RIEMANN MODULI SPACES ARE QUANTUM ERGODIC

Dean Baskin, Jesse Gell-Redman & Xiaolong Han

Abstract

In this note we show that the Riemann moduli spaces $\mathcal{M}_{\gamma,n}$ equipped with the Weil–Petersson metric are quantum ergodic for $3\gamma + n \geq 4$. We also provide other examples of singular spaces with ergodic geodesic flow for which quantum ergodicity holds.

1. Introduction

The aim of this note is to establish quantum ergodicity on a class of singular spaces; the main examples we address are the Riemann moduli spaces $\mathcal{M}_{\gamma,n}$ of Riemann surfaces of genus γ with n marked points equipped with the Weil-Petersson metric g_{WP}. We work in the stable range $3\gamma + n \geq 4$, so $\mathcal{M}_{\gamma,n}$ is a complex orbifold of complex dimension $3\gamma-3+n$ with smooth top dimensional stratum $\mathcal{M}_{\gamma,n,\mathrm{reg}}$. In this setting, we prove the following theorem:

Theorem 1.1 (Quantum ergodicity on Riemann moduli spaces)**.** *Let $3\gamma + n \geq 4$ and $\Delta_{g_{\mathrm{WP}}}$ be the positive Laplacian with respect to the Weil-Petersson metric g_{WP} on $M = \mathcal{M}_{\gamma,n,\mathrm{reg}}$. Suppose that $\{\phi_j\}$ is an orthonormal basis of eigenfunctions of $\Delta_{g_{\mathrm{WP}}}$ on M for the natural self-adjoint extension of $\Delta_{g_{\mathrm{WP}}}$ studied by Ji–Mazzeo–Müller–Vasy* [**JMMV14**]*. Then there is a density one subsequence $\{\phi_{j_k}\} \subset \{\phi_j\}$ such that*

$$\langle A\phi_{j_k}, \phi_{j_k}\rangle \to \int_{S^*M} \sigma_0(A)\, d\mu \quad \text{as } k \to \infty$$

for all zero order pseudodifferential operators A with Schwartz kernel compactly supported in the interior of $M \times M$ and $\sigma_0(A)$ is the principal symbol of A. Here, $d\mu$ is the Liouville measure on the cosphere bundle S^*M which is normalized such that $\mu(S^*M) = 1$.

In Theorem 4.1 below we prove a stronger result which allows for pseudodifferential operators A which are supported at the orbifold singularities.

In particular, the above theorem asserts the equidistribution of "almost all" eigenfunctions on the Riemann moduli spaces. An immediate consequence of taking $A = a(x) \in C_c^\infty(M)$ to approximate a characteristic function from above and below is that

$$\int_\Omega |\phi_{j_k}|^2 \to \frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(M)} \quad \text{as } k \to \infty$$

for all smooth domains $\Omega \Subset M$.

The ergodicity of the Weil-Petersson geodesic flow on Riemann moduli spaces is a celebrated result of Burns–Masur–Wilkinson [**BMW12**]. (See Section 4 for more background.) Therefore, the quantum ergodicity in Theorem 1.1 establishes the correspondence of the geodesic flow and Laplacian eigenfunctions (which are the stationary states of the quantized operator of the geodesic flow).

Quantum ergodicity on boundary-less compact manifolds with ergodic geodesic flow was first proved independently by Šnirel'man [**Šm74**], Zelditch [**Zel87**], and Colin de Verdière [**CdV85**]; on manifolds with boundary, if billiard flow (i.e., generalized geodesic flow that reflects on the boundary) is ergodic, then the corresponding quantum ergodicity was proved by Gérard-Leichtnam [**GL93**] and by Zelditch–Zworski [**ZZ96**].

Comparing with the boundary-less case [**Šm74**, **Zel87**, **CdV85**], the Riemann moduli spaces are incomplete and the Weil-Petersson geodesic flow is *not* defined everywhere. This difference is reflected in the structure assumptions (S1)-(S3) that we make later. Comparing with the manifolds with boundary [**GL93**, **ZZ96**], the required analysis for the proof of quantum ergodicity, e.g., the Egorov theorem in Theorem 2.5, is not available in the literature. We believe this formulation of Egorov's theorem may be of independent interest. (See also the analytic assumptions (A1)-(A5).)

In fact, we prove Theorem 1.1 for a more general class of singular spaces satisfying a number of structural and analytic hypotheses; in Section 4 we observe that the Riemann moduli spaces $\mathcal{M}_{\gamma,n}$ satisfies these hypotheses.

Let Φ_t denote the flow generated by the Hamilton vector field of the homogeneous degree 1 function $(x,\xi) \mapsto |\xi|_{g(x)}$. This function is (for now, formally) the principal symbol of the operator $P = \sqrt{\Delta}$.

The asymptotic behavior of Laplacian eigenfunctions is closely related to the dynamical properties of Φ_t. Notice that in our setting of singular spaces, the flow $\Phi_t(x,\xi)$ is not generally defined for all $(x,\xi) \in T^*M \setminus 0$, the cotangent space of M (removing the zero section). To clarify the notion of distance from the singular locus, it is convenient to assume M has a compactification. (In the examples considered in this paper, compactifications are readily available.)

In particular, we assume the following structural properties of M:

(S1). There is a compact metric measure space $\overline{M}$ such that $\overline{M} \supset M$ and the closure of M is $\overline{M}$. For $x \in M$ and neighborhoods U of x sufficiently small, the measure and distance function correspond with the Riemannian measure of (M, g).

(S2). The "singular locus" $\mathcal{P} = \overline{M} \setminus M$ is closed. Moreover, $\mathcal{P}$ has measure zero.

(S3). The distance function on $M \times M$ extends to a metric on $\overline{M} \times \overline{M}$. That is, the following function d on $\overline{M} \times \overline{M}$ is a metric:

$$d(x, y) = \inf \left\{ \int_0^1 |\gamma'(t)|_{g(\gamma(t))} \, dt \right\},$$

in which the infimum is taken from all smooth curve $\gamma : [0, 1] \to \overline{M}$ such that $\gamma(0) = x$, $\gamma(1) = y$, and $\gamma^{-1}(\mathcal{P})$ has measure zero.

In practice, many of the compactifications used are larger than required by our hypotheses and the distance function is degenerate on the boundary of $\overline{M}$, but assumption (S3) is satisfied after passing to the quotient by the equivalence relation defined by d.

We may therefore define, for $\epsilon > 0$, the spaces cut away from the singular locus $\mathcal{P}$:

$$M_\epsilon = \{x \in M : d(x, \mathcal{P}) > \epsilon\}.$$

Observe that for $(x, \xi) \in T^*M \setminus 0$, $\Phi_t(x, \xi)$ is defined a priori only for $t \in \mathbb{R}$ for which $d(\pi(\Phi_t(x, \xi)), \mathcal{P}) > \epsilon$ with some $\epsilon > 0$. Here, $\pi : T^*M \to M$ is the projection map. Note that our assumptions above imply that $M_\epsilon \Subset M$, since it is obviously compact in $\overline{M}$ and its closure (the points of distance at least ϵ from $\mathcal{P}$) is contained in M.

Due to the homogeneity of the geodesic flow, we need only study its restriction on the cosphere bundle $S^*M = \{(x, \xi) \in T^*M : |\xi|_{g(x)} = 1\}$. We define, for each $q = (x, \xi) \in S^*M$, the maximum lifespan T_q of the flow, i.e.,

$$T_q = \sup \{T' \in [0, \infty] : \pi(\Phi_t(q)) \in M \text{ for all } |t| \leq T'\}.$$

As in Zelditch-Zworski [**ZZ96**, Equation 2.5], we also define the permissible sets X_T and exceptional set $\mathcal{Y}$:

$$X_T = \{q \in S^*M \ : \ T_q \geq T\}, \quad \mathcal{Y} = S^*M \setminus \left(\bigcap_{T \in (0,\infty)} X_T \right). \tag{1.1}$$

The exceptional set $\mathcal{Y}$ can be thought of (in the cases considered below, quite concretely) as the flowout of the singular locus. If $(x, \xi) \notin \mathcal{Y}$, then $\Phi_t(x, \xi)$ exists for all $t \in \mathbb{R}$.

Now we make the following analytic assumptions about the manifold (M, g), which are verified for the examples of moduli spaces and manifolds with conic singularities in Sections 4 and 5.

(A1). $\mathrm{Vol}(M) < \infty$, where Vol is the volume with respect to the metric g.

(A2). For the (positive) Laplacian $\Delta = \Delta_g$ a self-adjoint extension $(\Delta_g, \mathcal{D})$ (which we fix and denote below also by Δ_g) with core domain the $C_0^\infty(M)$ is chosen so that Δ_g has compact resolvent, i.e. there is an operator $G\colon L^2 \longrightarrow \mathcal{D}$ such that $\Delta G - Id$ is compact and G is compact on $L^2(M)$. (As a result, its spectrum is discrete and consists only of eigenvalues $\lambda_j^2 \to \infty$ as $j \to \infty$.)

(A3). The eigenvalues of Δ obey a Weyl law, i.e.,

$$N(\Lambda) = \#\{\lambda_j : \lambda_j \leq \Lambda\} = \frac{\mathrm{Vol}(M)\,\mathrm{Vol}(B_n)}{(2\pi)^n}\Lambda^n + o(\Lambda^n),$$

in which $\mathrm{Vol}(B_n)$ denotes the volume of the unit ball in $\mathbb{R}^n$ with respect to the Euclidean metric.

(A4). The set $\mathcal{Y}$ has Liouville measure zero in S^*M.

(A5). The geodesic flow on $X_\infty = \mathcal{Y}^c = M \setminus \mathcal{Y}$ is ergodic.

We remark that Assumptions (A1), (A2), and (A3) are enough to ensure that the heat operator $e^{-t\Delta}$ can be built via the functional calculus; this is useful to show that $\sqrt{\Delta}$ is a pseudodifferential operator in the region of interest. See Section 2 for details. We also point out that assuming the Weyl law is only for notational convenience; it has already been verified for Riemann moduli spaces and is straightforward to verify (with current technology of heat kernels) on manifolds with conic singularities. We instead could impose an assumption on the small time behavior of the heat kernel; though this hypothesis implies the Weyl law, in practice it is sometimes easier to verify the Weyl law directly.

We may thus state our main theorem:

Theorem 1.2. *Suppose (M, g) satisfies the structural (S) and analytic (A) assumptions above. If $\{\phi_j\}$ is an orthonormal basis of eigenfunctions of Δ on M, then there is a density one subsequence $\{\phi_{j_k}\} \subset \{\phi_j\}$ so that*

$$\langle A\phi_{j_k}, \phi_{j_k}\rangle \to \int_{S^*M} \sigma_0(A)\, d\mu \quad \text{as } k \to \infty$$

for all order zero pseudodifferential operators A with Schwartz kernel compactly supported in $M \times M$.

Acknowledgements

DB was partially supported by NSF grant DMS-1500646 and NSF CAREER grant DMS-1654056. JGR was supported by the Australian Research Council through grant DP180100589. All the authors wish to thank the Australian Mathematical Sciences Institute and the Mathematical Sciences Institute at the Australian National University for their partial funding of the workshop "Microlocal Analysis and its Applications in Spectral Theory, Dynamical Systems, Inverse Problems, and PDE" at which part of this project was completed.

2. Preliminaries

Let M be a manifold that satisfies the structural and analytic assumptions defined in the introduction. In this section, we gather the facts about the microlocal analysis on such manifolds that are required to prove quantum ergodicity in Theorem 1.2. Because the singular structure on $\overline{M}$ (i.e., the presence of the singular locus $\mathcal{P}$) may be quite complicated, working near $\mathcal{P}$ in principle would require a specialized pseudodifferential calculus for each example (e.g., the b-calculus in the case of conic singularities; see Hillairet–Wunsch [**HW17**]). However, in Theorems 1.1 and 1.2, we restrict our analysis to pseudodifferential operators supported away from $\mathcal{P}$. Analysis in this region requires knowing little about the precise structure of the singularities.

We use the correspondence of the pseudodifferential operators $A \in \Psi^m(M)$ of order m and their principal symbols $\sigma_m(A) \in S^m(M)/S^{m-1}(M)$. We assume that the symbols have classical expansion at fiber infinity and therefore can be identified by functions in $C^\infty(S^*M)$ (so called the "classical symbols"). See, e.g., Hörmander [**Hör07**, Section 18.1] for detailed background.

As in Zelditch–Zworski [**ZZ96**, Lemma 4], we have a local Weyl law:

Lemma 2.1 (Local Weyl law)**.** *Let $K \Subset M$ be a smooth manifold with boundary compactly contained in M so that $K \setminus \partial K$ is an open domain, and let $A \in \Psi^0(K)$ have compactly supported Schwartz kernel. Then*

$$\frac{1}{N(\Lambda)} \sum_{\lambda_j \le \Lambda} \langle A\phi_j, \phi_j \rangle \to \int_{S^*M} \sigma_0(a)\, d\mu \quad \textit{as } \Lambda \to \infty.$$

Proof. This is a standard proof based on the short time estimate of the wave kernel $\cos(t\sqrt{\Delta})$. See Sogge [**Sog14**, Theorem 5.2.3] and also Hörmander [**Hör09**, Theorems 29.3.2 and 29.3.3] (for a proof of the Weyl law). Since the Schwartz kernel of A is compactly supported, finite speed of propagation implies that $A\cos(t\sqrt{\Delta})A^\star$ still has compactly supported Schwartz kernel (i.e., support away from the singular locus $\mathcal{P}$) when $|t|$ is small enough. Therefore, the result of Sogge [**Sog14**, Theorem 5.2.3] applies. q.e.d.

As a corollary, we have the following spatial version of the local Weyl law.

Corollary 2.2. *For every $f \in C_c^\infty(M)$, we have*

$$\frac{1}{N(\Lambda)} \sum_{\lambda_j \le \Lambda} \int_M f(x)|\phi_j(x)|^2\, dV \to \int_M f(x)\, dV \quad \textit{as } \Lambda \to \infty,$$

where dV is the volume measure associated to the metric g.

Remark. On compact manifolds, the Weyl law readily follows by taking $f = 1$ in the above corollary, c.f. Sogge [**Sog14**, Section 5.3]. However, in our case of manifolds with singular locus $\mathcal{P}$, $f \in C_c^\infty(M)$ has to stay away from $\mathcal{P}$. Hence, the local Weyl law in Lemma 2.2 does *not* immediately imply the Weyl law, explaining its presence as assumption (A3).

We next provide a supplement of Egorov's theorem in Theorem 2.5, which is sufficient for the proof of quantum ergodicity. We first require the following lemma establishing an analogue of the off-diagonal smoothing property of pseudodifferential operators. The statement and proof of the lemma are essentially from Hillairet–Wunsch [**HW17**, Appendix A]. There the authors assume that the Friedrichs extension for the Laplacian is chosen, and we include the proof here to clarify to the reader that the lemma holds for other extensions (under our analytic and structural assumptions.)

Lemma 2.3. *Recall that $\mathcal{P}$ is the singular locus and $M_\epsilon = \{x \in M : d(x, \mathcal{P}) > \epsilon\}$, i.e., the regular part of M with distance at least ϵ from $\mathcal{P}$.*

1) *Suppose $0 < \epsilon' < \epsilon$ and set $U = M_\epsilon$. For $V \subset M$ open with $\overline{V} \cap M_{\epsilon'} = \emptyset$, $\Delta^N \sqrt{\Delta}$ is a bounded operator $L^2(V) \to L^2(U)$ and $L^2(U) \to L^2(V)$ for any $N \in \mathbb{N}$.*
2) *For $\chi \in C_c^\infty(M_\epsilon)$, $\chi\sqrt{\Delta}\chi \in \Psi^1(M)$.*

Proof. As in Hillairet–Wunsch [**HW17**, Appendix A], both results follow from an understanding of the smoothing properties of the heat kernel and using the relationship[1] between the heat kernel and $\sqrt{\Delta}$:

$$\sqrt{\Delta} = \frac{\Delta}{\Gamma(\frac{1}{2})} \int_0^\infty e^{-t\Delta} t^{-\frac{1}{2}}\, dt.$$

Take $\rho \in C_c^\infty([0, \infty))$ so that $\rho \equiv 1$ on $[0, 2t_0]$ for some $t_0 > 0$ and write $\psi = 1 - \rho$. The contribution near infinity is smoothing because

$$\int_0^\infty e^{-t\Delta} \psi(t) t^{-\frac{1}{2}}\, dt = e^{-t_0\Delta} \int_0^\infty e^{-(t-t_0)\Delta} \psi(t) t^{-\frac{1}{2}}\, dt.$$

The boundedness of this term (and, indeed, its composition with any power of Δ) follows from the functional calculus.

We must thus show the results with $\sqrt{\Delta}$ replaced by

$$\Delta \int_0^\infty e^{-t\Delta} \rho(t) t^{-\frac{1}{2}}\, dt.$$

As multiplication by Δ does not change the first result (and changes the second statement in a straightforward way), it suffices to study

$$\int_0^\infty e^{-t\Delta} \rho(t) t^{-\frac{1}{2}}\, dt. \tag{2.1}$$

We now consider the first statement. Take $a \in L^2(U)$ and define the distribution $T_a \in \mathcal{D}'(\mathbb{R} \times V)$ by

$$(T_a, \phi(t) b(y))_{\mathcal{D}' \times \mathcal{D}} = \int_0^\infty \langle a, e^{-t\Delta} b \rangle_{L^2} \phi(t)\, dt.$$

[1] If Δ has finitely many non-postive eigenvalues (as may be the case for extensions other than the Friedrichs one), then one should project off of the non-positive eigenspaces. These projections satisfy the conclusions of the theorem and the rest of the argument carries through.

Take $b \in L^2(V)$. Since the supports of a and b are disjoint, $\lim_{t\downarrow 0}\langle a, e^{-t\Delta}b\rangle = 0$ and therefore

$$(\partial_t + \Delta_y)T_a = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}\times V).$$

We may thus conclude that T_a is smooth.

As $T_a \equiv 0$ for $t < 0$, for any $a \in L^2(U)$ and $b \in L^2(V)$, the function

$$t \mapsto \langle e^{-t\Delta}a, b\rangle$$

is smooth on $[0,\infty)$ and vanishes to infinite order at 0. In particular, for each N and k, the quantity

$$t^{-k}\langle \Delta^N e^{-t\Delta}a, b\rangle$$

is bounded on $(0,1]$. By the principle of uniform boundedness, we therefore know

$$\left\|\Delta^N e^{-t\Delta}\right\|_{L^2(U)\to L^2(V)} = O(t^k)$$

as $t \downarrow 0$ with a similar statement holding as a map $L^2(V) \to L^2(U)$. Substituting this bound into the integral above yields the first result.

For the second result, we fix a smooth Riemannian manifold $(\tilde{M}, \tilde{g})$ so that M_ϵ embeds isometrically as an open subset of $\tilde{M}$. Let e be the heat kernel on M and $\tilde{e}$ be the heat kernel on $\tilde{M}$. Let r denote the distribution on $\mathbb{R}\times M_\epsilon \times M_\epsilon$ defined by

$$(r,\phi) = \int_0^\infty \int_{M_\epsilon}\int_{M_\epsilon} \left(e(t,x,y) - \tilde{e}(t,x,y)\right)\phi(t,x,y)\,dy\,dx\,dt.$$

For any $\phi \in C_c^\infty(\mathbb{R}\times M_\epsilon\times M_\epsilon)$, we have

$$\lim_{t\downarrow 0}\int_{M_\epsilon}\int_{M_\epsilon}\left(e(t,x,y) - \tilde{e}(t,x,y)\right)\phi(t,x,y)\,dx\,dy = 0,$$

so, in $\mathcal{D}'(\mathbb{R}\times M_\epsilon\times M_\epsilon)$, we have

$$\left(2\partial_t + \Delta_x + \Delta_y\right) r = 0$$

and therefore r is smooth on $\mathbb{R}\times M_\epsilon \times M_\epsilon$. We may thus replace $e^{-t\Delta}$ with the heat kernel $\tilde{e}$ in (2.1) and incur only an error of the form

$$\int_0^\infty \rho(t) r(t,x,y) t^{-\frac{1}{2}}\,dt.$$

As r is smooth and vanishing to infinite order at $t = 0$, this integral is smoothing. It therefore follows that $\chi\sqrt{\Delta}\chi \in \Psi^1(M_\epsilon) \subset \Psi^1(M)$.

q.e.d.

Because Δ has compact resolvent by the analytic assumption (A2), we obtain the following corollary.

Corollary 2.4. *Fix $\epsilon > 0$ and let $\chi_1 \in C_c^\infty(M_\epsilon)$ and $\chi_2 \in C_c^\infty(M)$ be such that $\chi_2 \equiv 1$ on M_ϵ. The compositions $(1-\chi_2)P\chi_1$ and $\chi_1 P(1-\chi_2)$ are compact operators $L^2(M) \to L^2(M)$.*

We now discuss the crucial Egorov's theorem. In general, this theorem connects the quantum evolution $e^{-itP}Ae^{itP}$ and the classical evolution $\sigma_m(A)\circ\Phi_t$, where $A \in \Psi^m$ and recall that $P = \sqrt{\Delta}$. Indeed, $e^{-itP}Ae^{itP} \in \Psi^m$ and $\sigma_m(e^{-itP}Ae^{itP}) = \sigma_m(A)\circ\Phi_t$ on compact manifolds, see e.g. Sogge [**Sog14**, Theorem 4.3.6].

In our setting of the singular space $\overline{M}$, assume that A has compactly supported Schwartz kernel in $M \times M$. Observe that the conjugated operator $e^{-itP}Ae^{itP}$ may not have compactly supported Schwartz kernel (so can potentially be close to the singular locus). We provide the following supplement to Egorov's theorem to remedy this issue. It is also of independent interest in the context of singular spaces.

As is standard, we let $\mathrm{WF}(A)$ denote the microsupport of A (or equivalently, the essential support of its symbol) and κ_A be the Schwartz kernel of A. (See [**Hör07**, Section 18.1] for more background.) We also note that if $a \in C_c^\infty(S^*M)$, then there is $\tilde{A} \in \Psi^0(M)$ such that $\sigma_0(\tilde{A}) = a$ and $\kappa_{\tilde{A}}$ has compact support in $M \times M$. In fact, let $A \in \Psi^0(M)$ such that $\sigma_0(A) = a$. Take $\tilde{A} = \chi A \chi$ such that $\chi = 1$ on $\pi(\mathrm{supp}(a))$. Then $\tilde{A} - A$ is a smoothing operator.

Theorem 2.5. *Let $\epsilon > 0$ and $T > 0$. Suppose that $A \in \Psi^0(M)$ has $\mathrm{supp}\,\kappa_A \subset M_\epsilon \times M_\epsilon$ and $\mathrm{WF}(A) \subset X_{T+\epsilon}$ defined in 1.1. Let $\tilde{A}(t) \in \Psi^0(M)$ have compactly supported Schwartz kernel and $\sigma_0(\tilde{A}(t)) = a \circ \Phi_t$ for $|t| \le T + \epsilon$.*

Then for all $|t| \le T$,

$$e^{itP}Ae^{-itP} - \tilde{A}(t) : L^2(M) \to L^2(M)$$

is compact.

Proof. Let $\delta > 0$ be such that the Schwartz kernels of A and $\tilde{A}(t)$ lie in $M_\delta \times M_\delta$ for all $|t| \le T + \epsilon$. Fix $0 < \delta' < \delta$ and take $\chi_1 \in C_c^\infty(M_{\delta'})$ be so that $\chi_1 \equiv 1$ on M_δ. We also take $\chi_2 \in C_c^\infty(M)$ so that $\chi_2 \equiv 1$ on $M_{\delta'}$.

Consider the difference

$$E(t) = e^{-itP}\tilde{A}(t)e^{itP} - A.$$

It is then obvious that $E(0) : L^2(M) \to L^2(M)$ is smoothing. Because $(1-\chi_2)\tilde{A}(t) = \tilde{A}(t)(1-\chi_2) = 0$, we write

$$\begin{aligned} E'(t) &= e^{-itP}\left(\tilde{A}'(t) - i\left[P, \tilde{A}(t)\right]\right)e^{itP} \\ &= e^{-itP}\chi_2\left(\tilde{A}'(t) - i\left[P, \tilde{A}(t)\right]\right)\chi_2 e^{itP} \\ &\quad - ie^{-itP}(1-\chi_2)P\tilde{A}(t)\chi_2 e^{itP} \\ &\quad + ie^{-itP}\chi_2\tilde{A}(t)P(1-\chi_2)e^{itP}. \end{aligned}$$

Because the principal symbol of the inner part of the first term vanishes, we can write it as $e^{-itP}\chi_2 R_1(t)\chi_2 e^{itP}$, where $R_1(t) \in \Psi^{-1}(M)$.

As $\chi_1\chi_2 = \chi_1$ and $\tilde{A}(t)$ is supported where $\chi_1(x)\chi_1(y) \equiv 1$, the last two terms can be written

$$-ie^{-itP}(1-\chi_2)P\chi_1\tilde{A}(t)\chi_2 e^{itP} + ie^{-itP}\chi_2\tilde{A}(t)\chi_1 P(1-\chi_2)e^{itP}.$$

We may therefore write the difference of interest as

$$\begin{aligned} &e^{itP}Ae^{-itP} - \tilde{A}(t) \\ &= \int_0^s \chi_2 R_1(s)\chi_2\,ds \\ &\quad - i\int_0^t (1-\chi_2)P\chi_1\tilde{A}(s)\chi_2\,ds + i\int_0^t \chi_2\tilde{A}(s)\chi_1 P(1-\chi_2)\,ds. \end{aligned}$$

The first term lies in $\Psi^{-1}(M)$ and has compactly supported Schwartz kernel; it is therefore compact on L^2. The second two terms are both compact by Corollary 2.4. q.e.d.

Remark. From the proof above, we observe that the compact operator $e^{itP}Ae^{-itP} - \tilde{A}(t)$ is uniformly controlled for all $|t| \leq T + \epsilon$.

3. Proof of the main theorem

We now show that under our assumptions, a modified version of the argument of Zelditch–Zworski [**ZZ96**, Section 3] still holds. Recall that $P = \sqrt{\Delta}$.

We first establish some notation: For $B \in \Psi^0(M)$ with compactly supported Schwartz kernel and $T > 0$, set

$$\rho_j(B) = \langle B\phi_j, \phi_j\rangle \quad \text{and} \quad \langle B\rangle_T = \frac{1}{2T}\int_{-T}^{T} e^{-itP}Be^{itP}\,dt.$$

Note that by Lemma 2.5, if B has compactly supported Schwartz kernel and $\mathrm{WF}(B)$ is microsupported in $X_{2T+\epsilon}$, then with $\tilde{B}(t)$ as in Lemma 2.5 and

$$\widetilde{\langle B\rangle}_T = \frac{1}{2T}\int_{-T}^{T} \tilde{B}(t)\,dt, \tag{3.1}$$

we have that $\langle B\rangle_T - \widetilde{\langle B\rangle}_T : L^2(M) \to L^2(M)$ is compact.

Let $A \in \Psi^0(M)$ and write $a = \sigma(A)$, $a \in C_c^\infty(S^*M)$. We further assume that κ_A is compactly supported in $M \times M$. Set

$$\alpha = \int_{S^*M} a \quad \text{and} \quad \langle a\rangle_T = \frac{1}{2T}\int_{-T}^{T} a \circ \Phi_t\,dt,$$

where we are careful to use the second notation only for a supported in $X_{T+\epsilon}$. The theorem then follows from a standard extraction procedure (see e.g. Zelditch-Zworski [**ZZ96**]) if we can show that

$$\frac{1}{N(\Lambda)}\sum_{\lambda_j \leq \Lambda} |\langle A\phi_j, \phi_j\rangle - \alpha|^2 \to 0, \tag{3.2}$$

as $\Lambda \to \infty$.

In the case where $\alpha = 0$, the proof essentially proceeds by a series of approximations (the general case is proved fully below):

1) We replace A by a family $A_{\epsilon,T}$ that have microsupport in the set $X_{2T+\epsilon}$. The difference of (3.2) for A and $A_{\epsilon,T}$ can be estimated using the local Weyl law in Lemma 2.1.
2) We then replace $A_{\epsilon,T}$ by an averaged operator $\widetilde{\langle A_{\epsilon,T}\rangle}_T$ (as in in (3.1)) with compactly supported Schwartz kernel. By Egorov's theorem in Theorem 2.5, $\widetilde{\langle A_{\epsilon,T}\rangle}_T$ is (modulo a compact operator) a pseudodifferential operator with principal symbol $\langle \sigma_0(A_{\epsilon,T})\rangle_T$.
3) We finally use the dynamical condition of ergodicity in M to show that $\langle \sigma_0(A_{\epsilon,T})\rangle_T \to 0$ when $T \to \infty$.

We now let $T > 0$, which later is chosen large enough. Write $U_\epsilon = U_\epsilon(T)$ as

$$U_\epsilon = \{(x,\xi) \in X_{2T+\epsilon} \ : d(\pi(\Phi_t(x,\xi)), \mathcal{P}) > \epsilon \text{ for all } |t| < 2T+\epsilon\}.$$

Observe that if $\epsilon < \epsilon'$, then $U_{\epsilon'} \Subset U_\epsilon$. Moreover, $\bigcap_{\epsilon>0} U_\epsilon = X_{2T}$, which is defined in (1.1).

Because the U_ϵ have compact closure away from $\mathcal{P}$, we can find microlocal cutoffs to the U_ϵ. Namely, take $E_\epsilon \in \Psi^0(M)$ with compactly supported Schwartz kernels such that $\sigma(E_\epsilon) = 1$ on U_ϵ. Then

$\lim_{\epsilon\to 0}\sigma_0(E_\epsilon) = 1$ on X_{2T}. Let

$$A_\epsilon = E_\epsilon A, \quad \alpha_\epsilon = \int_{S^*M} \sigma_0(A_\epsilon), \quad R_\epsilon = I - E_\epsilon .$$

We now compare (3.2) for A and A_ϵ. Write

$$C(\epsilon,\Lambda) = \frac{1}{N(\Lambda)} \sum_{\lambda_j\le\Lambda} |\rho_j(A) - \alpha|^2 - \frac{1}{N(\Lambda)} \sum_{\lambda_j\le\Lambda} |\rho_j(A_\epsilon) - \alpha_\epsilon|^2 . \tag{3.3}$$

Note that $A = A_\epsilon + R_\epsilon A$. Letting $\beta_\epsilon = \int_{S^*M}\sigma_0(R_\epsilon A)$, we have by the Cauchy–Schwarz inequality,

$$\begin{aligned} & C(\epsilon,\Lambda) \\ \le\ & \frac{2}{N(\Lambda)} \left(\sum_{\lambda_j\le\Lambda} |\rho_j(A_\epsilon) - \alpha_\epsilon|^2 \right)^{1/2} \left(\sum_{\lambda_j\le\Lambda} |\rho_j(R_\epsilon A) - \beta_\epsilon)|^2 \right)^{1/2} \\ & + \frac{1}{N(\Lambda)} \sum_{\lambda_j\le\Lambda} |\rho_j(R_\epsilon A) - \beta_\epsilon|^2 \\ \le\ & 2 \left(\frac{1}{N(\Lambda)} \sum_{\lambda_j\le\Lambda} \rho_j((A_\epsilon - \alpha_\epsilon)^*(A_\epsilon - \alpha_\epsilon)) \right)^{1/2} \\ & \times \left(\frac{1}{N(\Lambda)} \sum_{\lambda_j\le\Lambda} \rho_j((R_\epsilon A - \beta_\epsilon)^*(R_\epsilon A - \beta_\epsilon)) \right)^{1/2} \\ & + \frac{1}{N(\Lambda)} \sum_{\lambda_j\le\Lambda} \rho_j((R_\epsilon A - \beta_\epsilon)^*(R_\epsilon A - \beta_\epsilon)). \end{aligned}$$

Because the Schwartz kernels of the products $R_\epsilon A$ are compactly supported (since A has the same property), the local Weyl law of Lemma 2.1 shows that, as $\Lambda\to\infty$,

$$\frac{1}{N(\Lambda)} \sum_{\lambda_j\le\Lambda} \rho_j((R_\epsilon A - \beta_\epsilon)^*(R_\epsilon A - \beta_\epsilon)) \to |\sigma_0(R_\epsilon A) - \beta_\epsilon|^2 .$$

Therefore, using the trivial bound that $\rho_j((A_\epsilon-\alpha_\epsilon)^*(A_\epsilon-\alpha_\epsilon)) \le 1$, we have that

$$C(T,\epsilon,\Lambda) = h_T(\epsilon) + r_{T,\epsilon}(\Lambda), \tag{3.4}$$

where $r_{T,\epsilon}(\Lambda)\to 0$ as $\Lambda\to\infty$. Because $\alpha(R_\epsilon A)\to 0$ and $\beta_\epsilon\to 0$ as $\epsilon\to 0$, we also know $h_T(\epsilon)\to 0$ as $\epsilon\to 0$.

We now turn our attention to the estimation of (3.2) involving A_ϵ and α_ϵ:

$$\begin{aligned} & \frac{1}{N(\Lambda)} \sum_{\lambda_j \le \Lambda} |\langle A_\epsilon \phi_j, \phi_j \rangle - \alpha_\epsilon|^2 \\ \le \; & \frac{1}{N(\Lambda)} \sum_{\lambda_j \le \Lambda} \rho_j \left(\langle A_\epsilon - \alpha_\epsilon \rangle_T^* \langle A_\epsilon - \alpha_\epsilon \rangle_T \right) \\ = \; & \frac{1}{N(\Lambda)} \sum_{\lambda_j \le \Lambda} \rho_j (B_{\epsilon,T}). \end{aligned} \tag{3.5}$$

Observe that because A_ϵ is microsupported in $X_{2T+\epsilon}$, Lemma 2.5 allows us to replace $B_{\epsilon,T}$ with

$$\tilde{B}_{\epsilon,T} = \widetilde{\langle A_\epsilon - \alpha_\epsilon \rangle}_T^* \widetilde{\langle A_\epsilon - \alpha_\epsilon \rangle}_T,$$

whose principal symbol is

$$\left| \frac{1}{2T} \int_{-T}^{T} (\sigma_0(A_\epsilon) \circ \Phi_t - \alpha_\epsilon) \, dt \right|^2,$$

moreover, $B_{\epsilon,T} - \tilde{B}_{\epsilon,T} : L^2(M) \to L^2(M)$ is compact. It then follows that

$$\frac{1}{N(\Lambda)} \sum_{\lambda_j \le \Lambda} \rho_j(B_{\epsilon,T}) \le \frac{1}{N(\Lambda)} \sum_{\lambda_j \le \Lambda} \rho_j(\tilde{B}_{\epsilon,T}) + f_{\epsilon,T}(\Lambda), \tag{3.6}$$

where $f_{\epsilon,T}(\Lambda) \to 0$ as $\Lambda \to \infty$.

Since $\tilde{B}_{\epsilon,T}$ has compactly supported Schwartz kernel, the local Weyl law in Lemma 2.1 implies that the difference

$$\frac{1}{N(\Lambda)} \sum_{\lambda_j \le \Lambda} \rho_j(\tilde{B}_{\epsilon,T}) - \int_{S^*M} \left| \frac{1}{2T} \int_{-T}^{T} (\sigma_0(A_\epsilon) \circ \Phi_t - \alpha_\epsilon) \, dt \right|^2 d\mu$$

is $o(1)$ as $\Lambda \to \infty$. Putting together with (3.3), (3.4), (3.5), and (3.6), we arrive at

$$\begin{aligned} & \frac{1}{N(\Lambda)} \sum_{\lambda_j \le \Lambda} |\rho_j(A) - \alpha|^2 \\ \le \; & \int_{S^*M} \left| \frac{1}{2T} \int_{-T}^{T} (\sigma_0(A_\epsilon) \circ \Phi_t - \alpha_\epsilon) \, dt \right|^2 d\mu + F_{\epsilon,T}(\Lambda) + h_T(\epsilon), \end{aligned}$$

in which $F_{\epsilon,T}(\Lambda) = r_{\epsilon,T}(\Lambda) + f_{\epsilon,T}(\Lambda) \to 0$ as $\Lambda \to \infty$ and $h_T(\epsilon) \to 0$ and $\epsilon \to 0$. To control the first time on the right-hand-side, notice that

$$\begin{aligned} g_T(\epsilon) &= \Big| \int_{S^*M} \left| \frac{1}{2T} \int_{-T}^{T} (a \circ \Phi_t - \alpha)\, dt \right|^2 d\mu \\ &\quad - \int_{S^*M} \left| \frac{1}{2T} \int_{-T}^{T} (\sigma_0(A_\epsilon) \circ \Phi_t - \alpha_\epsilon)\, dt \right|^2 d\mu \Big| \to 0 \end{aligned}$$

as $\epsilon \to 0$ by dominated convergence theorem, since $\sigma_0(A_\epsilon) \to a$ and $\alpha_\epsilon \to \alpha$ as $\epsilon \to 0$. We then use the ergodicity of the geodesic flow to conclude

$$e(T) = \int_{S^*M} \left| \frac{1}{2T} \int_{-T}^{T} (\sigma_0(A) - \alpha)\, dt \right|^2 d\mu \to 0 \quad \text{as } T \to \infty.$$

In total,

$$\frac{1}{N(\Lambda)} \sum_{\lambda_j \le \Lambda} |\rho_j(A) - \alpha|^2 \le e(T) + g_T(\epsilon) + F_{\epsilon,T}(\Lambda) + h_T(\epsilon).$$

Taking T large, ϵ small, and Λ large successively, we complete the proof.

4. Riemann moduli spaces with the Weil–Petersson metric

We now recall the definition and relevant properties of the Riemann moduli spaces and their Weil-Petersson metrics; in particular, we show that they satisfy assumptions (S) and (A) from the introduction, and thus, from Theorem 1.2, we conclude that Theorem 1.1 holds.

As in the introduction, let $\mathcal{M}_{\gamma,n}$ denote the space of equivalence classes of complex structures on a fixed, closed surface Σ of genus γ with n marked points $C = \{p_1, \dots, p_n\} \subset \Sigma$, where two complex structures on Σ are equivalent if one is the pullback of the other via a diffeomorphism Σ which fixes C. The set $\mathcal{M}_{\gamma,n}$ admits a natural compactification $\overline{\mathcal{M}}_{\gamma,n}$, the Deligne–Mumford compactification, which includes, in addition to complex structures on Σ, the nodal curves which can be obtained by degenerations of complex structures Σ. Then $\overline{\mathcal{M}}_{\gamma,n}$ is a compact, complex orbifold of complex dimension $3\gamma - 3 + n$. Within $\overline{\mathcal{M}}_{\gamma,n}$ there is a finite family of complex codimension 1 "normally crossing" divisors, i.e. complex codimension 1 sub-orbifolds, $D_1, \dots, D_\kappa$, such that $\bigcup_{i=1}^{\kappa} D_i = \overline{\mathcal{M}}_{\gamma,n} \setminus \mathcal{M}_{\gamma,n}$, and any finite intersection $\cap_{i \in J} D_i$ with $J \subset \{1, \dots, \kappa\}$, there is a neighborhood U of this intersection and a finite-to-one ramified holomorphic resolution $V \longrightarrow U$ with V an open complex manifold such the inverse image of $\cap_{i\in J} D_i$ is defined by the vanishing of $|J|$ non-degenerate holomorphic functions z_i with linearly independent differentials on the intersection. For further background on the definition of $\mathcal{M}_{\gamma,n}$ and its Deligne–Mumford compactification see for example the expository paper of Vakil [**Vak03**].

Let $M = \mathcal{M}_{\gamma,n,\mathrm{reg}}$ be the top dimensional stratum of $\mathcal{M}_{\gamma,n}$, i.e. the set $\mathcal{M}_{\gamma,n}$ minus the orbifold points. This is a dense open set in $\overline{\mathcal{M}}_{\gamma,n}$. Recall our assumption $3\gamma + n \geq 4$, which in the case $n = 0$ assures that $\gamma \geq 2$. The Weil-Petersson metric g_{WP}, typically defined initially on the Teichmüller space and descending to a smooth metric on M, is the Riemannian metric given locally by identification of the cotangent bundle of M at a point in M (i.e. an equivalence class of Riemann surfaces $[(\Sigma, c)]$), with the space of transverse-traceless holomorphic quadratic differentials on the uniformizing complete, hyperbolic metric g on $(\Sigma \backslash C, c)$ with cusp-type singularities at C; the inner product on this cotangent space is then given by the L^2-pairing defined by g. This metric has a well-known decomposition near the divisors; at the intersection $\cap_{i\in J} D_i$, for appropriately chosen (holomorphic) defining functions $z_i = |z_i| e^{\sqrt{-1}\theta_i}$ as in the previous paragraph and setting $s_i^2 = 1/\log(1/|z_i|)$, we have

$$g_{\mathrm{WP}} = \sum_{i\in J} c\, ds_i^2 + c' s_i^6 d\theta_i^2 + h_\cap + O(s^2) \tag{4.1}$$

where $c, c' > 0$ are constants, $h_\cap$ is an (orbifold) metric on $\cap_{i\in J} D_i$ and $s^2 = \sum_{i\in J} s_i^2$. This expansion was originally suggested by the work of Masur [**Mas76**] and established in the work of a number of authors, including by Liu–Sun–Yau [**LSY08**] Wolpert [**Wol85**, **Wol03**, **Wol08**, **Wol10**] and Yamada [**Yam04**]. The full polyhomogeneous regularity of the Weil-Petersson metric at the divisors is proven in Mazzeo–Swoboda [**MS17**] and Melrose–Zhu [**MZ17**].

We can now begin to address the structural and analytic assumptions. Indeed, for (S1) and (S2), $\overline{M} = \overline{\mathcal{M}}_{\gamma,n}$, so $\overline{M} - M$ is a closed measure zero subset of $\overline{M}$, and (S3) and (A1) follow from the local form of the metric. Skipping ahead to (A4) and (A5), consider the geodesic flow of for the Weil-Petersson metric, which is defined locally on M. A result of Wolpert [**Wol03**] implies (see [**BMW12**]) that the set $X_\infty \subset S^*M$ of points in the cosphere bundle on which the geodesic flow is defined for all times is full measure, so its complement $\mathcal{Y}$ is measure zero, i.e. (A4) holds, and as mentioned in the introduction, that (A5) holds is the well-known result of Burns–Masur–Wilkinson [**BMW12**].

It remains to discuss (A2) and (A3). Recall that, as is shown in [**Loo94**, **PdJ95**], $\overline{\mathcal{M}}_{\gamma,n}$ is in fact a "good" orbifold, meaning there is a complex manifold $\overline{M}'$ and a finite group S acting on $\overline{M}'$ by biholomorphic maps (possibly with fixed points) such that the quotient if $\overline{\mathcal{M}}_{\gamma,n} = \overline{M}'/S$ and the projection

$$\pi \colon \overline{M}' \longrightarrow \overline{\mathcal{M}}_{\gamma,n} \tag{4.2}$$

is a smooth (ramified) holomorphic map. The pullback of the Weil-Petersson metric $\pi^* g_{\mathrm{WP}}$ to $\overline{M}'$ is a smooth Riemannian metric on $M' := \pi^{-1}(\mathcal{M}_{\gamma,n})$, and elements of S are automatically isometries of this pullback metric. For γ fixed and n large, one can take $\overline{M}' = \overline{\mathcal{M}}_{\gamma,n}$ as there are no fixed points of the action of the mapping class group on Teichmüller space, see [**Vak03**, **JMMV14**].

Ji–Mazzeo–Müller–Vasy [**JMMV14**] study the general class of complex orbifolds $\overline{M}$ which have "crossing cusp-edge" singularities in the metric. These are exactly those complex Riemannian orbifolds whose metrics take the form described in the above paragraphs near a fixed set of normally intersecting complex codimension one divisors. In particular, they prove that Laplacian on $\mathcal{M}_{\gamma,n}$ is self-adjoint with core domain $C^\infty_{0,\mathrm{orb}}(\mathcal{M}_{\gamma,n})$, the Frechet space of smooth functions ϕ such that, with π the resolving map from the previous paragraph, $\phi \circ \pi \in C^\infty_0(M')$. In words, these are the functions which are compactly supported in $\mathcal{M}_{\gamma,n}$, smooth away from all orbifold singularities, and lift via the local resolutions of the orbifold singularities to smooth functions. They prove (see Theorem 3) that with this core domain, $\Delta_{g_{\mathrm{WP}}}$ is essentially self-adjoint, that the domain of this self-adjoint extension is compactly contained in L^2 (see below Theorem 3), and that Weyl asymptotics hold for the (necessarily discrete) spectrum (see Theorem 1). (We remark again that in [**JMMV14**] all the statements are for the non-pointed moduli spaces $\mathcal{M}_\gamma$ but all of the theorems in the body of the paper are for the general class of singular Riemannian space which include $\mathcal{M}_{\gamma,n}$.) In particular, assumptions (A2) and (A3) hold for this extension.

Thus the assumptions (S) and (A) hold for $\Delta_{g_{\mathrm{WP}}}$ on $\mathcal{M}_{\gamma,n}$ with its unique self-adjoint extension with core domain $C^\infty_{0,\mathrm{orb}}$, i.e. Theorem 1.1 follows from Theorem 1.2.

4.1. Orbifold regular PsiDO's on $\mathcal{M}_{\gamma,n}$. We now consider the Riemann moduli space and prove a stronger theorem. We continue with the notation of the previous section, in particular $M = \mathcal{M}_{\gamma,n,\mathrm{reg}}$, consider pseudodifferential operators $A \in \Psi^0_{0,\mathrm{orb}}(\mathcal{M}_{\gamma,n})$ which, by definition, are operators $A\colon C^\infty_0(M) \longrightarrow \mathcal{D}'(M)$ which have compactly supported Schwartz kernel in M and are regular under local orbifold resolutions; concretely, for the resolving map π in (4.2), $\pi^* A \in \Psi^0(M')$ (and $\pi^* A$ is compactly supported in M'.) Equivalently, working on the resolved space M', these are pseudodifferential operators $A \in \Psi^0(M')$ with compactly supported Schwartz kernels which are invariant under the action of S on M'. This family of pseudodifferential operators is defined independently of a choice of resolution M' as it is a equivalent to smoothness of the pullback of the A via any local resolution, but below we use a particular convenient choice of resolution, specifically the one used in [**BMW12**, Sec. 6],

$$M' = \mathcal{T}/\mathrm{MCG}[k],$$

where $\mathcal{T}$ is the Teichmüller space and

$$\mathrm{MCG}[k] = \{\psi \in \mathrm{MCG}(\Sigma) : \psi_* \equiv 0 \text{ acting on } H^1(\Sigma; \mathbb{Z}/k\mathbb{Z})\},$$

is a finite index subgroup of the mapping class group $\mathrm{MCG}(\Sigma)$ which is obviously normal. In [**BMW12**, Thm. 6.4], the authors prove that the Weil-Petersson geodesic flow is ergodic on this resolved space, so since the flow is defined for infinite times on the pullback of a full measure set, both assumptions (A4) and (A5) hold on M' with the Weil-Petersson metric. The moduli space is then the quotient of M' by the set of biholomorphic maps parametrized by (and identified with representatives of the set of) the group $S = \mathrm{MCG}(\Sigma)/\mathrm{MCG}[k]$. This will be useful to prove the following.

Theorem 4.1. *With the same assumptions as in Theorem 1.1, there is a density one subsequence $\{\phi_{j_k}\} \subset \{\phi_j\}$ such that for all $A \in \Psi^0_{0,\mathrm{orb}}(\mathcal{M}_{\gamma,n})$,*

$$\langle A\phi_{j_k}, \phi_{j_k}\rangle \to \int_{S^*M} \sigma_0(A)\, d\mu \quad \text{as } k \to \infty.$$

Proof. Since $(M', \pi^* g_{\mathrm{WP}})$ is a smooth crossing cusp-edge space, the results of [**JMMV14**] show that $\Delta_{\pi^* g_{\mathrm{WP}}}$ is essentially self-adjoint with core domain $C^\infty_0(M')$, and that assumptions (A2)–(A3) hold for this self-adjoint extension. The rest of the assumptions (S) and (A) also follow. Indeed, the assumptions (S) assumptions and (A1) hold automatically, and assumptions (A4) and (A5) follow as discussed prior to the statement of the theorem. Thus all the hypotheses are satisfied and the conclusion of Theorem 1.2 applies to $\Delta_{\pi^* g_{\mathrm{WP}}}$.

The theorem now follows easily from considering the identification of the eigenspaces E_λ of $\Delta_{g_{\mathrm{WP}}}$ on $\mathcal{M}_{\gamma,n}$ for the unique self-adjoint extension from $C^\infty_{0,\mathrm{orb}}$ with the S-invariant eigenspaces of $\Delta_{\pi^* g_{\mathrm{WP}}}$. Indeed, let $\tilde{E}_\lambda$ denote an eigenspace of $\Delta_{\pi^* g_{\mathrm{WP}}}$, and note that since S acts on $(M', \pi^* g_{\mathrm{WP}})$ by isometries, it acts by pull-back on $\tilde{E}_\lambda$. Letting $\phi \in \tilde{E}_\lambda$, then $\phi^S = |S|^{-1} \sum_{\psi \in S} \psi^* \phi$ is an S-invariant function on M' and thus descends to a function on $\mathcal{M}_{\gamma,n}$ which it is easy to see lies in the domain under consideration. The other direction of identification is automatic. Thus for all $\lambda \in \mathrm{spec}(\Delta_{\pi^* g_{\mathrm{WP}}})$,

$$\tilde{E}^S_\lambda := \{\phi \in \tilde{E}_\lambda : \psi^* \phi = \phi\} \subset \tilde{E}_\lambda$$

satisfies $\tilde{E}^S_\lambda = E_\lambda$ gives an identification of E_λ with a subset of $\tilde{E}_\lambda$. In particular, we may choose an orthonormal basis of eigenfunctions $\tilde{\phi}_j$ of $(M', \pi^* g_{\mathrm{WP}})$ which contains a subsequence of an orthonormal basis of $\tilde{E}^S_\lambda$.

On the other hand, the Weyl asymptotic formulas implies that, if $\tilde{N}(\lambda)$ is the eigenvalue counting function for $(M', \pi^* g_{\mathrm{WP}})$ and $N(\lambda)$ the counting function for $(\mathcal{M}_{\gamma,n}, g_{\mathrm{WP}})$,

$$\frac{\tilde{N}(\lambda)}{N(\lambda)} = \frac{\mathrm{Vol}(M', \pi^* g_{\mathrm{WP}})}{\mathrm{Vol}(\mathcal{M}_{\gamma,n}, g_{\mathrm{WP}})} + o(1) = |S| + o(1) \text{ as } \lambda \to \infty. \tag{4.3}$$

Hence any full density subsequence of eigenfunctions of $(M', \pi^* g_{\mathrm{WP}})$ contains a full density subsequence of eigenfunctions coming from the $\tilde{E}^S_\lambda$. Now there is a full density subsequence of eigenfunctions $(\tilde{\phi}_{j_k})$ which satisfy the conclusion of Theorem 1.1. It contains a subsequence of invariant eigenfunctions $(\tilde{\phi}^S_\ell) = (\tilde{\phi}_{j_k}) \cap L^2_S(\overline{M}')$ that also satisfy the conclusion fo Theorem 1.1 and in addition each $\tilde{\phi}^S_\ell = \pi^* \phi_\ell$ for some eigenfunction on $\mathcal{M}_{\gamma,n}$. Thus for any $B \in \Psi^0(M')$ with compact support, we have

$$\langle B \tilde{\phi}^S_\ell, \tilde{\phi}^S_\ell \rangle \to \int_{S^* M'} \sigma_0(B) \, d\mu \quad \text{as } \ell \to \infty.$$

Taking $B = \pi^* A$ and dividing by the area gives the result. q.e.d.

5. Hyperbolic surfaces with conic singularities

We consider the example of hyperbolic surfaces with conic singularities.[2] Concretely, consider a compact Riemann surface $\overline{M}$ of genus γ, a finite set of points $\mathcal{P}$. Suppose $\overline{M}$ is equipped with a Riemannian metric g smooth on the complement $M = \overline{M} \setminus \mathcal{P}$ and so that

1) for each $p \in \mathcal{P}$ there are conformal coordinates $\tilde{z}$ with $\tilde{z}(p) = 0$,
2) in the (non-smooth) coordinates $z = \alpha^{-1}\tilde{z}^{\alpha}$, we have
$$g = dr^2 + \alpha^2 \sinh^2 r \, d\theta^2,$$
where $z = re^{i\theta}$, and
3) g is hyperbolic on $\overline{M} \setminus \mathcal{P}$.

Here $\alpha = 1$ corresponds to a "phantom singularity"; in other words, when $\alpha = 1$, the metric extends to be smooth at the point p.

Given a finite set of points $\mathcal{P} = \{p_1, ..., p_k\}$ and numbers $\alpha_1, \cdots, \alpha_k \in (0, \infty)$, McOwen [**McO88**] showed the existence (and uniqueness) of a hyperbolic metric on M with conic singularities of the form above at the points p_j with constants α_j.

The spectral theory and heat kernel asymptotics of various self-adjoint extensions of the Laplacian Δ_g (and the Laplace operator on more general Riemannian spaces with conic singularities) were studied originally by Cheeger [**Che83**], with later works including Lesch [**Les97**], Mooers [**Moo99**], and Gil–Mendoza [**GM03**]. In particular, the first three analytic assumptions are well-known; see, for example, the book of Lesch [**Les97**, Page 72].

We verify assumption (A4) directly; assumption (A5) follows from the hyperbolicity of the metric (one can treat $M = \overline{M} \setminus \mathcal{P}$ as an open hyperbolic system). See e.g. Brin [**Bal95**, Appendix] for a short and nice proof for ergodicity of Anosov geodesic flows.

Lemma 5.1. *The set*
$$\mathcal{Y} = \{(x, \xi) \in S^*M : \pi(\Phi_t(x, \xi)) \in \mathcal{P} \text{ for some } t \in \mathbb{R}\}$$
has measure zero.

Proof. For $T > 0$, let
$$Y_{\pm,T} = \{(x, \xi) \in S^*M : \pi(\Phi_t(x, \xi)) \in \mathcal{P} \text{ for some } t, \pm t \in (0, T)\}.$$

[2] We consider only surfaces for the sake of brevity; the same method likely extends to hyperbolic cone manifolds of arbitrary dimension as described by McMullen [**McM17**].

For T sufficiently small, $Y_{\pm,T}$ has measure zero by the model form of the metric. We now realize $\mathcal{Y}$ as the countable union of flowouts of $Y_{\pm,T}$ and so it has measure zero. q.e.d.

As (M, g) satisfies the structural and analytic hypotheses, we have the following corollary:

Corollary 5.2. *If (M, g) is a hyperbolic surface with conic singularities, then it is quantum ergodic as in Theorem 1.2.*

References

[Bal95] Werner Ballmann. *Lectures on spaces of nonpositive curvature*, volume 25 of *DMV Seminar*. Birkhäuser Verlag, Basel, 1995. With an appendix by Misha Brin, MR1377265, Zbl 0834.53003.

[BMW12] K. Burns, H. Masur, and A. Wilkinson. The Weil-Petersson geodesic flow is ergodic. *Ann. of Math. (2)*, 175(2):835–908, 2012, MR2993753, Zbl 1254.37005.

[CdV85] Y. Colin de Verdière. Ergodicité et fonctions propres du Laplacien. *Comm. Math. Phys.*, 102(3):497–502, 1985, MR0818831, Zbl 0576.58032.

[Che83] Jeff Cheeger. Spectral geometry of singular Riemannian spaces. *J. Differential Geom.*, 18(4):575–657 (1984), 1983, MR0730920, Zbl 0529.58034.

[GL93] Patrick Gérard and Éric Leichtnam. Ergodic properties of eigenfunctions for the Dirichlet problem. *Duke Math. J.*, 71(2):559–607, 1993, MR1233448, Zbl 0788.35103.

[GM03] Juan B. Gil and Gerardo A. Mendoza. Adjoints of elliptic cone operators. *Amer. J. Math.*, 125(2):357–408, 2003, MR1963689, Zbl 1030.58012.

[Hör07] Lars Hörmander. *The analysis of linear partial differential operators. III.* Classics in Mathematics. Springer-Verlag, Berlin, 2007. Pseudo-differential Operators, Reprint of the 1994 edition, MR2304165, Zbl 1115.35005.

[Hör09] Lars Hörmander. *The analysis of linear partial differential operators. IV.* Classics in Mathematics. Springer-Verlag, Berlin, 2009. Fourier integral operators, Reprint of the 1994 edition, MR2512677, Zbl 1178.35003.

[HW17] Luc Hillairet and Jared Wunsch. On resonances generated by conic diffraction. To appear in Ann. Inst. Fourier. Preprint, arXiv:1706.07869, 2017.

[JMMV14] Lizhen Ji, Rafe Mazzeo, Werner Müller, and Andras Vasy. Spectral theory for the Weil-Petersson Laplacian on the Riemann moduli space. *Comment. Math. Helv.*, 89(4):867–894, 2014, MR3284297, Zbl 1323.35119.

[Les97] Matthias Lesch. *Operators of Fuchs type, conical singularities, and asymptotic methods*, volume 136 of *Teubner-Texte sur Math.* B.G. Teubner, Stuttgart, Leipzig, 1997, MR1449639, Zbl 1156.58302.

[Loo94] Eduard Looijenga. Smooth Deligne-Mumford compactifications by means of Prym level structures. *J. Algebraic Geom.*, 3(2):283–293, 1994, MR1257324, Zbl 0814.14030.

[LSY08] Kefeng Liu, Xiaofeng Sun, and Shing-Tung Yau. Good geometry on the curve moduli. *Publ. Res. Inst. Math. Sci.*, 44(2):699–724, 2008, MR2426362, Zbl 1219.14012.

[Mas76] Howard Masur. Extension of the Weil-Petersson metric to the boundary of Teichmuller space. *Duke Math. J.*, 43(3):623–635, 1976, MR0417456, Zbl 0358.32017.

[McM17] Curtis T. McMullen. The Gauss-Bonnet theorem for cone manifolds and volumes of moduli spaces. *Amer. J. Math.*, 139(1):261–291, 2017, MR3619915, Zbl 1364.53038.

[McO88] Robert C. McOwen. Point singularities and conformal metrics on Riemann surfaces. *Proc. Amer. Math. Soc.*, 103(1):222–224, 1988, MR0938672, Zbl 0657.30033.

[Moo99] Edith A. Mooers. Heat kernel asymptotics on manifolds with conic singularities. *J. Anal. Math.*, 78:1–36, 1999, MR1714065, Zbl 0981.58022.

[MS17] Rafe Mazzeo and Jan Swoboda. Asymptotics of the Weil-Petersson metric. *Int. Math. Res. Not. IMRN*, (6):1749–1786, 2017, MR3658182, Zbl 1405.32014.

[MZ17] Richard Melrose and Xuwen Zhu. Boundary Behaviour of Weil–Petersson and Fibre Metrics for Riemann Moduli Spaces. *Int. Math. Res. Not. IMRN*, (16):5012–5065, 2019, MR4001023, Zbl 07323094.

[PdJ95] M. Pikaart and A. J. de Jong. Moduli of curves with non-abelian level structure. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 483–509. Birkhäuser Boston, Boston, MA, 1995, MR1363068, Zbl 0860.14024.

[Šm74] A. I. Šnirel′ man. Ergodic properties of eigenfunctions. *Uspehi Mat. Nauk*, 29(6(180)):181–182, 1974, MR0402834, Zbl 0324.58020.

[Sog14] Christopher D. Sogge. *Hangzhou lectures on eigenfunctions of the Laplacian*, volume 188 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 2014, MR3186367, Zbl 1312.58001.

[Vak03] Ravi Vakil. The moduli space of curves and its tautological ring. *Notices Amer. Math. Soc.*, 50(6):647–658, 2003, MR1988577, Zbl 1082.14033.

[Wol85] Scott Wolpert. On the Weil-Petersson geometry of the moduli space of curves. *Amer. J. Math.*, 107(4):969–997, 1985, MR0796909, Zbl 0578.32039.

[Wol03] Scott A. Wolpert. Geometry of the Weil-Petersson completion of Teichmüller space. In *Surveys in differential geometry, Vol. VIII (Boston, MA, 2002)*, volume 8 of *Surv. Differ. Geom.*, pages 357–393. Int. Press, Somerville, MA, 2003, MR2039996, Zbl 1049.32020.

[Wol08] Scott A. Wolpert. Behavior of geodesic-length functions on Teichmüller space. *J. Differential Geom.*, 79(2):277–334, 2008, MR2420020, Zbl 1147.30032.

[Wol10] Scott A. Wolpert. *Families of Riemann surfaces and Weil-Petersson geometry*, volume 113 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2010, MR2641916, Zbl 1198.30049.

[Yam04] Sumio Yamada. On the geometry of Weil-Petersson completion of Teichmüller spaces. *Math. Res. Lett.*, 11(2-3):327–344, 2004, MR2067477, Zbl 1060.32005.

[Zel87] Steven Zelditch. Uniform distribution of eigenfunctions on compact hyperbolic surfaces. *Duke Math. J.*, 55(4):919–941, 1987, MR0916129, Zbl 0643.58029.

[ZZ96] Steven Zelditch and Maciej Zworski. Ergodicity of eigenfunctions for ergodic billiards. *Comm. Math. Phys.*, 175(3):673–682, 1996, MR1372814, Zbl 0840.58048.

Department of Mathematics, Texas A&M University
E-mail address: dbaskin@math.tamu.edu

School of Mathematics and Statistics, University of Melbourne
E-mail address: jgell@unimelb.edu.au

Department of Mathematics, Cal State University, Northridge
E-mail address: xiaolong.han@csun.edu