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Fibers of Monotone Maps of Finite Distortion

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Abstract

We study topologically monotone surjective $W^{1,n}$-maps of finite distortion $f\colon \Omega \to \Omega'$, where Ω, Ω' are domains in $\mathbb{R}^n$, $n \geq 2$. If the outer distortion function $K_f \in L^p_{\mathrm{loc}}(\Omega)$ with $p \geq n-1$, then any such map f is known to be homeomorphic, and hence the fibers $f^{-1}\{y\}$ are singletons. We show that as the exponent of integrability p of the distortion function K_f increases in the range $1/(n-1) \leq p < n-1$, then for increasingly many $k \in \{0, \dots, n\}$ depending on p, the k:th rational homology group $H_k(f^{-1}\{y\}; \mathbb{Q})$ of any reasonably tame fiber $f^{-1}\{y\}$ of f is equal to that of a point. In particular, if $p \geq (n-2)/2$ then this is true for all $k \in \{0, \dots, n\}$. We also formulate a Sobolev realization of a topological example by Bing of a monotone $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ with homologically non-trivial fibers. This example has $K_f \in L^{1/2-\varepsilon}_{\mathrm{loc}}(\mathbb{R}^3)$ for all $\varepsilon > 0$, which shows that our result is sharp in the case $n = 3$.

Keywords Mappings of finite distortion · MFD · Monotone · Fiber · Homology · Conformal cohomology

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1 Introduction

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$. Recall that a mapping $f : \Omega \to \mathbb{R}^n$ of Sobolev class $W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ has *finite distortion* if

$$|Df(x)|^n \leq K(x)\, J_f(x) \tag{1.1}$$

for some measurable function $1 \leq K(x) < \infty$. Here, $|Df(x)|$ stands for the operator norm of the differential $Df(x)$. Thus, the distortion inequality (1.1) simply asks that the Jacobian determinant $J_f(x) = \det Df(x)$ is positive at a.e. (almost every) point $x \in \Omega$ where $Df(x) \neq 0$. The smallest function $K(x) \geq 1$ for which the distortion inequality (1.1) holds is called the *(outer) distortion function* of f and is denoted by $K_f(x)$. When $K_f \in L^\infty(\Omega)$, we obtain the widely studied special case of *quasiregular mappings*; see, e.g., [23, 35, 36].

In the past 20 years, a systematic study of mappings of finite distortion has taken place in the field of geometric function theory (GFT). Many of the standard results of quasiregular mappings have been proven for mappings of finite distortion with sufficient integrability assumptions on K_f; see, e.g., [18, 23]. The theory finds concrete applications in the materials sciences, particularly in nonlinear elasticity (NE) and critical phase phenomena, and in the calculus of variations.

The mathematical models of NE [1, 2, 7], and a variational approach to GFT share common interests to study homeomorphisms of finite distortion and (topologically) monotone mappings of finite distortion. Here, a mapping $f : X \to Y$ between topological spaces is *(topologically) monotone* [29] if f is continuous and $f^{-1}\{y\}$ is connected for every $y \in Y$. Indeed, monotone mappings are well suited to model the *weak interpenetration of matter* where, roughly speaking, squeezing of a portion of the material can occur, but not folding or tearing.

To clarify our terminology, we note that in the study of mappings of finite distortion, it is also common to consider another form of monotonicity introduced by Manfredi [27], which we call the 1*-oscillation property*. Namely, a mapping $f : \Omega \to \mathbb{R}^n$ satisfies the 1-oscillation property if it satisfies the estimate $\operatorname{osc}_B(f) \leq \operatorname{osc}_{\partial B}(f)$ for every ball $B \subset \Omega$, where $\operatorname{osc}_K(f) = \sup_{x,x' \in K} \left|f(x) - f(x')\right|$. This is a weaker definition of monotonicity, as any $W^{1,n}$-Sobolev mapping of finite distortion enjoys the 1-oscillation property, see [21]; this includes even examples like the map $z \mapsto z^2$ on the complex plane, which is clearly not topologically monotone. As another example of the difference between these definitions, folding maps which cause *strong interpenetration of matter* are not topologically monotone, but may still satisfy the 1-oscillation property.

Our study is centered around the following general question: how does the integrability of K_f affect the possible shapes of the fibers $f^{-1}\{y\}$, when $f : \Omega \xrightarrow{\text{onto}} \Omega'$ is a monotone mapping in $W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$. We begin by recalling that, if a non-constant $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ has finite distortion, $K_f \in L^{n-1}_{\text{loc}}(\Omega)$, and f has essentially bounded multiplicity, then f is open and discrete by a version of Reshetnyak's theorem [33, 34] for mappings of finite distortion, see [17, 19]. Without assuming that the mapping f has essentially bounded multiplicity, a slightly higher integrability for the distortion

K_f is required for openness and discreteness to still hold, namely $K_f \in L^{n-1+\varepsilon}_{\mathrm{loc}}(\Omega)$ for some $\varepsilon > 0$, see [20, 39]. However, for non-constant monotone $W^{1,n}$-maps of finite distortion, this essential multiplicity bound always holds if $K_f \in L^{1/(n-1)}_{\mathrm{loc}}(\Omega)$, see Lemma 2.1. It follows that if a non-constant $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is monotone and $K_f \in L^{n-1}_{\mathrm{loc}}(\Omega)$, then f is homeomorphic, and therefore, all fibers $f^{-1}\{y\}$ with $y \in f(\Omega)$ are singletons.

The idea behind these Reshetnyak-type theorems is that if the required conditions are satisfied, then $\mathcal{H}^1(f^{-1}\{y\}) = 0$ for every $y \in \mathbb{R}^n$, which is then used to show openness and discreteness of f. A similar result also holds for other Hausdorff measures, as shown in [19] by Hencl and Malý. We highlight this result here, since it serves as a starting point for our investigation.

Theorem 1.1 ([19, Theorem 4]) *Let $\Omega \subset \mathbb{R}^n$ be a domain, $n \geq 2$. Suppose that $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is a non-constant mapping of finite distortion and the mapping f has essentially bounded multiplicity. Then for $p \in \big[(n-1)^{-1}, \infty\big)$, if $K_f \in L^p_{\mathrm{loc}}(\Omega)$, then $\mathcal{H}^{n/(p+1)}(f^{-1}\{y\}) = 0$ for all $y \in f(\Omega)$.*

1.1 Homeomorphic Approximation and Cellular Maps

Part of our motivation in studying this topic lies in questions related to approximating maps by homeomorphisms. For instance, such a question raises in the context of neo-Hookean materials. The *neoHookean material*, defined based on Hooke's law, refers to a stored energy function which increases to infinity when the Jacobian determinant J_f approaches zero, see, e.g., [3, 4, 8, 9, 13, 30–32, 40]. The model examples take the form

$$\mathsf{E}^p_q[f] = \int_\Omega \Big[|Df|^p + J_f^{-q}\Big]\, \mathrm{d}x\,, \quad p \geq n \quad q > 0 \quad \text{and} \quad \Omega \subset \mathbb{R}^n\,. \tag{1.2}$$

This model is also broadly studied by physicists, materials scientists, and engineers [38].

Nonetheless, establishing non-interpenetration of matter in this setting remains a mathematical challenge. Naturally the first step toward to understanding the injectivity of minimizers is to enlarge the class of admissible homeomorphisms. Adopting the class of monotone maps of finite distortion ensures the existence of minimizers. However, showing that there is no *Lavrentiev gap* between the classes of homeomorphisms and monotone maps, i.e., that the classes have the same infimal energy for a given energy minimization question, leads to a suitable approximation question. Before proceeding to illuminate the general problem of approximating a monotone map by homeomorphisms, we note that a mapping $f : \Omega \to \mathbb{R}^n$ with $\mathsf{E}^p_q[f] < \infty$ has finite distortion $K_f \in L^r_{\mathrm{loc}}(\Omega)$ where $n/p + 1/q = 1/r$.

Consider a continuous map $f : \overline{\Omega} \to \overline{\Omega'}$ between two simply connected planar Jordan domains $\Omega, \Omega' \subset \mathbb{C}$. By a theorem of Youngs [41], f can be approximated uniformly with homeomorphisms if and only if f is monotone. If moreover $f \in W^{1,p}(\Omega, \Omega')$ and the domain Ω' is Lipschitz regular, then a uniform homeomorphic approximation of f can be improved to also converge in the $W^{1,p}$-norm, $1 < p < \infty$, see [24].

Consider then a similar situation in higher dimensions. We restrict ourselves here to the simple case where $f : \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$ is continuous, $f(\mathbb{B}^n) = \mathbb{B}^n$, and $f : \partial\mathbb{B}^n \to \partial\mathbb{B}^n$ is a homeomorphism. Under which conditions can the map f be uniformly approximated by homeomorphisms $f_i : \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$?

In this higher-dimensional case, monotonicity of f is no longer sufficient. One additional necessary condition for uniform homeomorphic approximation of f is that every fiber $f^{-1}\{y\}$ is *cellular*; that is, $f^{-1}\{y\}$ is an intersection of a nested sequence of topological balls. Verifying the necessity of this condition is a simple exercise. Note that f is not cellular if $f^{-1}\{y\}$ is,e.g., a smoothly embedded copy of $\mathbb{S}^1$ for some $y \in \overline{\mathbb{B}^n}$; consequently, such maps can not be homeomorphically approximated.

It turns out that for the most part, this extra necessary condition of cellular fibers is sufficient for uniform approximation. Indeed, a result of Siebenmann [37] yields that if our map f is monotone, f has cellular fibers, and $n \neq 4$, then f can be uniformly approximated by homeomorphisms. We note that the result is formulated in terms of a more general definition of CE-maps, which in particular holds for continuous proper $f : \mathbb{B}^n \to \mathbb{B}^n$ with cellular fibers and homeomorphic boundary values.

1.2 An Example of a Non-cellular Monotone Map

In [5, Sect. 4], Bing gives a topological example of a monotone map $f : \mathbb{R}^3 \xrightarrow{\text{onto}} \mathbb{R}^3$ such that some of the fibers $f^{-1}\{y\}$ of f are topologically either $\mathbb{S}^1$ or $\mathbb{S}^1 \vee \mathbb{S}^1$. Here,$\mathbb{S}^1 \vee \mathbb{S}^1$ stands for a figure-eight formed by two disjoint copies of circle $\mathbb{S}^1$ that have been joined at a point. In particular, this example is topologically monotone, yet not cellular. The example is part of a detailed investigation into the failure of the higher-dimensional version of Moore's theorem [28], which states that each decomposition of $\mathbb{R}^2$ into continua which fail to separate $\mathbb{R}^2$ yields a decomposition space topologically equivalent to $\mathbb{R}^2$. We refer to a book of Daverman [10] for the development of monotone mappings as a part of the theory of decomposition spaces and manifold recognition problems.

In this paper, we construct an explicit Sobolev representation of Bing's mapping and study its properties as a mapping of finite distortion. Here, we recall that a map $f : \Omega \to \Omega'$ is *proper* if $f^{-1}K$ is compact for every compact $K \subset \Omega'$.

Theorem 1.2 *There exists a topologically monotone, proper, and surjective mapping* $h : \mathbb{R}^3 \to \mathbb{R}^3$ *of finite distortion which is h is not cellular. The map h is locally Lipschitz, and* $K_h \in L^p_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ *for every* $p < 1/2$*, but* $K_h \notin L^{1/2}_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3)$.

More precisely, if we use e_x, e_y *and* e_z *to denote the standard basis of* $\mathbb{R}^3$*, then* $h^{-1}\{0\}$ *and* $h^{-1}\{-e_x\}$ *are bilipschitz equivalent with* $\mathbb{S}^1$, $h^{-1}\{-te_x\}$ *for* $t \in (0, 1)$ *are bilipschitz equivalent with* $\mathbb{S}^1 \vee \mathbb{S}^1$, $h^{-1}\{-te_x\}$ *for* $t \in (1, \infty)$ *are bilipschitz equivalent with* $[0, 1]$*, and every remaining fiber* $h^{-1}\{w\}$, $w \in \mathbb{R}^3 \setminus \{-te_x : t \geq 0\}$ *is a point.*

A notable property of this example is that adjustments to the definition of h seem to fail to improve the integrability of K_h past the threshold of $p = 1/2$. For comparison, the threshold imposed by Theorem 1.1 at which 1-dimensional fibers are prevented is $p = 2$. Standard results instead imply that a monotone mapping of finite distortion $f \in W^{1,3}_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ with $K_f \in L^{1/2}_{\mathrm{loc}}(\mathbb{R}^3)$ satisfies the Lusin (N^{-1})-condition, and that $J_f > 0$ a.e., see [18, Theorem 4.13]. Here, we recall that a map $f: \mathbb{R}^3 \to \mathbb{R}^3$ satisfies the *Lusin* (N^{-1})*-condition* if $f^{-1}A$ has zero (Lebesgue) measure for every $A \subset \mathbb{R}^3$ of zero measure; conversely the *Lusin* (N)*-condition* is that $f(A)$ has measure zero for every $A \subset \mathbb{R}^3$ of measure zero. However, the aforementioned result presents no obstruction in our case, since the map h of Theorem 1.2 does have an a.e. positive J_h, and h therefore also satisfies the Lusin (N^{-1})-condition.

Hence, the existing results in the theory of mappings of finite distortion cannot seem to explain the apparent upper limit on the integrability of K_h. This suggests a potential missing result on the fact that the integrability of K_h limits the possible looping of fibers. Our main goal in this paper is to prove such a result.

1.3 Homological Obstructions

The most natural form of our main results is stated in terms of pre-images of open balls. In this setting, the statement is as follows.

Theorem 1.3 *Let $f: \Omega \to \Omega'$ be a proper, continuous, and monotone surjection in the Sobolev class $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, $n \geq 3$. Suppose that $k \in \{1, \dots, n-2\}$, and that*

$$K_f \in L^p_{\mathrm{loc}}(\Omega), \quad \textit{where } p = \begin{cases} \frac{n-(k+1)}{k+1}, & 1 \leq k < \frac{n}{2}, \\ 1, & k = \frac{n}{2}, \\ \frac{k-1}{n-(k-1)}, & \frac{n}{2} < k \leq n-2. \end{cases}$$

Then

$$H_k\left(f^{-1}\mathbb{B}^n(y,r); \mathbb{R}\right) = \{0\} \quad \textit{for every } \mathbb{B}^n(y,r) \Subset \Omega'.$$

Here $H_k(X; \mathbb{R})$ stands for the k:th *singular homology* group of X with coefficients in $\mathbb{R}$, and $U \Subset V$ denotes that the closure $\overline{U}$ is a compact subset of V. In the case $n = 3$, the map $h: \mathbb{R}^3 \to \mathbb{R}^3$ given in Theorem 1.2 shows the sharpness of Theorem 1.3. In particular, for $r \in (0, 1)$, the set $h^{-1}\mathbb{B}^n(0, r)$ is a solid torus for which $H_1(h^{-1}\mathbb{B}^n(0, r); \mathbb{R}) \cong \mathbb{R}$.

We note that Theorem 1.3 does not include the cases $k = 0, n-1, n$. In the case $k = n-1$ our argument in fact does give a critical exponent $p = (n-2)/2$. However, including these cases is unnecessary, as it follows purely from the topological properties of f that $H_k(f^{-1}\mathbb{B}^n(y,r); \mathbb{R}) = H_k(\mathbb{B}^n(y,r); \mathbb{R})$ for $k \in \{0, n-1, n\}$. See Proposition 4.4 and the accompanying proof for details. If $n = 2$, it notably follows that $H_k(f^{-1}\mathbb{B}^2(y,r); \mathbb{R}) = H_k(\mathbb{B}^2(y,r); \mathbb{R})$ for all k. See also, e.g., [24, Lemma 2.8], which as part of its statement implies that if $f: \overline{\Omega} \to \overline{\Omega'}$ is a monotone surjection between topologically equivalent compact planar Jordan domains, then for every $\mathbb{B}^2(y,r) \Subset \Omega'$ the pre-image $f^{-1}\mathbb{B}^2(y,r)$ is a simply connected domain, and therefore a topological disk by the Riemann mapping theorem.

$n=3$						$\frac{1}{2}$		$(\frac{1}{2})$					
$n=4$					1		1		(1)				
$n=5$				$\frac{3}{2}$		$\frac{2}{3}$		$\frac{2}{3}$		$(\frac{3}{2})$			
$n=6$			2		1		1		1		(2)		
$n=7$		$\frac{5}{2}$		$\frac{4}{3}$		$\frac{3}{4}$		$\frac{3}{4}$		$\frac{4}{3}$		$(\frac{5}{2})$	
$n=8$	3		$\frac{5}{3}$		1		1		1		$\frac{5}{3}$		(3)

Fig. 1 Values of p in Theorem 1.3 as $k = 1, \dots, n-2$. The unnecessary case $k = n-1$ is also listed in parenthesis to make the diagram symmetric

Next, let $f : \Omega \to \Omega'$ be a monotone map satisfying the assumptions of Theorem 1.3, where $\Omega, \Omega' \subset \mathbb{R}^n$ are domains. For every $y \in \Omega'$, the sets $f^{-1}B^n(y, i^{-1})$ for large enough $i \in \mathbb{Z}_+$ form a descending sequence of precompact neighborhoods of $f^{-1}\{y\}$, and the intersection of these neighborhoods is $f^{-1}\{y\}$. We are now interested in whether the triviality of the sets $H_k(f^{-1}B^n(y, i^{-1}); \mathbb{R})$ implies the triviality of $H_k(f^{-1}\{y\}; \mathbb{R})$.

One example of a situation in which this does occur is if $f^{-1}\{y\}$ is a *neighborhood retract*; that is, if there exists a neighborhood $U \subset \mathbb{R}^n$ of $f^{-1}\{y\}$ and a retraction $r : U \to f^{-1}\{y\}$. One class of examples of neighborhood retracts are closed manifolds with a tubular neighborhood, which for example include all embedded smooth closed submanifolds of $\mathbb{R}^n$.

Corollary 1.4 *Let $f : \Omega \to \Omega'$ be a proper, continuous, and monotone surjection in the Sobolev class $W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$, $n \geq 3$. Let $k \in \{1, \dots, n\}$. Moreover, if $k \leq n-2$, suppose also that*

$$K_f \in L^p_{\text{loc}}(\Omega), \quad \textit{where } p = \begin{cases} \frac{n-(k+1)}{k+1}, & 1 \leq k < \frac{n}{2}, \\ 1, & k = \frac{n}{2}, \\ \frac{k-1}{n-(k-1)}, & \frac{n}{2} < k \leq n-2. \end{cases}$$

If $y \in \Omega'$ is such that $f^{-1}\{y\}$ is a neighborhood retract, then $H_k(f^{-1}\{y\}; \mathbb{R}) = \{0\}$.

If, however, a compact connected set $K \subset \mathbb{R}^n$ is not a neighborhood retract, it is entirely possible that $H_k(K; \mathbb{R}) \neq \{0\}$ even if every neighborhood of K contains a topological ball B with $K \subset B$. A simple example of such a set is recalled in Remark 4.3. It is, however, unknown to us whether any such examples can occur as a fiber of a monotone $W^{1,n}$-map of finite distortion f, and if yes, what restrictions this would place on the degree of integrability of K_f.

Similarly to Theorem 1.3, the map h of Theorem 1.2 shows that Corollary 1.4 is sharp when $n = 3$. In this way, our results explain the difficulties in trying to improve the integrability of K_h beyond $K_h \in L^p_{\text{loc}}(\mathbb{R}^3)$, $p < 1/2$. It is unknown to us whether these bounds are sharp for other values of n, k. See Fig. 1 for a table of the critical exponents p.

We then consider questions on the cellularity of monotone mappings of finite distortion. Notably, by combining Theorem 1.3 with the corresponding purely topological results in the cases $k \in \{0, n-1, n\}$, we obtain the following partial result toward cellularity.

Corollary 1.5 *Let $f : \Omega \to \Omega'$ be a proper, continuous, and monotone surjection in $W^{1,n}(\Omega, \mathbb{R}^n)$, $n \geq 3$. Suppose that $K_f \in L^{(n-2)/2}_{\mathrm{loc}}(\Omega)$. Then for every $y \in \Omega'$, the set $f^{-1}\{y\}$ is an intersection of a nested sequence of neighborhoods U_i that are rational homology balls; that is, the neighborhoods U_i satisfy $H_k(U_i; \mathbb{Q}) = H_k(\mathbb{B}^n; \mathbb{Q})$ for all $k \in \mathbb{Z}_{\geq 0}$.*

In particular, if f satisfies the assumptions of Corollary 1.5 but fails to be cellular, then this failure cannot be seen through the lens of rational homology. We recall that an example of a rational homology ball U that is not homeomorphic to $\mathbb{B}^3$ is given by any homeomorphic copy of $U = \mathbb{S}^3 \setminus H$ where H is a filled-in Alexander horned ball; see, e.g., [15, pp. 169–172]. Note also that the condition we need to assume in Corollary 1.5 corresponds by Theorem 1.1 to the fibers having zero $\mathcal{H}^2$-measure.

Hence, our results suggest the following question on homeomorphic approximation in three dimensions. Similar questions can also be stated in higher dimensions, but the case $n = 3$ is especially notable since the example of Theorem 1.2 essentially shows that the assumptions cannot be improved.

Question 1.6 *Let $f : \mathbb{B}^3 \xrightarrow{\text{onto}} \mathbb{B}^3$ be a continuous, proper, and monotone mapping of finite distortion in $W^{1,3}(\mathbb{B}^3, \mathbb{R}^3)$. Suppose that $K_f \in L^{1/2}(\mathbb{B}^3)$, and that f extends to a continuous $f : \overline{\mathbb{B}^3} \to \overline{\mathbb{B}^3}$ with homeomorphic boundary values. Can f be uniformly approximated by homeomorphisms $f_i : \overline{\mathbb{B}^3} \to \overline{\mathbb{B}^3}$? If yes, can this approximation be improved to a uniform and $W^{1,3}$-approximation by $W^{1,3}$-homeomorphisms?*

1.4 Idea of the Proofs

Our strategy in showing Theorem 1.3 starts with ideas from the study of the discreteness and openness of mappings of finite distortion, and then combines these ideas with the use of Sobolev de Rham cohomology theories. In particular, the main idea of the proofs can be essentially condensed into a single diagram:

$$\begin{array}{ccc} C^\infty\left(\wedge^k f^{-1}\mathbb{B}^n(y,r)\right) & \xrightarrow{f_*} & L^{\frac{n}{k}}_{\mathrm{loc}}\left(\wedge^k \mathbb{B}^n(y,r)\right) \\ & & \big\downarrow \text{Sobolev–Poincaré} \\ L^1_{\mathrm{loc}}\left(\wedge^{k-1} f^{-1}\mathbb{B}^n(y,r)\right) & \xleftarrow[f^*]{} & L^{\frac{n}{k-1}}_{\mathrm{loc}}\left(\wedge^{k-1}\mathbb{B}^n(y,r)\right) \end{array}$$

Indeed, we take a smooth closed k-form ω on $f^{-1}\mathbb{B}^n(y,r)$, and push it forward in f to a form $f_*\omega$ on $\mathbb{B}^n(y,r)$. Since the conformal k-cohomology of $\mathbb{B}^n(y,r)$ is trivial, we have $f_*\omega = \mathrm{d}\tau$, where we may assume by the Sobolev–Poincaré inequality that the $(k-1)$-form τ is $L^{n/(k-1)}$-integrable. It then follows that $\omega = \mathrm{d}f^*\tau$, implying that ω is trivial in local L^1-cohomology.

In order for the push-forward map to have the correct target space, we need that $K_f \in L^{(n-k)/k}_{\mathrm{loc}}(\Omega)$. For the pull-back, we similarly need that $K_f \in L^{(k-1)/(n-k-1)}_{\mathrm{loc}}(\Omega)$. Hence, under these assumptions, the above computation and a de Rham theorem for L^p_{loc}-cohomologies lets us deduce that $H^k(f^{-1}\mathbb{B}^n(y,r);\mathbb{R})$ vanishes. The cases $k \geq n/2$ in Theorem 1.3 hence follow by the universal coefficient theorem. Note that the use of the conformal exponent on the image side is crucial for our argument, as a higher intermediary exponent will break the push-forward map, and a lower one will similarly break the pull-back map.

For the cases $k \leq n/2$ in Theorem 1.3, we use compactly supported cohomology. Indeed, since $f^{-1}\mathbb{B}^n(y,r)$ is an open subset of $\mathbb{R}^n$, its k-homology spaces are isomorphic to its compactly supported $(n-k)$-cohomology spaces by the Poincaré duality theorem. We can hence replace k with $n-k$ and perform the same argument with spaces of compactly supported forms, which yields our result in the cases $k \leq n/2$.

2 Differential Forms and Maps of Finite Distortion

We consider a continuous, proper, surjective, topologically monotone $W^{1,n}$-map $f\colon \Omega \to \Omega'$ between open domains in $\mathbb{R}^n$. We begin by recalling the following useful facts about such maps.

Lemma 2.1 *Let $f \in W^{1,n}(\Omega,\Omega')$ be a non-constant continuous monotone map between open domains. Suppose that f is a mapping of finite distortion with $K_f^{1/(n-1)} \in L^1(\Omega)$. Then $f^{-1}\{y\}$ is a singleton for a.e. $y \in f(\Omega)$, f satisfies both the Lusin (N) and (N^{-1})-conditions, and $J_f > 0$ almost everywhere.*

Proof We note that f satisfies the Lusin (N)-condition by [18, Theorem 4.5], and that f is differentiable almost everywhere by [18, Corollary 2.25b)].

Let B be the set of points $x \in \Omega$ where $f^{-1}(f(x))$ is not a singleton. If $x \in B$, then due to the monotonicity of f, we find a sequence of points $x_j \to x$ such that $f(x_j) = f(x)$ for every $j \in \mathbb{Z}_+$. If f is also differentiable at x, then we must have $Df(x)v = 0$ for some vector $v \in \mathbb{S}^{n-1}$. It follows that $J_f(x) = 0$ at every such x. We thus obtain that $J_f \equiv 0$ almost everywhere in B. Since f satisfies the Lusin (N)-condition, we may use the change of variables formula to conclude that

$$m_n(f(B)) \leq \int_B J_f = 0.$$

Since the set of points $y \in f(\Omega)$ where $f^{-1}\{y\}$ isn't a singleton is precisely $f(B)$, and since we've now shown this set to be of measure zero, we conclude that $f^{-1}\{y\}$ is a singleton for a.e. $y \in f(\Omega)$. The Lusin (N^{-1})-condition and the fact that $J_f > 0$ a.e. now follow from [18, Theorem 4.13] using the assumption $K_f^{1/(n-1)} \in L^1(\Omega)$, since the multiplicity function of f is essentially bounded from above by 1. □

We then consider how the integrability of K_f effects the pull-backs of differential forms by f. In the following lemma we are mainly interested in the case $p = n/k$, but we regardless give a more general statement.

Lemma 2.2 *Let $f \in W^{1,n}(\Omega, \Omega')$ be a continuous, proper, monotone surjection, where Ω, Ω' are open domains. Let $\omega \in L^p(\wedge^k \Omega')$ and let $K_f^q \in L^1(\Omega)$, with $k \in \{1, \dots, n\}$, $n/k \le p \le \infty$ and $(n-1)^{-1} \le q \le \infty$. Then*

$$\left|f^*\omega\right|^r \in L^1(\Omega) \qquad \text{where } r = \frac{n}{k + \frac{n}{pq}}.$$

More precisely, we have the estimate

$$\left\|f^*\omega\right\|_r \le \|\omega\|_p \left\|K_f\right\|_q^{\frac{1}{p}} \|Df\|_n^{\frac{qk-n}{q}}.$$

Proof The measurability of $f^*\omega$ follows from the Lusin (N^{-1})-condition. The case $q = \infty$ is an immediate consequence of the standard result that quasiregular maps preserve the (n/k)-integrability of k-forms; see, e.g., [26, Sect. 2.2]. The case $p = \infty$ is similarly simple, as we can then estimate

$$\int_\Omega \left|f^*\omega\right|^{\frac{n}{k}} \le \int_\Omega \left(|\omega|^{\frac{n}{k}} \circ f\right) |Df|^n \le \|\omega\|_\infty^{\frac{n}{k}} \int_\Omega |Df|^n < \infty.$$

We then consider the case $p \ne \infty \ne q$, where we use Hölder's inequality to estimate that

$$\begin{aligned}\int_\Omega \left|f^*\omega\right|^r &\le \int_\Omega \left(|\omega|^r \circ f\right) |Df|^{rk} = \int_\Omega \left(|\omega|^r \circ f\right) J_f^{\frac{r}{p}} K_f^{\frac{r}{p}} |Df|^{(k-p^{-1}n)r} \\ &\le \left(\int_\Omega \left(|\omega|^p \circ f\right) J_f\right)^{\frac{r}{p}} \left(\int_\Omega K_f^q\right)^{\frac{r}{pq}} \left(\int_\Omega |Df|^{\frac{(pqk-qn)r}{pq-qr-r}}\right)^{\frac{pq-qr-r}{pq}}.\end{aligned}$$

Note that our use of Hölder is valid if $pq - qr - r \ge 0$, which holds since

$$pq - qr - r = \frac{q(pk-n)}{k + \frac{n}{pq}}$$

and since we assumed that $p \ge n/k$. With the change of variables formula, we have

$$\int_\Omega \left(|\omega|^p \circ f\right) J_f = \int_{\Omega'} |\omega|^p < \infty.$$

Finally, we see using our choice of r that

$$\frac{(pqk-qn)r}{pq-qr-r} = \frac{(pqk-qn)n}{pq\left(k + \frac{n}{pq}\right) - qn - n} = \frac{(pqk-qn)n}{kpq + n - nq - n} = n,$$

and hence

$$\int_\Omega |Df|^{\frac{(pqk-qn)r}{pq-qr-r}} = \int_\Omega |Df|^n < \infty. \qquad \square$$

Next, we wish to define a push-forward map for a continuous monotone surjection $f \in W^{1,n}(\Omega, \Omega')$ with integrable $K_f^{1/(n-1)}$. By Lemma 2.1, for a.e. $y \in \Omega'$ there exists a unique point $f^{-1}(y) \in \Omega$ such that $f(f^{-1}(y)) = y$. Given a differential k-form ω on Ω, we define

$$(f_* \omega)_y = \omega_{f^{-1}(y)} \circ \wedge^k \left[Df\left(f^{-1}(y)\right)\right]^{-1} \tag{2.1}$$

for a.e. $y \in \Omega'$. Using the Lusin conditions of f, it can be seen that $f_* \omega$ is a measurable k-form whenever ω is a measurable k-form. We highlight the fact that under our assumptions, f_* is an inverse of the pull-back f^* up to a set of measure zero; this is in contrast to f_* only being a left inverse of f^* in most settings involving non-injective maps, such as in [26, Sect. 5].

We then prove a norm estimate for the push-forward map similar to Lemma 2.2.

Lemma 2.3 *Let $f \in W^{1,n}(\Omega, \Omega')$ be a continuous, proper, monotone surjection, where Ω, Ω' are open domains. Let $\omega \in L^p(\wedge^k \Omega)$ with $k \in \{1, \dots, n\}$ and $n/k \le p \le \infty$. Then*

$$|f_* \omega| \in L^{\frac{n}{k}}(\Omega') \quad \textit{if} \quad K_f \in L^{\frac{(n-k)p}{kp-n}}(\Omega).$$

More precisely, we have the estimate

$$\|f_* \omega\|_{\frac{n}{k}} \le \|\omega\|_p \left\|K_f\right\|_{\frac{(n-k)p}{kp-n}}^{\frac{n-k}{n}} .$$

Proof Note that $(n-k)p/(kp-n) \ge (n-k)/k \ge 1/(n-1)$, so the push-forward is well defined. We note that for a.e. $x \in \Omega$ and all $v \in \wedge^k T_x \Omega$, we have

$$\begin{aligned}
&\left\langle \left(\wedge^k Df(x)\right) v, (-1)^s \star \left(\wedge^{n-k} Df(x)\right) \star v' \right\rangle \\
&\quad = \star \left(\left((\wedge^k Df(x))v\right) \wedge \left((-1)^s \star\star \left(\wedge^{n-k} Df(x)\right) \star v'\right)\right) \\
&\quad = \star(\wedge^n Df(x))(v \wedge \star v') = \star(\wedge^n Df(x))\left(\left\langle v, v'\right\rangle e_{12\dots n}\right) \\
&\quad = \left\langle v, v'\right\rangle \star(\wedge^n Df(x)) e_{12\dots n} = \left\langle v, v'\right\rangle J_f(x)
\end{aligned}$$

with $s = k(n-k)$ and $e_{12\dots n} = \star 1$. Applying this with $v = (\wedge^k [Df(x)]^{-1})w$, $w \in T_{f(x)}\Omega'$, and $|w| = \left|v'\right| = 1$, leads to the estimate

$$\left|\wedge^k [Df(x)]^{-1}\right| \le J_f(x)^{-1} \left|\wedge^{n-k} Df(x)\right| \le J_f(x)^{-1} \left|Df(x)\right|^{n-k} .$$

Hence, by using the almost everywhere defined function f^{-1}, a change of variables in f gives us

$$\int_{\Omega'} |f_* \omega|^{\frac{n}{k}} \le \int_{\Omega'} \left(|\omega| \, J_f^{-1} \left|Df\right|^{n-k}\right)^{\frac{n}{k}} \circ f^{-1}$$

$$= \int_\Omega |\omega|^{\frac{n}{k}} J_f^{-\frac{n-k}{k}} |Df|^{\frac{(n-k)n}{k}} = \int_\Omega |\omega|^{\frac{n}{k}} K_f^{\frac{n-k}{k}}.$$

In the case $p = \infty$, we may estimate $|\omega| \le \|\omega\|_\infty$, and the result follows since our assumed degree of integrability from K_f is exactly $(n-k)/k$ in this case. In other cases, we use Hölder's inequality and get the desired bound

$$\int_\Omega |\omega|^{\frac{n}{k}} K_f^{\frac{n-k}{k}} \le \left(\int_\Omega |\omega|^p\right)^{\frac{n}{pk}} \left(\int_\Omega K_f^{\frac{n-k}{k}\cdot\frac{pk}{pk-n}}\right)^{\frac{pk-n}{pk}} < \infty.$$

□

2.1 Weak Differentials

We let $W^{d,p,q}(\wedge^k\Omega)$ denote the space of measurable differential k-forms $\omega \in L^p(\wedge^k\Omega)$ which have a weak differential $\mathrm{d}\omega \in L^q(\wedge^{k+1}\Omega)$. Recall that a $(k+1)$-form $\mathrm{d}\omega \in L^1_{\mathrm{loc}}(\wedge^{k+1}\Omega)$ is a weak differential of $\omega \in L^1_{\mathrm{loc}}(\wedge^k\Omega)$ if

$$\int_\Omega \omega \wedge \mathrm{d}\eta = (-1)^{k+1} \int_\Omega \mathrm{d}\omega \wedge \eta$$

for every compactly supported smooth $\eta \in C_c^\infty(\wedge^{n-k-1}\Omega)$. We use the shorthand $W^{d,p}(\wedge^k\Omega) = W^{d,p,p}(\wedge^k\Omega)$.

We similarly use $W^{d,p,q}_{\mathrm{loc}}(\wedge^k\Omega)$ to denote the space of measurable k-forms $\omega \in L^p_{\mathrm{loc}}(\wedge^k\Omega)$ with a weak differential $d\omega \in L^q_{\mathrm{loc}}(\wedge^{k+1}\Omega)$. We also denote by $W^{d,p,q}_c(\wedge^k\Omega)$ the space of compactly supported elements of $W^{d,p,q}(\wedge^k\Omega)$; recall that the *support* $\operatorname{spt}\omega$ of $\omega \in L^p_{\mathrm{loc}}(\wedge^k\Omega)$ is the set of all $x \in \Omega$ such that there exists no neighborhood U of x with $\omega = 0$ a.e. on U. Similarly as above, we use the shorthands $W^{d,p}_{\mathrm{loc}}(\wedge^k\Omega) = W^{d,p,p}_{\mathrm{loc}}(\wedge^k\Omega)$ and $W^{d,p}_c(\wedge^k\Omega) = W^{d,p,p}_c(\wedge^k\Omega)$.

We recall the following standard result which implies that $f^*\mathrm{d}\omega = \mathrm{d}f^*\omega$ when ω is smooth and f is a continuous $W^{1,p}_{\mathrm{loc}}$-map with suitably high p. For the case when ω is compactly supported, we refer to, e.g., [25, Lemma 2.2], and the general version follows using the continuity of f and a locally finite partition of unity.

Lemma 2.4 *Let $\Omega, \Omega' \subset \mathbb{R}^n$ be open domains. Suppose that $f \in C(\Omega, \Omega') \cap W^{1,p}_{\mathrm{loc}}(\Omega, \Omega')$. If $\omega \in C^\infty(\wedge^k M)$ and $p \ge k+1$, then $f^*\omega \in W^{d,p/k,p/(k+1)}_{\mathrm{loc}}(\wedge^k\Omega)$ and $\mathrm{d}f^*\omega = f^*\mathrm{d}\omega$.*

Using Lemma 2.4, we prove a similar result for the push-forward in our setting.

Lemma 2.5 *Let $f \in W^{1,n}(\Omega, \Omega')$ be a continuous, proper, monotone surjection, where $\Omega, \Omega' \subset \mathbb{R}^n$ are open, bounded domains. Let $\omega \in C^\infty(\wedge^k\Omega)$ with $k \in \{1, \ldots, n-1\}$, and let $K_f \in L^{(n-k)/k}(\Omega)$. Then, we have $f_*\omega \in W^{d,n/k,n/(k+1)}(\wedge^k\Omega')$ and $\mathrm{d}f_*\omega = f_*\mathrm{d}\omega$.*

Proof Due to Lemma 2.3, the only thing we have to check is that $\mathrm{d} f_* \omega = f_* \mathrm{d}\omega$ in the weak sense. Let $\eta \in C_c^\infty(\wedge^{n-k-1}\Omega')$. We use a Sobolev change of variables to conclude that

$$\int_{\Omega'} (f_* \mathrm{d}\omega) \wedge \eta = \int_{\Omega} f^*((f_* \mathrm{d}\omega) \wedge \eta) = \int_{\Omega} \mathrm{d}\omega \wedge f^*\eta.$$

By Lemma 2.4, $f^*\eta \in W^{d,n/(n-k-1),n/(n-k)}_{\mathrm{loc}}(\wedge^{n-k-1}\Omega)$ and $f^*\mathrm{d}\eta = \mathrm{d} f^*\eta$. Moreover, since f is proper, the form $f^*\eta$ is compactly supported. We let α_j be the convolutions of $f^*\eta$ with a sequence of mollifying kernels, in which case $\alpha_j \in C_c^\infty(\wedge^{n-k-1}\Omega)$ for large enough j, $\alpha_j \to f^*\eta$ in the $L^{n/(n-k-1)}$-norm, and $\mathrm{d}\alpha_j \to f^*\mathrm{d}\eta$ in the $L^{n/(n-k)}$-norm. A standard application of Hölder's inequality to the wedge products now implies that

$$\begin{aligned}\int_{\Omega} \mathrm{d}\omega \wedge f^*\eta &= \lim_{i\to\infty} \int_{\Omega} \mathrm{d}\omega \wedge \alpha_i = (-1)^{k+1} \lim_{i\to\infty} \int_{\Omega} \omega \wedge \mathrm{d}\alpha_i \\ &= (-1)^{k+1} \int_{\Omega} \omega \wedge f^*\mathrm{d}\eta.\end{aligned}$$

Finally, another change of variables gives us our result by

$$\int_{\Omega} \omega \wedge f^*\mathrm{d}\eta = \int_{\Omega} (f^* f_* \omega) \wedge f^*\mathrm{d}\eta = \int_{\Omega} f^*(f_* \omega \wedge \mathrm{d}\eta) = \int_{\Omega'} f_* \omega \wedge \mathrm{d}\eta.$$

□

We also require a result of the type that a k-form ω is weakly exact if its push-forward $f_* \omega$ is weakly exact. The proof is similar to the proof of the previous lemma.

Lemma 2.6 *Let $f \in W^{1,n}(\Omega, \Omega')$ be a continuous, proper, monotone surjection, where $\Omega, \Omega' \subset \mathbb{R}^n$ are open domains. Let $\omega \in C^\infty(\wedge^k\Omega)$ with $k \in \{2, \dots, n\}$, and let $K_f^q \in L^1(\Omega)$ with $(n-1)^{-1} \le q \le \infty$. Suppose that $f_* \omega = \mathrm{d}\tau$ weakly for some $\tau \in L^{n/(k-1)}_{\mathrm{loc}}(\wedge^{k-1}\Omega')$. If*

$$n \ge (k-1)(1+q^{-1}),$$

then $\omega = \mathrm{d} f^*\tau$ weakly.

Proof By Lemma 2.2, our assumption that $n \ge (k-1)(1+q^{-1})$ implies that $f^*\tau \in L^1_{\mathrm{loc}}(\wedge^{k-1}\Omega)$. Hence, it remains to check the exactness of ω when $f_* \omega$ is exact, by proving that ω is a weak exterior derivative of $f^*\tau$. Let $\eta \in C_c^\infty(\wedge^{n-k}\Omega)$. A Sobolev change of variables gives us

$$\int_{\Omega} \omega \wedge \eta = \int_{\Omega} f^*(f_* \omega \wedge f_* \eta) = \int_{\Omega'} \mathrm{d}\tau \wedge f_* \eta.$$

By Lemmas 2.5 and 2.3, we have that $f_* \eta \in W_c^{d,n/(n-k),n/(n-k+1)}(\wedge^{n-k}\Omega')$ and $f_* \mathrm{d}\eta = \mathrm{d} f_* \eta$. Hence, we may again take a sequence of mollifications α_j of $f_* \mathrm{d}\eta$, and we get that $\alpha_j \in C_c^\infty(\wedge^{n-k}\Omega')$ for large enough j, $\alpha_j \to f_* \eta$ in the $L^{n/(n-k)}$-norm, and $\mathrm{d}\alpha_j \to f_* \mathrm{d}\eta$ in the $L^{n/(n-k+1)}$-norm. Yet again a standard application of Hölder's inequality yields

$$\int_{\Omega'} \mathrm{d}\tau \wedge f_* \eta = \lim_{i\to\infty} \int_{\Omega'} \mathrm{d}\tau \wedge \alpha_i = (-1)^{k+1} \lim_{i\to\infty} \int_{\Omega'} \tau \wedge \mathrm{d}\alpha_i$$
$$= (-1)^{k+1} \int_{\Omega'} \tau \wedge f_* \mathrm{d}\eta.$$

Finally, we perform one more change of variables:

$$(-1)^{k+1} \int_{\Omega} \omega \wedge \eta = \int_{\Omega'} \tau \wedge f_* \mathrm{d}\eta = \int_{\Omega} f^*(\tau \wedge f_* \mathrm{d}\eta) = \int_{\Omega} f^*\tau \wedge \mathrm{d}\eta.$$

We conclude that ω is a weak exterior derivative of $f^*\tau$, completing the proof. □

3 Sobolev de Rham Cohomologies

Let M be an oriented Riemannian manifold without boundary. Note that for the purposes of this text, we only need the following results when M is an open domain in $\mathbb{R}^n$, but we state them more generally regardless. We use similar notation $W^{d,p,q}(\wedge^k M)$, $W^{d,p}(\wedge^k M)$, $W^{d,p,p}_{\mathrm{loc}}(\wedge^k M)$, $W^{d,p}_{\mathrm{loc}}(\wedge^k M)$, $W^{d,p,p}_c(\wedge^k M)$, and $W^{d,p}_c(\wedge^k M)$ for manifolds as we specified in the Euclidean setting.

We begin by recalling a *Poincaré lemma* in the Sobolev setting. It is closely tied with the *Sobolev–Poincaré inequalities* for differential forms. We give here the precise version we require. Note that the following result is very close to the one stated in [26, Lemma 4.2], but our statement also includes the exceptional L^1-case since we need it here.

Lemma 3.1 *Let M be an oriented Riemannian n-manifold without boundary with $n \geq 2$, let $k \in \{1, \dots, n\}$, let $x \in M$, and let $\omega \in L^q_{\mathrm{loc}}(\wedge^k M)$ for some $q \in [1, \infty)$. Suppose that $\mathrm{d}\omega = 0$ weakly. Then there exists a neighborhood U of x and a $(k-1)$-form $\tau \in W^{d,q}_{\mathrm{loc}}(\wedge^{k-1} U)$ such that $\omega|_U = \mathrm{d}\tau$. Moreover, if $q > 1$, then we also have $\tau \in L^p_{\mathrm{loc}}(\wedge^{k-1} U)$ for every $p \in [1, \infty)$ satisfying $p^{-1} + n^{-1} \geq q^{-1}$.*

Proof By restricting to a small enough neighborhood of x and using a smooth bilipschitz chart, we may assume that M is a convex Euclidean domain. Let U be a small ball around x.

We then refer to [22], where an integral operator T is constructed that is bounded $L^q(\wedge^k U) \to L^q(\wedge^{k-1} U)$ when $1 \leq q < \infty$, and that satisfies the chain homotopy condition $\alpha = T(\mathrm{d}\alpha) + \mathrm{d}T(\alpha)$ for all $\alpha \in W^{d,q}_{\mathrm{loc}}(\wedge^k U)$; see [22, (4.15–4.16)]. Due to $\mathrm{d}\omega = 0$ and the chain homotopy condition, we have $\mathrm{d}T(\omega) = \omega$, and due to the boundedness of T, we have $T(\omega) \in W^{d,q}(\wedge^{k-1} U)$.

Moreover, when $q > 1$, it is shown in [22, Proposition 4.1] that the $W^{1,q}$-norm of $T(\omega)$ is controlled by the L^q-norm of ω. Hence, the Sobolev embedding theorem also implies that $T(\omega) \in L^p(\wedge^{k-1} U)$ whenever $1 \leq p < \infty$ and $p^{-1} + n^{-1} \geq q^{-1}$, completing the proof. □

The L^p_{loc}-*cohomology* $H^*_p(M)$ of M is the cohomology of the chain complex

$$0 \to W^{d,p}_{\text{loc}}(\wedge^0 M) \xrightarrow{d} W^{d,p}_{\text{loc}}(\wedge^1 M) \xrightarrow{d} \dots .$$

That is, if $k \in \{0, 1, \dots\}$, then $H^k_p(M)$ is the quotient vector space

$$H^k_p(M) = \frac{\ker(d\colon W^{d,p}_{\text{loc}}(\wedge^k M) \to W^{d,p}_{\text{loc}}(\wedge^{k+1} M))}{\text{im}(d\colon W^{d,p}_{\text{loc}}(\wedge^{k-1} M) \to W^{d,p}_{\text{loc}}(\wedge^k M))},$$

where the same exponent of integrability $p \in [1, \infty)$ is used for every k. Similarly, we define the *compactly supported* L^p-*cohomology* $H^*_{p,c}(M)$ of M as the cohomology of the chain complex

$$0 \to W^{d,p}_c(\wedge^0 M) \xrightarrow{d} W^{d,p}_c(\wedge^1 M) \xrightarrow{d} \dots .$$

We then require the following standard result on equivalence of cohomologies.

Theorem 3.2 *For every $p \in [1, \infty)$ and $k \in \mathbb{N}$, we have*

$$H^k_p(M) \cong H^k_{dR}(M) \quad \textit{and} \quad H^k_{p,c}(M) \cong H^k_{dR,c}(M),$$

where $H^k_{dR}(M)$ is the de Rham cohomology of M, and $H^k_{dR,c}(M)$ is the de Rham cohomology of M with compact supports. Moreover, the above isomorphisms are induced by the inclusion maps $C^\infty(\wedge^* M) \hookrightarrow W^{d,p}_{\text{loc}}(\wedge^* M)$ and $C^\infty_c(\wedge^* M) \hookrightarrow W^{d,p}_c(\wedge^* M)$.

Theorem 3.2 follows via a standard argument from highly general results in sheaf theory. The essential ingredients required for the proof to work are the Poincaré lemma from Lemma 3.1, the fact that the spaces $W^{d,p}_{\text{loc}}(\wedge^k U)$ are defined locally, the fact that a function $u \in W^{d,p}_{\text{loc}}(\wedge^0 U)$ with $du = 0$ is locally constant, and the fact that the spaces $W^{d,p}_{\text{loc}}(\wedge^k U)$ are closed under multiplication by C^∞-functions. For the sake of readers less familiar with sheaf cohomology, we recall here a version of the general result which yields Theorem 3.2, where we try to minimize the use of sheaf-theoretic concepts in the statement.

Let X be a paracompact Hausdorff space. Note that this includes for example all metric spaces. A *presheaf* (of vector spaces) $\mathcal{S}$ on X is a choice of a vector space $\mathcal{S}(U)$ for every open set $U \subset X$, combined with restriction maps $u \mapsto u|_V : \mathcal{S}(U) \to \mathcal{S}(V)$ whenever $V \subset U$. The restriction maps are assumed to satisfy the following typical properties of the restriction of functions:

- $u|_U = u$ if $u \in \mathcal{S}(U)$;

- $(au + bv)|_V = a(u|_V) + b(v|_V)$ if $u, v \in \mathcal{S}(U)$ and $a, b \in \mathbb{R}$;
- $(u|_V)|_W = u|_W$ if $u \in \mathcal{S}(U)$ and $W \subset V \subset U$.

A presheaf $\mathcal{S}$ is a *sheaf* if it also satisfies the following two conditions

(S1) If $U = \bigcup_i U_i$ and $u \in \mathcal{S}(U)$ is such that $u|_{U_i} = 0$ for every i, then $u = 0$.
(S2) If $U = \bigcup_i U_i$, and we have elements $u_i \in \mathcal{S}(U_i)$ which coincide on intersections, i.e., $u_i|_{U_i \cap U_j} = u_j|_{U_i \cap U_j}$ if U_i and U_j intersect, then there exists an element $u \in \mathcal{S}(U)$ such that $u|_{U_i} = u_i$ for every i.

Notably, essentially every typical linear function space on a smooth manifold is a presheaf that satisfies (S1): C^∞, C^0, L^p, L^p_{loc}, etc. However, (S2) is only satisfied if the definition of the function space is in a sense local: for example $U \mapsto L^p_{\mathrm{loc}}(U)$ is a sheaf, but $U \mapsto L^p(U)$ fails to be a sheaf since the definition of $L^p(U)$ is global in nature.

A *presheaf morphism* $f : \mathcal{S} \to \mathcal{S}'$ between two presheaves on X is a collection of linear maps $f : \mathcal{S}(U) \to \mathcal{S}'(U)$ such that $f(u|_V) = (f(u))|_V$. A *family of supports* Φ on X is a collection of closed subsets of X such that

- If $C \in \Phi$, then every closed subset of C is also in Φ;
- If $C, C' \in \Phi$, then $C \cup C' \in \Phi$;
- If $C \in \Phi$, then there's a neighborhood V of C such that $\overline{V} \in \Phi$.

If $\mathcal{S}$ is a sheaf on X, then the *support* $\operatorname{spt} u$ of $u \in \mathcal{S}(X)$ is $X \setminus U$, where U is the largest open set on which $u|_U = 0$. We also use $\mathcal{S}_\Phi(X)$ to denote all elements of $\mathcal{S}(X)$ with $\operatorname{spt}(u) \in \Phi$.

We can now state the general result we use.

Theorem 3.3 *Let X be a paracompact Hausdorff space, let Φ be a family of supports, and suppose that we have a sequence*

$$0 \to \mathcal{S}^{-1} \xrightarrow{d_0} \mathcal{S}^0 \xrightarrow{d_1} \mathcal{S}^1 \xrightarrow{d_2} \mathcal{S}^2 \xrightarrow{d_3} \dots \tag{3.1}$$

such that the following conditions are satisfied.

- *Every $\mathcal{S}^i$ in (3.1) is a sheaf on X. Every d_i is a presheaf morphism.*
- *The sequence (3.1) is exact in the following local sense: if $U \subset X$ is open, then $d_0 : \mathcal{S}^{-1}(U) \to \mathcal{S}^0(U)$ is injective, and for every $x \in U$, $i \in \mathbb{Z}_{\geq 0}$, and $u \in \mathcal{S}^i(U)$ satisfying $d_{i+1}(u) = 0$, there exists a neighborhood $U_x \subset U$ of x such that $u|_{U_x} = d_i(v_x)$ for some $v_x \in \mathcal{S}^{i-1}(U_x)$.*
- *For $i \geq 0$, the sheaves $\mathcal{S}^i$ are Φ-soft: that is, for any $C \in \Phi$, any neighborhood U of C, and any $u \in \mathcal{S}^i(U)$, there exists $u' \in \mathcal{S}^i(X)$ and a neighborhood $V \subset U$ of C such that $u'|_V = u|_V$.*

Then the cohomology groups $H^i_\Phi(X; \mathcal{S}^{-1})$ of the sequence of vector spaces

$$0 \to \mathcal{S}^0_\Phi(X) \xrightarrow{d_1} \mathcal{S}^1_\Phi(X) \xrightarrow{d_2} \mathcal{S}^2_\Phi(X) \xrightarrow{d_2} \dots$$

are determined completely up to isomorphism by the sheaf $\mathcal{S}^{-1}$ and the family of supports Φ. Moreover, if we have a commutative diagram of sheaves and presheaf morphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S}^{-1} & \xrightarrow{d_0} & \mathcal{S}^0 & \xrightarrow{d_1} & \mathcal{S}^1 & \xrightarrow{d_2} & \dots \\ & & \downarrow \text{id} & & \downarrow s_0 & & \downarrow s_1 & & \\ 0 & \longrightarrow & \mathcal{S}^{-1} & \xrightarrow{d_0'} & \mathcal{T}^0 & \xrightarrow{d_1'} & \mathcal{T}^1 & \xrightarrow{d_2'} & \dots \end{array}$$

where both rows satisfy the above conditions, then the maps s_i induce an isomorphism between the cohomology groups of $\mathcal{S}^i_\Phi(X)$ and $\mathcal{T}^i_\Phi(X)$.

The proof can be found, e.g., in [6, Chapters I–II]. In particular, the proof of the above result is completed in [6, Theorem II.4.1 and Sect. II.4.2] with a different third assumption, where the sheaves are Φ-acyclic instead of Φ-soft. Afterward, Φ-soft sheaves are defined in [6, Sect. II.9], and they are shown to be Φ-acyclic in [6, Theorem II.9.11]. Note that [6] gives many definitions in terms of so-called stalks and étale spaces of sheaves, which we have elected not to discuss here. In most cases, any conversion of definitions is straightforward. However, converting the above definition of Φ-soft sheaves to the one in [6] is slightly trickier and requires the paracompactness assumption; the key step is [6, Theorem II.9.5].

We then prove Theorem 3.2.

Proof of Theorem 3.2 We wish to use Theorem 3.3. For this, we first select $\mathcal{S}^{-1}$ to be the *constant sheaf* $\mathcal{R}$ on M; that is $\mathcal{R}(U)$ is the space of locally constant real-valued functions on U. We then let $\mathcal{T}^i$ be the sheaves given by $U \mapsto W^{d,p}_{\mathrm{loc}}(\wedge^i U)$, and let $\mathcal{S}^i$ for $i \geq 0$ be the sheaves of smooth differential forms $U \mapsto C^\infty(\wedge^i U)$. The maps s_i are the inclusion maps $C^\infty(\wedge^i U) \hookrightarrow W^{d,p}_{\mathrm{loc}}(\wedge^i U)$. The maps d_0 and d_0' are given by the inclusions $\mathcal{R}(U) \hookrightarrow C^\infty(\wedge^0 U) \hookrightarrow W^{d,p}_{\mathrm{loc}}(\wedge^0 U)$ and are therefore injective. The other maps d_i, d_i' are given by the (weak) exterior derivative.

All of our chosen $\mathcal{S}^i$ and $\mathcal{T}^i$ are indeed sheaves; here, it is crucial to use the local spaces $W^{d,p}_{\mathrm{loc}}(\wedge^i U)$. Exactness at $\mathcal{S}^0$ and $\mathcal{T}^0$ follow from the fact that a real function with zero derivative is locally constant; for a Sobolev version, see, e.g., [16, Lemma 1.13]. Exactness at all other points of the sequence follows from the Poincaré lemma, or its Sobolev version given in 3.1. Finally softness follows from the fact that $C^\infty(\wedge^i U)$ and $W^{d,p}_{\mathrm{loc}}(\wedge^i U)$ are closed under multiplication by C^∞-functions. Indeed, if $C \subset U$ with C closed and U open, one can multiply any ω in $C^\infty(\wedge^k U)$ or $W^{d,p}_{\mathrm{loc}}(\wedge^k U)$ with a suitable smooth cut-off function $\eta \in C^\infty(M)$ satisfying spt $\eta \subset U$ and hence obtain an extension $\eta\omega$ on M that equals ω on a neighborhood of C.

Hence, Theorem 3.3 applies for any family of supports Φ. We get the compactly supported version by having Φ be the family of compact subsets of M, and the version without supports by having Φ be the family of all closed subsets of M. □

3.1 Conformal Cohomology

Besides the above cohomology theories, we also need to use a conformal cohomology theory. There are several variations of conformal cohomology theories in use: see, e.g., [11, 14, 26]. Since we generally do not assume higher integrability from our maps, the best suited one for our current application is the one from [26]. It is the cohomology of the chain complex $W^d_{\mathrm{CE,loc}}(\wedge^* M)$ given by

$$\begin{aligned} W^d_{\mathrm{CE,loc}}(\wedge^0 M) &= \bigcup_{p<\infty} W^{d,p,n}_{\mathrm{loc}}(\wedge^0 M) \\ W^d_{\mathrm{CE,loc}}(\wedge^k M) &= W^{d,\frac{n}{k},\frac{n}{k+1}}_{\mathrm{loc}}(\wedge^k M) \quad \text{for } 1 \le k \le n-2, \\ W^d_{\mathrm{CE,loc}}(\wedge^{n-1} M) &= \bigcap_{p>1} W^{d,\frac{n}{n-1},p}_{\mathrm{loc}}(\wedge^{n-1} M), \quad \text{and} \\ W^d_{\mathrm{CE,loc}}(\wedge^n M) &= \bigcap_{p>1} L^p_{\mathrm{loc}}(\wedge^n M). \end{aligned}$$

The resulting conformal cohomology spaces are denoted $H^k_{\mathrm{CE}}(M)$. We also use $W^d_{\mathrm{CE},c}(\wedge^k M)$ to denote the corresponding compactly supported spaces, which are defined analogously by replacing the Sobolev spaces $W^{d,p,q}_{\mathrm{loc}}(\wedge^k M)$ with $W^{d,p,q}_c(\wedge^k M)$. The cohomology spaces of the chain complex $W^d_{\mathrm{CE},c}(\wedge^* M)$ are in turn denoted $H^k_{\mathrm{CE},c}(M)$.

There is also a version of Theorem 3.2 for conformal cohomology. The proof is exactly the same as that of Theorem 3.2, where the choice of exponents in the cohomology theory is exactly such that Lemma 3.1 still applies. Hence, we refrain from repeating the argument. Note that for the part of the result involving $H^k_{\mathrm{CE}}(M)$, a detailed explanation of the proof has been given in [26, Sect. 4]; the compactly supported version has,however,not been stated previously to our knowledge.

Theorem 3.4 *For every $k \in \mathbb{N}$, we have*

$$H^k_{CE}(M) \cong H^k_{dR}(M) \quad \text{and} \quad H^k_{CE,c}(M) \cong H^k_{dR,c}(M),$$

where the isomorphisms are induced by the inclusion maps $C^\infty(\wedge^* M) \hookrightarrow W^d_{CE,loc}(\wedge^* M)$ and $C^\infty_c(\wedge^* M) \hookrightarrow W^d_{CE,c}(\wedge^* M)$.

4 Proof of Homological Obstructions

In this section, we prove our main obstruction results: Theorem 1.3, and Corollaries 1.4 and 1.5. We begin by recalling the statement of Theorem 1.3.

Theorem 1.3 *Let $f : \Omega \to \Omega'$ be a proper, continuous, and monotone surjection in the Sobolev class $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, $n \ge 3$. Suppose that $k \in \{1, \dots, n-2\}$, and that*

$$K_f \in L^p_{\mathrm{loc}}(\Omega), \quad \textit{where } p = \begin{cases} \frac{n-(k+1)}{k+1}, & 1 \le k < \frac{n}{2}, \\ 1, & k = \frac{n}{2}, \\ \frac{k-1}{n-(k-1)}, & \frac{n}{2} < k \le n-2. \end{cases}$$

Then

$$H_k(f^{-1}\mathbb{B}^n(y,r);\mathbb{R}) = \{0\} \quad \textit{for every } \mathbb{B}^n(y,r) \Subset \Omega'.$$

For the proof, we split Theorem 1.3 into two sub-theorems: a homological result proven with compactly supported L^p-cohomology, and a cohomological result proven with L^p_{loc}-cohomology. The proofs of these two results are essentially identical. We begin with the homological result.

Lemma 4.1 *Let $f \in W^{1,n}(\Omega,\Omega')$ be a proper, continuous, monotone surjection between open domains. Suppose that $k \in \{1,\dots,n-2\}$, and that*

$$K_f^p \in L^1(\Omega), \quad \textit{where } p \geq \frac{k}{n-k} \textit{ and } p \geq \frac{n-(k+1)}{k+1}.$$

Then

$$H_k(f^{-1}\mathbb{B}^n(y,r);\mathbb{R}) = \{0\} \quad \textit{for every } \mathbb{B}^n(y,r) \Subset \Omega'.$$

Proof Suppose toward contradiction that $H_k(f^{-1}\mathbb{B}^n(y,r);\mathbb{R}) \neq \{0\}$ for $\mathbb{B}^n(y,r) \Subset \Omega'$. Since $f^{-1}\mathbb{B}^n(y,r)$ is an oriented manifold, we have by Poincaré duality that

$$H^{n-k}_{\text{dR},c}(f^{-1}\mathbb{B}^n(y,r)) \neq \{0\}$$

where $H^l_{\text{dR},c}$ stands for the l:th compactly supported smooth de Rham cohomology space as in Theorem 3.2. We may hence fix a smooth $(n-k)$-form $\omega \in C_c^\infty(\wedge^{n-k} f^{-1}\mathbb{B}^n(y,r))$ for which the class $[\omega]$ of ω in $H^{n-k}_{\text{dR},c}(f^{-1}\mathbb{B}^n(y,r))$ is non-zero, In particular, the L^1-case of Theorem 3.2 implies that ω is not a weak differential of any $\tau' \in W^{d,1}_c(\wedge^{n-k-1} f^{-1}\mathbb{B}^n(y,r))$.

We then consider the push-forward $f_*\omega$. By Lemma 2.5 combined with our assumption that $p \geq k/(n-k)$, we have that $f_*\omega \in W^d_{\text{CE},c}(\wedge^{n-k} B^n(y,r))$ and $\mathrm{d}f_*\omega = f_*\mathrm{d}\omega = 0$. It follows that $f_*\omega$ defines a cohomology class in $H^{n-k}_{\text{CE},c}(B^n(y,r))$. By Theorem 3.4, we know that $H^{n-k}_{\text{CE},c}(B^n(y,r)) = \{0\}$, and therefore $f_*\omega = \mathrm{d}\tau$ for some $\tau \in W^d_{\text{CE},c}(\wedge^{n-k-1} B^n(y,r))$.

Now, since we assumed that $k \leq n-2$, we have $W^d_{\text{CE},c}(\wedge^{n-k-1} B^n(y,r)) \subset L_c^{n/(n-k-1)}(\wedge^{n-k-1} B^n(y,r))$. Hence, Lemma 2.2 and the assumption that f is proper yield that

$$f^*\tau \in L^r_c(\wedge^{n-k-1} f^{-1} B^n(y,r)), \quad \text{where } r = \frac{n}{(n-k-1)(1+p^{-1})}.$$

Our assumption that $p \geq (n-k-1)/(k+1)$ can be re-arranged as $1 + p^{-1} \leq n/(n-k-1)$. Hence, $r \geq 1$, and it also follows from Lemma 2.6 that $\mathrm{d}f^*\tau = f^*\mathrm{d}\tau = f^*f_*\omega = \omega$. This contradicts the fact that ω is not a weak differential of any $\tau' \in W^{d,1}_c(\wedge^{n-k-1} f^{-1}\mathbb{B}^n(y,r))$. We hence conclude that $H_k(f^{-1}\mathbb{B}^n(y,r);\mathbb{R}) = \{0\}$, completing the proof. □

We then give the cohomological version of Lemma 4.1. Note that this version has different assumptions on the integrability of K_f.

Lemma 4.2 *Let $f \in W^{1,n}(\Omega, \Omega')$ be a proper, continuous, monotone surjection between open domains. Suppose that $k \in \{2, \dots, n-1\}$, and that*

$$K_f^p \in L^1(\Omega), \quad \textit{where } p \geq \frac{n-k}{k} \textit{ and } p \geq \frac{k-1}{n-(k-1)}.$$

Then

$$H^k(f^{-1}\mathbb{B}^n(y,r); \mathbb{R}) = \{0\} \quad \textit{for every } \mathbb{B}^n(y,r) \Subset \Omega'.$$

Proof The proof is essentially the same as that of Lemma 4.1. Indeed, we suppose toward contradiction that $H^k_{\mathrm{dR}}(f^{-1}\mathbb{B}^n(y,r)) \neq \{0\}$, and hence there exists a k-form $\omega \in C^\infty(\wedge^k f^{-1}B^n(y,r))$ with $[\omega] \neq [0]$. The change from an $(n-k)$-form to a k-form causes the changes in our assumptions on k and p. The result then follows by repeating the rest of the argument of Lemma 4.1 on ω, where compactly supported cohomology is replaced with the corresponding cohomology without compact supports, and integrability results are applied locally using the continuity of f. □

Now, Theorem 1.3 follows from Lemmas 4.1 and 4.2. We give the few remaining details.

Proof of Theorem 1.3 If $1 \leq k < n/2$, then $p = (n-k-1)/(k+1) > k/(n-k)$, and hence $H_k(f^{-1}\mathbb{B}^n(y,r); \mathbb{R}) = \{0\}$ by Lemma 4.1, which is the desired result. If on the other hand $n/2 < k \leq n-2$ (or if we are in the unnecessary case $k = n-1$), then we similarly have $p = (k-1)\,/\,(n-k+1) > (n-k)/k$, in which case $H^k(f^{-1}\mathbb{B}^n(y,r); \mathbb{R}) = \{0\}$ by Lemma 4.2. Since $H^k(f^{-1}\mathbb{B}^n(y,r); \mathbb{R}) \cong H_k(f^{-1}\mathbb{B}^n(y,r); \mathbb{R})$ by the universal coefficient theorem, we hence have our claim also in this case.

The final case is $k = n/2$. In this case, our definition also gives $p = 1 = k/(n-k) > (n-k-1)/(k+1)$, and therefore Lemma 4.1 yields the claim. □

We then recall the version of the result for fibers given in Corollary 1.4 and give the short proof.

Corollary 1.4 *Let $f : \Omega \to \Omega'$ be a proper, continuous, and monotone surjection in the Sobolev class $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, $n \geq 3$. Let $k \in \{1, \dots, n\}$. Moreover, if $k \leq n-2$, suppose also that*

$$K_f \in L^p_{\mathrm{loc}}(\Omega), \quad \textit{where } p = \begin{cases} \frac{n-(k+1)}{k+1}, & 1 \leq k < \frac{n}{2}, \\ 1, & k = \frac{n}{2}, \\ \frac{k-1}{n-(k-1)}, & \frac{n}{2} < k \leq n-2. \end{cases}$$

If $y \in \Omega'$ is such that $f^{-1}\{y\}$ is a neighborhood retract, then $H_k(f^{-1}\{y\}; \mathbb{R}) = \{0\}$.

Proof Let $U \subset \mathbb{R}^n$ be a neighborhood of $f^{-1}\{y\}$ and let $r\colon U \to f^{-1}\{y\}$ be a retraction. The sets $U_i = f^{-1}B^n(y, i^{-1})$ for large enough i form a sequence of precompact neighborhoods of $f^{-1}\{y\}$ with $\overline{U_{i+1}} \subset U_i$ and $\bigcap_i U_i = f^{-1}\{y\}$. It follows that $U_i \subset U$ for some i. Now, if $\iota^i\colon f^{-1}\{y\} \hookrightarrow U_i$ and $\kappa^i\colon U_i \hookrightarrow U$ are inclusions and $c \in H_k(f^{-1}\{y\}; \mathbb{R})$, then Theorem 1.3 yields $c = r_* {\kappa^i}_* {\iota^i}_* c = r_* {\kappa^i}_* 0 = 0$, which yields the claim. □

Remark 4.3 As pointed out in the introduction, if $K \subset \mathbb{R}^n$ is a compact set, it is possible that $H_k(K; \mathbb{R}) \neq \{0\}$ even if $K = \bigcap_i U_i$, where (U_i) is a decreasing sequence of topological balls. For an example of this, consider

$$K = \overline{\left\{(x, y, z) \in \mathbb{R}^3 : 0 < x^2 + y^2 \leq 1, z = \sin\left(\pi/\sqrt{x^2 + y^2}\right)\right\}}.$$

That is, K is the closed topologist's sine curve that has been revolved around the z-axis. The set K is compact and connected, though it is not path connected. Moreover, the sets $\mathbb{B}^n(K, r) = \{x \in \mathbb{R}^n : d(x, K) < r\}$ are homeomorphic to $\mathbb{B}^n$ for $r > 0$, and K is their intersection. However, the loop $S = \{(x, y, z) \in \mathbb{R}^3 : z = 0, x^2 + y^2 = 1\} \subset K$ induces a non-zero homology class in $H_1(K; \mathbb{R})$, despite being homologically trivial in every $\mathbb{B}^n(K, r)$.

We then recall the remaining result we prove in this section, Corollary 1.5.

Corollary 1.5 *Let $f\colon \Omega \to \Omega'$ be a proper, continuous, and monotone surjection in $W^{1,n}(\Omega, \mathbb{R}^n)$, $n \geq 3$. Suppose that $K_f \in L^{(n-2)/2}_{\mathrm{loc}}(\Omega)$. Then for every $y \in \Omega'$, the set $f^{-1}\{y\}$ is an intersection of a nested sequence of neighborhoods U_i that are rational homology balls; that is, the neighborhoods U_i satisfy $H_k(U_i; \mathbb{Q}) = H_k(\mathbb{B}^n; \mathbb{Q})$ for all $k \in \mathbb{Z}_{\geq 0}$.*

Corollary 1.5 immediately follows by combining Theorem 1.3 with the following topological result which treats the cases $k \in \{0, n-1, n\}$.

Proposition 4.4 *Let $f\colon \Omega \to \Omega'$ be a proper, continuous, monotone surjection between open domains in $\mathbb{R}^n$, $n \geq 2$. Then for every $\mathbb{B}^n(y, r) \Subset \Omega'$, we have*

$$\begin{aligned} H_0(f^{-1}\mathbb{B}^n(y, r); \mathbb{R}) &\cong \mathbb{R}, \\ H_{n-1}(f^{-1}\mathbb{B}^n(y, r); \mathbb{R}) &\cong \{0\}, \\ H_n(f^{-1}\mathbb{B}^n(y, r); \mathbb{R}) &\cong \{0\}. \end{aligned}$$

Proof The case $k = n$ is simply due to the fact that the n-homology of any non-compact manifold vanishes. The case $k = 0$ uses the fact that if $g\colon X \to Y$ is a continuous monotone surjection between compact spaces, then $g^{-1}C$ is connected for every connected $C \subset X$; see, e.g., [12, Corollary 6.1.19]. In particular, we apply this to the map $g =\colon f^{-1}\overline{\mathbb{B}^n(y, r)} \to \overline{\mathbb{B}^n(y, r)}$ defined by $g = f|_{f^{-1}\overline{\mathbb{B}^n(y,r)}}$ and to the set $C = \mathbb{B}^n(y, r)$, where the domain of g is compact since f is proper. We conclude that $f^{-1}\mathbb{B}^n(y, r)$ is connected, and since $f^{-1}\mathbb{B}^n(y, r)$ is an open subset of $\mathbb{R}^n$, it hence follows that it is path connected. Therefore, $H_0(f^{-1}\mathbb{B}^n(y, r); \mathbb{R}) \cong \mathbb{R}$.

For the remaining case $k = n - 1$, suppose toward contradiction that $H_{n-1}(f^{-1}\mathbb{B}^n(y, r); \mathbb{R}) \ncong \{0\}$. It then follows from Alexander duality that $\tilde{H}^0((\mathbb{R}^n \cup \{\infty\}) \setminus f^{-1}\mathbb{B}^n(y, r); \mathbb{R}) \ncong \{0\}$, where $\tilde{H}^*(X; \mathbb{R})$ denotes the reduced Čech cohomology of X with coefficients in $\mathbb{R}$. Now, if r' is such that $r < r' < d(y, \partial\Omega')$, the reduced Mayer–Vietoris-sequence for the sets $(\mathbb{R}^n \cup \{\infty\}) \setminus f^{-1}\mathbb{B}^n(y, r)$ and $f^{-1}\overline{\mathbb{B}^n(y, r')}$ implies the exactness of the sequence

$$\begin{aligned}
0 &\to \tilde{H}^0(\mathbb{R}^n \cup \{\infty\}; \mathbb{R}) \\
&\to \tilde{H}^0((\mathbb{R}^n \cup \{\infty\}) \setminus f^{-1}\mathbb{B}^n(y, r); \mathbb{R}) \oplus \tilde{H}^0(f^{-1}\overline{\mathbb{B}^n(y, r')}; \mathbb{R}) \\
&\to \tilde{H}^0(f^{-1}(\overline{\mathbb{B}^n(y, r')} \setminus \mathbb{B}^n(y, r)); \mathbb{R}) \to \tilde{H}^1(\mathbb{R}^n \cup \{\infty\}; \mathbb{R}) \to \dots .
\end{aligned}$$

Since $\tilde{H}^1(\mathbb{R}^n \cup \{\infty\}; \mathbb{R}) \cong \tilde{H}^0(\mathbb{R}^n \cup \{\infty\}; \mathbb{R}) \cong \{0\}$ and since our counterassumption implies that $\tilde{H}^0((\mathbb{R}^n \cup \{\infty\}) \setminus f^{-1}\mathbb{B}^n(y, r); \mathbb{R}) \ncong \{0\}$, we have $\tilde{H}^0(f^{-1}(\overline{\mathbb{B}^n(y, r')} \setminus \mathbb{B}^n(y, r)); \mathbb{R}) \ncong \{0\}$. Since the 0:th Čech cohomology counts quasicomponents, and since quasicomponents are unions of ordinary connected components, it follows that $f^{-1}(\overline{\mathbb{B}^n(y, r')} \setminus \mathbb{B}^n(y, r))$ is disconnected. This is a contradiction, since $f^{-1}(\overline{\mathbb{B}^n(y, r')} \setminus \mathbb{B}^n(y, r))$ is connected due to the aforementioned result [12, Corollary 6.1.19]. □

With Theorem 1.3 and Proposition 4.4 proven, Corollary 1.5 hence follows.

5 The Example with Circular Fibers

We begin by recalling the statement of Theorem 1.2.

Theorem 1.2 *There exists a topologically monotone, proper, and surjective mapping $h: \mathbb{R}^3 \to \mathbb{R}^3$ of finite distortion such that $h^{-1}\{0\}$ is bilipschitz equivalent with $\mathbb{S}^1$. The map h is locally Lipschitz, and $K_h \in L^p_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ for every $p < 1/2$, but $K_h \notin L^{1/2}_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$.*

More precisely, if we use e_x, e_y and e_z to denote the standard basis of $\mathbb{R}^3$, then $h^{-1}\{0\}$ and $h^{-1}\{-e_x\}$ are bilipschitz equivalent with $\mathbb{S}^1$, $h^{-1}\{-te_x\}$ for $t \in (0, 1)$ are bilipschitz equivalent with $\mathbb{S}^1 \vee \mathbb{S}^1$, $h^{-1}\{-te_x\}$ for $t \in (1, \infty)$ are bilipschitz equivalent with $[0, 1]$, and every remaining fiber $h^{-1}\{w\}$, $w \in \mathbb{R}^3 \setminus \{-te_x : t \geq 0\}$ is a point.

We then begin the construction of the map h as above. We use cylindrical coordinates (r, θ, z) on the domain side, where $r \geq 0$ and $\theta \in (-\pi, \pi]$. On the target side, we use standard Euclidean coordinates (x, y, z). We also use $\operatorname{sgn} t = t/|t|$ to denote the sign of a real number $t \in \mathbb{R}$, with $\operatorname{sgn}(0) = 0$.

We partition the domain into a family of square torii T_c, $c \in [0, \infty)$, defined by

$$T_c = \{(r, \theta, z) \in \mathbb{R}^3 : |r - 1| + |z| = c\}.$$

For $c = 0$, T_c is the circle defined by $r = 1$ and $z = 0$. For $c \in (0, 1)$, T_c is a sharp-cornered topological torus. For $c = 1$, the hole in the center of the torus gets closed, and as c increases above 1, the surface becomes topologically $\mathbb{S}^2$.

Fig. 2 The set T_0 is just the unit circle in the xy-plane. The map h collapses it to the origin

For $c \in [0, 1]$, we map the slices $T_{c,\theta} = T_c \cap \{(r, \theta, z) \in \mathbb{R}^3 : r \in [0, \infty), z \in \mathbb{R}\}$ by the composition of three maps. First, we define a map $I_{c,\theta} : T_{c,\theta} \to \mathbb{R}^2$ which places the square $T_{c,\theta}$ into the xy-plane, centered at the origin. More precisely, the map $I_{c,\theta}$ is defined by

$$I_{c,\theta}(r, \theta, z) = (r - 1)e_x + ze_y.$$

For the second map, we let $S_{c,t} : \mathbb{R}^2 \to \mathbb{R}^2$ denote scaling by $t \in \mathbb{R}$, where the center of scaling is at the point $-ce_x$, $c \in \mathbb{R}$, which is where $I_{c,\theta}$ takes the tip of the square. This map is given by

$$S_{c,t}(xe_x + ye_y) = (t(x + c) - c)e_x + tye_y.$$

The third map $L_{c,t} : \mathbb{R}^2 \to \mathbb{R}^3$ then shifts the xy-plane in the z-direction so that the slope of the resulting plane is $t \in \mathbb{R}$, where the line $\{(-c, y) : y \in \mathbb{R}\}$ remains fixed in the xy-plane. The map is given by

$$L_{c,t}(xe_x + ye_y) = xe_x + ye_y + t(x + c)e_z.$$

Now, we may define h on $T_{c,\theta}$ by

$$h(r, \theta, z) = L_{c, \frac{(|\theta|-\pi)\,\mathrm{sgn}(\theta)}{\pi}} \circ S_{c, \frac{|\theta|}{\pi}} \circ I_{c,\theta}(r, \theta, z). \tag{5.1}$$

By setting $c = |r - 1| + |z|$ in (5.1), this defines h in the region of $\mathbb{R}^3$ with $|r - 1| + |z| \le 1$. It is reasonably easy to see that h is continuous, since the discontinuity at $\theta = 0$ introduced by the $\mathrm{sgn}(\theta)$-term in the $L_{c,t}$-part is eliminated by the fact that the scaling factor $|\theta|/\pi$ of the $S_{c,t}$-part is zero for $\theta = 0$. By a straightforward computation, we get an explicit formula for h in the region $|r - 1| + |z| \le 1$, given by

$$\begin{aligned} h(r, \theta, z) = &\left(\frac{|\theta|}{\pi} (r - 1 + |r - 1| + |z|) - (|r - 1| + |z|) \right) e_x \\ &+ \frac{|\theta|}{\pi} ze_y + \frac{(\pi - |\theta|)\theta}{\pi^2} (r - 1 + |r - 1| + |z|)e_z. \end{aligned} \tag{5.2}$$

See the following Figs. 2, 3 and 4 for an illustration of the resulting map h.

For $c > 1$, the slice $T_{c,\theta}$ is no longer a complete square, but instead gets cut-off at the z-axis. Hence, we modify $T_{c,\theta}$ into a square $T'_{c,\theta}$. We do this by uniformly scaling the two cut-off sides of $T_{c,\theta}$; see Fig. 5 for an illustration. More formally, in the region where $c = |r - 1| + |z| > 1$ and $r \ge 1$, the definition of h remains unchanged from

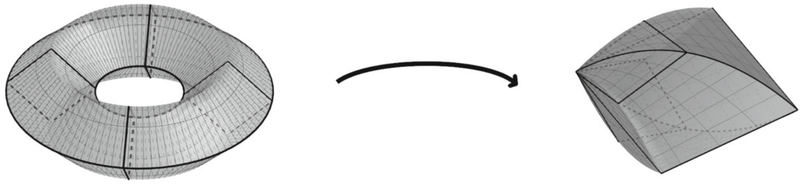


Fig. 3 For $0 < c < 1$, the set T_c is a square torus around the circle T_0. It is mapped into a surface centered at the origin, with the size of the image increasing with c. The inner ring of the torus and one of the square cross-sections get mapped to the single point at the tip of the surface. In the construction of h, the $S_{c,t}$-part is responsible for the shrinking of the square slices close to the tip, while the $L_{c,t}$-part is responsible for the increasingly steep slopes of the square slices close to the tip

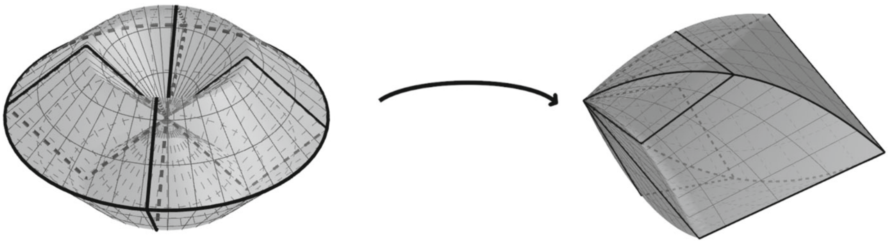


Fig. 4 For $c = 1$, the torus T_1 gets closed in the middle. However, our previous process of defining h remains valid, since the inner ring of the torus was mapped to a single point

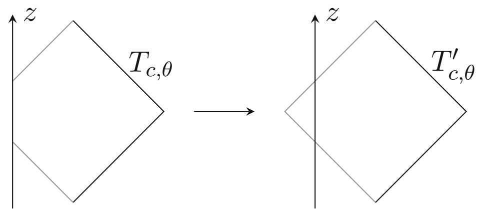


Fig. 5 How the cross-section sets $T_{c,\theta}$ are converted into squares $T'_{c,\theta}$ for $c > 1$. The gray part is scaled linearly, while the black part remains unchanged

(5.1) and (5.2). However, in the region where $c = |r - 1| + |z| > 1$ and $r \leq 1$, the previous definition (5.1) is precomposed with the map $s_c \colon [0, 1] \times (-\pi, \pi] \times \mathbb{R} \to [1 - c, 1] \times (-\pi, \pi] \times \mathbb{R}$ defined by

$$s_c(r, \theta, z) = (1 - c + cr, \theta, cr \operatorname{sgn}(z)),$$

where notably $|(1 - c + cr) - 1| + |cr \operatorname{sgn} z| = c$ when $0 \leq r \leq 1 < c$. That is, in the region defined by $c = |r - 1| + |z| > 1$ and $r \leq 1$, we have

$$h(r, \theta, z) = L_{c, \frac{(|\theta| - \pi) \operatorname{sgn}(\theta)}{\pi}} \circ S_{c, \frac{|\theta|}{\pi}} \circ I_{c,\theta} \circ s_c(r, \theta, z), \tag{5.3}$$

where $c = |r - 1| + |z| = 1 - r + |z|$. The resulting map h for $c > 1$ is shown in Fig. 6.

In the region where $|r - 1| + |z| > 1$ and $r \leq 1$, we can again obtain an explicit formula for the resulting map h by computing the composition (5.3) with $c = 1 - r + |z|$.

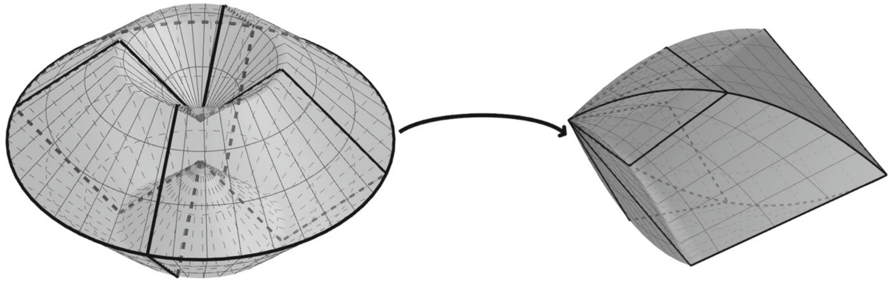
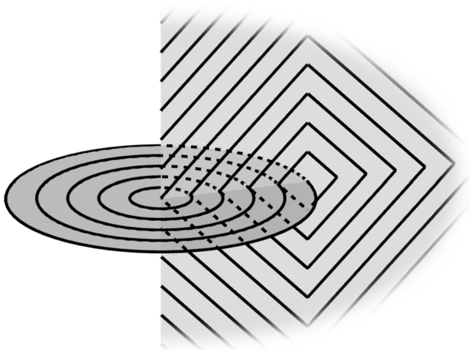


Fig. 6 The resulting map on T_c for $c > 1$. Now only one of the cut-off squares gets mapped to the tip on the image side

Fig. 7 The set where the map h is not a homeomorphism, with some of the fibers illustrated. The fibers $h^{-1}\{-ce_x\}$ with $0 < c < 1$ are figure-eights that interpolate between two linked loops. For $c > 1$, the fibers stop at the z-axis, and are hence topologically equivalent to a line segment



The resulting formula is

$$h(r, \theta, z) = \left(\frac{|\theta|}{\pi} r - 1\right)(|z| - r + 1)e_x \\ + \frac{|\theta|}{\pi} r(|z| - r + 1)\operatorname{sgn}(z)e_y + \frac{(\pi - |\theta|)\theta}{\pi^2} r(|z| - r + 1)e_z. \quad (5.4)$$

Hence, our map h is now defined on all of $\mathbb{R}^3$. Notably, the formulas (5.2) and (5.4) which define h are relatively simple polynomials in r, θ and z. This is our reason for using square torii, as well as the reason for our specific choice of shape for the sets $h(T_c)$. Indeed, having polynomial formulas for h vastly simplifies the computations required for the proof of Theorem 1.2.

We note that the set of points B_h where $h^{-1}\{h(x)\} \neq \{x\}$ consists of the disk $\{(r, \theta, z) : z = 0, r \leq 1\}$ combined with the half-plane $\{(r, \theta, z) : \theta = 0, r \geq 0\}$. An illustration of the non-trivial fibers of h is given in Fig. 7.

We then verify that our map h satisfies the required conditions.

Proof of Theorem 1.2 As stated above, our map h is given by (5.4) if $r \leq \min(1, |z|)$ and by (5.2) if $r > \min(1, |z|)$, where we assume $r \geq 0$ and $\theta \in (-\pi, \pi]$. It is clear from the geometry of the construction of h that h is a continuous surjection, that the fibers of h are as specified, and that h is hence topologically monotone.

It is clear from the formulas (5.2) and (5.4) that the coordinate functions h_x, h_y, and h_z of h are absolutely continuous on every line of the type $\{(r_0, \theta_0, z) : z \in \mathbb{R}\}$, $\{(r_0, \theta, z_0) : \theta \in [-\pi, \pi]\}$, and $\{(r, \theta_0, z_0) : r \in [0, \infty)\}$. Hence, the partial derivatives $\partial_r(h_x, h_y, h_z)$, $\partial_\theta(h_x, h_y, h_z)$, and $\partial_z(h_x, h_y, h_z)$ exist for almost all r, θ and z. We may also easily compute the partial derivatives from (5.2) and (5.4); for $r \geq \min(1, |z|)$, they are given by

$$\begin{aligned} &D^{x,y,z}_{r,\theta,z} h(r,\theta,z) \\ &= \begin{bmatrix} \partial_r h_x(r,\theta,z) & \partial_\theta h_x(r,\theta,z) & \partial_z h_x(r,\theta,z) \\ \partial_r h_y(r,\theta,z) & \partial_\theta h_y(r,\theta,z) & \partial_z h_y(r,\theta,z) \\ \partial_r h_z(r,\theta,z) & \partial_\theta h_z(r,\theta,z) & \partial_z h_z(r,\theta,z) \end{bmatrix} \\ &= \begin{bmatrix} \frac{|\theta|}{\pi} + \left(\frac{|\theta|}{\pi} - 1\right)\operatorname{sgn}(r-1) & \frac{\operatorname{sgn}(\theta)}{\pi}(r - 1 + |r-1| + |z|) & \left(\frac{|\theta|}{\pi} - 1\right)\operatorname{sgn}(z) \\ 0 & \frac{\operatorname{sgn}(\theta)}{\pi} z & \frac{|\theta|}{\pi} \\ \frac{(\pi - |\theta|)\theta}{\pi^2}(\operatorname{sgn}(r-1) + 1) & \frac{\pi - 2|\theta|}{\pi^2}(r - 1 + |r-1| + |z|) & \frac{(\pi - |\theta|)\theta}{\pi^2}\operatorname{sgn}(z) \end{bmatrix}, \end{aligned} \tag{5.5}$$

and for $r \leq \min(1, |z|)$, they are given by

$$\begin{aligned} &D^{x,y,z}_{r,\theta,z} h(r,\theta,z) \\ &= \begin{bmatrix} \frac{|\theta|}{\pi}(|z| - 2r + 1) + 1 & \frac{\operatorname{sgn}(\theta)}{\pi} r(|z| - r + 1) & \left(\frac{|\theta|}{\pi} r - 1\right)\operatorname{sgn}(z) \\ \frac{|\theta|}{\pi}(|z| - 2r + 1)\operatorname{sgn}(z) & \frac{\operatorname{sgn}(\theta)}{\pi} r(|z| - r + 1)\operatorname{sgn}(z) & \frac{|\theta|}{\pi} r \\ \frac{(\pi - |\theta|)\theta}{\pi^2}(|z| - 2r + 1) & \frac{\pi - 2|\theta|}{\pi^2} r(|z| - r + 1) & \frac{(\pi - |\theta|)\theta}{\pi^2} r \operatorname{sgn}(z) \end{bmatrix}. \end{aligned} \tag{5.6}$$

We then observe that $\partial_r h$, $\partial_z h$, and $r^{-1}\partial_\theta h$ are locally essentially bounded. Indeed, the only one for which this is not entirely obvious from (5.5) and (5.6) is $r^{-1}\partial_\theta h$. However, in the first case $r \leq \min(1, |z|)$,we have a common factor r in $\partial_\theta h$, in the second case $|z| \leq r \leq 1$,we have $r - 1 + |r - 1| = 0$ and hence

$$\begin{aligned} |\partial_\theta h| &= \left| \frac{\operatorname{sgn}(\theta)}{\pi} |z|\, e_x + \frac{\operatorname{sgn}(\theta)}{\pi} z e_y + \frac{\pi - 2|\theta|}{\pi^2} |z|\, e_z \right| \\ &\leq \left(2 \left| \frac{\operatorname{sgn}(\theta)}{\pi} \right| + \left| \frac{\pi - 2|\theta|}{\pi^2} \right| \right) |z| \leq \frac{3}{\pi} |z| \leq \frac{3}{\pi} r, \end{aligned}$$

and in the final case $r \geq 1$ the coefficient r^{-1} in $r^{-1}\partial_\theta h$ is bounded from above by 1. Now, since $\partial_r h$, $\partial_z h$, and $r^{-1}\partial_\theta h$ are locally essentially bounded, and since h is absolutely continuous on every cylindrical coordinate curve in $\mathbb{R}^3$ of the types $\{(r, \theta_0, z_0) : r \geq 0\}$, $\{(r_0, \theta, z_0) : \theta \in \mathbb{R}\}$ and $\{(r_0, \theta_0, z) : z \in \mathbb{R}\}$, it follows from a standard path integral estimate argument that h is locally Lipschitz.

It now remains to compute the Jacobian J_h of h. Note that we need an extra r^{-1}-term in front of the determinant of $D^{x,y,z}_{r,\theta,z} h$ to get the standard Jacobian, since $\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}z = r^{-1}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$. We split to the three cases $|z| \leq r \leq 1$, $r \leq \min(1, |z|)$, and $r \geq 1$.

In the case $|z| \leq r \leq 1$, we easily compute using (5.5) that

$$\begin{aligned} J_h(r,\theta,z) &= \frac{1}{r}\det\left(D^{x,y,z}_{r,\theta,z} h(r,\theta,z)\right) \\ &= \frac{1}{r}\det\begin{bmatrix} 1 & \frac{\operatorname{sgn}(\theta)}{\pi}|z| & \left(\frac{|\theta|}{\pi}-1\right)\operatorname{sgn}(z) \\ 0 & \frac{\operatorname{sgn}(\theta)}{\pi} z & \frac{|\theta|}{\pi} \\ 0 & \frac{\pi-2|\theta|}{\pi^2}|z| & \frac{(\pi-|\theta|)\theta}{\pi^2}\operatorname{sgn}(z) \end{bmatrix} \\ &= \frac{|z|}{r}\frac{|\theta|^2}{\pi^3}. \end{aligned}$$

In the case $r \leq \min(1, |z|)$, we similarly get J_h by dividing the determinant of (5.6) by r. Even though the matrix appears complicated, large parts of the first and third column are multiples of each other, leading to a great degree of simplification with the relatively tidy result

$$\begin{aligned} J_h(r,\theta,z) &= \frac{1}{r}\det\left(D^{x,y,z}_{r,\theta,z} h(r,\theta,z)\right) \\ &= \frac{(1+|z|-r)^2\,|\theta|^2}{\pi^3}. \end{aligned}$$

The remaining case $r \geq 1$, computed using (5.5), results in the least simplifications during the computation of J_h. Namely, the result in this case is

$$\begin{aligned} J_h(r,\theta,z) &= \frac{1}{r}\det\begin{bmatrix} 2\frac{|\theta|}{\pi}-1 & \frac{\operatorname{sgn}(\theta)}{\pi}(2r-2+|z|) & \left(\frac{|\theta|}{\pi}-1\right)\operatorname{sgn}(z) \\ 0 & \frac{\operatorname{sgn}(\theta)}{\pi} z & \frac{|\theta|}{\pi} \\ 2\frac{(\pi-|\theta|)\theta}{\pi^2} & \frac{\pi-2|\theta|}{\pi^2}(2r-2+|z|) & \frac{(\pi-|\theta|)\theta}{\pi^2}\operatorname{sgn}(z) \end{bmatrix} \\ &= \frac{|\theta|}{\pi^4 r}\left((4|\theta|^2-4\pi|\theta|+2\pi^2)(r-1)+(2|\theta|^2-3\pi|\theta|+2\pi^2)\,|z|\right). \end{aligned}$$

From the computed values of J_h, we see that $J_h > 0$ a.e. in $\mathbb{R}^3$; the fact that J_h does not change sign was also to be expected by the monotonicity of h. We conclude that h is a mapping of finite distortion.

Since h is locally Lipschitz, we obtain that $K_h \leq C J_h^{-1}$ a.e. locally. In the region $|z| \leq r \leq 1$,we have $J_h^{-1} = \pi^3 r\,|z|^{-1}\,|\theta|^{-2}$, which is locally L^p-integrable for $p < 1/2$. In the region $r \leq \min(1, |z|)$, we estimate by the arithmetic-geometric mean inequality that

$$J_h^{-1} = \pi^3\,|\theta|^{-2}\,((1-r)+|z|)^{-2} \leq \frac{\pi^3}{4}\,|\theta|^{-2}\,(1-r)^{-1}\,|z|^{-1},$$

where the upper bound is also clearly locally L^p-integrable for $p < 1/2$. Moreover, in the case $r \geq 1$, we can similarly estimate

$$J_h^{-1} \leq \pi^4 r\, |\theta|^{-1} \left(\pi^2(r-1) + \frac{7\pi^2}{8} |z| \right)^{-1} \leq \sqrt{\frac{2}{7}} \pi^2 r\, |\theta|^{-1} (r-1)^{-\frac{1}{2}} |z|^{-\frac{1}{2}},$$

where the upper bound is locally L^p-integrable for all $p < 1$. We conclude that $K_h \in L^p_{\text{loc}}(\mathbb{R}^3)$ for $p < 1/2$. Moreover, in the region $|z| \leq r \leq 1$,we have $\|Dh\| \geq |\partial_r h| = 1$, and J_h^{-1} is not locally $L^{1/2}$-integrable in this region near the plane $\{\theta = 0\}$. Hence, $K_h \notin L^{1/2}_{\text{loc}}(\mathbb{R}^3)$. □

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