

CONVERGENCE OF THE BOUNDARY INTEGRAL METHOD FOR INTERFACIAL STOKES FLOW

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ABSTRACT. Boundary integral numerical methods are among the most accurate methods for interfacial Stokes flow, and are widely applied. They have the advantage that only the boundary of the domain must be discretized, which reduces the number of discretization points and allows the treatment of complicated interfaces. Despite their popularity, there is no analysis of the convergence of these methods for interfacial Stokes flow. In practice, the stability of discretizations of the boundary integral formulation can depend sensitively on details of the discretization and on the application of numerical filters. We present a convergence analysis of the boundary integral method for Stokes flow, focusing on a rather general method for computing the evolution of an elastic capsule or viscous drop in 2D strain and shear flows. The analysis clarifies the role of numerical filters in practical computations.

1. INTRODUCTION

Boundary integral (BI) methods are among the most popular methods for computing interfacial fluid flow. They have been widely applied to compute the evolution of interfaces in potential flow, including the Kelvin-Helmholtz and Rayleigh-Taylor instabilities [5], [26], Hele-Shaw flow [23], [31], and water waves [7], [10]. They have also been extensively applied in Stokes flow to simulate the evolution of drops, bubbles, elastic capsules and vesicles [21], [40], [45], [49]. Boundary integral methods have been particularly important in micro- and bio-fluidic applications in which viscous forces are dominant over inertial ones. Such flows can therefore be accurately modeled by the Stokes equations. Overviews of the BI method applied to Stokes flow in micro- and bio-fluidic applications are given in [25], [42].

The main advantage of boundary integral methods is that they only involve surface quantities, thereby reducing the dimension of the problem. This simplifies the handling of complex geometries and reduces the number of discretization points. Another significant advantage is that they can be made to have high accuracy. Boundary integral methods use a sharp interface formulation, which allows accurate treatment of the discontinuity in normal stress due to surface tension forces or elastic membrane stress at the interface. Spectrally accurate discretizations of boundary integral formulations are now routinely implemented for 2D interfacial Stokes flow of drops and bubbles, see, e.g., [15], [27], [28], [38], [39], [52] and references therein. Spectral or high-order boundary integral methods for inextensible vesicles or elastic capsules are provided by [21], [35], [44], and [49]. High order

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accurate discretizations of axisymmetric and 3D flow problems, although still a subject of current research, are increasingly common [16], [46], [48], [50], [53]. As a result, boundary integral methods are a good choice in problems that demand high accuracy.

One of the greatest challenges in the practical implementation of boundary integral methods for time-evolution problems is that they are sensitive to numerical instabilities. If left uncontrolled, these instabilities will dominate and adversely affect the accuracy of computations. Numerical instabilities have been commonly observed in boundary integral computations for inviscid interfacial flow [8], [10], [17], [33], [43], and in spectrally accurate computations for Stokes flow [15], [27], [28], [38], [39], [52]. They are typically controlled by application of numerical filtering or by dealiasing through spectral padding. Computations of interfacial flow with surface or elastic membrane tension can be even more sensitive to numerical instabilities due to the presence of nonlinear terms with high-order spatial derivatives. A major aim of the analysis presented here is to clarify the role of spatial discretization and filtering in controlling these aliasing-type instabilities for interfacial Stokes flow.

There are relatively few analyses of the stability and convergence of BI methods for multi-phase flow simulations. This is in part due to the difficulty of the analysis involving nonstandard (nonlocal) governing equations. Hou, Lowengrub and Krasny [22] prove the convergence of a BI method for vortex sheets in inviscid flow without surface tension. Baker and Nachbin [6] identify common reasons for numerical instability in the same problem as [22] when surface tension is present. In fundamental work, Beale, Hou and Lowengrub [10] prove convergence in the fully nonlinear regime of a BI method for water waves in two dimensions both with and without surface tension. They discovered that delicate balances must be sustained among terms in singular integrals and derivatives at the discrete level in order to preserve numerical stability. They also noticed that numerical filtering is necessary at certain places to prevent the discretization from producing new instabilities in the high modes. Cenicerros and Hou [14] extend the analysis of [10] to include two-phase flow and surface tension, and Hou and Zhang [24] generalize the analysis of [10] to 3D. Other convergence analyses have been performed for Darcy-law flow problems. Hao et al. [20] show convergence of a boundary integral method for a generalized Darcy-law model of 2D tumor growth. Ambrose, Liu and Siegel [2] prove convergence of a boundary integral method for 3D Darcy-law flow with surface tension [4].

Despite the significance of the above-mentioned convergence studies for BI methods in inviscid and Darcy-law interfacial flow, there is no convergence analysis that we are aware of for the important case of interfacial Stokes flow. In this paper, we provide such an analysis. The main difficulty of this analysis, compared to previous convergence studies for water waves, is a more complicated boundary integral formulation for the Stokes problem, and the presence of high derivatives in the boundary condition for an elastic membrane.

In the analysis of the stability of our method, we make significant use of the stabilizing effects of the highest derivative or leading-order terms (so-called parabolic smoothing) to control lower-order terms. As in the water wave problem of [10], we find that a targeted application of numerical filtering is necessary to prove

stability in the Stokes-interface problem. This is consistent with numerical implementations of spectrally accurate methods for the evolution of drops, bubbles and elastic capsules in Stokes flow, e.g., [21], [27], [38], [39], [52], which also find the need for some form of numerical filtering or dealiasing for stability. However, to make use of the parabolic smoothing in the elastic capsule problem and to minimize the amount of numerical filtering, we find it important that filtering *not* be applied to the leading-order or highest derivative terms. Based on our analysis, we present a numerical scheme that utilizes a minimal amount of filtering yet is provably stable, even in the fully nonlinear regime. Note that the specific filtering applied in our method is not unique, and other filtering techniques (e.g., zero padding) may give stable schemes.

In our convergence analysis, we consider a rather general boundary integral formulation which governs the time-dependent evolution of a Hookean elastic capsule in 2D Stokes flow for an externally imposed straining or shearing flow, but also encompasses the deformation of a drop or bubble with constant surface tension. An elastic capsule is a drop or bubble that is enclosed by a thin, elastic membrane and suspended in an external fluid. It serves as a simple mechanical model of a cell or vesicle that is deformed by a fluid flow. Numerical studies of capsules in fluid flow performed with various membrane constitutive laws include [11], [16], [21], [49], [51]. If the parameter governing membrane bending stress is set to zero, one recovers the governing equations and numerical method for two-fluid flow about a drop or bubble with constant interfacial tension. This case is therefore also included in our convergence analysis.

The algorithm we analyze is closely based on a spectrally accurate numerical method for the evolution of a drop or bubble in an extensional flow that was developed by Kropinski [27], [28] and is extensively used, see, e.g., [38], [39], [52]. The method of Kropinski makes use of a complex-variable description of the problem known as the Sherman-Lauricella formulation, as well as a construction due to Hou, Lowengrub and Shelley [23] in which the interface is prescribed by its tangent angle $\theta(\alpha, t)$ and an equal-arclength parameter α , so that $\frac{\partial s}{\partial \alpha}$ is constant in α (here $s(\alpha, t)$ measures arclength from a reference point at $\alpha = 0$). This so-called arclength-angle formulation was originally developed to overcome numerical stiffness in the time-discretization, but is also convenient for analysis [1], [3]. We further adapt this formulation to Stokes flow with elastic surfaces. A complication of this approach is that the location of material points on the interface must be tracked to determine the elastic or “stretching” tension in the membrane. This is not readily available from the equal arclength parameterization. Following [21], we introduce a backwards map $\alpha_0(\alpha, t)$ which gives the location of the material parameter α_0 in terms of the equal arclength parameter α , and derive a time evolution equation for $\alpha_0(\alpha, t)$. The discrete version of this time-evolution equation is then incorporated into the energy estimates to show stability of our method. The algorithm presented here also generalizes a spectrally accurate method for the evolution of an elastic capsule in an extensional flow, which was developed in [21], to include a viscous interior fluid and nonzero membrane bending stress.

Following the earlier convergence studies for inviscid flow, our analysis is discrete in space and continuous in time. The main result is contained in Theorem 5.1, which proves that the numerical method with filtering converges to the exact solution with spectral accuracy. The convergence proof follows the general framework of [10] and

uses energy estimates in discrete Sobolev spaces. It relies on the smoothness of the underlying solution to the continuous evolution problem, see [32], [37] for relevant results on the existence and regularity of the continuous problem. When there is a jump in viscosity between the fluid in the interior of the capsule or drop and the exterior fluid, an additional system of Fredholm integral equations must be solved to obtain the density in the boundary integrals of the Sherman-Lauricella formulation. Analysis of the discrete version of this system of integral equations presents an additional complication of the proof. However, we are able to show that the discrete system is invertible and that the inverse operator is bounded for sufficiently small viscosity contrast between the interior and exterior fluids. This provides a convergence proof of the full evolution problem, including viscosity contrast, as long as that contrast is sufficiently small.

The governing equations for our problem are presented in Section 2, and the BI formulation is given in Section 3. For our BI formulation, we present in Section 4 a spectrally accurate numerical discretization. The statement of the main convergence theorem, Theorem 5.1, is given in Section 5. Several preliminary lemmas are presented in Section 6 which provide error estimates on numerical differentiation, integration, and filtering operators. We then prove consistency of our numerical method in Section 7. Some preliminary estimates used in the proof of stability are given in Sections 8 and 9. Evolution equations for the errors are presented in Section 10, and the proof of stability (and hence Theorem 5.1) by energy estimates in the special case of viscosity matched fluids and nonzero bending stress is given in Section 11. The case of unequal viscosities is discussed in Section 12. Modifications to the convergence analysis for the drop problem, with zero membrane bending stress and constant interfacial tension, are given in Section 13. Concluding remarks are provided in Section 14. Proofs of critical lemmas and estimates of nonlinear terms in the variation of velocity are given in Appendices A and B.

2. PROBLEM FORMULATION

We present the governing equations for a single elastic capsule in 2D Stokes flow. The exterior fluid domain is denoted by Ω , and we use a superscript ‘int’ for variables and parameters in the interior fluid. The membrane surface is given by $\partial\Omega = \gamma$.

The drop and exterior fluid are assumed to have the same density, so gravitational effects are absent. On γ the unit normal vector $\mathbf{n}$ points toward the exterior fluid. The unit tangent $\mathbf{t}$ points in the direction such that the interior fluid is to the right as γ is traversed clockwise. We define an angle θ measured counterclockwise positive from the positive x -axis to $\mathbf{t}$. The geometry is illustrated in Figure 1. The local curvature of the interface is $\kappa = -\frac{\partial\theta}{\partial s}$ and is positive when the shape is convex. Here, s is an arclength parameter that increases as γ is traversed clockwise.

In dimensionless form, the Stokes equations governing fluid flow are

$$\begin{aligned} (1) \quad & \begin{cases} \Delta \mathbf{u} = \nabla p, \nabla \cdot \mathbf{u} = 0, & \mathbf{x} \in \Omega, \\ \lambda \Delta \mathbf{u}^{\text{int}} = \nabla p^{\text{int}}, \nabla \cdot \mathbf{u}^{\text{int}} = 0, & \mathbf{x} \in \Omega^{\text{int}}, \end{cases} \\ (2) \quad & \end{aligned}$$

where $p(\mathbf{x})$ and $\mathbf{u}(\mathbf{x})$ are the pressure and velocity fields and $\lambda = \frac{\mu^{\text{int}}}{\mu}$ is the viscosity ratio. The fluid velocity is taken to be continuous across the interface, i.e., $\mathbf{u}(\mathbf{x}) = \mathbf{u}^{\text{int}}(\mathbf{x})$ for $\mathbf{x} \in \gamma$.

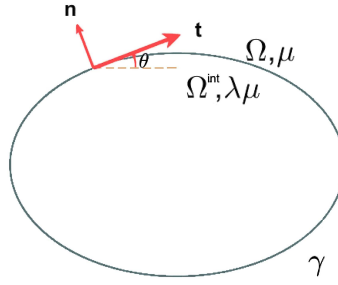


FIGURE 1. Fluid drop with viscosity $\lambda\mu$ occupying region Ω^{int} is immersed in a fluid with viscosity μ occupying region Ω

The area enclosed by the capsule is conserved, and lengths are nondimensionalized by the radius R of the circular capsule with the same area. Velocities are nondimensionalized by U , where U will be specified below. Time is nondimensionalized by $\frac{R}{U}$, and pressure by $\frac{U\mu}{R}$. At $t = 0$ the capsule can have arbitrary shape and membrane tension.

The no slip condition on the capsule surface is given by

$$(3) \quad \frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t) \quad \text{for } \mathbf{x} \in \gamma,$$

Equation (3) satisfies the kinematic condition that $\frac{d\mathbf{x}}{dt} \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n}$ on γ . The far-field boundary condition is taken to be a general incompressible linear flow

$$(4) \quad \lim_{|\mathbf{x}| \rightarrow \infty} \mathbf{u}(\mathbf{x}) = \mathbf{u}_\infty(\mathbf{x}) = \begin{pmatrix} Q & B + \frac{J}{2} \\ B - \frac{J}{2} & -Q \end{pmatrix} \mathbf{x} + O(|\mathbf{x}|^{-2}) \quad \text{as } \mathbf{x} \rightarrow \infty,$$

where the dimensionless parameters (Q, B, J) are equal to their dimensional counterparts Q_∞ , etc., times the time scale $\frac{R}{U}$; i.e. $(Q, B, J) = \frac{R}{U}(Q_\infty, B_\infty, J_\infty)$. The far-field flow is a pure strain if $B = J = 0$, and a linear shear flow if $Q = 0$ and $J = 2B$. At the elastic membrane interface, we have the additional boundary condition that the total interfacial stress $\mathbf{f}$ is balanced by the jump in fluid stress across the interface,

$$(5) \quad [T \cdot \mathbf{n}] = \mathbf{f},$$

where $T = -p + 2E_{ij}$ and $T^{\text{int}} = -p^{\text{int}} + 2\lambda E_{ij}^{\text{int}}$, and where

$$(6) \quad E_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

is the stress tensor. Here $[\cdot]$ denotes the jump

$$(7) \quad [\mathbf{g}] = \mathbf{g} - \mathbf{g}^{\text{int}} \quad \text{for } \mathbf{x} \in \gamma.$$

An expression for the interfacial stress $\mathbf{f}$ on the right hand side of (5) is obtained in [41] by an analysis of interfacial forces and torques. The result is given in equation (3.16) of [41], which in our notation is

$$(8) \quad \mathbf{f} = -\frac{\partial}{\partial s} (S\mathbf{t} + q_B\mathbf{n}),$$

where $\mathcal{S} = \mathcal{S}(s)$ is the membrane tension in terms of an arclength parameter s , and $q_B(s) = \frac{dm_B}{ds}$ with $m_B = m_B(s)$ the bending moment. The constitutive equation for the bending moment m is assumed to be the simple linear relation

$$(9) \quad m_B(s) = \kappa_B \kappa(s),$$

where κ_B is the (dimensionless) bending modulus, and $\kappa(s)$ is the interfacial curvature. For the sake of simplicity, we consider a membrane with a Hookean or linear elastic response, for which the dimensional tension is given by [40]

$$(10) \quad \tilde{\mathcal{S}} = E(\eta - 1), \quad \eta = \frac{\partial s}{\partial s_R}.$$

Here η is the stretch ratio between arclength s of the membrane at time t and arclength s_R in a reference configuration in which there is no tension in the membrane, and E is the modulus of elasticity. The tension is nondimensionalized by E , so that in dimensionless form

$$(11) \quad \mathcal{S} = \eta - 1.$$

We also now define the characteristic velocity which is used for nondimensionalization as $U = \frac{E}{\mu}$.

3. BOUNDARY INTEGRAL FORMULATION

For our boundary integral method, we adapt the Sherman-Lauricella formulation [18], [27], [36] to the capsule-membrane problem. This is a complex variable formulation for which the primitive variables are expressed in terms of an integral over a complex density that is defined on the drop interface and satisfies a second kind Fredholm equation. It has been extensively used to solve Stokes equations for multi-phase fluid flow, see e.g., [27], [28], [29], [38], [39], [52].

In the Sherman-Lauricella formulation, the complex fluid velocity $(u_1 + iu_2)(\tau, t)$ for $\tau = x_1 + ix_2$ on the time evolving interface γ is written in terms of Cauchy-type integrals that contain a single complex density $\omega(\zeta, t)$ [27], [52]:

$$(12) \quad \begin{aligned} u|_\gamma = (u_1 + iu_2)|_\gamma = & -\frac{1}{2\pi} \text{P.V.} \int_\gamma \omega(\zeta, t) \left(\frac{d\zeta}{\zeta - \tau} + \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{\tau}} \right) \\ & + \frac{1}{2\pi} \int_\gamma \overline{\omega(\zeta, t)} d\frac{\zeta - \tau}{\bar{\zeta} - \bar{\tau}} + (Q + iB)\bar{\tau} - \frac{iJ}{2}\tau. \end{aligned}$$

The apparent singularity for ζ near τ in the second integral is removable, but in the first integral the P.V. indicates that it is to be interpreted as a Cauchy principal value integral. Note that we use a Roman i for $\sqrt{-1}$, to distinguish from a script i used later for spatial indices.

The complex density ω satisfies an integral equation which is obtained by modifying the derivation of [27], [52] for drops to incorporate the more general elastic membrane stress in (8). If we denote the interface γ by $\tau(s, t)$, then the final form that the equation takes can be written as

$$(13) \quad \begin{aligned} \omega(\tau, t) + \frac{\beta}{2\pi i} \int_\gamma \omega(\zeta, t) d \ln \left(\frac{\zeta - \tau}{\bar{\zeta} - \bar{\tau}} \right) + \frac{\beta}{2\pi i} \int_\gamma \overline{\omega(\zeta, t)} d\frac{\zeta - \tau}{\bar{\zeta} - \bar{\tau}} \\ = -\frac{\chi}{2} [(\mathcal{S} + \kappa_B \kappa^2)\tau_s - \kappa_B \tau_{ss}] - \beta(B - iQ)\bar{\tau} - 2\beta H(t), \end{aligned}$$

where

$$\beta = \frac{1 - \lambda}{1 + \lambda} \text{ and } \chi = \frac{1}{1 + \lambda},$$

and a subscript s denotes derivative. The expression in brackets on the right hand side of (13) is the integral with respect to s of the interfacial stress in (8), represented using complex variables. When $\kappa_B = 0$ and $\mathcal{S} = \text{constant}$, (13) reduces to the corresponding equation for a drop interface [52]. The apparent singularity at $\zeta = \tau$ in the two integrals on the left-hand side is removable. We set

$$(14) \quad H(t) = \frac{1}{2} \int_{\gamma} \omega(\zeta, t) ds;$$

as demonstrated in [27], this choice removes a rank deficiency of the integral equation (13) in the limit $\lambda = 0$ of an inviscid drop and is consistent with $H(t) \equiv 0$, which is a consequence of the constant area of the interior region Ω^{int} .

The fluid velocity on the interface, in terms of its normal and tangential components u_n and u_s , is $\mathbf{u} = u_n \mathbf{n} + u_s \mathbf{t}$, where the complex counterparts of the unit vectors $\mathbf{n}$ and $\mathbf{t}$ are n and s_T with $s_T = -in = \partial_s \tau$. It follows that

$$(15) \quad u_n = \text{Re}\{u|_{\gamma} \bar{n}\} \quad \text{and} \quad u_s = -\text{Im}\{u|_{\gamma} \bar{n}\},$$

on the interface γ .

For the numerical discretization of (13), we introduce an equal arclength parametrization of the interface γ . This is constructed following Hou, Lowengrub and Shelley [23]. The spatial parametrization of the interface is given by $\alpha \in [-\pi, \pi]$, and a point τ on the interface has Cartesian coordinates (x_1, x_2) , so that $\tau(\alpha, t) = x_1(\alpha, t) + ix_2(\alpha, t)$. The unit tangent vector s_T and normal n in complex form are $s_T = \frac{\partial \tau}{\partial s} = \frac{\tau_{\alpha}}{s_{\alpha}} = \exp(i\theta)$ and $n = is_T = i \exp(i\theta)$. Differentiation of $\tau_{\alpha} = s_{\alpha} e^{i\theta}$ with respect to time implies that

$$(16) \quad \tau_{\alpha t} = s_{\alpha t} e^{i\theta} + s_{\alpha} \theta_t i e^{i\theta}.$$

When $\tau = \tau_m$ is a material point on the interface its velocity is equal to the local fluid velocity, per (3), so that differentiation with respect to time implies that

$$(17) \quad \frac{d\tau_m}{dt} = u_n i e^{i\theta} + u_s e^{i\theta},$$

where the subscript m is used to denote material point.

However, the shape of the evolving interface is determined by the normal velocity component u_n alone. Although u_s has physical meaning as the tangential component of the fluid velocity, if u_s is replaced by any other smooth function $\phi_s(\alpha, t)$ in (17), then τ still lies on the interface but is no longer a material point, and the role of ϕ_s is simply to implement a specific choice of the interface parametrization via α , without changing the interface shape or evolution. The interfacial velocity generated by using ϕ_s instead of u_s is denoted by v , and has complex form given by

$$(18) \quad v = \frac{d\tau}{dt} = u_n i e^{i\theta} + \phi_s e^{i\theta}.$$

Differentiation of (18) with respect to α gives a second relation for $\tau_{\alpha t}$,

$$(19) \quad \tau_{\alpha t} = ((\phi_s)_{\alpha} - u_n \theta_{\alpha}) e^{i\theta} + ((u_n)_{\alpha} + \phi_s \theta_{\alpha}) i e^{i\theta}.$$

Equating (16) and (19), we have

$$(20) \quad s_{\alpha t} = (\phi_s)_\alpha - u_n \theta_\alpha,$$

$$(21) \quad \theta_t = \frac{1}{s_\alpha} ((u_n)_\alpha - \phi_s \theta_\alpha),$$

where γ is now described parametrically by $s = s(\alpha, t)$ and $\theta = \theta(\alpha, t)$ instead of $x_1 = x_1(\alpha, t)$ and $x_2 = x_2(\alpha, t)$.

The equal arclength frame is chosen by setting $s_\alpha = s_\alpha(t)$ to be spatially constant along the interface, so that it varies in time only. Then since s_α is always equal to its mean around γ , it follows from (20) that

$$(22) \quad s_{\alpha t} = (\phi_s)_\alpha - u_n \theta_\alpha = -\frac{1}{2\pi} \int_{-\pi}^{\pi} u_n \theta_{\alpha'} d\alpha'.$$

Integration of the second of these equations with respect to α implies that

$$(23) \quad \phi_s(\alpha, t) = \partial_\alpha^{-1} (u_n \theta_\alpha - \langle u_n \theta_\alpha \rangle),$$

where

$$(24) \quad \langle f \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\alpha') d\alpha'$$

is the mean of f , ∂_α^{-1} is defined for a function f with zero mean as

$$(25) \quad \partial_\alpha^{-1} f = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{\hat{f}_k}{ik} e^{ik\alpha},$$

and $\hat{f}_k$ are the Fourier coefficients of f . In (23) an arbitrary function of time has been set so that $\phi_s(\alpha, t)$ has zero mean.

When (23) is substituted into (20) and (21), the system by which the dynamics of the interface is tracked becomes

$$(26) \quad s_{\alpha t} = -\frac{1}{2\pi} \int_{-\pi}^{\pi} u_n \theta_{\alpha'} d\alpha',$$

$$(27) \quad \theta_t = \frac{1}{\sigma} \left[\theta_\alpha \partial_\alpha^{-1} (u_n \theta_\alpha - \langle u_n \theta_\alpha \rangle) + (u_n)_\alpha \right].$$

At each time step (26) and (27) are integrated forward in time, and (s_α, θ) are mapped to the Cartesian coordinates (x_1, x_2) of points on γ . The map is given by integration of $\tau_\alpha = s_\alpha e^{i\theta}$ with respect to α and is

$$(28) \quad x_1(\alpha, t) = x_{1c}(t) + s_\alpha(t) \partial_\alpha^{-1} (\cos(\theta(\alpha', t))),$$

$$(29) \quad x_2(\alpha, t) = x_{2c}(t) + s_\alpha(t) \partial_\alpha^{-1} (\sin(\theta(\alpha', t))),$$

where $(x_{1c}(t), x_{2c}(t))$ is the constant Fourier mode of $(x_1(\alpha, t), x_2(\alpha, t))$, which is evolved from (18) as

$$(30) \quad \frac{d}{dt} (x_{1c}(t) + ix_{2c}(t)) = \hat{v}_0(t) = \langle v \rangle,$$

where $\hat{v}_0(t)$ is the $k = 0$ Fourier mode of interface velocity v .

Membrane tension.

A formula for the membrane tension $\mathcal{S}(\alpha, t)$ in terms of interface shape $\tau(\alpha, t)$ and the initial tension $\mathcal{S}(\alpha, 0)$ is required to close the system of equations. We obtain this formula by adapting the construction in [21].

Recall that $\tau(\alpha, t)$ is a general nonmaterial parameterization of the interface at time t . Introduce a parameterization $\tau(\alpha_p, 0)$ of the initial profile in terms of a Lagrangian or material coordinate α_p , and denote the location of the same material point at time $t > 0$ by $\tau(\alpha_m(\alpha_p, t), t)$; this serves as a definition of a ‘forward’ map $\alpha_m(\alpha_p, t)$. We also define the ‘backward’ map $\alpha_0(\alpha, t)$ such that $\tau(\alpha_0(\alpha, t), 0)$ is the location at $t = 0$ of the material point that at time t is located at $\tau(\alpha, t)$. It follows that α_m and α_0 are one-to-one and inverses.

A formula for $\mathcal{S}(\alpha, t)$ that gives the tension in terms of the initial state of the membrane and the backwards map $\alpha_0(\alpha, t)$ is given by (4.14) in [21]. In our notation, this formula takes the form

$$(31) \quad \mathcal{S}(\alpha, t) = \frac{s_\alpha(t)}{s_\alpha(0)\alpha'_0(\alpha, t)}(1 + \mathcal{S}(\alpha_0(\alpha, 0), 0)) - 1,$$

where $\alpha'_0(\alpha, t) = \frac{\partial \alpha_0}{\partial \alpha}(\alpha, t)$ and we have made use of the fact that $s_\alpha(\alpha, t) = s_\alpha(t)$ is spatially independent.

The formula for the membrane tension therefore requires an equation for the backward map $\alpha_0(\alpha, t)$. First, note that, by definition of α_m and α_p , the condition for the motion of a material particle becomes

$$(32) \quad \frac{d}{dt}\tau(\alpha_m(\alpha_p, t), t) = u,$$

that is,

$$(33) \quad \left. \frac{\partial \tau}{\partial t} \right|_\alpha + \frac{\partial \tau}{\partial \alpha} \frac{\partial \alpha_m}{\partial t} \Big|_{\alpha_p} = u,$$

at $\alpha = \alpha_m(\alpha_p, t)$. An expression for $\left. \frac{\partial \tau}{\partial t} \right|_\alpha$ is given by (18), and substituting this into (33) yields the evolution equation for the forward map $\alpha_m(\alpha_p, t)$

$$(34) \quad \left. \frac{\partial \alpha_m}{\partial t} \right|_{\alpha_p} = \frac{1}{\tau_\alpha} [u - (u_n i e^{i\theta} + \phi_s e^{i\theta})]$$

at $\alpha = \alpha_m(\alpha_p, t)$.

The evolution of the backward map $\alpha_0(\alpha, t)$ is obtained by noting that α_m and α_0 are inverses, so that differentiation of the identity $\alpha = \alpha_m(\alpha_0(\alpha, t), t)$ with respect to time keeping α fixed implies

$$(35) \quad \left. \frac{\partial \alpha_m}{\partial t} \right|_{\alpha_p} + \frac{\partial \alpha_p}{\partial \alpha_m} \frac{\partial \alpha_0}{\partial t} \Big|_\alpha = 0,$$

where we have set $\alpha_p = \alpha_0(\alpha, t)$ in the first two derivatives. Differentiation of the same identity with respect to α keeping t fixed gives

$$(36) \quad \frac{\partial \alpha_m}{\partial \alpha_p} = \left(\frac{\partial \alpha_0}{\partial \alpha} \right)^{-1}.$$

Eliminating α_m in favor of α_0 in (34), (35) and (36) gives the initial value problem for the backward map,

$$\begin{aligned} \left. \frac{\partial \alpha_0}{\partial t} \right|_{\alpha} &= \frac{\partial \alpha_0}{\partial \alpha} \frac{1}{\tau_{\alpha}} [u_n i e^{i\theta} + \phi_s e^{i\theta} - u] \\ (37) \quad &= \frac{\partial \alpha_0}{\partial \alpha} \frac{1}{\tau_{\alpha}} [(\phi_s - u_s) e^{i\theta}], \text{ with } \alpha_0(\alpha, 0) = \alpha, \end{aligned}$$

which together with (31) is the main result of this subsection. Note that the right hand sides of (34) and (37) are real valued.

In summary, the main equations that govern capsule evolution are given by (12)-(13), (26)-(30), (31), and (37). A spectrally accurate numerical method for solving this system of equations is presented in the next section.

4. NUMERICAL METHOD

We construct a continuous in time, discrete in space numerical scheme for the evolution equations by providing rules to approximate the spatial derivatives and singular integrals.

The spatial variable α is discretized as

$$\alpha_j = jh \text{ for } j = -\frac{N}{2} + 1, \dots, \frac{N}{2}, \text{ with mesh size } h = \frac{2\pi}{N},$$

and N assumed to be even. Define a discrete Fourier transform of a periodic function f whose values are known at α_j by

$$(38) \quad \hat{f}_k = \frac{1}{N} \sum_{j=-\frac{N}{2}+1}^{\frac{N}{2}} f(\alpha_j) e^{-ik\alpha_j}, \text{ for } k = -\frac{N}{2} + 1, \dots, \frac{N}{2},$$

with the inverse transform given by

$$(39) \quad f(\alpha_j) = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \hat{f}_k e^{ik\alpha_j}, \text{ for } j = -\frac{N}{2} + 1, \dots, \frac{N}{2}.$$

Spatial derivatives of f are computed using a pseudo-spectral approximation, which is denoted by $S_h f$ and defined by

$$(40) \quad \widehat{(S_h f)}_k = ik \hat{f}_k, \text{ for } k = -\frac{N}{2} + 1, \dots, \frac{N}{2} - 1,$$

$$(41) \quad = 0, \text{ for } k = \frac{N}{2}.$$

Due to the asymmetry of the discrete Fourier transform, we zero out the $k = \frac{N}{2}$ mode of $\widehat{(S_h f)}_k$. This will be important for stability.

Sometimes we need to apply numerical filtering to the discrete solution. Indeed, this will be critical for the stability of our method. Numerical filtering f_j^p is defined in Fourier space following [10] as

$$(42) \quad \widehat{(f^p)}_k = \rho(kh) \hat{f}_k,$$

where ρ is a cutoff function with the following properties:

- (i) $\rho(-x) = \rho(x)$; $\rho(x) \geq 0$,
- (ii) $\rho(x) \in C^r$; $r > 2$,
- (iii) $\rho(\pm\pi) = \rho'(\pm\pi) = 0$,
- (iv) $\rho(x) = 1$ for $|x| \leq \mu\pi$, $0 < \mu < 1$.

Condition (iv) ensures the spectral accuracy of the filtering. We also define a filtered derivative operator D_h by

$$(43) \quad D_h f = (S_h f)^p.$$

We denote by $\theta(\alpha_j)$, $\omega(\alpha_j)$, $\tau(\alpha_j)$, etc. the exact continuous solution evaluated at grid points α_j , and by θ_j , ω_j , τ_j , etc. the discrete approximation. Also, we use $\sigma(t)$ to denote the numerical approximation of $s_\alpha(t)$.

Discrete equations for velocity u and density ω .

The interface contour γ is parameterized by $\tau(\alpha, t)$. If we set $\tau = \tau(\alpha)$ (omitting the time dependence) and $\zeta = \tau(\alpha')$, then the integral equation (13) for ω becomes

$$(44) \quad \omega(\alpha) + \beta \int_{-\pi}^{\pi} F(\alpha, \alpha') d\alpha' = g(\alpha),$$

where

$$(45) \quad \begin{aligned} F(\alpha, \alpha') &= \frac{\omega(\alpha')}{2\pi i} \left(2i \operatorname{Im} \left[\frac{\tau_\alpha(\alpha')}{\tau(\alpha') - \tau(\alpha)} \right] \right) \\ &+ \frac{\overline{\omega(\alpha')}}{2\pi i} \left(\frac{\tau_\alpha(\alpha')}{\tau(\alpha') - \tau(\alpha)} - \frac{(\tau(\alpha') - \tau(\alpha))}{(\tau(\alpha') - \tau(\alpha))^2} \overline{\tau_\alpha(\alpha')} \right), \end{aligned}$$

and where we have written $\omega(\alpha)$ for $\omega(\tau, t)$ and $\omega(\alpha')$ for $\omega(\zeta, t)$. The function $g(\alpha)$ represents the right hand side of (13) and can be written in terms of θ , s_α and τ as

$$(46) \quad g(\alpha) = -\frac{\chi}{2} \left(S(\alpha) e^{i\theta(\alpha)} - \frac{\kappa_B \theta_{\alpha\alpha}(\alpha)}{s_\alpha^2} i e^{i\theta(\alpha)} \right) - \beta(B - iQ)\bar{\tau} - 2\beta H(t).$$

Although the apparent singularity $\tau(\alpha') = \tau(\alpha)$ is removable, we shall nonetheless discretize (44) using alternate point trapezoidal rule [22],

$$(47) \quad \int_{-\pi}^{\pi} f(\alpha, \alpha') d\alpha' \approx \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} f(\alpha_i, \alpha_j) (2h).$$

This quadrature rule is normally used for singular integrals, but for convenience we shall also apply it here for smooth kernels since it precludes the need for separate, analytical kernel evaluations at $\alpha = \alpha'$.

The real and imaginary parts of (44) form a system of Fredholm integral equations for $\omega_1 = \operatorname{Re}(\omega)$ and $\omega_2 = \operatorname{Im}(\omega)$. To write the corresponding discrete system, first decompose the density ω_i as the sum of the explicit quantity $\mathbf{g}_i$ and a ‘perturbation’ $\tilde{\omega}_i$:

$$(48) \quad \omega_i = \tilde{\omega}_i + \mathbf{g}_i,$$

where

$$(49) \quad \boldsymbol{\omega}_i = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_i, \quad \tilde{\boldsymbol{\omega}}_i = \begin{bmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \end{bmatrix}_i, \quad \mathbf{g}_i = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}_i,$$

with $g_1 = \operatorname{Re}(g)$ and $g_2 = \operatorname{Im}(g)$ (cf. (46)). Introduce the discrete function $g_i = g_{1i} + ig_{2i}$ which is the discretization of (46):

$$(50) \quad g_i = -\frac{\chi}{2} \left(S_i e^{i\theta_i} - \frac{\kappa_B S_h^2 \theta_i}{s_\alpha^2} i e^{i\theta_i} \right) - \beta(B - iQ) \overline{\tau_i} - 2\beta H_d.$$

Here H_d is the trapezoidal rule discretization of (14). We also introduce a partially filtered g_i in which the second derivative operator S_h^2 in (50) is replaced by its filtered version D_h^2 . This partially filtered discrete function is denoted by g_i^q . Then discretize (44), (45) as

$$(51) \quad (\mathbf{I} + \beta \mathbf{K}) \tilde{\boldsymbol{\omega}}_i = -\beta \mathbf{K} \mathbf{g}_i^q.$$

Here $\mathbf{K}$ is the discrete operator

$$(52) \quad \mathbf{K} \boldsymbol{\omega}_i = \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \begin{pmatrix} (K_R^{(1)})_{i,j} + (K_R^{(2)})_{i,j} & (K_I^{(2)})_{i,j} \\ (K_I^{(2)})_{i,j} & (K_R^{(1)})_{i,j} - (K_R^{(2)})_{i,j} \end{pmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_j (2h)$$

$$(53) \quad = \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} (\mathbf{K}_M)_{i,j} \boldsymbol{\omega}_j(2h),$$

where

$$(54) \quad (K_R^{(1)})_{i,j} = \frac{1}{\pi} \operatorname{Im} \left(\frac{S_h \tau_j}{\tau_j - \tau_i} \right) + \sigma,$$

$$(55) \quad (K_R^{(2)})_{i,j} = \operatorname{Re} \left\{ \frac{1}{2\pi i} \left(\frac{S_h \tau_j}{\overline{\tau_j} - \overline{\tau_i}} - \frac{\tau_j - \tau_i}{(\overline{\tau_j} - \overline{\tau_i})^2} \overline{S_h \tau_j} \right) \right\},$$

$$(56) \quad (K_I^{(2)})_{i,j} = \operatorname{Im} \left\{ \frac{1}{2\pi i} \left(\frac{S_h \tau_j}{\overline{\tau_j} - \overline{\tau_i}} - \frac{\tau_j - \tau_i}{(\overline{\tau_j} - \overline{\tau_i})^2} \overline{S_h \tau_j} \right) \right\},$$

and $(\mathbf{K}_M)_{i,j}$ is the matrix kernel in (52).

The invertibility of (51) is proven in Section 12. Note that $\tilde{\boldsymbol{\omega}} = 0$ if $\beta = 0$, in which case the density $\boldsymbol{\omega}_i$ in (48) is given explicitly.

We next consider the velocity equation (12). To obtain a stable scheme, a careful treatment of the principal value integral is required. We parameterize the contour by $\tau(\alpha)$ and then isolate the most singular part by adding and subtracting the periodic Hilbert transform

$$(57) \quad \mathcal{H}\omega(\alpha) = \frac{1}{2\pi} \text{P.V.} \int_{-\pi}^{\pi} \omega(\alpha') \cot \left(\frac{\alpha - \alpha'}{2} \right) d\alpha',$$

to obtain

$$(58) \quad u(\alpha) = \mathcal{H}\omega(\alpha) - \frac{1}{2\pi} \int_{-\pi}^{\pi} G(\alpha, \alpha') d\alpha' + (Q + iB) \overline{\tau(\alpha)} - \frac{iJ}{2} \tau(\alpha),$$

where

$$(59) \quad G(\alpha, \alpha') = \omega(\alpha') \left[2\operatorname{Re} \left(\frac{\tau_{\alpha'}(\alpha')}{\tau(\alpha') - \tau(\alpha)} \right) + \cot \left(\frac{\alpha - \alpha'}{2} \right) \right] - \overline{\omega(\alpha')} \left[\frac{\tau_{\alpha'}(\alpha')}{\overline{\tau(\alpha') - \tau(\alpha)}} - \frac{\tau(\alpha') - \tau(\alpha)}{(\tau(\alpha') - \tau(\alpha))^2} \tau_{\alpha'}(\alpha') \right].$$

It is easy to see that $G(\alpha, \alpha')$ is a smooth function of α and α' .

In our method, we need the discrete normal and tangential velocities,

$$(60) \quad (u_n)_i = \operatorname{Im}\{u_i e^{-i\theta_i}\}, \quad (u_s)_i = \operatorname{Re}\{u_i e^{-i\theta_i}\},$$

which follow from (15) with $n_i = i e^{i\theta_i}$. Care must be made in the discretization of $u_i e^{-i\theta_i}$, for reasons which will become apparent below. We first obtain a discrete velocity by applying alternate point trapezoidal rule to (58). This gives

$$(61) \quad u_i = \mathcal{H}_h \omega_i + (u_R)_i,$$

where $\mathcal{H}_h$ is the discrete Hilbert transform defined by

$$(62) \quad \mathcal{H}_h f_i = \frac{2h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \frac{f_j}{2} \cot \left(\frac{\alpha_i - \alpha_j}{2} \right),$$

and $(u_R)_i$ is given by

$$(63) \quad (u_R)_i = \frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \left\{ -\omega_j^q G_{ij}^{(1)} + \overline{\omega_j^q} G_{ij}^{(2)} \right\} + (Q + iB) \overline{\tau}_i - \frac{iJ\tau_i}{2},$$

in which

$$(64) \quad G_{ij}^{(1)} = 2\operatorname{Re} \left(\frac{S_h \tau_j}{\tau_j - \tau_i} \right) + \cot \left(\frac{\alpha_i - \alpha_j}{2} \right),$$

and

$$(65) \quad G_{ij}^{(2)} = \frac{S_h \tau_j}{\overline{\tau_j} - \overline{\tau_i}} - \frac{\tau_j - \tau_i}{(\overline{\tau_j} - \overline{\tau_i})^2} \overline{S_h \tau_j}.$$

In (61), we use the filtered density $\omega_i^q := \tilde{\omega}_i + g_i^q$ in the discretization $(u_R)_i$ of the regular integral but not in the leading order singular integral $\mathcal{H}_h(\omega_i)$. This targeted application of filtering is found to be necessary to prove stability of our method. In the discrete equations, $S_h \tau$ can be replaced by $\sigma e^{i\theta}$ (i.e., the actual application of the discrete derivative operator S_h is not required here), but for convenience we will continue to use $S_h \tau$ to represent the discrete version of τ_α . Next define the commutator

$$(66) \quad [\mathcal{H}_h, \phi_i](\psi_i) = \mathcal{H}_h(\phi_i \psi_i) - \phi_i \mathcal{H}_h(\psi_i).$$

Substitute (61) into $u_i e^{-i\theta_i}$ and move $e^{-i\theta_i}$ inside the discrete Hilbert transform, incurring a commutator, to obtain the discretization

$$(67) \quad u_i e^{-i\theta_i} = \mathcal{H}_h(\omega_i e^{-i\theta_i}) - [\mathcal{H}_h, e^{-i\theta_i}](\omega_i^q) + (u_R)_i e^{-i\theta_i}.$$

Note that the filter is applied in the argument of the commutator. This is combined with (60) to give $(u_n)_i$ and $(u_s)_i$, which are used in the evolution. The discrete normal and tangential velocities can be written in a particularly simple form in the

important special case of viscosity matched fluids ($\beta = 0$), which is now described.

Viscosity matched fluids ($\beta = 0$).

When $\beta = 0$, the nonlocal operator $\mathbf{K}$ in (51) drops out, leading to a considerable simplification. Taking $\beta = 0$ and $\chi = \frac{1}{2}$ in (44) and (50), we see that

$$(68) \quad \omega_i e^{-i\theta_i} = -\frac{1}{4} \left(\mathcal{S}_i - i \frac{\kappa_B}{\sigma^2} S_h^2 \theta_i \right).$$

Inserting this into the discrete Hilbert transform in (67) and taking the imaginary part per (60) gives

$$(69) \quad (u_n)_i = \frac{\kappa_B}{4\sigma^2} \mathcal{H}_h(S_h^2 \theta_i) + \text{Im} \left\{ -[\mathcal{H}_h, e^{-i\theta_i}](\omega_i^q) + (u_R)_i e^{-i\theta_i} \right\}.$$

The significance of the decomposition (67) is now apparent: by moving $e^{-i\theta_i}$ into the argument of discrete Hilbert transform, the leading order term of the normal velocity, namely $\frac{\kappa_B}{4\sigma^2} \mathcal{H}(S_h^2 \theta_i)$, becomes linear in θ_i with a spatially constant coefficient that has the right sign to take advantage of parabolic smoothing. This will be critical in energy estimates. We similarly decompose the tangential velocity as

$$(70) \quad (u_s)_i = -\frac{1}{4} \mathcal{H}_h(\mathcal{S}_i) + \text{Re} \left\{ -[\mathcal{H}_h, e^{-i\theta_i}](\omega_i^q) + (u_R)_i e^{-i\theta_i} \right\}.$$

It will later be shown that the nonlocal operator $\mathbf{K}$ in (51) does not affect the stability of the discretization. Henceforth, we focus the analysis on the special case of viscosity matched fluids, and later in Section 12 generalize to the full problem for nonzero β .

Discretization of evolution equations.

The semi-discrete (continuous in time, discrete in space) equations for θ, σ are

$$(71) \quad (\theta_t)_i = \frac{1}{\sigma} (S_h(u_n)_i + (\phi_s)_i S_h \theta_i),$$

$$(72) \quad \sigma_t = -\langle u_n S_h \theta \rangle_h,$$

where

$$(73) \quad \langle f \rangle_h = \frac{1}{N} \sum_{j=-\frac{N}{2}+1}^{\frac{N}{2}} f_j$$

is the discrete mean computed using trapezoid rule. In order to recover the interface location from θ_i and σ , we need to introduce the pseudo-spectral antiderivative operator defined in Fourier space on functions f of mean zero by

$$(74) \quad (\widehat{S_h^{-1} f})_k = \begin{cases} \frac{1}{ik} \hat{f}_k & \text{for } k \neq 0, \\ 0 & \text{for } k = 0. \end{cases}$$

Then the discretization of (28), (29) can be written as

$$(75) \quad \tau_i = \tau_c + S_h^{-1} (\sigma e^{i\theta} - \langle \sigma e^{i\theta} \rangle_h)_i,$$

where τ_c is the zero (constant) Fourier mode of τ_i . This is evolved from (30) as

$$(76) \quad \frac{d\tau_c}{dt} = \hat{v}_0 = \langle v \rangle_h,$$

where $\hat{v}_0$ is the zero Fourier mode of the discrete velocity v_i . Equation (23) is discretized as

$$(77) \quad (\phi_s)_i = S_h^{-1} (u_n S_h \theta - \langle u_n S_h \theta \rangle_h)_i,$$

and the membrane tension (31) as

$$(78) \quad \mathcal{S}_i = \frac{\sigma}{\sigma_0 D_h \alpha_{0i}} (1 + \mathcal{S}_{0i}) - 1,$$

where $\mathcal{S}_{0i}$ is the discrete initial tension, and σ_0 is the initial value of s_α . The semi-discrete equation for α_{0i} is obtained from (37) as

$$(79) \quad (\alpha_{0t})_i = \frac{D_h \alpha_{0i}}{\sigma e^{i\theta_i}} ((\phi_s - u_s) e^{i\theta})_i.$$

The principal equations for the semi-discrete scheme are now summarized. Assume the arclength-angle representation of the interface (θ_i, σ) , backward map α_{0i} , and shape constant τ_c are known at time t . Then the interface shape τ_i is determined from (75), the density ω_i from (48)-(56) and (78) (or (68) when $\beta = 0$), and velocities u_{ni} , u_{si} from (60) - (67). The time updates of θ_i , σ , α_{0i} and τ_c are given by (71), (72), (76), (79) using (77).

Discretization for a drop interface.

The discretization for a (nonelastic) drop interface with zero bending stress and constant surface tension, $\kappa_B = 0$ and $\mathcal{S}_i = 1$, is modified from the above. In this case the stability is more delicate, since we can no longer take advantage of the stabilizing properties of the (high derivative) bending stress term. Thus, the numerical method requires more filtering. We now filter the leading order term in the decomposition (61), so that

$$(80) \quad u_i = \mathcal{H}_h \omega_i^r + (u_R)_i,$$

where $\omega_i^r := \tilde{\omega}_i + g_i^p$ (cf. (48)), and g_i^p is the fully filtered g_i . Furthermore, we replace each occurrence of $S_h \tau_i$ in the kernels of (52) and (63) with its filtered version $D_h \tau_i$, or equivalently by $(\sigma e^{i\theta_i})^p$. Finally, we replace $(S_h u_n)_i$ in (71) with $(D_h u_n)_i$. Other aspects of the discretization remain the same as for an elastic capsule.

Numerical example.

An example numerical calculation is shown in Figure 2. We use the BI method of [21] for the elastic capsule computation. Their method is similar, but not identical to, that described in this section. In particular, the algorithm analyzed here generalizes that of [21] to include nonzero interior viscosity and membrane bending stress. The method for the drop computation is as described in this section, and essentially the same as in [27], [52]. More extensive numerical results using the method for capsules will be presented in later work.

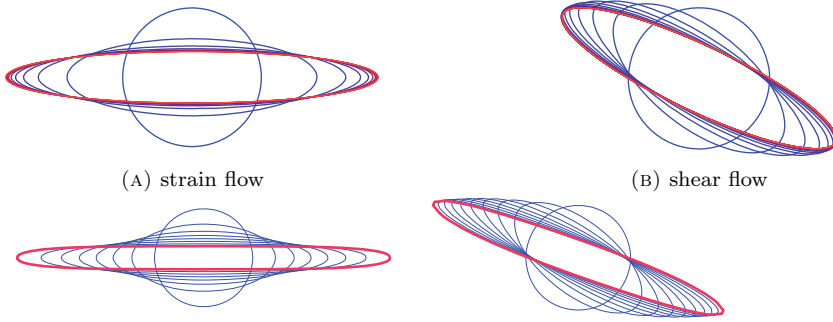


FIGURE 2. Time evolution of a Hookean elastic capsule (top) and fluid drop (bottom) in (A) a pure strain flow with $Q = 1$ and (B) a simple shear flow with $2B = J = -1$. Other parameters are $S_0 = 1$, $\kappa_B = \lambda = 0$ for the elastic capsule, and $S = 1$, $\lambda = 0.01$ for the fluid drop. The profiles are shown at intervals of $\Delta t = 1.0$ (top left) and $\Delta t = 0.5$ (top right and bottom).

5. STATEMENT OF MAIN CONVERGENCE THEOREM

Error estimates are given in terms of the discrete L^2 norm

$$(81) \quad \|u\|_{l^2} = \left(h \sum_{j=-\frac{N}{2}+1}^{\frac{N}{2}} |u_j|^2 \right)^{\frac{1}{2}}.$$

We now state the main convergence theorem for our numerical method, followed by some discussion:

Theorem 5.1. *Assume that for $0 \leq t \leq T$ there exists a smooth solution of the continuous problem with $\theta(\cdot, t)$, $\alpha_0(\cdot, t)$ in $C^{m+1}[-\pi, \pi]$ and $S(\cdot, 0) \in C^m[-\pi, \pi]$ for $m \geq 5$. Also assume that the interface shape $\tau(\alpha, t)$ satisfies*

$$(82) \quad |\tau(\alpha, t) - \tau(\alpha', t)| > d|\alpha - \alpha'|, \text{ for some } d > 0.$$

If $\sigma^{(h)}$, $\theta^{(h)}$ and $\alpha_0^{(h)}$ denote the numerical solution for s_α , θ and α_0 , then there exists $c(T) > 0$ such that for $h \leq h_0(T)$, $\beta \leq \beta_0(T)$ and all $0 \leq t \leq T$,

$$(83) \quad \begin{aligned} |\sigma^{(h)}(t) - s_\alpha(t)| &\leq c(T)h^r, \\ \|\theta^{(h)}(t) - \theta(\cdot, t)\|_{l^2} &\leq c(T)h^r, \\ \|\alpha_0^{(h)}(t) - \alpha_0(\cdot, t)\|_{l^2} &\leq c(T)h^r, \end{aligned}$$

where $r = m - 4$. In addition, the discrete interface shape $\tau^{(h)}(t)$ satisfies

$$(84) \quad \|\tau^{(h)}(t) - \tau(\cdot, t)\|_{l^2} \leq c(T)h^r.$$

The different levels of regularity that are assumed for the continuous solution follow from an analysis of the continuous problem. The condition in (82) guarantees that the interface does not self-intersect.

To prove Theorem 5.1, we need to establish the consistency and stability of the discrete scheme, and then demonstrate that these imply convergence. We first establish consistency of the method. Next, stability and convergence are shown for

the simpler special case of an elastic capsule with viscosity matched fluids in which $\beta = 0$. We then prove stability and convergence for an elastic capsule with viscosity contrast ($\beta \neq 0$), and finally for a fluid drop with constant surface tension $\mathcal{S}$.

6. PRELIMINARY LEMMAS

We define the Sobolev norm

$$(85) \quad \|f\|_s = \left(\sum_{k=-\infty}^{\infty} (1 + |k|^2)^s |\hat{f}_k|^2 \right)^{\frac{1}{2}}.$$

Lemma 6.1, a version of which is presented in [47], gives the accuracy of the pseudo-spectral derivative. For completeness, a proof is given in the appendix.

Lemma 6.1. *Let $f(\alpha)$ be a periodic $C^{s+1}[-\pi, \pi]$ function. Then*

$$(86) \quad |S_h f(\alpha_i) - f(\alpha_i)| \leq ch^{s-\frac{1}{2}} \|f\|_{s+1},$$

where S_h is defined in (40). The same inequality holds for the filtered derivative D_h (see (43)) in place of S_h . If $1 \leq n \leq s$ derivatives are applied, then a similar estimate holds with $h^{s-n+\frac{1}{2}}$ in place of $h^{s-\frac{1}{2}}$.

Similarly, for the pseudo-spectral antiderivative operator we have

Lemma 6.2. *Let f be a periodic, zero-mean, $C^s[-\pi, \pi]$ function. Then*

$$(87) \quad |\partial_\alpha^{-1} f(\alpha_j) - S_h^{-1} f(\alpha_j)| \leq ch^{s-\frac{1}{2}} \|f\|_s.$$

Proof. The proof is a simple adaptation of that for Lemma 6.1. $\square$

Lemma 6.3 is a well-known result on the accuracy of trapezoid rule for periodic functions.

Lemma 6.3. *Let $f(\alpha)$ be as in Lemma 6.1. Then*

$$(88) \quad \left| \sum_{j=-\frac{N}{2}+1}^{\frac{N}{2}} f(\alpha_j)h - \int_{-\pi}^{\pi} f(\alpha)d\alpha \right| \leq ch^{s+1} \|f\|_{s+1}.$$

Proof. This is derived from the Euler–Maclaurin formula. For more details, see [19]. $\square$

Lemma 6.4 provides a result on the accuracy of the filtering operator.

Lemma 6.4. *Let $f \in C^s[-\pi, \pi]$ be periodic, and let f^p be as defined in (42) with conditions (i)–(iv). Then*

$$(89) \quad |f^p(\alpha_i) - f(\alpha_i)| \leq ch^{s-\frac{1}{2}} \|f\|_s.$$

Proof. The proof is similar to that for Lemma 6.1, and is omitted here. $\square$

7. CONSISTENCY

We calculate the error when the exact solution is substituted into the discrete system of equations. The exact solution is assumed to have the same regularity as in the statement of Theorem 5.1. We denote by $u_h(\alpha_i)$, $(u_n)_h(\alpha_i)$, $\omega_h(\alpha_i)$, $(\phi_s)_h(\alpha_i)$, etc. quantities that are evaluated by substituting the exact solution $\theta(\cdot, t)$, $s_\alpha(t)$, $\alpha_0(\cdot, t)$ into the discrete equations. We make repeated use of the estimate

$$(90) \quad \tau_h(\alpha_i) = \tau_c + S_h^{-1}(s_\alpha e^{i\theta} - \langle s_\alpha e^{i\theta} \rangle_h)(\alpha_i) = \tau(\alpha_i) + O(h^{m+\frac{1}{2}}),$$

which follows from (75), Lemma 6.2, and the assumption on the regularity of the exact solution.

Consistency of ω equation.

We first assess the smoothness of the integrand $F(\alpha, \alpha')$ in the continuous equation for ω , (44). The apparent singularity in $F(\alpha, \alpha')$ is removable, and

$$(91) \quad \lim_{\alpha' \rightarrow \alpha} F(\alpha, \alpha') = i \left(\omega(\alpha) \kappa(\alpha) s_\alpha + \frac{\overline{\omega}(\alpha) \kappa(\alpha) \tau_\alpha^2(\alpha)}{s_\alpha} \right).$$

Recalling that $\omega(\cdot) \in C^{m-1}$, $\tau_\alpha(\cdot) = s_\alpha e^{i\theta(\cdot)} \in C^{m+1}$, and that s_α is bounded away from zero, it follows that

$$(92) \quad F(\alpha, \cdot) \in C^{m-1}.$$

The truncation error of the discrete operator $\mathbf{K}\omega = \mathbf{K}(\tilde{\omega} + \mathbf{g})$ in (52) is equivalent to the truncation error of the alternate point trapezoidal rule approximation of $\int_{-\pi}^{\pi} F(\alpha, \alpha') d\alpha'$ in (44), which is its complex counterpart. We therefore consider the alternate point sum $\sum F_h(\alpha_i, \alpha_j) 2h$, where the subscript h denotes evaluation of F using $\tau = \tau_h$, $\partial_\alpha = S_h$, and the exact θ , s_α , and ω . By (90), we can replace τ_h in this sum by τ incurring an order $O(h^{m+\frac{1}{2}}) [\min_j (\tau(\alpha_j) - \tau(\alpha_i))]^{-1} = O(h^{m-\frac{1}{2}})$ error. There is no error in $S_h \tau_h$ since the exact solution $s_\alpha e^{i\theta(\alpha_i)}$ is substituted for this term. The standard trapezoid rule discretization therefore satisfies

$$(93) \quad \sum_{j=-\frac{N}{2}+1}^{\frac{N}{2}} F_h(\alpha_i, \alpha_j) h = \sum_{j=-\frac{N}{2}+1}^{\frac{N}{2}} F(\alpha_i, \alpha_j) h + O(h^{m-\frac{1}{2}}),$$

where we have used the above remarks to replace F_h with F . Then by the error estimate for trapezoidal rule integration (Lemma 6.3), the truncation error is bounded as

$$(94) \quad \left| \sum_{j=-\frac{N}{2}+1}^{\frac{N}{2}} F_h(\alpha_i, \alpha_j) h - \int_{-\pi}^{\pi} F(\alpha_i, \alpha') d\alpha' \right| \leq ch^{m-1} \|F(\alpha_i, \cdot)\|_{m-1}.$$

A standard argument [10], [22] shows that the order of truncation error for the alternate point trapezoidal rule quadrature of F is the same as for trapezoidal rule, i.e.,

$$(95) \quad \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} F_h(\alpha_i, \alpha_j) 2h - \int_{-\pi}^{\pi} F(\alpha_i, \alpha') d\alpha' = O(h^{m-1}).$$

Error estimates for $\mathbf{g}_i$ in (48), (51) are obtained using Lemma 6.1. For example,

$$(96) \quad \frac{S_h^2 \theta(\alpha_i)}{s_\alpha^2} - \frac{\theta_{\alpha\alpha}(\alpha_i)}{s_\alpha^2} = O(h^{m-\frac{3}{2}})$$

(cf. (50)), which is the dominant source of truncation error (i.e., highest derivative) in $\mathbf{g}_i$. It follows that

$$(97) \quad (\mathbf{I} + \beta \mathbf{K}_h) \boldsymbol{\omega}(\alpha_i) - \mathbf{g}_h(\alpha_i) = O(h^{m-\frac{3}{2}}),$$

and the consistency of (51) results from substituting for $\boldsymbol{\omega}$ using (48) and noting that $\mathbf{K}_h(\mathbf{g}_h^q(\alpha_i) - \mathbf{g}(\alpha_i)) = O(h^{m-\frac{3}{2}})$.

We also need an estimate on $\boldsymbol{\omega}_h$, which is the solution of

$$(98) \quad (\mathbf{I} + \beta \mathbf{K}_h)(\boldsymbol{\omega}_h)_i = \mathbf{g}_h(\alpha_i).$$

This is obtained by subtracting (98) from (97) and using the boundedness of $(\mathbf{I} + \beta \mathbf{K}_h)^{-1}$ from Lemma 12.1 to find that

$$(99) \quad (\boldsymbol{\omega}_h)_i = \boldsymbol{\omega}(\alpha_i) + O(h^{m-\frac{3}{2}}).$$

Consistency of velocity.

We first consider the smoothness of the integrand $G(\alpha, \alpha')$ in (58). Note that

$$(100) \quad \lim_{\alpha' \rightarrow \alpha} G(\alpha, \alpha') = -\omega(\alpha) \operatorname{Re} \left(\frac{\tau_{\alpha\alpha}}{\tau_\alpha} \right) + i \overline{\omega(\alpha)} \kappa(\alpha) \frac{\tau_\alpha^2(\alpha)}{s_\alpha},$$

and it follows that $G(\alpha, \cdot) \in C^{m-1}$. Now, let $G_h(\alpha_i, \alpha_j)$ denote the discrete integrand in (63) but with τ_j replaced by $\tau_h(\alpha_j)$, ω_j^q by $\omega_h^q(\alpha_j)$, etc. Using the same argument as that which led to (95), we deduce

$$(101) \quad \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} G_h(\alpha_i, \alpha_j) 2h - \int_{-\pi}^{\pi} G(\alpha_i, \alpha') d\alpha' = O(h^{m-\frac{3}{2}}),$$

in which the dominant $O(h^{m-\frac{3}{2}})$ contribution to the error comes from replacing $\omega_h^q(\alpha_i)$ by $\omega(\alpha_i)$, invoking Lemma 6.4, and (99). This gives the truncation error of $(u_R)_i$ in (61). We next consider the discrete Hilbert transform in (61). It is shown in [10] that

$$(102) \quad \mathcal{H}_h \omega(\alpha_i) - \frac{1}{2\pi} \text{P.V.} \int_{-\pi}^{\pi} \omega(\alpha') \cot \left(\frac{\alpha - \alpha'}{2} \right) d\alpha' = O(h^{m-2}).$$

This is a special case of a result proven in Section 2 of [10], where it is shown that the order of accuracy of the discrete Hilbert transform is related to the regularity of $\omega_\alpha(\cdot)$, which here is C^{m-2} . It follows that

$$(103) \quad \begin{aligned} u_h(\alpha_i) &= \mathcal{H}_h \omega(\alpha_i) - \frac{1}{2\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} G_h(\alpha_i, \alpha_j) 2h + (Q + iB) \tau_h(\alpha_i), \\ &\quad - \frac{iJ \tau_h(\alpha_i)}{2} = u(\alpha_i) + O(h^{m-2}), \end{aligned}$$

which shows the consistency of the velocity discretization. From this it is easy to see that

$$(104) \quad (u_s)_h(\alpha_i) = u_s(\alpha_i) + O(h^{m-2}),$$

$$(105) \quad (u_n)_h(\alpha_i) = u_n(\alpha_i) + O(h^{m-2}),$$

$$(106) \quad S_h(u_n)_h(\alpha_i) = (u_n)_\alpha(\alpha_i) + O(h^{m-3}).$$

In addition, from (77),

$$(107) \quad (\phi_s)_h(\alpha_i) = S_h^{-1}((u_n)_h S_h \theta - \langle (u_n)_h S_h \theta \rangle_h)(\alpha_i) = \phi_s(\alpha_i) + O(h^{m-2}),$$

where the latter equality follows from (105). Combined, the above results show that the truncation errors for the $\theta, s_\alpha, \alpha_0$ evolution equations (71)-(72), (79) are given by

$$(108) \quad \frac{\partial}{\partial t} \theta(\alpha_i) = \frac{1}{s_\alpha} [S_h(u_n)_h(\alpha_i) + S_h \theta(\alpha_i)(\phi_s)_h(\alpha_i)] + O(h^{m-3}),$$

$$(109) \quad \frac{\partial}{\partial t} s_\alpha = -\langle (u_n)_h(\cdot) S_h \theta(\cdot) \rangle_h + O(h^{m-2}),$$

$$(110) \quad \frac{\partial \alpha_0}{\partial t}(\alpha_i) = \frac{S_h \alpha_0(\alpha_i)}{s_\alpha e^{i\theta(\alpha_i)}} ((u_n)_h i e^{i\theta} + (\phi_s)_h e^{i\theta} - u_h)(\alpha_i) + O(h^{m-2}).$$

We also need to check consistency of the discrete version of the kinematic condition (18). Differentiate (75) with respect to t to obtain

$$(111) \quad \frac{d\tau_i}{dt} = \frac{d\tau_c}{dt} + S_h^{-1} \left(\frac{d\sigma}{dt} e^{i\theta} + i\sigma e^{i\theta} \frac{d\theta}{dt} - \left\langle \frac{d\sigma}{dt} e^{i\theta} + i\sigma e^{i\theta} \frac{d\theta}{dt} \right\rangle_h \right)_i,$$

where from (76),

$$(112) \quad \frac{d\tau_c}{dt} = \langle v \rangle_h.$$

Then it is easy to see that

$$(113) \quad \frac{d\tau}{dt}(\alpha_i) = \frac{d\tau_h}{dt}(\alpha_i) + O(h^{m-3}).$$

Taken together, the above results prove the following consistency result:

Lemma 7.1. *Under the assumption that $\theta(\cdot, t)$ and $\alpha_0(\cdot, t)$ are in $C^{m+1}[-\pi, \pi]$, $S_0(\cdot)$ is in $C^m[-\pi, \pi]$, and $\omega(\cdot, t)$ is in $C^{m-1}[-\pi, \pi]$, the exact solutions of the evolution equations satisfy the discrete equations with a truncation error at most of size $O(h^{m-3})$.*

Lemma 7.1 demonstrates consistency at each t . Time bounds for the discrete-in-space, continuous-in-time system will follow by taking a ‘method of lines’ approach in the stability analysis (see e.g., [30]), as discussed below.

8. STABILITY: PRELIMINARIES

To show convergence of the numerical method, we need to establish the stability of the discrete scheme. We first do this for special case of viscosity matched fluids,

for which $\beta = 0$ and $\chi = \frac{1}{2}$. Define the errors between the exact and numerical solutions as

$$\begin{aligned} \dot{\theta}_j &= \theta_j - \theta(\alpha_j), \\ \dot{\omega}_j &= \omega_j - \omega_h(\alpha_j), \\ \dot{u}_j &= u_j - u_h(\alpha_j), \end{aligned} \quad (114)$$

and so forth. To show stability, we plan to obtain a system of evolution equations for these errors and perform energy estimates to show they remain bounded for $t \leq T$, where T is the assumed existence time for an exact solution to the continuous problem.

Therefore, our first task is to estimate quantities such as $(\dot{u}_n)_i$ and $(\dot{\phi}_s)_i$ in terms of the errors $\dot{\theta}_i, \dot{\sigma}_i, \dot{\alpha}_{0i}$. This can be done by identifying the most singular part in the variation $\dot{u}_i = u_i - u_h(\alpha_i)$ of the complex velocity. The estimates can be separated into linear and nonlinear terms in $\dot{\theta}_i, \dot{\sigma}_i, \dot{\alpha}_{0i}$. The nonlinear terms can be controlled by the high accuracy of the method for smooth solutions. Thus the leading order error contribution comes from the linear terms.

Following [10] and [14], we introduce notation for an n -th order smoothing operator A_{-n} which acts on a generic discrete function ϕ_j and satisfies

$$(115) \quad \|D_h^k(A_{-n}(\phi))\|_{l^2} \leq c\|\phi\|_{l^2} \text{ and } \|A_{-n}(S_h^k(\phi))\|_{l^2} \leq c\|\phi\|_{l^2} \text{ for } 0 \leq k \leq n,$$

where S_h is the spectral derivative (40) and D_h is the spectral derivative operator with filtering. When $n = 0$, $A_0(\phi_j)$ denotes a bounded operator in l^2 ,

$$(116) \quad \|A_0(\phi)\|_{l^2} \leq c\|\phi\|_{l^2}.$$

Remark 8.1. Note that if $f(\phi_i) = A_0(\phi_i)$, then $h^s f(\phi_i) = A_{-s}(\phi_i)$. However, $f(\phi_i) = A_{-s}(\phi_i)$ does not imply $f = O(h^s)$.

Remark 8.2. We use the expression $A_{-s}(\phi_i)$ to denote a generic high-order smoothing operator. Similarly, we denote by $O(h^s)$ a generic high-order discretization error. At the end of our proof, we choose m and s large enough so that all the estimates go through.

Remark 8.3. Unless otherwise noted, we use the phrase “smooth function” to denote a generic function $f \in C^s[-\pi, \pi]$ with high order regularity.

We define a time

$$(117) \quad T^* \equiv \sup \{t : 0 \leq t \leq T, |\dot{\sigma}|, \|\dot{\theta}\|_{l^2}, \|\dot{\tau}\|_{l^2}, \|\dot{\alpha}_0\|_{l^2} \leq h^{\frac{7}{2}}\},$$

where the power of h in (117) is chosen so that the estimates below easily go through. All the estimates we obtain are valid for $t \leq T^*$. We “close the argument” and prove Theorem 5.1 by showing at the end that $T^* = T$, the existence time of the continuous solution. We make repeated use of the inequalities

$$(118) \quad \|\dot{\theta}\|_{\infty} \leq h^3 \text{ and } \|\dot{\alpha}_0\|_{\infty} \leq h^3 \text{ for } t \leq T^*.$$

The above estimate on $\dot{\theta}$ follows from $h|\dot{\theta}_i|^2 \leq \|\dot{\theta}\|_{l^2}^2$ for $t \leq T^*$, so that $\|\dot{\theta}\|_{\infty} \leq h^{-\frac{1}{2}}\|\dot{\theta}\|_{l^2} \leq h^3$, with a similar estimate applying to $\|\dot{\alpha}_0\|_{\infty}$.

Preliminary lemmas.

We will frequently encounter a discrete operator of the form:

$$(119) \quad R_h(\phi_i) = \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} f(\alpha_i, \alpha_j) \phi_j(2h),$$

where $f(\alpha, \alpha')$ is a smooth periodic function in both variables, and ϕ is a discrete periodic function. Beale, Hou and Lowengrub [10] prove the following estimate on R_h applied to a filtered discrete function ϕ_i^p :

Lemma 8.4. *Assume $f(\alpha, \alpha')$ is a smooth periodic function in both α and α' , with $f(\cdot, \cdot)$ in C^r for $r > 3$. Then*

$$(120) \quad R_h(\phi_i^p) = A_{-2}(\phi_i).$$

We note that the application of the filter is essential in (120) due to aliasing error. To see this, consider the following example adapted from [10]. Let $g(\alpha) = e^{2i\alpha}$, define

$$(121) \quad f(\alpha, \alpha') = \frac{1}{2\pi} (g(\alpha) - g(\alpha')) \cot\left(\frac{\alpha - \alpha'}{2}\right), \quad f(\alpha, \alpha) = \frac{g_\alpha(\alpha)}{\pi},$$

and let $\phi_i = e^{i\alpha_i(\frac{N}{2}-1)}$. Then using Lemma 8.6 and the fact that $e^{i\alpha_i(\frac{N}{2}+1)}$ is aliased to $e^{i\alpha_i(-\frac{N}{2}+1)}$, we have

$$\begin{aligned} R_h(\phi_i) &= \frac{e^{2i\alpha_i}}{2\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} e^{i\alpha_j(\frac{N}{2}-1)} \cot\left(\frac{\alpha_i - \alpha_j}{2}\right) (2h) \\ &\quad - \frac{1}{2\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} e^{i\alpha_j(-\frac{N}{2}+1)} \cot\left(\frac{\alpha_i - \alpha_j}{2}\right) (2h) \\ &= -i(e^{2i\alpha_i} e^{i\alpha_i(\frac{N}{2}-1)} + e^{i\alpha_i(-\frac{N}{2}+1)}) \\ (122) \quad &= -2ie^{i\alpha_i(-\frac{N}{2}+1)} = -2ig(\alpha_i)\phi_i = A_0(\phi_i). \end{aligned}$$

Remark 8.5. If no filtering is applied, then it is easy to see that

$$(123) \quad R_h(\phi_i) = A_0(\phi_i).$$

Indeed we note that by the Schwarz inequality,

$$\begin{aligned} \|R_h(\phi)\|_{l^2} &= \left(h \sum_{i=-\frac{N}{2}+1}^{\frac{N}{2}} \left| \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} f(\alpha_i, \alpha_j) \phi_j(2h) \right|^2 \right)^{\frac{1}{2}} \\ &\leq 2h \|\phi\|_{l^2} \left(\sum_{i=-\frac{N}{2}+1}^{\frac{N}{2}} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} |f(\alpha_i, \alpha_j)|^2 \right)^{\frac{1}{2}} \\ (124) \quad &\leq 4\|f\|_{l^2} \|\phi\|_{l^2}, \end{aligned}$$

where

$$(125) \quad \|f\|_{l^2} = \left(h^2 \sum_{i=-\frac{N}{2}+1}^{\frac{N}{2}} \sum_{j=-\frac{N}{2}+1}^{\frac{N}{2}} |f(\alpha_i, \alpha_j)|^2 \right)^{\frac{1}{2}}.$$

The Hilbert transform $\mathcal{H}\omega$ is the leading order part (i.e., least regular term) in the velocity (58). This will be seen to play a crucial rule in the stability of our discretization. The continuous Hilbert transform satisfies

$$(126) \quad \widehat{(\mathcal{H}f)}_k = -i \operatorname{sgn}(k) \hat{f}_k,$$

$$(127) \quad \mathcal{H}(\mathcal{H}f(\alpha)) = -f(\alpha),$$

$$(128) \quad \mathcal{H}(f_\alpha)(\alpha) = (\mathcal{H}f)_\alpha(\alpha)$$

for a periodic function f with zero mean. Lemma 8.6 from [10] shows that the discrete transform acts in the same way.

Lemma 8.6. *Assume that f satisfies $\hat{f}_0 = \hat{f}_{\frac{N}{2}} = 0$. The discrete Hilbert transform (62) satisfies the following properties:*

$$(129) \quad \widehat{(\mathcal{H}_h f)}_k = -i \operatorname{sgn}(k) \hat{f}_k,$$

$$(130) \quad \mathcal{H}_h(\mathcal{H}_h f_i) = -f_i,$$

$$(131) \quad \mathcal{H}_h(S_h f_i) = S_h(\mathcal{H}_h f_i) = \frac{1}{\pi} \sum_{(j-i) \text{ odd}} \frac{f_i - f_j}{(\alpha_i - \alpha_j)^2} (2h),$$

where $\sum_{(j-i) \text{ odd}}$ is defined in (133). The first equality above also implies $\|\mathcal{H}_h f\|_{l^2} = \|f\|_{l^2}$.

Proof. We transform the kernel in (62) from a representation in the periodic domain to an equivalent representation in the infinite domain. This involves application of the formula [34]

$$(132) \quad \frac{1}{2} \cot\left(\frac{z}{2}\right) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - (2k\pi)^2}$$

from which it is easy to obtain (see [10] for details)

$$(133) \quad \begin{aligned} & \frac{1}{2} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} f(\alpha_j) \cot\left(\frac{\alpha_i - \alpha_j}{2}\right) (2h) = \lim_{M \rightarrow \infty} \sum_{\substack{j=-M(N+\frac{1}{2})+1 \\ (j-i) \text{ odd}}}^{N(M+\frac{1}{2})} \frac{f(\alpha_j)}{\alpha_i - \alpha_j} (2h) \\ & \equiv \sum_{(j-i) \text{ odd}} \frac{f(\alpha_j)}{\alpha_i - \alpha_j} (2h), \end{aligned}$$

where $\alpha_j = jh$ is extended outside the interval $-\frac{N}{2} + 1 \leq j \leq \frac{N}{2}$. We note that the first equality in (133) relies on the particular form of the bounds in the right hand sum, although we will use the notation $\sum_{(j-i) \text{ odd}}$ to denote more general infinite

sums. It follows from (133) with f_j in place of $f(\alpha_j)$ that

$$(134) \quad \mathcal{H}_h f_i = \frac{1}{\pi} \sum_{(j-i) \text{ odd}} \frac{f_j}{\alpha_i - \alpha_j} (2h)$$

is an equivalent form of (62). This form is proven to satisfy properties (129)-(131) in [10]. $\square$

We also need the following result on the commutator of the discrete Hilbert transform and a smooth function, from [10].

Lemma 8.7. *Let $g(\cdot) \in C^r$ for $r > 4$, and consider the commutator*

$$(135) \quad [\mathcal{H}_h, g](\phi_i^p) = \mathcal{H}_h(g(\alpha_i)\phi_i^p) - g(\alpha_i)\mathcal{H}_h(\phi_i^p).$$

Then $[\mathcal{H}_h, g](\phi_i^p) = A_{-2}(\phi_i)$.

Proof. Let

$$(136) \quad f(\alpha, \alpha') = \frac{1}{2\pi} (g(\alpha) - g(\alpha')) \cot\left(\frac{\alpha - \alpha'}{2}\right) \in C^{r-1}$$

and apply Lemma 8.4. The result follows from noting that for this f ,

$$(137) \quad R_h(\phi_i^p) = \mathcal{H}_h(g(\alpha_i)\phi_i^p) - g(\alpha_i)\mathcal{H}_h(\phi_i^p).$$

$\square$

We will also need a lemma on the commutator of the filtering operator and a smooth function. The proof can be found in [13].

Lemma 8.8. *Let $f(\alpha_i) \in C^r$ for $r \geq 2$, and $\phi \in l^2$. Define:*

$$(138) \quad G_h^p(\phi_i) = (f(\alpha_i)\phi_i)^p - f(\alpha_i)\phi_i^p.$$

Then $G_h^p(\phi_i) = A_{-1}(\phi_i)$.

In our stability analysis, we will need an analogue of the product rule for discrete derivative operators, proven in [10].

Lemma 8.9. *Assume $f(\cdot) \in C^3$ and $w \in l^2$. Then we have:*

$$(139) \quad D_h(f(\alpha_i)w_i) = f(\alpha_i)D_h(w_i) + w_i^b f_\alpha(\alpha_i) + A_{-1}(w_i),$$

where $\hat{w}_k^b = \hat{w}_k b(kh)$, $b(x) = \frac{\partial}{\partial x}(x\rho(x))$.

We will apply the following lemmas to obtain expressions for the variation of velocities and other quantities in our problem. Recall the definition of error between the exact and numerical solutions,

$$(140) \quad \dot{f}_i = f_i - f(\alpha_i).$$

It is straightforward to derive the following product rule for errors:

Lemma 8.10. *Let $\dot{f}_i, \dot{g}_i$ and $f(\alpha_i), g(\alpha_i)$ be as defined in (140). Then $(f_i g_i)' = \dot{f}_i g(\alpha_i) + f(\alpha_i) \dot{g}_i + \dot{f}_i \dot{g}_i$.*

Lemma 8.10 can be easily extended to products of three or more quantities. We also have

Lemma 8.11.

$$(141) \quad \left(\frac{1}{f_i}\right)' = -\frac{\dot{f}_i}{f^2(\alpha_i)} + \frac{\dot{f}_i^2}{f^2(\alpha_i)(f(\alpha_i) + \dot{f}_i)}.$$

Proof.

$$(142) \quad \left(\frac{1}{f_i}\right)' = \frac{1}{f_i} - \frac{1}{f(\alpha_i)} = \frac{f(\alpha_i) - f_i}{f_i f(\alpha_i)} = -\frac{\dot{f}_i}{f(\alpha_i)(f(\alpha_i) + \dot{f}_i)},$$

where in the last equality, we have eliminated f_i using $f_i = f(\alpha_i) + \dot{f}_i$. After decomposing the right hand side of (142) into a sum of linear and nonlinear terms in $\dot{f}_i$, we obtain the result. $\square$

We will need the following results on $\dot{\sigma} = \sigma - s_\alpha$.

Lemma 8.12. *Let f be a smooth function and $\langle f(\cdot) \rangle_h = 0$. Then:*

$$(143) \quad S_h^{-1}(f(\alpha_i)\dot{\sigma}) = \dot{\sigma}S_h^{-1}(f(\alpha_i)) = A_{-s}(\dot{\sigma}_i),$$

where $A_{-s}(\dot{\sigma}_i)$ is here interpreted for the spatially independent $\dot{\sigma}$ as

$$(144) \quad A_{-s}(\dot{\sigma}_i) = \dot{\sigma}g(\alpha_i)$$

for some smooth function g .

Proof. The relation (143) follows from the spatial independence of $\dot{\sigma}$. The second equality follows from the smoothness of f . $\square$

Similarly we have from Lemma 6.1

Lemma 8.13. *Let f be a smooth function. Then*

$$(145) \quad S_h(f(\alpha_i)\dot{\sigma}) = f_\alpha(\alpha_i)\dot{\sigma} + O(h^s) = A_{-s}(\dot{\sigma}_i) + O(h^s),$$

and the same is true for D_h instead of S_h .

We will make repeated use of the following result from [14]:

Lemma 8.14. *For $\dot{\theta}, \dot{\sigma}$ satisfying $\|\dot{\theta}\|_\infty \leq h^3, |\dot{\sigma}| < h^3$,*

$$(146) \quad (\sigma e^{i\theta})'_i = i s_\alpha e^{i\theta(\alpha_i)} \dot{\theta}_i + e^{i\theta(\alpha_i)} \dot{\sigma}_i + A_{-3}(\dot{\theta}_i) = O(h^3).$$

Proof. We have

$$(147) \quad \begin{aligned} (\sigma e^{i\theta})'_i &= \sigma e^{i\theta_i} - s_\alpha e^{i\theta(\alpha_i)} \\ &= s_\alpha e^{i\theta(\alpha_i)} (e^{i\dot{\theta}_i} - 1) + (\sigma - s_\alpha) e^{i\theta(\alpha_i)} e^{i\dot{\theta}_i} \\ &= s_\alpha e^{i\theta(\alpha_i)} (i\dot{\theta}_i + r(\dot{\theta}_i)) + \dot{\sigma}_i e^{i\theta(\alpha_i)} (1 + i\dot{\theta}_i + r(\dot{\theta}_i)), \end{aligned}$$

where $\|r(\dot{\theta})\|_{l^2} \leq \|\dot{\theta}^2\|_{l^2} \leq h^3 \|\dot{\theta}\|_{l^2}$. The first equality in (146) follows by using $|\dot{\sigma}| \leq h^3$ and the above estimate on $\|r(\dot{\theta})\|_{l^2}$ to show that the nonlinear terms in the variation are $A_{-3}(\dot{\theta})$. The second equality in (146) readily follows from (147), the bounds $\|\dot{\theta}\|_\infty \leq h^3, |\dot{\sigma}| < h^3$, and the above estimate on $\|r(\dot{\theta})\|_{l^2}$. $\square$

In estimating the variations, we will make use of the discrete Parseval equality. First, recall that the l^2 inner product is

$$(148) \quad (f, g)_h = h \sum_{i=-\frac{N}{2}+1}^{\frac{N}{2}} \overline{f_i} g_i,$$

for $f, g \in l^2$. Then, we have

Lemma 8.15 (Discrete Parseval's equality). *Let $f, g \in l^2$. Then*

$$(149) \quad (f, g)_h = 2\pi \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \overline{\hat{f}_k} \hat{g}_k.$$

In particular, when $g_i = f_i$

$$(150) \quad \|f\|_{l^2}^2 = 2\pi \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} |\hat{f}_k|^2.$$

A simple consequence of Parseval's equality is that derivatives can be transferred to a smooth function, similar to integration by parts.

Lemma 8.16. *Let $f(\cdot) \in C^{s+1}$ and $g \in l^2$. Then*

$$(151) \quad (f(\cdot), S_h g)_h = -(S_h f(\cdot), g)_h = A_{-s}(g).$$

The same result holds for the discrete average:

$$(152) \quad \langle f(\cdot), S_h g \rangle_h = -\langle S_h f(\cdot), g \rangle_h = A_{-s}(g).$$

Thus, when considered as an operator Q on g ,

$$(153) \quad Q(g) \equiv \langle f(\cdot), g \rangle_h = A_{-(s+1)}(g).$$

Proof. By Parseval's equality

$$\begin{aligned} (f(\cdot), S_h g)_h &= 2\pi \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \overline{\hat{f}_k} (ik) \hat{g}_k \\ &= 2\pi \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \overline{(-ik) \hat{f}_k} \hat{g}_k \\ (154) \quad &= -(S_h f(\cdot), g)_h. \end{aligned}$$

The result on discrete average follows similarly. $\square$

Note these results also hold for D_h . We will also make use of the fact that, for smooth f , $S_h^{-1}(f(\alpha_i)\phi_i)$ is a smoothing operator on ϕ_i . The proof of this includes, as a by-product, an 'integration by parts' formula for S_h^{-1} .

Lemma 8.17. *Let $f \in C^3$, $\phi \in l^2$, and assume $f(\alpha_i)\phi_i$ has zero mean, i.e., $\langle f(\cdot)\phi \rangle_h \equiv 0$. Then*

$$(155) \quad S_h^{-1}(f(\alpha_i)S_h\phi_i) = -S_h^{-1}(S_h f(\alpha_i)\phi_i) + f(\alpha_i)\phi_i + A_0(\phi_i),$$

and hence

$$(156) \quad S_h^{-1}(f(\alpha_i)\phi_i) = A_{-1}(\phi_i).$$

The proof of Lemma 8.17 is technical and is relegated to the appendix.

Lemma 8.17 can be used to prove the following estimate on τ_i :

Lemma 8.18. *Let τ_i represent the discrete interface. Then*

$$(157) \quad \dot{\tau}_i = iS_h^{-1}(\sigma e^{i\theta(\alpha_i)}\dot{\theta}_i) + A_{-3}(\dot{\theta}_i) + A_{-s}(\dot{\sigma}_i) + \dot{\tau}_c$$

$$(158) \quad = A_{-1}(\dot{\theta}_i) + A_{-s}(\dot{\sigma}_i) + \dot{\tau}_c.$$

Proof. Recall that

$$(159) \quad \tau_i = S_h^{-1}(\sigma e^{i\theta_i} - \langle \sigma e^{i\theta} \rangle_h) + \tau_c,$$

where $\tau_c = \hat{\tau}_0$ is the $k = 0$ Fourier mode of τ_i . Taking the variation, we have

$$(160) \quad \dot{\tau}_i = S_h^{-1}((\sigma e^{i\theta})_i^\cdot - \langle (\sigma e^{i\theta})^\cdot \rangle_h) + \dot{\tau}_c.$$

We now substitute the first equality in (146) and use Lemma 8.16 to find that

$$(161) \quad \begin{aligned} \dot{\tau}_i &= iS_h^{-1}(\sigma e^{i\theta(\alpha_i)}\dot{\theta}_i) + S_h^{-1}(e^{i\theta(\alpha_i)}\dot{\sigma}) \\ &\quad + \dot{\tau}_c + A_{-3}(\dot{\theta}_i) + A_{-s}(\dot{\sigma}_i). \end{aligned}$$

Finally, from Lemma 8.13 the second term on the right hand side of the above relation is $A_{-s}(\dot{\sigma}_i)$, and by Lemma 8.17 the first term is $A_{-1}(\dot{\theta})$, which gives the result. $\square$

Remark 8.19. Evidently, we also have from (160)

$$(162) \quad S_h \dot{\tau}_i = ((\sigma e^{i\theta})_i^\cdot - \langle (\sigma e^{i\theta})^\cdot \rangle_h),$$

and following the same reasoning as in the proof of Lemma 8.18

$$(163) \quad \begin{aligned} S_h \dot{\tau}_i &= i s_\alpha e^{i\theta(\alpha_i)}\dot{\theta}_i + A_{-3}(\dot{\theta}_i) + A_{-s}(\dot{\sigma}_i) \\ &= A_0(\dot{\theta}_i) + A_{-s}(\dot{\sigma}_i). \end{aligned}$$

Since $\|\dot{\theta}\|_{l^2} \leq h^{\frac{7}{2}}$, $|\dot{\sigma}| < h^{\frac{7}{2}}$ and $\|\dot{\theta}\|_\infty \leq h^3$, it also follows that $\|S_h \dot{\tau}\|_{l^2} \leq h^{\frac{7}{2}}$ and $\|S_h \dot{\tau}\|_\infty \leq h^3$.

9. ESTIMATES FOR THE VARIATION OF VELOCITIES

Recall the discrete equation for velocity has been decomposed as $u_i = \mathcal{H}_h \omega_i + (u_R)_i$ (cf. (61)). We further decompose $(u_R)_i$ as $(u_R)_i = U_{1,i} + U_{2,i} + U_{3,i} + U_{4,i}$, where

$$(164) \quad U_{1,i} = -\frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \omega_j^q \left(2\text{Re} \left(\frac{S_h \tau_j}{\tau_j - \tau_i} \right) + \cot \left(\frac{\alpha_i - \alpha_j}{2} \right) \right),$$

$$(165) \quad U_{2,i} = \frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \frac{\overline{\omega}_j^q S_h \tau_j}{\overline{\tau}_j - \overline{\tau}_i},$$

$$(166) \quad U_{3,i} = -\frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \frac{\overline{\omega}_j^q (\tau_j - \tau_i) \overline{S_h \tau_j}}{(\overline{\tau}_j - \overline{\tau}_i)^2},$$

$$(167) \quad U_{4,i} = (Q + iB)\tau_i - \frac{iJ\tau_i}{2},$$

for $i = -\frac{N}{2} + 1, \dots, \frac{N}{2}$.

Variations of the $U_{l,i}$, $l = 1, 2, \dots, 4$ are calculated using Lemmas 8.10 and 8.11. We represent these variations as the sum of linear and nonlinear quantities in the

variation, so that

$$\begin{aligned}
 \dot{U}_{1,i} = & -\frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \dot{\omega}_j^q \left(2\text{Re} \left(\frac{S_h \tau_h(\alpha_j)}{\tau_h(\alpha_j) - \tau_h(\alpha_i)} \right) + \cot \left(\frac{\alpha_i - \alpha_j}{2} \right) \right) \\
 & - \frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \left(\omega_h^q(\alpha_j) 2\text{Re} \left(\frac{S_h \dot{\tau}_j}{\tau_h(\alpha_j) - \tau_h(\alpha_i)} - \frac{S_h \tau_h(\alpha_j)}{(\tau_h(\alpha_j) - \tau_h(\alpha_i))^2} (\dot{\tau}_j - \dot{\tau}_i) \right) \right) \\
 (168) \quad & + \dot{U}_{1,i}^{NL},
 \end{aligned}$$

where $\dot{U}_{1,i}^{NL}$ represents the nonlinear terms in the variation, i.e., products of the dot quantities. Expressions for the nonlinear terms are given in the appendix.

Continuing,

$$\begin{aligned}
 \dot{U}_{2,i} = & \frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \left[\frac{\dot{\omega}_j^q S_h \tau_h(\alpha_j) + \overline{\dot{\omega}_j^q}(\alpha_j) S_h \dot{\tau}_j}{\overline{\tau_h}(\alpha_j) - \overline{\tau_h}(\alpha_i)} - (\overline{\dot{\tau}_j} - \overline{\dot{\tau}_i}) \frac{\overline{\dot{\omega}_j^q}(\alpha_j) S_h \tau_h(\alpha_j)}{[\overline{\tau_h}(\alpha_j) - \overline{\tau_h}(\alpha_i)]^2} \right] \\
 (169) \quad & + \dot{U}_{2,i}^{NL},
 \end{aligned}$$

and

$$\begin{aligned}
 \dot{U}_{3,i} = & -\frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \left[\left(\overline{\dot{\omega}_j^q} [\tau_h(\alpha_j) - \tau_h(\alpha_i)] \overline{S_h \tau_h}(\alpha_j) + \overline{\dot{\omega}_j^q}(\alpha_j) [\dot{\tau}_j - \dot{\tau}_i] \overline{S_h \tau_h}(\alpha_j) \right. \right. \\
 & + \overline{\dot{\omega}_j^q}(\alpha_j) [\tau_h(\alpha_j) - \tau_h(\alpha_i)] \overline{S_h \dot{\tau}_j} \left. \left(\frac{1}{\overline{\tau_h}(\alpha_j) - \overline{\tau_h}(\alpha_i)} \right)^2 \right. \\
 & \left. \left. - \frac{2 \overline{\dot{\omega}_j^q}(\alpha_j) [\tau_h(\alpha_j) - \tau_h(\alpha_i)] \overline{S_h \tau_h}(\alpha_j) [\overline{\dot{\tau}_j} - \overline{\dot{\tau}_i}]}{[\overline{\tau_h}(\alpha_j) - \overline{\tau_h}(\alpha_i)]^3} \right] \right] \\
 (170) \quad & + \dot{U}_{3,i}^{NL}.
 \end{aligned}$$

Finally,

$$(171) \quad \dot{U}_{4,i} = (Q + iB) \overline{\dot{\tau}_i} - \frac{iJ \dot{\tau}_i}{2}.$$

In the next section, we compute the leading order contribution to the variation in the velocity. This computation uses the following estimates.

Lemma 9.1. *Let $\dot{q} \in l^2$ be a variation of some quantity, and let*

$$(172) \quad I_h \dot{q}_i = \frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \frac{f(\alpha_j) \dot{q}_j}{\tau(\alpha_j) - \tau(\alpha_i)},$$

with $f(\alpha)$ and $\tau(\alpha)$ smooth, and $\tau_\alpha(\alpha) \neq 0$. Define

$$(173) \quad K_{1h}[f, \tau](\dot{q}_i) = \frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \dot{q}_j \left[\frac{f(\alpha_j)}{\tau(\alpha_j) - \tau(\alpha_i)} - \frac{f(\alpha_i)}{2\tau_\alpha(\alpha_i)} \cot\left(\frac{\alpha_j - \alpha_i}{2}\right) \right].$$

Then

$$(174) \quad I_h \dot{q}_i = -\frac{f(\alpha_i)}{2\tau_\alpha(\alpha_i)} \mathcal{H}_h \dot{q}_i + K_{1h}[f, \tau](\dot{q}_i),$$

and $K_{1h}[f, \tau](\dot{q}_i^p) = A_{-2}(\dot{q}_i)$, when filtering is applied.

Proof. The decomposition in (174) follows from adding and subtracting $-\frac{f(\alpha_i)}{2\tau_\alpha(\alpha_i)} \mathcal{H}_h \dot{q}_i$. The fact that K_{1h} is a smoothing operator on $\dot{q}^p$ then follows by noting that the kernel within brackets in K_{1h} is smooth, and applying Lemma 8.4. $\square$

Lemma 9.2. Under the same assumptions as in Lemma 9.1, let

$$(175) \quad J_h \dot{\tau}_i = \frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \frac{f(\alpha_j)(\dot{\tau}_j - \dot{\tau}_i)}{[\tau(\alpha_j) - \tau(\alpha_i)]^2},$$

and define

$$(176) \quad K_{2h}[f, \tau](\dot{\tau}_i) = \frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} (\dot{\tau}_j - \dot{\tau}_i) \left[\frac{f(\alpha_j)}{[\tau(\alpha_j) - \tau(\alpha_i)]^2} - \frac{f(\alpha_i)}{4\tau_\alpha^2(\alpha_i) \sin^2\left(\frac{\alpha_j - \alpha_i}{2}\right)} \right. \\ \left. - \frac{f_\alpha(\alpha_i)\tau_\alpha(\alpha_i) - f(\alpha_i)\tau_{\alpha\alpha}(\alpha_i)}{2\tau_\alpha^3(\alpha_i)} \cot\left(\frac{\alpha_j - \alpha_i}{2}\right) \right].$$

Then

$$(177) \quad J_h \dot{\tau}_i = -\frac{f(\alpha_i)}{2\tau_\alpha^2(\alpha_i)} \mathcal{H}_h(S_h \dot{\tau}_i) - \frac{f_\alpha(\alpha_i)\tau_\alpha(\alpha_i) - f(\alpha_i)\tau_{\alpha\alpha}(\alpha_i)}{2\tau_\alpha^3(\alpha_i)} \mathcal{H}_h \dot{\tau}_i \\ + K_{2h}[f, \tau](\dot{\tau}_i),$$

and

$$(178) \quad K_{2h}[f, \tau](\dot{\tau}_i^p) = A_{-2}(\dot{\tau}_i).$$

Proof. Write

$$(179) \quad \frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \frac{f(\alpha_j)(\dot{\tau}_j - \dot{\tau}_i)}{[\tau(\alpha_j) - \tau(\alpha_i)]^2} = \frac{h}{\pi} \left[\frac{f(\alpha_i)}{4\tau_\alpha^2(\alpha_i)} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \frac{\dot{\tau}_j - \dot{\tau}_i}{\sin^2\left(\frac{\alpha_j - \alpha_i}{2}\right)} \right. \\ \left. + \frac{f_\alpha(\alpha_i)\tau_\alpha(\alpha_i) - f(\alpha_i)\tau_{\alpha\alpha}(\alpha_i)}{2\tau_\alpha^3(\alpha_i)} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} (\dot{\tau}_j - \dot{\tau}_i) \cot\left(\frac{\alpha_j - \alpha_i}{2}\right) \right] \\ + K_2[f, \tau](\dot{\tau}^p)$$

by adding and subtracting $\frac{h}{\pi}$ times the quantity in brackets. To make use of the formula (131) for the derivative of the Hilbert transform, apply the identity [12]

$$(180) \quad \frac{1}{\sin^2 \frac{z}{2}} = 4 \sum_{n=-\infty}^{\infty} \frac{1}{(z - 2n\pi)^2}$$

to transform the representation of the first sum within brackets in (179) from a periodic to an infinite domain,

$$(181) \quad \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \frac{\dot{\tau}_j - \dot{\tau}_i}{\sin^2 \left(\frac{\alpha_j - \alpha_i}{2} \right)} = 4 \sum_{(j-i) \text{ odd}} \frac{\dot{\tau}_j - \dot{\tau}_i}{(\alpha_j - \alpha_i)^2},$$

where we have used the periodicity of $\dot{\tau}_j$. Identify

$$(182) \quad \mathcal{H}_h(S_h \dot{\tau}_i) = \frac{2h}{\pi} \sum_{(j-i) \text{ odd}} \frac{\dot{\tau}_i - \dot{\tau}_j}{(\alpha_i - \alpha_j)^2},$$

and

$$(183) \quad \mathcal{H}_h \dot{\tau}_i = -\frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} (\dot{\tau}_j - \dot{\tau}_i) \cot \left(\frac{\alpha_j - \alpha_i}{2} \right)$$

to obtain (177) (also using that the discrete Hilbert transform of a constant is zero). Finally, (178) follows from the observation that the quantity within brackets in the definition of K_{2h} , namely,

$$(184) \quad \begin{aligned} h(\alpha, \alpha') &= \frac{f(\alpha')}{[\tau(\alpha') - \tau(\alpha)]^2} - \frac{f(\alpha)}{4\tau_\alpha^2(\alpha) \sin^2 \left(\frac{\alpha' - \alpha}{2} \right)} \\ &\quad - \frac{f_\alpha(\alpha_i) \tau_\alpha(\alpha_i) - f(\alpha_i) \tau_{\alpha\alpha}(\alpha_i)}{2\tau_\alpha^3(\alpha_i)} \cot \left(\frac{\alpha' - \alpha}{2} \right) \end{aligned}$$

is a smooth function of α and α' . □

Lemmas 9.3 and 9.4 are derived similarly and are presented without proof.

Lemma 9.3. *Under the same assumptions as Lemma 9.1, let*

$$(185) \quad L_h \dot{q}_i = \frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \frac{f(\alpha_j)(\tau(\alpha_j) - \tau(\alpha_i)) \dot{q}_j}{[\tau(\alpha_j) - \tau(\alpha_i)]^2},$$

and define

$$(186) \quad K_{3h}[f, \tau](\dot{q}_i) = \frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \dot{q}_j \left[\frac{(\tau(\alpha_j) - \tau(\alpha_i)) f(\alpha_j)}{[\tau(\alpha_j) - \tau(\alpha_i)]^2} + \frac{f(\alpha_i) \tau_\alpha(\alpha_i)}{2\tau_\alpha^2(\alpha_i)} \cot \left(\frac{\alpha_i - \alpha_j}{2} \right) \right].$$

Then

$$(187) \quad L_h \dot{q}_i = -\frac{f(\alpha_i) \tau_\alpha(\alpha_i)}{2\tau_\alpha^2(\alpha_i)} \mathcal{H}_h \dot{q}_i + K_{3h}[f, \tau](\dot{q}_i),$$

and $K_{3h}[f, \tau](\dot{q}_i^p) = A_{-2}(\dot{q}_i)$.

Lemma 9.4. *Under the same assumption as Lemma 9.1, let*

$$(188) \quad M_h \bar{\tau}_i = \frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \frac{f(\alpha_j)(\tau(\alpha_j) - \tau(\alpha_i))(\bar{\tau}_j - \bar{\tau}_i)}{[\overline{\tau(\alpha_j) - \tau(\alpha_i)}]^3},$$

and define

$$(189) \quad K_{4h}[f, \tau](\bar{\tau}_j) = \frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} (\bar{\tau}_j - \bar{\tau}_i) \left[\frac{(\tau(\alpha_j) - \tau(\alpha_i))f(\alpha_j)}{[\overline{\tau(\alpha_j) - \tau(\alpha_i)}]^2} \right. \\ \left. - \frac{f(\alpha_i)\tau_\alpha(\alpha_i)}{4\tau_\alpha^3(\alpha_j)\sin^2\left(\frac{\alpha_j - \alpha_i}{2}\right)} - \frac{p[f](\alpha_i)}{2\tau_\alpha^3(\alpha_i)} \cot\left(\frac{\alpha_j - \alpha_i}{2}\right) \right],$$

where

$$(190) \quad p[f](\alpha_i) = \frac{\tau_\alpha(\alpha_i)f(\alpha_i)}{2} \left(\frac{\tau_{\alpha\alpha}(\alpha_i)}{\tau_\alpha(\alpha_i)} - \frac{3\overline{\tau_{\alpha\alpha}(\alpha_i)}}{\tau_\alpha(\alpha_i)} \right) + \tau_\alpha(\alpha_i)f_\alpha(\alpha_i).$$

Then

$$(191) \quad M_h \bar{\tau}_i = -\frac{f(\alpha_i)\tau_\alpha(\alpha_i)}{2\tau_\alpha^3(\alpha_i)} \mathcal{H}_h(\overline{S_h \dot{\tau}_i}) - \frac{p[f](\alpha_i)}{2\tau_\alpha^3(\alpha_i)} \mathcal{H}_h \bar{\tau}_i + K_{4h}[f, \tau](\bar{\tau}_i),$$

and $K_{4h}[f, \tau](\bar{\tau}_i^p) = A_{-2}(\dot{\tau}_i)$.

Remark 9.5. If filtering is not applied, then it is easy to see that $K_{1h}[\cdot, \cdot](\dot{q}) = A_0(\dot{q})$ and $K_{3h}[\cdot, \cdot](\dot{q}) = A_0(\dot{q})$. Similarly, $K_{2h}[\cdot, \cdot](\dot{\tau}) = A_0(\dot{\tau})$ and $K_{4h}[\cdot, \cdot](\bar{\tau}) = A_0(\dot{\tau})$. These estimates are a consequence of Remark 8.5.

9.1. Leading order velocity variations. We identify the most singular terms in the variation of the complex velocity, $\dot{u}_i = \mathcal{H}_h \dot{\omega} + \sum_{l=1}^4 \dot{U}_{l,i}$. First, note that in (168)-(170) we can replace ω_h with ω , τ_h with τ , and $S_h \tau_h$ with τ_α , incurring by consistency a high-order or $O(h^s)$ error. Now, consider $\dot{U}_1$ in (168), and denote the first sum (with the above replacements) by $K_{0h}(\dot{\omega}_i^q)$, so that

$$(192) \quad K_{0h}(\dot{\omega}_i^q) = \frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \dot{\omega}_j^q \left(2\operatorname{Re} \left(\frac{\tau_\alpha(\alpha_j)}{\tau(\alpha_j) - \tau(\alpha_i)} \right) + \cot \left(\frac{\alpha_i - \alpha_j}{2} \right) \right).$$

Next apply Lemmas 9.1 and 9.2 to find

$$(193) \quad \begin{aligned} \dot{U}_{1,i} = & -K_{0h}(\dot{\omega}_i^q) - K_{1h}[\omega^q, \tau](S_h \dot{\tau}_i) - K_{1h}[\omega^q, \bar{\tau}](\overline{S_h \dot{\tau}_i}) \\ & - \omega_\alpha^q(\alpha_i) \left(\frac{\mathcal{H}_h \dot{\tau}_i}{2\tau_\alpha(\alpha_i)} + \text{c.c.} \right) + K_{2h}[\omega^q \tau_\alpha, \tau](\dot{\tau}_i) + K_{2h}[\omega^q \overline{\tau_\alpha}, \bar{\tau}](\bar{\tau}_i) \\ & + \dot{U}_{1,i}^{NL} + O(h^s), \end{aligned}$$

where c.c. denotes the complex conjugate of the previous term. The leading order contribution to $\dot{U}_{2,i}$ in (169) is determined from Lemmas 9.1 and 9.2 as

$$\begin{aligned}
 \dot{U}_{2,i} = & -\frac{1}{2} \frac{\tau_\alpha(\alpha_i)}{\bar{\tau}_\alpha(\alpha_i)} \mathcal{H}_h \dot{\omega}_i^q + K_{1h}[\tau_\alpha, \bar{\tau}] (\dot{\omega}_i^q) - \frac{\bar{\omega}^q(\alpha_i)}{2\bar{\tau}_\alpha(\alpha_i)} \mathcal{H}_h(S_h \dot{\tau}_i) + K_{1h}[\bar{\omega}^q, \bar{\tau}] (S_h \dot{\tau}_i) \\
 & + \frac{\bar{\omega}^q(\alpha_i) \tau_\alpha(\alpha_i)}{2\bar{\tau}_\alpha^2(\alpha_i)} \mathcal{H}_h(\bar{S}_h \dot{\tau}_i) + \frac{1}{2\bar{\tau}_\alpha^3(\alpha_i)} (\bar{\tau}_\alpha(\alpha_i) (\bar{\omega}^q \tau_\alpha)_\alpha(\alpha_i) \\
 & - \bar{\omega}^q(\alpha_i) \tau_\alpha(\alpha_i) \bar{\tau}_{\alpha\alpha}(\alpha_i)) \mathcal{H}_h(\bar{\tau}_i) - K_{2h}[\bar{\omega}^q \tau_\alpha, \bar{\tau}] (\bar{\tau}_i) + \dot{U}_{2,i}^{NL} + O(h^s).
 \end{aligned}
 \tag{194}$$

Similarly, we find from (170) and Lemmas 9.2-9.4 that

$$\begin{aligned}
 \dot{U}_{3,i} = & \frac{\tau_\alpha(\alpha_i)}{2\bar{\tau}_\alpha(\alpha_i)} \mathcal{H}_h \dot{\omega}_i^q - K_{3h}[\bar{\tau}_\alpha, \tau] (\dot{\omega}_i^q) + \frac{\bar{\omega}^q(\alpha_i)}{2\bar{\tau}_\alpha(\alpha_i)} \mathcal{H}_h(S_h \dot{\tau}_i) + \frac{\bar{\omega}_\alpha^q(\alpha_i)}{2\bar{\tau}_\alpha(\alpha_i)} \mathcal{H}_h \dot{\tau}_i \\
 & - K_{2h}[\bar{\omega}^q \tau_\alpha, \bar{\tau}] (\dot{\tau}_i) - \frac{\bar{\omega}^q(\alpha_i) \tau_\alpha(\alpha_i)}{2\bar{\tau}_\alpha^2(\alpha_i)} \mathcal{H}_h(\bar{S}_h \dot{\tau}_i) - K_{3h}[\bar{\omega}^q, \tau] (\bar{S}_h \dot{\tau}_i) \\
 & - \frac{p[2\bar{\omega}^q \tau_\alpha](\alpha_i)}{2\bar{\tau}_\alpha^3(\alpha_i)} \mathcal{H}_h(\bar{\tau}_i) + K_{4h}[2\bar{\omega}^q \tau_\alpha, \tau] (\bar{\tau}_i) + \dot{U}_{3,i}^{NL} + O(h^s),
 \end{aligned}
 \tag{195}$$

where

$$p[2\bar{\omega}^q \tau_\alpha](\alpha_i) = \bar{\omega}^q(\alpha_i) \left(\tau_{\alpha\alpha}(\alpha_i) \bar{\tau}_\alpha(\alpha_i) - 3\bar{\tau}_{\alpha\alpha}(\alpha_i) \tau_\alpha(\alpha_i) \right) + 2\tau_\alpha(\alpha_i) (\bar{\omega} \bar{\tau}_\alpha)_\alpha(\alpha_i).$$

A simplified representation of the leading order velocity variations is provided in Section 9.3. Before presenting this, we consider the variation of ω .

9.2. Leading order variation of ω . For the case of viscosity matched fluids, in which $\beta = 0$ and $\chi = 1/2$, the equation for ω localizes. Taking the variation of (48) using (50) with $\tilde{\omega} = 0$, it is easy to see that

$$\begin{aligned}
 \dot{\omega}_i = & -\frac{1}{4} (\dot{S}_i e^{i\theta(\alpha_i)} + \mathcal{S}(\alpha_i) (e^{i\theta})_i) \\
 & + \frac{\kappa_B}{4s_\alpha^2} [\mathbf{i}\theta_{\alpha\alpha}(\alpha_i) (e^{i\theta})_i + \mathbf{i}e^{i\theta(\alpha_i)} S_h^2 \dot{\theta}_i] \\
 & + \frac{\kappa_B}{2s_\alpha} \mathbf{i}e^{i\theta(\alpha_i)} \theta_{\alpha\alpha}(\alpha_i) \left(\frac{1}{\sigma}\right)_i + \dot{\omega}_i^{NL},
 \end{aligned}
 \tag{197}$$

where $\dot{\omega}_i^{NL}$ contains nonlinear terms or products of the variations on the right hand side.

We now give estimates for each of the terms in (197). Lemma 8.14 provides the estimate (taking $\sigma = s_\alpha = 1$),

$$(e^{i\theta})_i = \mathbf{i}e^{i\theta(\alpha_i)} \dot{\theta}_i + A_{-3}(\dot{\theta}) = O(h^3).
 \tag{198}$$

It is easily seen from (78) and Lemmas 8.10 and 8.11, that

$$\dot{S}_i = \frac{1 + \mathcal{S}_0(\alpha_i)}{s_{0\alpha} \alpha_{0\alpha}(\alpha_i)} \dot{\sigma} - \frac{s_\alpha(1 + \mathcal{S}_0(\alpha_i))}{s_{0\alpha} \alpha_{0\alpha}^2(\alpha_i)} (D_h \dot{\alpha}_0)_i + \dot{S}_i^{NL} + O(h^s),
 \tag{199}$$

where we have assumed $\dot{S}_0 = 0$, i.e., the initial tension $\mathcal{S}_{0i}$ is exactly $\mathcal{S}_0(\alpha_i)$, and similarly $\dot{\sigma}_0 = 0$. An estimate for $\dot{S}_i^{NL}$ is readily obtained as

$$\dot{S}_i^{NL} = A_{-s}(\dot{\sigma}) + A_{-1}(\dot{\alpha}_0),
 \tag{200}$$

using $|\dot{\sigma}| \leq h^3$ and $\|D_h(\dot{\alpha}_0)\|_\infty \leq ch^2$. It follows that (199) can be written as

$$(201) \quad \dot{S}_i = A_{-s}(\dot{\sigma}_i) - \tilde{f}(\alpha_i)(D_h \dot{\alpha}_0)_i + A_{-1}(\dot{\alpha}_{0i}) + O(h^s),$$

where

$$(202) \quad \tilde{f}(\alpha) = \frac{s_\alpha(1 + S_0(\alpha))}{s_{0\alpha}\alpha_{0\alpha}^2(\alpha)} > 0$$

is a smooth, real and positive function. The positivity of $\tilde{f}(\alpha)$ will be seen to be critical. Indeed, it is found to be necessary for the well-posedness of the continuous equations. We note that $\tilde{f}(\alpha_i)(D_h \dot{\alpha}_0)_i = A_0(D_h \dot{\alpha}_{0i})$, but to make energy estimates we retain the specific form in (201).

An estimate on the nonlinear term,

$$(203) \quad \dot{\omega}_i^{NL} = A_{-1}(\dot{\theta}_i) + A_{-s}(\dot{\sigma}_i) + A_{-2}(\dot{\alpha}_{0i}) + O(h^s),$$

is derived at the end of this section. We also need the following estimate for $\left(\frac{1}{\sigma}\right)^\cdot$:

$$(204) \quad \left(\frac{1}{\sigma}\right)^\cdot = -\frac{\dot{\sigma}}{s_\alpha^2} + \frac{\dot{\sigma}^2}{s_\alpha^2(s_\alpha + \dot{\sigma})} = O(h^3),$$

which follows from Lemma 8.11, the boundedness s_α away from zero, and $|\dot{\sigma}| < h^3$. This also implies that

$$(205) \quad f(\alpha_i)\left(\frac{1}{\sigma}\right)^\cdot = A_{-s}(\dot{\sigma}_i) \text{ for smooth } f.$$

Together, (197), (198), (199), (203) and the above remarks tell us that

$$(206) \quad \begin{aligned} \dot{\omega}_i &= \kappa_B \frac{ie^{i\theta(\alpha_i)}}{4s_\alpha^2} S_h^2 \dot{\theta}_i + A_0(\dot{\theta}_i) + A_{-s}(\dot{\sigma}_i) + \frac{1}{4} e^{i\theta(\alpha_i)} \tilde{f}(\alpha_i)(D_h \dot{\alpha}_0)_i \\ &\quad + A_{-1}(\dot{\alpha}_{0i}) + O(h^s), \end{aligned}$$

where $\tilde{f}(\alpha) > 0$ is given by (202). The filtered quantity $\dot{\omega}_i^q$ satisfies the same estimate except $S_h^2 \dot{\theta}_i$ in the first term is replaced by $D_h^2 \dot{\theta}_i$, i.e.,

$$(207) \quad \begin{aligned} \dot{\omega}_i^q &= \kappa_B \frac{ie^{i\theta(\alpha_i)}}{4s_\alpha^2} D_h^2 \dot{\theta}_i + A_0(\dot{\theta}_i) + A_{-s}(\dot{\sigma}_i) + \frac{1}{4} e^{i\theta(\alpha_i)} \tilde{f}(\alpha_i)(D_h \dot{\alpha}_0)_i \\ &\quad + A_{-1}(\dot{\alpha}_{0i}) + O(h^s). \end{aligned}$$

We complete the derivation of (206) and (207) by giving some details of the estimate (203) for the nonlinear term $\dot{\omega}_i^{NL}$. This term contains, for example, products of $(e^{-i\theta})_i^\cdot$ and $\left(\frac{1}{\sigma}\right)^\cdot$, each of which is $O(h^3)$, with discrete derivatives of $\dot{\theta}$ and $\dot{\alpha}_{0i}$. It is easy to see that these products satisfy the estimate (203). This verifies (206) which is the main result of this section.

9.3. Summary of velocity variation. The velocity variation is the sum of the contributions from $\mathcal{H}_h \dot{\omega}_i$ and $(\dot{u}_R)_i = \sum_{l=1}^4 \dot{U}_{l,i}$ in (171), (193)-(195). We anticipate the main contribution will come from $\mathcal{H}_h \dot{\omega}_i$. The leading order $\mathcal{H}_h \overline{\dot{\omega}_i^q}$ term in the sum $\dot{U}_{2,i} + \dot{U}_{3,i}$ cancels out. This is related to the smoothness of the kernel (59) in the velocity equation. The next order terms in $\dot{U}_{2,i} + \dot{U}_{3,i}$ containing $\mathcal{H}_h S_h \dot{\tau}_i$ and its conjugate also cancel out. This has important consequences in the stability of the discrete equations in the drop evolution problem, i.e., with $S = \text{constant}$ and $\kappa_B = 0$.

To identify lower order terms in the velocity variation, we first apply Remarks 8.5 and 8.19 to see

$$(208) \quad K_{1h}[\cdot, \cdot](S_h \dot{\tau}_i) \text{ and } K_{3h}[\cdot, \cdot](S_h \dot{\tau}_i) = A_0(\dot{\theta}_i) + A_{-s}(\dot{\sigma}),$$

and similarly using Lemma 8.18,

$$(209) \quad K_{2h}[\cdot, \cdot](\dot{\tau}_i) \text{ and } K_{4h}[\cdot, \cdot](\dot{\tau}_i) = A_{-1}(\dot{\theta}_i) + A_{-s}(\dot{\sigma}).$$

Estimates for $K_{0h}[\cdot, \cdot](\dot{\omega}^q)$, $K_{1h}[\cdot, \cdot](\dot{\omega}_i^q)$, and $K_{3h}[\cdot, \cdot](\dot{\omega}_i^q)$ are obtained by substituting for $\dot{\omega}^q$ using (207) and absorbing smooth functions such as $e^{i\theta(\alpha_i)}$ into the kernels. This gives by Lemma 8.4

$$(210) \quad \begin{aligned} K_{0h}[\cdot, \cdot](\dot{\omega}^q) &= A_{-2}(D_h^2 \dot{\theta}_i) + A_{-2}(D_h \dot{\alpha}_{0i}) + A_{-s}(\dot{\sigma}_i) + O(h^s) \\ &= A_0(\dot{\theta}_i) + A_{-1}(\dot{\alpha}_{0i}) + A_{-s}(\dot{\sigma}_i) + O(h^s), \end{aligned}$$

and similarly for $K_{1h}[\cdot, \cdot](\dot{\omega}^q)$ and $K_{3h}[\cdot, \cdot](\dot{\omega}^q)$.

We also need to estimate U_4 defined in (171). From Lemma 8.18, we immediately see that

$$(211) \quad \dot{U}_{4,i} = A_{-1}(\dot{\theta}_i) + A_{-s}(\dot{\sigma}) + A_0(\dot{\tau}_c).$$

Note that U_4 is the only velocity term in which $\tau_c(t)$ appears. The other terms in our velocity decomposition, $U_1, \dots, U_3$ depend only on the difference $\tau_j - \tau_i$, for which τ_c cancels out. In the appendix, we show that the nonlinear term satisfies

$$(212) \quad \dot{u}_i^{NL} = \sum_{n=1}^4 \dot{U}_{n,i}^{NL} = A_0(\dot{\theta}_i) + A_{-1}(\dot{\alpha}_{0i}) + A_{-s}(\dot{\sigma}) + O(h^s).$$

In summary, the above remarks show that

$$(213) \quad (\dot{u}_R)_i = \sum_{l=1}^4 \dot{U}_{l,i} = A_0(\dot{\theta}_i) + A_{-1}(\dot{\alpha}_{0i}) + A_{-s}(\dot{\sigma}_i) + A_0(\dot{\tau}_c) + O(h^s),$$

which is the main result of this section.

9.4. Tangential and normal velocity variations. The variation of the discrete normal velocity is calculated from (69), rather than directly from the dot product (60), since we recall there is some subtlety in discretizing u_n . We find from (69)

$$(214) \quad \begin{aligned} (u_n)_i &= \frac{\kappa_B}{4s_\alpha^2} \mathcal{H}_h(S_h^2 \dot{\theta}_i) + \frac{\kappa_B}{4} \left(\frac{1}{\sigma^2} \right)' \mathcal{H}_h(\theta_{\alpha\alpha}(\alpha_i)) + \text{Im} \{ - [\mathcal{H}_h, e^{-i\theta(\alpha_i)}] (\dot{\omega}_i^q) \\ &\quad - [\mathcal{H}_h, (e^{-i\theta})_i] (\omega^q(\alpha_i)) + (u_R)_i e^{-i\theta(\alpha_i)} + u_R(\alpha_i) (e^{-i\theta})_i \} + (u_n)_i^{NL} + O(h^s), \end{aligned}$$

where $(u_n)_i^{NL}$ represents products of the variations on the right hand side.

It is straightforward to estimate each of the terms in (214). Clearly,

$$(215) \quad \frac{\kappa_B}{4} \left(\frac{1}{\sigma^2} \right)' \mathcal{H}_h(\theta_{\alpha\alpha}(\alpha_i)) = A_{-s}(\dot{\sigma}_i)$$

(cf. Lemma 8.13). Using Lemma 8.7 and the same arguments that lead to (210),

$$(216) \quad [\mathcal{H}_h, e^{-i\theta(\alpha_i)}] (\dot{\omega}_i^q) = A_0(\dot{\theta}_i) + A_{-1}(\dot{\alpha}_{0i}) + A_{-s}(\dot{\sigma}_i) + O(h^s).$$

It is also easy to see that

$$(217) \quad [\mathcal{H}_h, (e^{-i\theta})_i] (\omega^q(\alpha_i)) = A_0(\dot{\theta}).$$

Indeed, we can write

$$(218) \quad \begin{aligned} [\mathcal{H}_h, (e^{-i\theta})_i] (\omega^q(\alpha_i)) &= \mathcal{H}_h((e^{-i\theta})_i (\omega^q(\alpha_i))) - \omega^q(\alpha_i) \mathcal{H}_h((e^{i\theta})_i) \\ &+ \omega^q(\alpha_i) \mathcal{H}_h((e^{i\theta})_i) - (e^{i\theta})_i \mathcal{H}_h(\omega^q(\alpha_i)), \end{aligned}$$

and note that the first two terms combine to form an integral operator with a smooth kernel on the (unfiltered) $(e^{i\theta})_i$, while the latter two terms are clearly $A_0(\dot{\theta})$ functions. Remark 8.5 then implies (217). An estimate for $(\dot{u}_R)_i$ is given in (213). Finally, it is straightforward to show that the nonlinear term $(\dot{u}_n)_i^{NL}$ involves higher order smoothing operators. The above arguments demonstrate that

$$(219) \quad (\dot{u}_n)_i = \frac{\kappa_B}{4s_\alpha^2} \mathcal{H}_h(S_h^2 \dot{\theta}_i) + A_{-1}(\dot{\alpha}_{0i}) + A_0(\dot{\theta}_i) + A_{-s}(\dot{\sigma}) + A_0(\dot{\tau}_c) + O(h^s).$$

We will also need the variation in the tangential velocity $\dot{u}_s$, since this appears in the evolution equation for $\dot{\alpha}_0$. We leave it to the reader to show, from (70), using the same arguments as for $(\dot{u}_n)_i$, that

$$(220) \quad \begin{aligned} (\dot{u}_s)_i &= -\frac{1}{4} \mathcal{H}_h(\dot{S}_i) + A_{-1}(\dot{\alpha}_{0i}) + A_0(\dot{\theta}_i) + A_{-s}(\dot{\sigma}) \\ &+ A_0(\dot{\tau}_c) + O(h^s). \end{aligned}$$

The estimate in (201) implies $\dot{S}_i$ can be replaced by $-\tilde{f}(\alpha_i)(D_h \alpha_0)_i$, where $\tilde{f}(\alpha) > 0$ is defined in (202). It follows that

$$(221) \quad \begin{aligned} (\dot{u}_s)_i &= \frac{1}{4} \mathcal{H}_h(\tilde{f}(\alpha_i) D_h \alpha_{0i}) + A_0(\dot{\theta}_i) + A_{-s}(\dot{\sigma}) \\ &+ O(\dot{\tau}_c) + O(h^s). \end{aligned}$$

The next quantity we need to estimate is the variation of ϕ_s . Taking the variation of (77), we find

$$(222) \quad \begin{aligned} (\dot{\phi}_s)_i &= S_h^{-1}(u_n \theta_\alpha(\cdot) + u_n(\cdot) S_h \dot{\theta} - \langle u_n \theta_\alpha(\cdot) + u_n(\cdot) S_h \dot{\theta} \rangle_h)_i \\ &+ O(h^s) + (\dot{\phi}_s^{NL})_i, \end{aligned}$$

where

$$(223) \quad (\dot{\phi}_s^{NL})_i = S_h^{-1}(u_n S_h \dot{\theta} - \langle u_n S_h \dot{\theta} \rangle_h).$$

We readily obtain from Lemmas 8.16 and 8.17 with (219) the estimate

$$(224) \quad (\dot{\phi}_s)_i = A_0(S_h \dot{\theta}_i) + A_{-2}(\dot{\alpha}_{0i}) + A_{-s}(\dot{\sigma}) + A_0(\dot{\tau}_c) + O(h^s).$$

In obtaining this estimate, we have used $A_{-1}(S_h^2 \dot{\theta}) = A_0(S_h \dot{\theta})$, and the nonlinear terms are found to involve higher order smoothing operators than the terms already present in (224).

10. EVOLUTION EQUATIONS FOR THE ERROR

An evolution equation for $\dot{\theta}$ is formed by substituting the exact solution s_α , $\theta(\alpha_i)$ into (71), using consistency, and subtracting the result from (71). This gives

$$(225) \quad \begin{aligned} \frac{d\dot{\theta}_i}{dt} &= \frac{1}{s_\alpha} (S_h(\dot{u}_n)_i + \phi_s(\alpha_i) S_h(\dot{\theta}_i) + \theta_\alpha(\alpha_i) (\dot{\phi}_s)_i) \\ &+ \left(\frac{1}{\sigma} \right) ((u_n)_\alpha(\alpha_i) + \phi_s(\alpha_i) \theta_\alpha(\alpha_i)) \\ &+ \dot{\Theta}_i^{NL} + O(h^s), \end{aligned}$$

where the nonlinear term $\dot{\Theta}_i^{NL}$ contains products of the variations on the right hand side. In (225), we have also used consistency to replace, for example, $S_h \theta(\alpha_i)$ with $\theta_\alpha(\alpha_i)$, incurring an $O(h^s)$ error. It is easy to see that the nonlinear term satisfies

$$(226) \quad \dot{\Theta}_i^{NL} = A_0(\dot{\theta}_i) + A_{-s}(\dot{\sigma}) + h^3 A_0(\dot{\tau}_c) + O(h^s).$$

The relation (225) can be further simplified using (219), (224) and the first equation of (204), which give

$$(227) \quad \begin{aligned} \frac{d\dot{\theta}_i}{dt} &= \frac{\kappa_B}{4s_\alpha^3} \mathcal{H}_h(S_h^3 \dot{\theta}_i) + S_h A_0(\dot{\theta}_i) + A_0(S_h \dot{\theta}_i) + S_h A_{-1}(\dot{\alpha}_{0i}) \\ &+ A_{-s}(\dot{\sigma}) + A_0(\dot{\tau}_c) + O(h^s). \end{aligned}$$

The evolution equation for $\dot{\sigma}$ is derived similarly so that from (72),

$$(228) \quad \frac{d\dot{\sigma}}{dt} = -\langle (\dot{u}_n) \theta_\alpha(\cdot) \rangle_h - \langle u_n(\cdot) S_h \dot{\theta} \rangle_h - \langle (\dot{u}_n) S_h \dot{\theta} \rangle_h + O(h^s).$$

The nonlinear term is estimated as

$$(229) \quad \langle (\dot{u}_n) S_h \dot{\theta} \rangle_h = A_0(\dot{\theta}) + A_{-s}(\dot{\sigma}) + A_{-3}(\dot{\alpha}_{0i}) + h^2 A_0(\dot{\tau}_c) + O(h^s),$$

using (219) and the bound $\|S_h \dot{\theta}\|_{l^2} \leq h^2$. Equation (228) can be further simplified using Lemma 8.16 which together with (219) gives

$$(230) \quad \frac{d\dot{\sigma}}{dt} = A_0(\dot{\theta}) + A_{-s}(\dot{\sigma}) + A_{-1}(\dot{\alpha}_{0i}) + A_0(\dot{\tau}_c) + O(h^s).$$

We also need the variation of the evolution equation (79) for $\dot{\alpha}_0$. Let

$$(231) \quad u_t(\alpha) = (\phi_s(\alpha) - u_s(\alpha))e^{i\theta(\alpha)}$$

be the difference between the tangential interface velocity at a fixed α and the tangential fluid velocity at $\tau(\alpha)$. Taking the variation of (79) gives

$$(232) \quad \begin{aligned} \left(\frac{d\dot{\alpha}_0}{dt} \right)_i &= \frac{u_t(\alpha_i)}{s_\alpha e^{i\theta(\alpha_i)}} (D_h \dot{\alpha}_0)_i - \frac{\alpha_{0\alpha}(\alpha_i) u_t(\alpha_i)}{s_\alpha^2 e^{i\theta(\alpha_i)}} \dot{\sigma} - \frac{\alpha_{0\alpha}(\alpha_i) u_t(\alpha_i)}{s_\alpha (e^{i\theta(\alpha_i)})^2} (e^{i\theta_i}) \\ &+ \frac{\alpha_{0\alpha}(\alpha_i)}{s_\alpha e^{i\theta(\alpha_i)}} (\dot{u}_t)_i + (\dot{\alpha}_0^{NL})_i + O(h^s), \end{aligned}$$

where $\dot{\alpha}_0^{NL}$ contains nonlinear terms. From (221) and (224), we have (taking $\chi = \frac{1}{2}$),

$$(233) \quad \begin{aligned} (u_t)_i &= -\frac{e^{i\theta(\alpha_i)}}{4} \mathcal{H}_h(\tilde{f}(\alpha_i) D_h \dot{\alpha}_{0i}) + A_0(S_h \dot{\theta}_i) + A_{-s}(\dot{\sigma}) + A_{-1}(\dot{\alpha}_{0i}) \\ &+ A_0(\dot{\tau}_c) + O(h^s), \end{aligned}$$

where we have also used Lemma 8.14. Define the smooth functions

$$(234) \quad \tilde{f}_1(\alpha) = \frac{u_t(\alpha)}{s_\alpha e^{i\theta(\alpha)}} \text{ and } \tilde{f}_2(\alpha) = \frac{\tilde{f}(\alpha)(\alpha_0)_\alpha(\alpha)}{4s_\alpha},$$

where, crucially, $\alpha_{0\alpha}(\alpha)$, $\tilde{f}(\alpha)$ (cf. (202)) and hence $\tilde{f}_2(\alpha)$ are all positive functions. We note that $\alpha_{0\alpha}(\alpha) = 1$ at $t = 0$, and the positive definiteness of $\alpha_{0\alpha}$ is a consequence of $\alpha_0(\alpha)$ being a one-to-one mapping, which is related to the physical

property that material fluid points cannot overlap. Using (233), (234), and the commutator identity Lemma 8.7, equation (232) can be written as

$$(235) \quad \left(\frac{d\dot{\alpha}_0}{dt} \right)_i = \tilde{f}_1(\alpha_i)(D_h \dot{\alpha}_0)_i - \tilde{f}_2(\alpha_i) \mathcal{H}_h D_h \dot{\alpha}_{0i} + A_0(S_h \dot{\theta}_i) + A_{-s}(\dot{\sigma}) + A_{-1}(\dot{\alpha}_{0i}) + A_0(\dot{\tau}_c) + O(h^s).$$

The nonlinear terms are smoother or smaller than terms that are already present in (235), as is easily verified.

Next, we derive the evolution equation for $\dot{\tau}_c$. This immediately follows from (76), i.e.,

$$(236) \quad \left. \frac{d\dot{\tau}_c}{dt} \right|_\alpha = \hat{v}_0,$$

where $\hat{v}_0$ is the $k = 0$ Fourier mode of the interface velocity (18). Equations (227), (230), (235), and (236) are the main result of this section.

11. ENERGY ESTIMATES

Recall that we have defined a time T^* in (117), and all the estimates we obtain are valid for $t \leq T^*$. We close this so-called “ T^* argument” and prove Theorem 5.1 by showing at the end that T^* can be extended to T , the existence time for the continuous problem.

Define the energy

$$(237) \quad E(t) = \dot{\sigma}^2 + (\dot{\theta}, \dot{\theta})_h + (\dot{\alpha}_0, \dot{\alpha}_0)_h + |\dot{\tau}_c|^2,$$

and take the time derivative

$$(238) \quad \frac{1}{2} \frac{dE}{dt} = \dot{\sigma} \dot{\sigma}_t + (\dot{\theta}, \dot{\theta}_t)_h + (\dot{\alpha}_0, \dot{\alpha}_{0t})_h + 2\text{Re}(\overline{\dot{\tau}_c} \dot{\tau}_{ct}).$$

We will bound the above by $E(t) + O(h^s)$.

The first product on the right side of (238) is readily bounded using (230) and Young’s inequality,

$$(239) \quad \dot{\sigma} \dot{\sigma}_t \leq cE + O(h^s).$$

The other inner products in (238) are bounded by making use of parabolic smoothing. We have from (227)

$$(240) \quad (\dot{\theta}, \dot{\theta}_t)_h = \left(\dot{\theta}, \frac{\kappa_B}{4s_\alpha^3} \mathcal{H}_h(S_h^3 \dot{\theta}) + S_h A_0(\dot{\theta}) + A_0(S_h \dot{\theta}) + S_h A_{-1}(\dot{\alpha}_{0i}) + A_{-s}(\dot{\sigma}) + A_0(\dot{\tau}_c) + O(h^s) \right)_h.$$

The first term in this inner product is evaluated as

$$(241) \quad \left(\dot{\theta}, \frac{\kappa_B}{4s_\alpha^3} \mathcal{H}_h(S_h^3 \dot{\theta}) \right)_h = -\frac{\kappa_B \pi}{2s_\alpha^3} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} |k|^3 |\hat{\theta}_k|^2,$$

where we have used the discrete Parseval relation (Lemma 8.15) and (129). The sum extends to $k = \frac{N}{2} - 1$, in view of zeroing out the $\frac{N}{2}$ mode of $S_h \dot{\theta}$. The next term

is bounded using Lemma 8.16 and Young's inequality. First, introduce a generic discrete function $\dot{g}_{1i} = A_0(\dot{\theta}_i)$. Then

$$\begin{aligned}
 |(\dot{\theta}, S_h A_0(\dot{\theta}))_h| &= |(S_h \dot{\theta}, A_0(\dot{\theta}))_h| \\
 &\leq 2\pi \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} |k| |\hat{\theta}_k| |(\hat{g}_1)_k| \\
 &\leq \pi \left[\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} (k^2 |\hat{\theta}_k|^2 + |(\hat{g}_1)_k|^2) \right] \\
 (242) \quad &\leq c \left(\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} k^2 |\hat{\theta}_k|^2 + E \right),
 \end{aligned}$$

for a constant c . In the last inequality we have used

$$(243) \quad \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} |(\hat{g}_1)_k|^2 \leq \|\dot{g}_1\|_{l^2}^2 \leq c \|\dot{\theta}\|_{l^2}^2 \leq cE.$$

We similarly introduce a generic discrete function $\dot{g}_{2i} = A_0(S_h \dot{\theta}_i)$ and bound

$$\begin{aligned}
 |(\dot{\theta}, A_0(S_h \dot{\theta}))_h| &\leq 2\pi \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} |\hat{\theta}_k| |(\hat{g}_2)_k| \\
 &\leq \pi \left[\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} (|\hat{\theta}_k|^2 + |(\hat{g}_2)_k|^2) \right] \\
 (244) \quad &\leq c \left(\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} k^2 |\hat{\theta}_k|^2 + E \right),
 \end{aligned}$$

where $c > 0$ and the last inequality follows from the bound

$$(245) \quad \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} |(\hat{g}_2)_k|^2 \leq \|A_0(S_h \dot{\theta})\|_{l^2}^2 \leq c \|S_h \dot{\theta}\|_{l^2}^2 = c \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} k^2 |\hat{\theta}_k|^2.$$

The inner product $(\dot{\theta}, S_h A_{-1}(\dot{\alpha}_0)) = -(S_h \dot{\theta}, A_{-1}(\dot{\alpha}_0))$ is controlled following the same analysis as in (256).

The remaining terms in (238) are bounded as

$$(246) \quad |(\dot{\theta}, A_{-s}(\dot{\sigma}) + A_0(\dot{\tau}_c) + O(h^s))_h| \leq cE + O(h^s).$$

Combining (240)-(242), (244), and (246), we obtain

$$(247) \quad (\dot{\theta}, \dot{\theta}_t)_h \leq c \left(\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} k^2 |\hat{\theta}_k|^2 + E \right) + O(h^s).$$

Next, we estimate the inner product $(\dot{\alpha}_{0t}, \dot{\alpha}_0)_h$ in (238). Substitute (235) for $\dot{\alpha}_{0t}$ to obtain

$$(248) \quad (\dot{\alpha}_{0t}, \dot{\alpha}_0)_h = (\tilde{f}_1(\cdot) D_h \dot{\alpha}_0, \dot{\alpha}_0)_h - (\tilde{f}_2(\cdot) \mathcal{H}_h(D_h \dot{\alpha}_0), \dot{\alpha}_0)_h + (A_0(S_h \dot{\theta}) + A_{-s}(\dot{\sigma}) + A_{-1}(\dot{\alpha}_0) + A_0(\dot{\tau}_c), \dot{\alpha}_0)_h + O(h^s).$$

The first inner product on the right hand side of (248), which can be written as $(\tilde{f}_1(\cdot) \dot{\alpha}_0, D_h \dot{\alpha}_0)_h$, is estimated using Lemmas 8.9 and 8.16 as

$$(249) \quad (\tilde{f}_1(\cdot) \dot{\alpha}_0, D_h \dot{\alpha}_0)_h = -(D_h(\tilde{f}_1(\cdot) \dot{\alpha}_0), \dot{\alpha}_0)_h = -(\tilde{f}_1(\cdot) D_h \dot{\alpha}_0 + \dot{\alpha}_0^b \tilde{f}_{1\alpha}(\cdot) + A_{-1}(\dot{\alpha}_0), \dot{\alpha}_0)_h.$$

Move the first inner product on the right hand side of (249) to the left hand side (also moving the real function $\tilde{f}_1$ to the other side of the inner product) to obtain

$$(250) \quad 2(\tilde{f}_1(\cdot) \dot{\alpha}_0, D_h \dot{\alpha}_0)_h = -(\dot{\alpha}_0^b \tilde{f}_{1\alpha}(\cdot) + A_{-1}(\dot{\alpha}_0), \dot{\alpha}_0)_h.$$

This shows that the inner product on the left is bounded by the energy, i.e.,

$$(251) \quad |(\tilde{f}_1(\cdot) \dot{\alpha}_0, D_h \dot{\alpha}_0)_h| \leq cE.$$

The second inner product on the right hand side of (248) can be written as $-(\Lambda_h^p \dot{\alpha}_0, \tilde{f}_2(\cdot) \dot{\alpha}_0)_h$, where we have defined $\Lambda_h^p = \mathcal{H}_h D_h$. To bound this inner product, we make essential use of the positive definiteness of $\tilde{f}_2$. We first write:

$$(252) \quad -(\Lambda_h^p \dot{\alpha}_0, \tilde{f}_2(\cdot) \dot{\alpha}_0)_h = -(\sqrt{\tilde{f}_2(\cdot)} \Lambda_h^p \dot{\alpha}_0, \sqrt{\tilde{f}_2(\cdot)} \dot{\alpha}_0)_h,$$

then move $\sqrt{\tilde{f}_2(\cdot)}$ inside the argument of the operator Λ_h^p , which by Lemma 8.7 and the discrete product rule Lemma 8.9 introduces a commutator and other terms whose inner product with $\dot{\alpha}_0$ can be bounded by energy. If we define $\dot{\tilde{\alpha}}_0 = \sqrt{\tilde{f}_2(\cdot)} \dot{\alpha}_0$, then the preceding statements imply that

$$(253) \quad -(\sqrt{\tilde{f}_2(\cdot)} \Lambda_h^p \dot{\alpha}_0, \sqrt{\tilde{f}_2(\cdot)} \dot{\alpha}_0)_h = -(\Lambda_h^p \dot{\tilde{\alpha}}_0, \dot{\tilde{\alpha}}_0)_h + r,$$

where $r \in \mathbb{R}$ satisfies $|r| < cE$. The inner product on the right hand side of (253) satisfies

$$(254) \quad (\Lambda_h^p \dot{\tilde{\alpha}}_0, \dot{\tilde{\alpha}}_0)_h = 2\pi \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} |k| \rho(kh) |\dot{\tilde{\alpha}}_0|^2 > 0.$$

Combining (252)-(254) shows that

$$(255) \quad (\Lambda_h^p \dot{\alpha}_0, \tilde{f}_2(\cdot) \dot{\alpha}_0)_h \leq cE,$$

which gives the desired estimate on the second inner product in (248).

The third inner product that we need to estimate is $(A_0(S_h \dot{\theta}), \dot{\alpha}_0)_h$. This is bounded using Young's inequality as

$$(256) \quad \begin{aligned} |(A_0(S_h \dot{\theta}), \dot{\alpha}_0)_h| &\leq \frac{1}{2} (\|A_0(S_h \dot{\theta})\|_{l^2}^2 + \|\dot{\alpha}_0\|_{l^2}^2) \\ &\leq c (\|S_h \dot{\theta}\|_{l^2}^2 + \|\dot{\alpha}_0\|_{l^2}^2) \\ &\leq c \left(\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} |k|^2 |\hat{\theta}_k|^2 + E \right). \end{aligned}$$

The first sum above is controlled by parabolic smoothing (i.e. by the dominant contribution from the leading order term (241)).

The other inner products in (248) are clearly bounded by cE . Putting these estimates together, we obtain the bound

$$(257) \quad (\dot{\alpha}_{0t}, \dot{\alpha}_0)_h \leq c \left(\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} |k|^2 |\hat{\theta}_k|^2 + E \right) + O(h^s).$$

The final term in (238) is estimated using (236) and Young's inequality as

$$(258) \quad \begin{aligned} 2|\operatorname{Re}(\overline{\dot{\tau}_c} \dot{\tau}_{ct})| &\leq 2|\dot{\tau}_c| |\dot{\tau}_{ct}| \\ &\leq 2|\dot{\tau}_c| |\hat{v}_0| \\ &\leq |\dot{\tau}_c|^2 + \|\dot{v}\|_{l^2}^2, \end{aligned}$$

where we recall that $\dot{v} = (u_n i e^{i\theta} + \phi_s e^{i\theta})'$. From expressions for u_n and ϕ_s given in (219) and (224), it is easy to see that

$$(259) \quad \begin{aligned} \|\dot{v}\|_{l^2}^2 &\leq c(\|S_h \dot{\theta}\|_{l^2}^2 + \|A_{-1}(\dot{\alpha}_0)\|_{l^2}^2 + |\dot{\tau}_c|^2 + \|A_0(\dot{\theta})\|_{l^2}^2 \\ &\quad + \|A_{-s}(\dot{\sigma})\|_{l^2}^2 + O(h^s)). \end{aligned}$$

Therefore,

$$(260) \quad 2|\operatorname{Re}(\overline{\dot{\tau}_c} \dot{\tau}_{ct})| \leq c \left(\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} |k|^2 |\hat{\theta}_k|^2 + cE + O(h^s) \right),$$

which gives the desired bound on the last term in (238).

We now put these estimates together. First, set

$$(261) \quad d_1 = \min_{0 \leq t \leq T} \frac{\kappa_B \pi}{2s_\alpha},$$

and note by assumptions on s_α that d_1 is bounded away from zero and infinity. Then from (239), (247), (257), and (260) there exist positive constants d_2, d_3 , such that (238) can be bounded as

$$(262) \quad \frac{dE}{dt} \leq \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} (-d_1 |k|^3 + d_2 k^2) |\hat{\theta}_k|^2 + d_3 E + O(h^s).$$

Equation (262) can be written as

$$(263) \quad \frac{dE}{dt} \leq \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} d_4 |\hat{\theta}_k|^2 + d_3 E + O(h^s),$$

where

$$(264) \quad d_4 = \max_{-\frac{N}{2}+1 \leq k \leq \frac{N}{2}-1} (-d_1 |k|^3 + d_2 k^2),$$

and note that d_4 is bounded away from zero and infinity. It readily follows that there exists a positive constant c such that

$$(265) \quad \frac{dE}{dt} \leq cE + O(h^s), \text{ with } E(0) = 0$$

for $t \leq T^*$, which is the main result of this section.

Stability and convergence of our numerical method now follows from application of Gronwall's inequality to (265), which gives

$$(266) \quad E(t) \leq ch^s t(1 + e^t) \text{ for } t \leq T^*,$$

or

$$(267) \quad E(t) \leq c(T^*)h^s.$$

It follows that

$$(268) \quad |\dot{\sigma}|^2, \|\dot{\theta}\|_{l^2}^2, \|\dot{\alpha}_0\|_{l^2}^2, \|\dot{\tau}\|_{l^2}^2 \leq c(T^*)h^s,$$

where we have used $\|\dot{\tau}\|_{l^2}^2 \leq cE$, which follows from Lemma 8.18 and $|\dot{\tau}_c|^2 \leq cE$. We choose m large enough, so that s can be picked to satisfy $s \geq 8$. (Recall that m characterizes the smoothness of the continuous solution, and s is near m). Then

$$(269) \quad |\dot{\sigma}|, \|\dot{\theta}\|_{l^2}, \|\dot{\alpha}_0\|_{l^2}, \|\dot{\tau}\|_{l^2} \leq c(T^*)h^{\frac{s}{2}} < h^{\frac{7}{2}}$$

for h small enough. It follows from the definition (117) that we can extend T^* to $T^* = T$, so that the bounds (269) are valid throughout the entire interval $0 \leq t \leq T$ in which a smooth continuous solution exists. This completes the proof of the convergence of our method for $\beta = 0$, $\chi = \frac{1}{2}$ and $\kappa_B > 0$.

12. UNEQUAL VISCOSITIES ($\beta \neq 0$)

The case $\beta \neq 0$ corresponds to a viscosity contrast between the internal and external fluids. In this case, we must account for the additional nonlocal equation (51) and term $\tilde{\omega}$.

We start by taking the variation of (51), which is written as

$$(270) \quad (\mathbf{I} + \beta \mathbf{K}) \dot{\tilde{\omega}}_i = -\beta \dot{\mathbf{K}} (\tilde{\omega}(\alpha_i) + \mathbf{g}^q(\alpha_i)) - \beta \mathbf{K} \dot{\mathbf{g}}_i^q,$$

where, e.g.,

$$(271) \quad \dot{\mathbf{K}} \tilde{\omega}(\alpha_i) = \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \begin{pmatrix} (\dot{K}_R^{(1)})_{i,j} + (\dot{K}_R^{(2)})_{i,j} & -(\dot{K}_I^{(2)})_{i,j} \\ (\dot{K}_I^{(2)})_{i,j} & (\dot{K}_R^{(1)})_{i,j} - (\dot{K}_R^{(2)})_{i,j} \end{pmatrix} \begin{pmatrix} \tilde{\omega}_1(\alpha_j) \\ \tilde{\omega}_2(\alpha_j) \end{pmatrix} (2h),$$

and $\dot{\mathbf{g}}^q$ is the variation of the filtered version of (50). Here we have defined, e.g.,

$$(272) \quad \mathbf{K} \dot{\tilde{\omega}}_i = (\mathbf{K}_h + \dot{\mathbf{K}}) \dot{\tilde{\omega}}_i,$$

which contains both linear and nonlinear terms in the variation. We now use the fact that the kernels $(K_R^{(1)})_{i,j}$, $(K_R^{(2)})_{i,j}$, and $(K_I^{(2)})_{i,j}$ are simple modifications of the kernels in (164), (166), so that the same analysis as in Section 9.3 shows that

$$(273) \quad \dot{\mathbf{K}} (\tilde{\omega}(\alpha_i) + \mathbf{g}^q(\alpha_i)) = A_0(\dot{\theta}_i) + A_{-s}(\dot{\sigma}) + A_{-1}(\dot{\alpha}_{0i}) + O(h^s)$$

(the above relation is defined as holding for each component of the vector). Following the same argument as that leading to (210), we also have

$$(274) \quad \mathbf{K} \dot{\mathbf{g}}_i^q = A_0(\dot{\theta}_i) + A_{-s}(\dot{\sigma}) + A_{-1}(\dot{\alpha}_{0i}) + A_0(\dot{\tau}_c) + O(h^s).$$

Together, the above results imply that (270) can be written as

$$(275) \quad (\mathbf{I} + \beta \mathbf{K}) \dot{\tilde{\omega}}_i = A_0(\dot{\theta}) + A_{-s}(\dot{\sigma}) + A_{-1}(\dot{\alpha}_{0i}) + A_0(\dot{\tau}_c) + O(h^s).$$

Apparently, if $(\mathbf{I} + \beta \mathbf{K})^{-1}$ is bounded, then we can bound $\dot{\tilde{\omega}}$ by the right-hand side of (275). This implies that $\dot{\tilde{\omega}}$ does not contribute to the leading order analysis, and the energy estimates for $\beta \neq 0$ proceed exactly as in the case $\beta = 0$. The bound on $(\mathbf{I} + \beta \mathbf{K})^{-1}$ is given by Lemma 12.1:

Lemma 12.1. *Assume $\theta(\cdot, t) \in C^2$ and $s_\alpha \neq 0$. Then there exist constants $h_0 > 0$, $\beta_0 > 0$, and $c > 0$, such that for all h with $0 < h \leq h_0$, and β with $0 < \beta \leq \beta_0$, $\|(\mathbf{I} + \beta \mathbf{K})^{-1}\|_{l^2} \leq c$.*

The proof of Lemma 12.1 will be deferred to the appendix.

13. DROP WITH CONSTANT SURFACE TENSION

Finally, we consider the case of a drop interface with zero membrane bending stress and constant surface tension, $\kappa_B = 0$ and $\mathcal{S} = 1$, at first for equal interior and exterior viscosities or $\beta = 0$. As described in Section 4 after (80), the discretization is modified from that for an elastic interface. Since the highest, second derivative term is eliminated from (50), the stability analysis must also be modified.

When $\kappa_B = 0$ and $\mathcal{S} = 1$, we have from (197), (198), and Lemma 8.8,

$$(276) \quad \dot{\omega}_i^r = h(\alpha_i) \dot{\theta}_i^p + A_{-1}(\dot{\theta}_i) + A_{-s}(\dot{\sigma}),$$

where $h(\alpha) = -ie^{i\theta(\alpha)}/4$ is a smooth function. The velocity is decomposed as in (80), and we adapt the analysis in Section 9.3 to estimate $(\dot{u}_R)_i$. In particular, we make use of the additional filtering for the drop problem and the smoothing properties of convolutions to replace (208), (209), etc., with the improved estimates

$$K_{lh}[\cdot, \cdot](\dot{\omega}_i^r) = A_{-1}(\dot{\theta}_i) + A_{-s}(\dot{\sigma}),$$

for $l = 0, 1, 3$, and

$$K_{lh}[\cdot, \cdot](D_h \dot{\tau}_i) = A_{-2}(\dot{\theta}_i) + A_{-s}(\dot{\sigma}),$$

for $l = 1, 3$. In making the former estimate we absorb the smooth function $h(\alpha)$ into the kernel. The analysis of the nonlinear term in the velocity variation is also modified. Specifically, we use (276) to obtain the improved estimate

$$(277) \quad \dot{u}_i^{NL} = h^{3/2} A_0(\dot{\theta}_i) + A_{-s}(\dot{\sigma}) + O(h^s).$$

It follows that

$$(278) \quad (\dot{u}_R)_i = A_{-1}(\dot{\theta}_i) + A_{-s}(\dot{\sigma}) + A_0(\dot{\tau}_c) + O(h^s).$$

The variation of the normal velocity is given by (214) with $\kappa_B = 0$. The first commutator there is estimated as $A_{-2}(\dot{\theta}) + A_{-s}(\dot{\sigma})$ after inserting (276), absorbing $h(\alpha_i)$ into the kernel, and using the smoothing property in Lemma 8.7. We use (198) to write the second commutator as the sum $-(1/4)\mathcal{H}_h \dot{\theta}_i + h_1(\alpha_i)\dot{\theta}_i$ plus some smoother terms, where $h_1(\alpha)$ is a smooth real function. The remaining terms in (214) are smoother. The final estimate is

$$(279) \quad (\dot{u}_n)_i = -\frac{1}{4}\mathcal{H}_h \dot{\theta}_i + h_1(\alpha_i)\dot{\theta}_i + A_{-1}(\dot{\theta}_i) + A_{-s}(\dot{\sigma}) + A_0(\dot{\tau}_c) + O(h^s).$$

From (222) and Lemma 8.17, we also find that

$$(\dot{\phi}_s)_i = A_0(\dot{\theta}_i) + A_{-s}(\dot{\sigma}) + A_0(\dot{\tau}_c) + O(h^s),$$

and it is easy to see that $\dot{u}_s$ satisfies the same estimate as $\dot{\phi}_s$.

Energy estimates.

The energy is defined as

$$(280) \quad E(t) = \dot{\sigma}^2 + (\dot{\theta}, \dot{\theta})_h + |\dot{\tau}_c|^2,$$

and we form $\frac{1}{2} \frac{dE}{dt}$ as in (238). Substitute (225) for $\dot{\theta}_t$ in $(\dot{\theta}, \dot{\theta}_t)_h$ with $S_h \dot{u}_n$ replaced by $D_h \dot{u}_n$, per the comments following (80). Two inner products from this expression for $(\dot{\theta}, \dot{\theta}_t)_h$ involve a derivative of $\dot{\theta}$. These are $(\dot{\theta}, -(1/4)\Lambda_h^p \dot{\theta})_h$ and $(\dot{\theta}, h_1(\cdot) D_h \dot{\theta})_h$, where we recall that $h_1(\alpha)$ is a smooth real function. The first of these is negative definite and thus decreases the energy, and the second can be estimated with the aid of (139) and the skew-symmetry of D_h as

$$(281) \quad (\dot{\theta}, h_1(\cdot) D_h \dot{\theta}) = (\dot{\theta}, D_h(h_1(\cdot) \dot{\theta})) + (\dot{\theta}, A_0(\dot{\theta}))$$

$$(282) \quad = -(D_h \dot{\theta}, h_1(\cdot) \dot{\theta})_h + (\dot{\theta}, A_0(\dot{\theta})),$$

so that

$$(283) \quad (\dot{\theta}, h_1(\cdot) D_h \dot{\theta})_h = (\dot{\theta}, A_0(\dot{\theta})).$$

This shows that $(\dot{\theta}, h_1(\cdot) D_h \dot{\theta})_h$ is bounded by the energy. It is easily seen that all the other inner products in dE/dt can be bounded by the energy, from which the convergence of the method readily follows for $\kappa_B = 0$, $S = 1$, and $\beta = 0$.

Drop with viscosity contrast ($\beta \neq 0$).

The analysis for a drop with $\beta \neq 0$ follows that in Section 12 with a few minor changes. In view of the additional filtering in the numerical method for the drop problem, the right hand sides of the estimates (273)-(275) can be improved to $A_{-1}(\dot{\theta}_i) + A_{-s}(\dot{\sigma}) + A_0(\dot{\tau}_c)$. It readily follows that the additional term $\dot{\tilde{\omega}}$ from inverting (270) does not contribute to the leading order analysis, and the energy estimates for $\beta \neq 0$ proceed exactly as in the case $\beta = 0$.

14. CONCLUSIONS

A convergence proof has been presented for a boundary integral method for interfacial Stokes flow. While previous convergence studies of the boundary integral method exist for interfacial potential flow, this is the first work that we are aware of for the important case of interfacial Stokes flow. Our analysis has focused on a spectrally accurate numerical method, adapted in this paper from [21], [52], for a Hookean elastic capsule with membrane bending stress evolving in an externally applied strain or shear flow. The method is rather general, and setting the interfacial tension S to a constant and the bending stress κ_B to zero gives a method for simulating a drop or bubble in a shear or strain flow which has been widely applied. The method is based on an arclength-angle parameterization of the interface which was introduced in [23] and first adapted to Stokes flow in [27], [28].

The main task in the proof is to estimate the variations or errors such as $\dot{\theta} = \dot{\theta}_i - \dot{\theta}(\alpha_i)$ between the discrete and exact solutions at time t . This is done by estimating the most singular terms in the variations, and separating into linear and nonlinear terms. The nonlinear terms are controlled by the high (spectral) accuracy of the method for smooth solutions, and thus the crux of the proof is to

show the stability of linear terms in the variation, which is done with the aid of energy estimates.

The presence of high derivatives due to the bending forces requires a substantially different analysis from previous proofs of the convergence of the boundary integral method for potential flow. In particular, our energy estimates make significant use of the smoothing properties of the highest derivative term, or so-called ‘parabolic smoothing’, to control lower order derivatives. This allows us to close the energy estimates and prove stability of the method.

The proof also clarifies the role of numerical filtering. We find that targeted filtering is necessary to control the potentially destabilizing effect of aliasing errors and prove stability of the method. Crucially, however, our analysis shows that the filter should not be applied to the highest derivative term coming from the membrane bending stress, so that the smoothing properties of this term can be utilized.

Our work also provides a convergence analysis of a widely used boundary integral method for drops and bubbles without a surrounding elastic membrane, in which $S = \text{constant}$ and $\kappa_B = 0$ is zero.

Another important application of BI methods is in computing the evolution of vesicles with inextensible membranes, in which $s_\alpha \equiv 1$. BI methods for inextensible vesicles have been developed in, e.g., [44], [49]. The inextensibility constraint can be approximately satisfied in our method by choosing a large constant E in (10) (and using a different nondimensionalization), which keeps s_α near 1 [9]. In future work, we may consider the convergence analysis for an algorithm in which the tension $S(\alpha, t)$ is chosen to exactly enforce the inextensibility constraint.

APPENDIX A. PROOF OF LEMMAS

Proof of Lemma 6.1. Let $\hat{f}_k^e$ to be the exact Fourier coefficient of f . Then [47]

$$(284) \quad \hat{f}_k = \hat{f}_k^e + \sum_{j \neq 0} \hat{f}_{k+Nj}^e, \text{ for } k = -\frac{N}{2} + 1, \dots, \frac{N}{2},$$

is the computed Fourier coefficient from (38), where the sum represents high-wave-number modes that are aliased to $k \in [-\frac{N}{2} + 1, \frac{N}{2}]$. The sum in (284) is from $j = -\infty$ to ∞ , excluding $j = 0$. Introduce the notation $|k| \leq \frac{N'}{2}$ defined as $\{k : -\frac{N}{2} + 1 \leq k \leq \frac{N}{2}\}$ and $|k| > \frac{N'}{2}$, which is defined as $\{k : \frac{N}{2} < k \text{ or } k \leq -\frac{N}{2}\}$. Then we have the estimate

$$(285) \quad \begin{aligned} |S_h f(\alpha_i) - f_\alpha(\alpha_i)| &= \left| \sum_{|k| \leq \frac{N'}{2}} k(\hat{f}_k - \hat{f}_k^e) e^{ik\alpha_i} - \sum_{|k| > \frac{N'}{2}} k \hat{f}_k^e e^{ik\alpha_i} \right| \\ &\leq \sum_{|k| \leq \frac{N'}{2}} |k| |\hat{f}_k - \hat{f}_k^e| + \sum_{|k| > \frac{N'}{2}} |k| |\hat{f}_k^e|. \end{aligned}$$

The first term on the right hand side of (285) is the aliasing error, and the second term is the truncation error. We use (284) to bound the aliasing error as

$$\begin{aligned}
 \sum_{|k| \leq \frac{N'}{2}} |k| |\hat{f}_k - \hat{f}_k^e| &= \sum_{|k| \leq \frac{N'}{2}} |k| \left| \sum_{j \neq 0} \hat{f}_{k+Nj}^e \right| \\
 (286) \quad &\leq \sum_{|k| \leq \frac{N'}{2}} \sum_{j \neq 0} |k + jN| |\hat{f}_{k+Nj}^e| \leq \sum_{|\tilde{k}| \geq \frac{N}{2}} |\tilde{k}| |\hat{f}_{\tilde{k}}^e|,
 \end{aligned}$$

where $\tilde{k} = k + jN$, with $j \neq 0$. The last line of (286) follows from

$$(287) \quad |\tilde{k}| = |k + jN| \geq ||k| - jN| \geq \left| \frac{N}{2} - N \right| \geq \frac{N}{2}.$$

The aliasing error is further bounded by (dropping the tilde)

$$\begin{aligned}
 \sum_{|k| \geq \frac{N}{2}} |k| |\hat{f}_k^e| &\leq \sum_{|k| \geq \frac{N}{2}} \frac{|k|^{s+1}}{|k|^s} |\hat{f}_k^e| \\
 &\leq \left(\sum_{|k| \geq \frac{N}{2}} |k|^{2(s+1)} |\hat{f}_k^e|^2 \right)^{\frac{1}{2}} \left(\sum_{|k| \geq \frac{N}{2}} \frac{1}{|k|^{2s}} \right)^{\frac{1}{2}} \\
 &\leq c \|f\|_{s+1} (N^{-2s+1})^{\frac{1}{2}} \\
 (288) \quad &\leq ch^{s-\frac{1}{2}} \|f\|_{s+1}, \text{ using } h = \frac{2\pi}{N}.
 \end{aligned}$$

In the third inequality above we have used the bound

$$(289) \quad \sum_{|k| \geq \frac{N}{2}} \frac{1}{|k|^{2s}} < cN^{-2s+1}.$$

The truncation error is bounded as (starting from (285))

$$\begin{aligned}
 \sum_{|k| > \frac{N'}{2}} |k| |\hat{f}_k^e| &= \sum_{|k| > \frac{N'}{2}} \frac{|k|^{s+1}}{|k|^s} |\hat{f}_k^e| \\
 &\leq \left(\sum_{|k| > \frac{N'}{2}} |k|^{2(s+1)} |\hat{f}_k^e|^2 \right)^{\frac{1}{2}} \left(\sum_{|k| > \frac{N'}{2}} |k|^{-2s} \right)^{\frac{1}{2}} \\
 (290) \quad &\leq ch^{s-\frac{1}{2}} \|f\|_{s+1}.
 \end{aligned}$$

Combine the estimates of aliasing error and truncation error to obtain

$$(291) \quad |S_h f(\alpha_i) - f_\alpha(\alpha_i)| \leq ch^{s-\frac{1}{2}} \|f\|_{s+1}.$$

The proof of (291) for D_h instead of S_h , or for more than one derivative, is similar. $\square$

Proof of Lemma 8.17. Let

$$(292) \quad \hat{F}_k = \sum_{m=-\infty}^{\infty} \hat{f}_{k+mN}^e \text{ for } -\frac{N}{2} + 1 \leq k \leq \frac{N}{2}$$

denote the discrete Fourier coefficients of a smooth periodic function $f(\alpha)$ (taking aliasing into account), where $\hat{f}_k^e$ are the exact Fourier coefficients of f . Thus, $f(\alpha_i)$ and the discrete function ϕ_i have the representation

$$(293) \quad f(\alpha_i) = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \hat{F}_k e^{ik\alpha_i},$$

$$(294) \quad \phi_i = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \hat{\phi}_k e^{ik\alpha_i}.$$

Our interest is in obtaining an estimate for

$$(295) \quad S_h^{-1}(f(\alpha_i)S_h\phi_i) = \sum_{\substack{k=-\frac{N}{2}+1 \\ k \neq 0}}^{\frac{N}{2}} \frac{1}{ik} (\widehat{fS_h\phi})_k e^{ik\alpha_i},$$

where $(\widehat{fS_h\phi})_k$ denotes the discrete Fourier coefficients of the product $fS_h\phi$.

We shall need an expression for the Fourier coefficients of the product of a smooth function with a discrete function. For a given k , define the sets

$$(296) \quad \begin{aligned} I_{n,k} &= \left\{ n \in [-N/2+1, N/2] : -\frac{N}{2}+1 \leq k-n \leq \frac{N}{2} \right\}, \\ J_{n,k} &= \left\{ n \in [-N/2+1, N/2] : -\frac{N}{2}+1 \leq k+N-n \leq \frac{N}{2} \right\}, \\ K_{n,k} &= \left\{ n \in [-N/2+1, N/2] : -\frac{N}{2}+1 \leq k-N-n \leq \frac{N}{2} \right\}. \end{aligned}$$

Using (293) and (294), the product $f(\alpha_i)\phi_i$ can be written as

$$(297) \quad f(\alpha_i)\phi_i = \left(\sum_{k=-N+2}^{-\frac{N}{2}} + \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} + \sum_{k=\frac{N}{2}+1}^N \right) \sum_{n \in I_{n,k}} \hat{F}_{k-n} \hat{\phi}_n e^{ik\alpha_i}.$$

The wave numbers in the first and third sums in parentheses are aliased to $k \in [-\frac{N}{2}+1, \frac{N}{2}]$. Rewriting these two sums by replacing k with $k-N$ and $k+N$, respectively, we obtain the equivalent representation

$$(298) \quad \begin{aligned} f(\alpha_i)\phi_i &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \left(\sum_{n \in I_{n,k}} \hat{F}_{k-n} \hat{\phi}_n + \sum_{n \in J_{n,k}} \hat{F}_{k+N-n} \hat{\phi}_n \right. \\ &\quad \left. + \sum_{n \in K_{n,k}} \hat{F}_{k-N-n} \hat{\phi}_n \right) e^{ik\alpha_i}, \end{aligned}$$

where the requirement $n \in J_{n,k}$ in the second double sum of (298) and $n \in K_{n,k}$

in the third has allowed us to replace $\sum_{k=-\frac{N}{2}+1}^0$ and $\sum_{k=2}^{\frac{N}{2}}$, respectively, with $\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}}$.

From (298), we therefore have for $-\frac{N}{2} + 1 \leq k \leq \frac{N}{2}$

$$(299) \quad \begin{aligned} (\widehat{fS_h\phi})_k &= \sum_{n \in I_{n,k}} i n \hat{F}_{k-n} \hat{\phi}_n + \sum_{n \in J_{n,k}} i n \hat{F}_{k+N-n} \hat{\phi}_n \\ &+ \sum_{n \in K_{n,k}} i n \hat{F}_{k-N-n} \hat{\phi}_n, \end{aligned}$$

and similarly,

$$(300) \quad \begin{aligned} (\widehat{\phi S_h f})_k &= \sum_{n \in I_{n,k}} i(k-n) \hat{F}_{k-n} \hat{\phi}_n + \sum_{n \in J_{n,k}} i(k+N-n) \hat{F}_{k+N-n} \hat{\phi}_n \\ &+ \sum_{n \in K_{n,k}} i(k-N-n) \hat{F}_{k-N-n} \hat{\phi}_n. \end{aligned}$$

Combining (299) with the negative of (300) gives for $-\frac{N}{2} + 1 \leq k \leq \frac{N}{2}$

$$(301) \quad \begin{aligned} (\widehat{fS_h\phi})_k &= -(\widehat{\phi S_h f})_k + \sum_{n \in I_{n,k}} i k \hat{F}_{k-n} \hat{\phi}_n \\ &+ \sum_{n \in J_{n,k}} i(k+N) \hat{F}_{k+N-n} \hat{\phi}_n + \sum_{n \in K_{n,k}} i(k-N) \hat{F}_{k-N-n} \hat{\phi}_n. \end{aligned}$$

We next recognize from (298) that

$$(302) \quad \begin{aligned} (\widehat{f\phi})_k &= \sum_{n \in I_{n,k}} \hat{F}_{k-n} \hat{\phi}_n + \sum_{n \in J_{n,k}} \hat{F}_{k+N-n} \hat{\phi}_n \\ &+ \sum_{n \in K_{n,k}} \hat{F}_{k-N-n} \hat{\phi}_n, \end{aligned}$$

and combining this with (301) shows that the Fourier coefficients in (295) can be written as

$$(303) \quad \begin{aligned} \frac{(\widehat{fS_h\phi})_k}{ik} &= -\frac{(\widehat{\phi S_h f})_k}{ik} + (\widehat{f\phi})_k \\ &+ \sum_{n \in J_{n,k}} \frac{N}{k} \hat{F}_{k+N-n} \hat{\phi}_n - \sum_{n \in K_{n,k}} \frac{N}{k} \hat{F}_{k-N-n} \hat{\phi}_n, \end{aligned}$$

for $-\frac{N}{2} + 1 \leq k \leq \frac{N}{2}$ with $k \neq 0$. The first three terms in (303) are the Fourier coefficients of the first three terms in (155). To finish the derivation of (155), we simply need to estimate the two sums on the right hand side of (303). The main difficulty is to overcome the large factor of N .

Each of the two sums in (303) is a discrete convolution which represents the k -th Fourier coefficient of the product a smooth function with ϕ_i . Denoting the smooth functions by f_1 and f_2 , we form the l^2 -norm of the products with the aid of the discrete Parseval equality (Lemma 8.15),

$$(304) \quad \|f_1(\cdot)\phi\|_{l^2} = \left(2\pi \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \left| \sum_{n \in J_{n,k}} \frac{N}{k} \hat{F}_{k+N-n} \hat{\phi}_n \right|^2 \right)^{\frac{1}{2}},$$

and

$$(305) \quad \|f_2(\cdot)\phi\|_{l^2} = \left(2\pi \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \left| \sum_{n \in K_{n,k}} \frac{N}{k} \hat{F}_{k-N-n} \hat{\phi}_n \right|^2 \right)^{\frac{1}{2}}.$$

We estimate these l^2 -norms by decomposing the wavenumber range into

$$(306) \quad \mathfrak{K}_1 = \left\{ k : \frac{N}{4} \leq |k| \leq \frac{N}{2} \right\},$$

and

$$(307) \quad \mathfrak{K}_2 = \left\{ k : 0 < |k| < \frac{N}{4} \right\}.$$

The sum over $k \in \mathfrak{K}_1$ is bounded using $|\frac{N}{k}| \leq 4$. For example,

$$(308) \quad \left[2\pi \sum_{k \in \mathfrak{K}_1} \left| \frac{N}{k} \sum_{n \in J_{n,k}} \hat{F}_{k+N-n} \hat{\phi}_n \right|^2 \right]^{\frac{1}{2}} \leq 4 \left[2\pi \sum_{k \in \mathfrak{K}_1} \left| \sum_{n \in J_{n,k}} \hat{F}_{k+N-n} \hat{\phi}_n \right|^2 \right]^{\frac{1}{2}} \\ \leq 4 \left[2\pi \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \left| \sum_{n \in J_{n,k}} \hat{F}_{k+N-n} \hat{\phi}_n \right|^2 \right]^{\frac{1}{2}}$$

$$(309) \quad = 4 \left[2\pi \sum_{k=\frac{N}{2}+1}^N \sum_{n \in I_{n,k}} |\hat{F}_{k-n} \hat{\phi}_n|^2 \right]^{\frac{1}{2}},$$

where in the latter equality we have first replaced $\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}}$ in (308) with $\sum_{k=-\frac{N}{2}+1}^0$ (per the comment following (298)) and then substituted $k-N$ for k . The expression in (309) is clearly bounded by a constant times the extended l^2 norm of (297),

$$(310) \quad \|f(\cdot)\phi\|_{l^2_{ext}} \equiv \left[2\pi \sum_{k=-N+2}^N \left| \sum_{n \in I_{n,k}} \hat{F}_{k-n} \hat{\phi}_n \right|^2 \right]^{\frac{1}{2}},$$

for which

$$(311) \quad \|f(\cdot)\phi\|_{l^2_{ext}} \leq \|f\|_{\infty} \|\phi\|_{l^2_{ext}} = \|f\|_{\infty} \|\phi\|_{l^2}$$

(where $\|\phi\|_{l^2_{ext}}$ is defined by zero padding).

Hence, in (304) (and similarly in (305)), the sum over $k \in \mathfrak{K}_1$ is bounded by $c\|\phi\|_{l^2}$.

The sum over $k \in \mathfrak{K}_2$ requires a different estimate. For $k \in \mathfrak{K}_2$, we have

$$(312) \quad k + N - n \geq \frac{N}{4} \text{ when } n \in \left[-\frac{N}{2} + 1, \frac{N}{2} \right].$$

Moreover, $\hat{f}_k^e$ decay like $O(k^{-s})$, where s is the number of continuous derivatives of f , and it is easily seen that when $s > 1$, $\hat{F}_k$ also decays like $O(k^{-s})$. Hence,

$$(313) \quad |\hat{F}_{k+N-n}| \leq c|k+N-n|^{-s} \leq c\left(\frac{N}{4}\right)^{-s},$$

per (312). Considering (304), it follows that the sum over $k \in \varkappa_2$ satisfies the bound

$$\begin{aligned}
 \left[2\pi \sum_{k \in \varkappa_2} \left| \sum_{n \in J_{n,k}} \frac{N}{k} \hat{F}_{k+N-n} \hat{\phi}_n \right|^2 \right]^{\frac{1}{2}} &\leq c N^{-s+1} \left[\sum_{|k| \leq \frac{N}{4}} \left| \sum_{n \in J_{n,k}} \hat{\phi}_n \right|^2 \right]^{\frac{1}{2}} \\
 &\leq c N^{-s+2} \left[\sum_{|k| \leq \frac{N}{4}} \sum_{n \in J_{n,k}} |\hat{\phi}_n|^2 \right]^{\frac{1}{2}} \\
 (314) \qquad \qquad \qquad &\leq c N^{-s+\frac{5}{2}} \|\phi\|_{l^2}.
 \end{aligned}$$

Here we have used (313) in the first inequality,

$$(315) \qquad \left| \sum_{n \in J_{n,k}} \phi_n \right|^2 \leq N^2 \sum_{n \in J_{n,k}} |\phi_n|^2$$

in the second, and replaced the double sum $\sum_{|k| \leq \frac{N}{4}} \sum_{n \in J_{n,k}}$ by $\frac{N}{4} \sum_{n=-N/2+1}^{N/2}$ in the

third. It follows that when $s \geq \frac{5}{2}$, the sum over $k \in \varkappa_2$ in (304) and similarly in (305) are bounded by $c\|\phi\|_{l^2}$. Thus, both (304) and (305) are bounded by $c\|\phi\|_{l^2}$, which finishes our estimate of the two sums in (303). This completes the derivation of (155).

Equation (156) readily follows from the definition (115) of a smoothing operator by noting that both $D_h S_h^{-1}(f(\alpha_i)\phi_i)$ and $S_h^{-1}(f(\alpha_i)S_h\phi_i)$ are $A_0(\phi_i)$ operators, the latter of which is a consequence of (155). $\square$

Proof of Lemma 12.1. First, we provide the result for the corresponding continuous equation, which is denoted by

$$(316) \qquad (\mathbf{I} + \beta \mathbf{K}_c) \tilde{\omega}(\alpha) = -\beta \mathbf{K}_c \mathbf{g}(\alpha).$$

The kernel for this system is smooth and therefore $\mathbf{K}_c$ is bounded. It follows that there exists a value $\beta_0(T)$ such that for $0 \leq \beta < \beta_0(T)$ the continuous operator $(\mathbf{I} + \beta \mathbf{K}_c)$ is invertible and has a bounded inverse [36], i.e.,

$$(317) \qquad \|(\mathbf{I} + \beta \mathbf{K}_c)^{-1}\|_{L^2} \leq c.$$

We summarize the argument, following the analysis in [10], that the discrete operator $\mathbf{I} + \beta \mathbf{K}$ is likewise invertible.

For any discrete l^2 function ω_i , define

$$(318) \qquad \Phi_i = \omega_i + \beta \mathbf{K} \omega_i,$$

where ω_i is the vector counterpart of ω_i (see (49)). We show that $\Phi = 0$ implies $\omega = 0$, which demonstrates the invertibility of $\mathbf{I} + \beta \mathbf{K}$. The strategy is to exactly represent the discrete equation (318) as a continuous (integral) equation with a piecewise continuous integrand. The invertibility result for a continuous equation then will imply the invertibility of the discrete equation. This construction is essentially the same as in the appendix of [10], and relies on the smoothness of the continuous kernel $\mathbf{K}_c(\alpha, \alpha')$ and the consistency of our discretization. We refer the reader to [10] for details. $\square$

APPENDIX B. ESTIMATES FOR THE NONLINEAR TERMS IN THE VELOCITY VARIATION

We first present expressions for the nonlinear terms in the variation of the velocity, i.e., $\dot{U}_{l,i}^{NL}$ for $l = 1, \dots, 3$. The variation $\dot{U}_{1,i}^{NL}$ is given by

$$(319) \quad \begin{aligned} \dot{U}_{1,i}^{NL} = & -\frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \left\{ \dot{\omega}_j^q \left(2\text{Re} \left(\frac{S_h \tau_j}{\tau_j - \tau_i} \right) \right) + \omega_h^q(\alpha_j) \left[2\text{Re} \left(S_h \dot{\tau}_j \left(\frac{1}{\tau_j - \tau_i} \right) \right) \right] \right. \\ & \left. + \omega_h^q(\alpha_j) 2\text{Re} \left(\frac{S_h \tau_h(\alpha_j) (\dot{\tau}_j - \dot{\tau}_i)^2}{[\tau_h(\alpha_j) - \tau_h(\alpha_i)]^2 (\tau_h(\alpha_j) - \tau_h(\alpha_i) + \dot{\tau}_j - \dot{\tau}_i)} \right) \right\}. \end{aligned}$$

Note that the third term within braces above comes from the nonlinear term in the variation of $\left(\frac{1}{\tau_j - \tau_i} \right)'$, via (141).

To compactly represent the other nonlinear terms, introduce the notation

$$(320) \quad [f_i, g_i, h_i]' = \dot{f}_i \dot{g}_i h(\alpha_i) + \dot{f}_i g(\alpha_i) \dot{h}_i + f(\alpha_i) \dot{g}_i \dot{h}_i + \dot{f}_i \dot{g}_i \dot{h}_i,$$

which gives the nonlinear terms in the variation of the product $f_i g_i h_i$. A similar notation is used for the nonlinear terms in the variation of a product with four or more discrete functions. Then,

$$(321) \quad \begin{aligned} \dot{U}_{2,i}^{NL} = & \frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \left\{ \left[\overline{\omega}_j^q, S_h \tau_j, \frac{1}{\overline{\tau}_j - \overline{\tau}_i} \right]' \right. \\ & \left. + \frac{\overline{\omega}_h^q(\alpha_j) S_h \tau_h(\alpha_j) (\overline{\tau}_j - \overline{\tau}_i)^2}{[\overline{\tau}_h(\alpha_j) - \overline{\tau}_h(\alpha_i)]^2 (\overline{\tau}_h(\alpha_j) - \overline{\tau}_h(\alpha_i) + \overline{\tau}_j - \overline{\tau}_i)} \right\}, \end{aligned}$$

where for example in $\left[\overline{\omega}_j^q, S_h \tau_j, \frac{1}{\overline{\tau}_j - \overline{\tau}_i} \right]'$, if $g_j = S_h \tau_j$, then $g(\alpha_j) = S_h \tau_h(\alpha_j)$. Continuing,

$$(322) \quad \begin{aligned} \dot{U}_{3,i}^{NL} = & -\frac{h}{\pi} \sum_{\substack{j=-\frac{N}{2}+1 \\ (j-i) \text{ odd}}}^{\frac{N}{2}} \left\{ \left[\overline{\omega}_j^q, \tau_j - \tau_i, \overline{S_h \tau_j}, \frac{1}{\overline{\tau}_j - \overline{\tau}_i}, \frac{1}{\overline{\tau}_j - \overline{\tau}_i} \right]' \right. \\ & \left. + \frac{\overline{\omega}_h^q(\alpha_j) [\tau_h(\alpha_j) - \tau_h(\alpha_i)] \overline{S_h \tau_h(\alpha_j)} [\overline{\tau}_j - \overline{\tau}_i]^2}{[\overline{\tau}_h(\alpha_j) - \overline{\tau}_h(\alpha_i)]^2 (\overline{\tau}_h(\alpha_j) - \overline{\tau}_h(\alpha_i) + \overline{\tau}_j - \overline{\tau}_i)} \right\}. \end{aligned}$$

We now estimate these nonlinear terms. Consider first the expression (319) for $\dot{U}_1^{NL}$. We expand some of the variations in this expression using Lemmas 8.10 and 8.11, for example,

$$(323) \quad \begin{aligned} \left(\frac{S_h \tau_j}{\tau_j - \tau_i} \right)' = & -\frac{\tau_\alpha(\alpha_j)}{(\tau(\alpha_j) - \tau(\alpha_i))^2} (\dot{\tau}_j - \dot{\tau}_i) + \frac{S_h \dot{\tau}_j}{\tau(\alpha_j) - \tau(\alpha_i)} \\ & + \frac{\tau_\alpha(\alpha_j) (\dot{\tau}_j - \dot{\tau}_i)^2}{(\tau(\alpha_j) - \tau(\alpha_i))^2 (\tau(\alpha_j) - \tau(\alpha_i) + \dot{\tau}_j - \dot{\tau}_i)} \\ & + S_h \dot{\tau}_j \left(\frac{1}{\tau_j - \tau_i} \right)' + O(h^s). \end{aligned}$$

To estimate the terms involving differences of variations in (323), we make use of the Fourier series representation

$$(324) \quad \frac{\dot{\tau}_j - \dot{\tau}_i}{\alpha_j - \alpha_i} = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \hat{\tau}_k \frac{e^{ik\alpha_j} - e^{ik\alpha_i}}{\alpha_j - \alpha_i},$$

so that

$$(325) \quad \begin{aligned} \left| \frac{\dot{\tau}_j - \dot{\tau}_i}{\alpha_j - \alpha_i} \right| &\leq \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} |\hat{\tau}_k| \left| \frac{e^{ik\alpha_j} - e^{ik\alpha_i}}{\alpha_j - \alpha_i} \right| \\ &\leq \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} |k| |\hat{\tau}_k| \\ &\leq \left(\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} 1 \right)^{\frac{1}{2}} \|S_h \dot{\tau}\|_{l^2} \\ &\leq \frac{c}{h^{\frac{1}{2}}} \|S_h \dot{\tau}\|_{l^2}, \end{aligned}$$

for any $-\frac{N}{2} + 1 \leq j \leq \frac{N}{2}$ and $-\frac{N}{2} + 1 \leq i \leq \frac{N}{2}$. The first term on the right hand side of (323) is estimated by multiplying and dividing by $(\alpha_j - \alpha_i)^2$, and using (325) to obtain

$$(326) \quad \left| \frac{\tau_\alpha(\alpha_j)(\dot{\tau}_j - \dot{\tau}_i)}{(\tau(\alpha_j) - \tau(\alpha_i))^2} \right| \leq \frac{c}{h^{\frac{3}{2}}} \|S_h \dot{\tau}\|_{l^2}.$$

The extra factor $\frac{1}{h}$ comes from an extra factor of $\frac{1}{\alpha_j - \alpha_i}$. It is easy to see that the second, third and fourth terms on the right hand side of (323) are also bounded in magnitude by $\frac{c}{h^{\frac{3}{2}}} \|S_h \dot{\tau}\|_{l^2}$.

Returning to the expression for $\dot{U}_1^{NL}$ in (319), it follows that the first sum of the right hand side of (319) is bounded by

$$(327) \quad ch^2 \|\dot{\omega}^q\|_{l^2} + O(h^s),$$

using the bound $\frac{c}{h^{\frac{3}{2}}} \|S_h \dot{\tau}\|_{l^2}$ on the magnitude of (323) along with $\|S_h \dot{\tau}\|_{l^2} = O(h^{\frac{7}{2}})$ (see Remark 8.19) to obtain the factor of h^2 in (327). It is easy to see that the second and third sums in equation (319) are bounded by $h^{\frac{3}{2}} \|S_h \dot{\tau}\|_{l^2}$, which, in view of (163), implies that these terms are $h^{\frac{3}{2}} A_0(\dot{\theta}) + A_{-s}(\dot{\sigma}) + O(h^s)$. Putting these estimates together, we find that

$$(328) \quad \dot{U}_{1,i}^{NL} = A_0(\dot{\theta}) + A_{-1}(\dot{\alpha}_{0i}) + A_{-s}(\dot{\sigma}) + O(h^s),$$

using the estimate (207) for $\dot{\omega}^q$. Estimates for $\dot{U}_2^{NL}$ and $\dot{U}_3^{NL}$ are performed similarly to $\dot{U}_1^{NL}$, and verify that

$$(329) \quad \dot{u}^{NL} = A_0(\dot{\theta}) + A_{-1}(\dot{\alpha}_{0i}) + A_{-s}(\dot{\sigma}) + O(h^s).$$

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