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Cluster construction of the second motivic Chern class

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Abstract

Let G be a split, simple, simply connected, algebraic group over $\mathbb{Q}$. The degree 4, weight 2 motivic cohomology group of the classifying space BG of G is identified with $\mathbb{Z}$. We construct cocycles representing the generator, known as the second universal motivic Chern class. If $\mathrm{G} = \mathrm{SL(m)}$, there is a canonical cocycle, defined by Goncharov (Explicit construction of characteristic classes. Advances in Soviet mathematics, 16, vol 1. Special volume dedicated to I.M.Gelfand's 80th birthday, pp 169–210, 1993). For any group G, we define a collection of cocycles parametrised by cluster coordinate systems on the space of G-orbits on the cube of the principal affine space G/U. Cocycles for different clusters are related by explicit coboundaries, constructed using cluster transformations relating the clusters. The cocycle has three components. The construction of the last one is canonical and elementary; it does not use clusters, and provides the motivic generator of $\mathrm{H}^3(\mathrm{G}(\mathbb{C}), \mathbb{Z}(2))$. However to lift it to the whole cocycle we need cluster coordinates: construction of the first two components uses crucially the cluster structure of the moduli spaces $\mathcal{A}(\mathrm{G}, \mathbb{S})$ related to the moduli space of G-local systems on $\mathbb{S}$. In retrospect, it partially explains why cluster coordinates on the space $\mathcal{A}(\mathrm{G}, \mathbb{S})$ should exist. The construction has numerous applications, including explicit constructions of the universal extension of the group G by K_2, the line bundle on Bun(G) generating its Picard group, Kac–Moody groups, etc. Another application is an explicit combinatorial construction of the second motivic Chern class of a G-bundle. It is a motivic analog of the work of Gabrielov et al. (1974), for any G. We show that the cluster construction of the measurable group 3-cocycle for the group $\mathrm{G}(\mathbb{C})$, provided by our motivic cocycle, gives rise to the quantum deformation of its exponent.

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1 Summary

Given a regular variety X, there is the weight two motivic cohomology complex $\mathbb{Z}^\bullet_{\mathcal{M}}(X; 2)$. It is defined via the Gersten resolution of the Bloch complex [13], see also (31). One has by the very definition (31):

$$\mathrm{H}^{i+2}(\mathbb{Z}^\bullet_{\mathcal{M}}(X; 2)) = \mathrm{H}^i(X, \underline{K}_2), \quad i \geq 0. \tag{1}$$

The definition of the complex $\mathbb{Z}^\bullet_{\mathcal{M}}(X; 2)$ extends to the case when X is a regular simplicial scheme.

Let G be a split, simple, simply connected, algebraic group over $\mathbb{Q}$. Let BG be the classifying space of G. We use its Milnor's simplicial model $\mathrm{BG}_\bullet$. There are canonical isomorphisms:

$$\mathrm{H}^4(\mathbb{Z}_{\mathcal{M}}(\mathrm{BG}_\bullet; 2)) \stackrel{\tau}{=} \mathrm{H}^2(\mathrm{BG}_\bullet, \underline{K}_2) = \mathrm{H}^1(\mathrm{G}, \underline{K}_2) = \mathrm{H}^3_{\mathrm{Betti}}(\mathrm{G}, \mathbb{Z}(2)) = \mathbb{Z}. \tag{2}$$

The last is well known. The third isomorphism was established by Brylinsky–Deligne [4]. The second is the transgression in K_2-cohomology for the universal G-bundle. The first follows from (1) when $i = 2$. See Lemma 2.2 for further details.

We construct cocycles $\mathrm{C}^\bullet$ representing the second universal motivic Chern class, i.e. an element

$$\mathrm{c}_2 \in \mathrm{H}^4(\mathbb{Z}_{\mathcal{M}}(\mathrm{BG}_\bullet; 2)). \tag{3}$$

such that $\tau(c_2) = 1 \in \mathbb{Z}$ in (2). If $\mathrm{G} = \mathrm{SL}_m$, there is a canonical cocycle, defined in [14]. Given a representation V of G, it induces a cocycle for BG. Yet this way we can get only multiples of c_2, e.g. $60 \cdot c_2$ for E_8.

For any group G, we define a collection of cocycles $\mathrm{C}^\bullet$, parametrised by cluster coordinate systems on the space of G-orbits on the cube of the principal affine space G/U. Cocycles for different clusters are related by explicit coboundaries, constructed using cluster transformations relating the clusters.

A cocycle $\mathrm{C}^\bullet$ has three components: $\mathrm{C}^{(1)}, \mathrm{C}^{(2)}, \mathrm{C}^{(3)}$. The construction of the component $\mathrm{C}^{(3)}$ is canonical and elementary; it does not use clusters, and provides a

canonical cocycle for the generator of $\mathrm{H}^3_{\mathrm{Betti}}(\mathrm{G}, \mathbb{Z}(2))$. However to lift $\mathrm{C}^{(3)}$ to a cocycle $\mathrm{C}^\bullet$ we need cluster coordinates: the construction of the first two components uses crucially the cluster structure of the moduli spaces $\mathcal{A}_{\mathrm{G},\mathbb{S}}$, related to the G-character varieties for decorated surfaces $\mathbb{S}$ [17].

In retrospect, it partially explains why the cluster coordinates on the space $\mathcal{A}_{\mathrm{G},\mathbb{S}}$ should exist.

This construction has numerous applications, including an explicit construction of the universal extension of the group G by K_2, the determinant line bundle on $\mathrm{Bun}_{\mathrm{G}}$, Kac–Moody groups, etc.

Another application is an explicit combinatorial construction of the second motivic Chern class of a G-bundle. It is a motivic analog of the work of Gabrielov–Gelfand–Losik [12], for any G.

The cluster construction of the second motivic Chern class also provides its quantum deformation. In Sect. 9 we explain the quantum deformation of the exponent of third measurable cohomology class

$$\beta_3 \in \mathrm{H}^3_{\mathrm{meas}}(\mathrm{G}(\mathbb{C}), \mathbb{R}).$$

2 Introduction and main results

1. The group $\mathrm{H}^3(\mathrm{G}, \mathbb{Z})$**.** In this paper G is a split, simple, simply connected algebraic group over $\mathbb{Q}$. Its Lie algebra $\mathfrak{g}$ is a Lie algebra over $\mathbb{Q}$. The de Rham cohomology group $\mathrm{H}^3_{\mathrm{DR}}(\mathrm{G}; \mathbb{Q})$ is identified with invariant bilinear symmetric forms $\langle *, *\rangle$ on $\mathfrak{g}$:

$$\mathrm{H}^3_{\mathrm{DR}}(\mathrm{G}; \mathbb{Q}) = S^2(\mathfrak{g}^*)^{\mathrm{G}} \stackrel{\sim}{=} \mathbb{Q}. \tag{4}$$

Namely, a form $\langle *, *\rangle \in S^2(\mathfrak{g}^*)^{\mathrm{G}}$ gives rise to the Ad_{G}-invariant 3-form on $\mathfrak{g}$:

$$\varphi_{\langle *,*\rangle} \in \Lambda^3(\mathfrak{g}^*)^{\mathrm{G}}, \qquad \varphi_{\langle *,*\rangle}(A, B, C) := \langle A, [B, C]\rangle. \tag{5}$$

It determines a closed biinvariant differential 3-form on G, providing isomorphism (4). For example, for $\mathrm{G} = \mathrm{SL}_m$ we get rational multiples of the form $\mathrm{Tr}(g^{-1}dg)^3$.

Let $\mathfrak{h}$ be the Lie algebra of the Cartan group H of G, and W the Weyl group of G. Then

$$S^2(\mathfrak{g}^*)^{\mathrm{G}} = S^2(\mathfrak{h}^*)^W. \tag{6}$$

It is known that the canonical generator of $\mathrm{H}^3_{\mathrm{DR}}(\mathrm{G}; \mathbb{Z})$ is provided by the Killing form normalized so that its value on the shortest coroot is equal to 1. We call it the DeRham generator.

Denote by $\mathrm{H}^*_{\mathrm{B}}$ the singular (Betti) cohomology of a topological space. The integration provides an isomorphism between the DeRham and Betti cohomology, and identifies the generators:

$$\int : \mathrm{H}^3_{\mathrm{DR}}(\mathrm{G}; \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^3_{\mathrm{B}}(\mathrm{G}(\mathbb{C}); \mathbb{Z}(2)), \quad \mathbb{Z}(2) := (2\pi i)^2\mathbb{Z}. \tag{7}$$

Denote by BG the classifying space for the algebraic group G. It is well known that

$$\mathrm{H}^4(\mathrm{BG}, \mathbb{Z}(2)) = \mathrm{H}^3(\mathrm{G}, \mathbb{Z}(2)). \tag{8}$$

To introduce the motivic upgrade of this isomorphism, we recall the weight two motivic complex.

2. The K_2-cohomology. Given a field F, the Milnor K_2-group of F is the abelian group given by the quotient of the wedge square $\Lambda^2\mathrm{F}^\times$ of the multiplicative group $\mathrm{F}^\times$ by the subgroup generated by the Steinberg relations $(1-x) \wedge x$, where $x \in \mathrm{F}^\times - \{1\}$:

$$K_2(\mathrm{F}) := \Lambda^2\mathrm{F}^\times / \langle (1-x) \wedge x \rangle. \tag{9}$$

Let X be a regular algebraic variety over a field k, with the field of functions $k(X)$. Denote by X_d the set of irreducible subvarieties of codimenion d on X. Then there is a complex of abelian groups:

$$\underline{K_2^\bullet} := K_2(k(X)) \xrightarrow{\text{res}} \bigoplus_{D \in X_1} k(D)^\times \xrightarrow{\text{val}} \bigoplus_{X_2} \mathbb{Z}. \tag{10}$$

We place it in the degrees [0, 2]. The right map is the valuation map. The left map is the tame symbol:

$$\text{res} : f \wedge g \longmapsto \sum_{D \in X_1} (-1)^{\text{val}_D(f)\text{val}_D(g)} \frac{f^{\text{val}_D(g)}}{g^{\text{val}_D(f)}} | D. \tag{11}$$

We denote its cohomology by $\mathrm{H}^*(X, \underline{K_2})$.

3. The Hodge regulator map. For a regular complex algebraic variety X, the group $\mathrm{H}^1(X, \underline{K_2})$ provides *some* elements of $\mathrm{H}^3(X(\mathbb{C}); \mathbb{Z}(2))$ of the Hodge type (2, 2), defined as currents of algebraic–geometric origin as follows. Given a divisor $D \subset X$ and a rational function f on D, there is a 3-current $\psi_{D,f}$ on $X(\mathbb{C})$ whose value on a smooth differential form ω is

$$\psi_{D,f}(\omega) := 2\pi i \cdot \int_{D(\mathbb{C})} d\log(f) \wedge \omega. \tag{12}$$

Its differential is the δ-current, given by the integration along the codimension two cycle on X provided by the divisor $\text{div}(f)$ of f:

$$d\psi_{D,f} = (2\pi i)^2 \delta_{\text{div}(f)}. \tag{13}$$

The cycles in the complex calculating $\mathrm{H}^1(X, \underline{K}_2)$ are given by linear combinations

$$\sum_i (D_i, f_i), \quad \sum_i \mathrm{div}(f_i) = 0. \tag{14}$$

Here D_i is an irreducible divisor in X, and f_i a rational function on D_i. The cocycle condition implies that the 3-current $\sum_i \psi_{D_i, f_i}$ is closed, defining an element of $\mathrm{H}^3(X(\mathbb{C}), \mathbb{Z}(2))$ of the Hodge type $(2, 2)$. Denote the subgroup of such classes as $\mathrm{H}^3_{2,2}(X(\mathbb{C}), \mathbb{Z}(2))$. So we get the *Hodge regulator map*

$$\mathrm{reg}_{\mathcal{H}} : \mathrm{H}^1(X; \underline{K}_2) \longrightarrow \mathrm{H}^3_{2,2}(X(\mathbb{C}); \mathbb{Z}(2)). \tag{15}$$

Beilinson's generalized Hodge conjecture [1] predicts that it is an isomorphism modulo torsion. This generalises the Hodge conjecture isomorphism for the codimension two cycles:

$$\mathrm{H}^2(X; \underline{K}_2) \otimes \mathbb{Q} = \mathrm{CH}^2(X) \otimes \mathbb{Q} \xrightarrow{\sim} \mathrm{H}^4_{2,2}(X(\mathbb{C}); \mathbb{Q}(2)). \tag{16}$$

Our next goal is an explicit description of the group $\mathrm{H}^3(\mathrm{G}(\mathbb{C}), \mathbb{Z}(2))$ via the Hodge regulator map.

4. The generator of the group $\mathrm{H}^1(\mathrm{G}, \underline{K}_2) = \mathbb{Z}$**.** Denote by I the set of vertices of the Dynkin diagram for the group G. Let C_{ij}, $i, j \in \mathrm{I}$, be the Cartan matrix. Recall the Bruhat decomposition of G:

$$G = \coprod_{w \in W} \mathcal{B}_w, \quad \mathcal{B}_w := \mathrm{UH}\overline{w}\mathrm{U}. \tag{17}$$

Here $\overline{w}$ is the canonical lift of a Weyl group element w to G. Therefore, given a Weyl group element $w \in W$ and a character χ of the Cartan group H, we get a regular function χ_w on the Bruhat cell $\mathcal{B}_w$:

$$\chi_w \in \mathcal{O}^\times(\mathcal{B}_w), \qquad \chi_w(u_1 h \overline{w} u_2) := \chi(h). \tag{18}$$

The dominant weight Δ_k gives rise to a regular function on the Bruhat cell $\mathcal{B}_w$, denoted by $\Delta_{k,w}$.

Recall the longest element w_0 of W. The Bruhat divisor $\mathcal{B}_{s_k w_0}$ is determined by the equation $\Delta_{k,w_0} = 0$. Let us introduce the following rational function on this divisor. Denote by $i_k : \mathcal{B}_{s_k w_0} \subset \mathrm{G}$ the natural embedding. Set

$$F_k := i_k^* \left(\Delta^{-1}_{k, s_k \omega_0} \prod_{i \in \mathrm{I} \setminus \{k\}} (\Delta_{i,w_0})^{\frac{C_{ki}}{2}} \right)^{d_k}. \tag{19}$$

Here the integers $\{d_i\}$ are the symmetrizers: $d_i \mathrm{C}_{ij} = d_j \mathrm{C}_{ji}$. Let us consider the following formal sum of the pairs (a Bruhat divisor, a rational function on it):

$$\mathrm{C}^{(3)} = \bigoplus_{k \in \mathrm{I}} \Big(\mathcal{B}_{s_k w_0}, F_k\Big). \tag{20}$$

Theorem 2.1 *The element* $\mathrm{C}^{(3)}$ *is a 1-cocycle in the complex* $\underline{K}_2^\bullet \otimes \mathbb{Z}[\frac{1}{2}]$. *Its cohomology class* $[\mathrm{C}^{(3)}]$ *generates the group* $\mathrm{H}^1(\mathrm{G}, \underline{K}_2) = \mathbb{Z}$. *Its Hodge realization* $\mathrm{reg}_{\mathcal{H}}[\mathrm{C}^{(3)}]$ *generates the group* $\mathrm{H}^3(\mathrm{G}(\mathbb{C}), \mathbb{Z}(2))$.

5. An example: $\mathrm{H}^3(\mathrm{SL}_2(\mathbb{C}))$**.** There are three ways to describe this group:

1. *Betti*. One has $\mathrm{H}^3_{\mathrm{B}}(SL_2(\mathbb{C}); \mathbb{Z}) = \mathbb{Z}$ since $SU(2) = S^3$ is a retract of $SL_2(\mathbb{C})$.
2. *De Rham*. The generator of $H^3_{\mathrm{DR}}(\mathrm{SL}_2; \mathbb{Z})$ is given by the form $\mathrm{Tr}(g^{-1}dg)^3$ on SL_2. The coefficient $\mathbb{Z}(2)$ in the comparison isomorphism (7) reflects the volume formula $\mathrm{vol}(S^3) = 2\pi^2$.
3. *Motivic*. A line L in a 2-dimensional vector space V provides a divisor B_L with a function f:

$$\mathrm{B}_L := \{g \in SL_2 \mid gL = L\}, \qquad gl = f(g)l, \quad \forall g \in \mathrm{B}_L, \;\; l \in L.$$

The 3-current $\psi_{\mathrm{B}_L, f}$ generates $\mathrm{H}^3_{\mathrm{B}}(SL_2(\mathbb{C}); \mathbb{Z}(2))$.

Theorem 2.1 is proved in Sect. 7. The group $\mathrm{H}^1(\mathrm{G}, \underline{K}_2)$ was described by Brylinsky–Deligne [4]. Theorem 2.1 provides a specific cocycle for the generator of $\mathrm{H}^1(\mathrm{G}, \underline{K}_2)$. Such a cocycle, of course, is not unique. Our cocycle is tied up with the cluster structure of the space $\mathcal{A}_{\mathrm{G},\mathbb{S}}$. Let us elaborate on this.

6. The key feature of the cocycle $\mathrm{C}^{(3)}$**.** We identify $\mathrm{H}^1(\mathrm{G}, \underline{K}_2)$ with the G-invariants $\mathrm{H}^1(\mathrm{G} \times \mathrm{G}, \underline{K}_2)^{\mathrm{G}}$, for the left diagonal action of G. There are three projections

$$p_{ij} : \mathrm{G}^3 \longrightarrow \mathrm{G}^2, \quad 1 \leq i < j \leq 3, \quad p_{ij}(g_1, g_2, g_3) := (g_i, g_j). \tag{21}$$

We claim that

$$p_{12}^*[\mathrm{C}^{(3)}] + p_{23}^*[\mathrm{C}^{(3)}] - p_{13}^*[\mathrm{C}^{(3)}] = 0. \tag{22}$$

Our goal is to prove this on the level of complexes, constructing explicitly a G-invariant element of $K_2(\mathrm{G}^3)$ whose residue is the cocycle representing the cohomology class on the left. This boils down to a construction of a certain G-invariant element $\mathrm{C}^{(2)}$ in $\mathbb{Q}(\mathrm{G}^3)^* \wedge \mathbb{Q}(\mathrm{G}^3)^*$.

7. The element $\mathrm{C}^{(2)}$**.** Observe that $\mathrm{U}\backslash\mathrm{G}/\mathrm{U} = \mathrm{G}\backslash(\mathrm{G}/\mathrm{U} \times \mathrm{G}/U)$. By the construction, the cocycle $\mathrm{C}^{(3)}$ is invariant under the right action of the group $\mathrm{U} \times \mathrm{U}$ on $\mathrm{G} \times \mathrm{G}$. Note that projections (21) determine three similar canonical projections involving $\mathcal{A} := \mathrm{G}/\mathrm{U}$ which are denoted, abusing notation, by

$$p_{ij} : \mathcal{A}^3 \longrightarrow \mathcal{A}^2 \quad 1 \leq i < j \leq j. \tag{23}$$

So we are looking for an element

$$\begin{aligned} &\mathrm{C}^{(2)} \in \mathbb{Q}(\mathcal{A}^3)^* \wedge \mathbb{Q}(\mathcal{A}^3)^* \\ &\mathrm{res}(\mathrm{C}^{(2)}) = p_{12}^*\mathrm{C}^{(3)} + p_{23}^*\mathrm{C}^{(3)} - p_{13}^*\mathrm{C}^{(3)}. \end{aligned} \tag{24}$$

Explicitely, we can write

$$\mathrm{C}^{(2)} = \sum_{i,j} \widetilde{\varepsilon}_{ij} \cdot \mathrm{A}_i \wedge A_j, \quad \mathrm{A}_i \in \mathbb{Q}(\mathcal{A}^3)^{\mathrm{G}} \quad \widetilde{\varepsilon}_{ij} = -\widetilde{\varepsilon}_{ji} \in \mathbb{Z}. \tag{25}$$

Here $\{A_j\}$ is a collection of G-invariant regular functions on $\mathcal{A}^3$. So to construct $\mathrm{C}^{(2)}$ we must exibit a collection of such functions. This is exactly what the cluster structure on the space $\mathrm{Conf}_3(\mathcal{A}) := \mathrm{G}\backslash\mathcal{A}^3$ does: the functions $\{A_j\}$ are the cluster coordinates, and $\widetilde{\varepsilon}_{ij}$ is the skew-symmetrized exchange matrix.

The element $\mathrm{C}^{(2)}$ is defined in the end of Sect. 5, where we recall the construction of a cluster for the space $\mathrm{Conf}_3(\mathcal{A})$. Different cluster coordinate systems deliver elements $\mathrm{C}^{(2)}$ which differ by explicitly given sums of Steinberg relations, and therefore define the same class in K_2.

Note that the cluster structure does more: it delivers elements where the number of functions A_i equals to the dimension of $\mathrm{Conf}_3(\mathcal{A})$, and these functions are regular coordinates on this space.

On the other hand, this partially explains why the cluster coordinates on $\mathrm{Conf}_3(\mathcal{A})$ should exist: we know that an element (24) must exist.

8. Remark. A similar problem for the deRham cocycle is much easier, and has a canonical solution:

$$3 \cdot d\mathrm{Tr}\Big(g_1g_2dg_2^{-1}dg_1^{-1}\Big) = \mathrm{Tr}(g_1^{-1}dg_1)^3 + \mathrm{Tr}(g_2^{-1}dg_2)^3 - \mathrm{Tr}\Big((g_1g_2)^{-1}d(g_1g_2)\Big)^3. \tag{26}$$

To explain the general problem, and how the elements $\mathrm{C}^{(2)}, \mathrm{C}^{(3)}$ fit in the motivic framework, we recall two basic ingredients of the construction: the weight two motivic complex, and Milnor's model for BG.

9. The weight two motivic complex. Recall the cross-ratio of four points on $\mathbb{P}^1(\mathrm{F})$:

$$r(s_1, s_2, s_3, s_4) := \frac{(s_1 - s_2)(s_3 - s_4)}{(s_1 - s_4)(s_2 - s_3)}, \quad r(\infty, -1, 0, z) = z. \tag{27}$$

Given any five distinct points $s_1, \dots, s_5$ on $\mathbb{P}^1(\mathrm{F})$, consider the element

$$\sum_{i=1}^{5}\{-r(s_i, s_{i+1}, s_{i+2}, s_{i+3})\} \in \mathbb{Z}[\mathrm{F}], \quad i \in \mathbb{Z}/5\mathbb{Z}. \tag{28}$$

Denote by $\mathrm{R}_2(\mathrm{F})$ the subgroup of $\mathbb{Z}[\mathrm{F}^* - \{1\}]$ generated by elements (28) for all 5-tuples of distinct points. The Bloch group $\mathrm{B}_2(\mathrm{F})$ is the quotient

$$\mathrm{B}_2(\mathrm{F}) := \frac{\mathbb{Z}[\mathrm{F}^* - \{1\}]}{\mathrm{R}_2(\mathrm{F})}. \tag{29}$$

The key point is that there is a well defined map

$$\begin{aligned} \delta : \mathrm{B}_2(\mathrm{F}) &\longrightarrow \mathrm{F}^* \wedge \mathrm{F}^*. \\ \{x\} &\longmapsto (1-x) \wedge x. \end{aligned} \tag{30}$$

This complex, placed in the degrees $[1, 2]$, is called the *Bloch complex*. Note that $\mathrm{Coker}(\delta) = K_2(\mathrm{F})$.

Let X be a regular algebraic variety over a field k. Then there is a complex of abelian groups placed in the degrees $[1, 4]$, and called the *weight two motivic complex* of X:

$$\mathbb{Z}^\bullet_{\mathcal{M}}(X; 2) := \mathrm{B}_2(k(X)) \stackrel{\delta}{\longrightarrow} k(X)^* \wedge k(X)^* \stackrel{\mathrm{res}}{\longrightarrow} \bigoplus_{D \in X_1} k(D)^\times \stackrel{\mathrm{val}}{\longrightarrow} \bigoplus_{X_2} \mathbb{Z}. \tag{31}$$

It is a good time now to prove the following Lemma which we refer to discussing the definition of the second motivic Chern class c_2.

Lemma 2.2 *There are canonical epimorphisms*

$$\mathrm{H}^4(\mathbb{Z}_{\mathcal{M}}(\mathrm{BG}_\bullet; 2)) \stackrel{\tau}{=} \mathrm{H}^2(\mathrm{BG}_\bullet, \underline{K}_2) = \mathrm{H}^1(\mathrm{G}, \underline{K}_2). \tag{32}$$

Proof The second isomorphism is the transgression in the universal G-bundle on $\mathrm{BG}_\bullet$. It can be defined as follows. Consider the following diagram.

$$\begin{tikzcd} \bigoplus_{D \in \mathrm{X}_1(\mathrm{G}^2)} \mathbb{Q}(D)^\times & \bigoplus_{D \in \mathrm{X}_1(\mathrm{G})} \mathbb{Q}(D)^\times \arrow[l, "s^*"'] \\ K_2(\mathbb{Q}(\mathrm{G}^2)) \arrow[u, "\mathrm{res}"] & \end{tikzcd}$$

Then the second map is given by the restriction of the cocycle on the diagonal to its top right part.

The map τ is defined similarly, by using the diagram in Sect. 2, paragraph 10, where the principal affine space $\mathcal{A}$ is replaced by the group G. Then the map τ is given by the restriction of the circled cocycle to its two-component part. The map τ is an isomorphism due to isomorphism (1), evident from (31). □

Recall that

$$\mathrm{H}^4(\mathbb{Q}^\bullet_{\mathcal{M}}(\mathrm{BG}_\bullet; 2) = \mathrm{H}^3(\mathbb{Q}^\bullet_{\mathcal{M}}(\mathrm{G}; 2)) = S^2(\mathfrak{h}^*)^W = \mathbb{Q}. \tag{33}$$

Definition 2.3 The second universal motivic Chern class

$$\mathrm{c}_2 \in \mathrm{H}^4(\mathrm{BG}_\bullet, \mathbb{Z}_{\mathcal{M}}(2)) \tag{34}$$

is the integral generator which corresponds, under isomorphisms (33), to the Killing form on $\mathfrak{g}$ normalized so that its values on the shortest coroot is equal to 1.

10. Milnor's simplicial model $\mathrm{BG}_\bullet$ **of the classifying space** BG**.** Recall the simplicial realization $\mathrm{EG}_\bullet$ of the space EG:

$$\begin{tikzcd} \cdots & G^3 \arrow[r, shift left=4] \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] \arrow[r, shift right=4] & G^2 \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & G \end{tikzcd}$$

In particular, there are the $n+1$ standard maps

$$s_{n,i} : \mathrm{G}^{n+1} \longrightarrow \mathrm{G}^n, \ (g_0, \ldots, g_n) \longmapsto (g_0, \ldots, \widehat{g_i}, \ldots, g_n), \ \ i = 0, \ldots, n. \tag{35}$$

Then we set $\mathrm{BG}_\bullet := \mathrm{G}\backslash\mathrm{EG}_\bullet$:

$$\begin{tikzcd} \cdots & G^2 \arrow[r, shift left=4] \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] \arrow[r, shift right=4] & G \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & * \end{tikzcd}$$

Let $X \longmapsto \mathcal{F}^\bullet(X)$ be an assignment to an algebraic variety X a complex of abelian groups $\mathcal{F}^\bullet(X)$, contravariant under surjective maps $X \to Y$. We define the complex $\mathcal{F}^\bullet(\mathrm{EG}_\bullet)$ as the total complex associated with the bicomplex

$$\begin{gathered} \cdots \stackrel{s^*}{\longleftarrow} \mathcal{F}^\bullet(\mathrm{G}^4) \stackrel{s^*}{\longleftarrow} \mathcal{F}^\bullet(\mathrm{G}^3) \stackrel{s^*}{\longleftarrow} \mathcal{F}^\bullet(\mathrm{G}^2) \stackrel{s^*}{\longleftarrow} \mathcal{F}^\bullet(\mathrm{G}) \stackrel{s^*}{\longleftarrow} \mathcal{F}^\bullet(*). \\ s^* := \sum_{i=0}^{n} (-1)^i s^*_{n,i} : \mathcal{F}^\bullet(\mathrm{G}^n) \longrightarrow \mathcal{F}^\bullet(\mathrm{G}^{n+1}). \end{gathered} \tag{36}$$

Applying this construction to the weight two motivic complex $\mathbb{Z}^\bullet_{\mathcal{M}}(*; 2)$, and taking the G-invariants, we get the complex

$$\mathbb{Z}_{\mathcal{M}}(\mathrm{BG}_\bullet; 2) := \mathbb{Z}_{\mathcal{M}}(\mathrm{EG}_\bullet; 2)^{\mathrm{G}}.$$

Let N be a maximal unipotent subgroup. Recall the principal affine space $\mathcal{A} := \mathrm{G}/\mathrm{N}$.

The canonical projection $\mathrm{G}^n \longrightarrow \mathcal{A}^n$ induces a map of complexes, denoted $\varphi_{\mathcal{A}\to\mathrm{G}}$:

$$
\begin{tikzcd}
\Big(\ldots & \mathbb{Z}^{\bullet}_{\mathcal{M}}(\mathcal{A}^4;2) \arrow[l, "s^*"'] \arrow[d] & \mathbb{Z}^{\bullet}_{\mathcal{M}}(\mathcal{A}^3;2) \arrow[l, "s^*"'] \arrow[d] & \mathbb{Z}^{\bullet}_{\mathcal{M}}(\mathcal{A}^2;2) \arrow[l, "s^*"'] \arrow[d] & \mathbb{Z}^{\bullet}_{\mathcal{M}}(\mathcal{A};2)\Big)^{\mathrm{G}} \arrow[l, "s^*"'] \arrow[d] \\
\Big(\ldots & \mathbb{Z}^{\bullet}_{\mathcal{M}}(G^4;2) \arrow[l, "s^*"'] & \mathbb{Z}^{\bullet}_{\mathcal{M}}(\mathrm{G}^3;2) \arrow[l, "s^*"'] & \mathbb{Z}^{\bullet}_{\mathcal{M}}(\mathrm{G}^2;2) \arrow[l, "s^*"'] & \mathbb{Z}^{\bullet}_{\mathcal{M}}(\mathrm{G};2)\Big)^{\mathrm{G}} \arrow[l, "s^*"']
\end{tikzcd}
$$

We define a degree 4 cycle in the total complex associated with the bicomplex illustarted on the diagram. It is given by the encircled in the bicomplex degree 4 cocycle $\mathrm{C}^{\bullet} = (\mathrm{C}^{(1)}, \mathrm{C}^{(2)}, \mathrm{C}^{(3)})$:

$$
\begin{aligned}
&\mathrm{C}^{(1)} \in \mathrm{B}_2\Big(\mathbb{Q}(\mathrm{Conf}_4(\mathcal{A}))\Big), \quad \mathrm{C}^{(2)} \in \bigwedge^2 \mathbb{Q}(\mathrm{Conf}_3(\mathcal{A}))^{\times}, \\
&\mathrm{C}^{(3)} \in \bigoplus_{D\in \mathrm{X}_1(\mathrm{Conf}_2(\mathcal{A}))} \mathcal{O}_D^{\times}.
\end{aligned} \tag{37}
$$

The cocycle property just means that

$$
s^*(\mathrm{C}^{(1)}) = 0, \quad \delta(\mathrm{C}^{(1)}) = s^*(\mathrm{C}^{(2)}), \quad \mathrm{res}(\mathrm{C}^{(2)}) = s^*(\mathrm{C}^{(3)}), \quad \mathrm{div}(\mathrm{C}^{(3)}) = 0. \tag{38}
$$

The cocycle will be well defined up to a coboundary. It provides a cocycle $\varphi_{\mathcal{A}\to\mathrm{G}}(\mathrm{C}^{\bullet})$.

$$
\begin{tikzcd}
\cdots & \cdots \arrow[l, "s^*"'] & \bigoplus\limits_{D\in \mathrm{X}_2(\mathrm{Conf}_2(\mathcal{A}))} \mathbb{Z} \arrow[l, "s^*"'] \\
\cdots \arrow[u, "\mathrm{div}"] & \bigoplus\limits_{D\in \mathrm{X}_1(\mathrm{Conf}_3(\mathcal{A}))} \mathbb{Q}(D)^{\times} \arrow[l, "s^*"'] \arrow[u, "\mathrm{div}"] & \bigoplus\limits_{D\in \mathrm{X}_1(\mathrm{Conf}_2(\mathcal{A}))} \mathbb{Q}(D)^{\times} \arrow[l, "s^*"'] \arrow[u, "\mathrm{div}"] \\
\bigwedge^2 \mathbb{Q}(\mathrm{Conf}_4(\mathcal{A}))^{\times} \arrow[u, "\mathrm{res}"] & \bigwedge^2 \mathbb{Q}(\mathrm{Conf}_3(\mathcal{A}))^{\times} \arrow[l, "s^*"'] \arrow[u, "\mathrm{res}"] & \bigwedge^2 \mathbb{Q}(\mathrm{Conf}_2(\mathcal{A}))^{\times} \arrow[l, "s^*"'] \arrow[u, "\mathrm{res}"] \\
B_2\Big(\mathbb{Q}(\mathrm{Conf}_4(\mathcal{A}))\Big) \arrow[u, "\delta"] & B_2\Big(\mathbb{Q}(\mathrm{Conf}_3(\mathcal{A}))\Big) \arrow[l, "s^*"'] \arrow[u, "\delta"] & B_2\Big(\mathbb{Q}(\mathrm{Conf}_2(\mathcal{A}))\Big) \arrow[l, "s^*"'] \arrow[u, "\delta"]
\end{tikzcd}
$$

Theorem 2.4 *There is a cocycle* $\mathrm{C}^{\bullet} = (\mathrm{C}^{(1)}, \mathrm{C}^{(2)}, \mathrm{C}^{(3)})$ *such that the induced cocycle* $\varphi_{\mathcal{A}\to\mathrm{G}}(\mathrm{C}^{\bullet})$ *represents the second motivic Chern class*

$$
\mathrm{c}_2 \in \mathrm{H}^4\mathbb{Z}_{\mathcal{M}}(\mathrm{BG}_{\bullet}; 2). \tag{39}
$$

If $\mathrm{G} = \mathrm{SL}_m$, there is a canonical cocycle $\mathrm{C}^{\bullet}$, defined in [14]. Given a non-trivial representation V of the group G, the pull back of this cocycle via the embedding

$G \hookrightarrow SL(V)$ is a non-trivial cocycle for G. However in general we can not get the generator of the group H^4 this way. For example, for the group of type E_8, the closest we get this way is $60 \cdot c_2$ for the adjoint representation.

11. Cluster nature of the construction. Our construction is cluster. The construction of the components $C^{(1)}, C^{(2)}$ uses essentially the construction of the cluster structure on the moduli space $\mathcal{A}_{G,\mathbb{S}}$ [8], closely related to the moduli space of G-local systems on a decorated surface $\mathbb{S}$, in the case when $\mathbb{S}$ is a triangle or a quadrilateral. For $G = SL_m$ this is explained in [8, Section 12].

On the other hand, the construction of the cluster structure for the general moduli space $\mathcal{A}_{G,\mathbb{S}}$ follows immediately from the one for a triangle and rectangle, provided that we prove that these cluster structures are invariant under the twisted cyclic rotations of these polygons. The latter is the most challenging part of the proof in [17], which takes about 30 pages of elaborate calculations, with the final result coming as a pleasant surprise. Our approach explains why the cluster structure should be invariant under the twisted cyclic shift, and establishes a key step of the proof without any elaborate computations.

The last component $C^{(3)}$ is crucial to prove that the cohomology class $[C^\bullet]$ coincides with the motivic Chern class c_2.

12. Applications. This construction has numerous applications. Here are some of them.

1. An explicit construction on the level of cocycles of the universal extension of the group G by K_2.
   Thus we get an explicit construction of the Kac–Moody group $\widehat{G}$ given by a central extension of the loop group:

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \widehat{G} \longrightarrow G((t)) \longrightarrow 1. \tag{40}$$

2. We get an explicit construction of the line bundle generating the Picard group of $\mathrm{Bun}_G(\Sigma)$, where Σ is a Riemann surface with punctures. See [20] for the background on the generating line bundle.
3. Using the dilogarithm and the weight two exponential complex [15], we get an explicit combinatorial formula for the second Chern class of a G-bundle on a manifold, with values in the Beilinson–Deligne complex. In particular we get a combinatorial formula for the second integral Chern class, in the spirit of the Gabrielov–Gelfand–Losik combinatorial formula [12] for the first Pontryagin class.
4. Given a punctured surface S, let $\mathcal{U}_{G,S}$ be the moduli space parametrizing *framed* unipotent G-local systems on S, that is G-local systems with unipotent monodromies around the punctures, equipped with a reduction to the Borel subgroup at each puncture.
   Let M be a threefold whose boundary is the surface $\overline{S}$ with filled punctures. We prove that the subspace $\mathcal{M}_{G,M} \subset \mathcal{U}_{G,S}$ parametrising framed unipotent G-local systems on S which can be extended to M is a K_2-Lagrangian. We define the *motivic volume* map on its generic part

$$\mathrm{Vol}_{\mathrm{mot}} : \mathcal{M}^\circ_{G,M} \longrightarrow B_2(\mathbb{C}) \tag{41}$$

valued in the Bloch group of $\mathbb{C}$. Its composition with the map $B_2(\mathbb{C}) \to \mathbb{R}$ provided by the Bloch–Wigner dilogarithm is a volume map generalising the volume of a hyperbolic threefold. For $G = GL_m$ these results were obtained in [6] using the canonical cocycle for GL_m.
5. The cluster construction of the second motivic Chern class provides at the same time its quantum deformation, see Sect. 9.

3 The simplest example: $G = SL_2$

The cocycle $C^{(\bullet)}$ for the generator of $H^4(B_{SL_2\bullet}, \mathbb{Z}_{\mathcal{M}}(2))$ has three components. Using $G = SL_2$, they are:

$$\begin{aligned} C^{(1)} &\in B_2\Big(\mathbb{Q}(G^4)\Big)^G, \\ C^{(2)} &\in \Big(\mathbb{Q}(G^3)^\times \bigwedge \mathbb{Q}(G^3)^\times\Big)^G, \\ C^{(3)} &\in \Big(\mathbb{Q}(D)^\times\Big)^G, \qquad D \in \mathrm{div}(G^2)^G. \end{aligned} \tag{42}$$

Fix a complex two dimensional vector space V_2 with an area form Δ. Then a flag is a 1-dimensional subspace of V_2, and a decorated flag is a non-zero vector $v \in V_2$. Two decorated flags are in generic position if $\Delta(v_1 v_2) \neq 0$. To construct a cocycle we pick a non-zero vector $v \in V_2$.
The cycle $C^{(3)}$**.** There is G-invariant divisor

$$D_v \subset G^2, \qquad D_v := \{(g_1, g_2) \in G^2 \mid \Delta(g_1 v, g_2 v) = 0\}. \tag{43}$$

It carries a function

$$\lambda_v(g_1, g_2) := \frac{g_1 v}{g_2 v}, \qquad (g_1, g_2) \in D_v \subset G^2. \tag{44}$$

Note that the residue of this function is equal to zero. So we set

$$C^{(3)} := (D_v, \lambda_v). \tag{45}$$

The G-invariant divisor with a function (D_v, λ_v) in G^2 is the same thing as a divisor with a function (D'_v, λ'_v) for the quotient $G^2/G = G$. Namely, we identify G with the section $\{e\} \times G \subset G^2$.

To check that the current $2\pi i \cdot d\log(\lambda'_v)\delta(D'_v)$ generates $H^3(SL_2(\mathbb{C}), \mathbb{Z}(2))$, we integrate it over the cycle generating the 3-dimensional homology of $SL_2(\mathbb{C})$, given by the subgroup $SU(2)$. Precisely, pick a Hermitian form $\langle , \rangle$ in V_2 and an orthonornal basis (v, w) containing v. Then

$$SU(2) = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \qquad \alpha, \beta \in \mathbb{C}, \ \ |\alpha|^2 + |\beta|^2 = 1.$$
$$D'_v = \begin{pmatrix} a^{-1} & b \\ 0 & a \end{pmatrix}, \quad D'_v \cap SU(2) = \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix}, |\alpha| = 1, \quad \lambda'_v = \alpha. \tag{46}$$

Integrating the current over $SU(2)$ we get $2\pi i \cdot \int_{|\alpha|=1} d\log\alpha = (2\pi i)^2$. So its cohomology class generates the group $\mathrm{H}^3(\mathrm{SL}_2(\mathbb{C}), \mathbb{Z}(2))$.

The component $\mathrm{C}^{(2)}$**.** Below we use the notation $v_i := g_i v$. We define $\mathrm{C}^{(2)}$ by setting

$$\mathrm{C}^{(2)} \in \Big(\mathbb{Q}(\mathrm{SL}_2^3)^\times \wedge \mathbb{Q}(\mathrm{SL}_2^3)^\times\Big)^{\mathrm{SL}_2}$$
$$\mathrm{C}^{(2)} = \Delta(v_1v_2) \wedge \Delta(v_1v_3) + \Delta(v_1v_3) \wedge \Delta(v_2v_3) + \Delta(v_2v_3) \wedge \Delta(v_1v_2). \tag{47}$$

Let us compute the residue of $\mathrm{C}^{(2)}$. The divisors supporting the residue are:

$$D_{ij} = \{\Delta(v_iv_j) = 0\}.$$

The residue of $\mathrm{C}^{(2)}$ at the divisor D_{12} is

$$\begin{aligned} &\mathrm{res}_{\Delta(v_1v_2)=0}(\mathrm{C}^{(2)}) \\ &\quad = \mathrm{res}_{\Delta(v_1v_2)=0}\Big(\Delta(v_1v_2) \wedge \frac{\Delta(v_1v_3)}{\Delta(v_2v_3)}\Big) = \frac{\Delta(v_1v_3)}{\Delta(v_2v_3)} = \Big(\frac{v_1}{v_2}\Big) = (D_{12}, \lambda_{1/2}). \end{aligned} \tag{48}$$

The result does not depend on v_3 since on the divisor $\{\Delta(v_1v_2) = 0\}$ the vectors v_1 and v_2 are parallel. The total residue is

$$\mathrm{res}(\mathrm{C}^{(2)}) = (D_{12}, \lambda_{1/2}) + (D_{23}, \lambda_{2/3}) + (D_{31}, \lambda_{3/1}) = s^*\mathrm{C}^{(3)}.$$

It splits into three parts, one for each edge of the triangle. So we can set

The component $\mathrm{C}^{(1)}$**.** Consider the cross-ratio

$$\mathrm{C}^{(1)} := \{r_2(v_1, v_2, v_3, v_4)\}_{B_2} = \left\{-\frac{\Delta(v_1v_2)\Delta(v_3v_4)}{\Delta(v_1v_4)\Delta(v_2v_3)}\right\}_{B_2}. \tag{49}$$

The 5-term relation in the definition of the Bloch group implies that $s^*\mathrm{C}^{(1)} = 0$.

The key step is the calculation of the differential in the Bloch complex:

$$\delta\mathrm{C}^{(1)} = \delta\{r_2(v_1, v_2, v_3, v_4)\} = -\frac{1}{2}\mathrm{Alt}_4\Big(\Delta(v_1v_2) \wedge \Delta(v_1v_3)\Big). \tag{50}$$

where Alt_4 means that we take the alternating sum over all permutations of vectors v_1, v_2, v_3, v_4 (Fig. 1).

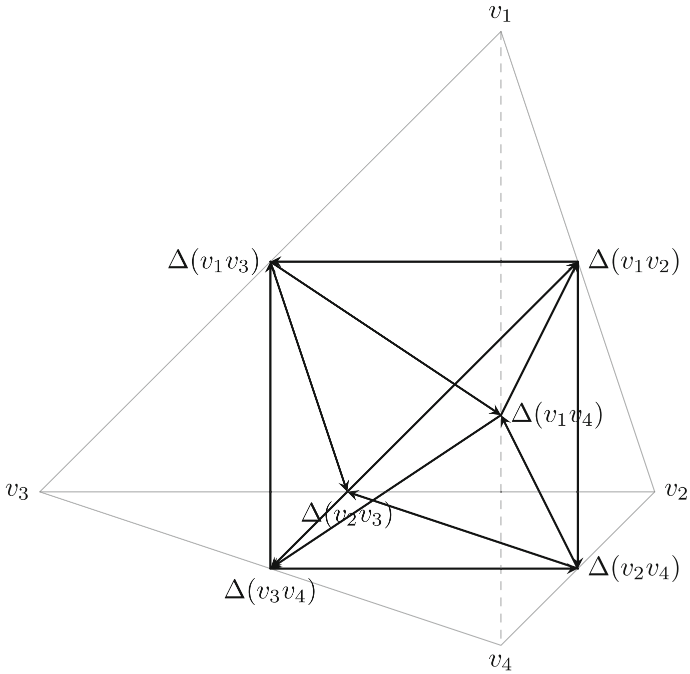


Fig. 1 Calculating $\delta\mathrm{C}^{(1)}$ for the group SL_2, and the octahedron

We picture $\delta\mathrm{C}^{(1)}$ on Fig. 1 as a 3-dimensional simplex with four flags at the vertices, and elements $\Delta(v_i v_j)$ at the centers of the corresponding edges. Each arrow represents a summand in (50). For example the arrow $\Delta(v_1v_2) \to \Delta(v_1v_3)$ represents $\Delta(v_1v_2) \wedge \Delta(v_1v_3)$. The terms in (50) split into parts that live on the faces, i.e. depend only on three flags.

4 The components $\mathrm{C}^{(1)}$, $\mathrm{C}^{(2)}$ of the cocycle

1. Cluster varieties set-up. Let us recall quivers, also known as seeds, see [9, Definition 1.4].

Definition 4.1 A quiver **c** is a data $\{\Lambda, (*, *), \{e_i\}, \{d_i\}, i \in \mathrm{I}, \mathrm{I}_0 \subset \mathrm{I}\}$, where:

- Λ is an integral lattice; $(*, *)$ is a bilinear symmetric $\mathbb{Q}$-valued form form on Λ; $\{e_i\}$ is a basis for Λ parametrised by a is a finite set I—we call its elements vertices, I_0 is the subset of frozen vertices; and $\{d_i\}$ is a set of positive integers such that: $\varepsilon_{ij} = (e_i, e_j)d_j \in \mathbb{Z}$ unless $i, j \in \mathrm{I}_0$, when $\varepsilon_{ij} \in \frac{1}{2}\mathbb{Z}$.

We describe a quiver geometrically by drawing a vertex for each basis element e_i, and $n = \varepsilon_{ij}$ arrows from the vertex e_i to the vertex e_j if $n > 0$ or in the opposite direction if $n < 0$.

Definition 4.2 For each unfrozen vertex e_k of a quiver $\mathbf{c}$ there is a quiver mutation $\mu_k : \mathbf{c} \to \mathbf{c}'$ defined as the change of the basis for Λ:

$$e_i' = \begin{cases} -e_k, & i = k \\ e_i + [\varepsilon_{ik}]_+ e_k, & i \neq k, \qquad [a]_+ := max(a, 0). \end{cases}$$

Let $\{f_i\} \in \mathrm{Hom}(\Lambda, \mathbb{Q})$ be the quasidual to $\{e_i\}$ basis: $\langle f_i, e_j \rangle = d_i^{-1}\delta_{ij}$, and Λ° the sublattice generated by $\{f_i\}$. Consider the split torus:

$$\mathcal{A}_\Lambda := \mathrm{Hom}(\Lambda^\circ, \mathbb{G}_m).$$

It comes with cluster $\mathcal{A}$-coordinates $\{\mathrm{A}_i\}$ provided by the basis $\{f_i\}$.

One associates with the mutation $\mu_k : \mathbf{c} \to \mathbf{c}'$ a transformation of the cluster coordinates, acting by

$$\begin{aligned} &\mu_k^* A_i = A_i, \quad i \neq k, \\ &\mu_k^* A_k = \frac{1}{A_k}\left(\prod_{\varepsilon_{ik}>0} A_i^{\varepsilon_{ik}} + \prod_{\varepsilon_{ik}<0} A_i^{-\varepsilon_{ik}}\right). \end{aligned} \tag{51}$$

The cluster variety $\mathcal{A}$ with the initial quiver $\mathbf{c}$ is obtained by gluing the tori $\mathcal{A}_\Lambda$ assigned to quivers obtained from $\mathbf{c}$ by sequences of mutations via the corresponding composition of the transformations (51). By the Laurent Phenomena theorem [11], each element A_i is a regular function on $\mathcal{A}$. The algebra of regular functions $\mathcal{O}(\mathcal{A})$ is nothing else but the Fomin–Zelevinsky upper cluster algebra.

Therefore each cluster $\mathbf{c}$ on a cluster variety $\mathcal{A}$ is given by a collection of cluster coordinates $(\mathrm{A}_1, \ldots, \mathrm{A}_m)$ and an exchange matrix ε_{ij} with the skewsymmetrizers d_i. This data is encoded in a single element

$$\mathrm{W}_\mathbf{c} := \frac{1}{2} \cdot \sum_{i,j \in \mathrm{I}} d_i \varepsilon_{ij} \cdot \mathrm{A}_i \wedge \mathrm{A}_j \in \mathcal{O}^\times(\mathcal{A}) \wedge \mathcal{O}^\times(\mathcal{A}) \otimes \mathbb{Z}\left[\frac{1}{2}\right]. \tag{52}$$

Note that $2 \cdot \mathrm{W}_\mathbf{c}$ has integral coefficients, and $\mathrm{W}_\mathbf{c}$ has coefficients in $\mathbb{Z}$ if I_0 is empty.

Let us assign to a cluster mutation $\mu_k : \mathbf{c} \to \mathbf{c}'$ a rational function, written in the coordinate system $\{A_i\}$ for the cluster $\mathbf{c}$ as

$$\widehat{X}_k^\mathbf{c} := \prod_{j \in \mathrm{I}} A_j^{\varepsilon_{ij}}. \tag{53}$$

Then the elements $\mathrm{W}_\mathbf{c}$ and $\mathrm{W}_{\mathbf{c}'}$ differ by the Steinberg relation [9, Proposition 6.3]:

$$\mathrm{W}_{\mathbf{c}'} - \mathrm{W}_\mathbf{c} = d_k \cdot \left(1 + \widehat{X}_k^\mathbf{c}\right) \wedge \widehat{X}_k^\mathbf{c}. \tag{54}$$

2. The moduli space $\mathcal{A}_{\mathrm{G},\mathbb{S}}$. Let us recall the definition of the moduli space $\mathcal{A}_{\mathrm{G},\mathbb{S}}$ [8].

Definition 4.3 Let $\mathbb{S}$ be a decorated surface. Let G be a simply-connected split semi-simple group.

The moduli space $\mathcal{A}_{\mathrm{G},\mathbb{S}}$ parametrises twisted G-local system $\mathcal{L}$ on $\mathbb{S}$ together with a flat section of the local system $\mathcal{L} \times_{\mathrm{G}} \mathcal{A}$ near the special points and punctures.

According to the main result of [17], the moduli space $\mathcal{A}_{\mathrm{G},\mathbb{S}}$ has a cluster $\mathcal{A}$-variety structure.

In particular, when the decorated surface $\mathbb{S}$ is an oriented n-gon p_n, we get the space

$$\mathcal{A}_{\mathrm{G},\mathrm{p}_n} = \mathrm{Conf}_n(\mathcal{A}) := \mathrm{G}\backslash\mathcal{A}^n, \qquad \mathcal{A} := \mathrm{G}/\mathrm{U}.$$

The isomorphism depends on the choice of a vertex of the polygon. For example, for the triangle t:

- The space $\mathcal{A}_{\mathrm{G},t}$ is the configuration space of three decorated flags - $\mathrm{Conf}_3(\mathcal{A})$.

3. An element $\mathrm{C}^{(2)}$**.** Pick a reduced decomposition of the longest element w_0 of the Weyl group:

$$\mathbf{i} = (i_1, \ldots, i_n), \quad w_0 = s_{i_1} \ldots s_{i_n}.$$

In [17], there is a construction of the cluster coordinate system on the space $\mathrm{Conf}_3(\mathcal{A})$, given by a collection of the regular functions, called the cluster coordinates

$$(\mathrm{A}_1, \ldots, \mathrm{A}_m), \quad \mathrm{A}_i \in \mathcal{O}(\mathrm{Conf}_3(\mathcal{A})) := \mathcal{O}(\mathcal{A}^3)^{\mathrm{G}}, \tag{55}$$

together with the exchange matrix $\varepsilon_{ij} \in \frac{1}{2}\mathbb{Z}$. We recall the construction of the cluster assigned to the reduced decomposition of $\mathbf{i}$ in Sect. 5. Then the element $\mathrm{C}^{(2)}$ is defined (see Definition 5.6) by

$$\mathrm{C}^{(2)} := \mathrm{W}_{\mathbf{c}} = \frac{1}{2} \cdot \sum_{i,j} d_i \varepsilon_{ij} \cdot \mathrm{A}_i \wedge \mathrm{A}_j. \tag{56}$$

4. An element $\mathrm{C}^{(1)}$**.** Consider two cluster coordinate systems $\mathbf{c}_{1,3}$ and $\mathbf{c}_{2,4}$ on the space $\mathrm{Conf}_4(\mathcal{A})$:

1. The one $\mathbf{c}_{2,4}$, obtained by amalgamating triangles $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ and $(\mathcal{F}_1, \mathcal{F}_3, \mathcal{F}_4)$.
2. The one $\mathbf{c}_{1,3}$, given by amalgamating triangles $(\mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)$ and $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_4)$.

According to one of the main results of [17], there exists an ordered sequence of mutations $\mu_1, \ldots, \mu_n$ providing a cluster transformation between the two cluster coordinate systems above. For each mutation μ_i there is a rational function $\widehat{X}_i$ on $\mathrm{Conf}_4(\mathcal{A})$. So we get a collection of rational functions

$$(\widehat{X}_1, \ldots, \widehat{X}_n), \quad \widehat{X}_i \in \mathbb{Q}(\mathrm{Conf}_4(\mathcal{A}))^{\times}. \tag{57}$$

5. The first cocycle condition. Thanks to (54), the difference of the elements W assigned to the cluster coordinate systems $\mathbf{c}_{1,3}$ and $\mathbf{c}_{2,4}$ is the sum of the Steinberg relations provided by functions (57):

$$\mathrm{W}_{\mathbf{c}_{1,3}} - \mathrm{W}_{\mathbf{c}_{2,4}} = \sum_{k=1}^{N} d_k \cdot (1+\widehat{X}_k) \wedge \widehat{X}_k. \tag{58}$$

This just means that setting

$$\mathrm{C}^{(1)} := \sum_{k=1}^{N} d_k \cdot \{-\widehat{X}_k\} \in \mathrm{B}_2\Big(\mathbb{Q}(\mathrm{Conf}_4(\mathcal{A}))\Big). \tag{59}$$

we get, at least modulo 2-torsion, the first cocycle identity in (38):

$$\delta(\mathrm{C}^{(1)}) = s^*(\mathrm{C}^{(2)}). \tag{60}$$

6. Altering the cluster transformation. According to [17], changing a reduced decomposition $\mathbf{i}$ we alter the chain $(\mathrm{C}^1, \mathrm{C}^2, \ldots)$ by a coboundary of an element of $\mathrm{B}_2(\mathcal{O}_{\mathrm{G}^3})$.

Theorem 4.4 *Changing a cluster transformation $\mathbf{c}_{1,3} \to \mathbf{c}_{2,4}$ does not affect the element $C^{(1)}$, since it is changed by a sum of the five-term relations, modulo an order 6 cyclic subgroup.*

Proof Thanks to (58), for a different cluster transformation $\mathbf{c}_{1,3} \to \mathbf{c}_{2,4}$ provided by a sequence of mutations associated with the functions $\widehat{Y}_1, \ldots, \widehat{Y}_M$ we have

$$\sum_{k=1}^{N} d_k \cdot (1+\widehat{X}_k) \wedge \widehat{X}_k - \sum_{k=1}^{M} d_k \cdot (1+\widehat{Y}_k) \wedge \widehat{Y}_k = 0. \tag{61}$$

Denote by β_{F} the kernel of the differential $\delta : \mathrm{B}_2(\mathrm{F}) \longrightarrow \mathrm{F}^\times \wedge \mathrm{F}^\times$ in the Bloch complex (30). Then identity (61) just means that we get, modulo 2-torsion, an element of the group $\beta_{\mathbb{F}}$, where $\mathbb{F} := \mathbb{Q}(\mathrm{Conf}_4(\mathcal{A}))$ is the function field on the configuration space:

$$\sum_{k=1}^{N} d_k \cdot \{\widehat{X}_k\} - \sum_{k=1}^{M} d_k \cdot \{\widehat{Y}_k\} \ \in \ \beta_{\mathbb{F}}, \tag{62}$$

Let $\widetilde{\mathrm{Tor}}(\mathrm{F}^\times, \mathrm{F}^\times)$ be the unique non-trivial extension of the group $\mathrm{Tor}(\mathrm{F}^\times, \mathrm{F}^\times)$ by $\mathbb{Z}/2\mathbb{Z}$. By Suslin's theorem [22], for any field F, there is an exact sequence

$$0 \longrightarrow \widetilde{\mathrm{Tor}}(\mathrm{F}^\times, \mathrm{F}^\times) \longrightarrow K_3^{\mathrm{ind}}(\mathrm{F}) \longrightarrow \beta_{\mathrm{F}} \longrightarrow 0. \tag{63}$$

Note that $\mathbb{Z}/2\mathbb{Z} = \mathrm{Tor}(\mathbb{Q}^\times, \mathbb{Q}^\times) = \mathrm{Tor}(\mathbb{Q}(t_1, \ldots, t_n)^\times, \mathbb{Q}(t_1, \ldots, t_n)^\times)$. Next, $K_3^{\mathrm{ind}}(\mathrm{F}(t)) = K_3^{\mathrm{ind}}(\mathrm{F})$. Therefore, since the configuration spaces are rational varieties, the element (62) provides an element of $K_3^{\mathrm{ind}}(\mathbb{Q})/(\mathbb{Z}/4\mathbb{Z})$. Suslin proved [22, Corollary 5.3] that the latter group is isomorphic to $\mathbb{Z}/6\mathbb{Z}$—this uses the Lee and Szczarba theorem [21]. Therefore the element (62) belongs to the subgroup $\mathbb{Z}/6\mathbb{Z}$. □

5 Cluster structure of the space $\mathrm{Conf}_3(\mathcal{A})$

For the convenience of the reader, we reproduce the definition of the clusters, that is cluster coordinates and quivers, describing the cluster structure of the space $\mathrm{Conf}_3(\mathcal{A})$, borrowing the construction of the cluster coordinates from [17, Section 5], and the construction of quivers from [17, Section 7.2].

1. The set-up. Recall that G is a split semi-simple simply-connected algebraic group with the Cartan group H, the Weyl group W, and the Cartan matrix $\{\mathrm{C}_{ij}\}_{i,j\leq r}$, simple positive roots α_i and coroots $\alpha_j^\vee$:

$$\alpha_i : \mathrm{H} \to \mathbb{G}_m, \ \alpha_i^\vee : \mathbb{G}_m \to \mathrm{H}, \ \ \alpha_i \circ \alpha_j^\vee = \mathrm{C}_{ij}. \tag{64}$$

There is a set of the fundamental weights $\Lambda_1, \ldots, \Lambda_r$:

$$\Lambda_i : \mathrm{H} \to \mathbb{G}_m, \qquad \Lambda_i \circ \alpha_j^\vee = \delta_{ij}. \tag{65}$$

The length and reduced decomposition of the Weyl group elements induce the Bruhat order of Bruhat cells. If elements $w, w' \in W$ have reduced decompositions such that the one for w' is a substring of the one for w then $w \succ w'$. If in addition $l(w) = l(w') + 1$ then the cell $\mathrm{B}w'\mathrm{B}$ is a boundary divisor of $\mathrm{B}w\mathrm{B}$.

A pinning for a generic pair of flags $\{\mathrm{B}, \mathrm{B}^-\}$ provides maps $x_i : \mathbb{A}^1 \to \mathrm{U}$ and $y_i : \mathbb{A}^1 \to \mathrm{U}^-$ for every simple root α_i, where U is the maximal unipotent subgroup of B and U^- is the maximal unipotent in B^-, such that each pair x_i, y_i can be extended to a standard embedding $\gamma_i : \mathrm{SL}_2 \to \mathrm{G}$. A pinning allows to lift to the group G the generators of the Weyl group W corresponding to simple roots:

$$\overline{s_i} := y_i(1)x_i(-1)y_i(1).$$

These elements satisfy the braid relations. Therefore we define the lift for all other elements of W by using any reduced decomposition $w = s_1 \cdot \cdots \cdot s_m$, setting: $\overline{w} = \overline{s_1} \cdot \cdots \cdot \overline{s_m}$. Using this, we define the Bruhat decomposition of any element $g \in \mathrm{G}$:

$$g = uh\overline{n_w}v, \qquad h \in \mathrm{H} = \mathrm{B} \cap \mathrm{B}^-, \qquad u, v \in U. \tag{66}$$

Therefore any G-orbit in the space of pairs of decorated flags

$$(\mathcal{F}, \mathcal{G}) \in \mathrm{Conf}_2(\mathcal{A}) = \mathrm{G}\backslash(\mathrm{G}/\mathrm{U})^2 = \mathrm{U}\backslash\mathrm{G}/\mathrm{U} \tag{67}$$

has two invariants: the $\omega-distance\,\omega(\mathcal{F}, \mathcal{G}) := w$, and the $h-distance\,h(\mathcal{F}, \mathcal{G}) := h$, where $g \in \mathrm{G}$ is decomposed as in (66). Each fundamental weight Λ_i gives rise to a regular function on every Bruhat cell:

$$\Delta_{i,w}(uh\overline{n_w}v) := \Lambda_i(h). \tag{68}$$

2. Cluster $\mathcal{A}$-coordinates for the space $\mathcal{A}_{\mathrm{G},t} = \mathrm{Conf}_3(\mathcal{A})$. For each reduced word $\mathbf{i} = (i_1, \ldots, i_m)$ of w_0 there are chains of distinct positive roots and coroots:

$$\alpha_k^{\mathbf{i}} := s_{i_m} \ldots s_{i_{k+1}} \cdot \alpha_{i_k}, \qquad \beta_k^{\mathbf{i}} := s_{i_m} \ldots s_{i_{k+1}} \cdot \alpha_{i_k}^{\vee}, \qquad k \in \{1, \ldots, m\}. \tag{69}$$

Lemma 5.1 [17, Lemma 5.3]. *Given any generic pair of decorated flags $\{\mathcal{F}, \mathcal{G}\}$, i.e. $\omega(\mathcal{F}, \mathcal{G}) = w_0$, and a reduced decomposition $\mathbf{i} = \{i_1, \ldots . i_m\}$ of w, there exists a unique chain of decorated flags*

$$\{\mathcal{F} = \mathcal{F}^0 \xleftarrow{s_{i_1}} \mathcal{F}^1 \xleftarrow{s_{i_2}} \cdots \xleftarrow{s_{i_m}} \mathcal{F}^m = \mathcal{G}\} \tag{70}$$

such that that for the consequtive decorated flags, counted from the right to the left, we have:

$$\omega\big(\mathcal{F}^k, \mathcal{F}^{k-1}\big) = s_{i_k}, \qquad h\big(\mathcal{F}^k, \mathcal{F}^{k-1}\big) \in \begin{cases} \alpha_{i_k}^{\vee}(\mathbb{G}_m), & \text{if } \beta_k^{\mathbf{i}} \text{ is simple}, \\ 1, & \text{otherwise}. \end{cases} \tag{71}$$

We also note that

$$h_k := s_{i_m} \ldots s_{i_{k+1}} \Big(h(\mathcal{F}^k, \mathcal{F}^{k-1})\Big) = \begin{cases} \alpha_i^{\vee}(b_i), & \text{if } \beta_k^{\mathbf{i}} = \alpha_i^{\vee}, \\ 1, & \text{otherwise}. \end{cases} \tag{72}$$

Recall the involution $* : \mathrm{I} \to \mathrm{I}$ such that $\alpha_{i^*}^{\vee} = -\omega_0(\alpha_i^{\vee})$. Let $w^* := \omega_0 w \omega_0^{-1}$. Then any reduced decomposition $w = s_{i_1} \cdot \cdots \cdot s_{i_k}$ provides a reduced decomposition $w^* = s_{i_1^*} \cdot \cdots \cdot s_{i_k^*}$. Note that $\omega_0^* = \omega_0$.

Definition 5.2 The cluster $\mathcal{A}$-coordinates on the space $\mathcal{A}_{\mathrm{G},t}$ are defined as follows. Pick a vertex of the triangle t with a decorated flag $\mathcal{F}_1$, and a reduced decomposition $\mathbf{i} = (i_1, \ldots, i_m)$ of w_0. Then:

- The *frozen cluster coordinates* are:

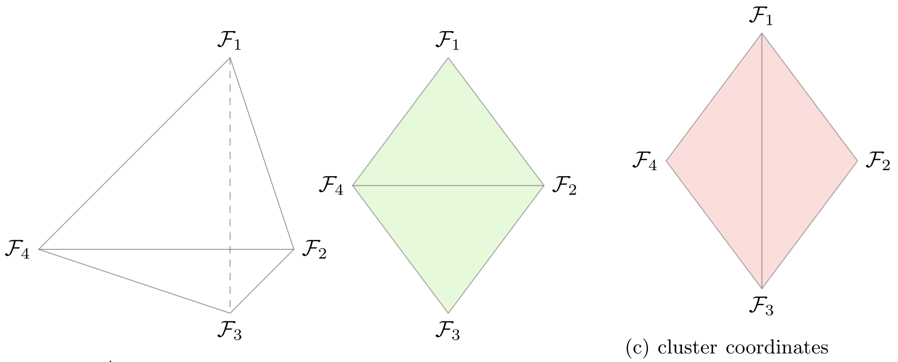
$$\Delta_i(\mathcal{F}_1, \mathcal{F}_2), \quad \Delta_i(\mathcal{F}_1, \mathcal{F}_3), \quad \Delta_i(\mathcal{F}_3, \mathcal{F}_2), \qquad \forall i \in \mathrm{I}.$$

- Let $\mathbf{i}^* = (i_1^*, \ldots, i_m^*)$. By Lemma 5.1, there is a unique chain of decorated flags, see Fig. 2, with respect to the reduced decomposition $\mathbf{i}^*$:

$$\{\mathcal{F}_2 = \mathcal{F}_{23}^0 \longleftarrow \mathcal{F}_{23}^1 \longleftarrow \ldots \longleftarrow \mathcal{F}_{23}^m = \mathcal{F}_3\}.$$

  Then the *unfrozen cluster coordinates* are:

$$A_p = \Delta_{i_p}\big(\mathcal{F}_1, \mathcal{F}_{23}^p\big),$$



(a) $\mathcal{A}^4$ configuration. (b) cluster coordinates $\mathbf{c}_{1,3}$. (c) cluster coordinates $\mathbf{c}_{2,4}$.

Fig. 2 Illustrating cluster coordinates choices and the flip of the configuration of for decorated flags.

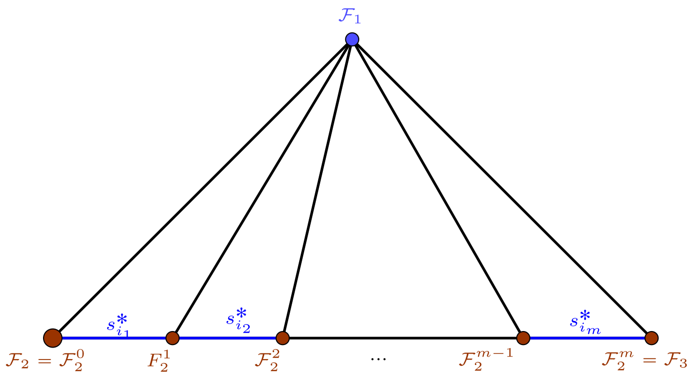


Fig. 3 The $\mathcal{A}_{G,t}$ triangle

where p runs through indices $1, \ldots, m$ such that i_p is not the rightmost simple reflection i in $\mathbf{i}$, $\forall i \in \mathrm{I}$.

We stress that:

- Unfrozen vertices depend on all three decorated flags; we picture them inside of the triangle.
  Frozen vertices depend only on two decorated flags; we picture them on the sides of the triangle.
  Cluster coordinates on the space $\mathrm{Conf}_2(\mathcal{A})$ are labeled by the vertices $i \in \mathrm{I}$ of the Dynkin diagram. The twisted cyclic shift $(\mathcal{F}_1, \mathcal{F}_2) \longmapsto (\mathcal{F}_2, s_G\mathcal{F}_1)$ amounts to the automorphism $i \longmapsto i^*$ of I.

Let us define the quiver $\mathbf{Q}(\mathbf{i})$ for $\mathrm{Conf}_3(\mathcal{A})$, assigned to the reduced word $\mathbf{i} = (i_1, \ldots, i_m)$ for w_0 (Fig. 3).

3. Elementary quivers $\mathbf{J}(i)$. Let us define the quiver $\mathbf{J}(i)$, where $i \in \mathrm{I}$. Its underlying set is:

$$J(i) := (\mathrm{I} - \{i\}) \cup \{i_l\} \cup \{i_r\} \cup \{i_e\}. \tag{73}$$

There is a *decoration* map $\pi : J(i) \to \mathrm{I}$ which sends i_l, i_r and i_e to i, and is the identity map on $\mathrm{I} - \{i\}$. The multipliers on $J(i)$ are defined by pulling back the multipliers on I. The skew-symmetrizable matrix $\varepsilon(i)$ is indexed by $J(i) \times J(i)$, and defined as follows:

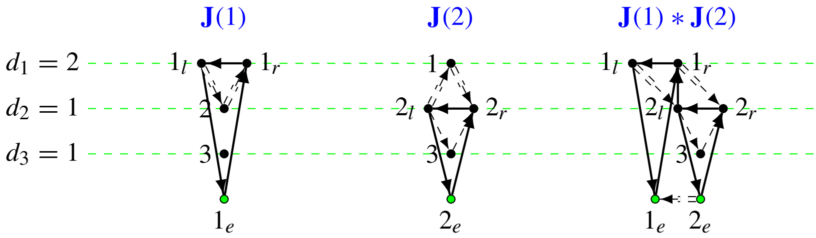
$$\begin{aligned} &\varepsilon(i)_{i_l,j} = \frac{-C_{ij}}{2}, \quad \varepsilon(i)_{i_r,j} = \frac{C_{ij}}{2}, \\ &\varepsilon(i)_{i_r,i_l} = \varepsilon(i)_{i_l,i_e} = \varepsilon(i)_{i_e,i_r} = 1; \quad \varepsilon(i)_{jk} = 0 \ \text{ if } i \notin \{j,k\}. \end{aligned} \tag{74}$$

A quiver $\mathbf{J}(i)$ is pictured by a directed graph with vertices labelled by the set (73) and arrows encoding the exchange matrices $\varepsilon = (\varepsilon_{jk})$, where

$$\varepsilon_{jk} = \#\{\text{arrows from } j \text{ to } k\} - \#\{\text{arrows from } k \text{ to } j\}.$$

Here $\#\{$arrows from a to $b\}$ is the *total weight* of the arrows from a to b, which is a half-integer. The arrows from a to b are either dashed, and counted with the weight $\frac{1}{2}$, or solid, and t counted with the weight 1. For non simply laced cases we use special arrows, see Example 5.3.

Example 5.3 The quivers $\mathbf{J}(1)$, $\mathbf{J}(2)$ for type B_3, and their amalgamation $\mathbf{J}(1) * \mathbf{J}(2)$, described below:



4. The quiver $\mathbf{H}(\mathbf{i})$. Recall the pairing $(*, *)$ between the root and coroot lattices, the Cartan matrix $C_{ij} = (\alpha_i, \alpha_j^\vee)$, and the multipliers $d_j = \langle \alpha_j^\vee, \alpha_j^\vee \rangle \in \{1, 2, 3\}$, so that $d_i C_{ij}$ is symmetric.

Given a reduced word $\mathbf{i} = (i_1, \ldots, i_m)$ of w_0, recall the chains of distinct positive roots $\alpha_j^{\mathbf{i}}$ and coroots $\beta_k^{\mathbf{i}}$ in (69). Let us define first an auxiliary quiver $\mathbf{K}(\mathbf{i})$. It consists of m frozen vertices labeled by $(i_1, \ldots, i_m)$, with the multiplier for the kth vertex given by $d_k = \langle \alpha_{i_k}^\vee, \alpha_{i_k}^\vee \rangle$, and the exchange matrix

$$\varepsilon_{jk} = \begin{cases} \frac{\mathrm{sgn}(k-j)}{2} \left(\alpha_j^{\mathbf{i}}, \beta_k^{\mathbf{i}} \right) & \text{if } i_j, i_k \in \mathrm{I}, \\ 0 & \text{otherwise.} \end{cases} \tag{75}$$

Then $\mathbf{H}(\mathbf{i})$ is a full subquiver of $\mathbf{K}(\mathbf{i})$ with the vertices k such that $\beta_k^{\mathbf{i}}$, and hence $\alpha_k^{\mathbf{i}}$, are simple.

5. The quiver $\mathbf{Q}(\mathbf{i})$. We use the amalgamation of quivers, introduced in [10, Section 2.2].

Definition 5.4 Given a reduced word $\mathbf{i} = (i_1, \ldots, i_m)$ for $w_0 \in W$, $i_k \in \mathrm{I}$, the quiver $\mathbf{Q}(\mathbf{i})$ is the amalgamation of quivers $\mathbf{J}(i_k)$ and $\mathbf{H}(\mathbf{i})$:

$$\mathbf{Q}(\mathbf{i}) := \mathbf{J}(i_1) * \ldots * \mathbf{J}(i_m) * \mathbf{H}(\mathbf{i}).$$

Precisely, the amalgamated quiver is defined as follows:

(i) For every $i \in \mathrm{I}$ and for every $j = 1, \ldots, m-1$, the right element of $\mathbf{J}(i_j)$ at level i is glued with the left element of $\mathbf{J}(i_{j+1})$ at level i. The extra vertex of each $\mathbf{J}(i_k)$ is glued with the kth vertex of $\mathbf{H}(\mathbf{i})$.

(ii) The weight of an arrow obtained by gluing two arrows is the sum of the weights of those arrows.

The unfrozen part of the quiver $\mathbf{Q}(\mathbf{i})$ is the full subquiver obtained by deleting the leftmost and rightmost vetices at every level $i \in \mathrm{I}$, and the vertices of $\mathbf{H}(\mathbf{i})$.

The following Theorem is one of the main results of [17].

Theorem 5.5 *Given a reduced decomposition* $\mathbf{i}$ *of* $w_0 \in W$*, the coordinates* $\{A_i\}$ *from Definition 5.2 and the quiver* $\mathbf{Q}(\mathbf{i})$ *from Definition 5.4 describe an* $\mathcal{A}$*-cluster for the space* $\mathrm{Conf}_3(\mathcal{A})$*. The clusters assigned to different reduced decompositions are related by cluster* $\mathcal{A}$*-transformations. The obtained cluster structure is invariant under the twisted cyclic shift* $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \longmapsto (\mathcal{F}_2, \mathcal{F}_3, s_0\mathcal{F}_1)$.

Definition 5.6 Given a reduced ecomposition $\mathbf{i}$ of w_0, the element $\mathrm{C}^{(2)}$ is given by

$$\mathrm{C}^{(2)} := \frac{1}{2} \cdot \sum_{i,j} d_i \varepsilon_{ij} \cdot \mathrm{A}_i \wedge A_j, \tag{76}$$

where $\{A_i\}$ are the cluster coordinates from Definition 5.2, and ε_{ij} is the exchange matrix for the quiver $\mathbf{Q}(\mathbf{i})$ from Definition 5.4.

6 The tame symbol of $\mathrm{C}^{(2)}$ and the component $\mathrm{C}^{(3)}$

Recall the tame symbol (11), also known as the residue. The cluster coordinates $\{A_k\}$ are regular functions on $\mathrm{Conf}_3(\mathcal{A})$. So for the element $W_{\mathbf{c}}$, see (52), its tame symbol is supported on the divisors $\{\mathrm{A}_k = 0\}$.

The Bruhat divisor $\mathcal{B}_{s_k w_0} \subset \mathrm{Conf}_2(\mathcal{A})$ is determined by the equation $\Delta_{k,w_0} = 0$.

Denote by i_k the embedding $\mathcal{B}_{s_k w_0} \subset \mathrm{Conf}_2(\mathcal{A})$. Recall the function $\Delta_{k,s_k\omega_0}$ on the divisor $\mathcal{B}_{s_k w_0}$:

$$\Delta_{k,s_k\omega_0} = \Lambda_k(h_{s_k\omega_0}(\mathcal{F}_2, \mathcal{F}_3)), \qquad (\mathcal{F}_2, \mathcal{F}_3) \in \mathrm{Conf}_2(\mathcal{A}). \tag{77}$$

Recall the rational function F_k on $\mathcal{B}_{s_k w_0}$:

$$F_k := i_k^* \left(\Delta_{k,s_k\omega_0}^{-1} \prod_{j \in \mathrm{I} \setminus \{k\}} (\Delta_{j,w_0})^{\frac{C_{kj}}{2}} \right)^{d_k}. \tag{78}$$

Definition 6.1 The component $\mathrm{C}^{(3)}$ of the cocycle $\mathrm{C}^{(\bullet)}$ is defined as

$$\mathrm{C}^{(3)} := \sum_{k \in \mathrm{I}} \Big(\mathcal{B}_{s_k w_0}, F_k\Big) \in \bigoplus_{D \in \mathrm{divConf}_2(\mathcal{A})} \mathcal{O}(D)^*. \tag{79}$$

Let E be an oriented edge of the triangle t. Then there is a map

$$\beta_E : \mathrm{Conf}_3(\mathcal{A}) \longrightarrow \mathrm{Conf}_2(\mathcal{A}).$$

which forgets the element of $\mathcal{A}$ at the vertex of t opposite to the edge E. It induces a map

$$\begin{aligned} \beta_E^* : \bigoplus_{D \in \mathrm{divConf}_2(\mathcal{A})} \mathcal{O}(D)^* &\to \bigoplus_{D' \in \mathrm{divConf}_3(\mathcal{A})} \mathcal{O}(D')^*, \\ (D, f_D) &\longmapsto (\beta_E^* D, \beta_E^* f_D). \end{aligned} \tag{80}$$

We count the vertices labeled by the decorated flags counterclockwise: $(\mathcal{F}_1, \mathcal{F}_3, \mathcal{F}_2)$. The edges E of the triangle are labeled by the ordered pairs of flags $(\mathcal{F}_i, \mathcal{F}_j)$ assigned to them: $E = (i, j)$.

Theorem 6.2 *The tame symbol of the element* $W_{\mathbf{c}}$ *on* $\mathrm{Conf}_3(\mathcal{A})$ *is the sum over the edges of the triangle:*

$$\mathrm{res}(W_{\mathbf{c}}) = \big(\beta_{1,3}^* + \beta_{3,2}^* - \beta_{1,2}^*\big)(\mathrm{C}^{(3)}). \tag{81}$$

Corollary 6.3 $\mathrm{div}(\mathrm{C}^{(3)}) = 0$.

Proof We know that $\mathrm{div} \circ \mathrm{res}(W_{\mathbf{c}}) = 0$ and

$$\mathrm{div}\beta_{1,3}^*(\mathrm{C}^{(3)}) + \mathrm{div}\beta_{3,2}^*(\mathrm{C}^{(3)}) - \mathrm{div}\beta_{1,2}^*(\mathrm{C}^{(3)}) = 0. \tag{82}$$

The codimension two cycles $\mathrm{div}\beta_{i,j}^*(\mathrm{C}^{(3)})$ can not share a common codimension two component. This is clear for the pull back to $\mathcal{A}^3$, since a point $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ which lies in two cycles $\mathrm{div}\beta_{i,j}^*(\mathrm{C}^{(3)})$ satisfies codimension two condition for each of the two pars of decorated flags, which gives the codimension > 2 intersection. Since their sum is zero, the claim follows. □

Proof of the Theorem Recall the element $W_{\mathbf{c}}$. Denote by $i_k^*(f)$ the pull back of a function f to the divisor $\{\mathrm{A}_k = 0\}$. Then the tame symbol of $W_{\mathbf{c}}$ is

$$\begin{aligned} \mathrm{res}_{\mathrm{A}_k=0}(W_{\mathbf{c}}) &= \mathrm{res}_{\mathrm{A}_k=0}\left(\frac{1}{2} d_i \varepsilon_{ij} \cdot \sum_{i,j} \mathrm{A}_i \wedge \mathrm{A}_j\right) = i_k^* \prod_{j \neq k} A_j^{d_k \varepsilon_{kj}}, \\ \mathrm{res}(W_{\mathbf{c}}) &= \bigoplus_k \left(\{\mathrm{A}_k = 0\}, i_k^* \prod_{j \in \mathrm{I} - \{k\}} A_j^{d_k \varepsilon_{kj}}\right). \end{aligned} \tag{83}$$

To check the last equality in the top formula here note that $d_i\varepsilon_{ij}$ is skew-symmetric, and thus we count twice the contribution of $\frac{1}{2}d_k\varepsilon_{kj}\cdot A_j = A_j^{d_k\varepsilon_{kj}/2}$; note the multiplicative notation used here: $n \cdot A = A^n$.

There are two cases for the vertex v_k related to the coordinate A_k.

1. *The coordinate A_k corresponds to a non-frozen vertex.* This is the general case, and fortunately we can handle without going into details what is the coordinate A_k. Indeed, since the coordinate A_k is non-frozen, we can mutate A_k, getting a new cluster coordinate A'_k, which satisfies the exchange relation:

$$A_k \cdot A'_k = \prod_{\varepsilon_{kj}>0} A_j^{\varepsilon_{kj}} + \prod_{\varepsilon_{kj}<0} A_j^{-\varepsilon_{kj}}.$$

All we need to know is that there exists at least one non-trivial mutation at A_k, providing a different regular function A'_k on $\mathrm{Conf}_3(\mathcal{A})$. Restricting it to the divisor $A_k = 0$, we have

$$0 = \prod_{\varepsilon_{kj}>0} A_j^{\varepsilon_{kj}} + \prod_{\varepsilon_{kj}<0} A_j^{-\varepsilon_{kj}}.$$

Therefore

$$\prod_j A_j^{\varepsilon_{kj}} = -1.$$

This is a 2-torsion in the multiplicative group. So the residue on the divisor $A_k = 0$ is a 2-torsion.

For example, for the moduli space $\mathcal{A}_{SL_3,t}$ with the special cluster coordinates illustrated on the Fig. 4, the only non-frozen coordinate is the one in the center of the triangle. The exchange relation is

$$\begin{aligned} &\Delta_{\omega^*}(e_1 \wedge e_2, f_1 \wedge f_2, g_1 \wedge g_2)\Delta_\omega(e_1, f_1, g_1) \\ &\quad = \Delta_\omega(e_1, e_2, f_1)\Delta_\omega(f_1, f_2, g_1)\Delta_\omega(g_1, g_2, e_1) \\ &\quad\quad + \Delta_\omega(e_1, e_2, g_1)\Delta_\omega(f_1, f_2, e_1)\Delta_\omega(g_1, g_2, f_1). \end{aligned} \tag{84}$$

Here ω is a volume form in a three dimensional vector space V, ω^* is the dual volume form in V^*, and $\mathcal{F}_1 = (e_1, e_1 \wedge e_2)$, $\mathcal{F}_2 = (f_1, f_1 \wedge f_2)$ and $\mathcal{F}_3 = (g_1, g_1 \wedge g_2)$ are decorated flags in V.

2. *The coordinate A_k is frozen.* Then it corresponds to a vertex located on a side of the triangle t. This is the difficult case. Since the definition of the quiver $\mathbf{c}$ depends on the choice of the vertex of the triangle, referred to as the top vertex, we consider the residue for each of the three sides of the triangle.

We start from the right edge $(\mathcal{F}_1, \mathcal{F}_3)$. Since the K_2-class $[W_\mathbf{c}]$ does not depend on the choice of the reduced decomposition $\mathbf{i}$ of ω_0, and the tame symbol depends only on the K_2-class, we can assume that:

$$\text{The decomposition } \mathbf{i} \text{ ends by } s_k. \tag{85}$$

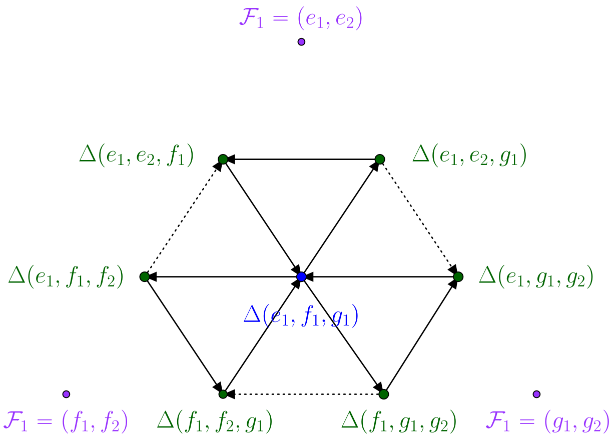
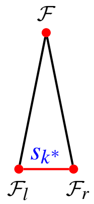


Fig. 4 The canonical coordinates on the moduli space $\mathcal{A}_{SL_3,t}$ of triples of decorated flags

The elementary configuration space $\mathcal{A}(k)$ [17, Section 7.5]. Let $k \in \mathrm{I}$. Consider the space $\mathcal{A}(k)$ parametrizing G-orbits of triples of decorated flags $(\mathcal{F}, \mathcal{F}_l, \mathcal{F}_r)$ such that

$$w(\mathcal{F}, \mathcal{F}_l) = w(\mathcal{F}, \mathcal{F}_r) = w_0, \qquad w(\mathcal{F}_r, \mathcal{F}_l) = s_{k^*}, \quad h(\mathcal{F}_r, \mathcal{F}_l) \in \mathrm{H}(s_{k^*}). \tag{86}$$



There is a cluster $\mathcal{A}$-coordinate system on the space $\mathcal{A}(k)$ parametrized by $J(k)$ defined by:

$$\forall(\mathcal{F}, \mathcal{F}_l, \mathcal{F}_r) \in \mathcal{A}(k), \qquad A_j := \begin{cases} \Delta_j(\mathcal{F}, \mathcal{F}_l) & \text{if } j \in \mathrm{I} - \{k\} \\ \Delta_k(\mathcal{F}, \mathcal{F}_l) & \text{if } j = k_l \\ \Delta_k(\mathcal{F}, \mathcal{F}_r) & \text{if } j = k_r \\ \Lambda_{k^*}(h(\mathcal{F}_r, \mathcal{F}_l)) & \text{if } j = k_e. \end{cases} \tag{87}$$

If we fix a pinning in G, then we have

$$(\mathcal{F}, \mathcal{F}_l, \mathcal{F}_r) := (\mathrm{U}_-, h\mathrm{U}, gh\mathrm{U}), \quad h \in \mathrm{H}, g \in \varphi_k(\mathrm{SL}_2/\mathrm{U}_{\mathrm{SL}_2}). \tag{88}$$

In particular, by [17, Lemma 7.13], for any $(\mathcal{F}, \mathcal{F}_l, \mathcal{F}_r) \in \mathcal{A}(k)$, we have

$$\Delta_j(\mathcal{F}, \mathcal{F}_l) = \Delta_j(\mathcal{F}, \mathcal{F}_r). \quad \forall j \neq k. \tag{89}$$

There is a canonical projection

$$\tau_k : \mathcal{A}(k) \longrightarrow \mathrm{Conf}_3(\mathcal{A}_{\mathrm{SL}_2}). \tag{90}$$

It assigns to a triple of decorated flags $(\mathcal{F}, \mathcal{F}_l, \mathcal{F}_r)$ the intersections of the corresponding maximal unipotent subgroups with the subgroup $\varphi_k(\mathrm{SL}_2) \subset \mathrm{G}$ corresponding to the simple coroot $\alpha_k^\vee$. The coordinates $\mathrm{A}_{k_l}, \mathrm{A}_{k_r}, \mathrm{A}_{k_e}$ are the pull backs of the standard coordinates on $\mathrm{Conf}_3(\mathcal{A}_{\mathrm{SL}_2})$.

Recall the matrix $\varepsilon(k)$ of $\mathbf{J}(k)$ in (74), and the *canonical* element describing cluster (87) on $\mathcal{A}(k)$:

$$\mathrm{W}(k) := \sum_{i,j \in J(k)} d_i \varepsilon(k)_{ij} \cdot A_i \wedge A_j. \tag{91}$$

Denote by $\mathrm{Conf}_3^\times(\mathcal{A})$ the subspace of $\mathrm{Conf}_3(\mathcal{A})$ given by the condition that each pair of the decorated flags are in the generic position. The amalgamation provides an embedding of the space $\mathrm{Conf}_3^\times(\mathcal{A})$ obtained by the amalgamation into the product of the elementary spaces $\mathcal{A}(i_j)$ used for the amalgamation:

$$\mathrm{Conf}_3^\times(\mathcal{A}) \hookrightarrow \prod_{j=1}^m \mathcal{A}(i_j).$$

Denote by $\eta_k : \mathrm{Conf}_3(\mathcal{A}) \longrightarrow \mathcal{A}(k)$ the composition of this map with the projection onto the rightmost factor $\mathcal{A}(i_j)$ with $i_j = k$. Thanks to assumption (85), the cluster coordinate A_k on $\mathrm{Conf}_3(\mathcal{A})$ is the pull back $\eta_k^* \mathrm{A}_k$ of the cluster coordinate A_k on $\mathcal{A}(k)$. This immediately implies that

$$\mathrm{res}_{\mathrm{A}_k=0}(W_{\mathbf{c}}) = \eta_k^* \mathrm{res}_{\mathrm{A}_k=0}(W(k)). \tag{92}$$

So the calculation of $\mathrm{res}_{\mathrm{A}_k=0}(W_{\mathbf{c}})$ boils down to the calculation of the residue of W_k at the divisor $\mathrm{A}_k = 0$ on the elementary space $\mathcal{A}(k)$. Let us write $W(k)$ as a sum, see (93):

$$W(k) = W'(k) + W_\Delta(k), \quad W_\Delta(k) := \mathrm{A}_{k_r} \wedge \mathrm{A}_{k_l} + \mathrm{A}_{k_l} \wedge A_{k_e} + \mathrm{A}_{k_e} \wedge \mathrm{A}_{k_r}. \tag{93}$$

Note that by the definition of the amalgamation, $\mathrm{A}_k = \mathrm{A}_{k_l}$. Next, we evidently have:

$$\mathrm{res}_{\{\mathrm{A}_k=0\}} \mathrm{W}'(k) = \prod_{j \notin \{k_l, k_r, k_e\}} A_j^{d_k \varepsilon(k)_{kj}}. \tag{94}$$

Since $\varepsilon(k)_{k_r,j} = \frac{C_{kj}}{2}$ by (74), the factor $(\Delta_{j,w_0})^{\frac{d_k C_{kj}}{2}}$ in (78) match the factor $A_j^{d_k \varepsilon(k)_{kj}}$ in (94):

$$(\Delta_{j,w_0})^{\frac{d_k C_{kj}}{2}} = A_j^{d_k \varepsilon(k)_{kj}}. \tag{95}$$

Therefore the product in (94) match the product over $j \neq k$ in (78). So it remains to show that

$$\mathrm{res}_{\{\mathrm{A}_k=0\}} \mathrm{W}_\Delta(k) = \Delta^{-1}_{k,s_k\omega_0} \tag{96}$$

Note that $W_\Delta(k) = \tau_k^* W_{\mathrm{Conf}_3(\mathcal{A}_{\mathrm{SL}_2})}$, and the divisor $\mathrm{A}_k = 0$ on $\mathrm{Conf}_3(\mathcal{A})$ is the pull back of the divisor $\mathrm{A}_{i_r} = 0$ on $\mathrm{Conf}_3(\mathcal{A}_{\mathrm{SL}_2})$ under the map τ_k^*. So parametrisation (88) and the calculation of the residue for SL_2 from Sect. 3 implies (96). So we calculated the residue for the right edge of the triangle.

For the left edge the calculation is similar. We claim that the residue corresponding to the left edge is given by $-\beta^*_{1,2}(\mathrm{C}_3)$, see (81). Indeed, this agrees with the fact that $\varepsilon_{k_l,j} = -\mathrm{C}_{kj}/2$ while $\varepsilon_{k_r,j} = \mathrm{C}_{kj}/2$ in (74), as well as with the calculation of the residue for SL_2.

Computation for the bottom side $(\mathcal{F}_3, \mathcal{F}_2)$ follows easily using $W_\Delta(k) = \tau_k^* W_{\mathrm{Conf}_3(\mathcal{A}_{\mathrm{SL}_2})}$ and (75). □

7 Proof of Theorems 2.1 and 2.4

Brylinsky–Deligne results [4, Section 4]**.** Let $W^{(p)} \subset W$ the subset parametrising Bruhat cells $\mathrm{B}w\mathrm{B}$ of codinension p. In particular,

$$W^{(1)} = \{w_0 s_k \in W\}, k \in \mathrm{I}; \quad W^{(2)} = \{w_0 s_i s_j \in W\}, \; i, j \in \mathrm{I}, \; i \neq j. \tag{97}$$

Let $X = \mathrm{Hom}(\mathrm{H}, \mathbb{G}_m)$ be the character group of the Cartan group H. Consider the following complex

$$\bigwedge^2 X \longrightarrow \bigoplus_{W^{(1)}} X \longrightarrow \bigoplus_{W^{(2)}} \mathbb{Z}. \tag{98}$$

Here, using the notation $(w, -)$ for an element of $\bigoplus_{W^{(1)}} X$, the differentials are:

$$\begin{aligned} x_1 \wedge x_2 &\longmapsto \sum_{i \in \mathrm{I}} \Big(\langle x_1, \alpha_i^\vee \rangle \cdot (w_0 s_i, x_2) - \langle x_2, \alpha_i^\vee \rangle \cdot (w_0 s_i, x_1) \Big). \\ (\omega_0 s_j, x) &\longmapsto \sum_{i \neq j} \Big((\omega_0 s_i s_j, \langle x, s_j(\alpha_i^\vee) \rangle) + (\omega_0 s_j s_i, \langle x, \alpha_i^\vee \rangle) \Big). \end{aligned} \tag{99}$$

Proposition 7.1 *There exists a natural map of complexes*

$$\begin{tikzcd} \bigwedge^2 X \arrow[r] \arrow[d] & \bigoplus_{W^{(1)}} X \arrow[r] \arrow[d] & \bigoplus_{W^{(2)}} \mathbb{Z} \arrow[d] \\ \bigwedge^2 \mathbb{Q}(G)^* \arrow[r, "\mathrm{res}"] & \bigoplus_{D \in \mathrm{G}^{(1)}} \mathbb{Q}(D)^* \arrow[r, "\mathrm{val}"] & \bigoplus_{\mathrm{G}^{(2)}} \mathbb{Z} \end{tikzcd}$$

Proof The right vertical map is induced by the canonical embedding $W^{(2)} \hookrightarrow \mathrm{G}^{(2)}$. We assign to a character χ of the Cartan group and an element $w \in W$ a regular function $\chi'_{\overline{\omega}}$ on the Bruhat cell $\mathcal{B}_w$:

$$\chi'_{\overline{\omega}}(u\overline{w}hv) := \chi(h). \tag{100}$$

We warn the reader that the functions $\chi'_{\overline{\omega}}$, and $\chi_{\overline{\omega}}$ from (18), are not the same since $\chi_{\overline{\omega}}(uh\overline{w}v) := \chi(h)$. We need now both since [4, Section 4] uses $\chi'_{\overline{\omega}}$, while [17] uses $\chi_{\overline{\omega}}$.

The two left vertical maps are given by

$$\begin{aligned} \chi \wedge \psi &\longmapsto \chi'_{\overline{\omega_0}} \wedge \psi'_{\overline{\omega_0}} \\ (w_0 s_k, \chi) &\longmapsto \chi'_{\overline{\omega_0 s_k}} \end{aligned} \tag{101}$$

Let us prove that we get a map of complexes. The valuations of the function $\chi'_{\overline{\omega_0}}$ on a divisor can be non-zero only if it is a Bruhat divisor. In this case they are calculated as follows. For every Bruhat divisor $\mathrm{B}\omega_1\mathrm{B}$, we can choose reduced Weyl decompositions of ω and ω_1 so that: $\omega = \omega' s_i \omega''$ and $\omega_1 = \omega'\omega''$. Using the valuation formula of Demazure, [4, Lemma 4.2] tells:

$$\mathrm{val}_{\omega_1}(\chi'_{\overline{\omega}}) = \langle \chi, \omega''^{-1}(\alpha_i^\vee)\rangle. \tag{102}$$

This is consistent with formulas (99). The Proposition is proved. $\square$

Denote by Y_{sc} the lattice generated by the simple coroots. Consider the dual lattice $X_{sc} \subset \mathcal{O}^*(H)$. We identify $X_{sc} = \bigoplus_{D\in W^{(1)}} \mathbb{Z}$ by $x \longmapsto \sum_{s_i} x(\alpha_i^\vee)\omega_0 s_i$. Then we identify the complex (98) with

$$\bigwedge^2 X \longrightarrow X_{sc} \otimes X \longrightarrow \bigoplus_{W^{(2)}} \mathbb{Z}. \tag{103}$$

By [4, Lemma in 4.4.4] or (99), the $\omega_0 s_i s_j$-component of the differential of $C \in X_{sc} \otimes X$ is given by:

$$C \longrightarrow C\big(\alpha_i^\vee, \alpha_j^\vee\big) + C\big(\alpha_j^\vee, s_j\big(\alpha_i^\vee\big)\big).$$

We write the right hand side via the quadratic form $Q(y) := \mathrm{C}(y, y)$ on Y_{sc} and the associated symmetric bilinear form $\mathrm{B}(\alpha_i^\vee, \alpha_j^\vee) = C(\alpha_i^\vee, \alpha_j^\vee) + C(\alpha_j^\vee, \alpha_i^\vee)$. Namely, using $s_j(\alpha_i^\vee) = \alpha_i^\vee - \alpha_j(\alpha_i^\vee)\alpha_j^\vee$, we get:

$$C\big(\alpha_i^\vee, \alpha_j^\vee\big) + C\big(\alpha_j^\vee, s_j\big(\alpha_i^\vee\big)\big) = \mathrm{B}\big(\alpha_i^\vee, \alpha_j^\vee\big) - \alpha_j\big(\alpha_i^\vee\big)Q\big(\alpha_j^\vee\big). \tag{104}$$

Therefore the element C is killed by the second differential if and only if the quadratic from $Q(y)$ on Y_{sc} is W-invariant [4, Lemma 4.5]. So the cohomology class of a cocycle in $\bigoplus_{W^{(1)}} X$ is non-trivial if and only if the corresponding quadratic form is non-zero.

Now let us look at the cocycle that we constructed using the cluster coordinates:

$$\mathrm{C}^{(3)} = \sum_{k \in I} d_k \cdot \left(\{A_k = 0\}, \;\; \Delta^{-1}_{k, s_k \omega_0} \prod_{i \in I \setminus \{k\}} A_i^{\frac{C_{ki}}{2}} \right).$$

Here is a caveat. There are two ways to write the Bruhat decompostions of an element $g \in G$:

$$g = u_1 h_l \overline{\omega} u_2 \;\text{ or }\; g = u_1 \overline{\omega} h_r u_2, \quad h_l, h_r \in \mathrm{H}.$$

Following [17], we defined $\mathrm{C}^{(3)}$ using the left one, while [4] use the right one. Since $\overline{\omega} h_r = \omega(h_r)\overline{\omega}$, we have $h_l = \omega(h_r)$. This impacts our formulas as follows. The function $\Delta_i(g) = \Lambda_i(h_l)$ on the Bruhat cell $B\omega_0 B$ is equal to the function

$$\Lambda_i(\omega_0(h_r)) = \Lambda_{i^*}(h_r).$$

Since $s_k \omega_0 = \omega_0 s_k^*$, and $\Delta_{k, s_k \omega_0} = \Lambda_k(h_l)$ on the divisor $B s_k \omega_0 B = B \omega_0 s_k^* B$, we have $h_l = s_k \omega_0(h_r)$, and so as $s_k \alpha_k^\vee(t) = \alpha_k^\vee(t)^{-1}$, we have

$$\Lambda_k(h_l) = \Lambda_k(s_k \omega_0(h_r)) = \Lambda_k^{-1}(\omega_0(h_r)) = \Lambda_{k^*}^{-1}(h_r).$$

Since $*$ is an involution, we have $C_{ij} = C_{i^* j^*}, d_k = d_{k^*}$. So the element $\mathrm{C}^{(3)} \in \bigoplus_{W^{(1)}} X$ is:

$$\mathrm{C}^{(3)} = \sum_{k \in I} d_k \cdot \left(\mathcal{B}_{\omega_0 s_k}, \;\; \Delta_{k, \omega_0 s_k} \prod_{i \in I \setminus \{k\}} (\Delta_{i, w_0})^{\frac{C_{ki}}{2}} \right). \tag{105}$$

In the Brylinski–Deligne format the Bruhat divisor $\mathcal{B}_{w_0 s_k}$ corresponds to $\omega_0 s_k \in W^{(1)}$, and identified with the basis element dual to the coroot $\alpha_k^\vee$. Then the element $\mathrm{C}^{(3)}$ is mapped to

$$\mathrm{C}^{(3)} \longmapsto \sum_{k \in \mathrm{I}} d_k \cdot \Lambda_k \otimes \Lambda_k + \sum_{i \in \mathrm{I} \setminus \{k\}} \frac{d_k C_{ki}}{2} \Lambda_k \otimes \Lambda_i. \tag{106}$$

Since $d_k \mathrm{C}_{ki}$ is symmetric, we get a symmetric tensor, which is the Killing quadratic form in the basis Λ_k:

$$\mathrm{C}^{(3)} \longmapsto \sum_{k \in \mathrm{I}} d_k \cdot (\Lambda_k)^2 + \sum_{i \neq k \in \mathrm{I}} d_k C_{ki} \cdot \Lambda_k \Lambda_i. \tag{107}$$

The corresponding quadratic form at any simple coroot $\alpha_i^\vee$ corresponding to a short root α_i ($d_i = 1$) is:

$$Q(\alpha_i^\vee) = h_{sc}(\alpha_i^\vee, \alpha_i^\vee) = 1.$$

Note that if simple roots α and β are not orthogonal, and α is not shorter than β, then $\beta(\alpha^\vee) = -1$.

Although the cocycle $\mathrm{C}^{(3)}$ has half-integral coefficients, we can alter it by a coboundary and get an integral cocycle. Therefore the cohomology class $[\mathrm{C}^{(3)}]$ is the canonical generator. Theorem 2.1 is proved.

Since we proved that $\mathrm{C}^{(\bullet)}$ is a cocycle by combining Theorem 6.2, formula (60), and Theorem 4.4, and its cohomology class is a generator by Theorem 2.1 and isomorphisms (2), Theorem 2.4 is proved.

8 Applications

It was proved in [17] that the class $[W] \in K_2(\mathrm{Conf}_3(\mathcal{A}))$ is dihedrally sign-invariant, that is invariant under the cyclic shifts, and skew invariant under the permutation of two vertices $(1, 2, 3) \longmapsto (2, 1, 3)$. Therefore the class $[\mathrm{C}^{(2)}] \in K_2(\mathrm{Conf}_3^\times(\mathcal{A}))$ is dihedrally sign-invariant. This implies the important

Proposition 8.1 *The element*

$$\mathrm{C}^{(1)} \in \mathrm{B}_2(\mathbb{Q}(\mathrm{Conf}_4(\mathcal{A}))) \otimes \mathbb{Q} \tag{108}$$

is sign-invariant under the action of the permutation group S_4 on $\mathrm{Conf}_4(\mathcal{A})$.

Proof The dihedral sign-invariance of $[\mathrm{C}^{(2)}]$ implies that $\delta\mathrm{C}^{(1)}$ is dihedrally sign-invariant. Since $\mathrm{Conf}_4(\mathcal{A})$ is a rational variety, the group $K_3^{\mathrm{ind}} \otimes \mathbb{Q}$ of its function field is the same as for $\mathbb{Q}$, and thus trivial. □

1. The universal K_2-extension of G. Its existence was proved by Matsumoto, and revisited by Brylinsky–Deligne [4]. However no explicit cocycle description was known before. Here is one.

Pick a decorated flag $\mathcal{F} \in \mathcal{A}_{\mathrm{G}}$. Then given a generic triple $(g_1, g_2, g_3) \in \mathrm{G}^3(\mathrm{F})$ we set

$$\mathcal{C}_2(g_1, g_2, g_3) := \mathrm{C}^2(g_1 \cdot \mathcal{F}, g_2 \cdot \mathcal{F}, g_3 \cdot \mathcal{F}) \in K_2(\mathrm{F}). \tag{109}$$

Then for any generic quadruple (g_1, g_2, g_3, g_4) it satisfies the 2-cocycle condition. It is well known that a 2-cocycle of $\mathrm{G}(\mathrm{F})$ with values in $K_2(\mathrm{F})$, defined at the generic point, determines the group extension

$$1 \longrightarrow K_2(\mathrm{F}) \longrightarrow \widehat{\mathrm{G}}(\mathrm{F}) \longrightarrow \mathrm{G}(\mathrm{F}) \longrightarrow 1. \tag{110}$$

2. The Kac–Moody group extension related to a Riemann surface. Let Σ be a Riemann surface with punctures. Then there is a group extension

$$1 \longrightarrow \mathrm{H}^1(\Sigma, \mathbb{C}^*) \longrightarrow \widehat{\mathrm{G}}_\Sigma \longrightarrow \mathrm{G}(\mathrm{Hol}(\Sigma)) \longrightarrow 1. \tag{111}$$

Here $\mathrm{Hol}(\Sigma)$ stands for the field of all holomorphic functions with arbitrary singularities at the punctures, including the functions with essential singularities at the punctures, e.g. $e^{c_1/z+c_2/z^2+\cdots}$. There is an algebraic variant where we take the field of rational functions on Σ and the corresponding group $\mathrm{G}(\mathbb{C}(\Sigma))$.

The extension is the push down of the universal extension of $\mathrm{G}(\mathrm{Hol}(\Sigma))$ by $K_2(\mathrm{Hol}(\Sigma))$ by the Beilinson-Deligne regulator map

$$\mathrm{reg} : K_2(\mathrm{Hol}(\Sigma)) \longrightarrow \mathrm{H}^1(\Sigma, \mathbb{C}/\mathbb{Z}(2)). \tag{112}$$

Namely, following Beilinson [1, Lemma 1.3.1] and Deligne [5], given an element $f \wedge g$ and a loop γ on Σ, the value of the cohomology class $\mathrm{reg}(f \wedge g)$ on the homology class $[\gamma]$ is given by the integral

$$\langle \mathrm{reg}(f \wedge g), \gamma \rangle := \exp \frac{1}{2\pi i} \cdot \left(\int_\gamma \log f d \log g - g(p) \int_\gamma d \log f \right) \in \mathbb{C}^\times. \tag{113}$$

Here p is a point on γ and the integrals start from p. The result is independent of the choices of the branch of $\log f$ and the initial point p.

It is important for some applications that the construction works for the group defined using all holomorphic functions on a punctured Riemann surface, rather than just the meromorphic ones.

In particular, in the special case when $\Sigma = \mathbb{C}^\times$ and γ is a loop around zero, we get a holomorphic variant of the Kac–Moody group extension:

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \widehat{\mathrm{G}}(\mathrm{Hol}(\mathbb{C}^\times)) \longrightarrow \mathrm{G}(\mathrm{Hol}(\mathbb{C}^\times)) \longrightarrow 1, \tag{114}$$

3. The determinant line bundles. Using (114), we get an explicit construction of the determinant line bundle on the affine Grassmannian $\widehat{\mathrm{G}}((t))/\mathrm{G}(\mathcal{O})$. Similarly, we get an explicit construction of the determinant line bundle on $\mathrm{Bun}_{\mathrm{G}}$.

4. K_2-Lagrangians in moduli spaces of G-local systems on S. Recall the moduli space $\mathcal{U}_{\mathrm{G},S}$ of G-local systems on a punctured surface S, with unipotent monodromies around the punctures, and a reduction to a Borel subgroup at each puncture, called a framing.

Let M be a threefold whose boundary $\overline{S}$ is obtained by filling the punctures on S. Consider the subspace $\mathcal{M}_{\mathrm{G},M} \subset \mathcal{U}_{\mathrm{G},S}$ parametrising framed unipotent G-local systems on S which extend to M.

Theorem 8.2 (i) *The moduli space $\mathcal{U}_{\mathrm{G},S}$ is K_2-symplectic.*

(ii) *The moduli subspace $\mathcal{M}_{\mathrm{G},M}$ is a K_2-Lagrangian subspace of the moduli space $\mathcal{U}_{\mathrm{G},S}$.*

(iii) *There is the motivic volume map, defined at the generic point* $\mathcal{M}^{\circ}_{\mathrm{G,M}}$ *of* $\mathcal{M}_{\mathrm{G,M}}$*, with values in the Bloch group of* $\mathbb{C}$*:*

$$\mathrm{Vol}_{\mathrm{mot}} : \mathcal{M}^{\circ}_{\mathrm{G,M}} \longrightarrow \mathrm{B}_2(\mathbb{C}). \tag{115}$$

Proof Pick a cocycle $\mathrm{C}^{(\bullet)}$ representing the class c_2.

(i) Take an ideal triangulation $\mathcal{T}$ of $\overline{S}$, i.e. a triangulation with the vertices at the punctures. Take a generic framed G-local system $\mathcal{L}$ on S. Since its monodromy around each puncture is a regular unipotent element, there exists a unique decorated flags $\mathcal{F}_p$ near every puncture p which is invariant under the monodromy around p. For each triangle t of $\mathcal{T}$, there is a configuration of three decorated flags $(\mathcal{F}^t_1, \mathcal{F}^t_2, \mathcal{F}^t_3) \in \mathrm{Conf}_3(\mathcal{A}_{\mathrm{G}})$ obtained by restricting the $\mathcal{L}$ and the three flat sections of the associated to $\mathcal{L}$ local system of decorated flags near each vertex of t to the triangle t. Then we have an element

$$W^{\mathcal{T}}_S := \sum_{t\in\mathcal{T}} \mathrm{C}^{(2)}\big(\mathcal{F}^t_1, \mathcal{F}^t_2, \mathcal{F}^t_3\big) \in K_2(\mathbb{Q}(\mathcal{U}_{\mathrm{G},S})). \tag{116}$$

Its tame symbol is zero. Indeed, the tame symbol of each of the terms is a sum of the three standard terms provided by the element $\mathrm{C}^{(3)}$, associated with the edges of the triangle t, but for each edge E, the contributions of the two triangles cancel each other. The element $W^{\mathcal{T}}_S$ does not depend on the choice of the triangulation $\mathcal{T}$ since a flip of the triangulation $\mathcal{T} \to \mathcal{T}'$ at an edge E amounts to

$$W^{\mathcal{T}}_S - W^{\mathcal{T}'}_S = \delta\mathrm{C}^{(1)}\big(\mathcal{F}^r_1, \mathcal{F}^r_2, \mathcal{F}^r_3, \mathcal{F}^r_4\big), \qquad . \tag{117}$$

where r is the rectangle of the triangulation associated with the edge E, and $(\mathcal{F}^r_1, \mathcal{F}^r_2, \mathcal{F}^r_3, \mathcal{F}^r_4) \in \mathrm{Conf}_4(\mathcal{A}_{\mathrm{G}})$ is the quadruple of flags associated to the rectangle.

(ii) Take a triangulation $\mathcal{T}_{\mathrm{M}}$ of the threefold M extending the triangulation $\mathcal{T}$ of $\overline{S}$. Then just as above, one assigns to each tetrahedron T of this triangulation a configuration of 4 decorated flags $(\mathcal{F}^{\mathrm{T}}_1, \mathcal{F}^{\mathrm{T}}_2, \mathcal{F}^{\mathrm{T}}_3, \mathcal{F}^{\mathrm{T}}_4) \in \mathrm{Conf}_4(\mathcal{A}_{\mathrm{G}})$ and apply to it the element $\mathrm{C}^{(1)}$:

$$\mathrm{Vol}_{\mathrm{mot}} := \sum_{\mathrm{T}\in\mathcal{T}_{\mathrm{M}}} \mathrm{C}^{(1)}\big(\mathcal{F}^{\mathrm{T}}_1, \mathcal{F}^{\mathrm{T}}_2, \mathcal{F}^{\mathrm{T}}_3, \mathcal{F}^{\mathrm{T}}_4\big) \in \mathrm{B}_2(\mathbb{Q}(\mathcal{M}_{\mathrm{G,M}})). \tag{118}$$

This element is sign-invariant under the action of the group S_4 by Proposition 8.1, and thus does not depend on does not depend on the choice of the order of the four flags. It also does not depend on the triangulation. Indeed, altering a triangulation by a 2 by 3 Pachner move related to the five tetrahedra whose vertices are decorated by the five flags $\mathcal{F}_1, \ldots, \widehat{\mathcal{F}_i}, \ldots, \mathcal{F}_5$ amounts to changing element (118) by

$$\sum_{i=1}^{5} \mathrm{C}^{(1)}(\mathcal{F}_1, \ldots, \widehat{\mathcal{F}_i}, \ldots, \mathcal{F}_5) \in \mathrm{B}_2(\mathbb{Q}(\mathcal{M}_{\mathrm{G,M}})). \tag{119}$$

The cocycle property of $C^\bullet$ implies that applying the Bloch complex differential δ to (119) we get zero:

$$\delta \sum_{i=1}^{5} \mathrm{C}^{(1)}(\mathcal{F}_1, \ldots, \widehat{\mathcal{F}_i}, \ldots, \mathcal{F}_5) = 0.$$

Therefore (119)= 0 by a K-theoretic argument very similar to the one in the proof of Proposition 8.1.

Next, denote by $j : \mathcal{M}_{\mathrm{G,M}} \subset \mathcal{U}_{\mathrm{G},S}$ the natural inclusion. Since $\mathrm{C}^{(\bullet)}$ is a cocycle:

$$\begin{aligned}\delta \mathrm{Vol}_{\mathrm{mot}} &= \sum_{\mathrm{T}\in\mathcal{T}_{\mathrm{M}}} \delta \mathrm{C}^{(1)}\big(\mathcal{F}_1^{\mathrm{T}}, \mathcal{F}_2^{\mathrm{T}}, \mathcal{F}_3^{\mathrm{T}}, \mathcal{F}_4^{\mathrm{T}}\big)\\ &= \sum_{\mathrm{t}\in\mathcal{T}} \mathrm{C}^{(2)}\big(\mathcal{F}_1^{\mathrm{t}}, \mathcal{F}_2^{\mathrm{t}}, \mathcal{F}_3^{\mathrm{t}}\big)\\ &\stackrel{\mathrm{def}}{=} j^* W_S^{\mathcal{T}} \in \Lambda^2 \mathbb{Q}(\mathcal{M}_{\mathrm{G,M}})^*. \end{aligned} \tag{120}$$

The second = is because the contributions of the internal triangles cancel out. The third equality is valid by the definition of j^*. Therefore $[j^* W_S^{\mathcal{T}}] = 0$ in $K_2(\mathrm{B}_2(\mathbb{Q}(\mathcal{M}_{\mathrm{G,M}})))$. The claim ii) is proved.

(iii) Specializing the element (118) to any generic complex point of x we get the motivic volume map (115). Its composition with the map $\mathrm{B}_2(\mathbb{C}) \longrightarrow \mathbb{R}$, provided by the Bloch–Wigner dilogarithm, is a volume map, generalizing the volume of a hyperbolic threefold. □

5. A local combinatorial formula for the second Chern class of a G-bundle. Recall the weight two exponential complex of sheaves on a complex manifold X [15]:

$$\mathbb{Z}(2) \longrightarrow \mathcal{O}(1) \longrightarrow \Lambda^2\mathcal{O} \stackrel{\wedge^2 \exp}{\longrightarrow} \Lambda^2\mathcal{O}^*. \tag{121}$$

Here the second arrow is $2\pi i \otimes f \longmapsto 2\pi i \wedge f$, and the last one is $f \wedge g \longmapsto \exp(f) \wedge \exp(g)$. It is a complex of sheaves in the analytic topology on X, exact modulo torsion.

We sheafify the Bloch complex to a complex of sheaves $B_2(\mathcal{O}) \longrightarrow \Lambda^2\mathcal{O}^*$ and define a map of complexes

$$\begin{tikzcd} & \mathrm{R}_2(\mathcal{O}) \arrow[r] \arrow[dl] \arrow[d] & \mathbb{Z}[\mathcal{O}] \arrow[r, "\delta"] \arrow[d, "\mathbb{L}_2"'] & \Lambda^2\mathcal{O}^* \arrow[d, "="] \\ \mathbb{Q}(2) \arrow[r] & \mathcal{O}(1) \arrow[r] & \Lambda^2\mathcal{O} \arrow[r, "\wedge^2\exp"] & \Lambda^2\mathcal{O}^* \end{tikzcd}$$

To define the map $\mathbb{L}_2$, recall the dilogarithm function, with the two accompanying logarithms:

$$\mathrm{Li}_2(x) := \int_0^x \frac{dt}{1-t} \circ \frac{dt}{t}, \quad -\log(1-x) = \int_0^x \frac{dt}{1-t}, \quad \log x := \int_0^x \frac{dt}{t}. \tag{122}$$

Here all integrals are along the same path from 0 to x. The last one is regularised using the tangential base point at 0 dual to dt. Then we set, modifying slightly the original construction of Bloch [2, 3],

$$\begin{aligned} \mathrm{L}_2(x) &:= \mathrm{Li}_2(x) + \frac{1}{2} \cdot \log(1-x) \log x + \frac{(2\pi i)^2}{24}, \\ \mathbb{L}_2(\{x\}_2) &:= \frac{1}{2} \cdot \log(1-x) \wedge \log x + 2\pi i \wedge \frac{1}{2\pi i} \mathrm{L}_2(x). \end{aligned} \tag{123}$$

We keep the summand $\frac{(2\pi i)^2}{24}$ in $\mathrm{L}_2(x)$, although it does not change $2\pi i \wedge \frac{1}{2\pi i}\mathrm{L}_2(x)$ since $2\pi i \wedge \frac{2\pi i}{24} = 0$ in $\Lambda^2\mathbb{C}$. The key fact is [15, Lemma 1.6] the map $\mathbb{L}_2$ is well defined on $\mathbb{Z}[\mathcal{O}]$, i.e. does not depend on the monodromy of the logarithms and the dilogarithm along the path γ in (122). It evidently provides a map of complexes. So one has $\mathbb{L}_2 : \mathrm{Ker}\ \delta \longrightarrow \mathcal{O}(1)$. Furthermore, we have

$$\mathbb{L}_2(\mathrm{Ker}\ \delta) \subset \mathbb{C}(1), \qquad \mathbb{L}_2(\mathrm{R}_2(\mathcal{O})) \subset \mathbb{Q}(2). \tag{124}$$

Given a G-bundle $\mathcal{L}$ over a complex manifold X, pick an open by discs U_i and choose a section g_i of $\mathcal{L}$ over U_i. Then we define a 4-cocycle for the Chech cover $\{\mathrm{U}_i\}$ with values in $\mathbb{Q}(2)$ by setting

$$\mathrm{U}_{i_1} \cap \ldots \cap \mathrm{U}_{i_5} \longmapsto \sum_{k=1}^{5} (-1)^k \mathbb{L}_2\Big(\mathrm{C}^{(1)}(g_{i_1}, \ldots, \widehat{g_{i_k}}, \ldots, g_{i_5})\Big) \in \mathbb{Q}(2). \tag{125}$$

The main result of this paper implies that it represents the second Chern class $c_2(\mathcal{L})$. This is a local combinatorial formula for $c_2(\mathcal{L})$, in the spirit of the Gabrielov–Gelfand–Losik combinatorial formula [12] for the first Pontryagin class. See an elaborate discussion of the simplest example in [15, Section 1.7].

We conclude that, although given a cocycle $\mathrm{C}^\bullet$ the above constructions are very transparent, the cocycle itself for $\mathrm{G} \neq \mathrm{SL_m}$ is rather complicated, and can not be written without the cluster technology. On the other hand, for $\mathrm{G} = \mathrm{SL_n}$ the cocycle is simple and canonical, see [14], [15, Sections 4.2−4.3].

9 Quantum deformation of the cohomology group $\mathrm{H}^3_{\mathrm{meas}}(\mathrm{G}(\mathbb{C}), \mathbb{R})$.

1. Measurable cocycles of $\mathrm{G}(\mathbb{C})$**.** The measurable cohomology $\mathrm{H}^*_{\mathrm{meas}}(\mathrm{G}, \mathbb{R})$ of a Lie group G are the cohomology of the complex of G-invariants of measurable functions on $\mathrm{Meas}(\mathrm{G}^n)$:

$$\begin{gathered} \ldots \longrightarrow \mathrm{Meas}(\mathrm{G}^{n-1})^{\mathrm{G}} \longrightarrow \mathrm{Meas}(\mathrm{G}^{n})^{\mathrm{G}} \longrightarrow \mathrm{Meas}(\mathrm{G}^{n+1})^{\mathrm{G}} \longrightarrow \ldots \\ df(g_1, \ldots, g_n) := \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, \widehat{g}_i, \ldots, g_n). \end{gathered} \tag{126}$$

Denote by $\{d_m\}$ the degrees of the generators of the ring $\mathrm{S}^*(\mathfrak{h})^W$. So, when G is of type A_r, we have $(d_1, \ldots, d_r) = (2, 3, \ldots, r)$. Then $\mathrm{H}^*_{\mathrm{meas}}(\mathrm{G}(\mathbb{C}), \mathbb{R})$ is a graded commutative algebra generated by the classes

$$b_{\mathrm{G}, 2d_m - 1} \in \mathrm{H}^{2d_m - 1}_{\mathrm{meas}}(\mathrm{G}(\mathbb{C}), \mathbb{R}). \tag{127}$$

In particular, one always has $d_1 = 2$. So we have a class

$$b_3 = b_{\mathrm{G},3} \in \mathrm{H}^3_{\mathrm{meas}}(\mathrm{G}(\mathbb{C}), \mathbb{R}). \tag{128}$$

Below we quantize the exponent of the class b_3, for any G, using crucially the fact that $\mathrm{G}\backslash\mathrm{G}^n$ and $\mathrm{Conf}_n(\mathrm{G}/\mathrm{B}) := \mathrm{G}\backslash(\mathrm{G}/\mathrm{B})^n$ have a cyclically invariant cluster Poisson structure [17].

2. The quantum set-up. The cluster Poisson structure on $\mathrm{G}\backslash\mathrm{G}^n$ gives rise to an algebra of q-deformed functions

$$\mathcal{O}_q(\mathrm{G}\backslash\mathrm{G}^n).$$

It is the non-commutative version of the algebra $\mathrm{Meas}(\mathrm{G}^n)^{\mathrm{G}}$. There are natural maps

$$s_i^* : \mathcal{O}_q(\mathrm{G}\backslash\mathrm{G}^n) \longrightarrow \mathcal{O}_q(\mathrm{G}\backslash\mathrm{G}^{n+1}); \tag{129}$$

induced by the cluster Poisson maps

$$\mathrm{G}\backslash\mathrm{G}^{n+1} \longrightarrow \mathrm{G}\backslash\mathrm{G}^n, \qquad s_i : (g_1, \ldots, g_{n+1}) \longmapsto (g_1, \ldots, \widehat{g}_i, \ldots, g_{n+1}).$$

Similarly there is a non-commutative algebra of q-deformed functions

$$\mathcal{O}_q(\mathrm{Conf}_n(\mathrm{G}/\mathrm{B}))$$

together with maps
of algebras

$$s_i^* : \mathcal{O}_q(\mathrm{Conf}_n(\mathrm{G}/\mathrm{B})) \longrightarrow \mathcal{O}_q(\mathrm{Conf}_{n+1}(\mathrm{G}/\mathrm{B})).$$

These algebras and the maps between them are related by the maps of algebras

$$\pi^* : \mathcal{O}_q(\mathrm{Conf}_n(\mathrm{G}/\mathrm{B})) \longrightarrow \mathcal{O}_q(\mathrm{G}\backslash\mathrm{G}^n). \tag{130}$$

For any cluster Poisson variety $\mathcal{X}$, the completion $\widehat{\mathcal{O}}_q(\mathcal{X})$ is defined as collections of formal quantum power series in each of the cluster coordinate systems, related by quantum cluster transformations. The maps of algebras above provide maps of completed algebras.

3. The class $\mathcal{B}_3$ quantizing β_3 in (128). The cluster construction of the second motivic Chern class provides at the same time its quantum deformation. Let us explain the quantum deformation of the class

$$\beta_3 \in \mathrm{H}^3_{\mathrm{meas}}(\mathrm{G}(\mathbb{C}), \mathbb{R}).$$

A 3-cocycle for the class β_3 is a measurable G-invariant function $\beta_3(g_1, \ldots, g_4)$ on $\mathrm{G}(\mathbb{C})^4$. Our construction of the element $\mathrm{C}^{(1)}$ gives an explicit formula for this function as a sum of Bloch–Wigner dilogarithms:

$$\beta_3(g_1, \ldots, g_4) = \sum_j \mathcal{L}_2(z_j), \quad g_i \in \mathrm{G}(\mathbb{C}). \tag{131}$$

Here z_i are certain rational functions on $\mathrm{G}\backslash(\mathrm{G}/\mathrm{B})^4$. Let us define a quantum deformation of this cocycle.

The quantum analog of the exponent of the cocycle β_3 lies in the formal completion:

$$\mathcal{B}_3^{\mathbf{i}} \in \widehat{\mathcal{O}}_q(\mathrm{G}^4). \tag{132}$$

Recall the maps of algebras

$$s_i^* : \widehat{\mathcal{O}}_q(\mathrm{G}^4) \longrightarrow \widehat{\mathcal{O}}_q(\mathrm{G}^5).$$

Pick a reduced decomposition $\mathbf{i}$ of the longest element w_0 of the Weyl group of G.

Theorem 9.1 *There is an element $\mathcal{B}_3^{\mathbf{i}}$ in (132) which satisfies the multiplicative quantum cocycle relation*

$$\prod_{j=1}^{5} s_{2j+1}^* \mathcal{B}_3^{\mathbf{i}} = 1. \tag{133}$$

Changing the reduced decomposition $\mathbf{i}$ of w_0 amounts to changing the cocycle $\mathcal{B}_3^{\mathbf{i}}$ by a coboundary.

Proof Recall the quantum dilogarithm power series, convergent if $|q| < 1$ for any $Z \in \mathbb{C}$:

$$\Psi_q(Z) = \frac{1}{(1+qZ)(1+q^3Z)(1+q^5Z)\dots}$$

We define the element $\mathbf{B}_3^{\mathbf{i}}$ as a product of the quantum dilogarithm power series

$$\mathbf{B}_3^{\mathbf{i}} = \prod_j \Psi_q(Z_j), \quad Z_j \in \mathcal{O}_q((\mathrm{G/B})^4). \tag{134}$$

The functions $\{Z_j\}$ are q-deformations of functions z_j in (131), $\lim_{q\to 1} Z_j = z_j$, defined as follows.

Given an oriented triangle t and a choice of one of its angles, the reduced decomposition $\mathbf{i}$ of w_0 provides a cluster Poisson coordinate system on the moduli space $\mathcal{P}_{\mathrm{G},3}$ of triples of flags with pinnings [17] assigned to the triangle t.

Consider a quadrilateral Q with a special side F and a diagonal E. It has two marked angles: the one opposite to F in the triangle with the base F, and the one in the second triangle, opposite to E. Therefore amalgamating along the diagonal E the cluster Poisson structures which we assigned to each of the two triangles of Q using this data we get a cluster Poisson structure on the space $\mathcal{P}_{\mathrm{G},4}$ assigned to the quadrilateral, and thus on $\mathrm{Conf}_4(\mathcal{B})$.

Take an oriented convex pentagon P_5 whose vertices are decorated by the flags $\mathrm{B}_1, \dots, \mathrm{B}_5$, providing a point of $\mathrm{Conf}_5(\mathcal{B})$. Take a triangulation of the pentagon. Pick one of the diagonals and denote it by F. The diagonal F cuts the pentagon into a quadrilateral Q_F with a base F, and a triangle t_F:

$$\mathrm{P}_5 = Q_F \cup t_{\mathrm{F}}.$$

Mark the angle of the triangle t_F opposite to F, and mark the two angles in the quadrangle Q_F as above, using the base F and the diagonal E, as shown on Fig. 5. Then each of the three triangles of the pentagon has a marked angle, marked the red point on Fig. 5.

Therefore the reduced decomposition $\mathbf{i}$ provides a cluster Poisson coordinate system on the space $\mathcal{P}_{\mathrm{G},3}$ assigned to each of the three triangles, and hence by the amalgamation a cluster Poisson system on $\mathrm{Conf}_5(\mathcal{B})$. Now flip the triangulation at the edge F, getting a new edge F'. Label the new edges as (E_1, F_1), setting $E_1 := F'$, $F_1 := E$, see Fig. 6. Assign to the triangulation (E_1, F_1) a similar cluster Poisson coordinate system on $\mathrm{Conf}_5(\mathcal{B})$ using the marked angles in each of the three triangles of the new triangulation, and the reduced decomposition $\mathbf{i}$ of w_0.

The flip of triangulation at the edge F alters cluster Poisson coordinates only in the quadrileteral Q'_F containing the edge F as the diagonal. It is realized as an ordered sequence of mutations, provided by the cluster Poisson rational functions $Z_1, \dots, Z_N$ on $\mathrm{Conf}_4(\mathcal{B})$ related to the quadrilateral Q'_F. Each mutation is given by the conjugation by $\Psi_q(Z_i)$. We use the sequence $\{Z_j\}$ to define the element $\mathbf{B}_3^{\mathbf{i}}$ in (134). The elements $\{z_j\}$ in (131) are defined as the $q = 1$ specialization of the elements $\{Z_j\}$.

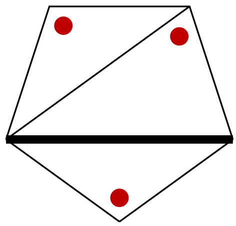


Fig. 5 A triangulated pentagon P_5 with a boldface diagonal F, which cuts the pentagon into a quadrilateral with a diagonal E, and triangle. The data $(P_5; F, E)$ determines three red marked angles

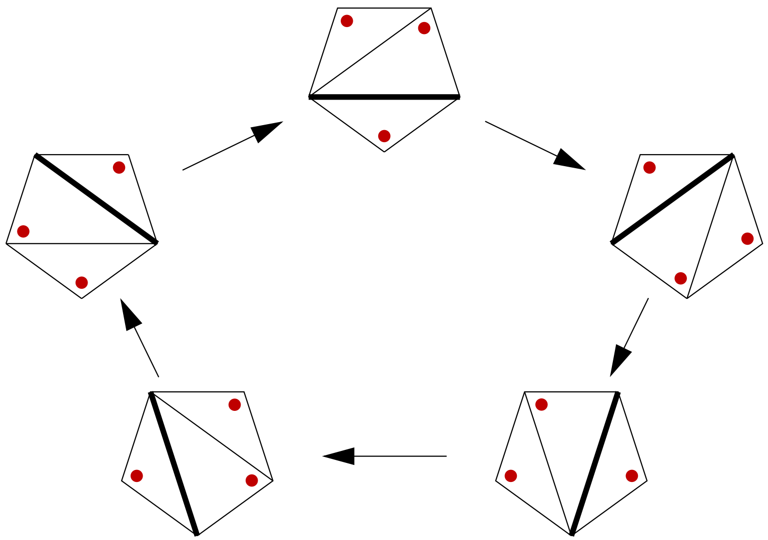


Fig. 6 The five triangulations of the pentagon, related by flips of the boldface diagonals, and the red marked angles for each triangulation

The main difference between the classical and quantum cocycles β_3 and $\mathbf{B}_3^{\mathbf{i}}$ is that the elements $\{Z_i\}$ do not commute, and so their order is an essential part of the definition of the element $\mathbf{B}_3^{\mathbf{i}}$.

Traditionally each mutation is given by the conjugation by $\Psi_q(Z)$ followed by a monomial transformation, and a cluster Poisson transformation is defined as a composition of such elementary transformations. However one can also define a *reduced mutation* as just the conjugation by $\Psi_q(Z)$, and define the *reduced cluster Poisson transformation* as the composition of reduced mutations [16, Proposition 2.4].

Performing this procedure five times, as shown on Fig. 6, we get the original triangulation (E, F), and the original cluster Poisson coordinate system. The sequence of cluster Poisson coordinates given by the sequence of mutations realizing the flip of the diagonal F_i on the step i is denoted by $Z_1^{(i)}, \ldots, Z_N^{(i)}$. Then the ordered sequence of cluster Poisson coordinates we need is given by the $5N$ functions

$$Z_1^{(1)}, \ldots, Z_N^{(1)};\ Z_1^{(1)}, \ldots, Z_N^{(1)};\ \ldots;\ Z_1^{(5)}, \ldots, Z_N^{(5)}. \tag{135}$$

Proposition 9.2 *The following product is equal to* 1*:*

$$\prod_j \Psi_q\big(Z_j^{(5)}\big) \cdot \prod_j \Psi_q\big(Z_j^{(4)}\big) \cdot \ldots \cdot \prod_j \Psi_q\big(Z_j^{(1)}\big) = 1. \tag{136}$$

Proof If a reduced cluster transformation is the identity map, then the product of the corresponding $\Psi_q(Z_i)$ in the completed q-deformed algebra is equal to 1 [18, 19], cf [16, Theorem 3.2]. □

Therefore the element $\mathbf{B}_3^{\mathbf{i}}$ satisfies the multiplicative analog of the cocycle condition

$$\prod_{j=1}^{5} s_{2j+1}^{*}\mathbf{B}_3^{\mathbf{i}} = 1. \tag{137}$$

Indeed, relation (136) is equivalent to relation (137) on elements (134). Its pull back $\mathcal{B}_3^{\mathbf{i}}$ automatically satisfies the cocycle relation (133). The first part of Theorem 9.1 is proved.

4. Changing the reduced decomposition i alters the cocycle $\mathcal{B}_3^{\mathbf{i}}$ by a coboundary. The element $\mathcal{B}_3^{\mathbf{i}}$ depends on the choice of a reduced decomposition $\mathbf{i}$ of w_0, providing the Poisson cluster $\mathbf{c_i}$ on $\mathcal{P}_{\mathrm{G},3}$. Let $\mathbf{i}'$ be another reduced decomposition of w_0. Then there is a sequence of cluster mutations providing a cluster Poisson transformations $\mathbf{c_i} \to \mathbf{c_{i'}}$. Let us denote by $Y_1, \ldots, Y_k$ the related ordered sequence of cluster Poisson functions.

Let P_4 be a convex quadrilateral with the vertices labeled cyclically by $\{1, 2, 3, 4\}$. Forgewtting a vertex $I \in \mathbb{Z}/4\mathbb{Z}$ we get a triangle with one distinguished vertex - the one opposite to the forgotten vertex i in P_4. Therefore the cluster transformation $\mathbf{c_i} \to \mathbf{c_{i'}}$ provides the collection of quantum functions $Y_1^{(i)}, \ldots, Y_k^{(i)}$ above on the space $\mathcal{P}_{\mathrm{G},3}$ provided by the triangle. Let us introduce the notation

$$\Psi(v_i) := \Psi_q\big(Y_1^{(i)}\big) \cdot \ldots \cdot \Psi_q\big(Y_k^{(i)}\big) \in \widehat{\mathcal{O}}_q(\mathcal{P}_{\mathrm{G},3}). \tag{138}$$

Observe the key point: the elements $Y_1^{(i)}, \ldots, Y_k^{(i)}$ commute with the ones $Y_1^{(i+2)}, \ldots, Y_k^{(i+2)}$. Indeed, the factors of each of them correspond to the non-frozen cluster poisson coordinates in each of the triangles, and thus commute after the cluster Poisson amalgamation. So the elements $\Psi(v_i)$ and $\Psi(v_{i+2})$ commute:

$$\Psi(v_i) \cdot \Psi(v_{i+2}) = \Psi(v_{i+2}) \cdot \Psi(v_i). \tag{139}$$

Therefore changing the reduced decomposition $\mathbf{i}$ to $\mathbf{i}'$ we alter the cocycle $\mathbf{B}_3^{\mathbf{i}}$ by

$$\mathbf{B}_3^{\mathbf{i}} \longmapsto \mathbf{B}_3^{\mathbf{i}'} = \Psi(v_2)^{-1}\Psi(v_4)^{-1}\mathbf{B}_3^{\mathbf{i}}\Psi(v_1)\Psi(v_3).$$

We can interprete this as follows: the cocycles $\mathbf{B}_3^{\mathbf{i}}$ and $\mathbf{B}_3^{\mathbf{i}'}$ differ by the non-commutative coboundary of the element (138). Theorem 9.1 is proved. □

Our definition of a non-commutative multiplicative cocycle is specific for 3-cocycles. Our definition of the coboundary even more specific: we use the fact (139).

5. The relation with the dilogarithm. To justify the name *quantum dilogarithm* for the formal power series $\Psi_q(z)$, recall the following version of the classical dilogarithm function:

$$\mathrm{L}_2(x) := \int_0^x \log(1+t)\frac{dt}{t}.$$

It has a q-deformation:

$$\mathrm{L}_2(x;q) := \sum_{n=1}^{\infty} \frac{x^n}{n(q^n - q^{-n})}.$$

One has the identity

$$\log \mathbf{\Psi}_q(x) = \mathrm{L}_2(x;q).$$

If $|q| < 1$ the power series $\mathbf{\Psi}_q(x)$ converge, providing an analytic function in $x \in \mathbb{C}$. If in addition to this $|x| < 1$, the q-dilogarithm power series also converge. There are asymptotic expansions when $q \to 1^-$:

$$\mathrm{L}_2(x;q) \sim \frac{\mathrm{L}_2(x)}{\log q^2}, \qquad \mathbf{\Psi}_q(x) \sim \exp\Big(\frac{\mathrm{L}_2(x)}{\log q^2}\Big). \tag{140}$$

Using this one can show that the quantum cocycle relation (136) implies the classical one if $q \to 1$.

In the case when $\mathrm{G} = \mathrm{PGL}_2$, the element $\mathbf{B}_3$ is just the quantum dilogarithm $\Psi_3(Z)$, and our cocycle relation reduces to the Faddeev–Kashaev [7] pentagon relation for the quantum dilogarithm.

The main difference between classical and quantum cocycles is that the latter is a sum of commutative expressions, while the former is an ordered product of non-commuting expressions. The order is crucial, and provided by the cluster Poisson transformation describing the flip of a triangulation [17].

Note also that there is a version of the quantum cocycle where the role of the power series $\Psi_q(Z)$ is played by the quantum modular dilogarithm $\Phi_\hbar(z)$. The main difference is that now the cocycle is well defined for any $q \in \mathbb{C}$, and is understood as an operator acting in a Hilbert space.

6. Further perspectives. One can hope that there are quantum deformation of the exponents of the cocycles representing the basic classes

$$b_{2m-1} \in \mathrm{H}^{2m-1}_{\mathrm{meas}}(\mathrm{PGL}_m(\mathbb{C}), \mathbb{R}).$$

These cocycles are expressed via certain m-logarithm functions. The cocycle condition is provided by the functional equation for these functions. Cocycles for the classes in $\mathrm{H}^{2m-1}_{\mathrm{meas}}(\mathrm{PGL}_N(\mathbb{C}), \mathbb{R})$ for $N > m$ are defined once we know the ones for $N = m$ via the configuration of partial flag construction [14].

The simplest class after the dilogarithm class b_3 is

$$b_5 \in \mathrm{H}^5_{\mathrm{meas}}(\mathrm{PGL}_3(\mathbb{C}), \mathbb{R}).$$

This class was defined in [13] by the following function on configurations of 6 points $(x_1, \dots, x_6)$ in $\mathbb{C}\mathrm{P}^2$:

$$\beta_5(x_1, \dots, x_6) := \mathrm{Alt}_6 \mathcal{L}_3 \left(\frac{\Delta(1,2,3)\Delta(2,3,4)\Delta(3,1,5)}{\Delta(1,2,4)\Delta(2,3,5)\Delta(3,1,6)} \right). \tag{141}$$

Here $\mathcal{L}_3$ is the single-valued version of the trilogarithm function, $\Delta(i, j, k) := \langle \Omega_3, l_i \wedge l_j \wedge l_k \rangle$ where $l_i \in \mathbb{C}^3 - \{0\}$ lifts the point x_i, and Ω_3 is a volume form in $\mathbb{C}^3$. The function β_5 satisfies the relation

$$\sum_{i=1}^{7} (-1)^i \beta_5(x_1, \dots, \widehat{x_i}, \dots, x_7) = 0.$$

The 5-cocycle is defined by

$$b_5(g_1, \dots, g_6) := \beta_5(g_1 \cdot x, \dots, g_6 \cdot x), \quad x \in \mathbb{C}\mathrm{P}^2, \; g_i \in \mathrm{G}(\mathbb{C}).$$

It become clear later [8] that the mysterious triple ratio in formula (141) is a cluster Poisson coordinate on the moduli space $\mathrm{Conf}_6(\mathrm{P}^2)$ parametrising 6-tuples points on P^2 modulo the action of PGL_3. The latter is a cluster Poisson variety of the finite type D_4. The very fact that this function is defined on a space which carries a cluster Poisson structure suggests that one should have a quantum deformation of the exponent of β_5, provided by an element

$$\mathbf{B}_5 \in \widehat{\mathcal{O}}_q(\mathrm{Conf}_6(\mathrm{P}^2)).$$

More generally, for any $m > 1$ one should have an element

$$\mathbf{B}_{2m-1} \in \widehat{\mathcal{O}}_q(\mathrm{Conf}_{2m}(\mathrm{P}^{m-1})) \tag{142}$$

which satisfies a multiplicative $(2m+1)$-term cocycle relation. Its pull back $\mathcal{B}_{2m-1} := \pi^* \mathbf{B}_{2m-1}$ should be the quantum deformation of the exponent of cocycle for the class b_{2m-1} in (128).

10 Cluster structures and motivic cohomology: conclusion

1. Conclusions. 1. Formula (106) tells that the cocycle $\mathrm{C}^{(3)}$ is just the Killing form (107), written as a bilinear form (106), translated isomorphically into the middle group in (103), thus interpreted as a cocycle for $\mathrm{H}^1(\mathrm{G}, \underline{K}_2)$. To make the bilinear form (106) from the quadratic form (107) we need the coefficients $\frac{1}{2}$ in front of $d_k\mathrm{C}_{kj}$. Indeed, the left and the right factors in the bilinear expression (106) have entirely different meanings in (103) as, respectively, Bruhat divisors and functions on them. A posteriori this explains why the exchange matrix ε_{ij} has half integral values between the frozen variables.

2. The cluster structure of the elementary variety $\mathcal{A}(k)$, $k \in \mathrm{I}$, is determined by the following facts:

i) The corresponding element $W(k)$ is decomposed into a sum of two terms

$$W(k) = W'(k) + W_\Delta(k),$$

where $W_\Delta(k)$ is the pull back τ_k^* of the element W from the space $\mathrm{Conf}_3(\mathcal{A}_{\mathrm{SL}_2})$ for the canonical projection

$$\tau_k : \mathcal{A}(k) \longrightarrow \mathrm{Conf}_3(\mathcal{A}_{\mathrm{SL}_2}). \tag{143}$$

ii) The residue of $W(k)$ at the “right side of the quiver” is given by the cocycle $\mathrm{C}^{(3)}$. Equivalently, the cocycle $\mathrm{C}^{(3)}$ is the residue of $W_\mathbf{c}$ at the right side of the triangle t.

Indeed, the tame symbol calculation (81) nails the shape of the quiver $\mathrm{J}(k)$ of $\mathcal{A}(k)$. Namely, the exchange matrix for the right side of the quiver $\mathrm{J}(k)$ is the negative of the one for the left edge, as the argument in the end of the proof of Theorem 6.2 shows. It is determined by the cocycle C_3, and the latter is fixed by the Killing form, as discussed above.

3. The element $W_\mathbf{c}$ on $\mathrm{Conf}_3(\mathcal{A})$ determines the cluster structure on this space. The element $W_\mathbf{c}$ is forced onto us as the one whose tame symbol is given by formula (81). Therefore its existence follows from $\mathrm{H}^4(\mathrm{BG}, \mathbb{Z}_\mathcal{M}(2)) = \mathbb{Z}$.

Although such an element $W_\mathbf{c}$ is not unique, the difference between any two $W_\mathbf{c}$ and $W'_\mathbf{c}$ of them is a cocycle, providing a class $[W_\mathbf{c} - W'_\mathbf{c}] \in \mathrm{H}^0(\mathrm{G}^2, \underline{K}_2)/K_2(\mathbb{Z})$. Note that $K_2(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. On the other hand,

$$\mathrm{H}^0(\mathrm{G}^2, \underline{K}_2)/K_2(\mathbb{Z}) = 0. \tag{144}$$

This implies the crucial, and one of the most challenging, properties of the element $W_\mathbf{c}$: its class in the group K_2 of the field of functions $\mathbb{Q}(\mathrm{Conf}_3(\mathcal{A}))$ is *twisted cyclically invariant* [17, Section 7]. Indeed, it follows from (144), since the tame symbol of $W_\mathbf{c}$, given by (81), is twisted cyclic shift invariant on the nose.

4. One also has

$$\mathrm{H}^0(\mathrm{G}^3, \underline{K}_2)/K_2(\mathbb{Z}) = 0. \tag{145}$$

This makes evident another crucial fact, this time about the cluster structure of the space $\mathrm{Conf}_4(\mathcal{A})$: the *flip invariance* of the K_2-class of the element W on $\mathrm{Conf}_4(\mathcal{A})$, see paragraph 4 in Sect. 4. Indeed, the vanishing (145) implies that this K_2-class its determined by its tame symbol. The latter, as follows from (81), is the sum of the contributions of the four sides of the rectangle, and thus evidently flip invariant.

5. The cluster structure of the moduli space $\mathcal{A}_{\mathrm{G},\mathbb{S}}$ is constructed by starting from the cluster structure of the space $\mathrm{Conf}_3(\mathcal{A})$. Next, using its twisted cyclic invariance, we introduce the cluster structure on $\mathrm{Conf}_4(\mathcal{A})$ via the amalgamation. The flip invariance of the latter allows to extend the construction of the cluster structure by the amalgamation to the whole surface, and guarantees its $\Gamma_{\mathbb{S}}$-equivariance. The cluster Poisson structure of the space $\mathcal{P}_{\mathrm{G},\mathbb{S}}$ is deduced from this. Therefore the discussion above explains, for the first time, why the cluster structure on the dual pair of moduli spaces $(\mathcal{A}_{\mathrm{G},\mathbb{S}}, \mathcal{P}_{\mathrm{G},\mathbb{S}})$ should exist.

6. The fact that the number of functions entering $W_{\mathbf{c}}$ is the same as the dimension of $\mathrm{Conf}_3(\mathcal{A})$ is irrelevant for the motivic considerations described in this paper, although the collection of different clusters was used essentially to prove relation (81).

However what is needed for many applications, e.g. for the cluster quantization, is not just the fact that the K_2-class $[W_{\mathbf{c}}]$ is twisted cyclically invariant, but that the equivalence between different elements $W_{\mathbf{c}}$ is achieved by cluster transformations. This, and the amazing fact that the number of functions entering $W_{\mathbf{c}}$ is equal to $\dim\mathrm{Conf}_3(\mathcal{A})$, shows that the construction of the second motivic Chern class capture many, but not all, cluster features of the space $\mathrm{Conf}_3(\mathcal{A})$.

2. Generalizations. The truncated cocycle $(\mathrm{C}^{(2)}, \mathrm{C}^{(3)})$ gives the second Chern class in the K_2-cohomology:

$$\mathrm{c}_2^{\mathrm{M}} \in \mathrm{H}^2(\mathrm{BG}_\bullet, \underline{K}_2). \tag{146}$$

For $\mathrm{G} = \mathrm{SL}_m$, there is an explicit construction of all Chern classes in the Milnor K-theory [14]:

$$\mathrm{c}_m^{\mathrm{M}} \in \mathrm{H}^m(\mathrm{BGL}_\bullet, \underline{K}_m^{\mathrm{M}}). \tag{147}$$

Its analogs for other groups G is not known for $m > 2$. Note that these are the classes

$$\mathrm{c}_{d_m}^{\mathrm{M}} \in \mathrm{H}^{d_m}(\mathrm{BG}_\bullet, \underline{K}_{d_m}^{\mathrm{M}}). \quad m \in \{1, \ldots, \mathrm{rk}(\mathrm{G})\}. \tag{148}$$

where $\{d_m\}$ are the exponents of G. It would be very interesting to find them. An interesting question is whether we would need a more general notion than the cluster structure to do this.

Furthermore, there is an explicit construction of the third motivic Chern class, see [15]:

$$\mathrm{c}_3 \in \mathrm{H}^6(\mathrm{BGL}_{m\bullet}, \underline{\mathbb{Z}}_{\mathcal{M}}(3)). \tag{149}$$

This class is crucial to understand the Beilinson regulator for the weight 3. However, strangely enough, the class c_3 did not appear yet in any geometric/Physics applications like the ones in Sect. 1.11.

It would be interesting to construct explicitly the third motivic Chern class for any classical group G. Note that although $d_1 = 2$, for the classical G we have $d_2 = 3$, while otherwise $d_2 > 3$.

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