

FACETS AND FACET SUBGRAPHS OF SYMMETRIC EDGE POLYTOPES

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ABSTRACT. Symmetric edge polytopes, a.k.a. PV-type adjacency polytopes, associated with undirected graphs have been defined and studied in several seemingly independent areas including number theory, discrete geometry, and dynamical systems. In particular, the authors are motivated by the study of the algebraic Kuramoto equations of unmixed form whose Newton polytopes are the symmetric edge polytopes.

The interplay between the geometric structure of symmetric edge polytopes and the topological structure of the underlying graphs has been a recurring theme in recent studies. In particular, “facet/face subgraphs” have emerged as one of the central concepts in describing this symmetry. Continuing along this line of inquiry we provide a complete description of the correspondence between facets/faces of a symmetric edge polytope and maximal bipartite subgraphs of the underlying connected graph.

1. INTRODUCTION

For a simple connected graph G with nodes $\mathcal{V}(G) = \{1, \dots, N\}$ and edge set $\mathcal{E}(G)$, its *symmetric edge polytope* [17] (a.k.a. PV-type *adjacency polytope* [4]) is the convex polytope $\text{conv}\{\pm(\mathbf{e}_i - \mathbf{e}_j) \mid \{i, j\} \in \mathcal{E}(G)\}$ where $\mathbf{e}_0 = \mathbf{0}$ and $\mathbf{e}_i$ is the i^{th} standard basis. In the context of Kuramoto models [16], the geometric structure of such polytopes has been instrumental in understanding the root count for algebraic Kuramoto equations [5, 6, 16]. In the broader context, they have been studied by number theorists, combinatorialists, and discrete geometers motivated by several seemingly independent problems [10, 13, 14, 17, 18, 19, 20]. These viewpoints are consolidated in recent work by D’Alì, Delucchi, and Michałek [9] which, among other contributions, sheds new light on the structure of symmetric edge polytopes of bipartite graphs, cycles, wheels, and graphs consisting of two subgraphs sharing a single edge. Using Gröbner basis methods, they provided explicit formulae for the number of facets and the volume of the symmetric edge polytopes associated with several classes of graphs.

One recurring theme in these recent works is the symmetry between the geometric structure of symmetric edge polytopes and topological structure of the underlying graphs. In particular, the concept of “facet/face subgraphs” is defined and studied [3, 9]. The present

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work is a continuation along this line of research. The main contributions of this paper include the following descriptions of the correspondence between faces of a symmetric edge polytope and face subgraphs of the underlying connected simple graph:

- Connected face subgraphs are exactly the maximal bipartite subgraphs in their corresponding induced subgraphs. (Theorem 3 part (1))
- Facet subgraphs are exactly the maximal bipartite subgraphs. (Theorem 3 part (2))
- The map from facets to facet subgraphs is surjective but not injective: each facet subgraph corresponds to an equivalence class of facets and each can be described as an assignment of edge orientations for the cut-set induced by the bipartite facet subgraph. From the view point of cut-set vectors, we provide a complete description of the equivalence class of facets corresponding to a given facet of the symmetric edge polytope, i.e., the fiber over a given facet subgraph. (Theorem 9)
- We establish equivalences between geometric properties of faces and topological properties of the corresponding face subgraphs. (Theorem 12)
- Armed with these results, we compute the number of facets of a symmetric edge polytope derived from a graph formed by joining an even and an odd cycle along a shared edge, generalizing a recent result of D’Alì, Delucchi, and Michałek [9].

This paper is structured as follows. Section 2 states necessary definitions and notation. Section 3 reviews the construction of symmetric edge polytopes and face subgraphs. Section 4.1 provides a brief overview of existing results on the interplay between facets/faces of symmetric edge polytopes and their corresponding subgraphs. Then, in Sections 4.2 to 4.5, we develop the main results. Section 5 illustrates how these results apply to a concrete class of non-bipartite graphs. Appendix A highlights the important implications of the results in the study of Kuramoto equations from dynamical systems and electrical engineering.

2. PRELIMINARIES AND NOTATION

All graphs we consider will be simple. For a graph G , let $\mathcal{V}(G)$ and $\mathcal{E}(G)$ denote its sets of nodes and edges respectively. We say G is *trivial* if $|\mathcal{V}(G)| \leq 1$. A graph H is a *subgraph* of G , and use the notation $H \leq G$, if $\mathcal{V}(H) \subseteq \mathcal{V}(G)$ and $\mathcal{E}(H) \subseteq \mathcal{E}(G)$. The edge connecting i and j is denoted $\{i, j\}$ or $i \leftrightarrow j$. A graph is *bipartite* if it is 2-colorable; equivalently, if it has no odd cycles. By a *maximal* bipartite subgraph H of G , we mean a subgraph of G that is bipartite and inclusion-maximal. Such subgraphs are necessarily connected and *spanning* (i.e., $\mathcal{V}(H) = \mathcal{V}(G)$). For a subset $V \subseteq \mathcal{V}(G)$, the induced subgraph $G[V]$ is the subgraph consisting of all edges $\{i, j\} \in \mathcal{E}(G)$ where both $i, j \in V$. With respect to a spanning tree T of a connected graph G , a *fundamental cycle* of G is the unique cycle formed by an edge outside T and a path in T . For a digraph $\vec{G}$, the arrowhead emphasizes the distinction between $\vec{G}$ and its underlying undirected graph G . A directed edge, called an *arc*, from i to j is denoted (i, j) , and we use the notation $(i, j)^{-1} = (j, i)$. The *converse* of $\vec{G}$, which reverses the orientation of all its edges, is denoted $\vec{G}^{-1}$.

A *point configuration* $X = \{\mathbf{x}_1, \dots, \mathbf{x}_m\} \subset \mathbb{R}^n$ is a finite collection of distinct points. Its (affine) *dimension* is the dimension of the smallest affine subspace containing X . A nonempty *face* of X is a subset of X for which a linear functional $\langle \cdot, \boldsymbol{\alpha} \rangle$ is minimized. In this case, $\boldsymbol{\alpha}$ is an *inner normal* of the face. The emptyset $\emptyset$ is also a face of X . Note that faces are themselves point configurations. The 0-dimensional faces are *vertices* and maximal proper faces are *facets*. The set of all facets of X is denoted by $\mathcal{F}(X)$. We say X

is *(affinely) dependent* if there are $\lambda_1, \dots, \lambda_m \in \mathbb{R}$, not all zero, with $\sum_{i=1}^m \lambda_i = 0$ such that $\sum_{i=1}^m \lambda_i \mathbf{x}_i = \mathbf{0}$. Otherwise, it is *(affinely) independent*. Call X a *simplex* if $|X| = \dim(X) + 1$ and a *circuit* if it is an inclusion-minimal dependent set. Its *corank*¹ is $|X| - \dim(X) - 1$; Its convex hull $\text{conv}(X) = \{\sum_{i=1}^m \lambda_i \mathbf{x}_i \mid \lambda_1, \dots, \lambda_m \geq 0, \sum_{i=1}^m \lambda_i = 1\}$ is a *convex polytope*.

3. SYMMETRIC EDGE POLYTOPES

For a connected graph G with nodes $\mathcal{V}(G) = \{1, \dots, N\}$, its *symmetric edge polytope* [9, 17] is the convex polytope $\text{conv}\{\pm(\mathbf{e}_i - \mathbf{e}_j) \mid \{i, j\} \in \mathcal{E}(G)\} \subset \mathbb{R}^N$, where $\mathbf{e}_i \in \mathbb{R}^N$ is the vector with 1 in the i -th entry and zero elsewhere. Since we are mostly interested in combinatorial aspects of symmetric edge polytopes, it is more convenient to focus on the underlying point configuration. We define

$$\begin{aligned}\check{\nabla}_G &= \{\pm(\mathbf{e}_{i-1} - \mathbf{e}_{j-1}) \mid \{i, j\} \in \mathcal{E}(G)\} \subset \mathbb{R}^n = \mathbb{R}^{N-1} \quad \text{and} \\ \bar{\nabla}_G &= \{\pm(\mathbf{e}_i - \mathbf{e}_j) \mid \{i, j\} \in \mathcal{E}(G)\} \subset \mathbb{R}^N\end{aligned}$$

with the convention that $\mathbf{e}_0 = \mathbf{0}$. These two point configurations have the same intrinsic geometric properties, and they only differ in the ambient space in which they are embedded: $\bar{\nabla}_G$ is a full-dimensional point configuration in $\mathbb{R}^{N-1}$ whereas $\check{\nabla}_G$ is a codimension-1 point configuration in $\mathbb{R}^N$, and $\check{\nabla}_G$ is precisely the projection of $\bar{\nabla}_G$ onto the last $N-1$ coordinates. The check mark notation in $\check{\nabla}_G$ is a reminder that it is a projection to a lower-dimensional subspace. We also extend this notation to their subsets, e.g., we identify any subset $X \subseteq \bar{\nabla}_G$ with its projection $\check{X} \subseteq \check{\nabla}_G$. When describing subsets of $\check{\nabla}_G$ or $\bar{\nabla}_G$, the “codimension” of a subset always refers to the codimension relative to $\check{\nabla}_G$ or $\bar{\nabla}_G$ themselves, regardless of the ambient space. The projection that maps $\bar{\nabla}_G$ to $\check{\nabla}_G$ is a unimodular equivalence between the two configurations, and the two are simply different embeddings of the same polytope into the Euclidean space. The distinction between the two will not be relevant when referencing intrinsic geometric properties, in which case we will simply use ∇_G .

In Ref. [12], Higashitani extended this construction to digraphs: for a digraph $\vec{G}$, we define

$$\begin{aligned}\check{\nabla}_{\vec{G}} &= \{\mathbf{e}_{i-1} - \mathbf{e}_{j-1} \mid (i, j) \in \mathcal{E}(\vec{G})\} \subset \mathbb{R}^{N-1} \quad \text{and} \\ \bar{\nabla}_{\vec{G}} &= \{\mathbf{e}_i - \mathbf{e}_j \mid (i, j) \in \mathcal{E}(\vec{G})\} \subset \mathbb{R}^N.\end{aligned}$$

Here, $\mathbf{e}_i - \mathbf{e}_j \in \bar{\nabla}_{\vec{G}}$ no longer implies $\mathbf{e}_j - \mathbf{e}_i \in \bar{\nabla}_{\vec{G}}$, thus $\text{conv}(\bar{\nabla}_{\vec{G}})$ may not be a symmetric edge polytope. The notation $\check{\nabla}_G, \bar{\nabla}_G$ and $\check{\nabla}_{\vec{G}}, \bar{\nabla}_{\vec{G}}$ extend naturally to subgraphs of G and subdigraphs of a digraph $\vec{G}$, respectively, by restriction.

By construction, $\mathbf{0}$ is always an interior point of $\text{conv}(\check{\nabla}_G)$, which allows the inner normals to be normalized to the following certain form.

Lemma 1. *For a connected nontrivial graph G , a nonempty subset $F \subsetneq \check{\nabla}_G$ is a face if and only if there is a nonzero vector $\check{\alpha} \in \mathbb{R}^n$ such that*

$$\begin{aligned}\langle \mathbf{x}, \check{\alpha} \rangle &= -1 \quad \text{for any } \mathbf{x} \in F, \text{ and} \\ \langle \mathbf{x}, \check{\alpha} \rangle &> -1 \quad \text{for any } \mathbf{x} \in \check{\nabla}_G \setminus F\end{aligned}$$

¹Here, the term “corank” is not to be confused with the different possible notions of corank in matroid theory – what we are calling “corank” is what a matroid theorist would call “nullity”.

The same description extends to inner normals for faces of $\bar{\nabla}_G$ since each inner normal of $\bar{\nabla}_G$ projects down to an inner normal of $\check{\nabla}_G$, and each inner normal of $\check{\nabla}_G$ lifts to an equivalence class of inner normals of $\bar{\nabla}_G$.

4. FACETS, FACES, AND ASSOCIATED SUBGRAPHS

Faces and facets of a symmetric edge polytope have been studied from several different viewpoints [9]. We continue this line of inquiry through a graph-theoretical approach. The central theme of this paper is the interplay between combinatorial properties of faces of ∇_G and graph-theoretic properties of subgraphs of G through the connection of face subgraphs. Throughout this section, we fix G to be a nontrivial, connected, and simple graph.

Definition 2. For a nonempty subset $X \subseteq \bar{\nabla}_G$, we define $\vec{G}_X$ and G_X to be the subgraphs with node and edge sets

$$\begin{aligned}\mathcal{V}(\vec{G}_X) &= \{i \mid \mathbf{e}_i - \mathbf{e}_j \in X \text{ or } \mathbf{e}_j - \mathbf{e}_i \in X \text{ for some } j\} \\ \mathcal{E}(\vec{G}_X) &= \{(i, j) \mid \mathbf{e}_i - \mathbf{e}_j \in X\} \\ \mathcal{V}(G_X) &= \{i \mid \mathbf{e}_i - \mathbf{e}_j \in X \text{ or } \mathbf{e}_j - \mathbf{e}_i \in X \text{ for some } j\} \\ \mathcal{E}(G_X) &= \{\{i, j\} \mid \mathbf{e}_i - \mathbf{e}_j \in X \text{ or } \mathbf{e}_j - \mathbf{e}_i \in X\},\end{aligned}$$

respectively. If F is a face (resp. facet) of ∇_G , then $\vec{G}_F$ is the *face (resp. facet) subdigraph* associated with F , and G_F is the associated *face (resp. facet) subgraph*.

These concepts are defined and studied in recent works [3, 9]. The notational conventions are chosen to mirror the connection between $\vec{G}$ and $\nabla_{\vec{G}}$, and they interact in an expected way: for any $X \subseteq \nabla_G$, we have $\nabla_{\vec{G}_X} = X$, and for any subgraph $\vec{H} \leq \vec{G}$, we have $\vec{G}_{\nabla_{\vec{H}}} = \vec{H}$.

Note that points in X are exactly the columns in the incidence matrix of $\vec{G}_X$, which will be denoted by $Q(\vec{G}_X)$. The *truncated incidence* matrix $\check{Q}(\vec{G}_X)$, obtained by removing the first row of $Q(\vec{G}_X)$, corresponds to points in the projection $\check{X} \subset \check{\nabla}_G$.

Face and facet subgraphs were studied in Refs. [9, 11, 13]. Section 4.1 reviews recent results on the interplay between facets and their corresponding subgraphs. Then, in Sections 4.2 to 4.5, we develop the main results. In particular, we provide a complete description of facet and face subgraphs as well as the equivalence classes of facet subdigraphs. Implications of these results in algebraic Kuramoto equations are highlighted in Appendix A.

4.1. Recent results on facets and facet subgraphs. Facets of ∇_G associated with even cycles are described from the viewpoint of Ehrhart theory by Ohsugi and Shibata [19] and from the viewpoint of Lipschitz polytope by Gordon and Petrov [11]. Explicit descriptions of the facets of ∇_G , when G is a tree or cycle, are also established [6]. Using Gröbner basis methods, D'Alì, Delucchi, and Michałek [9] provided more detailed descriptions for the faces of ∇_G . In particular, they showed that unimodular simplices contained in a facet $F \in \mathcal{F}(\nabla_G)$ correspond to spanning trees of G_F [13, Corollary 3.3], and for a connected bipartite graph G , the total number of facets is bounded by $2^{|\mathcal{V}(G)|-1}$. This bound is exact if G is a tree. Indeed, in this special case, the numbers of faces of any dimension are offered by Delucchi and Hoessly [10]. In Section 4.4, we provide graph-theoretic refinements to these results.

For an even cycle C_{2k} , the number of faces of each dimension of $\nabla_{C_{2k}}$, is also computed in [9, Proposition 30]. Moreover, if G_1 and G_2 are both connected bipartite graphs, and

G is formed by identifying an edge of G_1 with an edge of G_2 , then it has been shown that $|\mathcal{F}(\nabla_G)| = \frac{1}{2}f_1f_2$ where $f_1 = |\mathcal{F}(\nabla_{G_1})|$ and $f_2 = |\mathcal{F}(\nabla_{G_2})|$ [9, Proposition 37]. This result can be applied recursively and extended to graphs formed by joining multiple even cycles consecutively by an edge [9, Corollary 38]. We will extend these to non-bipartite graphs.

4.2. Characterizing face and facet subgraphs. Our goal is to clarify the structure of the map $F \mapsto G_F$ between facets and faces of ∇_G and subgraphs of G . We first show that connected face and facet subgraphs associated with faces and facets of ∇_G are exactly the maximal bipartite subgraphs of induced subgraphs of G (Theorem 3). In particular, the map $F \mapsto G_F$ is a surjective map from $\mathcal{F}(\nabla_G)$ to the set of maximal bipartite subgraphs of G . Corollary 4 generalizes this description to components of face subgraphs.

Theorem 3. *Let H be a nontrivial connected subgraph of G .*

- (1) *H is a face subgraph of G if and only if it is a maximal bipartite subgraph of $G[\mathcal{V}(H)]$.*
- (2) *H is a facet subgraph of G if and only if it is a maximal bipartite subgraph of G .*

Note that part (2) of this theorem can be derived from [14, Theorem 3.1], which provides a general description of facet-defining labels for nodes in G . Here, we provide an alternative derivation as a special case of part (1).

Proof. First, note that facet subgraphs and maximal bipartite subgraphs are necessarily connected, nontrivial, and spanning, therefore part (2) is a special case of part (1). If $F \neq \emptyset$ is a proper face of $\bar{\nabla}_G$, then, by Lemma 1, there exists an $\alpha \in \{\mathbf{1}\}^\perp \subset \mathbb{R}^N$ such that

$$\langle \mathbf{x}, \alpha \rangle = -1 \quad \text{for all } \mathbf{x} \in F \quad \text{and} \quad \langle \mathbf{x}, \alpha \rangle > -1 \quad \text{for all } \mathbf{x} \in \bar{\nabla}_G \setminus F.$$

To show G_F is bipartite, suppose $i_1 \leftrightarrow \dots \leftrightarrow i_\ell \leftrightarrow i_1$ is a cycle in G_F . Then there are $\lambda_1, \dots, \lambda_\ell \in \{\pm 1\}$ such that $\lambda_j(\mathbf{e}_{i_j} - \mathbf{e}_{i_{j+1}}) \in F$ for $j = 1, \dots, \ell$ with $i_{\ell+1} = i_1$. By the above,

$$\langle \lambda_j(\mathbf{e}_{i_j} - \mathbf{e}_{i_{j+1}}), \alpha \rangle = -1 \quad \text{i.e.,} \quad \langle \mathbf{e}_{i_j} - \mathbf{e}_{i_{j+1}}, \alpha \rangle = -\lambda_j \quad \text{for } j = 1, \dots, \ell.$$

Summing both sides over all j produces

$$0 = \langle \mathbf{0}, \alpha \rangle = \left\langle \sum_{j=1}^{\ell} \mathbf{e}_{i_j} - \mathbf{e}_{i_{j+1}}, \alpha \right\rangle = -\sum_{j=1}^{\ell} \lambda_j.$$

Since $\lambda_j \in \{\pm 1\}$, ℓ must be even, i.e., any cycle in G_F must be even, and G_F is bipartite.

To show G_F is a maximal bipartite subgraph of $G[\mathcal{V}(G_F)]$, consider a spanning tree T of G_F , which is necessarily a spanning tree of $G[\mathcal{V}(G_F)]$. If B is a bipartite graph such that $G_F < B \leq G[\mathcal{V}(G_F)]$, then any edge $\{i, i'\} \in \mathcal{E}(B) \setminus \mathcal{E}(G_F)$ is also outside T . The fundamental cycle formed by $\{i, i'\}$ and the unique path $i = i_1 \leftrightarrow \dots \leftrightarrow i_\ell = i'$ in T is contained in B and hence must be an even cycle. That is, ℓ is even. Since the path $i_1 \leftrightarrow \dots \leftrightarrow i_\ell$ is in $T \leq G_F$, there are $\lambda_1, \dots, \lambda_{\ell-1} \in \{\pm 1\}$ such that $\lambda_j(\mathbf{e}_{i_j} - \mathbf{e}_{i_{j+1}}) \in F$ for $j = 1, \dots, \ell - 1$. As in the paragraph above,

$$\langle \mathbf{e}_i - \mathbf{e}_{i'}, \alpha \rangle = \left\langle \sum_{j=1}^{\ell-1} (\mathbf{e}_{i_j} - \mathbf{e}_{i_{j+1}}), \alpha \right\rangle = -\sum_{j=1}^{\ell-1} \lambda_j.$$

Also, by Lemma 1, $\langle \pm(\mathbf{e}_i - \mathbf{e}_{i'}), \alpha \rangle = \mp \sum_{j=1}^{\ell-1} \lambda_j \geq -1$. Recall that $\lambda_j \in \{\pm 1\}$ and ℓ is even. So either $\langle +(\mathbf{e}_i - \mathbf{e}_{i'}), \alpha \rangle$ or $\langle -(\mathbf{e}_i - \mathbf{e}_{i'}), \alpha \rangle$ must be -1 , and hence either $+(\mathbf{e}_i - \mathbf{e}_{i'})$ or

$-(\mathbf{e}_i - \mathbf{e}_{i'})$ is also contained in F . This implies $\{i, i'\} \in \mathcal{E}(G_F)$, contradicting our assumption. Therefore, G_F is not contained in any larger bipartite subgraphs of $G[\mathcal{V}(G_F)]$.

For the converse, suppose a connected subgraph $B \leq G$ is a maximal bipartite subgraph of $G[\mathcal{V}(B)]$ with the partition $\mathcal{V}(B) = V_+ \cup V_-$. Define $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N)$ with

$$\alpha_i = \begin{cases} +1/2 & \text{if } i \in V_+ \\ -1/2 & \text{if } i \in V_- \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\langle \mathbf{e}_i - \mathbf{e}_j, \boldsymbol{\alpha} \rangle = \begin{cases} 0 & \text{if } i, j \in V_+ \text{ or } i, j \in V_- \text{ or } i, j \notin V_+ \cup V_- \\ \pm 1 & \text{if } i \in V_{\pm} \text{ and } j \in V_{\mp} \\ \pm 1/2 & \text{if exactly one of } i, j \text{ is in } V_+ \cup V_- \end{cases}$$

for any $i, j \in \mathcal{V}(G)$. In particular, since V_+ and V_- partition the nodes of B , which is bipartite, $\langle \mathbf{e}_i - \mathbf{e}_j, \boldsymbol{\alpha} \rangle = \pm 1$ for every $\{i, j\} \in \mathcal{E}(B)$.

Let

$$F = \{\mathbf{e}_i - \mathbf{e}_j \mid \langle \mathbf{e}_i - \mathbf{e}_j, \boldsymbol{\alpha} \rangle = -1\}.$$

The linear functional $\langle \cdot, \boldsymbol{\alpha} \rangle$ attains its minimum over ∇_G on F , and thus F is a face of ∇_G . Moreover, G_F is a bipartite subgraph of $G[\mathcal{V}(B)]$ that contains B . But B is assumed to be a maximal bipartite subgraph of $G[\mathcal{V}(B)]$, so $B = G_F$, i.e., B is a face subgraph. $\square$

The connectedness condition in Theorem 3 is important as disconnected face subgraphs may not be maximal bipartite subgraph of their associated induced subgraphs. For example, Figure 1 shows a face subgraph associated with the 4-cycle $G = C_4$, which is not a maximal bipartite subgraph of its associated induced subgraph $G[\{1, 2, 3, 4\}] = G$. Nonetheless, the argument still applies to individual connected components of a face subgraph. From this observation we can derive the following generalization.

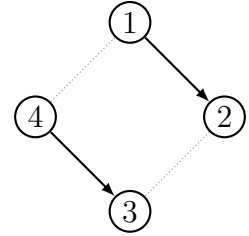


FIGURE 1. A disconnected face subgraph.

Corollary 4. *Let F be a face of ∇_G . Then each connected component H of G_F is a maximal bipartite subgraph of $G[\mathcal{V}(H)]$.* $\square$

The proof of Theorem 3 also shows that for a maximal bipartite subgraph, there is a canonical choice of edge orientations that defines a facet subdigraph and hence a facet.

Corollary 5. *For a maximal bipartite subgraph $B \leq G$ with the partition $\mathcal{V}(B) = V_+ \cup V_-$,*

$$F = \{\mathbf{e}_i - \mathbf{e}_j \mid \{i, j\} \in \mathcal{E}(B), i \in V_- \text{ and } j \in V_+\}$$

is a facet of $\bar{\nabla}_G$, defined by the facet inner normal $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N)^\top$ with

$$\alpha_i = \begin{cases} +1/2 & \text{if } i \in V_+ \\ -1/2 & \text{if } i \in V_-, \end{cases}$$

and the associated facet subdigraph is the digraph with edges set

$$\mathcal{E}(\vec{G}_F) = \{(i, j) \mid \{i, j\} \in \mathcal{E}(B), i \in V_-, j \in V_+\}.$$

The choice of the labels V_+ and V_- is, of course, arbitrary: permuting the two will result in the facet $-F$ defined by $-\alpha$ associated with the facet subdigraph $\vec{B}^{-1}$. This partition defines a unique *cut-set*, which is the set of edges that go across the partition. The construction in the above corollary can be interpreted as a special type of cut-set.

Remark 6. For a maximal bipartite subgraph $B \leq G$, the canonical choice of edge orientations that produces $\vec{B}$ in Corollary 5 are exactly the edge orientation assignments that ensure the cut-set (which includes all edges in B) has a uniform direction across the cut (V_+, V_-) of B , i.e., all arcs of $\vec{B}$ are from V_- to V_+ . Such an assignment will be referred to as a canonical edge orientation for B as well as its spanning subgraphs.

4.3. Cyclic constraints on facet subdigraphs. Theorem 3 shows that the map $F \mapsto G_F$ is a surjective map from $\mathcal{F}(\nabla_G)$ to the set of maximal bipartite subgraphs of G . However, this map is not injective. Indeed, as clarified in Corollary 5, there is at least a pair of canonical choices of facets F and $-F$ associated with a given maximal bipartite subgraph B of G which correspond to cut-sets having either of the two uniform directions. In general, if we let $V_+ \cup V_- = \mathcal{V}(B) = \mathcal{V}(G)$ be the partition in B , then each facet $F \in \mathcal{F}(\nabla_G)$ such that $G_F = B$ can be described as an assignment of edge orientations for the cut-set induced by the cut (V_+, V_-) in B . However, not every such assignment will produce a facet of ∇_G . The constraints on such assignments has been studied in [11, Section 6] from the viewpoint of metric spaces. In the following, we describe constraints on the possible choices that will result in facets in terms of *oriented cycles*: directed cycles in which each node is the head of exactly one arc and the tail of exactly one arc.

Theorem 7. If F is a facet of ∇_G , then for any cycle $\vec{O}$ in G with an assigned orientation,

$$|\mathcal{E}(\vec{G}_F) \cap \mathcal{E}(\vec{O})| = |\mathcal{E}(\vec{G}_F) \cap \mathcal{E}(\vec{O}^{-1})|.$$

Note that G_F being a maximal bipartite subgraph already implies that $|\mathcal{E}(G_F) \cap \mathcal{E}(O)|$ is even for any cycle O in G . This theorem states that $\mathcal{E}(\vec{G}_F)$ consists of two subsets of arcs of equal size having opposite orientations. Later in this section, we will show that, under an additional dimensional condition, the converse is also true.

Proof. Recall that the incidence matrix $Q(\vec{G}_F)$ is totally unimodular [11, Lemma 2.6], and the reduced inner normal $\check{\alpha}$, being the vector satisfying $\check{\alpha}^\top \check{Q}(\vec{G}_F) = -\mathbf{1}^\top$, must be an integer vector. Then $\alpha = (0, \check{\alpha})$ is also an integer vector, and $\pm \langle \mathbf{e}_i - \mathbf{e}_j, \alpha \rangle > -1$ is an integer for any $\mathbf{e}_i - \mathbf{e}_j \notin F$. This implies that $\langle \mathbf{e}_i - \mathbf{e}_j, \alpha \rangle = 0$ for any $\mathbf{e}_i - \mathbf{e}_j \notin \pm F$.

Suppose G contains a cycle O of length m having edges $i_1 \leftrightarrow i_2 \leftrightarrow \dots \leftrightarrow i_{m+1}$ with $i_{m+1} = i_1$, and let $E = \mathcal{E}(G_F) \cap \mathcal{E}(O)$. Then $\mathbf{0} = \sum_{r=1}^m (\mathbf{e}_{i_r} - \mathbf{e}_{i_{r+1}})$ implies

$$\begin{aligned} 0 &= \left\langle \sum_{r=1}^m (\mathbf{e}_{i_r} - \mathbf{e}_{i_{r+1}}), \alpha \right\rangle \\ &= \sum_{\{j, j+1\} \in E} \langle \mathbf{e}_{i_{r_j}} - \mathbf{e}_{i_{r_{j+1}}}, \alpha \rangle + \sum_{\{j, j+1\} \in \mathcal{E}(O) \setminus E} \langle \mathbf{e}_{i_{r_j}} - \mathbf{e}_{i_{r_{j+1}}}, \alpha \rangle \\ &= \sum_{\{j, j+1\} \in E} \langle \mathbf{e}_{i_{r_j}} - \mathbf{e}_{i_{r_{j+1}}}, \alpha \rangle, \end{aligned}$$

since $\langle \mathbf{e}_i - \mathbf{e}_j, \boldsymbol{\alpha} \rangle = 0$ for any $\mathbf{e}_i - \mathbf{e}_j \notin \pm F$. Moreover, $\langle \mathbf{e}_i - \mathbf{e}_j, \boldsymbol{\alpha} \rangle = \pm 1$ for any $\mathbf{e}_i - \mathbf{e}_j \in F$, i.e., each term in the above sum is ± 1 . We conclude that $|E| = |\mathcal{E}(G_F) \cap \mathcal{E}(O)|$ is even.

Let $\vec{O}$ with arcs $i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_{m+1}$ be the corresponding oriented cycle. Then $\langle \mathbf{e}_{i_{r_j}} - \mathbf{e}_{i_{r_j+1}}, \boldsymbol{\alpha} \rangle = -1$ implies $(i_{r_j}, i_{r_j+1}) \in \mathcal{E}(\vec{G}) \cap \mathcal{E}(\vec{O})$ and $\langle \mathbf{e}_{i_{r_j}} - \mathbf{e}_{i_{r_j+1}}, \boldsymbol{\alpha} \rangle = 1$ implies $(i_{r_j+1}, i_{r_j}) \in \mathcal{E}(\vec{G}) \cap \mathcal{E}(\vec{O}^{-1})$. Therefore,

$$|\mathcal{E}(\vec{G}_F) \cap \mathcal{E}(\vec{O})| = |\mathcal{E}(\vec{G}_F) \cap \mathcal{E}(\vec{O}^{-1})|. \quad \square$$

Theorem 7 shows necessary conditions for a subset F of $\bar{\nabla}_G$ to be a facet: the intersection between $\vec{G}_F$ and any cycle of G itself must satisfy a balancing condition, i.e., $|\mathcal{E}(\vec{G}_F) \cap \mathcal{E}(\vec{O})| = |\mathcal{E}(\vec{G}_F) \cap \mathcal{E}(\vec{O}^{-1})|$ for any oriented cycle $\vec{O}$. By itself, however, this condition is not sufficient to define a facet. In the following, we will show that under additional assumptions, this balancing condition is also a sufficient condition.

Theorem 8. *Let F be a codimension 1 subset of ∇_G such that $\mathbf{0} \notin \text{conv}(F)$. Then F is a facet of ∇_G if, for any oriented cycle $\vec{O}$ in $\vec{G}$,*

$$|\mathcal{E}(\vec{G}_F) \cap \mathcal{E}(\vec{O})| = |\mathcal{E}(\vec{G}_F) \cap \mathcal{E}(\vec{O}^{-1})|.$$

Proof. It is sufficient to consider the embedding $\check{\nabla}_G \subset \mathbb{R}^n$ and assume $F \subset \check{\nabla}_G$. Since F is a codimension 1 subset of $\check{\nabla}_G$ that does not contain $\mathbf{0}$ in its convex hull,

$$N - 2 = \dim(F) = \text{rank}(Q(\vec{G}_F)) - 1 = |\mathcal{V}(\vec{G}_F)| - k - 1$$

where $Q(\vec{G}_F)$ is the incidence matrix whose columns are points in F and k is the number of weakly connected components in $\vec{G}_F$ [11, Theorem 2.3]. Therefore $\vec{G}_F$ is necessarily (weakly) connected and spanning, and G_F is connected and spanning.

If T is a spanning tree of G_F , then T is also a spanning tree of G . Let $\vec{T}$ be the corresponding directed subgraph of $\vec{G}_F$. By [13, Corollary 3.2] ([9, Corollary 13]), points in $\Delta = \check{\nabla}_{\vec{T}} \subset \check{F}$ form a simplex, and $\check{Q}(\vec{T})$ is nonsingular. Additionally, if $\check{\boldsymbol{\alpha}}$ is the unique solution to $\check{\boldsymbol{\alpha}}^\top \check{Q}(\vec{T}) = -\mathbf{1}^\top$, then

$$\langle \mathbf{e}_i - \mathbf{e}_j, \check{\boldsymbol{\alpha}} \rangle = -1 \quad \text{for all } (i, j) \in \vec{T}.$$

Since F is assumed to be a codimension 1 subset in $\check{\nabla}_G$, i.e., $\dim(F) = \dim(\Delta)$, F must be contained in the affine span of Δ . Consequently,

$$\langle \mathbf{e}_i - \mathbf{e}_j, \check{\boldsymbol{\alpha}} \rangle = -1 \quad \text{for all } \mathbf{e}_i - \mathbf{e}_j \in F.$$

Recall that $\mathbf{0}$ is not contained in the convex hull of F , i.e., F cannot contain both $\pm(\mathbf{e}_i - \mathbf{e}_j)$ for any $\{i, j\}$, so F and $-F$ are disjoint. For any $\mathbf{e}_i - \mathbf{e}_j \in -F$ and hence outside F , it is clear that $\langle \mathbf{e}_i - \mathbf{e}_j, \check{\boldsymbol{\alpha}} \rangle = +1$.

For any $\mathbf{e}_i - \mathbf{e}_j \in \check{\nabla}_G \setminus (F \cup (-F))$, the corresponding undirected edge $\{i, j\}$ is outside G_F and hence outside T . Consider the fundamental cycle O formed by $\{i, j\}$ and the path $i = i_1 \leftrightarrow \cdots \leftrightarrow i_m \leftrightarrow i_{m+1} = j$ in T . We have

$$\mathbf{e}_i - \mathbf{e}_j = \sum_{j=1}^m (\mathbf{e}_{i_j} - \mathbf{e}_{i_{j+1}}),$$

and there are $\lambda_1, \dots, \lambda_m \in \{\pm 1\}$ such that $\lambda_j(\mathbf{e}_{i_j} - \mathbf{e}_{i_{j+1}}) \in \Delta \subseteq F$. By the assumption that $|\mathcal{E}(\vec{G}_F) \cap \mathcal{E}(\vec{O})| = |\mathcal{E}(\vec{G}_F) \cap \mathcal{E}(\vec{O}^{-1})|$, m must be even, and $\lambda_1 + \dots + \lambda_m = 0$. Therefore,

$$\langle \mathbf{e}_i - \mathbf{e}_j, \boldsymbol{\alpha} \rangle = \sum_{j=1}^m \langle \mathbf{e}_{i_j} - \mathbf{e}_{i_{j+1}}, \boldsymbol{\alpha} \rangle = \sum_{j=1}^m \lambda_j \langle \lambda_j(\mathbf{e}_{i_j} - \mathbf{e}_{i_{j+1}}), \boldsymbol{\alpha} \rangle = \sum_{j=1}^m \lambda_j(-1) = 0.$$

That is, the linear functional $\langle \cdot, \boldsymbol{\alpha} \rangle$ takes the value of -1 on F , and it is nonnegative on $\nabla_G \setminus F$. Therefore F is a facet. $\square$

4.4. Parameterizing facets with cut-set vectors. Facets of ∇_G correspond to maximal bipartite subgraphs of G through the map $F \mapsto G_F$. In general, multiple facets will be mapped to the same facet subgraph. Theorems 7 and 8 gave necessary and sufficient conditions to identify facets in the fiber over a given facet subgraph in terms of oriented cycles. In this section, we refine these constraints into “independent” equations and thereby provide a complete description of the equivalence class of facets corresponding to the same facet subgraph. It will form the foundation for counting and generating facets of ∇_G .

The description makes use of the fundamental cycle vectors and cut-set vectors. For a facet subgraph G_F and a spanning tree T of G_F , let $\vec{T}$ be the corresponding subdigraph of $\vec{G}_F$. Since a facet subgraph is necessarily connected and spanning, T is also a spanning tree of G . Any edge $e \in \mathcal{E}(G) \setminus \mathcal{E}(T)$ induces a fundamental cycle O with respect to T . With an arbitrary choice of the orientation, the oriented cycle $\vec{O}$ can be expressed as an incidence vector $\mathbf{c}_{\vec{T}}(e) = (c_1, \dots, c_n)^\top$ with respect to the ordered list of arcs in $\vec{T}$ so that $\vec{e}^{-1}$ corresponds to the point $Q(\vec{T})\mathbf{c}_{\vec{T}}(e) \in \check{\nabla}_G$. In other words, $\mathbf{c}_{\vec{T}}(e) \in \{+1, 0, -1\}^{|\mathcal{E}(\vec{T})|}$, with each entry indicating whether the orientation of the corresponding edge of $\vec{T}$ agrees or disagrees (or is not involved) with the orientation of $\vec{O}$.

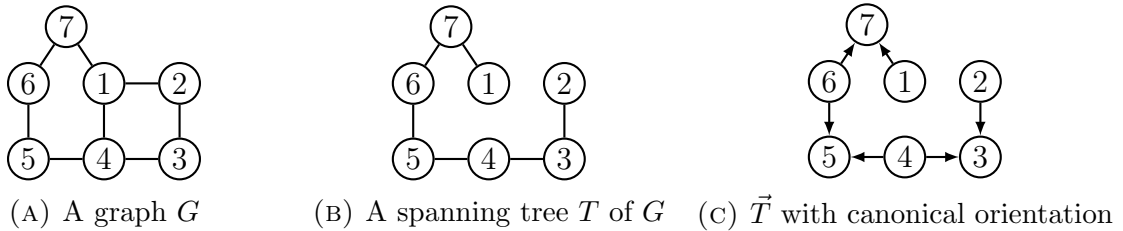


FIGURE 2. A running example

Consider, for example, in the graph G shown in Figure 2a, which will serve as a running example. The subgraph T in Figure 2b is a spanning tree. In Figure 2c, we choose a canonical orientation and produce $\vec{T}$. If we arrange its arcs into the ordered list

$$(1) \quad (2 \rightarrow 3, 4 \rightarrow 3, 4 \rightarrow 5, 6 \rightarrow 5, 6 \rightarrow 7, 1 \rightarrow 7),$$

then the oriented fundamental cycle $1 \rightarrow 2 \rightarrow \dots \rightarrow 7 \rightarrow 1$, induced by $e = \{1, 2\}$, can be expressed as the incidence vector $(+1, -1, +1, -1, +1, -1)^\top$.

Similarly, fixing an ordering of the arcs, a cut-set defined by a cut can be expressed as the vector with entries in $\{-1, 0, +1\}$ indicating the direction in which each arc goes across the partition (0 for not crossing the partition). In the example shown in Figure 2c, with respect to the cut $\mathcal{V}(G) = \{3, 5, 7\} \cup \{1, 2, 4, 6\}$, the cut set, which include arcs listed in (1), can be encoded as $(-1, -1, -1, -1, -1, -1)$, as they all go from $\{1, 2, 4, 6\}$ to $\{3, 5, 7\}$.

Theorem 9. For a maximal bipartite subgraph $B \leq G$ with partition $V_+ \cup V_- = \mathcal{V}(B)$, let T be a spanning tree of B and $\vec{T}$ be the corresponding digraph with canonical orientations (see Remark 6). There is a bijection between the facets $\{F \in \mathcal{F}(\vec{\nabla}_G) \mid G_F = B\}$ and the set of cut-set vectors $\mathbf{d} \in \{\pm 1\}^n$ of T with respect to the cut (V_+, V_-) satisfying the system

$$(2) \quad \begin{cases} \mathbf{c}_{\vec{T}}(e)^\top \mathbf{d} = \pm 1 & \text{for } e \in \mathcal{E}(B) \setminus \mathcal{E}(T) \\ \mathbf{c}_{\vec{T}}(e)^\top \mathbf{d} = 0 & \text{for } e \in \mathcal{E}(G) \setminus \mathcal{E}(B) \end{cases}$$

Note that the $\pm$ sign is the consequence of the inherent ambiguity in the orientation assignment for a given fundamental cycle.

Proof. Since the maximal bipartite subgraph B must span G , T also spans G , and (V_+, V_-) is a partition of $\mathcal{V}(G)$. Recall that $\vec{T}$ (see Remark 6) has the arc set

$$\mathcal{E}(\vec{T}) = \{(i, j) \mid \{i, j\} \in \mathcal{E}(T), i \in V_-, j \in V_+\}.$$

Let $\mathbf{d} \in \{\pm 1\}^n$ be a cut-set vector satisfying the system of equations (2), and let $D = \text{diag}(\mathbf{d})$. Then there exists a vector $\boldsymbol{\alpha} \in \mathbb{R}^N$ such that $\boldsymbol{\alpha}^\top Q(\vec{T}) = -\mathbf{d}$, i.e.

$$\boldsymbol{\alpha}^\top Q(\vec{T})D = -\mathbf{1}.$$

Define Δ' to be the set of points that are the columns of $Q(\vec{T}) \text{diag}(\mathbf{d})$. Then $G_{\Delta'} = T$ and thus Δ' is a codimension-1 simplex in $\vec{\nabla}_G$. Moreover, the above equations show that

$$\langle \mathbf{x}, \boldsymbol{\alpha} \rangle = -1 \quad \text{for any } \mathbf{x} \in \Delta'.$$

If $T \neq B$, then each edge $e \in \mathcal{E}(B) \setminus \mathcal{E}(T)$ determines a fundamental cycle with respect to T represented by the vector $\mathbf{c} = \mathbf{c}_T(e)$ such that e corresponds to a point $\mathbf{x} = -Q(\vec{T})\mathbf{c}$. By assumption (2), $\mathbf{c}^\top \mathbf{d} = \pm 1$. Therefore,

$$\boldsymbol{\alpha}^\top \mathbf{x} = \boldsymbol{\alpha}^\top (-Q(\vec{T})\mathbf{c}) = -\boldsymbol{\alpha}^\top Q(\vec{T})DD\mathbf{c} = \mathbf{1}^\top D\mathbf{c} = \mathbf{d}^\top \mathbf{c} = \pm 1.$$

Similarly, for any $e \in \mathcal{E}(G) \setminus \mathcal{E}(B)$, the fundamental cycle with respect to T is represented by a vector $\mathbf{c} = \mathbf{c}_T(e)$ such that e corresponds to a point $\mathbf{x} = -Q(\vec{T})\mathbf{c}$. By assumption (2), $\mathbf{c}^\top \mathbf{d} = 0$. Following the same calculations above,

$$\boldsymbol{\alpha}^\top \mathbf{x} = \boldsymbol{\alpha}^\top (-Q(\vec{T})\mathbf{c}) = -\boldsymbol{\alpha}^\top Q(\vec{T})DD\mathbf{c} = \mathbf{1}^\top D\mathbf{c} = \mathbf{d}^\top \mathbf{c} = 0.$$

We have shown $\langle \mathbf{x}, \boldsymbol{\alpha} \rangle$ is -1 for all $\mathbf{x}$ in the codimension-1 simplex Δ' of $\vec{\nabla}_G$ and ± 1 or 0 for any non-interior point in $\vec{\nabla}_G$. Therefore, $\boldsymbol{\alpha}$ defines a unique facet $\text{aff}(\Delta') \cap \vec{\nabla}_G \in \mathcal{F}(\vec{\nabla}_G)$. That is, each solution $\mathbf{d}$ to the system (2), determines a unique facet of $\vec{\nabla}_G$.

Conversely, any facet $F \in \mathcal{F}(\vec{\nabla}_G)$ such that $G_F = B \geq T$ must contain the subset $\{d_1 \mathbf{x}_1, \dots, d_n \mathbf{x}_n\}$ for some $\mathbf{d} = (d_1, \dots, d_n) \in \{\pm 1\}^n$, where $\{\mathbf{x}_1, \dots, \mathbf{x}_n\} = \vec{\nabla}_{\vec{T}}$. The vector $\mathbf{d}$ is uniquely determined by the choice of $\vec{T}$. By Theorem 7, $\mathbf{d}$ must satisfy the equations in (2). That is, each facet of $\vec{\nabla}_G$ corresponds to a unique solution $\mathbf{d}$ to the system (2). $\square$

D'Alì, Delucchi, and Michałek showed that for a connected bipartite graph G , $|\mathcal{F}(\vec{\nabla}_G)|$ is bounded by 2^{N-1} [9, Corollary 33]. It is then noted that this upper bound no longer holds when the graph is not bipartite. From the above proof, we can derive a refinement of this result: this upper bound always holds for the number of facets in an equivalence class of facets associated with a given facet subgraph.

Corollary 10 (A refinement of Corollary 33 of Ref. [9]). *For a facet $F \in \mathcal{F}(\nabla_G)$,*

$$|\{F' \in \mathcal{F}(\nabla_G) \mid G_{F'} = G_F\}| \leq 2^{N-1}.$$

With this, we can derive an upper bound for the total number of facets, which is a generalization of [9, Corollary 33] to all connected graphs.

Corollary 11. *If β is the number of maximal bipartite subgraphs of G , then*

$$|\mathcal{F}(\nabla_G)| \leq \beta \cdot 2^{N-1}.$$

When G is bipartite, the only facet subgraph (the unique maximal bipartite subgraph) is G itself, and the result reduces to the previously established upper bound [9, Corollary 33].

4.5. Properties of faces and their face subgraphs. We now establish connections between the geometric properties of faces of ∇_G and the graph-theoretical properties of their corresponding face subgraphs. Recall that the *cyclomatic number* of a graph G is the minimum number of edges that can be deleted from G such that the resulting graph is acyclic.

Theorem 12. *For a proper face F of ∇_G ,*

- (i) *F is independent if and only if G_F is a forest;*
- (ii) *F is a circuit if and only if G_F is a chordless cycle;*
- (iii) *$\dim(F) = |\mathcal{V}(G_F)| - k - 1$ where k is the number of connected components in G_F ;*
- (iv) *$\text{corank}(F)$ is the cyclomatic number of G_F .*

If G_F is spanning, its Betti numbers are the codimension and corank of F . Some properties have been studied in different context. E.g., part (iii) was established in [11, Lemma 1].

Proof.

(i) Let F' be a facet of ∇_G containing F . If G_F contains a cycle with edges $i_1 \leftrightarrow \dots \leftrightarrow i_m \leftrightarrow i_1$, then this cycle is also contained in $G_{F'}$. By Theorem 7, it must be an even cycle, and there exist $\lambda_1, \dots, \lambda_m \in \{\pm 1\}$ with $\sum_{j=1}^m \lambda_j = 0$ such that $\lambda_j(\mathbf{e}_{i_j} - \mathbf{e}_{i_{j+1}}) \in F$ for all j . This gives us the affine dependence relation

$$\sum_{j=1}^m \lambda_j(\lambda_j(\mathbf{e}_{i_j} - \mathbf{e}_{i_{j+1}})) = \sum_{j=1}^m (\mathbf{e}_{i_j} - \mathbf{e}_{i_{j+1}}) = \mathbf{0}$$

with the coefficients $\lambda_1, \dots, \lambda_m$. Therefore F itself cannot be independent.

Conversely, if G_F is a forest, then the incidence matrix $Q(\vec{G}_F)$, whose columns are points in F , has full column rank [1, Lemma 2.5] [21]. Therefore F is independent.

(ii) If F is a circuit, then F is dependent by definition. By part (i), G_F contains a cycle and the corresponding subset of points in F is dependent. However, the circuit F , being a minimal affinely dependent set, must be exactly this set. Therefore G_F is exactly this cycle.

Conversely, if F is not a circuit, then either F is independent or F contains a proper dependent subset F' . Again by part (i), G_F is either a forest or it contains a strictly smaller cycle, and thus G_F is not a chordless cycle.

(iii) Let k be the number of connected components in G_F . Then

$$\dim(F) = \text{rank}(Q(\vec{G}_F)) - 1 = |\mathcal{V}(G_F)| - k - 1.$$

(iv) Let k be as defined above and μ be the cyclomatic number of G_F . By part (iii),

$$\mu = |\mathcal{E}(G_F)| - |\mathcal{V}(G_F)| + k = |F| - (\dim(F) + 1) = \text{corank}(F). \quad \square$$

Remark 13. Theorem 12 highlights the connection through which independent faces correspond to forests, dependent faces correspond to cyclic graphs, and circuit faces correspond to chordless cycles. The precise description emerges from the viewpoint of matroid theory [10].

Combining Theorems 3 and 12 part (i), we get a simple alternative proof to the fact that ∇_G is simplicial (i.e., all of its facets are simplices) if and only if all maximal bipartite subgraphs of G are trees, which was first established by Higashitani in [12, Corollary 2.3] and has important consequences in the study of facet systems of Kuramoto equations (see Appendix A) and the structure of certain metric spaces [11].

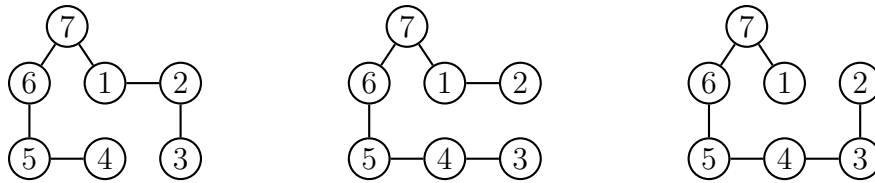
Corollary 14 ([12, Corollary 2.3]). ∇_G is simplicial if and only if G has no even cycles. $\square$

5. CASE STUDY: JOINING TWO CYCLES ALONG A SHARED EDGE

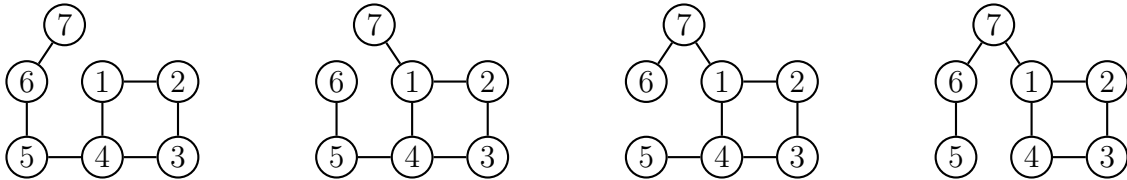
D’Alì, Delucchi, and Michałek showed that for a graph formed by joining two bipartite graphs along an edge, the number of facets of the associated symmetric edge polytope is $\frac{1}{2}f_1f_2$ where f_1 and f_2 are the number facets in the symmetric edge polytopes associated with the two bipartite subgraphs respectively [9, Proposition 37]. In the following we explore the more general situation in which one of the subgraph is *not* bipartite.

The running example G , from Figure 2a, is a non-bipartite graph formed by joining a 4-cycle and a 5-cycle along a single shared edge. As we will calculate, the facet count described above ([9, Proposition 37]) no longer applies, yet Theorem 9 provides a concrete recipe for calculating the number of facets and describing their combinatorial structures.

Figure 3 shows the seven maximal bipartite subgraphs of G . As established in Theorem 3, they correspond to the five equivalence classes of facets in $\mathcal{F}(\nabla_G)$. Among these subgraphs, three of them are trees (Figure 3a), and, according to Theorem 12 part (i), they correspond to corank-0 (simplicial) facets. The other four each contain a unique 4-cycle (Figure 3b), and, according to Theorem 12 part (iv), they correspond to corank-1 facets.



(A) Corank-0 facet subgraphs of G



(B) Corank-1 facet subgraphs of G

FIGURE 3. Facet subgraphs (maximal bipartite subgraphs) of G

As shown in Figure 2, we pick a corank-0 facet subgraph, a spanning tree of this subgraph, and an assignment of edge orientations. Up to a recording of the edges, the two fundamental cycles can be expressed as vectors $[+1, -1, +1, -1, +1, -1]$ and $[0, 0, +1, -1, +1, -1]$. Therefore the defining equation (2) in Theorem 9 for the parametrization of the facets in this equivalence class is

$$\begin{bmatrix} +1 & -1 & +1 & -1 & +1 & -1 \\ 0 & 0 & +1 & -1 & +1 & -1 \end{bmatrix} \mathbf{d} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

in the unknowns $\mathbf{d} = (d_1, \dots, d_6)^\top \in \{\pm 1\}^6$. This equation is equivalent to

$$\begin{bmatrix} +1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & +1 & -1 \end{bmatrix} \mathbf{d} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From this we can see that (d_1, d_2) and (d_3, d_4, d_5) can be described independently, and there are two possible choices for (d_1, d_2) , namely $(+1, +1)$ and $(-1, -1)$. Similarly, there are six possible choices for (d_3, d_4, d_5, d_6) :

$$\begin{array}{lll} (+1, +1, +1, +1) & (+1, +1, -1, -1) & (+1, -1, -1, +1) \\ (-1, -1, +1, +1) & (-1, -1, -1, -1) & (-1, +1, +1, -1). \end{array}$$

Altogether, there are 12 possible choice for the vector $\mathbf{d} \in \{\pm 1\}^6$ for the equation (2). These produce 12 distinct facets in the equivalence class of facets corresponding to a corank-0 maximal bipartite subgraph of G shown in Figure 3a.

Similarly, we pick a facet subgraph G_F of corank 1 in Figure 3b, a spanning tree T of this subgraph, and a canonical assignment of edge orientations $\vec{T}$ shown in Figure 4, where the dotted edge is in $\vec{G}_F$ but not in $\vec{T}$. With respect to this choice, and up to a reordering of the edges in $\vec{T}$, the two fundamental cycles can be expressed as vectors $[-1, -1, +1, 0, 0, 0]$ and $[0, 0, +1, -1, +1, -1]$. Therefore the fundamental cycle equations are

$$\begin{bmatrix} -1 & -1 & +1 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & +1 & -1 \end{bmatrix} \mathbf{d} = \begin{bmatrix} \pm 1 \\ 0 \end{bmatrix}.$$

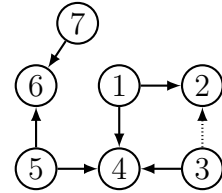


FIGURE 4. A corank-1 facet subgraph and its spanning tree.

Through direct calculations we can see there are 18 solutions for $\mathbf{d} \in \{\pm 1\}^6$ corresponding to the 18 facets in the equivalence class.

The same argument can be applied to each of the four corank-1 facet subgraphs in Figure 3b. Therefore there are 72 corank-1 facets in $\mathcal{F}(\nabla_G)$. All together there are $36 + 72 = 108$ facets in $\mathcal{F}(\nabla_G)$. Among them, 36 facets are simplicial and the remaining 72 facets are (affinely) dependent and of corank 1.

The calculation shown in this concrete example can be easily generalized to graphs formed by joining an even cycle and an odd cycle along a shared edge. The counting argument involved makes use of the following elementary formulas from combinatorics.

Lemma 15. *For a positive integer n , there are exactly $\binom{2n}{n}$ distinct choices of vectors $\mathbf{d} \in \{\pm 1\}^{2n}$ that satisfy the equation*

$$\left[\begin{array}{cccccc} \overbrace{+1 \cdots +1}^n & \overbrace{-1 \cdots -1}^n \end{array} \right] \mathbf{d} = 0.$$

Lemma 16. For a positive integer n , there are exactly $\binom{2n}{n-1}$ distinct choices of vectors $\mathbf{d} \in \{\pm 1\}^{2n}$ that satisfy the equation

$$\left[\begin{array}{cccccc} \overbrace{+1 \ \cdots \ +1}^n & \overbrace{-1 \ \cdots \ -1}^n \\ +1 & \cdots & +1 & -1 & \cdots & -1 \end{array} \right] \mathbf{d} = 2.$$

Proof. The solutions $\mathbf{d} = (d_1, \dots, d_{2n})^\top \in \{\pm 1\}^{2n}$ correspond to the different ways of choosing only i entries among $(d_1, \dots, d_n)$ and $i + 1$ entries among $(d_{n+1}, \dots, d_{2n})$ to be -1 for $i = 0, \dots, n - 1$. By applying the Vandermonde identity, the total number of possibilities is

$$\sum_{i=0}^{n-1} \binom{n}{i} \binom{n}{i+1} = \binom{2n}{n-1}. \quad \square$$

Proposition 17. Let G be the graph formed by joining two cycles of size $2m_1$ and $2m_2 + 1$ respectively along a shared edge. The total number of facets of ∇_G is

$$(2m_1 - 1) \binom{2m_1 - 2}{m_1 - 1} \binom{2m_2}{m_2} + (2m_2) \binom{2m_1 - 1}{m_1} \binom{2m_2}{m_2},$$

and the two summands are the number of corank-0 (simplicial) facets and corank-1 facets of ∇_G , respectively.

Proof. First note that G contains $N = 2m_1 + 2m_2 - 1$ nodes and $2m_1 + 2m_2$ edges. Since G has a unique even cycle, by Theorem 3 and Theorem 12, the coranks of facets of ∇_G are either 0 or 1. We shall count them separately.

(Corank-0) We first count the corank-0 facets. By Theorem 12 part (iv), the facet subgraph of G associated with a corank-0 facet must be a spanning tree of G that is also a maximal bipartite subgraph of G . There are exactly $2m_1 - 1$ such spanning trees of G , obtained by deleting the edge shared by the $2m_1$ -cycle and the $(2m_2 + 1)$ -cycle and by deleting exactly one additional edge of the $2m_1$ -cycle (exemplified in Figure 3a). Note that these corank-0 facet subgraphs are all paths of the same length, hence isomorphic to one another. Up to a re-indexing of the $n = 2m_1 + 2m_2 - 2$ edges, the counting arguments for each of these spanning trees are identical. It is therefore sufficient to calculate the number of facets associated with one such spanning tree (path) T of G .

There are exactly two edges in $\mathcal{E}(G) \setminus \mathcal{E}(T)$: the edge shared by the two cycles and another edge of the even cycle. Let $\vec{T}$ be the corresponding digraph resulting from the canonical choice of edge orientations as described in Remark 6. The fundamental cycle C induced by the first edge includes all edges of the odd cycle, and the other fundamental cycle C' includes all edges of $\vec{T}$. Therefore, equation (2) is of the form

$$(3) \quad \left[\begin{array}{ccccccccc} \overbrace{0 \ 0 \ \cdots \ 0 \ 0}^{2m_1 - 2} & \overbrace{+1 \ -1 \ \cdots \ +1 \ -1}^{2m_2} \\ +1 \ -1 \ \cdots \ +1 \ -1 & +1 \ -1 \ \cdots \ +1 \ -1 \end{array} \right] \mathbf{d} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and the number of facets whose facet subgraph is T is exactly the total number of solutions $\mathbf{d} = (d_1, \dots, d_n)^\top \in \{\pm 1\}^n$ to the above equation. Note that $(d_1, \dots, d_{2m_1-2}) \in \{\pm 1\}^{2m_1-2}$ and $(d_{2m_1-1}, \dots, d_n) \in \{\pm 1\}^{2m_2}$ can be solved independently.

By applying Lemma 15 to these two groups of coordinates, we can see the number of solutions $\mathbf{d} \in \{\pm 1\}^n$ to (3), being the product of solutions for $(d_1, \dots, d_{2m_1-2}) \in \{\pm 1\}^{2m_1-2}$ and $(d_{2m_1-1}, \dots, d_n) \in \{\pm 1\}^{2m_2}$ is $\binom{2m_1-2}{m_1-1} \binom{2m_2}{m_2}$. By Theorem 9,

$$|\{F \in \mathcal{F}(\nabla_G) \mid G_F = T\}| = \binom{2m_1-2}{m_1-1} \binom{2m_2}{m_2}.$$

Recall that there are $2m_1 - 1$ distinct spanning tree that are maximal bipartite subgraphs of G , each having the same number of associated facets, therefore, the total number of corank-0 facets ∇_G has is

$$(4) \quad (2m_1 - 1) \binom{2m_1-2}{m_1-1} \binom{2m_2}{m_2}.$$

(Corank-1) Now we count the corank-1 facets. Since G contains a unique even cycle, by Theorem 3 and Theorem 12 part (iv), the facet subgraph G_F of a facet $F \in \mathcal{F}(\nabla_G)$ of corank 1, being a maximal bipartite subgraph of G of cyclomatic number 1, must contain this even cycle as well as all except one edge of the odd cycle. Therefore, there are exactly $2m_2$ distinct subgraphs of G corresponding to facets of corank 1, as exemplified in Figure 3b. As in the previous case, since all of these corank-1 facet subgraphs have the same fundamental cycles, up to a re-indexing of edges, the counting arguments for each of these subgraph are identical, and it is therefore sufficient to calculate the number of facets associated with a single corank-1 subgraph $B < G$.

Fix any spanning tree $T < B$, and let $\vec{T}$ be the corresponding digraph resulting from the canonical choice of edge orientations. There is only one edge in $\mathcal{E}(B) \setminus \mathcal{E}(T)$, and the corresponding fundamental cycle involves all edges of the even cycle (as shown in Figure 4). Up to a re-indexing of the edges, the corresponding fundamental cycle vector can be expressed as $[+1, -1, +1, \dots, -1, +1, 0, \dots, 0]$ with the last $2m_2 - 1$ coordinates being zero.

Similarly, there is a unique edge in $\mathcal{E}(G) \setminus \mathcal{E}(B)$, and the corresponding fundamental cycle involves all edges of the $(2m_2 + 1)$ -cycle. Its fundamental cycle vector can be expressed as $[0, \dots, 0, +1, -1, +1, \dots, -1, +1]$ with the first $2m_1 - 2$ coordinates being zero. Therefore, equation (2) is of the form

$$(5) \quad \begin{bmatrix} \overbrace{+1 \ \cdots \ -1}^{2m_1-2} & \overbrace{+1 \ 0 \ \cdots \ 0 \ 0}^{2m_2} \\ 0 \ \cdots \ 0 & +1 \ -1 \ \cdots \ +1 \ -1 \end{bmatrix} \mathbf{d} = \begin{bmatrix} \pm 1 \\ 0 \end{bmatrix},$$

and its solutions $\mathbf{d} = (d_1, \dots, d_n)^\top \in \{\pm 1\}^n$ are in one-to-one correspondence with the facets whose facet subgraph is B .

We can solve for $(d_{2m_1-1}, d_{2m_1}, \dots, d_n) \in \{\pm 1\}^{2m_2}$, subjects to the constraint $d_{2m_1-1} - d_{2m_1}, \dots, +d_n = 0$, independently. Following the counting argument from the previous case, we can verify that there are exactly $\binom{2m_2}{m_2}$ choices for $(d_{2m_1-1}, d_{2m_1}, \dots, d_n) \in \{\pm 1\}^{2m_2}$.

The choices of $(d_1, \dots, d_{2m_1-2})$, however, depend on the value of d_{2m_1-1} , since they are related by the equation $d_1 - d_2 + \dots - d_{2m_1-2} + d_{2m_1-1} = p$, where $p \in \{\pm 1\}$. We consider the two cases depending on the sign of d_{2m_1-1}/p .

If $p = d_{2m_1-1}$, the equation is equivalent to $d_1 - d_2 + \dots - d_{2m_1-2} = 0$, and there are exactly $\binom{2m_1-2}{m_1-1}$ distinct choices for $(d_1, \dots, d_{2m_1-2}) \in \{\pm 1\}^{2m_1-2}$. If $p = -d_{2m_1-1}$, the equation is

equivalent to $d_1 - d_2 + \dots - d_{2m_1-2} = 2(-d_{2m_1-1})$, and, by Lemma 16, there are exactly $\binom{2m_1-2}{m_1-2}$ distinct choices for $(d_1, \dots, d_{2m_1-2}) \in \{\pm 1\}^{2m_1-2}$. In total, the number of possibilities for $(d_1, \dots, d_{2m_1-2}) \in \{\pm 1\}^{2m_1-2}$ is

$$\binom{2m_1-2}{m_1-1} + \binom{2m_1-2}{m_1-2} = \binom{2m_1-1}{m_1-1} = \binom{2m_1-1}{m_1}.$$

Therefore, the total number of distinct choices of $\mathbf{d} \in \{\pm 1\}^n$ that satisfy (5) is

$$\binom{2m_1-1}{m_1} \binom{2m_2}{m_2}.$$

This number is also the number of facets of ∇_G whose facet subgraph is B , i.e.,

$$|\{F \in \mathcal{F}(\nabla_G) \mid G_F = B\}| = \binom{2m_1-1}{m_1} \binom{2m_2}{m_2}.$$

Recall that there are $2m_2$ corank-1 facet subgraphs. Therefore, the total number of corank-1 facets ∇_G has is

$$2m_2 \binom{2m_1-1}{m_1} \binom{2m_2}{m_2},$$

which completes the proof. $\square$

Note that this proof is constructive in the sense that the facets, encoded as cut-set vectors, can be enumerated as solutions to (4) and (5).

We conclude with an alternative formulation for the facet count provided by the proposition above, similar to the result established in [9, Proposition 37]. Ohsugi and Shibata showed that for an even cycle C_{2k} , the number of facets of $\nabla_{C_{2k}}$ is $\binom{2k}{k}$ [19]. Using this formula, we can relate the facet count presented above and the facet counts for symmetric edge polytopes associated with even cycles.

Corollary 18. *Let G be the graph formed by joining two cycles of size $2m_1$ and $2m_2 + 1$ respectively along a shared edge. Then*

$$|\mathcal{F}(\nabla_G)| = \frac{m_1 + 2m_2}{2} f_{C_{2m_1}} f_{C_{2m_2}},$$

where $f_{C_{2m_1}}$ and $f_{C_{2m_2}}$ are the number of facets $\nabla_{C_{2m_1}}$ and $\nabla_{C_{2m_2}}$ have respectively.

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APPENDIX A. APPLICATIONS TO ALGEBRAIC KURAMOTO EQUATIONS

Facets of ∇_G play important roles in the study of algebraic Kuramoto equations [3, 4, 7], which has attracted interests from electrical engineering, biology, and chemistry. The original Kuramoto equations model the synchronization behaviors of a network of coupled oscillators [16], which can be represented by a weighted graph G with the nodes $\mathcal{V}(G) = \{0, \dots, n\}$ representing the oscillators, the edges $\mathcal{E}(G)$ representing the connections among the oscillators, and the weights $K = \{k_{ij}\}$ representing the *coupling strengths* along the edges. Each oscillator i has its own natural frequency ω_i . The dynamics of the network can be described by the differential equations

$$(6) \quad \frac{d\theta_i}{dt} = \omega_i - \sum_{j \in \mathcal{N}_G(i)} k_{ij} \sin(\theta_i - \theta_j), \quad \text{for } i = 0, \dots, n,$$

where each $\theta_i \in [0, 2\pi)$ is the phase angle that describes the status of the i -th oscillator, and $\mathcal{N}_G(i)$ is the set of its neighbors. *Frequency synchronization* occurs when the competing forces reach equilibrium and all oscillators are tuned to the same frequency, i.e., $\frac{d\theta_i}{dt} = c$ for

a common constant c for all i . They are precisely the solutions to the system of equations

$$(7) \quad \omega_i - \sum_{j \in \mathcal{N}_G(i)} k_{ij} \sin(\theta_i - \theta_j) = c \quad \text{for } i = 1, \dots, n$$

in the variables $\theta_1, \dots, \theta_n$. Here $\theta_0 = 0$ is fixed as the reference phase angle. With the substitution $x_i := e^{i\theta_i}$ ($x_0 = 1$), (7) can be transformed into the algebraic system

$$(8) \quad F_{G,i}(x_1, \dots, x_n) = \omega_i - c - \sum_{j \in \mathcal{N}_G(i)} \frac{k_{ij}}{2i} \left(\frac{x_i}{x_j} - \frac{x_j}{x_i} \right) = 0 \quad \text{for } i = 1, \dots, n.$$

The system $F_G = (F_{G,1}, \dots, F_{G,n})$ consists of n Laurent polynomial equations in the n nonzero complex variables $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$.

Considering F_G as a column vector, for any nonsingular $n \times n$ matrix R , the systems $F_G^R = R \cdot F_G$ and F_G have the same zero set. For generic choices of the matrix R , there is no complete cancellation of the terms, and thus F_G^R is of the form

$$(9) \quad F_{G,k}^R = c_k^R - \sum_{\{i,j\} \in \mathcal{E}(G)} \left(a_{ijk}^R \frac{x_i}{x_j} + a_{jik}^R \frac{x_j}{x_i} \right) \quad \text{for } k = 1, \dots, n,$$

where c_k^R and a_{ijk}^R are the resulting nonzero coefficients after collection of similar terms. This is the *algebraic Kuramoto system* in its *unmixed form*. To see the connections to symmetric edge polytopes more clearly, we shall use the vector exponent notation

$$(x_1, \dots, x_n) \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = x_1^{a_1} \cdots x_n^{a_n}.$$

We can then write (9) as

$$F_{G,k}^R(\mathbf{x}) = \sum_{\mathbf{a} \in \check{\nabla}_G} c(\mathbf{a}) \mathbf{x}^{\mathbf{a}} \quad \text{for } k = 1, \dots, n,$$

where the function $c : \check{\nabla}_G \rightarrow \mathbb{C}$ captures the coefficients. That is, $\check{\nabla}_G$ is exactly the support of the unmixed form of the algebraic Kuramoto system. Facets and faces, in general, of $\check{\nabla}_G$ play particularly important roles in the study of this algebraic system. In the following, we highlight three of them, namely the roles in toric deformation homotopy method (appendix A.1), root counting (appendix A.2), and construction of homogeneous coordinates (appendix A.3).

A.1. Toric deformation homotopy. The toric deformation homotopy for unmixed algebraic Kuramoto equations is a specialized *polyhedral homotopy* [15] construction for locating *all* complex zeros of (9), which includes *all* frequency synchronization configurations. Utilizing the topological information extracted from the underlying graph, this homotopy construction has the potential to avoid the computationally expensive preprocessing steps associated with polyhedral homotopy (e.g. mixed cell computations). In the most basic form, it is defined by the function $H_G : \mathbb{C}^n \times \mathbb{C} \rightarrow \mathbb{C}^n$ with $H_G(\mathbf{x}, t) = (H_{G,1}, \dots, H_{G,n})$ given by

$$(10) \quad H_{G,k} = \sum_{\mathbf{a} \in \check{\nabla}_G} c(\mathbf{a}) \mathbf{x}^{\mathbf{a}} t^{\omega(\mathbf{a})} \quad \text{for } k = 1, \dots, n, \quad \text{where } \omega(\mathbf{a}) = \begin{cases} 0 & \text{if } \mathbf{a} = \mathbf{0} \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, $H_G(\mathbf{x}, 1) = F_G^R(\mathbf{x})$. As t varies between 0 and 1 within the interval $(0, 1)$, $H_G(\mathbf{x}, t)$ represents a smooth deformation of the system F_G^R , and the corresponding complex roots also vary smoothly, forming smooth paths reaching *all* complex zeros of F_G^R [3, 15].

The starting points of the smooth paths, however, are not well defined, as the limit points of these paths as $t \rightarrow 0$ are not contained in $(\mathbb{C}^*)^n$. Yet, with the change of variables

$$x_k = y_k t^{\alpha_k} \quad \text{for } k = 1, \dots, n,$$

where $\check{\alpha} = (\alpha_1, \dots, \alpha_n)$ is a normalized inner normal of a facet $F \in \mathcal{F}(\check{\nabla}_G)$, the limit points of certain paths, in the y -coordinates, are exactly the $(\mathbb{C}^*)$ -solutions to the subsystem

$$0 = \sum_{\mathbf{a} \in F \cup \{0\}} c(\mathbf{a}) \mathbf{x}^{\mathbf{a}} \quad \text{for } k = 1, \dots, n$$

defined by the facet F . As the pair $(F, \check{\alpha})$ runs through the set of facets and their corresponding normalized inner normals, the solutions of the subsystems of the above form include limit points, in the y -coordinate, of *all* paths defined by $H_{G,k} = 0$. Therefore, explicit descriptions to the facets of $\check{\nabla}_G$ as well as their inner normals are crucially important in bootstrapping the toric deformation homotopy method for solving the Kuramoto equations.

A.2. Facet systems and the root counting problem. A closely related application is the root counting problem for the algebraic Kuramoto equations [9]. It is shown that for generic choices of the coefficients, the total complex root count for algebraic Kuramoto equations induced by cycles and trees is exactly the adjacency polytope bound. When there are algebraic relations among the coefficients, the actual root count may be strictly less. An algebraic certificate for such decrease in root count is provided by “face systems”. For a positive-dimensional face F of $\check{\nabla}_G$, the corresponding *face system* of [9] is given by

$$(11) \quad 0 = \sum_{\mathbf{a} \in F} c(\mathbf{a}) \mathbf{x}^{\mathbf{a}} \quad \text{for } k = 1, \dots, n.$$

By Bernshtein’s Second Theorem [2], if (11) has nontrivial solutions ($\mathbb{C}^*$ -solutions), then the root count for [9] is strictly less than the adjacency polytope bound. Consequently, the descriptions of faces, especially facets, of $\check{\nabla}_G$ given in Theorems 3 and 9 provide a foundation for studying the root count of the algebraic Kuramoto equations [9].

A.3. Construction of homogeneous coordinates. One issue users of homotopy continuation methods have to face is the existence of divergent paths, e.g., solution paths defined by [10] that do not have limit points, as $t \rightarrow 1$, in the work space $\mathbb{C}^n$. A common solution is to compactify the work space. Homogeneous coordinates, a special case of Cox’s homogeneous ring [8], provide the most general construction for achieving this goal. Even though the use of homogeneous coordinates tends to introduce a large number of auxiliary variables and hence is impractical in actual calculations, they remain important tools for theoretical analysis of algebraic Kuramoto equations. The foundation of homogeneous coordinates construction is the full description of facets $\mathcal{F}(\nabla_G)$ and their inner normals.

Let $m = |\mathcal{F}(\check{\nabla}_G)|$, and let $\alpha_1, \dots, \alpha_m$ be the primitive inner normals for the facets of $\check{\nabla}_G$. Define the matrix V to be the $m \times n$ matrix whose rows are $\alpha_1^\top, \dots, \alpha_m^\top$ and let $\mathbf{h} = (h_1, \dots, h_m)^\top$ be the column vector with entries

$$h_i = \min_{\mathbf{x} \in \check{\nabla}_G} \langle \mathbf{x}, \alpha_i \rangle.$$

The homogenization of $F_{G,k}^R$ (9) is the polynomial $\hat{F}_{G,k}^R(\mathbf{y})$ in the complex variables $\mathbf{y} = (y_1, \dots, y_m)$ given by

$$\hat{F}_{G,k}^R(\mathbf{y}) = \sum_{\mathbf{a} \in \tilde{\nabla}_G} c(\mathbf{a}) \mathbf{y}^{V\mathbf{a} - \mathbf{h}}.$$

The system $\hat{F}_G^R = (\hat{F}_{G,1}^R, \dots, \hat{F}_{G,n}^R)$ represents a lifting of the system (9) to a compact topological space (a compact toric variety). Consequently, each solution path defined by the homotopy (10) has a limit point corresponding to an equivalence class of zero of $\hat{F}_G^R$ in this compact space, even if it has no limit point in $\mathbb{C}^n$.