FLUCTUATIONS OF THE WINDING NUMBER OF A DIRECTED POLYMER ON A CYLINDER

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Abstract. We prove a central limit theorem for the winding number of a directed polymer on a cylinder, which is equivalent with proving the Gaussian fluctuations of the endpoint of the directed polymer in a spatial periodic environment.

Keywords: directed polymer, central limit theorem, homogenization.

1. Introduction

1.1. **Main result.** We consider the problem of a directed polymer on a cylinder and study the fluctuations of the winding number, that is, the algebraic number of turns the polymer path does around the cylinder. The problem is equivalent to studying the fluctuations of the endpoint of a directed polymer in a random periodic environment. To state the main result, we first give an informal description of the model. The reference path measure is chosen to be the Wiener measure, and the random environment is modeled by a Gaussian space-time white noise $\xi(t,x)$ on $\mathbb{R}_+ \times [0,1]$, with periodic boundary condition, and we periodically extend it to $\mathbb{R}_+ \times \mathbb{R}$.

For each realization of the random environment, the partition function of the directed polymer is given by

$$Z_T = \mathbb{E}\exp(\beta \int_0^T \xi(t,w_t)dt), \tag{1.1}$$

where $\{w_t\}_{t\geq 0}$ is a one-dimensional standard Brownian motion starting from the origin, independent of ξ, and $\mathbb{E}$ is the expectation over the realizations of the Brownian motion w. Here $\beta > 0$ is a fixed parameter playing the role of the inverse temperature. Since ξ is a space-time white noise, the above expression should be interpreted carefully, see Section 2 below for more details.

The quenched density of the polymer endpoint w_T is then given by

$$\rho(T,x) = \frac{\mathbb{E}\exp(\beta \int_0^T \xi(t,w_t)dt)\delta(w_T - x)}{\mathbb{E}\exp(\beta \int_0^T \xi(t,w_t)dt)}. \tag{1.2}$$

Since the random environment ξ is periodic in space, another perspective is to view the polymer path as lying on a cylinder, in which case it is the trajectory of $\{w_t - \lfloor w_t \rfloor\}_{t\geq 0}$ we are tracking. The winding number of the polymer path around the cylinder, denoted by W_T, then equals to

$$W_T = \lfloor w_T \rfloor \mathbb{1}_{w_T\geq 0} + \lceil w_T \rceil \mathbb{1}_{w_T<0}. \tag{1.3}$$

Here $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and ceiling functions, respectively. Thus, to study the large time behavior of w_T is equivalent to that of W_T. Denote the quenched probability measure by $\hat{\mathbb{P}}_T$ and the expectation with respect to it by $\hat{\mathbb{E}}_T$, so for any bounded function $f:\mathbb{R}\to\mathbb{R}$, we have $\hat{\mathbb{E}}_T f(w_T) = \int_{\mathbb{R}} f(x)\rho(T,x)dx$. Let $\mathbf{P},\mathbf{E}$ be the probability and expectation with respect to the noise ξ. Now we can state the main result of the paper, namely, under the annealed polymer measure $\mathbf{P}\otimes\hat{\mathbb{P}}_T$, $\{\frac{w_T}{\sqrt{T}}\}_{T>0}$, or equivalently, $\{\frac{W_T}{\sqrt{T}}\}_{T>0}$, satisfies a central limit theorem.

Theorem 1.1. *There exists $\sigma_{\text{eff}}^2 \in (0,\infty)$, given in (3.27) below, such that for any $\theta\in\mathbb{R}$, we have*

$$\mathbf{E}\hat{\mathbb{E}}_T \exp(i\theta\frac{w_T}{\sqrt{T}}) \to \exp(-\frac{1}{2}\sigma_{\text{eff}}^2\theta^2), \qquad \text{as } T\to\infty.$$

1.2. **Context and motivation.** Our study of the winding number is motivated by the work of Brunet [2], where the same problem was investigated by the replica method. What is particularly interesting is the exact formula he derived for σ_{eff}^2 and how it depends on the size of the period, see [2, Eq. (19)-(20)]. It is not hard to convince oneself that Theorem 1.1 holds for any spatial period, with the effective diffusion constant depending on the size of the cell – we chose the length $L=1$ only to simplify the notations. Denote the corresponding variance by $\sigma_{\text{eff}}^2(L)$. It is known that the polymer path is super-diffusive with the exponent $2/3$ when $L=\infty$, i.e., if there is no periodic structure, $T^{-2/3}w_T$ is of order $O(1)$ for $T\gg 1$, see [5, Theorem 1.11] for relevant results on this particular model. To go from the diffusive to super-diffusive scaling as $L\to\infty$, it is natural to expect $\sigma_{\text{eff}}^2(L)$ to blow up. This was indeed predicted in [2]: as $L\to\infty$, $\sigma_{\text{eff}}^2(L)\sim\sqrt{L}$. The blow up rate is related to the $2/3$ super-diffusion exponent, and here is a heuristic explanation: for cells of size L, the displacement of the endpoint w_T is of the order $\sigma_{\text{eff}}(L)\sqrt{T}$, provided that $L\sim O(1)$ and $T\gg 1$. As we keep $T\gg 1$ fixed and slowly increase L, the polymer path would still visit many cells provided that $\sigma_{\text{eff}}(L)\sqrt{T}\gg L$. In this case we still expect to see a homogenization phenomenon and the central limit theorem as in Theorem 1.1 holds. So, it is natural to guess that the critical scale comes from balancing the two terms, $\sigma_{\text{eff}}(L)\sqrt{T}$ and L. This leads to $L\sim T^{2/3}$, under the assumption of $\sigma_{\text{eff}}^2(L)\sim\sqrt{L}$.

It was our hope to prove the above heuristics rigorously, and to confirm (or disprove) the replica calculations in [2]. Theorem 1.1 can be viewed as a small step towards this goal, through which we confirmed the diffusive scaling and the Gaussian fluctuations. The formula derived for σ_{eff}^2, see (3.27) below, is of Green-Kubo type which involves the integral of some covariance function, and is too implicit to perform any asymptotic analysis. This is not surprising though, since the homogenization constant is usually given by the solution to some cell problem and precise estimates on it are not easy to obtain (see a very recent contribution along this line for a different model of diffusion in random environment [4]). Guided by the same philosophy, a similar study was carried out for the fluctuations of the free energy $\log Z_T$, which leads to the optimal size of fluctuations in certain regimes where L, T go to infinity together, see [7].

For the connections between the winding number of the directed polymer in random environment and other models in statistical physics, such as vortices in superconductors and strongly correlated fermions, we refer to [2] and the references cited there.

One can also formulate the problem as a diffusion in a distribution-valued random environment and study the corresponding SDE with a singular drift, see e.g. [6, 3, 10] and the references therein. In this framework, making sense of the singular diffusion is already highly nontrivial, and is intimately related to the study of singular SPDE [12, 13, 11]. For our specific problem of directed polymer, one can view it as a passive scalar with the velocity field given by the solution of a stochastic Burgers equation, see [6, Theorem 31] which gives a rigorous meaning of it. Although the velocity field is spatial periodic, which is sometimes viewed as the simple case in the study of homogenization or invariance principle of diffusion in a random environment, the problem does not fall into any classical framework. It might be possible to employ the tools developed for singular diffusions and combine with homogenization type of arguments to study similar problems and to prove central limit type results. For this particular problem, we make use of the structure of the Gibbs measure and give a proof using a classical argument for the central limit theorem for weakly dependent random variables.

1.3. **Sketch of proof.** Our approach relies heavily on the previous work of studying the periodic KPZ equation [8], where we showed the endpoint distribution of the directed polymer on a cylinder mixes exponentially fast. The proof in [8] was inspired by the classical work of Sinai [16], which was on the stochastic Burgers equation on the torus. Similar results were also obtained in [15], using a random version of the Krein-Rutman theorem. As mentioned previously, one could view the polymer path as lying on the cylinder by considering the path $\{w_t - \lfloor w_t \rfloor\}_{t\geq 0}$. Assuming that at each integer time k, the position of the path is x_k, i.e., $w_k - \lfloor w_k \rfloor = x_k$, our previous result implies a strong mixing property of $\{x_k\}_{k\geq 1}$ under the polymer measure. If we denote η_k the winding number of the polymer path accumulated during the interval $[k-1, k]$, then the total winding number is simply $\sum_k \eta_k$. It is not hard to deduce that, given the positions of $\{x_k\}_{k\geq 1}$, the sequence of random variables $\{\eta_k\}_{k\geq 1}$ are independent, so, the correlation only comes from the correlation in those x_k. Our strategy will be to first consider the case when the starting and ending points x_0 and x_T are both sampled from the stationary measure so that $\{\eta_k\}_{k\geq 1}$ is a sequence of stationary random variables, and we will prove a ρ–mixing (correlation mixing) condition to apply the general central limit theorem for the sum of stationary random variables. Then, to finish the proof, we will show that the error induced by resampling x_0 and x_T is asymptotically small, again using the strong mixing property of $\{x_k\}_{k\geq 1}$.

The same proof applies verbatim to the high dimensional setting when the random environment is assumed to be white in time and smooth in space.

Organization of the paper. In Section 2, we formulate the problem rigorously, define the endpoint distribution through a stochastic heat equation, and construct a Markov chain which keeps tracking the winding number of the polymer path as time increases. Sections 3 and 4 are devoted to proving the main result, by first reducing it to the stationary setting, then proving the ρ–mixing condition for the stationary sequence $\{\eta_k\}$. In Section 5, we prove the nondegeneracy of the variance σ_{eff}^2. Some further discussion is left in Section 6.

Notations. We will sometimes use the shorthand integral notation $\int$ when the domain of integration is clear from the context. If we do not specify the range of the summation in $\sum_j$, it stands for $\sum_{j\in\mathbb{Z}}$.

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2. Preparations

2.1. Stochastic heat equation and endpoint density on $\mathbb{R}$.

As mentioned previously, the expression $Z_T = \mathbb{E}\exp(\int_0^T \xi(t, w_t)dt)$ is only formal since ξ is a space-time white noise, and we actually need to consider the so-called Wick exponential. In this section, we define the endpoint density ρ rigorously, through the stochastic heat equation (SHE). For an excellent introduction to the theory of the stochastic heat equation, we refer to the monograph [14].

Consider the equation of the form

$$\begin{aligned} &\partial_t u(t,x;\nu) = \frac{1}{2}\Delta u(t,x;\nu) + \beta\xi(t,x)u(t,x;\nu), \qquad t>0, x\in\mathbb{R}, \\ &u(0,dx) = \nu(dx), \end{aligned} \tag{2.1}$$

where $\beta > 0$, the product between u and ξ is interpreted in the Ito-Walsh sense, and $\nu \in \mathcal{M}_1(\mathbb{R})$ - the set of Borel probability measures on $\mathbb{R}$. Denote the propagator of the above equation by $Z_{t,s}(x,y)$, i.e. for each $(s,y)\in\mathbb{R}_+\times\mathbb{R}$ fixed, we have

$$\begin{aligned}
&\partial_t Z_{t,s}(x,y) = \frac{1}{2}\Delta_x Z_{t,s}(x,y) + \beta\xi(t,x)Z_{t,s}(x,y), \qquad t>s, x\in\mathbb{R},\\
&Z_{s,s}(x,y) = \delta_y(x).
\end{aligned} \tag{2.2}$$

Due to the 1-periodicity of the noise, we obviously have

$$Z_{t,s}(x+j,y+j) = Z_{t,s}(x,y), \quad j\in\mathbb{Z},\ x,y\in\mathbb{R},\ t>s. \tag{2.3}$$

Let $\mathbb{T}=[0,1]$ be the unit torus with the end points identified in the usual way. Since ξ is periodic, we can consider the same equation on $\mathbb{T}$ with the periodic boundary condition. Then its propagator is given by

$$G_{t,s}(x,y) = \sum_j Z_{t,s}(x+j,y) = \sum_j Z_{t,s}(x,y-j), \quad x,y\in\mathbb{T}. \tag{2.4}$$

In other words, $G_{t,s}(x,y)$ is the periodic solution to (2.2) with the initial data $G_{s,s}(x,y)=\sum_j \delta_{y-j}(x)$.

With the above notations, the random density ρ, which is the density of w_T under the quenched polymer measure $\hat{\mathbb{P}}_T$ and was formally defined in (1.2), takes the form

$$\rho(T,x;\nu) = \frac{u(T,x;\nu)}{\int_{\mathbb{R}} u(T,x';\nu)dx'}. \tag{2.5}$$

From now on, we choose $\nu(dx)$ to be the Dirac measure at the origin. To simplify the notation, we will omit the dependence on ν when there is no confusion.

2.2. **Endpoint density on the cylinder.**

Besides studying the polymer endpoint on the whole line, we also consider its periodic counterpart:

$$\rho_{\mathrm{per}}(t,x;\nu) = \frac{v(t,x;\nu)}{\int_{\mathbb{T}} v(t,x';\nu)dx'},$$

where $\nu\in\mathcal{M}_1(\mathbb{T})$ and v solves the equation

$$\begin{aligned}
&\partial_t v(t,x;\nu) = \frac{1}{2}\Delta v(t,x;\nu) + \beta\xi(t,x)v(t,x;\nu), \qquad t>0, x\in\mathbb{T},\\
&v(0,dx) = \nu(dx).
\end{aligned} \tag{2.6}$$

Using the propagator, the solution can be written as

$$v(t,x;\nu) = \int_{\mathbb{T}} G_{t,0}(x,y)\nu(dy).$$

It turns out, see [8, Lemma 2.2], that $\{\rho_{\mathrm{per}}(t)\}_{t\ge 0} = \{\rho_{\mathrm{per}}(t,\cdot;\nu)\}_{t\ge0,\nu\in\mathcal{M}_1}$ is a Markov family. For any $t>0$, the random element $\rho_{\mathrm{per}}(t,\cdot;\nu)$ takes values in $\mathbb{D}_c(\mathbb{T})$, which we use to denote the space of continuous probability densities on $\mathbb{T}$.

To simplify the notation, for any $t > s$, we define the forward and backward polymer densities starting from ν by

$$\begin{aligned}\rho_{\rm per}(t,x;s,\nu) &= \frac{\int_{\mathbb{T}} G_{t,s}(x,y)\nu(dy)}{\int_{\mathbb{T}^2} G_{t,s}(x',y')\nu(dy')dx'},\\ \tilde\rho_{\rm per}(t,\nu;s,x) &= \frac{\int_{\mathbb{T}} G_{t,s}(y,x)\nu(dy)}{\int_{\mathbb{T}^2} G_{t,s}(y',x')\nu(dy')dx'}.\end{aligned} \tag{2.7}$$

We have $\rho_{\rm per}(t,x;\nu) = \rho_{\rm per}(t,x;0,\nu)$. By the time reversal of the space-time white noise, for any $t > s$ and ν fixed, we have

$$\{\rho_{\rm per}(t,x;s,\nu)\}_{x\in\mathbb{T}} \stackrel{\rm law}{=} \{\tilde\rho_{\rm per}(t,\nu;s,x)\}_{x\in\mathbb{T}}$$

We emphasize that, throughout the paper, the notation $\rho(t,\cdot;\nu)$ is the endpoint density of the polymer on the whole line, while the notations $\rho_{\rm per}(t,\cdot;s,\nu)$ and $\tilde\rho_{\rm per}(t,\nu;s,\cdot)$ are reserved for the endpoint density on the torus.

Now we summarize a few results concerning the properties of the Markov family $\{\rho_{\rm per}(t)\}_{t\geq 0}$, see [8, Theorem 2.3, Eq. (4.17), Lemma 4.1, Proposition 4.6].

Proposition 2.1. *There exists a unique invariant measure π_∞ for $\{\rho_{\rm per}(t)\}_{t\geq0}$, supported on $\mathbb{D}_c(\mathbb{T})$. For any $p \geq 1$, there exists $C,\lambda > 0$ such that for all $t > 1$,*

$$\mathbf{E} \sup_{\nu,\nu'\in\mathcal{M}_1(\mathbb{T})} \sup_{x\in\mathbb{T}} |\rho_{\rm per}(t,x;\nu) - \rho_{\rm per}(t,x;\nu')|^p \leq Ce^{-\lambda t}, \tag{2.8}$$

and

$$\mathbf{E} \sup_{\nu\in\mathcal{M}_1(\mathbb{T})} \sup_{x\in\mathbb{T}} \{\rho_{\rm per}(t,x;\nu)^p + \rho_{\rm per}(t,x;\nu)^{-p}\} \leq C. \tag{2.9}$$

2.3. **A Markov chain for the winding number.** As the random environment is periodic in space, to study the displacement of the polymer endpoint w_T, it is equivalent to studying the winding number of the polymer path when we view it as lying on a cylinder by considering the trajectory $\{w_t - \lfloor w_t \rfloor\}_{t\in[0,T]}$. This is the perspective we will take from now on. The idea is to first sample the trajectory of the polymer path on the cylinder at integer times, then consider the winding of the path between successive integer times.

For any $N \in \mathbb{Z}_+$, consider $u(N, j_N + x_N)$ where $x_N \in [0,1)$ and $j_N \in \mathbb{Z}$. By the definition of the propagator, we have

$$\begin{aligned}u(N, j_N + x_N) &= \int_{\mathbb{R}} Z_{N,N-1}(j_N + x_N, y)u(N-1,y)dy\\ &= \sum_{j_{N-1}} \int_{\mathbb{T}} Z_{N,N-1}(j_N + x_N, j_{N-1} + x_{N-1})u(N-1, j_{N-1} + x_{N-1})dx_{N-1}.\end{aligned}$$

Iterate the above relation, we reach at (recall that $u(0,x) = \delta(x)$)

$$\begin{aligned}&u(N, j_N + x_N)\\ &= \sum_{j_1,\ldots,j_{N-1}} \int_{\mathbb{T}^{N-1}} \prod_{k=1}^{N} Z_{k,k-1}(j_k + x_k, j_{k-1} + x_{k-1}) d\mathbf{x}_{1,N-1},\end{aligned}$$

where we used the simplified notation $d\mathbf{x}_{1,N-1} = dx_1 \ldots dx_{N-1}$ and the convention $j_0 = x_0 = 0$. In other words, in the above integration, we have decomposed the domain $\mathbb{R}$ as $\mathbb{R} = \cup_j [j, j+1)$, then integrate in each interval and sum them up. One should think of the variable $j_k + x_k$ as representing the location of the polymer path at time k, with j_k the integer part and x_k the fractional part, i.e., $j_k = \lfloor w_k \rfloor$ and $x_k = w_k - \lfloor w_k \rfloor$.

Now we make use of the periodicity and observe that

$$\begin{aligned}&\sum_{j_k} Z_{k,k-1}(j_k + x_k, j_{k-1} + x_{k-1}) = \sum_{j_k} Z_{k,k-1}(j_k - j_{k-1} + x_k, x_{k-1})\\ &= \sum_{j_k} Z_{k,k-1}(j_k + x_k, x_{k-1}) = G_{k,k-1}(x_k, x_{k-1}),\end{aligned} \tag{2.10}$$

where G is the periodic propagator defined in (2.4). Then we can write

$$\begin{aligned}u(N, j_N + x_N) = \int_{\mathbb{T}^{N-1}} &\left(\sum_{j_1,\ldots,j_{N-1}} \frac{\prod_{k=1}^{N} Z_{k,k-1}(j_k + x_k, j_{k-1} + x_{k-1})}{\prod_{k=1}^{N} G_{k,k-1}(x_k, x_{k-1})} \right)\\ &\times \prod_{k=1}^{N} G_{k,k-1}(x_k, x_{k-1}) dx_{1,N-1}.\end{aligned} \tag{2.11}$$

Fix the realization of the random noise and

$$\mathbf{x} = (x_0, x_1, \ldots, x_N) \in \mathbb{T}^{N+1}. \tag{2.12}$$

We construct an integer-valued, time inhomogeneous Markov chain $\{Y_j\}_{j=1}^N$, with

$$\begin{aligned}&\mathbb{P}_{\mathbf{x}}[Y_1 = j_1] = \frac{Z_{1,0}(j_1 + x_1, x_0)}{G_{1,0}(x_1, x_0)},\\ &\mathbb{P}_{\mathbf{x}}[Y_2 = j_2 | Y_1 = j_1] = \frac{Z_{2,1}(j_2 + x_2, j_1 + x_1)}{G_{2,1}(x_2, x_1)},\\ &\ldots\\ &\mathbb{P}_{\mathbf{x}}[Y_N = j_N | Y_{N-1} = j_{N-1}] = \frac{Z_{N,N-1}(j_N + x_N, j_{N-1} + x_{N-1})}{G_{N,N-1}(x_N, x_{N-1})}.\end{aligned} \tag{2.13}$$

With the Markov chain, one can write the summation in (2.11) as

$$\sum_{j_1,\ldots,j_{N-1}} \frac{\prod_{k=1}^{N} Z_{k,k-1}(j_k + x_k, j_{k-1} + x_{k-1})}{\prod_{k=1}^{N} G_{k,k-1}(x_k, x_{k-1})} = \mathbb{P}_{\mathbf{x}}[Y_N = j_N],$$

where, to emphasize the dependence of the Markov chain on $x_0, x_1, \ldots, x_N$, we have denoted the probability by $\mathbb{P}_{\mathbf{x}}$.

In this way, (2.11) is rewritten as

$$u(N, j_N + x_N) = \int_{[0,1]^{N-1}} \mathbb{P}_{\mathbf{x}}[Y_N = j_N] \prod_{k=1}^{N} G_{k,k-1}(x_k, x_{k-1}) d\mathbf{x}_{1,N-1}. \tag{2.14}$$

Recall that $\hat{\mathbb{P}}_N$ is the quenched probability of the polymer measure on paths of length N, and $\lfloor w_N \rfloor$ is the integer part of the endpoint w_N. Then

$$\begin{aligned}\hat{\mathbb{P}}_N[\lfloor w_N \rfloor = j_N] &= \int_{\mathbb{T}} \rho(N, j_N + x_N) dx_N = \frac{\int_{\mathbb{T}} u(N, j_N + x_N) dx_N}{\int_{\mathbb{R}} u(N, x') dx'} \\ &= \frac{\int_{\mathbb{T}^N} \mathbb{P}_{\mathbf{x}}[Y_N = j_N] \prod_{k=1}^{N} G_{k,k-1}(x_k, x_{k-1}) d\mathbf{x}_{1,N}}{\int_{\mathbb{T}^N} \prod_{k=1}^{N} G_{k,k-1}(x_k, x_{k-1}) d\mathbf{x}_{1,N}}.\end{aligned} \tag{2.15}$$

In other words, the quenched distribution of $\lfloor w_N \rfloor$ is a weighted average of the distribution of Y_N (the average is over the $\mathbf{x}$ variable).

We introduce another notation: suppose that $f, g \in \mathbb{D}_c(\mathbb{T})$, define

$$\mu_N(\mathbf{x}; f, g) := \frac{f(x_N) \prod_{k=1}^{N} G_{k,k-1}(x_k, x_{k-1}) g(x_0)}{G_{N,0}(f, g)}, \tag{2.16}$$

with $\mathbf{x} = (x_0, \ldots, x_N)$ and the normalization factor

$$G_{N,0}(f, g) := \int_{\mathbb{T}^{N+1}} f(x_N) \prod_{k=1}^{N} G_{k,k-1}(x_k, x_{k-1}) g(x_0) d\mathbf{x}_{0,N}, \tag{2.17}$$

where $d\mathbf{x}_{0,N} := dx_0 \ldots dx_N$. For each realization of the random environment, one should view $\mu_N(\mathbf{x}; f, g)$ as the joint density of the polymer on the cylinder, evaluated at $(0, x_0), (1, x_1), \ldots, (N, x_N)$, with the starting and ending points sampled from the densities g, f respectively. For any $\nu, \nu' \in \mathcal{M}_1(\mathbb{T})$, we abuse the notation and write $\mu_N(\mathbf{x}; \nu, \nu')$ as well, meaning that the starting and ending points are sampled from ν', ν. In this case, $G_{N,0}(\nu, \nu')$ equals to

$$G_{N,0}(\nu, \nu') := \int_{\mathbb{T}^{N+1}} \prod_{k=1}^{N} G_{k,k-1}(x_k, x_{k-1}) \nu'(dx_0) d\mathbf{x}_{1,N-1} \nu(dx_N).$$

With the above new notation, we can rewrite

$$\hat{\mathbb{P}}_N[\lfloor w_N \rfloor = j_N] = \int_{\mathbb{T}^{N+1}} \mathbb{P}_{\mathbf{x}}[Y_N = j_N] \mu_N(\mathbf{x}; \mathrm{m}, \delta_0) d\mathbf{x}_{0,N}, \tag{2.18}$$

where m is the Lebesgue measure on $\mathbb{T}$ (note that in (2.15), the convention is $x_0 = 0$).

By (2.10) and (2.13), it is clear that Y_N is a sum of independent random variables, for each fixed realization of the noise and $\mathbf{x}$. We rewrite it as

$$Y_N = \sum_{k=1}^{N} \eta_k, \qquad \eta_k = Y_k - Y_{k-1}, \qquad Y_0 = 0. \tag{2.19}$$

One should interpret η_k as the winding number accumulated during the time interval $[k-1,k]$, and we have

$$\mathbb{P}_{\mathbf{x}}[\eta_k = j] = \frac{Z_{k,k-1}(x_k + j, x_{k-1})}{G_{k,k-1}(x_k, x_{k-1})}, \qquad j \in \mathbb{Z}. \tag{2.20}$$

To prove Theorem 1.1 for the winding number W_N, see (1.3), or the endpoint w_N, it is equivalent to proving it for $\lfloor w_N \rfloor$. From now on, we will focus on the law of $\lfloor w_N \rfloor$ and the rest of the analysis starts from the representation (2.18).

3. Proof of the central limit theorem

The goal is to prove the central limit theorem for $\frac{w_T}{\sqrt{T}}$ under the annealed polymer measure $\mathbf{P} \otimes \hat{\mathbb{P}}_T$. For $\theta \in \mathbb{R}$, define

$$\varphi_T(\theta) := \mathbf{E}\hat{\mathbb{E}}_T e^{i\theta w_T/\sqrt{T}} = \mathbf{E}\int_{\mathbb{R}} \exp\left\{\frac{i\theta x}{\sqrt{T}}\right\}\rho(T,x)dx,$$

where ρ was defined in (1.2) with ν chosen to be the Dirac measure at the origin. In this section, we will consider those T taking integer values, and the main goal is to show

Theorem 3.1. *We have*

$$\lim_{N\to\infty} \varphi_N(\theta) = \exp\left\{-\frac{(\sigma_{\rm eff}\theta)^2}{2}\right\}, \quad \theta \in \mathbb{R}, \tag{3.1}$$

with $\sigma_{\rm eff}$ given by (3.27) below.

To show the above theorem it suffices to consider the integer part of w_N. Define

$$\psi_N(\theta) = \mathbf{E}\int \exp\left\{\frac{i\theta\lfloor x\rfloor}{\sqrt{N}}\right\}\rho(N,x)dx. \tag{3.2}$$

We focus on finding the limit of $\psi_N(\theta)$. From the construction of the Markov chain in Section 2.3 and (2.18), we have

$$\begin{aligned}
\psi_N(\theta) &= \mathbf{E}\int_{\mathbb{T}^N} \mathbb{E}_{\mathbf{x}} \exp\left\{\frac{i\theta Y_N}{\sqrt{N}}\right\}\mu_N(\mathbf{x};\mathrm{m},\delta_0)d\mathbf{x}_{0,N} \\
&= \mathbf{E}\int_{\mathbb{T}^N} \mathbb{E}_{\mathbf{x}} \exp\left\{\sum_{k=1}^N \frac{i\theta\eta_k}{\sqrt{N}}\right\}\mu_N(\mathbf{x};\mathrm{m},\delta_0)d\mathbf{x}_{0,N} \\
&= \mathbf{E}\int_{\mathbb{T}^N} \prod_{k=1}^N \mathbb{E}_{\mathbf{x}} \exp\left\{\frac{i\theta\eta_k}{\sqrt{N}}\right\}\mu_N(\mathbf{x};\mathrm{m},\delta_0)d\mathbf{x}_{0,N}.
\end{aligned}$$

Here we used $\mathbb{E}_{\mathbf{x}}$ to denote the expectation with respect to $\mathbb{P}_{\mathbf{x}}$.

3.1. **Estimates of the moments of increments.** For any $\nu,\nu'\in\mathcal{M}_1(\mathbb{T})$ and $p\ge 1$, define

$$E^{(p)}_{N,k}(\nu,\nu') := \mathbf{E}\int_{\mathbb{T}^{N+1}}(\mathbb{E}_{\mathbf{x}}|\eta_k|^p)\mu_N(\mathbf{x};\nu,\nu')d\mathbf{x}_{0,N}. \tag{3.3}$$

Recall that η_k was defined in (2.19) and its law is given by (2.20). The following result holds.

Lemma 3.2. *For any $p>1$, we have*

$$\mathfrak{E}_p := \sup_{\nu,\nu'\in\mathcal{M}_1(\mathbb{T})}\sup_{N\ge 1}\sup_{k=1,\dots,N} E^{(p)}_{N,k}(\nu,\nu') < \infty. \tag{3.4}$$

Proof. By the definition we have

$$\begin{aligned}E^{(p)}_{N,k}(\nu,\nu') = \mathbf{E}\int_{\mathbb{T}^2} dx_k dx_{k-1}\sum_j |j|^p Z_{k,k-1}(x_k+j,x_{k-1})\\ \times\frac{G_{N,k}(\nu;x_k)G_{k-1,0}(x_{k-1};\nu')}{\int_{\mathbb{T}^2}G_{N,k}(\nu;y')G_{k,k-1}(y',y)G_{k-1,0}(y;\nu')dy'dy}.\end{aligned}$$

Here we used the simplified notation $G_{t,s}(x;\nu)=\int_{\mathbb{T}}G_{t,s}(x,y)\nu(dy)$ and $G_{t,s}(\nu;x)=\int_{\mathbb{T}}G_{t,s}(y,x)\nu(dy)$. The above expression is bounded from above by

$$\begin{aligned}&\mathbf{E}\Big\{\Big(\inf_{z,z'}G_{k,k-1}(z',z)\Big)^{-1}\int_{\mathbb{T}^2}dx_kdx_{k-1}\sum_j|j|^pZ_{k,k-1}(x_k+j,x_{k-1})\\ &\times\frac{G_{N,k}(\nu;x_k)G_{k-1,0}(x_{k-1};\nu')}{\int_{\mathbb{T}^2}G_{N,k}(\nu;y')G_{k-1,0}(y;\nu')dy'dy}\Big\}\\ &=\int_{\mathbb{T}^2}\mathbf{E}\Big\{F_k(x_k,x_{k-1})\tilde\rho_{\rm per}(N,\nu;k,x_k)\rho_{\rm per}(k-1,x_{k-1};\nu')\Big\}dx_kdx_{k-1}.\end{aligned}$$

Here

$$F_k(x_k,x_{k-1}) := \Big(\inf_{z,z'}G_{k,k-1}(z',z)\Big)^{-1}\Big(\sum_j|j|^pZ_{k,k-1}(x_k+j,x_{k-1})\Big). \tag{3.5}$$

Note that F_k, $\tilde\rho_{\rm per}(N,\nu;k,x_k)$ and $\rho_{\rm per}(k-1,x_{k-1};\nu')$ are independent. Therefore, we have

$$E^{(p)}_{N,k}(\nu,\nu')\le\int_{\mathbb{T}^2}\mathbf{E}F_k(x_k,x_{k-1})\mathbf{E}\tilde\rho_{\rm per}(N,\nu;k,x_k)\mathbf{E}\rho_{\rm per}(k-1,x_{k-1};\nu')dx_kdx_{k-1}.$$

For any $q>1$, we conclude by the Hölder inequality

$$\begin{aligned}\mathbf{E}F_k(x_k,x_{k-1}) &= \sum_j|j|^p\mathbf{E}\Big\{\Big(\inf_{z,z'}G_{k,k-1}(z',z)\Big)^{-1}Z_{k,k-1}(x_k+j,x_{k-1})\Big\}\\ &\le\sum_j|j|^p\Big\{\mathbf{E}\Big(\inf_{z,z'}G_{k,k-1}(z',z)\Big)^{-q}\Big\}^{1/q}\Big\{\mathbf{E}Z^{q'}_{k,k-1}(x_k+j,x_{k-1})\Big\}^{1/q'},\end{aligned}$$

with $1/q + 1/q' = 1$. There exists a constant $C_q > 0$ depending only on q and such that

$$\mathbf{E}\left(\inf_{z,z'} G_{k,k-1}(z',z)\right)^{-q} \le C_q,$$

$$\sup_{x',x}\left\{\mathbf{E}Z^{q'}_{k,k-1}(x'+j,x)\right\}^{1/q'} \le C_q \exp\left\{-\frac{j^2}{C_q}\right\},$$

see [8, Lemma 4.1] and Lemma 3.3 below. Hence

$$\mathfrak{F} := \sup_{k\ge 1}\sup_{x',x} \mathbf{E}F_k(x',x) < \infty$$

and $E^{(p)}_{N,k}(\nu,\nu') \le \mathfrak{F}$. This completes the proof of the lemma. □

The following lemma, concerning the moments estimate of the propagator of SHE, is quite standard. For completeness sake we present its proof in Section A.

Lemma 3.3. *For any $p \ge 1$, there exists $C_p > 0$ such that*

$$\mathbf{E}Z_{t,0}(x,0)^p \le \frac{C_p}{t^{p/2}} \exp\left\{-\frac{x^2}{C_p t}\right\}, \qquad \textit{for all } t \in (0,2], x \in \mathbb{R}.$$

3.2. **Characteristic function at equilibrium.** Recall from Section 2.2 that, there exists a unique probability measure π_∞ on the space $\mathbb{D}_c(\mathbb{T})$ - continuous probability densities on $\mathbb{T}$ - that is invariant under the dynamics of the polymer endpoint process.

Suppose that ϱ and $\tilde{\varrho}$ are two independent copies of $\mathbb{D}_c(\mathbb{T})$-valued random fields, distributed according to π_∞. They are also assumed to be independent of the noise ξ. We will use $\mathbb{E}_\varrho, \mathbb{E}_{\tilde{\varrho}}$ to denote the expectation with respect to them respectively. From [8, Theorem 2.3] and (2.9), we know that (2.9) also holds for ϱ, i.e. for any $p \ge 1$ we have

$$\mathfrak{R}_p := \mathbb{E}_\varrho \sup_x \{\varrho(x)^p + \varrho(x)^{-p}\} < +\infty. \tag{3.6}$$

Define

$$\tilde{\psi}_N(\theta) = \mathbb{E}_\varrho \mathbb{E}_{\tilde{\varrho}} \mathbf{E} \int_{\mathbb{T}^{N+1}} \mathbb{E}_{\mathbf{x}} \exp\left\{\sum_{k=1}^N \frac{i\theta\eta_k}{\sqrt{N}}\right\} \mu_N(\mathbf{x}; \tilde{\varrho}, \varrho) d\mathbf{x}_{0,N},$$

and recall that

$$\psi_N(\theta) = \mathbf{E} \int_{\mathbb{T}^{N+1}} \mathbb{E}_{\mathbf{x}} \exp\left\{\sum_{k=1}^N \frac{i\theta\eta_k}{\sqrt{N}}\right\} \mu_N(\mathbf{x}; \mathrm{m}, \delta_0) d\mathbf{x}_{0,N}.$$

The only difference between ψ_N and $\tilde{\psi}_N$ comes from the distributions of the starting and ending points of the directed polymer on the cylinder: for ψ_N, the starting point is the origin and the ending point is "free" and distributed according to the Lebesgue measure on $\mathbb{T}$, while for $\tilde{\psi}_N$, the starting and ending points are sampled independently from the stationary distribution.

The purpose of the present section is to show the following proposition, which reduces the proof of central limit theorem to the stationary setting.

Proposition 3.4. *For any $\theta \in \mathbb{R}$ we have*

$$\lim_{N\to\infty}[\tilde{\psi}_N(\theta) - \psi_N(\theta)] = 0. \tag{3.7}$$

Proof. Consider a sequence $\{k_N\}_N$ such that $k_N \to \infty$ and $\frac{k_N}{\sqrt{N}} \to 0$. Define

$$\begin{aligned}\psi_{N,o}(\theta) &:= \mathbf{E}\int_{\mathbb{T}^{N+1}} \mathbb{E}_{\mathbf{x}} \exp\left\{\sum_{k=k_N}^{N-k_N} \frac{i\theta\eta_k}{\sqrt{N}}\right\} \mu_N(\mathbf{x}; \mathrm{m}, \delta_0) d\mathbf{x}_{0,N},\\ \tilde{\psi}_{N,o}(\theta) &:= \mathbb{E}_{\varrho}\mathbb{E}_{\tilde{\varrho}}\mathbf{E}\int_{\mathbb{T}^{N+1}} \mathbb{E}_{\mathbf{x}} \exp\left\{\sum_{k=k_N}^{N-k_N} \frac{i\theta\eta_k}{\sqrt{N}}\right\} \mu_N(\mathbf{x}; \tilde{\varrho}, \varrho) d\mathbf{x}_{0,N}.\end{aligned} \tag{3.8}$$

We have

$$\begin{aligned}&\Big|\psi_{N,o}(\theta) - \psi_N(\theta)\Big|\\ &\le \mathbf{E}\int_{\mathbb{T}^{N+1}} \mathbb{E}_{\mathbf{x}}\Big|\exp\left\{\sum_{k=k_N}^{N-k_N} \frac{i\theta\eta_k}{\sqrt{N}}\right\} - \exp\left\{\sum_{k=1}^{N} \frac{i\theta\eta_k}{\sqrt{N}}\right\}\Big|\mu_N(\mathbf{x}; \mathrm{m}, \delta_0) d\mathbf{x}_{0,N}\\ &\le \mathbf{E}\int_{\mathbb{T}^{N+1}} \left(\sum_{k=1}^{k_N-1} + \sum_{k=N-k_N+1}^{N}\right)\mathbb{E}_{\mathbf{x}}\Big|\frac{\theta\eta_k}{\sqrt{N}}\Big|\mu_N(\mathbf{x}; \mathrm{m}, \delta_0) d\mathbf{x}_{0,N}\\ &\le \frac{2k_N|\theta|\sqrt{\mathfrak{E}_2}}{\sqrt{N}} \to 0,\end{aligned}$$

as $N \to \infty$. In the last step, we have applied Lemma 3.2 with $\mathfrak{E}_2$ defined in (3.4). By the same proof, we have $|\tilde{\psi}_{N,o}(\theta) - \tilde{\psi}_N(\theta)| \to 0$. Thus, to prove the proposition, it suffices to show that

$$\Big|\tilde{\psi}_{N,o}(\theta) - \psi_{N,o}(\theta)\Big| \to 0. \tag{3.9}$$

Recall that for any $t > s$ and $\nu \in \mathcal{M}_1(\mathbb{T})$, $\rho_{\mathrm{per}}(t, \cdot; s, \nu)$ and $\tilde{\rho}_{\mathrm{per}}(t, \nu; s, \cdot)$ were defined in (2.7). We also have $\rho_{\mathrm{per}}(t, \cdot; \nu) = \rho_{\mathrm{per}}(t, \cdot; 0, \nu)$. In $\mu_N(\mathbf{x}; \mathrm{m}, \delta_0)$, we integrate out the variables

$$x_0, \dots, x_{k_N-2}, x_{N-k_N+1}, \dots, x_N$$

to obtain

$$\begin{aligned}&\int \mu_N(\mathbf{x}; \mathrm{m}, \delta_0) d\mathbf{x}_{0,k_N-2} d\mathbf{x}_{N-k_N+1,N}\\ &= \frac{\tilde{\rho}_{\mathrm{per}}(N, \mathrm{m}; N-k_N, x_{N-k_N})\left(\prod_{k=k_N}^{N-k_N} G_{k,k-1}(x_k, x_{k-1})\right)\rho_{\mathrm{per}}(k_N-1, x_{k_N-1}; \delta_0)}{\mathcal{G}_{N-k_N,k_N}(\mathrm{m}, \delta_0)},\end{aligned}$$

with

$$\mathcal{G}_{m_2,m_1}(\nu, \nu') := \int \tilde{\rho}_{\mathrm{per}}(N, \nu; m_2, x) G_{m_2,m_1-1}(x, y) \rho_{\mathrm{per}}(m_1-1, y; \nu') dx dy. \tag{3.10}$$

This leads to the following expression
(3.11)

$$\psi_{N,o}(\theta) = \mathbf{E}\int \tilde{\rho}_{\rm per}(N,\mathrm{m};N-k_N,x_{N-k_N})\frac{\prod_{k=k_N}^{N-k_N} G_{k,k-1}(x_k,x_{k-1})}{\mathcal{G}_{N-k_N,k_N}(\mathrm{m},\delta_0)} \times \rho_{\rm per}(k_N-1,x_{k_N-1};\delta_0)\mathbb{E}_{\mathbf{x}}\exp\left\{\sum_{k=k_N}^{N-k_N}\frac{i\theta\eta_k}{\sqrt{N}}\right\}d\mathbf{x}_{k_N-1,N-k_N}.$$

For brevity sake we write $d\mathbf{x}_{m,M} = dx_m\dots dx_M$ for any $m \le M$. With the above notations, we can also write $\tilde{\psi}_{N,o}$ as
(3.12)

$$\tilde{\psi}_{N,o}(\theta) = \mathbb{E}_{\varrho}\mathbb{E}_{\tilde{\varrho}}\mathbf{E}\int \tilde{\rho}_{\rm per}(N,\tilde{\varrho};N-k_N,x_{N-k_N})\frac{\prod_{k=k_N}^{N-k_N} G_{k,k-1}(x_k,x_{k-1})}{\mathcal{G}_{N-k_N,k_N}(\tilde{\varrho},\varrho)} \times \rho_{\rm per}(k_N-1,x_{k_N-1};\varrho)\mathbb{E}_{\mathbf{x}}\exp\left\{\sum_{k=k_N}^{N-k_N}\frac{i\theta\eta_k}{\sqrt{N}}\right\}d\mathbf{x}_{k_N-1,N-k_N}.$$

The idea is that, since $k_N \gg 1$, we expect $\rho_{\rm per}(k_{N-1},\cdot;\delta_0)$ and $\tilde{\rho}_{\rm per}(N,\mathrm{m};N-k_N,\cdot)$ to be close to the stationary distribution, so that in the expression of $\psi_{N,o}$, we can first "replace" δ_0 by ϱ, then "replace" m by $\tilde{\varrho}$. In this way, we reach at the expression of $\tilde{\psi}_{N,o}$

We define an intermediate version:
(3.13)

$$\psi^{(1)}_{N,o}(\theta) = \mathbb{E}_{\varrho}\mathbf{E}\int \tilde{\rho}_{\rm per}(N,\mathrm{m};N-k_N,x_{N-k_N})\frac{\prod_{k=k_N}^{N-k_N} G_{k,k-1}(x_k,x_{k-1})}{\mathcal{G}_{N-k_N,k_N}(\mathrm{m},\varrho)} \times \rho_{\rm per}(k_N-1,x_{k_N-1};\varrho)\mathbb{E}_{\mathbf{x}}\exp\left\{\sum_{k=k_N}^{N-k_N}\frac{i\theta\eta_k}{\sqrt{N}}\right\}d\mathbf{x}_{k_N-1,N-k_N}.$$

Compare (3.13) with (3.11), the only difference comes from replacing δ_0 with the stationary measure ϱ. Next we write

$$\psi_{N,o}(\theta) - \psi^{(1)}_{N,o}(\theta) = \mathrm{I}_N + \mathrm{II}_N,$$

where
(3.14)

$$\mathrm{I}_N = \mathbb{E}_{\varrho}\mathbf{E}\int \tilde{\rho}_{\rm per}(N,\mathrm{m};N-k_N,x_{N-k_N})\frac{\prod_{k=k_N}^{N-k_N} G_{k,k-1}(x_k,x_{k-1})}{\mathcal{G}_{N-k_N,k_N}(\mathrm{m},\delta_0)} \times \mathcal{E}_1(k_N-1,x_{k_N-1};\delta_0,\varrho)\mathbb{E}_{\mathbf{x}}\exp\left\{\sum_{k=k_N}^{N-k_N}\frac{i\theta\eta_k}{\sqrt{N}}\right\}d\mathbf{x}_{k_N-1,N-k_N},$$

with

$$\mathcal{E}_1(k_N-1,\cdot;\delta_0,\varrho) := \rho_{\rm per}(k_N-1,\cdot;\delta_0) - \rho_{\rm per}(k_N-1,\cdot;\varrho), \tag{3.15}$$

and
(3.16)

$$\mathrm{II}_N = \mathbb{E}_\varrho \mathbf{E} \int \tilde{\rho}_{\rm per}(N, \mathrm{m}; N - k_N, x_{N-k_N}) \left(\prod_{k=k_N}^{N-k_N} G_{k,k-1}(x_k, x_{k-1}) \right) \rho_{\rm per}(k_N - 1, x_{k_N-1}; \varrho) \times \mathcal{E}_2(k_N - 1; \delta_0, \varrho) \mathbb{E}_{\mathbf{x}} \exp\left\{ \sum_{k=k_N}^{N-k_N} \frac{i\theta\eta_k}{\sqrt{N}} \right\} d\mathbf{x}_{k_N-1, N-k_N},$$

with

$$\mathcal{E}_2(k_N - 1; \delta_0, \varrho) := \mathcal{G}_{N-k_N,k_N}(\mathrm{m}, \delta_0)^{-1} - \mathcal{G}_{N-k_N,k_N}(\mathrm{m}, \varrho)^{-1}.$$

In the following, we will estimate I_N and II_N separately.

Estimates on I_N. First we note that the term $|\mathbb{E}_{\mathbf{x}} \exp\left\{ \sum_{k=k_N}^{N-k_N} \frac{i\theta\eta_k}{\sqrt{N}} \right\}| \le 1$. Secondly, by the definition of $\mathcal{G}$ in (3.10), it is straightforward to check that

$$\frac{\tilde{\rho}_{\rm per}(N, \mathrm{m}; N - k_N, x_{N-k_N})}{\mathcal{G}_{N-k_N,k_N}(\mathrm{m}, \delta_0)} \le \frac{\sup_x \tilde{\rho}_{\rm per}(N, \mathrm{m}; N - k_N, x)}{\inf_x \tilde{\rho}_{\rm per}(N, \mathrm{m}; N - k_N, x) \inf_x \rho_{\rm per}(k_N - 1, x; \delta_0)} \times \frac{1}{G_{N-k_N,k_N-1}(\mathrm{m}, \mathrm{m})}.$$

Here we used the simplified notation $G_{t,s}(\nu, \nu') = \int G_{t,s}(x,y)\nu(dx)\nu'(dy)$. Thus, we have

$$\begin{aligned} |\mathrm{I}_N| \le & \mathbb{E}_\varrho \mathbf{E} \left[\frac{\sup_x \tilde{\rho}_{\rm per}(N, \mathrm{m}; N - k_N, x)}{\inf_x \tilde{\rho}_{\rm per}(N, \mathrm{m}; N - k_N, x) \inf_x \rho_{\rm per}(k_N - 1, x; \delta_0)} \sup_x |\mathcal{E}_1(k_N - 1, x; \delta_0, \varrho)| \right] \\ \le & \sqrt{ \mathbb{E}_\varrho \mathbf{E} \left(\frac{\sup_x \tilde{\rho}_{\rm per}(N, \mathrm{m}; N - k_N, x)}{\inf_x \tilde{\rho}_{\rm per}(N, \mathrm{m}; N - k_N, x) \inf_x \rho_{\rm per}(k_N - 1, x; \delta_0)} \right)^2 } \\ & \times \sqrt{ \mathbb{E}_\varrho \mathbf{E} \sup_x |\mathcal{E}_1(k_N - 1, x; \delta_0, \varrho)|^2 }. \end{aligned}$$

According to Proposition 2.1, there exist constants $C, \lambda > 0$ such that the above expression is bounded by

$$|\mathrm{I}_N| \le Ce^{-\lambda k_N}.$$

Estimates on II_N. The proof is similar to that of I_N. By (3.10) and the definition of $\mathcal{E}_1$ in (3.15), we have

$$\begin{aligned} & \mathcal{E}_2(k_N - 1; \delta_0, \varrho) \\ & = - \int \tilde{\rho}_{\rm per}(N, \mathrm{m}; N - k_N, x_{N-k_N}) \frac{G_{N-k_N,k_N-1}(x_{N-k_N}, x_{k_N-1})}{\mathcal{G}_{N-k_N,k_N}(\mathrm{m}, \delta_0)\mathcal{G}_{N-k_N,k_N}(\mathrm{m}, \varrho)} \\ & \times \mathcal{E}_1(k_N - 1, x_{k_N-1}; \delta_0, \varrho) dx_{N-k_N} dx_{k_N-1}, \end{aligned}$$

which implies that

$$
\begin{aligned}
&|\mathcal{E}_2(k_N-1;\delta_0,\varrho)|\\
&\le \frac{\sup_x \tilde\rho_{\rm per}(N,\mathrm{m};N-k_N,x)\sup_x|\mathcal{E}_1(k_N-1,x;\delta_0,\varrho)|}{(\inf_x\tilde\rho_{\rm per}(N,\mathrm{m};N-k_N,x))^2\inf_x\rho_{\rm per}(k_N-1,x;\delta_0)\inf_x\rho_{\rm per}(k_N-1,x;\varrho)}\\
&\times\frac{1}{G_{N-k_N,k_N-1}(\mathrm{m},\mathrm{m})}.
\end{aligned}
$$

This leads to

$$
\begin{aligned}
&|\mathrm{II}_N|\\
&\le \mathbb{E}_\varrho\mathbf{E}\frac{(\sup_x\tilde\rho_{\rm per}(N,\mathrm{m},N-k_N,x))^2\sup_x\rho_{\rm per}(k_N-1,x;\varrho)\sup_x|\mathcal{E}_1(k_N-1,x;\delta_0,\varrho)|}{(\inf_x\tilde\rho_{\rm per}(N,\mathrm{m},N-k_N,x))^2\inf_x\rho_{\rm per}(k_N-1,x;\delta_0)\inf_x\rho_{\rm per}(k_N-1,x;\varrho)}.
\end{aligned}
$$

Applying Hölder inequality as before, we also obtain that

$$|\mathrm{II}_N|\le Ce^{-\lambda k_N},\quad N=1,2,\dots.$$

Thus, we have

$$|\psi_{N,o}(\theta)-\psi^{(1)}_{N,o}(\theta)|\le Ce^{-\lambda k_N}\to 0$$

as $N\to\infty$.

It remains to show that $\tilde\psi_{N,o}(\theta)-\psi^{(1)}_{N,o}(\theta)\to 0$ as $N\to\infty$. Comparing (3.12) and (3.13), the only difference is m being replaced by $\tilde\varrho$. By following the same proof for $\psi_{N,o}(\theta)-\psi^{(1)}_{N,o}(\theta)$ verbatim, we conclude the proof of the proposition. □

3.3. **Construction of the path measure.** Recall that by the construction of the Markov chain in Section 2.3, the study of the winding number W_N, see (1.3), or equivalently $\lfloor w_N\rfloor$, reduces to that of $Y_N=\sum_{k=1}^N\eta_k$:

$$\hat{\mathbb{P}}_N[\lfloor w_N\rfloor=j]=\int_{\mathbb{T}^{N+1}}\mathbb{P}_{\mathbf{x}}[Y_N=j]\mu_N(\mathbf{x};\mathrm{m},\delta_0)d\mathbf{x}_{0,N},\qquad\text{for all } j\in\mathbb{Z}.$$

By the result in Section 3.2, to study the law of $\lfloor w_N\rfloor$, we can further replace $\mu_N(\mathbf{x};\mathrm{m},\delta_0)$ in the above expression by the stationary density $\mu_N(\mathbf{x};\tilde\varrho,\varrho)$. In this section, we construct a path measure to realize $\{\eta_k\}_{k\in\mathbb{Z}}$ as a sequence of stationary random variables, and the proof of the central limit theorem for $\lfloor w_N\rfloor$ reduces to that of $\sum_{k=1}^N\eta_k$.

The space $\mathbb{Z}^{\mathbb{Z}}$ consists of all functions $\sigma:\mathbb{Z}\to\mathbb{Z}$. For any $k\in\mathbb{Z}$, we denote by $\eta_k:\mathbb{Z}^{\mathbb{Z}}\to\mathbb{Z}$ the k-th coordinate map, i.e. $\eta_k(\sigma):=\sigma(k)$. Recall that for any $f,g\in\mathbb{D}_c(\mathbb{T})$, we have defined

$$\mu_N(\mathbf{x};f,g)=\frac{f(x_N)\prod_{k=1}^N G_{k,k-1}(x_k,x_{k-1})g(x_0)}{\int_{\mathbb{T}^{N+1}}f(x'_N)\prod_{k=1}^N G_{k,k-1}(x'_k,x'_{k-1})g(x'_0)d\mathbf{x}'_{0,N}}.$$

In the following, we construct a probability measure $\mathcal{P}$ on $\mathbb{Z}^{\mathbb{Z}}$ such that

1) for each $N\ge 1$ and $j_1,\dots,j_N\in\mathbb{Z}$, we have

$$
\begin{aligned}
(3.17)\qquad &\mathcal{P}\Big[\eta_1=j_1,\dots,\eta_N=j_N\Big]\\
&:=\mathbb{E}_{\varrho}\mathbb{E}_{\tilde\varrho}\mathbf{E}\int_{\mathbb{T}^{N+1}}\mathbb{P}_{\mathbf{x}}\Big[\eta_1=j_1,\dots,\eta_N=j_N\Big]\mu_N(\mathbf{x};\tilde\varrho,\varrho)d\mathbf{x}_{0,N}\\
&=\mathbb{E}_{\varrho}\mathbb{E}_{\tilde\varrho}\mathbf{E}\int_{\mathbb{T}^{N+1}}\prod_{k=1}^{N}\frac{Z_{k,k-1}(j_k+x_k,x_{k-1})}{G_{k,k-1}(x_k,x_{k-1})}\mu_N(\mathbf{x};\tilde\varrho,\varrho)d\mathbf{x}_{0,N}
\end{aligned}
$$

2) for any $\ell\in\mathbb{Z}$, we have

$$
(3.18)\qquad \mathcal{P}\Big[\eta_1=j_1,\dots,\eta_N=j_N\Big]=\mathcal{P}\Big[\eta_{\ell+1}=j_1,\dots,\eta_{\ell+N}=j_N\Big].
$$

This is done as follows. First, we define the family of measures $(\mathcal{P}_N)_{N\ge1}$ on $\mathbb{Z}^N$ by (3.17). They induce a finite additive set function $\mathcal{P}$ on the algebra $\mathcal{C}$ of cylindrical subsets of $\mathbb{Z}^{\mathbb{Z}}$. To show that $\mathcal{P}$ extends to the σ-algebra $\sigma(\mathcal{C})$, it suffices to prove the following consistency condition: for each $N\ge1$, $\ell\ge1$ and $j_1,\dots,j_N\in\mathbb{Z}$,

$$
\begin{aligned}
(3.19)\qquad &\mathcal{P}\Big[\eta_1=j_1,\dots,\eta_N=j_N\Big]\\
&=\mathbb{E}_{\varrho}\mathbb{E}_{\tilde\varrho}\mathbf{E}\int\mathbb{P}_{\mathbf{x}}\Big[\eta_1=j_1,\dots,\eta_N=j_N\Big]\mu_N(\mathbf{x};\tilde\varrho,\varrho)d\mathbf{x}_{0,N}\\
&=\mathbb{E}_{\varrho}\mathbb{E}_{\tilde\varrho}\mathbf{E}\int\sum_{j_{N+1},\dots,j_{N+\ell}}\mathbb{P}_{\mathbf{x}}\Big[\eta_1=j_1,\dots,\eta_N=j_N,\eta_{N+1}=j_{N+1},\dots,\eta_{N+\ell}=j_{N+\ell}\Big]\\
&\times\mu_{N+\ell}(\mathbf{x};\tilde\varrho,\varrho)d\mathbf{x}_{0,N+\ell}\\
&=\sum_{j_{N+1},\dots,j_{N+\ell}}\mathcal{P}\Big[\eta_1=j_1,\dots,\eta_N=j_N,\eta_{N+1}=j_{N+1},\dots,\eta_{N+\ell}=j_{N+\ell}\Big].
\end{aligned}
$$

The right hand side of (3.19) equals

$$
\begin{aligned}
&\mathbb{E}_{\varrho}\mathbb{E}_{\tilde\varrho}\mathbf{E}\int\sum_{j_{N+1},\dots,j_{N+\ell}}\prod_{k=1}^{N+\ell}\frac{Z_{k,k-1}(j_k+x_k,x_{k-1})}{G_{k,k-1}(x_k,x_{k-1})}\\
&\times\frac{\prod_{k=1}^{N+\ell}G_{k,k-1}(x_k,x_{k-1})\varrho(x_0)\tilde\varrho(x_{N+\ell})}{\int\prod_{k=1}^{N+\ell}G_{k,k-1}(x'_k,x'_{k-1})\varrho(x'_0)\tilde\varrho(x'_{N+\ell})d\mathbf{x}'_{0,N+\ell}}d\mathbf{x}_{0,N+\ell}\\
&=\mathbb{E}_{\varrho}\mathbb{E}_{\tilde\varrho}\mathbf{E}\int d\mathbf{x}_{0,N}\prod_{k=1}^{N}\frac{Z_{k,k-1}(j_k+x_k,x_{k-1})}{G_{k,k-1}(x_k,x_{k-1})}\\
&\times\frac{\tilde\rho_{\rm per}(N+\ell,\tilde\varrho;N,x_N)\left(\prod_{k=1}^{N}G_{k,k-1}(x_k,x_{k-1})\varrho(x_0)\right)}{\int\prod_{k=1}^{N}G_{k,k-1}(x'_k,x'_{k-1})\varrho(x'_0)\tilde\rho_{\rm per}(N+\ell,\tilde\varrho;N,x'_N)d\mathbf{x}'_{0,N}}.
\end{aligned}
$$

Here $\tilde\rho_{\rm per}(N+\ell,\tilde\varrho;N,\cdot)$ is the reverse time, polymer endpoint process starting at stationarity, see the definition of $\tilde\rho_{\rm per}$ in (2.7). So we know that $\tilde\rho_{\rm per}(N+\ell,\tilde\varrho;N,\cdot)$ has the same law as $\tilde\varrho$ and is independent of ϱ and the random environment in the interval $[0, N]$. We can therefore write that the right hand side of (3.19) equals

$$\mathbb{E}_{\varrho}\mathbb{E}_{\tilde{\varrho}}\mathbf{E}\int d\mathbf{x}_{0,N}\prod_{k=1}^{N}\frac{Z_{k,k-1}(j_k+x_k,x_{k-1})}{G_{k,k-1}(x_k,x_{k-1})}$$
$$\times\frac{\prod_{k=1}^{N}G_{k,k-1}(x_k,x_{k-1})\tilde{\varrho}(x_N)\varrho(x_0)}{\int\prod_{k=1}^{N}G_{k,k-1}(x'_k,x'_{k-1})\varrho(x'_0)\tilde{\varrho}(x'_N)d\mathbf{x}'_{0,N}},$$

which proves (3.19). The argument for (3.18) is similar (for $\ell \geq 1$). This way we construct a stationary measure $\mathcal{P}$ on $\mathbb{Z}^{\mathbb{N}}$. Its extension to $\mathbb{Z}^{\mathbb{Z}}$ is standard.

From now on, we use E to denote the expectation with respect to $\mathcal{P}$.

Proposition 3.5. *We have*

$$E\eta_j = 0, \tag{3.20}$$

$$E\eta_j^2 < \infty, \qquad \text{for all } j \in \mathbb{Z}. \tag{3.21}$$

Proof. First, (3.21) is a direct consequence of Lemma 3.2. Now we prove (3.20). Define

(3.22)
$$M_N := \mathbf{E}\int\mathbb{E}_{\mathbf{x}}\frac{\sum_{k=1}^{N}\eta_k}{N}\mu_N(\mathbf{x};\mathrm{m},\delta_0)d\mathbf{x}_{0,N} = \mathbf{E}\int\mathbb{E}_{\mathbf{x}}\frac{Y_N}{N}\mu_N(\mathbf{x};\mathrm{m},\delta_0)d\mathbf{x}_{0,N},$$
$$\tilde{M}_N := \mathbb{E}_{\varrho}\mathbb{E}_{\tilde{\varrho}}\mathbf{E}\int\mathbb{E}_{\mathbf{x}}\frac{\sum_{k=1}^{N}\eta_k}{N}\mu_N(\mathbf{x};\tilde{\varrho},\varrho)d\mathbf{x}_{0,N} = E\eta_1.$$

By following the proof of Proposition 3.4 and applying Lemma 3.2, we have $M_N - \tilde{M}_N \to 0$ as $N \to \infty$. On the other hand, note that $M_N = N^{-1}\mathbf{E}\hat{\mathbb{E}}_N\lfloor w_N\rfloor$. By symmetry we have $\mathbf{E}\hat{\mathbb{E}}_N w_N = 0$, which implies that $M_N \to 0$ as $N \to \infty$, since $w_N - 1 < \lfloor w_N\rfloor \leq w_N$. This further implies that $\tilde{M}_N = E\eta_1 = 0$. The proof is complete. □

3.4. **The correlation mixing.** In this section, we will show that the stationary sequence $\{\eta_k\}_{k\in\mathbb{Z}}$ constructed in Section 3.3 satisfies a central limit theorem:

$$\frac{\sum_{k=1}^{N}\eta_k}{\sqrt{N}} \Rightarrow N(0,\sigma_{\mathrm{eff}}^2), \qquad \text{as } N \to \infty. \tag{3.23}$$

With (3.23), Proposition 3.4, we conclude the proof of Theorem 3.1.

Consider the probability space space $(\mathbb{Z}^{\mathbb{Z}}, \sigma(\mathcal{C}), \mathcal{P})$. Let $\mathcal{F}_j$ and $\mathcal{F}^j$ be the σ-algebras generated by $\{\eta_k\}_{k\leq j}$ and $\{\eta_k\}_{k\geq j}$, respectively. Define the correlation coefficient between two square integrable random variables X, Y as

$$\mathrm{corr}[X,Y] = \frac{\mathrm{Cov}[X,Y]}{(EX^2)^{1/2}(EY^2)^{1/2}}.$$

We have the following definition of the ρ–mixing coefficient:

Definition 3.6. *(ρ-mixing coefficients, see [1, Section 19]) The ρ-mixing coefficients for the stationary sequence $\{\eta_k\}_{k\in\mathbb{Z}}$ are defined as*

$$r(n) := \sup\left\{|\mathrm{corr}[F,G]| : F \in \mathcal{F}_j,\ G \in \mathcal{F}^{j+n}\right\}, \quad n = 1, 2, \dots. \tag{3.24}$$

Note that, due to the stationarity, the definition of $r(n)$ in (3.24) does not depend on j. The main result of this section is the following:

Proposition 3.7. *There exist $C, \lambda > 0$ such that*

$$|\mathrm{Cov}[F,G]| \le Ce^{-\lambda n}\|F\|_{L^2}\|G\|_{L^2} \tag{3.25}$$

for any $n \ge 1$ and F, G that are $\mathcal{F}_j$ and $\mathcal{F}^{j+n}$ measurable respectively. As a consequence, we have

$$r(n) \le Ce^{-\lambda n}. \tag{3.26}$$

From (3.26), Proposition 3.5 and [1, Theorem 19.2], we immediately conclude the proof of (3.23), with

$$\sigma_{\mathrm{eff}}^2 = \sum_{j\in\mathbb{Z}} E[\eta_0\eta_j]. \tag{3.27}$$

Note that $\sigma_{\mathrm{eff}}^2 < \infty$ is a direct consequence of (3.26) and (3.21). We will show $\sigma_{\mathrm{eff}}^2 > 0$ in Section 5 below.

The rest of the section is devoted to the proof of Proposition 3.7.

Proof. Recall that E denote the expectation with respect to $\mathcal{P}$. Suppose that

$$F = f(\eta_1, \dots, \eta_{m_1}), \quad G = g(\eta_{m_1+n}, \dots, \eta_{m_2+n})$$

for some $m_1, m_2 \in \mathbb{N}$ and Borel measurable functions

$$f : \mathbb{R}^{m_1} \to \mathbb{R}, \quad g : \mathbb{R}^{m_2-m_1+1} \to \mathbb{R}.$$

Throughout the proof, to simplify the notation, define

$$\mathbf{x} = (x_0, \dots, x_N), \qquad N = m_2 + n.$$

We have

$$E[FG] = \mathbb{E}_\varrho \mathbf{E} \int \mathbb{E}_{\mathbf{x}}[f(\eta_1, \dots, \eta_{m_1})g(\eta_{m_1+n}, \dots, \eta_{m_2+n})]\mu_N(\mathbf{x}; \varrho_1, \varrho_2)d\mathbf{x}_{0,N}, \tag{3.28}$$

where ϱ_1, ϱ_2 are sampled independently from π_∞, also independent from ξ, and $\mathbb{E}_\varrho$ is the expectation on them. We recall that

$$\mu_N(\mathbf{x}; \varrho_1, \varrho_2) := \varrho_1(x_N)\frac{\prod_{k=1}^N G_{k,k-1}(x_k, x_{k-1})}{G_{N,0}(\varrho_1, \varrho_2)}\varrho_2(x_0)$$

is a density function, and

$$G_{N,0}(\varrho_1, \varrho_2) = \int \varrho_1(x_N)\prod_{k=1}^N G_{k,k-1}(x_k, x_{k-1})\varrho_2(x_0)d\mathbf{x}_{0,N}.$$

The goal is to show that, when N is large, the density $\mu_N(\mathbf{x};\varrho_1,\varrho_2)$ factorizes into two independent ones, with an error that is exponentially small in n. The proof consists of several steps.

Step 1. Rewriting $E[FG]$. Since $\mathbb{E}_{\mathbf{x}}[f(\eta_1,\ldots,\eta_{m_1})g(\eta_{m_1+n},\ldots,\eta_{m_2+n})]$ only depends on the variables $\mathbf{x}_{0,m_1},\mathbf{x}_{m_1+n-1,N}$, we will first integrate out other variables in $\mu_N(\cdot;\varrho_1,\varrho_2)$. We keep a "middle" one for a future purpose: define

$$\ell = m_1 + \lfloor n/2 \rfloor.$$

After integrating out the variables $\mathbf{x}_{m_1+1,\ell-1},\mathbf{x}_{\ell+1,m_1+n-2}$, we obtain

$$\begin{aligned}&\int \mu_N(\mathbf{x};\varrho_1,\varrho_2)d\mathbf{x}_{m_1+1,\ell-1}d\mathbf{x}_{\ell+1,m_1+n-2}\\&=G_{N,0}(\varrho_1,\varrho_2)^{-1}\varrho_1(x_N)\prod_{k=m_1+n}^{N}G_{k,k-1}(x_k,x_{k-1})G_{m_1+n-1,\ell}(x_{m_1+n-1},x_\ell)\\&\quad\times G_{\ell,m_1}(x_\ell,x_{m_1})\prod_{k=1}^{m_1}G_{k,k-1}(x_k,x_{k-1})\varrho_2(x_0).\end{aligned}\tag{3.29}$$

Recall the definition of forward and backward density in (2.7), we rewrite the two factors in (3.29) that contain x_ℓ as

$$\begin{aligned}G_{m_1+n-1,\ell}(\cdot,x_\ell)&=\rho_{\mathrm{per}}(m_1+n-1,\cdot;\ell,\delta_{x_\ell})\int G_{m_1+n-1,\ell}(y,x_\ell)dy,\\G_{\ell,m_1}(x_\ell,\cdot)&=\tilde\rho_{\mathrm{per}}(\ell,\delta_{x_\ell};m_1,\cdot)\int G_{\ell,m_1}(x_\ell,y)dy.\end{aligned}\tag{3.30}$$

Further define the normalization constant

$$\begin{aligned}h_1(x_\ell)&=\int \varrho_1(x_N)\prod_{k=m_1+n}^{N}G_{k,k-1}(x_k,x_{k-1})\rho_{\mathrm{per}}(m_1+n-1,x_{m_1+n-1};\ell,\delta_{x_\ell})d\mathbf{x}_{m_1+n-1,N},\\h_2(x_\ell)&=\int \tilde\rho_{\mathrm{per}}(\ell,\delta_{x_\ell};m_1,x_{m_1})\prod_{k=1}^{m_1}G_{k,k-1}(x_k,x_{k-1})\varrho_2(x_0)d\mathbf{x}_{0,m_1},\end{aligned}\tag{3.31}$$

and the densities

$$\begin{aligned}&p_1(\mathbf{x}_{m_1+n-1,N},x_\ell)=h_1(x_\ell)^{-1}\varrho_1(x_N)\prod_{k=m_1+n}^{N}G_{k,k-1}(x_k,x_{k-1})\\&\times\rho_{\mathrm{per}}(m_1+n-1,x_{m_1+n-1};\ell,\delta_{x_\ell}),\\&p_2(x_\ell,\mathbf{x}_{0,m_1})=h_2(x_\ell)^{-1}\tilde\rho_{\mathrm{per}}(\ell,\delta_{x_\ell};m_1,x_{m_1})\prod_{k=1}^{m_1}G_{k,k-1}(x_k,x_{k-1})\varrho_2(x_0).\end{aligned}\tag{3.32}$$

Using the above notations, one can rewrite (3.29) as

$$\begin{aligned}&\int \mu_N(\mathbf{x};\varrho_1,\varrho_2)d\mathbf{x}_{m_1+1,\ell-1}d\mathbf{x}_{\ell+1,m_1+n-2}\\&=p_1(\mathbf{x}_{m_1+n-1,N},x_\ell)p_2(x_\ell,\mathbf{x}_{0,m_1})\mathbf{p}(x_\ell),\end{aligned}$$

where $\mathbf{p}(\cdot)$ is a density that takes the form

$$\begin{aligned}\mathbf{p}(x_\ell) =& G_{N,0}(\varrho_1,\varrho_2)^{-1}h_1(x_\ell)h_2(x_\ell)\\ &\times\Big(\int G_{m_1+n-1,\ell}(y,x_\ell)dy\Big)\Big(\int G_{\ell,m_1}(x_\ell,y)dy\Big).\end{aligned}$$

It is clear that $\mathbf{p}$ is the marginal density of x_ℓ, since

$$\int \mu_N(\mathbf{x};\varrho_1,\varrho_2)d\mathbf{x}_{0,\ell-1}d\mathbf{x}_{\ell+1,N} = \mathbf{p}(x_\ell).$$

In this way, the expectation of the product is rewritten as

$$\begin{aligned}E[FG] =& \mathbb{E}_\varrho \mathbf{E}\int \mathbb{E}_{\mathbf{x}}[f\big(\eta_1,\ldots,\eta_{m_1}\big)g\big(\eta_{m_1+n},\ldots,\eta_{m_2+n}\big)]\\ &\times p_1(\mathbf{x}_{m_1+n-1,N},x_\ell)p_2(x_\ell,\mathbf{x}_{0,m_1})\mathbf{p}(x_\ell)d\mathbf{x}_{m_1+n-1,N}d\mathbf{x}_{0,m_1}dx_\ell.\end{aligned} \tag{3.33}$$

Step 2. Rewriting $E[F]E[G]$. Let ϱ_3,ϱ_4 be sampled independently from π_∞, which are also independent from ϱ_1,ϱ_2 and the random environment. Define

$$\begin{aligned}&p_3(\mathbf{x}_{m_1+n-1,N})\\ &= \frac{\varrho_1(x_N)\prod_{k=m_1+n}^N G_{k,k-1}(x_k,x_{k-1})\rho_{\rm per}(m_1+n-1,x_{m_1+n-1};\ell,\varrho_3)}{\int \varrho_1(x_N')\prod_{k=m_1+n}^N G_{k,k-1}(x_k',x_{k-1}')\rho_{\rm per}(m_1+n-1,x_{m_1+n-1}';\ell,\varrho_3)d\mathbf{x}_{m_1+n-1,N}'},\\ &p_4(\mathbf{x}_{0,m_1})\\ &= \frac{\tilde\rho_{\rm per}(\ell,\varrho_4;m_1,x_{m_1})\prod_{k=1}^{m_1} G_{k,k-1}(x_k,x_{k-1})\varrho_2(x_0)}{\int \tilde\rho_{\rm per}(\ell,\varrho_4;m_1,x_{m_1}')\prod_{k=1}^{m_1} G_{k,k-1}(x_k',x_{k-1}')\varrho_2(x_0')d\mathbf{x}_{0,m_1}'}.\end{aligned} \tag{3.34}$$

In other words, in the expressions of p_1,p_2, we have replaced δ_{x_ℓ} with ϱ_3,ϱ_4 to obtain p_3,p_4 respectively. Now it is straightforward to check that

$$\begin{aligned}E[F]E[G] =& \mathbb{E}_\varrho \mathbf{E}\int \mathbb{E}_{\mathbf{x}}[f\big(\eta_1,\ldots,\eta_{m_1}\big)g\big(\eta_{m_1+n},\ldots,\eta_{m_2+n}\big)]\\ &\times p_3(\mathbf{x}_{m_1+n-1,N})p_4(\mathbf{x}_{0,m_1})\mathbf{p}(x_\ell)d\mathbf{x}_{m_1+n-1,N}d\mathbf{x}_{0,m_1}dx_\ell.\end{aligned} \tag{3.35}$$

In the above expression, the term $\mathbf{p}(x_\ell)$ actually plays no role since one can integrate it out and $\int \mathbf{p}(x_\ell)dx_\ell = 1$ – we kept it there to compare with the expression of $E[FG]$.

Combining (3.33) and (3.35), we have

$$\begin{aligned}\mathrm{Cov}[F,G] =& \mathbb{E}_\varrho \mathbf{E}\int \mathbb{E}_{\mathbf{x}}[f\big(\eta_1,\ldots,\eta_{m_1}\big)g\big(\eta_{m_1+n},\ldots,\eta_{m_2+n}\big)]\\ &\times[p_1(\mathbf{x}_{m_1+n-1,N},x_\ell)p_2(x_\ell,\mathbf{x}_{0,m_1})-p_3(\mathbf{x}_{m_1+n-1,N})p_4(\mathbf{x}_{0,m_1})]\\ &\times \mathbf{p}(x_\ell)d\mathbf{x}_{m_1+n-1,N}d\mathbf{x}_{0,m_1}dx_\ell.\end{aligned} \tag{3.36}$$

Step 3. Approximation. Now we decompose $\mathrm{Cov}[F,G] = Err_1(n) + Err_2(n)$, with

$$Err_1(n) = \mathbb{E}_\varrho \mathbf{E} \int \mathbb{E}_{\mathbf{x}}[f(\eta_1,\dots,\eta_{m_1}) g(\eta_{m_1+n},\dots,\eta_{m_2+n})]$$
$$\times [p_1(\mathbf{x}_{m_1+n-1,N}, x_\ell) - p_3(\mathbf{x}_{m_1+n-1,N})] p_2(x_\ell, \mathbf{x}_{0,m_1}) \mathbf{p}(x_\ell) d\mathbf{x}_{m_1+n-1,N} d\mathbf{x}_{0,m_1} dx_\ell,$$

and

$$Err_2(n) = \mathbb{E}_\varrho \mathbf{E} \int \mathbb{E}_{\mathbf{x}}[f(\eta_1,\dots,\eta_{m_1}) g(\eta_{m_1+n},\dots,\eta_{m_2+n})]$$
$$\times p_3(\mathbf{x}_{m_1+n-1,N}) [p_2(x_\ell, \mathbf{x}_{0,m_1}) - p_4(\mathbf{x}_{0,m_1})] \mathbf{p}(x_\ell) d\mathbf{x}_{m_1+n-1,N} d\mathbf{x}_{0,m_1} dx_\ell.$$

It suffices to show that $|Err_i(n)| \le Ce^{-\lambda n} \|F\|_{L^2} \|G\|_{L^2}$ for $i = 1,2$. The two cases are handled in the same way, so we will only focus on $Err_1(n)$.

The rest of the proof is very similar to that of Proposition 3.4. First, from (3.32) and (3.34) we have

$$|p_1(\mathbf{x}_{m_1+n-1,N}, x_\ell) - p_3(\mathbf{x}_{m_1+n-1,N})| \le I_1 + I_2,$$

with

$$I_1 = \frac{\varrho_1(x_N) \prod_{k=m_1+n}^N G_{k,k-1}(x_k, x_{k-1})}{\int \varrho_1(x'_N) \prod_{k=m_1+n}^N G_{k,k-1}(x'_k, x'_{k-1}) d\mathbf{x}'_{m_1+n-1,N} \cdot \inf_y \rho_{\mathrm{per}}(m_1+n-1, y; \ell, \delta_{x_\ell})}$$
$$\times \sup_y |\rho_{\mathrm{per}}(m_1+n-1, y; \ell, \delta_{x_\ell}) - \rho_{\mathrm{per}}(m_1+n-1, y; \ell, \varrho_3)|,$$

$$I_2 = \frac{\varrho_1(x_N) \prod_{k=m_1+n}^N G_{k,k-1}(x_k, x_{k-1})}{\int \varrho_1(x'_N) \prod_{k=m_1+n}^N G_{k,k-1}(x'_k, x'_{k-1}) d\mathbf{x}'_{m_1+n-1,N}}$$
$$\times \frac{\sup_y \rho_{\mathrm{per}}(m_1+n-1, y; \ell, \varrho_3)}{\inf_y \rho_{\mathrm{per}}(m_1+n-1, y; \ell, \delta_{x_\ell}) \inf_y \rho_{\mathrm{per}}(m_1+n-1, y; \ell, \varrho_3)}$$
$$\times \sup_y |\rho_{\mathrm{per}}(m_1+n-1, y; \ell, \delta_{x_\ell}) - \rho_{\mathrm{per}}(m_1+n-1, y; \ell, \varrho_3)|.$$

Thus, $Err_1(n)$ can be bounded from above by

$$Err_1(n) \le \mathbb{E}_\varrho \mathbf{E} \int \mathbb{E}_{\mathbf{x}}[|f(\eta_1,\dots,\eta_{m_1}) g(\eta_{m_1+n},\dots,\eta_{m_2+n})|]$$
$$\times [I_1 + I_2] p_2(x_\ell, \mathbf{x}_{0,m_1}) \mathbf{p}(x_\ell) d\mathbf{x}_{m_1+n-1,N} d\mathbf{x}_{0,m_1} dx_\ell =: J_1 + J_2.$$

Consider the term J_1. We first bound I_1 by

$$I_1 \le \frac{\varrho_1(x_N) \prod_{k=m_1+n}^N G_{k,k-1}(x_k, x_{k-1})}{\int \varrho_1(x'_N) \prod_{k=m_1+n}^N G_{k,k-1}(x'_k, x'_{k-1}) d\mathbf{x}'_{m_1+n-1,N}}$$
$$\times \frac{\sup_{y,x} |\rho_{\mathrm{per}}(m_1+n-1, y; \ell, \delta_x) - \rho_{\mathrm{per}}(m_1+n-1, y; \ell, \varrho_3)|}{\inf_{y,x} \rho_{\mathrm{per}}(m_1+n-1, y; \ell, \delta_x)}.$$

Using (3.32), we bound $p_2(x_\ell, \mathbf{x}_{0,m_1})$ by

$$p_2(x_\ell, \mathbf{x}_{0,m_1}) \le \frac{\prod_{k=1}^{m_1} G_{k,k-1}(x_k, x_{k-1}) \varrho_2(x_0)}{\int \prod_{k=1}^{m_1} G_{k,k-1}(x'_k, x'_{k-1}) \varrho_2(x'_0) d\mathbf{x}'_{0,m_1}} \times \frac{\sup_{x,y} \tilde\rho_{\mathrm{per}}(\ell, \delta_x; m_1, y)}{\inf_{x,y} \tilde\rho_{\mathrm{per}}(\ell, \delta_x; m_1, y)}.$$

In this way we got rid of the dependence on x_ℓ in all other terms except for $\mathbf{p}(x_\ell)$ which can be integrated out. By the independence of the random environment in separate time intervals, we obtain $J_1 \le \prod_{i=1}^4 K_i$, with

$$K_1 := \mathbb{E}_\varrho \mathbf{E} \int \mathbb{E}_{\mathbf{x}} |f(\eta_1, \dots, \eta_{m_1})| \frac{\prod_{k=1}^{m_1} G_{k,k-1}(x_k, x_{k-1}) \varrho_2(x_0)}{\int \prod_{k=1}^{m_1} G_{k,k-1}(x'_k, x'_{k-1}) \varrho_2(x'_0) d\mathbf{x}'_{0,m_1}} d\mathbf{x}_{0,m_1},$$

$$K_2 := \mathbb{E}_\varrho \mathbf{E} \int \mathbb{E}_{\mathbf{x}} |g(\eta_{m_1+n}, \dots, \eta_{m_2+n})| \\ \times \frac{\varrho_1(x_N) \prod_{k=m_1+n}^{N} G_{k,k-1}(x_k, x_{k-1})}{\int \varrho_1(x'_N) \prod_{k=m_1+n}^{N} G_{k,k-1}(x'_k, x'_{k-1}) d\mathbf{x}'_{m_1+n-1,N}} d\mathbf{x}_{m_1+n-1,N},$$

$$K_3 := \mathbb{E}_\varrho \mathbf{E} \frac{\sup_{y,x} |\rho_{\rm per}(m_1+n-1, y; \ell, \delta_x) - \rho_{\rm per}(m_1+n-1, y; \ell, \varrho_3)|}{\inf_{y,x} \rho_{\rm per}(m_1+n-1, y; \ell, \delta_x)},$$

$$K_4 := \mathbb{E}_\varrho \mathbf{E} \frac{\sup_{x,y} \tilde\rho_{\rm per}(\ell, \delta_x; m_1, y)}{\inf_{x,y} \tilde\rho_{\rm per}(\ell, \delta_x; m_1, y)}.$$

Applying Proposition 2.1, we have $K_3 \le Ce^{-\lambda n}$ and $K_4 \le C$. For K_1, we can bound it from above by

$$K_1 \le \mathbb{E}_\varrho \mathbf{E} \int \mathbb{E}_{\mathbf{x}} |f(\eta_1, \dots, \eta_{m_1})| \\ \times \frac{\varrho_4(x_{m_1}) \prod_{k=1}^{m_1} G_{k,k-1}(x_k, x_{k-1}) \varrho_2(x_0)}{\int \varrho_4(x'_{m_1}) \prod_{k=1}^{m_1} G_{k,k-1}(x'_k, x'_{k-1}) \varrho_2(x'_0) d\mathbf{x}'_{0,m_1}} d\mathbf{x}_{0,m_1} \times \frac{\sup_y \varrho_4(y)}{\inf_y \varrho_4(y)}.$$

Applying the Cauchy-Schwarz inequality, we have

$$K_1^2 \le C \mathbb{E}_\varrho \mathbf{E} \Bigg(\int \mathbb{E}_{\mathbf{x}} |f(\eta_1, \dots, \eta_{m_1})| \\ \times \frac{\varrho_4(x_{m_1}) \prod_{k=1}^{m_1} G_{k,k-1}(x_k, x_{k-1}) \varrho_2(x_0)}{\int \varrho_4(x'_{m_1}) \prod_{k=1}^{m_1} G_{k,k-1}(x'_k, x'_{k-1}) \varrho_2(x'_0) d\mathbf{x}'_{0,m_1}} d\mathbf{x}_{0,m_1} \Bigg)^2.$$

The r.h.s. can be further bounded from above by

$$C \mathbb{E}_\varrho \mathbf{E} \int \mathbb{E}_{\mathbf{x}} |f(\eta_1, \dots, \eta_{m_1})|^2 \\ \times \frac{\varrho_4(x_{m_1}) \prod_{k=1}^{m_1} G_{k,k-1}(x_k, x_{k-1}) \varrho_2(x_0)}{\int \varrho_4(x'_{m_1}) \prod_{k=1}^{m_1} G_{k,k-1}(x'_k, x'_{k-1}) \varrho_2(x'_0) d\mathbf{x}'_{0,m_1}} d\mathbf{x}_{0,m_1} = C\|F\|_{L^2}^2.$$

The same proof shows that $K_2 \le C\|G\|_{L^2}$. So we have $J_1 \le Ce^{-\lambda n}\|F\|_{L^2}\|G\|_{L^2}$. The term J_2 is dealt with in the same way, and this combines to show that $Err_1(n) \le Ce^{-\lambda n}\|F\|_{L^2}\|G\|_{L^2}$. Since $Err_2(n)$ is proved in exactly the same way, we complete the proof of the proposition. □

4. Proof of Theorem 1.1

Recall the goal was to show that, as $T \to \infty$, $\frac{w_T}{\sqrt{T}}$ converges to a nondegenerate Gaussian distribution under the measure $\mathbf{P} \otimes \hat{\mathbb{P}}_T$. As we have already observed in Section 3.4 the laws of $\frac{w_N}{\sqrt{N}}$ under the measure $\mathbf{P} \otimes \hat{\mathbb{P}}_N$ weakly converge to $N(0, \sigma_{\text{eff}}^2)$, as $N \to \infty$. In this section we show how to extend the conclusion to the laws of $\frac{w_T}{\sqrt{T}}$, under the measure $\mathbf{P} \otimes \hat{\mathbb{P}}_T$, when $T \to \infty$ (not necessarily taking integer values).

For a general $T > 0$, define $N := \lfloor T \rfloor$. Let $X_1(T)$ and $X_2(T)$ be random variables with the same laws as w_N under $\mathbf{P} \otimes \hat{\mathbb{P}}_N$ and w_T under $\mathbf{P} \otimes \hat{\mathbb{P}}_T$, respectively. We have the following lemma, which is the last piece that is needed to complete the proof of Theorem 1.1.

Lemma 4.1. *There exists $C > 0$ such that for any $T \ge 1$ one can find a coupling $\Big(X_1(T), X_2(T)\Big)$ such that*

$$\mathbb{E}\Big|X_2(T) - X_1(T)\Big|^2 \le C. \tag{4.1}$$

Proof. To lighten the notation we write X_1 and X_2, instead of $X_1(T)$ and $X_2(T)$. Recall that $\rho(T, \cdot)$ is the density of w_T under $\hat{\mathbb{P}}_T$, and we have the relation

$$\rho(T, x) = \frac{\int Z_{T,N}(x, y)\rho(N, y)dy}{\int Z_{T,N}(x', y')\rho(N, y')dx'dy'}. \tag{4.2}$$

From (4.2), we know that, given the value of X_1, if we sample X_2 from the density $\frac{Z_{T,N}(\cdot, X_1)}{\int Z_{T,N}(x', X_1)dx'}$, then X_2 has the same law as w_T under $\mathbf{P} \otimes \hat{\mathbb{P}}_T$. By the construction of X_1, X_2, we have for any $y \in \mathbb{R}$ that

$$\mathbb{E}[|X_2 - X_1|^2 | X_1 = y] = \mathbf{E}\frac{\int (x - y)^2 Z_{T,N}(x, y)dx}{\int Z_{T,N}(x', y)dx'}.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\mathbb{E}[|X_2 - X_1|^2 | X_1 = y] \\
&\le \int (x - y)^2 \sqrt{\mathbf{E} Z_{T,N}(x, y)^2} dx \sqrt{\mathbf{E}(\int Z_{T,N}(x', y)dx')^{-2}}.
\end{aligned}$$

For the first term on the r.h.s., we apply Lemma 3.3 to derive (note that $T - N < 1$)

$$\int (x - y)^2 \sqrt{\mathbf{E} Z_{T,N}(x, y)^2} dx \le C.$$

For the second term, we have for each fixed y that

$$\int Z_{T,N}(x', y)dx' \stackrel{\text{law}}{=} v_{\mathbb{1}}(T - N, 0),$$

with $v_{\mathbb{1}}$ solving the SHE (2.6), with the initial data $v_{\mathbb{1}}(0, x) \equiv 1$. Thus, applying [8, Lemma B.7], we have $\mathbf{E}(\int Z_{T,N}(x', y)dx')^{-2} \le C$. The proof is complete. □

5. Nondegeneracy of the diffusion constant

The goal of this section is to show that $\sigma^2_{\text{eff}} \neq 0$, which is a nontrivial fact. In general, it needs not be true that $\sum_{j\in Z} r(j) > 0$ when $r(\cdot)$ is the covariance function of a stationary sequence. We will need to make use of some specific structure of our model.

We first show a stability result on the approximation of the diffusion constant. Define

$$\sigma_N^2 := \frac{1}{N}\mathbf{E}\int \mathbb{E}_{\mathbf{x}}\Big(\sum_{k=1}^N \eta_k\Big)^2 \mu_N(\mathbf{x};\mathrm{m},\delta_0)d\mathbf{x}_{0,N} = \frac{1}{N}\mathbf{E}\hat{\mathbb{E}}_N\lfloor w_N\rfloor^2, \tag{5.1}$$

and

$$\tilde{\sigma}_N^2 := \frac{1}{N}\mathbb{E}_{\varrho}\mathbb{E}_{\tilde{\varrho}}\mathbf{E}\int \mathbb{E}_{\mathbf{x}}\Big(\sum_{k=1}^N \eta_k\Big)^2 \mu_N(\mathbf{x};\tilde{\varrho},\varrho)d\mathbf{x}_{0,N} = \frac{1}{N}E\Big(\sum_{k=1}^N \eta_k\Big)^2. \tag{5.2}$$

We have

Proposition 5.1. *$\sigma_N^2 - \tilde{\sigma}_N^2 \to 0$ as $N \to \infty$.*

Proof. The proof is similar to that of Propositions 3.4 and 3.7, so we only sketch the argument here.

First, for some $k_N \to \infty$, yet to be determined, we decompose $\sigma_N^2 = \sum_{\ell=1}^4 I_\ell$ and $\tilde{\sigma}_N^2 = \sum_{\ell=1}^4 J_\ell$, with

$$I_\ell = \frac{1}{N}\sum_{(i,j)\in A_\ell}\mathbf{E}\int_{\mathbb{T}^{N+1}}\big(\mathbb{E}_{\mathbf{x}}\eta_i\eta_j\big)\mu_N(\mathbf{x};\mathrm{m},\delta_0)d\mathbf{x}_{0,N},$$

$$J_\ell = \frac{1}{N}\sum_{(i,j)\in A_\ell}\mathbb{E}_{\varrho}\mathbb{E}_{\tilde{\varrho}}\mathbf{E}\int_{\mathbb{T}^{N+1}}\big(\mathbb{E}_{\mathbf{x}}\eta_i\eta_j\big)\mu_N(\mathbf{x};\tilde{\varrho},\varrho)d\mathbf{x}_{0,N},$$

with

$$A_1 = B_1\times B_1, \qquad A_2 = B_2\times B_2,$$
$$A_3 = B_1\times B_2, \qquad A_4 = B_2\times B_1,$$

and $B_1 = [k_N, N-k_N]$ and $B_2 = [1, k_N-1]\cup[N-k_N+1, N]$. In the following, we will show that $I_\ell - J_\ell \to 0$, for each $\ell = 1,\ldots,4$.

The case of $\ell = 1$. By following closely the proof of Proposition 3.4 and with the help of Lemma 3.2, we derive that, for each $(i,j)\in A_1$,

$$\Big|\mathbf{E}\int\big(\mathbb{E}_{\mathbf{x}}\eta_i\eta_j\big)\mu_N(\mathbf{x};\mathrm{m},\delta_0)d\mathbf{x}_{0,N} - \mathbb{E}_{\varrho}\mathbb{E}_{\tilde{\varrho}}\mathbf{E}\int\big(\mathbb{E}_{\mathbf{x}}\eta_i\eta_j\big)\mu_N(\mathbf{x};\tilde{\varrho},\varrho)d\mathbf{x}_{0,N}\Big|$$
$$\leq Ce^{-\lambda k_N},$$

which implies that

$$|I_1 - J_1| \leq CN^{-1}N^2e^{-\lambda k_N} = CNe^{-\lambda k_N}.$$

The case of $\ell = 2$. In this case, we directly apply Lemma 3.2 to derive that

$$|I_2 - J_2| \leq CN^{-1}k_N^2.$$

The case of $\ell = 3$ and 4. By symmetry, we only need to consider $\ell = 3$. First, for J_3, we have

$$J_3 = \frac{1}{N} \sum_{(i,j)\in A_3} E(\eta_i \eta_j)$$

By Proposition 3.7, we know that $|E(\eta_i\eta_j)| \le Ce^{-\lambda|i-j|}$, and this implies that

$$|J_3| \le CN^{-1} \sum_{i\in B_1, j\in B_2} e^{-\lambda|i-j|} \le CN^{-1}k_N.$$

It remains to study I_3, which we rewrite as

$$\begin{aligned} I_3 =& \frac{1}{N} \sum_{(i,j)\in A_3} \left(\mathbf{E} \int \big(\mathbb{E}_{\mathbf{x}} \eta_i\eta_j\big) \mu_N(\mathbf{x}; \mathrm{m}, \delta_0) d\mathbf{x}_{0,N} - \prod_{k=i,j} \mathbf{E} \int \big(\mathbb{E}_{\mathbf{x}} \eta_k\big) \mu_N(\mathbf{x}; \mathrm{m}, \delta_0) d\mathbf{x}_{0,N} \right) \\ &+ \frac{1}{N} \sum_{(i,j)\in A_3} \prod_{k=i,j} \mathbf{E} \int (\mathbb{E}_{\mathbf{x}}\eta_k) \mu_N(\mathbf{x}; \mathrm{m}, \delta_0) d\mathbf{x}_{0,N} =: I_{31} + I_{32}. \end{aligned}$$

For the term I_{31}, by following the proof for Proposition 3.7, one can show that

$$|I_{31}| \le CN^{-1} \sum_{(i,j)\in A_3} e^{-\lambda|i-j|} \le CN^{-1}k_N.$$

For the other term, since $(i,j) \in A_3$, we have $i \in [k_N, N-k_N]$. By following the proof for Proposition 3.4, one can show that

$$|\mathbf{E} \int (\mathbb{E}_{\mathbf{x}}\eta_i) \mu_N(\mathbf{x}; \mathrm{m}, \delta_0) d\mathbf{x}_{0,N} - \mathbf{E} \int (\mathbb{E}_{\mathbf{x}}\eta_i) \mu_N(\mathbf{x}; \tilde{\varrho}, \varrho) d\mathbf{x}_{0,N}| \le Ce^{-\lambda k_N}.$$

We have

$$\mathbf{E} \int (\mathbb{E}_{\mathbf{x}}\eta_i) \mu_N(\mathbf{x}; \tilde{\varrho}, \varrho) d\mathbf{x}_{0,N} = E\eta_i = 0.$$

We can apply Lemma 3.2 again and conclude

$$|I_{32}| \le CN^{-1} \sum_{(i,j)\in A_3} e^{-\lambda k_N} \le Ck_N e^{-\lambda k_N}.$$

To summarize, we have

$$|\sigma_N^2 - \tilde{\sigma}_N^2| \le \sum_{\ell=1}^4 |I_\ell - J_\ell| \le C\big(Ne^{-\lambda k_N} + N^{-1}k_N^2 + N^{-1}k_N + k_N e^{-\lambda k_N}\big).$$

Choosing $k_N = N^\alpha$ for any $\alpha \in (0, 1/2)$, the proof is complete. □

Since

$$\tilde{\sigma}_N^2 = \frac{1}{N} E(\sum_{k=1}^N \eta_k)^2 \to \sigma_{\text{eff}}^2, \qquad \text{as } N \to \infty,$$

applying Proposition 5.1, we derive that, as $N \to \infty$,

$$\sigma_N^2 \to \sigma_{\text{eff}}^2.$$

The following proposition completes the proof of the nondegeneracy of σ_{eff}^2:

Proposition 5.2. *We have*

$$\liminf_{N\to\infty} \sigma_N^2 \geq 1.$$

As a result, $\sigma_{\text{eff}}^2 \geq 1$.

Proof. By definition, we have

$$\sigma_N^2 = \frac{1}{N}\mathbf{E}\hat{\mathbb{E}}_N \lfloor w_N \rceil^2.$$

Since $|w_N| \leq |\lfloor w_N \rceil| + 1$, we have via a triangle inequality that

$$\sqrt{\frac{1}{N}\mathbf{E}\hat{\mathbb{E}}_N w_N^2} \leq \sqrt{\frac{1}{N}\mathbf{E}\hat{\mathbb{E}}_N \lfloor w_N \rceil^2} + \sqrt{\frac{1}{N}}.$$

Sending $N \to \infty$ and applying Lemma 5.3 below, the proof is complete. □

Lemma 5.3. *For any* $N \in \mathbb{Z}_+$, *we have*

$$\mathbf{E}\hat{\mathbb{E}}_N w_N^2 \geq N. \tag{5.3}$$

Proof. Recall that $\rho(N,\cdot)$ is the density of w_N under the quenched polymer measure $\hat{\mathbb{P}}_N$, we have

$$\begin{aligned}\mathbf{E}\hat{\mathbb{E}}_N w_N^2 &= \mathbf{E}\int x^2\rho(N,x)dx\\ &\geq \mathbf{E}\Big[\int x^2\rho(N,x)dx - \Big(\int x\rho(N,x)dx\Big)^2\Big].\end{aligned}$$

Note that the last expression is just the average of the quenched variance. It remains to show

$$\mathbf{E}\Big[\int x^2\rho(N,x)dx - \Big(\int x\rho(N,x)dx\Big)^2\Big] = N, \tag{5.4}$$

which is a standard folklore for the directed polymer when the random environment is statistically invariant under shear transformations. We sketch the argument below.

For any $\theta \in \mathbb{R}$, consider the solution to SHE

$$\begin{aligned}&\partial_t Z_\theta(t,x) = \frac{1}{2}\Delta Z_\theta(t,x) + \beta\xi(t,x)Z_\theta(t,x), \qquad t>0,\\ &Z_\theta(0,x) = e^{\theta x},\end{aligned} \tag{5.5}$$

then we know that, for fixed N,

$$\int x^2\rho(N,x)dx - \Big(\int x\rho(N,x)dx\Big)^2 \overset{\text{law}}{=} \partial_\theta^2 \log Z_\theta(N,0)\big|_{\theta=0}.$$

For a proof of this fact, we refer to e.g. [9, Eq. (2.15)]. Since ξ is a Gaussian process that is white in time and stationary in space, we have

$$\{\xi(t,x)\}_{t,x} \overset{\text{law}}{=} \{\xi(t,x+\theta t)\}_{t,x}, \tag{5.6}$$

and this implies

$$\{Z_\theta(t,x)\}_{t,x} \overset{\text{law}}{=} \{Z_0(t,x+\theta t)e^{\theta x+\frac{1}{2}\theta^2 t}\}_{t,x}, \tag{5.7}$$

which comes from the fact that $Z_0(t, x+\theta t)e^{\theta x+\frac{1}{2}\theta^2 t}$ solves (5.5) with $\{\xi(t,x)\}$ replaced by $\{\xi(t, x+\theta t)\}$. Therefore, we have

$$\begin{aligned}\mathbf{E}\log Z_\theta(N,0) &= \frac{1}{2}\theta^2 N + \mathbf{E}\log Z_0(N,\theta N)\\ &= \frac{1}{2}\theta^2 N + \mathbf{E}\log Z_0(N,0),\end{aligned}$$

where the last step comes from the fact that $Z_0(N,\cdot)$ is a stationary random field. Therefore, we have

$$\mathbf{E}\partial_\theta^2 \log Z_\theta(N,0)\big|_{\theta=0} = N.$$

The proof is complete. □

6. Further discussion

We list two problems here.

Quenched behavior. Theorem 1.1 concerns the behavior of w_T under the annealed measure $\mathbf{P}\otimes\hat{\mathbb{P}}_T$, and our approach does not give information on the quenched behavior. For the problem on the whole line with no periodic structure, the annealed and quenched behaviors are quite different, due to the localization phenomenon. It would be interesting to study the quenched asymptotics of w_T in our setting.

Relation between two diffusion constants. Recall that Z_T is the partition function formally defined in (1.1). It was shown in [8], under the same assumption as here, that the free energy $\log Z_T$ satisfies a central limit theorem: there exists $\gamma, \Sigma > 0$ such that under $\mathbf{P}$,

$$\frac{\log Z_T + \gamma T}{\sqrt{T}} \Rightarrow N(0,\Sigma), \qquad \text{as } T\to\infty.$$

Different expressions of Σ were derived, see [8, Eq. (5.58)] which involves the solution to an abstract cell problem and [7, Eq. (2.10)] which takes the form of an average of a Brownian bridge functional. A surprising relation between Σ and σ_{eff}^2 was derived by Brunet through the replica method, see [2, Eq. (20)]. It is unclear at all why they should be related, and we believe it is an important problem to unravel the connection here.

Appendix A. Proof of Lemma 3.3

We start with the following.

Lemma A.1. *For each $t > s$ and $y \in \mathbb{R}$, the process $\{Z_{t,s}(x,y)/q_{t-s}(x-y)\}_{x\in\mathbb{R}}$ is stationary.*

Proof. The argument is rather standard, so we only sketch the proof. We consider first the case when $\xi_R(t,x)$ is a 1-periodic Gaussian noise that is white in time and colored in space, with the covariance function $R(\cdot)\in C^\infty(\mathbb{T})$. Recall that $q_t(x)=(2\pi t)^{-1/2}\exp(-\frac{x^2}{2t})$ denotes the standard heat kernel. The propagator of equation (2.1), corresponding to this noise, shall be denoted by $Z^{(R)}_{t,s}(x,y)$ and is given by the formula

$$Z^{(R)}_{t,s}(x,y)=q_{t-s}(x-y)\mathbb{E}\left[\exp\left\{\beta\int_s^t \xi_R(\sigma,B^{y,x}_{s,t}(\sigma))d\sigma-\frac{1}{2}\beta^2R(0)(t-s)\right\}\right],$$

where $(B^{y,x}_{s,t}(\sigma))_{s\le\sigma\le t}$ is the Brownian bridge between (s,y) and (t,x). It is clear from the above formula that $\{Z^{(R)}_{t,s}(x,y)/q_{t-s}(x-y)\}_{x\in\mathbb{R}}$ is stationary, using the fact that ξ_R also satisfies the relation (5.6). The conclusion can be extended to the case of the Gaussian space-time white noise by approximation of $\delta(x-y)$ by a sequence of $R_n(\cdot)\in C^\infty(\mathbb{T})$ as $n\to\infty$. □

The end of the proof of Lemma 3.3. Using Lemma A.1 and the definition of the propagator of the SHE on a torus, see (2.4), we can write

$$\mathbf{E}Z_{t,0}(x,0)^p=\Big(\frac{q_t(x)}{q_t(0)}\Big)^p\mathbf{E}Z_{t,0}(0,0)^p\le\Big(\frac{q_t(x)}{q_t(0)}\Big)^p\mathbf{E}G_{t,0}(0,0)^p.$$

Applying [8, Lemma B.1], we complete the proof.

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