

Volume of convex polytopes equals mixed volume of simplices

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Abstract. We provide a simple proof for the equality between the normalized volume of a convex polytope with m vertices and the mixed volume of m simplices and thus show the seemingly restrictive problem of computing mixed volume of simplices is still at least as hard as computing volumes of convex polytopes in general.

1. Introduction

Let V and MV denote the normalized volume and mixed volume functions for convex polytopes respectively. It is well known that for an n -simplex with vertices $\mathbf{p}_1, \dots, \mathbf{p}_{n+1} \in \mathbb{R}^n$,

$$\begin{aligned} V(\text{conv}\{\mathbf{p}_1, \dots, \mathbf{p}_{n+1}\}) &= \left| \det \begin{bmatrix} 1 & \cdots & 1 \\ \mathbf{p}_1 & \cdots & \mathbf{p}_{n+1} \end{bmatrix} \right| \\ &= MV(\{\hat{\mathbf{p}}_1, \mathbf{e}_{n+1}\}, \dots, \{\hat{\mathbf{p}}_{n+1}, \mathbf{e}_{n+1}\}), \end{aligned}$$

where the $\mathbf{p}_i$'s are expressed as column vectors, each $\hat{\mathbf{p}}_i$ is the embedding of $\mathbf{p}_i$ into $\mathbb{R}^{n+1} \times \{\mathbf{0}\} \subset \mathbb{R}^m$, and $\mathbf{e}_{n+1} \in \mathbb{R}^{n+1}$ is the $(n+1)$ -th standard basis vector. This equation tells us that the normalized volume of an n -simplex in $\mathbb{R}^n$ can be turned into the mixed volume of $n+1$ line segments in $\mathbb{R}^{n+1}$. Using root counting results from algebraic geometry, we provide a simple proof for the generalization of this observation to the case of convex hull of m points $\mathbf{p}_1, \dots, \mathbf{p}_m$ in $\mathbb{R}^n$ for $m > n$:

$$V(\text{conv}\{\mathbf{p}_1, \dots, \mathbf{p}_m\}) = MV(\Delta(\mathbf{p}_1), \dots, \Delta(\mathbf{p}_m))$$

where each $\Delta(\mathbf{p}_i)$ is a simplex of dimension $m - n$. From the view point of complexity, this equation shows that the seemingly specialized problem of computing the mixed volume of only simplices is still at least as hard as the general problem of computing the volume of a convex polytope.

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2. Notations and preliminaries

$\mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}$ are the sets of complex numbers, reals, rationals, and integers respectively. $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. For a ring R , $M_{m \times n}(R)$ is the set of $m \times n$ matrices with entries in R .

For a set $S \subseteq \mathbb{R}^n$, $\text{conv}(S)$ is the convex hull of S , i.e., the smallest convex set containing S . A *convex polytope* in $\mathbb{R}^n$ is a subset P such that $P = \text{conv}\{\mathbf{p}_1, \dots, \mathbf{p}_m\}$ for some $\mathbf{p}_1, \dots, \mathbf{p}_m \in \mathbb{R}^n$. Its dimension $\dim(P)$ is the dimension of the smallest affine space containing it, and its *normalized volume* $V(P)$ is $n! \text{Vol}_n(P)$ where Vol_n is the Euclidean volume form for $\mathbb{R}^n$. Note that this definition is not restricted to lattice polytopes and implies the convention that $V(P) = 0$ if $\dim(P) < n$.

For $A, B \subset \mathbb{R}^n$, their *Minkowski sum* is $A + B = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A, \mathbf{b} \in B\}$. For convex polytopes $P_1, \dots, P_n \subset \mathbb{R}^n$ the volume of the Minkowski sum $\lambda_1 P_1 + \dots + \lambda_n P_n$ under the scaling by positive factors $\lambda_1, \dots, \lambda_n$, is a homogeneous polynomial in $\lambda_1, \dots, \lambda_n$ [5]. The coefficient of the term $\lambda_1 \dots \lambda_n$ is known as the *mixed volume* of these polytopes, denoted $\text{MV}(P_1, \dots, P_n)$.

For $\mathbf{x} = [x_1 \dots x_n]$ and $\mathbf{a} = [a_1 \dots a_n]^\top \in \mathbb{Z}^n$, $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \dots x_n^{a_n}$. For $A \in M_{n \times m}(\mathbb{Z})$ with columns $\mathbf{a}_1, \dots, \mathbf{a}_m \in \mathbb{Z}^n$, $\mathbf{x}^A = [\mathbf{x}^{\mathbf{a}_1} \dots \mathbf{x}^{\mathbf{a}_m}]$, which defines functions over $(\mathbb{C}^*)^n$. A *Laurent polynomial* is an expression of the form $f(\mathbf{x}) = \sum_{\mathbf{a} \in S} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ where each $c_{\mathbf{a}} \in \mathbb{C}^*$, $S \subset \mathbb{Z}^n$ is the *support* of f , denoted $\text{supp}(f)$, and $\text{newt}(f) := \text{conv}(S)$ is its *Newton polytope*. For a system of Laurent polynomials $F = (f_1, \dots, f_n)$ in $x_1, \dots, x_n$, its set of common isolated zeros in $(\mathbb{C}^*)^n$ is denoted $\mathcal{Z}_0^*(F)$, and the following theorems provide us bound on $|\mathcal{Z}_0^*(F)|$.

Theorem 1 (Kushnirenko [4]). *If $P = \text{newt}(f_i)$ for all i , $|\mathcal{Z}_0^*(F)| \leq V(P)$.*

Theorem 2 (Bernshtein [1]). $|\mathcal{Z}_0^*(F)| \leq \text{MV}(\text{newt}(f_1), \dots, \text{newt}(f_n))$.

Moreover, for generic choices of the coefficients, both bounds are exact. This exactness can be tested via “initial systems”. For a Laurent polynomial $f(\mathbf{x}) = \sum_{\mathbf{a} \in S} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ and $\boldsymbol{\alpha} \in \mathbb{R}^n$, $\text{init}_{\boldsymbol{\alpha}}(f)(\mathbf{x}) = \sum_{\mathbf{a} \in (S)_{\boldsymbol{\alpha}}} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$, where $(S)_{\boldsymbol{\alpha}}$ is the subset of S on which the linear functional $\langle \boldsymbol{\alpha}, \cdot \rangle$ is minimized. For a Laurent polynomial system $F = (f_1, \dots, f_n)$ in $x_1, \dots, x_n$, its *initial system* $\text{init}_{\boldsymbol{\alpha}}(F)$ with respect to a vector $\boldsymbol{\alpha} \in \mathbb{R}^n$, is $(\text{init}_{\boldsymbol{\alpha}}(f_1), \dots, \text{init}_{\boldsymbol{\alpha}}(f_n))$.

Theorem 3 (Bernshtein [1]). *If for all $\mathbf{0} \neq \boldsymbol{\alpha} \in \mathbb{R}^n$, $\text{init}_{\boldsymbol{\alpha}}(F)$ has no $\mathbb{C}^*$ -zero, then all zeros of F in $(\mathbb{C}^*)^n$ are isolated and their total number (counting multiplicity) is $\text{MV}(\text{newt}(f_1), \dots, \text{newt}(f_n))$. The converse is also true.*

Here, a $\mathbb{C}^*$ -zero is simply a complex zero for which all coordinates are nonzero. We avoid defining “multiplicity” as only zeros of multiplicity 1 (non-singular zeros, a.k.a. regular zeros) are considered in this paper.

3. The main result

Using the root counting theorems listed above, collectively known as the theory of BKK bound [1, 3, 4] we now state and prove the main result.

Theorem 4. For m distinct points $\{\mathbf{p}_1, \dots, \mathbf{p}_m\} \subset \mathbb{R}^n$, with $m > n > 0$,

$$V(\text{conv}\{\mathbf{p}_1, \dots, \mathbf{p}_m\}) = \text{MV}(\Delta(\mathbf{p}_1), \dots, \Delta(\mathbf{p}_m))$$

where

$$\Delta(\mathbf{p}) = \text{conv}\{\hat{\mathbf{p}}, \mathbf{e}_{n+1}, \dots, \mathbf{e}_m\} \quad \text{and} \quad \hat{\mathbf{p}} = \begin{bmatrix} \mathbf{p} \\ \mathbf{0}_{m-n} \end{bmatrix} \in \mathbb{R}^m.$$

Here, $\Delta(\mathbf{p})$ is a simplex of dimension $m - n$. So the equation above states that the normalized volume of the convex hull of m points in $\mathbb{R}^n$ can always be expressed as the mixed volume of m simplices in $\mathbb{R}^m$. Also recall the convention that $V(X) = 0$ if and only if X is not full-dimensional.

Proof. In the trivial case where $\dim(\text{conv}\{\mathbf{p}_1, \dots, \mathbf{p}_m\}) < n$, by our convention, $V(\text{conv}\{\mathbf{p}_1, \dots, \mathbf{p}_m\}) = 0$. The simplices $\Delta(\mathbf{p}_i)$ for $i = 1, \dots, m$ are all contained in a common affine hyperplane in $\mathbb{R}^m$, and thus $\text{MV}(\Delta(\mathbf{p}_1), \dots, \Delta(\mathbf{p}_m))$ is also zero. Conversely, if $\text{MV}(\Delta(\mathbf{p}_1), \dots, \Delta(\mathbf{p}_m)) = 0$, then there is no m -tuple of line segments $(L_1, \dots, L_m)$ with $L_i \subset \Delta(\mathbf{p}_i)$ for $i = 1, \dots, m$ that represent linearly independent vectors [6, Theorem 5.1.7]. In that case, $\{\mathbf{p}_1, \dots, \mathbf{p}_m\}$ must be affinely dependent, so $V(\text{conv}\{\mathbf{p}_1, \dots, \mathbf{p}_m\}) = 0$.

For cases where $\text{conv}\{\mathbf{p}_1, \dots, \mathbf{p}_m\}$ is full-dimensional, we first assume $\{\mathbf{p}_1, \dots, \mathbf{p}_m\} \subset \mathbb{Z}^n$. Let

$$P = [\mathbf{p}_1 \quad \dots \quad \mathbf{p}_m] \in M_{n \times m}(\mathbb{Z}).$$

For a point $\mathbf{x} = [x_1 \quad \dots \quad x_n] \in (\mathbb{C}^*)^n$, let $\mathbf{x}^P$ denotes its image $[\mathbf{x}^{\mathbf{p}_1} \quad \dots \quad \mathbf{x}^{\mathbf{p}_m}]$ under the monomial map determined by $\mathbf{p}_1, \dots, \mathbf{p}_m$ (expressed as a row vector). For a generic choice of $A \in M_{n \times m}(\mathbb{C})$, we consider the square system F of n Laurent polynomials in the n variables $x_1, \dots, x_n$, given by

$$F(x_1, \dots, x_n) = A(\mathbf{x}^P)^\top,$$

where $\mathbf{x} = [x_1 \quad \dots \quad x_n]$. By the genericity assumption, the zero set of F in $(\mathbb{C}^*)^n$ consists of nonsingular isolated points, and, by Kushnirenko's Theorem (Theorem 1), $|\mathcal{Z}_0^*(F)| = V(\text{conv}\{\mathbf{p}_1, \dots, \mathbf{p}_m\})$.

Since $m > n$, and the choice of $A \in M_{n \times m}(\mathbb{C})$ is generic, the nullity of A is exactly $d := m - n > 0$. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$ be a basis of the null space of A and $K = [\mathbf{v}_1 \quad \dots \quad \mathbf{v}_d] \in M_{m \times d}(\mathbb{C})$, then,

$$F(x_1, \dots, x_n) = A(\mathbf{x}^P)^\top = \mathbf{0} \quad \text{if and only if} \quad (\mathbf{x}^P)^\top = K(\mathbf{y})^\top$$

for some $\mathbf{y} = [y_1 \quad \dots \quad y_d] \in \mathbb{C}^d$. Moreover, since $\mathcal{Z}_0^*(F)$ is finite, without loss of generality, we can choose the basis $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$ so that $\mathbf{y} \in (\mathbb{C}^*)^d$ for every corresponding $\mathbf{x} \in \mathcal{Z}_0^*(F)$. Define the Laurent polynomial system $G = (g_1, \dots, g_m) : (\mathbb{C}^*)^n \times (\mathbb{C}^*)^d \rightarrow \mathbb{C}^m$, given by

$$G(x_1, \dots, x_n, y_1, \dots, y_d) = (\mathbf{x}^P)^\top - K(\mathbf{y})^\top,$$

where $\mathbf{x} = [x_1 \quad \dots \quad x_n]$ and $\mathbf{y} = [y_1 \quad \dots \quad y_d]$ as above. Then G is a system of m nonzero Laurent polynomials in the m variables $x_1, \dots, x_n, y_1, \dots, y_d$, and there is a bijection between $\mathcal{Z}_0^*(F)$ and $\mathcal{Z}_0^*(G)$. Moreover, under the genericity assumption, this map preserves multiplicity.

The Newton polytopes of G are exactly the simplexes $(\Delta(\mathbf{p}_1), \dots, \Delta(\mathbf{p}_m))$. Therefore, by Theorem 2

$$|\mathcal{Z}_0^*(G)| \leq \text{MV}(\Delta(\mathbf{p}_1), \dots, \Delta(\mathbf{p}_m)).$$

To establish equality, we shall show the system G satisfies the conditions in Theorem 3, even though there are algebraic relations among its coefficients. That is, we aim to show that for any nonzero vector $\hat{\alpha} \in \mathbb{R}^m$, the initial system $\text{init}_{\hat{\alpha}}(G) = (\text{init}_{\hat{\alpha}}(g_1), \dots, \text{init}_{\hat{\alpha}}(g_m))$ has no common $\mathbb{C}^*$ -zeros.

Fix a nonzero $\hat{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$. We define $\alpha = (\alpha_1, \dots, \alpha_n)$, $\ell = \min\{\alpha_{n+j} \mid j = 1, \dots, d\}$ and $L = \{j \in \{1, \dots, d\} \mid \alpha_{n+j} = \ell\}$, then

$$\text{init}_{\hat{\alpha}}(g_i)(\mathbf{x}, \mathbf{y}) = \begin{cases} \mathbf{x}^{\mathbf{p}_i} & \text{if } \langle \alpha, \mathbf{p}_i \rangle < \ell \\ \mathbf{x}^{\mathbf{p}_i} - \sum_{j \in L} v_{ij} y_j & \text{if } \langle \alpha, \mathbf{p}_i \rangle = \ell \\ - \sum_{j \in L} v_{ij} y_j & \text{if } \langle \alpha, \mathbf{p}_i \rangle > \ell \end{cases} \quad \text{for } i = 1, \dots, m.$$

If $\langle \alpha, \mathbf{p}_i \rangle < \ell$, i.e., the first case appears, for at least one $i \in \{1, \dots, m\}$, then $\text{init}_{\hat{\alpha}}(G)$ clearly has no $\mathbb{C}^*$ -zero. Similarly, if $\alpha = 0$, the initial system $\text{init}_{\hat{\alpha}}(G)$ also has no $\mathbb{C}^*$ -zeros. Therefore, we only need to consider the cases where $\alpha \neq 0$ and $\langle \alpha, \mathbf{p}_i \rangle \geq \ell$ for all i .

Since $\text{conv}\{\mathbf{p}_1, \dots, \mathbf{p}_m\}$ is assumed to be full-dimensional, the set $\{\mathbf{p}_i \mid i \in \{1, \dots, m\}, \langle \alpha, \mathbf{p}_i \rangle = \ell\}$ cannot equal to $\{\mathbf{p}_1, \dots, \mathbf{p}_m\}$ itself. Consequently, any $(\mathbf{x}, \mathbf{y}) \in (\mathbb{C}^*)^n \times (\mathbb{C}^*)^d$ satisfying $\text{init}_{\hat{\alpha}}(G)(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ would give rise to a $\mathbb{C}^*$ -zero to $\text{init}_{\alpha}(F)(\mathbf{x}) = \mathbf{0}$, which would contradict the assumption that $|\mathcal{Z}_0^*(F)| = V(\text{conv}\{\mathbf{p}_1, \dots, \mathbf{p}_m\})$, according to Theorem 3. Therefore, we can conclude that $\text{init}_{\hat{\alpha}}(G)(\mathbf{x}, \mathbf{y})$ has no $\mathbb{C}^*$ -zeros for any nonzero $\hat{\alpha} \in \mathbb{R}^n$. By Theorem 3

$$V(\text{conv}\{\mathbf{p}_1, \dots, \mathbf{p}_m\}) = |\mathcal{Z}_0^*(F)| = |\mathcal{Z}_0^*(G)| = \text{MV}(\Delta(\mathbf{p}_1), \dots, \Delta(\mathbf{p}_m))$$

for a set $\{\mathbf{p}_1, \dots, \mathbf{p}_m\}$ of distinct points in $\mathbb{Z}^n$.

Since both $V(\text{conv}\{\mathbf{p}_1, \dots, \mathbf{p}_m\})$ and $\text{MV}(\Delta(\mathbf{p}_1), \dots, \Delta(\mathbf{p}_m))$ are homogeneous of degree n with respect to a uniform scaling $\mathbf{p}_i \mapsto \lambda \mathbf{p}_i$, the equality extends to cases where $\{\mathbf{p}_1, \dots, \mathbf{p}_m\} \subset \mathbb{Q}^n$. By continuity with respect to the Hausdorff metric, the result further extends to point sets in $\mathbb{R}^n$. $\square$

4. Complexity implications

It is well known that the mixed volume of n line segments in $\mathbb{R}^n$ equals to the determinant of an $n \times n$ matrix and hence can be computed in polynomial time. Yet, M. Dyer, P. Gritzmann, and A. Hufnagel showed that even the relatively simple problem of computing the mixed volume of n “boxes”, each formed as the Minkowski sum of n line segments of nonuniform length along the axes, is $\#\mathbb{P}$ -hard [2, Theorem 3]. Indeed, this problem is equivalent to the problem of permanent computation, which is shown to be $\#\mathbb{P}$ -hard by L. Valiant [7]. It is reasonable to speculate that one source of the $\#\mathbb{P}$ -hardness is the geometric complexity of the “boxes”, since a d -dimensional box can be subdivided into $d!$ full-dimensional simplices. This is not the case. Theorem 4 shows that even when the “boxes” are replaced by simplices, the $\#\mathbb{P}$ -hardness

persists: Even the much more restrictive problem of computing the mixed volume of m simplices of dimension $d < m$ is at least as hard as the problem of computing the volume of the convex hull of m points in $\mathbb{R}^{m-d}$, which is known to be $\#\mathbb{P}$ -hard [2, Theorem 1] (under a binary Turing machine model with oracles).

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