((0,\infty))$. □

From Theorem 3 and Proposition 10 we obtain the following regularity result.

Theorem 11. *Let $n\ge 1$, $\delta>0$, $\epsilon>0$ and $\beta<n+4$. Assume that $f\in H^s(T^n)$ and its Fourier coefficients are summable. Then*

(1) $u\in C^\infty((0,\infty); H^s(T^n))$ *for* $\beta<n$,

(2) $u\in C^\infty((0,\infty); H^{s+\frac{4nt}{\delta^2}(1-\epsilon)}(T^n))$ *for* $\beta=n$,

(3) $u\in C^\infty((0,\infty); H^\infty(T^n))$ *for* $\beta>n$.

The following lemma will be used to prove Theorem 13 on the convergence of solutions of the nonlocal diffusion equation as $\delta\to 0^+$.

Lemma 12. *Let $n<\beta<n+2$ and $\delta\le 1$. Then, there exist $c_1>0$ and $c_2>0$ such that, for all $\nu\in\mathbb{R}^n$,*

$$m^{\delta,\beta}(\nu)\le\max\{-c_1\|\nu\|^{\beta-n}, -c_2\|\nu\|^2\}.$$

Proof. From Theorem 2, we have

$$m^{1,\beta}(\nu)\sim c\|\nu\|^{\beta-n},$$

where

$$c = (2)2^{n+2-\beta}\frac{\Gamma\left(\frac{n+4-\beta}{2}\right)\Gamma\left(\frac{n+2}{2}\right)}{(\beta-n)\Gamma\left(\frac{\beta}{2}\right)} > 0.$$

This is equivalent to

$$\lim_{\|\nu\|\to\infty}\frac{m^{1,\beta}(\nu)}{-\|\nu\|^{\beta-n}} = c,$$

which implies that there is $c_1>0$ and $N>0$ such that for all $\|\nu\|>N$

$$m^{1,\beta}(\nu)\le -c_1\|\nu\|^{\beta-n}. \tag{23}$$

On the other hand, from [1], we have

$$\lim_{\|\nu\|\to 0}\frac{m^{1,\beta}(\nu)}{-\|\nu\|^{\beta-n}} = 1.$$

Thus, there exists $c_2 > 0$ such that, for all $\|\nu\| < N$,

$$m^{1,\beta}(\nu) \le -c_2\|\nu\|^2. \tag{24}$$

Combining (23) and (24), we have

$$m^{1,\beta}(\nu) \le \max\{-c_1\|\nu\|^{\beta-n}, -c_2\|\nu\|^2\}, \tag{25}$$

for all $\nu \in \mathbb{R}^n$. Using (25) and the fact that $m^{\delta,\beta}(\nu) = \frac{1}{\delta^2}m^{1,\beta}(\delta\nu)$, which follows from (5), we obtain

$$\begin{aligned} m^{\delta,\beta}(\nu) = \frac{1}{\delta^2}m^{1,\beta}(\delta\nu) &\le \frac{1}{\delta^2}\max\{-c_1\|\delta\nu\|^{\beta-n}, -c_2\|\delta\nu\|^2\} \\ &= \max\{-c_1\|\nu\|^{\beta-n}\delta^{\beta-(n+2)}, -c_2\|\nu\|^2\}. \end{aligned}$$

Since $\delta \le 1$, we have $-\delta^{\beta-(n+2)} \le -1$ and hence

$$m^{\delta,\beta}(\nu) \le \max\{-c_1\|\nu\|^{\beta-n}, -c_2\|\nu\|^2\}.$$ □

Convergence of solutions of the nonlocal diffusion equation (7) to the solution of the corresponding classical diffusion equation is given next in Theorem 13 and Theorem 14.

Theorem 13. *Let $n \ge 1$, $s \in \mathbb{R}$ and let $f \in H^s(T^n)$. Suppose u is the solution of the classical diffusion equation $u_t = \Delta u$ with initial condition $u|_{t=0} = f$. For any $\delta > 0$, let $u^{\delta,\beta}$ be the solution of the nonlocal diffusion equation in (7). Then, for $t > 0$ and $\beta \le n$,*

$$\lim_{\delta\to 0^+} u^{\delta,\beta}(\cdot\,,t) = u(\cdot\,,t) \quad \textit{in } H^s(T^n),$$

and, for $n < \beta \le n+2$,

$$\lim_{\delta\to 0^+} u^{\delta,\beta}(\cdot\,,t) = u(\cdot\,,t) \quad \textit{in } H^\infty(T^n).$$

Proof. The Fourier coefficients satisfy $\hat{u}_k^{\delta,\beta} = \hat{f}_k e^{m(\nu_k)t}$ and $\hat{u}_k = \hat{f}_k e^{-\|\nu_k\|^2 t}$. When $\beta \le n$, we observe

$$\begin{aligned} \|u^{\delta,\beta}(\cdot\,,t) - u(\cdot\,,t)\|^2_{H^s(T^n)} &= \sum_{0\ne k\in\mathbb{Z}^n} (1+\|k\|^2)^s |\hat{u}_k^{\delta,\beta} - \hat{u}_k|^2 \\ &= \sum_{0\ne k\in\mathbb{Z}^n} (1+\|k\|^2)^s |e^{m(\nu_k)t} - e^{-\|\nu_k\|^2 t}|^2 |\hat{f}_k|^2. \end{aligned}$$

To pass the limit $\delta \to 0^+$ inside the sum, it is sufficient to show that $|e^{m(\nu_k)t} - e^{-\|\nu_k\|^2 t}|^2$ as a function of k is uniformly bounded. Using (4), it is straightforward to see that $m(\nu) \le 0$ for $\nu \in \mathbb{R}^n$. Thus,

$$|e^{m(\nu_k)t} - e^{-\|\nu_k\|^2 t}|^2 \le (e^{m(\nu_k)t} + e^{-\|\nu_k\|^2 t})^2 \le 4.$$

Since $\lim_{\delta\to 0^+} m(\nu_k) = -\|\nu_k\|^2$, we have

$$\lim_{\delta\to 0^+} u^{\delta,\beta}(\cdot\,,t) = u(\cdot\,,t) \quad \text{in } H^s(T^n).$$

For the case $\beta > n$, fix an arbitrary $p \in \mathbb{R}$. Then,

$$\begin{aligned}\|u^{\delta,\beta}(\,\cdot\,,t) - u(\,\cdot\,,t)\|^2_{H^p(T^n)} &= \sum_{0\neq k\in\mathbb{Z}^n} (1+\|k\|^2)^p |\hat{u}_k^{\delta,\beta} - \hat{u}_k|^2 \\ &= \sum_{0\neq k\in\mathbb{Z}^n} (1+\|k\|^2)^s\,(1+\|k\|^2)^{p-s} |e^{m(\nu_k)t} - e^{-\|\nu_k\|^2 t}|^2\,|\hat{f}_k|^2.\end{aligned}$$

Since $(1+\|k\|^2)^s|\hat{f}_k|^2$ is summable, to pass the limit inside the above sum, we show that the following function in k, that is given by

$$(1+\|k\|^2)^{p-s}|e^{m(\nu_k)t} - e^{-\|\nu_k\|^2 t}|^2,$$

is uniformly bounded. First, we rewrite the above expression as

$$(1+\|k\|^2)^{p-s}|e^{m(\nu_k)t} - e^{-\|\nu_k\|^2 t}|^2 = (1+\|k\|^2)^{p-s} e^{2m(\nu_k)t}(1-e^{-(m(\nu_k)+\|\nu_k\|^2)t})^2.$$

Then, we observe that

$$\begin{aligned}m^{\delta,\beta}(\nu_k) + \|\nu_k\|^2 &= -\|\nu_k\|^2 {}_2F_3\big(1, \tfrac{1}{2}(n+2-\beta); 2, \tfrac{1}{2}(n+2), \tfrac{1}{2}(n+4-\beta); -\tfrac{1}{4}\|\nu_k\|^2\delta^2\big) + \|\nu_k\|^2 \\ &= \|\nu_k\|^2\big(1 - {}_2F_3\big(1, \tfrac{1}{2}(n+2-\beta); 2, \tfrac{1}{2}(n+2), \tfrac{1}{2}(n+4-\beta); -\tfrac{1}{4}\|\nu_k\|^2\delta^2\big)\big).\end{aligned}$$

Since ${}_2F_3\big(1, \frac{n+2-\beta}{2}; 2, \frac{n+2}{2}, \frac{n+4-\beta}{2}; x\big) \le 1$ for all $x \le 0$, we have $m^{\delta,\beta} + \|\nu_k\|^2 \ge 0$. Therefore,

$$1 - e^{-(m(\nu_k)+\|\nu_k\|^2)t} < 1.$$

Using this fact, we have

$$\begin{aligned}(1+\|k\|^2)^{p-s}|e^{m(\nu_k)t} - e^{-\|\nu_k\|^2}| &= (1+\|k\|^2)^{p-s} e^{2m(\nu_k)t}(1-e^{-(m(\nu_k)+\|\nu_k\|^2 t})^2 \\ &< (1+\|k\|^2)^{p-s} e^{2m(\nu_k)t}.\end{aligned}$$

Lemma 12 implies that there exist $c_1 > 0$ and $c_2 > 0$ such that

$$e^{m(\nu_k)t} \le \max\{e^{-c_1\|\nu_k\|^{\beta-n}\,t}, e^{-c_2\|\nu_k\|^2 t}\}.$$

Consequently,

$$(1+\|k\|^2)^{p-s} e^{2m(\nu_k)t} \le \frac{(1+\|k\|^2)^{p-s}}{\min\big\{\exp(2c_1\|\nu_k\|^{\beta-n}\,t), \exp(2c_2\|\nu_k\|^2 t)\big\}},$$

which is bounded for sufficiently large k for all $\delta \in [0, 1]$. ☐

Theorem 14. *Let $n \ge 1$, $s \in \mathbb{R}$ and let $f \in H^s(T^n)$. Suppose u is the solution of the classical diffusion equation $u_t = \Delta u$, with $u|_{t=0} = f$, and for any $\beta < n+4$, let $u^{\delta,\beta}$ be the solution of the nonlocal diffusion equation (7). Then for $t > 0$*

$$\lim_{\beta\to n+2} u^{\delta,\beta}(\,\cdot\,,t) = u(\,\cdot\,,t) \quad \textit{in } H^\infty(T^n).$$

Proof. Let $q \in \mathbb{R}$ be arbitrary. Consider

$$\|u^{\delta,\beta}(\,\cdot\,,t) - u(\,\cdot\,,t)\|^2_{H^q(T^n)} = \sum_{k\in\mathbb{Z}^n} (1+\|k\|^2)^{q-s}|e^{m(\nu_k)t} - e^{-\|\nu_k\|^2 t}|^2 (1+\|k\|^2)^s\,\|\hat{f}_k\|^2.$$

We observe that for β near $n+2$, $m(\nu_k) \to -\infty$ as $\|k\| \to \infty$. Thus, we can pass the limit in β inside the sum above, since $f \in H^s(T^n)$ and the expression

$$(1+\|k\|^2)^{q-s}|e^{m(\nu_k)t} - e^{-\|\nu_k\|^2 t}|^2$$

is uniformly bounded. Since $\lim_{\beta\to n+2} m(\nu_k) = -\|\nu_k\|^2$, the result follows. □

4. Nonlocal diffusion equation with a diffusion source

In this section, we focus on the nonlocal diffusion equation with a diffusion source and zero initial data

$$\begin{cases} u_t(x,t) = L^{\delta,\beta}u(x,t) + b(x), & x \in T^n, t > 0, \\ u(x,0) = 0. \end{cases} \tag{26}$$

In order to study the existence, uniqueness, and regularity of solutions to (26) over the space of periodic distributions, we consider the identification $U(t) = u(\,\cdot\,, t)$, with $U : [0,\infty) \to H^s(T^n)$.

Let $b \in H^s(T^n)$ and define $U, V : [0,\infty) \to H^q(T^n)$, for some $q \in \mathbb{R}$, by

$$U(t) = \hat{b}_0 t + \sum_{0\neq k\in\mathbb{Z}^n} \frac{e^{m(\nu_k)t} - 1}{m(\nu_k)} \hat{b}_k e^{i\nu_k\cdot x}, \tag{27}$$

$$V(t) = \sum_{k\in\mathbb{Z}^n} e^{m(\nu_k)t} \hat{b}_k e^{i\nu_k\cdot x}. \tag{28}$$

Observe that for any $t \geq 0$, $U(t)$ and $V(t)$ are well-defined periodic distributions, since $(e^{m(\nu_k)t} - 1)/m(\nu_k)$ and $e^{m(\nu_k)t}$ are bounded functions in k.

Theorem 15. *Let $n \geq 1$, $\delta > 0$, and $\beta < n+4$. Assume that $\epsilon_1 > 0$ and $b \in H^s(T^n)$ for some $s \in \mathbb{R}$. Then, for any fixed $t > 0$, $U(t) \in H^p(T^n)$ and $V(t) \in H^r(T^n)$, where*

$$p = \begin{cases} s, & \text{if } \beta \leq n, \\ s+\beta-n, & \text{if } \beta > n, \end{cases} \quad \text{and} \quad r = \begin{cases} s, & \text{if } \beta < n, \\ s + \dfrac{4nt}{\delta^2}(1-\epsilon_1), & \text{if } \beta = n, \\ \infty, & \text{if } \beta > n. \end{cases} \tag{29}$$

Proof. We observe that

$$\begin{aligned} \|U(t) - \hat{b}_0 t\|^2_{H^p(T^n)} &= \sum_{0\neq k\in\mathbb{Z}^n} (1+\|k\|^2)^p |\hat{U}_k(t)|^2 \\ &= \sum_{0\neq k\in\mathbb{Z}^n} \frac{(1+\|k\|^2)^{p-s}}{|m(\nu_k)|^2} (e^{m(\nu_k)} - 1)^2 (1+\|k\|^2)^s |\hat{b}_k|^2. \end{aligned}$$

Since $b \in H^s(T^n)$ and $e^{m(\nu_k)t}$ is bounded because $m(\nu_k) < 0$, then the result follows by showing that

$$\frac{(1+\|k\|^2)^{p-s}}{|m(\nu_k)|^2}$$

is bounded for $k \neq 0$. When $\beta \leq n$, we have $p - s = 0$, and by using Theorem 2, there exist $C_1 > 0$ and $r_1 > 0$ such that $|m(\nu_k)| \geq C_1$, for all $\|k\| \geq r_1$. Thus,

$$\frac{(1+\|k\|^2)^{p-s}}{|m(\nu_k)|^2} \leq \frac{1}{C_1^2}.$$

When $\beta > n$, we have $p - s = \beta - n$, and by using Theorem 2, there exist $C_2 > 0$ and $r_2 > 0$ such that $|m(\nu_k)| \geq C_2\|k\|^{\beta-n}$, for all $\|k\| \geq r_2$. This implies that

$$\frac{(1+\|k\|^2)^{p-s}}{|m(\nu_k)|^2} \leq \frac{1}{C_2^2}\left(\frac{1+\|k\|^2}{\|k\|^2}\right)^{\beta-n},$$

which is bounded. The proof of $V(t) \in H^r(T^n)$ is similar to the proof of Theorem 3. □

Lemma 16. *Let U and V be as defined in (27) and (28), respectively. Then*

$$V(t) = L^{\delta,\beta}U(t) + b.$$

Proof. By using (17), for any $x \in T^n$ and $t > 0$, we have

$$\begin{aligned} L^{\delta,\beta}U(t)(x) &= L^{\delta,\beta}(\hat{b}_0 t) + \sum_{0\neq k\in\mathbb{Z}^n} m(\nu_k)\hat{U}_k(t)e^{i\nu_k\cdot x} \\ &= \sum_{0\neq k\in\mathbb{Z}^n} m(\nu_k)\frac{(e^{m(\nu_k)t}-1)}{m(\nu_k)}\hat{b}_k e^{i\nu_k\cdot x} \\ &= \sum_{0\neq k\in\mathbb{Z}^n} \hat{b}_k e^{m(\nu_k)t}e^{i\nu_k\cdot x} - \sum_{0\neq k\in\mathbb{Z}^n} \hat{b}_k e^{i\nu_k\cdot x} \\ &= V(t)(x) - b(x). \end{aligned}$$

□

Proposition 17. *Let $U(t)$ and $V(t)$ be as defined in (27) and (28), respectively. Then,*

$$\frac{dU}{dt} = V(t).$$

Moreover, for $N \geq 1$,

$$\frac{d^N U}{dt^N} = \sum_{k\in\mathbb{Z}^n}\hat{b}_k m(\nu_k)^{N-1}e^{m(\nu_k)t}e^{i\nu_k\cdot x},$$

for all $t \in (0,\infty)$, where the differentiation here is in the sense of Gateaux differentiation.

Proof. We show that

$$\lim_{h\to 0}\left\|\frac{1}{h}[U(t+h)-U(t)] - V(t)\right\|^2_{H^q(T^n)} = 0,$$

where $q = s$ when $\beta \leq n$ and q is arbitrary when $\beta > n$. Equivalently, we show that

$$\lim_{h\to 0}\sum_{0\neq k\in\mathbb{Z}^n}(1+\|k\|^2)^q|\hat{b}_k|^2\left[\frac{1}{h}(e^{m(\nu_k)h}-1)\frac{e^{m(\nu_k)t}}{m(\nu_k)} - e^{m(\nu_k)t}\right]^2 = 0. \tag{30}$$

This result follows from passing the limit inside the sum, which we justify next by the dominated convergence theorem.

When $\beta < n$, we have $q = s$ and similar to (19), there exists a constant $C_2 > 0$ such that

$$\left| \frac{e^{m(\nu_k)h} - 1}{h} \right| < C_2,$$

for all k and for sufficiently small h. Moreover, $e^{m(\nu_k)t}$ and $1/m(\nu_k)$, for $k \neq 0$, are bounded. Combining this with the fact that $b \in H^s(T^n)$, it follows that the summand in the left-hand side of (30) is uniformly bounded.

For the cases $\beta > n$ and $\beta = n$, we first note that the summand in (30) can be written as

$$(1 + \|k\|^2)^s |b_k|^2 (1 + \|k\|^2)^{q-s} \frac{e^{2m(\nu_k)t}}{m(\nu_k)^2} \left(\frac{e^{m(\nu_k)} - 1 - m(\nu_k)h}{h} \right)^2.$$

Since $b \in H^s(T^n)$, it is left to show that

$$(1 + \|k\|^2)^{q-s} \frac{e^{2m(\nu_k)t}}{m(\nu_k)^2} \left(\frac{e^{m(\nu_k)} - 1 - m(\nu_k)h}{h} \right)^2 \tag{31}$$

is uniformly bounded. Using (21), we have

$$(1 + \|k\|^2)^{q-s} \frac{e^{2m(\nu_k)t}}{m(\nu_k)^2} \left(\frac{e^{m(\nu_k)} - 1 - m(\nu_k)h}{h} \right)^2 \leq \tfrac{1}{4}(1 + \|k\|^2)^{q-s} m(\nu_k)^2 e^{2m(\nu_k)t}.$$

When $\beta = n$, we have $q - s = 0$ and since $m(\nu_k) \to -\infty$ as $\|k\| \to \infty$, we find

$$\tfrac{1}{4} m(\nu_k)^2 e^{2m(\nu_k)t} \to 0$$

as $\|k\| \to \infty$ showing that (31) is uniformly bounded.

When $\beta > n$, by using Theorem 2, we have that

$$\tfrac{1}{4}(1 + \|k\|^2)^{q-s} m(\nu_k)^2 e^{2m(\nu_k)t} = \frac{(1 + \|k\|^2)^{q-s} m(\nu_k)^2}{4e^{2|m(\nu_k)|t}} \to 0$$

as $\|k\| \to \infty$, showing that (31) is uniformly bounded and therefore completing the proof of the first part. The second part of this proposition follows from arguments similar to those in the proof of Proposition 5. □

The following regularity theorem summarizes the results of this subsection.

Theorem 18. *Let $b \in H^s(T^n)$ with $s \in \mathbb{R}$. Then there exists a unique solution U to the nonlocal diffusion equation*

$$\begin{cases} \dfrac{dU}{dt} = L^{\delta,\beta} U(t) + b, \\ U(0) = 0. \end{cases} \tag{32}$$

Moreover, $U \in C^\infty((0, \infty); H^p(T^n))$, where p is as defined in (29).

Remark 19. The temporal regularity is in the sense of Gateaux differentiation.

Proof. The existence follows from Lemma 16 and Proposition 17. For the uniqueness, the proof is similar to the proof of uniqueness in Theorem 7. The spatial regularity follows from Theorem 15 and the temporal regularity follows from Proposition 17. □

4.1. Regular functions as solutions of nonlocal diffusion equations with diffusion source. In this section, we focus on functions b with absolutely summable Fourier coefficients, that is, $\sum_{k\in\mathbb{Z}^n}|\hat{b}_k| < \infty$.

Proposition 20. *Let $b \in H^s(T^n)$ such that $\sum_{k\in\mathbb{Z}^n}|\hat{b}_k| < \infty$ and let $\beta < n+4$. Then*

$$u(x,\cdot) \in C^\infty((0,\infty)),$$

for all $x \in T^n$.

Proof. We use the Leibniz rule to differentiate under the sum. Let $g_k(t) = ((e^{m(\nu_k)t}-1)/m(\nu_k))\hat{b}_k e^{i\nu_k\cdot x}$ and consider

$$\sum_{0\neq k\in\mathbb{Z}^n}|g_k(t)| = \sum_{0\neq k\in\mathbb{Z}^n}\left|\frac{e^{m(\nu_k)t}-1}{m(\nu_k)}\hat{b}_k e^{i\nu_k\cdot x}\right| = \sum_{0\neq k\in\mathbb{Z}^n}|\hat{b}_k|\left|\frac{e^{m(\nu_k)t}-1}{m(\nu_k)}\right| \le \sum_{0\neq k\in\mathbb{Z}^n}|\hat{b}_k|\frac{1}{|m(\nu_k)|},$$

where in the last inequality, we used the fact that $m(\nu_k) < 0$. Since $1/|m(\nu_k)|$, $k \neq 0$, is bounded and $\sum_{k\in\mathbb{Z}^n}|\hat{b}_k| < \infty$, we have $g_k(t)$ is summable for any fixed t. Moreover,

$$\frac{dg_k}{dt} = \hat{b}_k e^{m(\nu_k)t} e^{i\nu_k\cdot x}$$

is continuous for all k. Now fix $t > 0$. Then there exists τ such that $0 < \tau < t$. We define $\theta_k := |\hat{b}_k|e^{m(\nu_k)\tau}$. Since $m(\nu_k) \le 0$, for all k, we have

$$\theta_k = |\hat{b}_k|e^{m(\nu_k)\tau} \le |\hat{b}_k|,$$

showing that θ_k is summable. Moreover,

$$\left|\frac{dg_k}{dt}\right| = |\hat{b}_k|e^{m(\nu_k)t} \le |\hat{b}_k|e^{m(\nu_k)\tau} = \theta_k.$$

Therefore,

$$\frac{\partial u(x,t)}{\partial t} = \hat{b}_0 + \frac{d}{dt}\sum_{0\neq k\in\mathbb{Z}^n} g_k(t) = \hat{b}_0 + \sum_{0\neq k\in\mathbb{Z}^n}\frac{dg_k}{dt} = \sum_{k\in\mathbb{Z}^n}\hat{b}_k e^{m(\nu_k)t}e^{i\nu_k\cdot x}.$$

This shows that u is differentiable with respect to t. For higher derivatives, let $N \ge 2$ be an integer. We observe that

$$\frac{d^N g_k}{dt^N} = \hat{b}_k(m(\nu_k))^{N-1}e^{m(\nu_k)t}e^{i\nu_k\cdot x}.$$

Define $\theta_k = |\hat{b}_k||m(\nu_k)|^{N-1}e^{m(\nu_k)\tau}$. When $\beta < n$, there exists $C > 0$ such that $|m(\nu_k)| \le C$. Thus,

$$\theta_k = |\hat{b}_k||m(\nu_k)|^{N-1}e^{m(\nu_k)\tau} \le C^{N-1}|\hat{b}_k|,$$

showing that θ_k is summable. When $\beta \ge n$, then $e^{m(\nu_k)\tau} \to 0$ as $\|k\| \to \infty$. Thus, $|m(\nu_k)|^{N-1}e^{m(\nu_k)\tau} \le 1$ for sufficiently large $\|k\|$. Hence $\theta_k \le |\hat{b}_k|$, showing that θ_k is summable. Furthermore,

$$\left|\frac{d^N g_k}{dt^N}\right| = |\hat{b}_k||m(\nu_k)|^{N-1}e^{m(\nu_k)t} \le |\hat{b}_k||m(\nu_k)|^{N-1}e^{m(\nu_k)\tau} = \theta_k.$$

This implies that $u(x,\cdot)$ is N-times continuously differentiable and

$$\frac{\partial^N u(x,t)}{\partial t^N} = \frac{d^N}{dt^N}\sum_{k\in\mathbb{Z}^n} g_k(t) = \sum_{k\in\mathbb{Z}^n}\frac{d^N g_k}{dt^N} = \sum_{k\in\mathbb{Z}^n}\hat{b}_k |m(\nu_k)|^{N-1} e^{m(\nu_k)t} e^{i\nu_k\cdot x}.$$

Since N is arbitrary, it follows that $u(x,\cdot)\in C^\infty((0,\infty))$. □

From Theorem 15 and Proposition 20 we obtain the following regularity result.

Theorem 21. *Let $n \ge 1$, $\delta > 0$ and $\beta < n+4$. Assume that $b \in H^s(T^n)$ and its Fourier coefficients are summable. Then*

(1) $u \in C^\infty((0,\infty); H^s(T^n))$, *for* $\beta \le n$,

(2) $u \in C^\infty((0,\infty); H^{s+\beta-n}(T^n))$, *for* $\beta > n$.

Convergence of solutions of the nonlocal diffusion equation (26) to the solution of the corresponding classical diffusion equation is given next in Theorem 22 and Theorem 24.

Theorem 22. *Let $n \ge 1$ and $b \in H^s(T^n)$, with $s \in \mathbb{R}$. Suppose u is the solution of the classical diffusion equation $u_t = \triangle u + b$ with initial condition $u|_{t=0} = 0$. For any $\delta > 0$, let $u^{\delta,\beta}$ be the solution of the nonlocal diffusion equation*

$$\begin{cases} u_t^{\delta,\beta}(x,t) = L^{\delta,\beta}u^{\delta,\beta}(x,t) + b(x), & x\in T^n, t>0,\\ u^{\delta,\beta}(x,0)=0, & x\in T^n.\end{cases} \tag{33}$$

Then, for $t > 0$ and $\beta \le n$,

$$\lim_{\delta\to 0^+} u^{\delta,\beta}(\cdot,t) = u(\cdot,t) \quad \text{in } H^s(T^n),$$

and, for $n < \beta \le n+2$,

$$\lim_{\delta\to 0^+} u^{\delta,\beta}(\cdot,t) = u(\cdot,t) \quad \text{in } H^{s+\beta-n}(T^n).$$

Proof. The Fourier coefficients satisfy $\hat{u}_k^{\delta,\beta} = ((e^{m(\nu_k)t}-1)/m(\nu_k))\hat{b}_k$ and $\hat{u}_k = ((e^{-\|\nu_k\|^2 t}-1)/(-\|\nu_k\|^2))\hat{b}_k$, for $k \ne 0$. For $\beta \le n$, we have

$$\begin{aligned}\|u^{\delta,\beta}(\cdot,t)-u(\cdot,t)\|^2_{H^s(T^n)} &= \sum_{0\ne k\in\mathbb{Z}^n}(1+\|k\|^2)^s|\hat{u}_k^{\delta,\beta}-\hat{u}_k|^2\\ &= \sum_{0\ne k\in\mathbb{Z}^n}(1+\|k\|^2)^s\left|\frac{(e^{m(\nu_k)t}-1)}{m(\nu_k)} - \frac{(e^{-\|\nu_k\|^2 t}-1)}{-\|\nu_k\|^2}\right|^2|\hat{b}_k|^2.\end{aligned}$$

To pass the limit $\delta \to 0^+$ inside the sum, it is sufficient to show that

$$\left|\frac{(e^{m(\nu_k)t}-1)}{m(\nu_k)} - \frac{(e^{-\|\nu_k\|^2 t}-1)}{-\|\nu_k\|^2}\right|$$

is uniformly bounded. Using (4), $m(\nu) \le 0$ for $\nu \in \mathbb{R}^n$, and thus,

$$\left|\frac{(e^{m(\nu_k)t}-1)}{m(\nu_k)} - \frac{(e^{-\|\nu_k\|^2 t}-1)}{-\|\nu_k\|^2}\right| \le \frac{1}{|m(\nu_k)|} + \frac{1}{\|\nu_k\|^2}.$$

Since $1/m(\nu_k)$, $k \neq 0$, and $1/(\|\nu_k\|^2)$, $k \neq 0$, are bounded, we conclude that

$$\left|\frac{(e^{m(\nu_k)t}-1)}{m(\nu_k)} - \frac{(e^{-\|\nu_k\|^2 t}-1)}{-\|\nu_k\|^2}\right|$$

is uniformly bounded. For the case $\beta > n$, consider

$$\begin{aligned}\|u^{\delta,\beta}(\cdot,t)-u(\cdot,t)\|^2_{H^{s+\beta-n}(T^n)} &= \sum_{0\neq k\in\mathbb{Z}^n}(1+\|k\|^2)^{s+\beta-n}|\hat{u}_k^{\delta,\beta}-\hat{u}_k|^2\\ &= \sum_{0\neq k\in\mathbb{Z}^n}(1+\|k\|^2)^{s}(1+\|k\|^2)^{\beta-n}\left|\frac{(e^{m(\nu_k)t}-1)}{m(\nu_k)} - \frac{(e^{-\|\nu_k\|^2 t}-1)}{-\|\nu_k\|^2}\right|^2|\hat{b}_k|^2.\end{aligned}$$

Since $(1+\|k\|^2)^s|\hat{b}_k|^2$ is summable, to pass the limit inside the above sum, we show that the following function in k that is given by

$$(1+\|k\|^2)^{\beta-n}\left|\frac{(e^{m(\nu_k)t}-1)}{m(\nu_k)} - \frac{(e^{-\|\nu_k\|^2 t}-1)}{-\|\nu_k\|^2}\right|^2$$

is uniformly bounded. Since $m(\nu) \leq 0$ for all $\nu \in \mathbb{R}^n$, we have

$$(1+\|k\|^2)^{\beta-n}\left|\frac{(e^{m(\nu_k)t}-1)}{m(\nu_k)} - \frac{(e^{-\|\nu_k\|^2 t}-1)}{-\|\nu_k\|^2}\right|^2 \leq (1+\|k\|^2)^{\beta-n}\left(\frac{1}{|m(\nu_k)|}+\frac{1}{\|\nu_k\|^2}\right)^2.$$

By using Theorem 2, there exists constant $C > 0$ such that $|m(\nu_k)| > C\,\|k\|^{\beta-n}$. Furthermore, using the fact that $A\,\|k\| \leq \|\nu_k\| \leq B\,\|k\|$ for positive constants A and B and since $\beta-n \leq 2$, we have $\|\nu_k\|^2 \geq A^2\|k\|^{\beta-n}$. Therefore,

$$\begin{aligned}(1+\|k\|^2)^{\beta-n}\left(\frac{1}{|m(\nu_k)|}+\frac{1}{\|\nu_k\|^2}\right)^2 &\leq (1+\|k\|^2)^{\beta-n}\left(\frac{1}{C\,\|k\|^{\beta-n}}+\frac{1}{A^2\,\|k\|^{\beta-n}}\right)^2\\ &\leq \max\left(\frac{1}{C^2},\frac{1}{A^4}\right)\left(\frac{1+\|k\|^2}{\|k\|^2}\right)^{\beta-n},\end{aligned}$$

showing uniform boundedness. Whether $\beta \leq n$ or $n < \beta \leq n+2$, we have $\lim_{\delta\to 0^+} m(\nu_k) = -\|\nu_k\|^2$, which implies that $\lim_{\delta\to 0^+}\|u^{\delta,\beta}(\cdot,t)-u(\cdot,t)\|_{H^s(T^n)} = 0$, or $\lim_{\delta\to 0^+}\|u^{\delta,\beta}(\cdot,\ t)-u(\cdot,t)\|_{H^{s+\beta-n}(T^n)} = 0$, respectively, and therefore completes the proof. □

A proof of the following lemma on the monotonicity of the multipliers can be found in [2].

Lemma 23. *Let $\beta' < \beta \leq n+2$. Then, for all $\nu \neq 0$, $m^{\delta,\beta}(\nu) < m^{\delta,\beta'}(\nu)$.*

Theorem 24. *Let $n \geq 1$, $s \in \mathbb{R}$ and let $b \in H^s(T^n)$. Suppose u is the solution of the classical diffusion equation $u_t = \Delta u + b$, with $u|_{t=0} = 0$, and for any $\beta < n+2$, let $u^{\delta,\beta}$ be the solution of the nonlocal diffusion equation (33). Then, for $t > 0$ and $0 < \epsilon < 2$,*

$$\lim_{\beta\to(n+2)^-} u^{\delta,\beta}(\cdot,t) = u(\cdot,t) \quad \text{in } H^{s+2-\epsilon}(T^n).$$

Proof. For $0 < \epsilon < 2$, define $\beta' = n+2-\epsilon$. For any $\beta > \beta'$, we have from Theorem 15 that $u^{\delta,\beta} \in H^{s+\beta-n}(T^n) \subset H^{s+2-\epsilon}(T^n)$. Furthermore, $u \in H^{s+2}(T^n) \subset H^{s+2-\epsilon}(T^n)$. Thus the limit makes sense. Consider

$$\|u^{\delta,\beta}(\cdot\,,t)-u(\cdot\,,t)\|^2_{H^{s+2-\epsilon}(T^n)} = \sum_{0\neq k\in\mathbb{Z}^n} (1+\|k\|^2)^{2-\epsilon}\left|\frac{e^{m^{\delta,\beta}(\nu_k)t}-1}{m^{\delta,\beta}(\nu_k)} - \frac{e^{-\|\nu_k\|^2 t}-1}{-\|\nu_k\|^2}\right|^2 (1+\|k\|^2)^s\|\hat{b}_k\|^2.$$

Since $b\in H^s(T^n)$, in order to pass the limit in β inside the sum, we show that the expression

$$(1+\|k\|^2)^{2-\epsilon}\left|\frac{e^{m^{\delta,\beta}(\nu_k)t}-1}{m^{\delta,\beta}(\nu_k)} - \frac{e^{-\|\nu_k\|^2 t}-1}{-\|\nu_k\|^2}\right|^2$$

is uniformly bounded for $k\neq 0$ and $\beta\in[\beta',n+2)$. Applying Lemma 23,

$$\begin{aligned}(1+\|k\|^2)^{2-\epsilon}\left|\frac{e^{m^{\delta,\beta}(\nu_k)t}-1}{m^{\delta,\beta}(\nu_k)} - \frac{e^{-\|\nu_k\|^2 t}-1}{-\|\nu_k\|^2}\right|^2 &\leq (1+\|k\|^2)^{2-\epsilon}\left(\frac{1}{|m^{\delta,\beta}(\nu_k)|}+\frac{1}{\|\nu_k\|^2}\right)^2\\ &\leq (1+\|k\|^2)^{2-\epsilon}\left(\frac{1}{|m^{\delta,\beta'}(\nu_k)|}+\frac{1}{\|\nu_k\|^2}\right)^2.\end{aligned}$$

From Theorem 2, there exists $C>0$ such that $|m^{\delta,\beta'}(\nu_k)|\geq C\,\|k\|^{2-\epsilon}$. Furthermore, there exists $A>0$ such that $\|\nu_k\|\geq A\,\|k\|$. Thus,

$$\begin{aligned}(1+\|k\|^2)^{2-\epsilon}\left(\frac{1}{|m^{\delta,\beta'}(\nu_k)|}+\frac{1}{\|\nu_k\|^2}\right)^2 &\leq (1+\|k\|^2)^{2-\epsilon}\left(\frac{1}{C\,\|k\|^{2-\epsilon}}+\frac{1}{A^2\,\|k\|^2}\right)^2\\ &\leq (1+\|k\|^2)^{2-\epsilon}\left(\frac{1}{C\,\|k\|^{2-\epsilon}}+\frac{1}{A^2\,\|k\|^{2-\epsilon}}\right)^2,\end{aligned}$$

which is uniformly bounded. Since $\lim_{\beta\to(n+2)^-} m(\nu_k) = -\|\nu_k\|^2$, the result follows. □

5. Propagation of discontinuities for the nonlocal diffusion equation

In this section, we study the propagation of discontinuities for the nonlocal diffusion equation in (7). We emphasize that Theorem 11 implies that the nonlocal diffusion equation satisfies an instantaneous smoothing effect when the integral kernel is singular with $\beta>n$ and a gradual (over time) smoothing effect for when $\beta=n$. However, for integrable kernels ($\beta<n$), the nonlocal diffusion equation is nonsmoothing. We investigate this latter case further by studying the propagation of discontinuities. To this end, given a discontinuous initial data $f\in L^2(T^n)$, we show that for certain conditions on f and β, discontinuities persist and propagate. In particular, we show that in one-dimension, if f is piecewise continuous, then the solution u is piecewise continuous and both f and u share the same locations of jumps.

To study the propagation of discontinuities, we look for a decomposition of u, the solution of (7), of the form

$$u(x,t)=v(x,t)+g(t)f(x),$$

for some function $v(x,t)$, which is continuous in x and satisfies $v(x,0)=0$, and some function g satisfying $g(0)=1$. This would imply that any discontinuity in f will persist to be a discontinuity in u for all $t>0$. We show that the magnitude of a jump discontinuity decays as t increases.

We observe that v satisfies

$$v_t = u_t - g'(t)f(x) = L^{\delta,\beta}u - g'(t)f(x) = L^{\delta,\beta}v + g(t)L^{\delta,\beta}f(x) - g'(t)f(x).$$

Since $\beta < n$, we observe that

$$L^{\delta,\beta}f(x) = c^{\delta,\beta}\int_{B_\delta(x)} \frac{f(y)-f(x)}{\|y-x\|^\beta}\,dy = h(x) - \alpha f(x),$$

where

$$h(x) = c^{\delta,\beta}\int_{B_\delta(x)} \frac{f(y)}{\|y-x\|^\beta}\,dy = \left(\frac{c^{\delta,\beta}}{\|\cdot\|^\beta}\chi_{B_\delta(0)}(\,\cdot\,)\right) * f(x), \tag{34}$$

and α is a constant given by

$$\alpha = c^{\delta,\beta}\int_{B_\delta(x)} \frac{1}{\|y-x\|^\beta}\,dy = \frac{2n(n+2-\beta)}{\delta^2(n-\beta)}. \tag{35}$$

Therefore,

$$\begin{aligned} v_t &= L^{\delta,\beta}v + g(t)h(x) - \alpha g(t)f(x) - g'(t)f(x) \\ &= L^{\delta,\beta}v + g(t)h(x) - f(x)(\alpha g(t) - g'(t)). \end{aligned}$$

Since $g(t)$ is unknown and we are looking for $v(x,t)$, a continuous function in x which is independent of $f(x)$, we set $\alpha g(t) - g'(t) = 0$. Thus $g(t) = e^{-\alpha t}$ and v solves

$$\begin{cases} v_t = L^{\delta,\beta}v + e^{-\alpha t}h(x), & x \in T^n, t > 0, \\ v(x,0) = 0, \end{cases} \tag{36}$$

and therefore,

$$u(x,t) = v(x,t) + e^{-\alpha t}f(x). \tag{37}$$

Hence,

$$\hat{v}_k = \hat{u}_k - e^{-\alpha t}\hat{f}_k = \hat{f}_k e^{m(\nu_k)t} - e^{-\alpha t}\hat{f}_k = \hat{f}_k(1 - e^{-(m(\nu_k)+\alpha)t})e^{m(\nu_k)t}. \tag{38}$$

It remains to find conditions on f and β to guarantee the continuity of v. Towards this end, we make use of the following lemma, whose proof is similar to the proof of Theorem 3.2 in [2]. We note that the constant α appears in the asymptotics formula (6).

Lemma 25. *Let $\nu \in \mathbb{R}^n$ and let $\beta < n$. Suppose α is as defined in (35). Then*

$$m(\nu) + \alpha \sim \begin{cases} 4(n+2-\beta)\Gamma\left(\frac{n}{2}+1\right)\delta^2 \dfrac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+2-\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)}\left(\frac{\delta\|\nu\|}{2}\right)^{\beta-n} & \text{if } \frac{n-1}{2} < \beta < n, \\ 4(n+2-\beta)\Gamma\left(\frac{n}{2}+1\right)\delta^2 \dfrac{(n-\beta)\Gamma\left(\frac{n}{2}\right)}{4\sqrt{\pi}}\left(\frac{\delta\|\nu\|}{2}\right)^{-\frac{n+1}{2}} & \text{if } \beta \le \frac{n-1}{2}. \end{cases}$$

In addition, we make use of the following lemma.

Lemma 26. *Let α be as defined in (35). Then,*

$$(1-e^{-(m(\nu_k)+\alpha)t})e^{m(\nu_k)t} \sim \begin{cases} \dfrac{C_1 t e^{-\alpha t}}{\|k\|^{n-\beta}} & \text{if } \dfrac{n-1}{2} < \beta < n, \\ \dfrac{C_2 t e^{-\alpha t}}{\|k\|^{\frac{n+1}{2}}} & \text{if } \beta \le \dfrac{n-1}{2}, \end{cases}$$

for some positive constants C_1 and C_2.

Proof. When $\frac{n-1}{2} < \beta < n$, by using Lemma 25 and the definition of ν_k, there exists $C_1 > 0$ such that

$$\lim_{\|k\|\to\infty} (m(\nu_k)+\alpha)\|k\|^{n-\beta} = C_1,$$

which implies that, for any $\epsilon > 0$,

$$\frac{C_1(1-\epsilon)}{\|k\|^{n-\beta}} < m(\nu_k)+\alpha < \frac{C_1(1+\epsilon)}{\|k\|^{n-\beta}},$$

for sufficiently large $\|k\|$. Thus

$$\|k\|^{n-\beta}(1-e^{-\frac{C_1(1-\epsilon)t}{\|k\|^{n-\beta}}}) < \|k\|^{n-\beta}(1-e^{-(m(\nu_k)+\alpha)t}) < \|k\|^{n-\beta}(1-e^{-\frac{C_1(1+\epsilon)t}{\|k\|^{n-\beta}}}),$$

and consequently,

$$C_1(1-\epsilon)t < \lim_{\|k\|\to\infty} \|k\|^{n-\beta}(1-e^{-(m(\nu_k)+\alpha)t}) < C_1(1+\epsilon)t.$$

Since ϵ is arbitrary, we obtain

$$\lim_{\|k\|\to\infty} \|k\|^{n-\beta}(1-e^{-(m(\nu_k)+\alpha)t}) = C_1 t,$$

and thus,

$$(1-e^{-(m(\nu_k)+\alpha)t})e^{m(\nu_k)t} \sim \frac{C_1 t e^{-\alpha t}}{\|k\|^{n-\beta}}.$$

The proof is similar for the case $\beta \le \frac{n-1}{2}$. □

Conditions on f and β to guarantee the continuity of v are given in the following result.

Theorem 27. *Let $v(x,t)$ be as in (37) and assume that $\hat{f}_k$ satisfies*

$$\hat{f}_k \sim \begin{cases} \dfrac{1}{\|k\|^{\beta+\epsilon}} & \text{if } \dfrac{n-1}{2} < \beta < n, \\ \dfrac{1}{\|k\|^{\frac{n-1}{2}+\zeta}} & \text{if } \beta \le \dfrac{n-1}{2}, \end{cases}$$

for $\epsilon, \zeta > 0$. Then, $v(x,t)$ is continuous.

Proof. For $\frac{n-1}{2} < \beta < n$ with $\hat{f}_k \sim (1/(\|k\|^{\beta+\epsilon}))$, by using Lemma 26, we have

$$\hat{v}_k = \hat{f}_k(1-e^{-(m(\nu_k)+\alpha)t})e^{m(\nu_k)t} \sim \frac{Cte^{-\alpha t}}{\|k\|^{n+\epsilon}}.$$

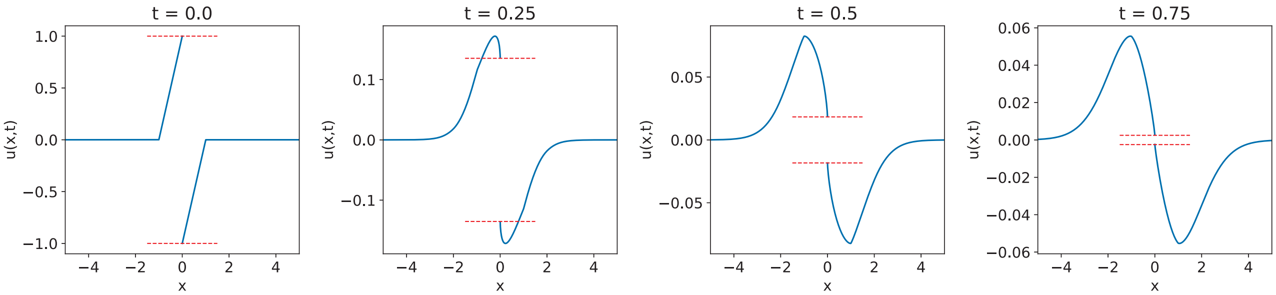

Figure 1. The slow decay of a discontinuity in the nonlocal diffusion equation with $\beta < n$.

Similarly, for $\beta \le \frac{n-1}{2}$ with $\hat{f}_k \sim (1/(\|k\|^{\frac{n-1}{2}+\zeta}))$, we have

$$\hat{v}_k = \hat{f}_k(1 - e^{-(m(\nu_k)+\alpha)t})e^{m(\nu_k)t} \sim \frac{Cte^{-\alpha t}}{\|k\|^{n+\zeta}}.$$

By Proposition 3.3.12 in [18], we conclude that, for $t > 0$, $v(\,\cdot\,, t)$ is continuous in both cases. □

The following theorem summarizes the results in this section.

Theorem 28. *Let $\beta < n$ and let u be as given in (37) and assume that*

$$\hat{f}_k \sim \begin{cases} \dfrac{1}{\|k\|^{\beta+\epsilon}} & \text{if } \frac{1}{2}(n-1) < \beta < n, \\ \dfrac{1}{\|k\|^{\frac{n-1}{2}+\zeta}} & \text{if } \beta \le \frac{1}{2}(n-1), \end{cases}$$

for some $\epsilon, \zeta > 0$. Then, if f is discontinuous at x then u is discontinuous at x.

Corollary 29. *If $f \in L^2(T)$ is piecewise continuous, then u is piecewise continuous and f and u share the same locations of jumps. Furthermore, the magnitude of a jump decays as t increases.*

This is an immediate consequence of Theorem 28, since, for a piecewise continuous function $f \in L^2(T)$, we have $\hat{f}_k \sim \frac{C}{|k|}$, for some $C > 0$.

A one-dimensional example for the propagation of a discontinuity in the nonlocal diffusion equation is described below. Figure 1 shows the results of a numerical solution to the periodic nonlocal diffusion problem $u_t = L^{\delta,\beta}u$ on the interval $(-10, 10)$ with $\delta = 1$, $\beta = \frac{1}{3}$, and initial condition

$$u(x,0) = \begin{cases} x+1 & \text{if } -1 < x \le 0, \\ x-1 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

In Figure 1, function values for $x < 0$ were plotted separately from those for $x > 0$ so that the jump is apparent. The dashed lines indicate the values $\pm e^{-\alpha t}$, showing the theoretical extremes of the jump.

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