THE RADIATION FIELD ON PRODUCT CONES

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Abstract. We consider the wave equation on a product cone and find a joint asymptotic expansion for solutions near null and future infinities. The rates of decay seen in the expansion at future infinity are the resonances of a hyperbolic cone and were computed by the authors in [BM19]. The expansion treats an asymptotic regime not considered in the influential work of Cheeger and Taylor [CT82a, CT82b]. The main result follows the blueprint laid out in the [BVW15, BVW18] with key new elements including propagation estimates near the conic singularities. The proof of the propagation estimates extends prior work of Melrose–Vasy–Wunsch [MVW08] and Gannot–Wunsch [GW18].

1. Introduction

For a given compact connected Riemannian manifold (Z, k), we say that the cone $C(Z)$ over Z is the manifold

$$(0,\infty)_r \times Z,$$

equipped with the (singular) Riemannian metric

$$dr^2 + r^2 k.$$

We consider the wave equation

$$\begin{cases} \Box w = (D_t^2 - \Delta_{C(Z)})w = 0 \in C_c^\infty(\mathbb{R} \times C(Z)), \\ (w, \partial_t w)|_{t=0} \in C_c^\infty(C(Z)) \times C_c^\infty(C(Z)), \end{cases} \tag{1}$$

on $\mathbb{R} \times C(Z)$. Here $\Delta_{C(Z)}$ represents the Friedrichs extension of the Laplacian on $C(Z)$.

In order to simplify the statement of our main result, we introduce the (forward) *Friedlander radiation field*, which is given in terms of $s = t - r$, r, and z by

$$\mathcal{R}_+[w](s,z) = \lim_{r\to\infty} r^{(n-1)/2} w(s+r, r, z).$$

The function $\mathcal{R}_+[w]$ measures the radiation pattern seen by a distant observer and is an explicit realization of the Lax–Phillips translation representation as well as a generalization of the Radon transform. Our main theorem can then be stated in terms of the radiation field as s, the "lapse" parameter, tends toward infinity (a more detailed theorem is stated later as Theorem 9.1):

Theorem 1.1. *Suppose w is a solution of the wave equation on a cone with smooth initial data compactly supported away from the conic singularity, i.e., that w is a solution of equation (1). The radiation field $\mathcal{R}_+[w](s,z)$ of w admits an asymptotic expansion of the form*

$$\mathcal{R}_+[w](s,z) \sim \sum_j a_{j,\kappa}(z) s^{-i\sigma_j}$$

as $s \to +\infty$.

In short, we find a complete asymptotic expansion for the radiation field of a solution. The exponents in the expansion are the resonances of the spectral family of the Laplacian on a related "hyperbolic cone" and were computed in a previous paper [BM19]. In fact, the σ_j can be computed explicitly in terms of the eigenvalues μ_j^2 of Δ_k. Because each eigenvalue μ_j^2 leads to an entire family of resonances, it is easier to rename them $\sigma_{j,k}$ in terms of two parameters, which we call j and k. Here j refers to the eigenvalue in question and $k \in \mathbb{N} = \{0, 1, \dots\}$.

$$\sigma_{j,k} = -i\left(\frac{1}{2} + k + \sqrt{\left(\frac{n-2}{2}\right)^2 + \mu_j^2}\right) \tag{2}$$

provided that

$$\sqrt{\left(\frac{n-2}{2}\right)^2 + \mu_j^2} \notin \frac{1}{2} + \mathbb{Z}.$$

The resonance σ_j has the same multiplicity as the eigenvalue μ_j^2 of Δ_k.

In fact, we prove a stronger theorem showing that w in fact enjoys a joint asymptotic expansion in r and s of the form

$$w \sim r^{-\frac{n-1}{2}} \sum_j \sum_{\ell=0}^{\infty} a_{j\kappa\lambda}(z) s^{-i\sigma_j} (s/r)^\ell.$$

A precise statement of the theorem is given at the beginning of Section 9 below. The hyperbolic cone above is naturally realized as a boundary face of the spacetime compactification introduced below in Section 2; the joint asymptotics describe the behavior near the intersection of two faces.

We note further that the hypotheses of Theorem 1.1 may be relaxed somewhat; it is not strictly necessary that we consider the static wave equation on a product cone; we stick to this setting for pedagogical reasons but describe straightforward generalizations below (see Section 2). Although the argument simplifies in the product setting, the complications arising in the general setting can be treated using more refined microlocal techniques, though logarithmic terms might appear in the expansion. See for instance the previous papers [BVW15, BVW18] for relaxing the static hypothesis and Melrose–Vasy–Wunsch [MVW08] to relax the product hypothesis. Recent work of Yang [Yan20] further connects this paper with the work of Cheeger–Taylor by linking the scattering matrix (whose structure can be obtained from our result) with the principal symbol of the diffracted wave.

The results in Theorem 1.1 extend the foundational work initiated by Cheeger–Taylor in [CT82a, CT82b], though our aim is different. Cheeger and Taylor were more interested in the propagation of wavefront set for the wave equation on product cones; in particular their main aim was to show the existence (and calculate the symbol) of the diffracted wave arising from the metric singularity. In the process, they also found the asymptotic behavior of solutions of the wave equation away from $\mathcal{I}^+$; we recover their result in this region. Although in principle Theorem 1.1 can be recovered using the methods of Cheeger–Taylor [CT82a, CT82b] provided one could extend their asymptotic expansion uniformly to the boundary of the light cone, we provide an alternative microlocal proof.

The novelty of this paper involves several advances on existing technology for the study of waves in a diffractive setting. Not only do we essentially finish the project of Cheeger–Taylor in a fashion that gives a complete asymptotic description, we find that cones provide an additional class of examples where the expansion of the radiation field can be computed

explicitly using special functions methods as in our previous work [BM19]. In particular, the exponents in the expansion are resonance poles of the Laplacian on the hyperbolic cone and the coefficients in the expansion arise as boundary values of the resonant states; the structure of the resonant states gives insight into the structure of the scattering matrix. A similar observation underlies the work of Yang mentioned above [Yan20].

From a more technical perspective, we have extended a number of microlocal tools to our setting. In the bulk spacetime, we adapt and extend the propagation results of Melrose–Wunsch [MW04] and Vasy [Vas08] to our compactification. This extension requires putting the differential–pseudodifferential interactions at the core of those papers on a more global footing.

The technical heart of the paper, however, lies in our treatment of the normal operator on the boundary in Section 8. We extend the differential–pseudodifferential interactions to a class of variable order Sobolev spaces on which the boundary operator is Fredholm. We also establish *semiclassical* propagation estimates on these spaces; to our knowledge analogous results have not yet appeared in the literature. The work [GW18] of Gannot–Wunsch establishes similar semiclassical estimates for conormal potentials, which in this case can be viewed as a one-dimensional cone.

Finally, an additional technical novelty encountered is that solutions of the wave equation are not polyhomogeneous on the final compactified spacetime. Indeed, they are conormal to all boundary hypersurfaces but only polyhomogeneous at a subset of them. To this end, we formalize the notion of *partial polyhomogeneity* in Definition 3.1.

In addition to advances in analysis, the results we obtain here have several direct applications to important physical models. Diffractive systems arise naturally in physical settings where singular potentials appear, such as in the cases of inverse square potential or the Dirac-Coulomb system. This framework has been adapted to study the long time asymptotics directly for the massless Dirac-Coulomb system in recent work [BBGR21]. In addition, further advances building upon this work have appeared in studying the resolvent and/or scattering matrix for Laplacian on a manifold with conic singularities in the works [Hin20, Hin21a, Yan20], as well as in development of propagation of singularities for conic operators. The partial polyhomogeneity of solutions also implies a novel version of the so-called Price's law explored recently using similar tools in the non-diffractive setting by Hintz [Hin21b]. In more singular settings, the observed decay rates change in an interesting fashion directly related to the diffractive component of the problem. This particular application will be explored further in a forthcoming work.

1.1. **A sketch of the proof of Theorem 1.1.** To prove the main theorem, we show that solutions to equation (1) are partially polyhomogeneous on a compactification of the spacetime $\mathbb{R} \times C(Z)$ and then identify the exponents seen in the expansions. As this proof is somewhat involved, we provide a sketch here.

We compactify the spacetime $\mathbb{R} \times C(Z)$ to a manifold with corners we call M, which has two boundary hypersurfaces: one, denoted mf, corresponds to the "boundary at infinity", while the other, denoted cf, corresponds to the world line of the conic singularity. We refer the reader to Figure 1 in the next section for a fuller picture of the geometry.

An instructive example is the case of a "phantom cone". One can view $\mathbb{R}^n$ as a conic manifold by equipping it with polar coordinates; in this case the link is $Z = \mathbb{S}^{n-1}$. The compactification to M in this case can be blown down along cf to yield the compactification of the Minkowski spacetime considered in previous work [BVW15, BVW18].

The proof of the main theorem roughly follows the blueprint laid out in previous work of the first author [BVW15, BVW18], which in turn builds on the influential work of Vasy [Vas13]. In particular, our aim is to reduce the problem of finding an asymptotic expansion to the inversion of a family of Fredholm operators on mf; the residues of the poles of this family generate the terms in the expansion. Showing that the family is Fredholm (and that the argument can begin) reduces to a sequence of propagation of singularities arguments.

The forward radiation field encodes the behavior of the solution u near the intersection of the future light cone with infinity (i.e., the face mf); we denote this intersection S_+. To find the asymptotics of the radiation field, we therefore ultimately *blow up* S_+ in mf to obtain a third boundary hypersurface corresponding to "future null infinity". Locally near the interior of this new front face (denoted $\mathcal{I}^+$), the blow-up amounts to introducing new coordinates $\rho = (1 + t^2 + r^2)^{-1/2}$, $s = t - r$, and z; the front face is given by $\rho = 0$.

We begin with the solution of equation (1); by smoothly cutting off the solution for $t < 0$, we consider instead the forward solution of $\Box w = f$, where $f \in C_c^\infty(M^\circ)$ vanishes identically for $t < 0$. We consider then the function $u = \rho^{-(n-1)/2} w$ and set

$$L = \rho^{-2}\rho^{-(n-1)/2}\Box\rho^{(n-1)/2},$$

so that u satisfies $Lu = f'$ for some other function $f' \in C_c^\infty(M^\circ)$ vanishing for $t < 0$. Note that the asymptotic properties of w and u are linked by a simple relationship. A propagation of singularities argument (proved in Section 8) shows that u is conormal to $S_\pm$. The conormality of the solution at the conic singularity cf is one of the consequences of the work of Melrose–Wunsch [MW04, Proposition 11.1]; we extend that result to the corners $\mathrm{mf} \cap \mathrm{cf}$.

We then set $P_\sigma = \hat{N}(L)$ where $\hat{N}$ is the reduced normal operator, i.e., the family of operators on mf obtained by the Mellin transform in the normal variable ρ. We set $\widetilde{u}_\sigma$ and $\widetilde{f}_\sigma$ to be the Mellin transforms of u and f', so that $\widetilde{u}_\sigma$ solves

$$P_\sigma \widetilde{u}_\sigma = \widetilde{f}_\sigma.$$

In general, one would expect additional correction terms, but the dilation invariance of the model problem simplifies the argument considerably and accounts for the absence of logarithmic terms in the expansion in Theorem 1.1. We show that we can propagate regularity from the past "radial points" of P_σ to the future ones. Away from the conic singularity, this argument is contained in the previous papers [BVW15, BVW18, Vas13]; the main missing piece is the propagation near the conic singularity (proved in Section 8). This argument shows that P_σ is Fredholm on variable-order Sobolev-type spaces and P_σ^{-1} has finitely many poles in any horizontal strip. In fact, the poles of P_σ^{-1} can be identified with the resonances of the corresponding hyperbolic cone.

Once these pieces are in place, we can adapt the argument from the prequel [BVW18] to prove the main theorem. As parts of it are somewhat more complicated in the present context, we provide a sketch of that argument below (Section 9).

Section 2 provides an introduction to the specific geometry we consider, and Section 3 provides a brief review of the geometry of manifolds with corners and asymptotic expansions on them. We discuss the model operators we consider in Section 4, then in Section 5 we present the pseudodifferential calculi employed. Section 6 develops the function spaces in which the various arguments take place. Sections 7 and 8 then establish the key propagation of singularities results we need to prove the main theorem in Section 9.

1.2. **Notation.** As the proof of this paper relies on quite a few interacting differential and pseudodifferential objects, we provide here a short list of notational conventions employed.

We adopt the convention that $D = \frac{1}{i}\partial$. The differential operator Δ_h is the nonnegative Laplacian for a Riemannian metric h; in a coordinate system it is given by

$$\Delta_h = \sum_{i,j} \frac{1}{\sqrt{h}} D_i \left(h^{ij} \sqrt{h} D_j \right).$$

In indexing spaces of pseudodifferential operators and Sobolev spaces, m is typically the differential order while ℓ represents an order of growth or decay.

The L^2 spaces employed in this manuscript are always taken with respect to a density induced by a metric near $x = 0$; in coordinates, these densities are given by

$$x^{n-1} \frac{d\rho}{\rho}\, dx \ \mathrm{dvol}_k \ \text{ in the bulk spacetime } M, \quad \text{and} \quad x^{n-1} dx \ \mathrm{dvol}_k \ \text{ on the boundary } \mathrm{mf}\,.$$

We aim to use the same Greek letter to denote a dual coordinate in the cotangent bundle to a coordinate on the base; we use τ to denote a dual to ρ, ξ a dual to x, and ζ a dual to z. We use different typographical conventions to denote covectors in different cotangent bundle constructions. An undecorated covector (such as ξ) refers to that coordinate in the standard cotangent bundle, an underline ($\underline{\xi}$) is reserved for the b-cotangent bundle, and the subscript e is used with the edge cotangent bundle.

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2. Conic geometry

As our primary concern is the wave equation on a cone, we describe this setting in detail. Remark 2.1 describes natural extensions to this setting on which versions of our main result still hold.

Let (Z, k) be a compact, connected, $(n-1)$-dimensional Riemannian manifold. The metric cone $C(Z)$ over Z is the manifold

$$(0, \infty)_r \times Z$$

equipped with the warped product metric

$$dr^2 + r^2 k.$$

This metric is singular and incomplete at $r = 0$; we refer to the natural boundary $\{0\} \times Z$ as the *cone point*.[1]

Our main result concerns solutions of the wave equation on the spacetime $M^\circ = \mathbb{R}_t \times C(Z)$, which is equipped with the Lorentzian metric

$$g = -dt^2 + dr^2 + r^2 k.$$

[1]We regard the conic singularity as being purely metric; one can think of the underlying manifold as having been previously resolved by blowing up a conic singularity.

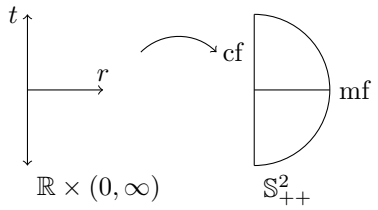

FIGURE 1. The compactification of $\mathbb{R} \times (0, \infty)$ to $\mathbb{S}^2_{++}$

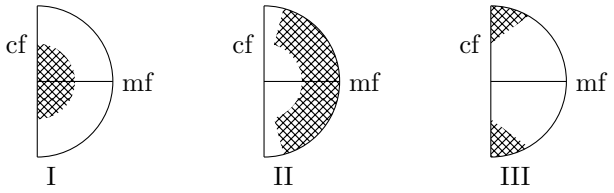

FIGURE 2. Regions I, II, and III in $\mathbb{S}^2_{++}$

We may regard M° as the interior of a compact manifold with corners. For clarity, we first describe this compactification in the $(1+1)$-dimensional setting (i.e., when Z is a single point) even though Theorem 1.1 is trivial in this case.

We compactify $\mathbb{R}_t \times (0, \infty)_r$ by stereographic projection to a quarter-sphere $\mathbb{S}^2_{++}$ as depicted in Figure 1. In other words, the map $\mathbb{R}_t \times (0, \infty)_r \to \mathbb{S}^2 \subset \mathbb{R}^3$ given by

$$(t, r) \mapsto \frac{(t, r, 1)}{\sqrt{1 + t^2 + r^2}}$$

sends M° to the interior of the quarter-sphere given by

$$\mathbb{S}^2_{++} = \left\{(z_1, z_2, z_3) \in \mathbb{S}^2 \subset \mathbb{R}^3 \mid z_2 \geq 0, z_3 \geq 0\right\}.$$

The quarter-sphere $\mathbb{S}^2_{++}$ is a manifold with corners and has two boundary hypersurfaces defined by the boundary defining functions z_2 and z_3. We let cf (or the *conic face*) be the hypersurface defined by the function

$$z_2 = \frac{r}{\sqrt{1 + t^2 + r^2}}$$

and we let mf (or the *main face*) be the face defined by

$$z_3 = \frac{1}{\sqrt{1 + t^2 + r^2}}.$$

The boundary hypersurface mf plays an outsized role in the manuscript and is often referred to as X when considered on its own.

Having defined the smooth structure of this compactification, it is often convenient to work with other equivalent boundary defining functions in different regions. We define regions I, II, and III (the shaded regions in Figure 2) as follows: We let region I denote a fixed neighborhood in $\mathbb{S}^2_{++}$ bounded away from mf; region II is a neighborhood of mf bounded away from cf; finally, region III is a neighborhood of the corners $\mathrm{mf} \cap \mathrm{cf}$. For concreteness, we can take region I to be given by $\{|t|, r \leq 10\}$, region II to be $\{r \geq 2, r \geq |t|/2\}$, and region III to be $\{|t| \geq 2, |t| \geq r/2\}$. Note that region III has two connected components; we typically work with only one component at a time.

In defining the Mellin transform below, it is useful to have a fixed boundary defining function for mf. For this purpose, we let ρ denote a defining function for mf that is equal to $\frac{1}{t}$ for $t/r > 1/2$ and equal to $\frac{1}{-t}$ for $t/r < -1/2$.

We now describe several convenient boundary defining functions valid in the different regions. For notational convenience, we always use ρ (or $\tilde{\rho}$) to denote a defining function for mf and x to denote a defining function for cf. In region I, (where we are bounded away from mf), it is convenient to take $x = r$, while in region II (where we are bounded away from cf), we can take $\tilde{\rho} = 1/r$. Finally, in region III (the source of most of the new technical work in this manuscript), it is typically convenient to take $\rho = \pm 1/t$ and $x = r/|t|$. *Because polyhomogeneity is independent of the choice of equivalent boundary defining functions, one can typically use whichever boundary defining functions are most convenient at the time.*

On the $(1+1)$-dimensional Lorentzian manifold $\mathbb{S}^2_{++}$, we employ coordinate systems specialized to the different regions. In region I, we employ $x = r$ and use coordinates (t, x); the Lorentzian metric here has the familiar form

$$-dt^2 + dx^2.$$

In region II, the metric has the form of a short-range asymptotically Minkowski metric as employed by the first author and collaborators [BVW15]; we use $(\tilde{\rho}, y)$ as coordinates, where $\tilde{\rho} = 1/r$ and $y = t/r$. The metric in this coordinate system has the form

$$-\frac{dy^2}{\tilde{\rho}^2} + 2y\frac{dy}{\tilde{\rho}}\frac{d\tilde{\rho}}{\tilde{\rho}^2} + (1-y^2)\frac{d\tilde{\rho}^2}{\tilde{\rho}^4}.$$

Near the corner (region III), in terms of (ρ, x) the metric has the form

$$-(1-x^2)\frac{d\rho^2}{\rho^4} - 2x\frac{dx}{\rho}\frac{d\rho}{\rho^2} + \frac{dx^2}{\rho^2}.$$

For the more general case of $M^\circ = \mathbb{R} \times C(Z)$, we take M to be the closure of the image of M° under the map $\mathbb{R} \times (0,\infty) \times Z \to \mathbb{S}^2 \times Z$ given by

$$(t, r, z) \mapsto \left(\frac{(t, r, 1)}{\sqrt{1 + t^2 + r^2}}, z\right).$$

In other words, we take $M = \mathbb{S}^2_{++} \times Z$ to be the compactification of M° to a manifold with corners.

In region I, the metric is the spacetime metric on a conic manifold studied by Melrose–Wunsch [MW04] (and later by Melrose–Vasy–Wunsch [MVW08]). In region II, g has the form

$$g = -\frac{dy^2}{\tilde{\rho}^2} + 2y\frac{dy}{\tilde{\rho}}\frac{d\tilde{\rho}}{\tilde{\rho}^2} + (1-y^2)\frac{d\tilde{\rho}^2}{\tilde{\rho}^4} + \frac{k}{\tilde{\rho}^2}, \tag{3}$$

which is again a short-range asymptotically Minkowski metric (though written in somewhat different coordinates than those in [BVW15]).

Near the corner (region III), the metric has the form

$$g = -(1-x^2)\frac{d\rho^2}{\rho^4} - 2x\frac{dx}{\rho}\frac{d\rho}{\rho^2} + \frac{dx^2}{\rho^2} + \frac{x^2 k}{\rho^2}. \tag{4}$$

This metric is a hybrid of a Lorentzian scattering metric (in that it is built from 1-forms of the type $d\rho/\rho^2$ and α/ρ) and a conic type metric (in that it degenerates as $x \to 0$).

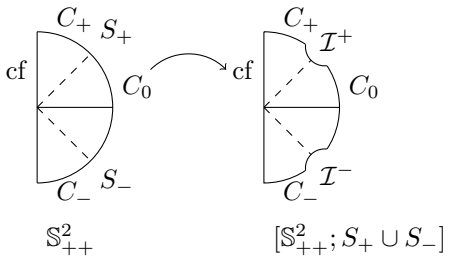

FIGURE 3. A schematic view of the radiation field blow-up. The lapse function s increases along $\mathcal{I}^+$ towards C_+.

Remark 2.1. There are a number of natural extensions to the product cone setting that require little additional work. All of the results and proofs in this manuscript (other than the explicit characterization of exponents) apply to the setting where g is a Lorentzian metric on $M = \mathbb{S}^2_{++} \times Z$ that is

(1) a spacetime conic metric (so that the results of Melrose–Wunsch [MW04] apply) in region I,
(2) a (long-range or short-range) asymptotically Minkowski metric in region II, and
(3) a hybrid in region III. In other words, in region III, we demand that g is built out of $d\rho^2/\rho^2$, dx/ρ, and dz/ρ and that its leading order behavior as $x \to 0$ (in terms of these objects) is

$$-\frac{d\rho^2}{\rho^4} + \frac{dx^2}{\rho^2} + \frac{x^2 k}{\rho^2}.$$

2.1. **The radiation field blow-up.** In this section we recall from previous work [BVW15, BVW18] the construction of the manifold with corners on which the radiation field naturally lives.

Consider the submanifold $S = \{\rho = 0, y = \pm 1\}$ in region II (S is given by $\{\rho = 0, x = 1\}$ in region III). This submanifold naturally splits into two components according to whether $\pm t > 0$ near the component. We use $S_\pm$ to denote these two pieces; they split the complement of S in mf into three connected components. We use C_0 to denote those points in mf where $y \in (-1, 1)$, while the subset of mf in region III where $x < 1$ has two components, denoted $C_\pm$ according to whether $\pm t > 0$ nearby.

We now *blow up* S in M by replacing it with its inward pointing spherical normal bundle.[2] In the product cone setting, this is equivalent to blowing up a pair of points in $\mathbb{S}^2_{++}$ and then taking the product with Z. This process replaces M with a new manifold $\overline{M} = [M; S]$ on which polar coordinates around the submanifold are smooth; the smooth structure of this manifold with corners depends only on the submanifold S (and not on the particular choice of defining functions). The blow-up is equipped with a natural blow-down map $\overline{M} \to M$; this map is a diffeomorphism on the interior. Figure 3 depicts this blow-up construction.

The new space $\overline{M}$ is again a manifold with corners and has six boundary hypersurfaces: the closure of the lifts of the interiors of C_0 and $C_\pm$ to $\overline{M}$, which are again denoted by C_0 and $C_\pm$; the lift of cf, again denoted cf, and two new boundary hypersurfaces consisting of the pre-images of $S_\pm$ under the blow-down map. These two new hypersurfaces are called future/past null infinity and denoted by $\mathcal{I}^\pm$. Moreover, $\mathcal{I}^\pm$ is naturally a (trivial) fiber bundle

[2]The reader may wish to consult Melrose's book [Mel93] for more details of the blow-up construction.

over $S_\pm$ with fibers diffeomorphic to intervals. Indeed, the interior of each fiber is naturally an affine space (i.e., $\mathbb{R}$ acts by translations, but there is no natural origin). In terms of y and ρ, the fibers of the interior of $\mathcal{I}^\pm$ in $\overline{M}$ can be identified with $\mathbb{R}\times Z$ via the coordinate $s=\pm(y\mp 1)/\rho$. In other words, $s=t-r$ provides a coordinate along $\mathcal{I}^+$ and $s=r+t$ is a coordinate along $\mathcal{I}^-$.

In our setting, Friedlander's argument [Fri80, Fri01] shows that for solutions w of the wave equation $\Box_g w=0$ with smooth, compactly supported initial data, the restriction

$$\mathcal{R}_\pm[w](s,z)=\rho^{-\frac{n-1}{2}}w|_{\mathcal{I}^\pm}$$

is well-defined and smooth. This is *Friedlander's radiation field*.[3]

3. Basics of b-geometry

The main results in this paper require an understanding of the interaction between Melrose's b-calculus and differential operators on cones. In light of the compactification described above, we begin by recalling results about analysis on manifolds with corners. Some of the discussion in the next few sections is adapted from prior work of the first author [BVW15, BVW18], while a more thorough discussion of b-geometry can be found in Melrose's book [Mel93, Chapter 4]. In the context of manifolds with corners, we refer the reader to Melrose's unpublished book [Mel96] and to Vasy's work [Vas08].

Throughout the paper we let M denote a compact $(n+1)$-dimensional manifold with corners and X a compact n-dimensional manifold with boundary. A function $\rho\in C^\infty(M)$ is a boundary defining function for a boundary hypersurface H of M if ρ vanishes simply at H and is non-vanishing elsewhere. A codimension k corner is the intersection of k boundary hypersurfaces of M. Near a codimension k corner $H_1\cap\cdots\cap H_k$, we may use

$$(\rho_1,\dots,\rho_k,y)\in[0,1)^k\times\mathbb{R}^{n+1-k}$$

as coordinates on M, where ρ_j is a boundary defining function for H_j and y are coordinates along the corner $H_1\cap\cdots\cap H_k$.

As our main applications involve corners of codimension no greater than two, we now specialize to that case. We assume now that M has corners of codimension two and that ρ and x are boundary defining functions (to keep consistent with notation above) in a neighborhood of a codimension two corner. Further keeping consistent with our notation, we use z to denote the remaining coordinates.

The space of b-vector fields on M, denoted $\mathcal{V}_{\mathrm{b}}(M)$, is the space of smooth vector fields on M tangent to ∂M. Near a codimension 2 corner $\{\rho=x=0\}$, $\mathcal{V}_{\mathrm{b}}(M)$ is spanned over $C^\infty(M)$ by the vector fields $\rho\partial_\rho, x\partial_x, \partial_z$. The vector field $\rho\partial_\rho$ is called the b-normal vector field to the boundary hypersurface $\{\rho=0\}$ and is independent of choice of coordinate system as an element of $\mathcal{V}_{\mathrm{b}}(M)/\rho\mathcal{V}_{\mathrm{b}}(M)$.

In fact, $\mathcal{V}_{\mathrm{b}}(M)$ is a Lie algebra and is the space of smooth sections of a vector bundle (called the b-tangent bundle) ${}^{\mathrm{b}}TM$ over M. The sections of its dual bundle ${}^{\mathrm{b}}T^*M$ are locally spanned near a codimension 2 corner over $C^\infty(M)$ by the 1-forms $d\rho/\rho, dx/x$, and dz.

The b-cotangent bundle ${}^{\mathrm{b}}T^*M$ is equipped with a canonical 1-form, which can be written

$$\underline{\tau}\frac{d\rho}{\rho}+\underline{\xi}\frac{dx}{x}+\underline{\zeta}\cdot dz \tag{5}$$

[3]Note that our definition differs from Friedlander's by the absence of a derivative.

in local coordinates near a codimension 2 corner. The fiber compactification $\overline{{}^{\mathrm{b}}T^*}M$ of ${}^{\mathrm{b}}T^*M$ is given by radially compactifying each fiber. A defining function for the "boundary at infinity" of a fiber is given by

$$\nu = \left(\underline{\tau}^2 + \underline{\xi}^2 + |\underline{\zeta}|^2\right)^{1/2},$$

and near infinity we may use

$$\nu, \widehat{\underline{\tau}} = \nu\underline{\tau}, \widehat{\underline{\xi}} = \nu\underline{\xi}, \widehat{\underline{\zeta}} = \nu\underline{\zeta}$$

as a redundant set of coordinates on each fiber near $\{\nu = 0, \rho = 0, x = 0\}$.[4] We let ${}^{\mathrm{b}}S^*M$ denote the boundary at infinity of $\overline{{}^{\mathrm{b}}T^*}M$, i.e., $\{\nu = 0\}$.

The b-cotangent bundle further inherits a canonical symplectic structure where the symplectic form is given by the exterior derivative of the canonical 1-form. (In other words, the natural symplectic structure on T^*M extends to ${}^{\mathrm{b}}T^*M$.) If we write covectors in ${}^{\mathrm{b}}T^*M$ in local coordinates as

$$\underline{\tau}\frac{d\rho}{\rho} + \underline{\xi}\frac{dx}{x} + \underline{\zeta}\cdot dz,$$

then the symplectic form is given by

$$d\underline{\tau} \wedge \frac{d\rho}{\rho} + d\underline{\xi} \wedge \frac{dx}{x} + d\underline{\zeta} \wedge dz. \tag{6}$$

As $\mathcal{V}_{\mathrm{b}}(M)$ is a Lie algebra, we also consider its universal enveloping algebra, denoted $\mathrm{Diff}^*_{\mathrm{b}}(M)$. Near the codimension 2 corner defined by $\{\rho = x = 0\}$, an operator $A \in \mathrm{Diff}^m_{\mathrm{b}}(M)$ has the form

$$A = \sum_{j+k+|\alpha|\leq m} a_{jk\alpha}(\rho, x, z)\,(\rho D_\rho)^j\,(xD_x)^k\,D_z^\alpha, \tag{7}$$

where $a_{jk\alpha} \in C^\infty(M)$. The principal symbol of such an operator is given by

$$\sigma_{\mathrm{b}}(A) = \sum_{j+k+|\alpha|=m} a_{jk\alpha}(\rho, x, z)\underline{\tau}^j\underline{\xi}^k\underline{\zeta}^\alpha.$$

The semiclassical version of $\mathrm{Diff}^m_{\mathrm{b}}(M)$, denoted $\mathrm{Diff}^m_{\mathrm{b},h}(M)$, is similarly defined with a parametric dependence on a small parameter $h > 0$. In local coordinates, an operator $A \in \mathrm{Diff}^m_{\mathrm{b},h}(M)$ has the form

$$A = \sum_{j+k+|\alpha|\leq m} a_{jk\alpha}(\rho, x, z; h)\,(h\rho D_\rho)^j\,(hxD_x)^k\,(hD_z)^\alpha \tag{8}$$

where $a_{jk\alpha} \in C^\infty(M)$ are bounded in h. In fact we require $\mathrm{Diff}^*_{\mathrm{b},h}$ only in the context of the manifold with boundary X. The semiclassical principal symbol of such an operator captures the leading order behavior, i.e., up to $h\,\mathrm{Diff}^{m-1}_{\mathrm{b},h}(M)$.

While the principal symbol of a differential operator captures its high-frequency behavior, it fails to characterize the asymptotic behavior at the boundary. At each boundary face, there is a dilation-invariant model operator, called the *normal operator* that captures this behavior. We require this operator only at the face given by $\{\rho = 0\}$, where it is obtained by freezing the

[4]Strictly speaking, we should regard $(\widehat{\underline{\tau}}, \widehat{\underline{\xi}}, \widehat{\underline{\zeta}}) \in \mathbb{S}^n$ and then regard $(\nu, \widehat{\underline{\tau}}, \widehat{\underline{\xi}}, \widehat{\underline{\zeta}})$ as "polar coordinates" near infinity.

coefficients of ρD_ρ, xD_x, and D_z at $\rho = 0$. In other words, $N(A) \in \mathrm{Diff}^m_{\mathrm{b}}([0,\infty) \times \{\rho = 0\})$ and is given by

$$N(A) = \sum_{j+k+|\alpha|\leq m} a_{jk\alpha}(0,x,z)(\rho D_\rho)^j (xD_x)^k D_z^\alpha. \tag{9}$$

Just as the Fourier transform is central to the study of approximately translation-invariant operators, the *Mellin transform* is useful in the study of approximately dilation-invariant operators. For the main application of this paper, we need only the Mellin transform associated to a single boundary hypersurface $H = \{\rho = 0\}$. Suppose u is a distribution on M suitably localized near the boundary hypersurface H defined by ρ. The Mellin transform of u associated to H is defined by

$$\widetilde{u}_\sigma = \mathcal{M}u(\sigma,x,z) = \int_0^\infty \chi(\rho) u(\rho,x,z) \rho^{-i\sigma-1}\, d\rho,$$

where χ is a smooth compactly supported function that is equal to 1 near $\rho = 0$.

The Mellin conjugate of the operator $N(A)$ is known as the *reduced normal operator*.[5] For $N(A)$ given by the formula (9) above, the reduced normal operator is the family of operators on the boundary hypersurface H given by

$$\widehat{N}(A) = \sum_{j+k+|\alpha|\leq m} a_{jk\alpha}(0,x,z)\sigma^j (xD_x)^k D_z^\alpha. \tag{10}$$

The Mellin transform is particularly useful in the study of asymptotic expansions in powers of ρ and $\log\rho$. We first discuss the case where M has only a single boundary hypersurface, i.e., when M is a manifold with boundary. In particular, we recall from Melrose [Mel93, Section 5.10] that if u is a distribution on a manifold with boundary, we say that u *is polyhomogeneous with index set* E if and only if u is conormal to ∂M (in particular, u is smooth away from the boundary), and

$$u \sim \sum_{(z,k)\in E} \rho^{iz} (\log\rho)^k a_{zk},$$

where a_{zk} are smooth functions on ∂M. Here the expansion should be interpreted as an asymptotic series as $\rho \to 0$ and E is an *index set* and therefore must satisfy[6]

- $E \subset \mathbb{C} \times \{0,1,2,\dots\}$,
- E is discrete,
- if $(z_j,k_j) \in E$ with $|(z_j,k_j)| \to \infty$, then $\operatorname{Im} z_j \to -\infty$,
- if $(z,k) \in E$, then $(z,l) \in E$ for all $l = 0,1,\dots,k-1$, and
- if $(z,k) \in E$, then $(z-ij,k) \in E$ for all $j = 1,2,\dots$.

We refer the reader to Melrose's book [Mel93, Section 5.10] for a discussion of the naturality of these conditions. As an example, the functions that are smooth up to ∂M are polyhomogeneous with index set $E = \{(-ij,0) : j = 0,1,2,\dots\}$.

Polyhomogeneous distributions are characterized in two different ways: by the Mellin transform and by the application of scaling (or *radial*) vector fields. To see the former, we recall a characterization of this space given by Melrose [Mel93, Proposition 5.27]. For a given

[5]We require this construction only for differential operators, though it extends to b-pseudodifferential operators as well.

[6]We have adopted the index set convention of Melrose's unpublished book [Mel96] rather than the other reference [Mel93] to remain consistent with the first author's prior work [BVW15, BVW18].

index set E, a distribution u is polyhomogeneous with index set E if and only if its Mellin transform is meromorphic with poles of order k only at points z for which $(z, k-1) \in E$ (together with appropriate decay estimates in σ).

Alternatively, we may test for polyhomogeneity by using radial vector fields. Let R denote the radial vector field ρD_ρ. We characterize a polyhomogeneous distribution u with index set E by the requirement that for all A, there is a γ_A with $\gamma_A \to +\infty$ as $A \to +\infty$ so that

$$\left(\prod_{(z,k)\in E, \operatorname{Im} z > -A} (R - z) \right) u \in \rho^{\gamma_A} H_{\mathrm{b}}^{\infty}(M), \tag{11}$$

where $H_{\mathrm{b}}^{\infty}(M)$ denotes the space of distributions conormal to the boundary.

Our main theorem concerns polyhomogeneity at two boundary hypersurfaces on a manifold with codimension 2 corners. We apply this characterization below to the manifold $\overline{M}$, which has six boundary hypersurfaces $\mathrm{cf}, C_\pm, C_0$, and $\mathcal{I}^\pm$. The distributions we consider vanish identically near $\mathcal{I}^-, C_-$, and C_0, so there are three hypersurfaces of interest.

In the characterization that follows, we let $H_1 = \mathcal{I}^+$, $H_2 = C_+$, and $H_3 = \mathrm{cf}$ denote the relevant hypersurfaces; for now we let ρ_1 define H_1, ρ_2 define H_2, and x define H_3. We now define the space of *partially polyhomogeneous* distributions with index sets $\mathcal{E} = (E_1, E_2)$.

Definition 3.1. A distribution u lies in $\mathcal{A}^{\mathcal{E}}_{\mathrm{pphg}}(\overline{M})$, the space of partially polyhomogeneous distributions with index sets $\mathcal{E}$, if u is conormal to all boundary hypersurfaces, and, for each $j = 1, 2$, we have

$$u \sim \sum_{(z,k)\in E_j} a_{jzk} \rho^{iz} (\log \rho)^k \mod \rho_j^{\infty} \rho_{3-j}^{-A} x^{-B} H_{\mathrm{b}}^{\infty}(\overline{M}),$$

where A is some fixed number greater than $\sup\{\operatorname{Im} z \mid (z,k) \in E_j, j = 1, 2\}$, B is some fixed number, and a_{jzk} are smooth at the hypersurface defined by ρ_j, conormal at H_3, and polyhomogeneous (with index set E_{3-j}) at the other one.

When testing for (partial) polyhomogeneity at multiple boundary hypersurfaces, it suffices to test individually at each one with uniform estimates at the others. This result is due independently to Mazzeo [Eco93, Appendix] and Melrose [Mel96, Chapter 4] and is a consequence of a characterization by multiple Mellin transforms. In particular, we appeal to the following proposition.

Proposition 3.2 (cf. Mazzeo, Melrose)**.** *Let R_j denote $\rho_j D_{\rho_j}$, the radial vector field at the boundary hypersurfaces defined by ρ_j. For $\mathcal{E} = (E_1, E_2)$, a distribution u lies in $\mathcal{A}^{\mathcal{E}}_{\mathrm{pphg}}(\overline{M})$ if and only if it is conormal to all boundary hypersurfaces and for each $j = 1, 2$ there are fixed weights α_j, β_j and for all A, there is a $\gamma_{j,A}$ with $\gamma_{j,A} \to +\infty$ as $A \to +\infty$, so that*

$$\left(\prod_{(z,k)\in E_j, \operatorname{Im} z > -A} (R_j - z) \right) u \in \rho_j^{\gamma_{j,A}} \rho_{3-j}^{\alpha_j} x^{\beta_j} H_{\mathrm{b}}^{\infty}(\overline{M}).$$

In other words, applying the test (11) above at the boundary hypersurface H_j defined by ρ_j improves the decay at H_j at no cost to the growth/decay at the other hypersurfaces. Note that there is no requirement that the coefficients be polyhomogeneous; their joint polyhomogeneity at $H_1 \cap H_2$ follows automatically when the condition is imposed individually at H_1 and H_2.

4. The operators L and $\widetilde{P}_\sigma$

Friedlander's argument for the existence of the radiation field motivates the definition below of the operator

$$L = \rho^{-2-(n-1)/2} \Box_g \rho^{(n-1)/2},$$

and its reduced normal operator $\widetilde{P}_\sigma = \widehat{N}(L)$. Because changing the boundary defining functions by a smooth non-vanishing multiple changes L and $\widetilde{P}_\sigma$ by a lower order term, we freely work with whichever forms of the boundary defining functions are most convenient.

For later reference, we record the forms of the operators L and $\widetilde{P}_\sigma$ in region III, where the metric has the form as in equation (4).[7] Indeed, we write (using $D = \frac{1}{i}\partial$):

$$L = (\rho D_\rho + x D_x)^2 - ni\,(\rho D_\rho + x D_x) - D_x^2 + \frac{(n-1)i}{x} D_x - \frac{1}{x^2}\Delta_k - \frac{n^2-1}{4},$$
$$\widetilde{P}_\sigma = (x D_x + \sigma) - ni\,(x D_x + \sigma) - D_x^2 + \frac{(n-1)i}{x} D_x - \frac{1}{x^2}\Delta_k - \frac{n^2-1}{4}.$$

In the main propagation results of Sections 7 and 8, we require an understanding of the Hamilton flow of the principal symbols of the operators L and $\widetilde{P}_\sigma$. We start by describing this flow near cf (in M) and near $\partial X = \mathrm{mf} \cap \mathrm{cf}$ (in $X = \mathrm{mf}$).

4.1. **Broken bicharacteristics for the operator L.** We now aim to describe set of broken bicharacteristics along which singularities may propagate. Perhaps the shortest path to their characterization involves the *edge cotangent bundle*, which we describe shortly. Moreover, the propagation arguments in the bulk spacetime M in Section 7 require commuting b-pseudodifferential operators through the differential operators naturally associated to the conic metric. It is therefore convenient to introduce a small amount of the edge calculus machinery (namely, the bundles and the differential operators) introduced by Mazzeo [Maz91]. We specialize our description to the specific setting in which we work, though the calculus applies in much more general settings. In an abuse of notation, we use the term "edge" to refer to objects that behave as edge objects at cf and as b-objects at mf. The reader wishing to skip this section need only note that the space of edge differential operators $\mathrm{Diff}_{\mathrm{e}}^*$ and the compressed characteristic set $\dot{\Sigma}$ are referred to later.

Our use of the edge machinery is limited to a neighborhood of the boundary hypersurface cf corresponding to the conic singularity. This boundary hypersurface is the total space of a trivial fiber bundle:

$$\begin{array}{ccc} Z & \longrightarrow & \mathrm{cf} \\ & & \downarrow \\ & & I \end{array}$$

Here I is a compactification of $\mathbb{R}$ to an interval; t is locally a coordinate on the interior of I while ρ provides a coordinate near each endpoint of I.

The set of edge-vector fields, typically denoted $\mathcal{V}_{\mathrm{e}}$, consists of those b-vector fields that are tangent to the leaves of the fibration. In local coordinates (ρ, x, z) where x is the boundary defining function for cf and z is a coordinate along Z, $\mathcal{V}_{\mathrm{e}}$ is spanned over C^∞ by

$$x\partial_x, \quad x\rho\partial_\rho, \quad \text{and} \quad \partial_{z_j}.$$

[7] We do not use the explicit form of the operators in region II and instead appeal standard hyperbolic propagation estimates as in previous work [BVW15] there.

The Lie algebra $\mathcal{V}_{\mathrm{e}}$ is the space of smooth sections of a vector bundle (called the e-tangent bundle) ${}^{\mathrm{e}}TM$ over M.[8] Its dual is the e-cotangent bundle ${}^{\mathrm{e}}T^*M$.

We let $\mathrm{Diff}^*_{\mathrm{e}}(M)$ denote the universal enveloping algebra of $\mathcal{V}_e(M)$. An element $A \in \mathrm{Diff}^m_{\mathrm{e}}(M)$ near $\mathrm{mf} \cap \mathrm{cf}$ has the form

$$A = \sum_{j+k+|\alpha|\leq m} a_{jk\alpha}(\rho, x, z)(x\rho D_\rho)^j (xD_x)^k D_z^\alpha,$$

where the $a_{jk\alpha}$ are smooth on M. The operator L is an element of $x^{-2}\,\mathrm{Diff}^2_{\mathrm{e}}(M)$; this relationship is exploited below in Section 7.

Canonical coordinates on ${}^{\mathrm{e}}T^*M$ induced by the coordinates (ρ, x, z) are $(\rho, x, z, \tau_{\mathrm{e}}, \xi_{\mathrm{e}}, \zeta_{\mathrm{e}})$, which corresponds to writing covectors as

$$\tau_{\mathrm{e}} \frac{d\rho}{x\rho} + \xi_{\mathrm{e}} \frac{dx}{x} + \zeta_{\mathrm{e}} \cdot dz.$$

One then obtains a bundle map $\pi : {}^{\mathrm{e}}T^*M \to {}^{\mathrm{b}}T^*M$ given in these coordinates by

$$\pi\left(\rho, x, z, \tau_{\mathrm{e}}, \xi_{\mathrm{e}}, \zeta_{\mathrm{e}}\right) = \left(\rho, x, z, \underline{\tau} = \tau_{\mathrm{e}}, \underline{\xi} = x\xi_{\mathrm{e}}, \underline{\zeta} = x\zeta_{\mathrm{e}}\right).$$

In other words, the map π is given by $\omega \mapsto x\omega$, which is an isomorphism ${}^{\mathrm{e}}T^*M \to {}^{\mathrm{b}}T^*M$ away from $x = 0$.

Away from $x = 0$, the bicharacteristics (in this case lifts of geodesics to the b-cotangent bundle) of L are the integral curves of the b-Hamilton vector field of the b-principal symbol of L. As (M, g) is incomplete owing to the conic singularity of $C(Z)$, we must clarify what we mean by bicharacteristics that hit the cone point. As we are interested in wave equations, we restrict our attention to *null bicharacteristics*, i.e., those lying in the characteristic set of L.

We define now the *compressed cotangent bundle* by

$${}^{\mathrm{b}}\dot{T}^*M = \pi({}^{\mathrm{e}}T^*M)/Z, \quad \dot{\pi} : {}^{\mathrm{e}}T^*M \to {}^{\mathrm{b}}\dot{T}^*M,$$

where the quotient by Z acts only over the boundary; the topology is given by the quotient topology. Observe that ${}^{\mathrm{b}}\dot{T}^*_{\mathrm{cf}}M$ can be identified with ${}^{\mathrm{b}}T^*I$; in terms of coordinates $(\rho, x, z, \tau_{\mathrm{e}}, \xi_{\mathrm{e}}, \zeta_{\mathrm{e}})$ on ${}^{\mathrm{e}}T^*M$, $\pi({}^{\mathrm{e}}T^*_{\mathrm{cf}}M)$ is given by points of the form $(\rho, 0, z, \underline{\tau}, 0, 0)$. After the quotient, ρ and $\underline{\tau}$ provide coordinates on ${}^{\mathrm{b}}T^*I$.

In an abuse of notation (but following Melrose–Vasy–Wunsch [MVW08, Section 7]), we introduce

$$\pi({}^{\mathrm{e}}S^*M) = \left(\pi({}^{\mathrm{e}}T^*M) \setminus 0\right)/\mathbb{R}^+ \subset {}^{\mathrm{b}}S^*M,$$
$$\dot{\pi}\left({}^{\mathrm{e}}S^*M\right) = \left(\dot{\pi}({}^{\mathrm{e}}T^*M) \setminus 0\right)/\mathbb{R}^+ \subset {}^{\mathrm{b}}\dot{S}^*M,$$

where ${}^{\mathrm{b}}S^*M$ and ${}^{\mathrm{e}}S^*M$ are quotients of their respective cotangent bundles by the natural scaling action and ${}^{\mathrm{b}}\dot{S}^*M = {}^{\mathrm{b}}S^*M/Z$ with the quotient acting over cf.

We now observe that $x^2L \in \mathrm{Diff}^2_{\mathrm{e}}(M)$; near $\mathrm{mf} \cap \mathrm{cf}$, its edge-principal symbol is

$$\sigma_{\mathrm{e}}(x^2L) = (\tau_{\mathrm{e}} + x\xi_{\mathrm{e}})^2 - \xi_{\mathrm{e}}^2 - |\zeta_{\mathrm{e}}|^2$$

[8]Strictly speaking, as a global object, we are considering a mixed edge-b-tangent bundle, but our arguments are essentially local so we do not stress this point.

As cf is noncharacteristic for L, nonzero covectors in the edge-characteristic set of x^2L (i.e., the vanishing set for $\sigma_e(x^2L)$) are mapped to nonzero covectors by π and $\dot\pi$. We can thus define the *compressed characteristic set*

$$\dot\Sigma = \dot\pi(\Sigma),$$

where $\Sigma \subset {}^{\mathrm{e}}S^*M$ is the edge-characteristic set of x^2L. Over $x=0$, $\dot\Sigma = {}^{\mathrm{b}}\dot S^*_{\mathrm{cf}}M$, i.e.,

$$\dot\Sigma|_{\mathrm{cf}} = \{(\rho, x=0, z, \underline{\tau}, 0, 0) \mid \tau \neq 0, z \in Z\}/Z.$$

In the parlance of Melrose–Vasy–Wunsch, all of the points of $\dot\Sigma$ lying over cf are hyperbolic.

There are many equivalent and nearly-equivalent definitions of generalized broken bicharacteristics (see, e.g., Melrose–Vasy–Wunsch [MVW08] or Vasy [Vas08]), but in the present context they can instead be described more simply. Away from cf they are lifts to ${}^{\mathrm{b}}S^*M$ of maximally extended light-like geodesics of $\rho^2 g$. At cf, they are concatenations of bicharacteristics that are continuous as functions to $\dot\Sigma$.

In particular, at cf, the broken bicharacteristics are concatenations of lifts of light-like geodesics entering and exiting cf; the continuity condition requires that they enter and leave "at the same time" (i.e., with the same ρ or t coordinate along I) and with the same "time momentum" (i.e., the same value of $\underline{\tau}$). More precisely, straightforward ODE analysis shows that in the edge cotangent bundle, null bicharacteristics enter ${}^{\mathrm{e}}S^*_{\mathrm{cf}}M$ with coordinates

$$(\rho, 0, z_0, \tau_{\mathrm{e}}, \xi_{\mathrm{e}}, 0),$$

with $\tau_{\mathrm{e}}^2 = \xi_{\mathrm{e}}^2$. They then leave ${}^{\mathrm{e}}S^*_{\mathrm{cf}}M$ from the point

$$(\rho, 0, z_1, \tau_{\mathrm{e}}, -\xi_{\mathrm{e}}, 0),$$

where z_1 is a possibly different point in Z.[9] The main result of Section 7 below is to show that singularities of L propagate only along these broken bicharacteristics.

4.2. **Broken bicharacteristics for the operator $\widetilde P_\sigma$.** The classical propagation for the operator $\widetilde P_\sigma$ near $\partial X = \{x=0\}$ is simpler to describe as $\widetilde P_\sigma$ is classically elliptic there: there is no propagation. On the other hand, the related semiclassical operator

$$\widetilde P_h = h^2 \widetilde P_\sigma, \quad h = |\sigma|^{-1},$$

is not semiclassically elliptic.

We consider the characteristic set Σ_h of the operator $\widetilde P_h$ near $x=0$. The principal symbol of $\widetilde P_h$ in this region is

$$\sigma_{\mathrm{b},h}(\widetilde P_h) = \big(\lambda + \underline{\xi}\big)^2 - \frac{\underline{\xi}^2}{x^2} - \frac{1}{x^2}|\underline{\zeta}|^2$$

where $\lambda = \sigma/|\sigma| = \pm 1 + \mathcal{O}(h)$. Its Hamilton vector field is

$$\frac{2}{x^2}\left((x^2\underline{\xi} + x^2\lambda - \underline{\xi})x\partial_x - (\underline{\xi}^2 + |\underline{\zeta}|^2)\partial_\xi\right) - \frac{1}{x^2}H_{|\underline{\zeta}|^2},$$

where $H_{|\underline{\zeta}|^2}$ is the Hamilton vector field of the metric function k^{-1}. Within the characteristic set of $\widetilde P_h$, the only trajectories reaching $x=0$ reach points of the form

$$(x=0, z\in Z, \underline{\xi}=0, \underline{\zeta}=0),$$

[9] In other words, the direction in which the bicharacteristic leaves the cone point need not have any relation to the direction in which it entered. In the parlance of Melrose–Wunsch [MW04], these are the "diffractive" bicharacteristics.

i.e., the analogue of the compressed characteristic set for the semiclassical operator is the zero section over the boundary.

An analogous construction to the one described for the operator L shows that over $\partial X = \mathrm{mf} \cap \mathrm{cf}$, we have

$$\Sigma_h|_{\partial X} = \left\{(x = 0, z, \underline{\xi} = 0, \underline{\zeta} = 0) \mid z \in Z\right\},$$

and that the broken bicharacteristics for the Hamilton flow of the principal symbol of $\widetilde{P}_h$ must enter and leave through a point of this form with possibly different z values.

4.3. **The radial sets.** We finally describe the *radial sets* for the Hamilton flow associated to the operators L and $\widetilde{P}_h$; these are the sets where the Hamilton vector field of the principal symbol is a multiple of the radial vector field $\underline{\xi}\partial_{\underline{\xi}} + \underline{\zeta}\partial_{\underline{\zeta}}$. In both cases, the radial sets are identical to those described in prior work [BVW15]; we include them here for the purpose of completeness but refer to that work for their characterization.

The radial sets for $\widetilde{P}_\sigma$ and $\widetilde{P}_h$ occur at $N^*S_\pm$; boundary of the fiber-compactification acts as a source or sink for the global flow. We define $\Lambda^\pm$ to be the fiber-infinity boundary of $N^*S_\pm$ in $\overline{{}^{\mathrm{b}}T^*}X$; in our analysis below we aim to propagate regularity from the radial set living over S_- to the one living over S_+.

The radial points $\mathcal{R}^\pm$ of L also lie over $S_\pm$; in terms of coordinates $(\rho, x, z, \underline{\tau}, \underline{\xi}, \underline{\zeta})$ in region III, their image $\partial\mathcal{R}^\pm \subset \overline{{}^{\mathrm{b}}T^*}M$ in the cosphere bundle is given by

$$\left\{(\rho = 0, x = 1, z, \underline{\tau} = 0, \underline{\xi}, \underline{\zeta} = 0) \mid z \in Z, \underline{\xi} = \pm 1\right\}.$$

5. Pseudodifferential operators

The main results of this paper all rely on the interaction between spaces defined using both b-pseudodifferential operators and conic differential operators. While these interactions were key in the analysis of Melrose–Wunsch in [MW04], their structure was codified and explained by Vasy in [Vas08]. We now describe the spaces of b-pseudodifferential operators employed below as well as their interactions with the generators of the conic differential operators.

5.1. **The homogeneous** b**-calculus.** We now briefly describe the spaces Ψ_{b}^m, $\Psi_{\mathrm{b},\infty}^m$, and $\Psi_{\mathrm{b}}^{m,\ell}$ of b-pseudodifferential operators on the bulk spacetime M. Rather than provide detailed definitions and proofs, we instead provide a list of their properties and refer the reader to Melrose's unpublished book [Mel96] and Vasy's paper [Vas08] for details.

Our discussion in this section is specialized to a neighborhood of $\mathrm{mf} \cap \mathrm{cf}$ (region III) in M; the relevant results in region I can be quoted, while the results in region II can be recovered by assuming that x is bounded away from 0.

The space of b-pseudodifferential operators $\Psi_{\mathrm{b}}^*(M)$ is the "quantization" of the Lie algebra of vector fields tangent to the boundary of M and formally consists of operators of the form

$$b\left(\rho, x, z, \rho D_\rho, xD_x, D_z\right),$$

where b is a classical symbol (i.e., it is smooth on ${}^{\mathrm{b}}T^*M$ and has a complete asymptotic expansion at fiber infinity). In terms of coordinates (ρ, x, z) near the corner $\mathrm{mf} \cap \mathrm{cf}$, we may write an explicit quantization of the symbol b by

$$\begin{aligned}\mathrm{Op}(b)u(\rho, x, z) = \frac{1}{(2\pi)^{n+1}} \int\int e^{i(\rho-\rho')\underline{\tau}+i(x-x')\underline{\xi}+i(z-z')\cdot\underline{\zeta}}\phi\left(\frac{\rho-\rho'}{\rho}\right)\phi\left(\frac{x-x'}{x}\right)\psi(z) \\ \cdot b(\rho, x, z, \rho\underline{\tau}, x\underline{\xi}, \underline{\zeta})u(\rho', x', z')\, d\underline{\tau}\, d\underline{\xi}\, d\underline{\zeta}\, d\rho'\, dx'\, dz',\end{aligned}$$

where $\phi \in C_c^\infty((-1/2,1/2))$ is identically 1 near 0, ψ localizes to a region of Z where the local coordinate z is valid, and the integrals in ρ' and x' are over $[0,\infty)$.

We further define the multi-filtered algebra $\Psi_{\mathrm{b}}^{m,\ell}(M) = \rho^{-\ell}\Psi_{\mathrm{b}}^{m}(M)$. The index ℓ refers only to the filtration in ρ; we do not explicitly rely on a filtration in x later in the text.

Our regularization arguments in Section 7.2 rely (in a similar way to those of Melrose–Vasy–Wunsch [MVW08]) on a slightly larger algebra we call $\Psi_{\mathrm{b},\infty}^{*}(M)$. It is defined in the same way but with symbols satisfying Kohn–Nirenberg estimates (rather than having complete asymptotic expansions).

The algebra $\Psi_{\mathrm{b}}^{m,\ell}(M)$ satisfies the following properties:

i. The principal symbol of a b-differential operator, defined by

$$\sigma_{\mathrm{b},m,\ell}\left(\rho^{-\ell}\sum_{j+k+|\alpha|\leq m} a_{jk\alpha}(\rho D_\rho)^j(xD_x)^k D_z^\alpha\right) = \rho^{-\ell}\sum_{j+k+|\alpha|=m} a_{jk\alpha}\underline{\tau}^j\underline{\xi}^k\zeta^\alpha,$$

extends continuously to give a map

$$\sigma_{\mathrm{b},m,\ell} : \Psi_{\mathrm{b}}^{m,\ell}(M) \to \rho^{-\infty}C^\infty({}^{\mathrm{b}}S^*M).$$

The principal symbol map is multiplicative, i.e., $\sigma(AB) = \sigma(A)\sigma(B)$.

In the case of $\Psi_{\mathrm{b},\infty}^{m}(M)$, the principal symbol instead takes values in the quotient of the symbol spaces

$$S^m({}^{\mathrm{b}}T^*M)/S^{m-1}({}^{\mathrm{b}}T^*M),$$

which in the case of classical symbols can be identified with $C^\infty({}^{\mathrm{b}}S^*M)$.

The principal symbol captures the top order behavior (in m) of elements of $\Psi_{\mathrm{b}}^{m,\ell}(M)$. In other words, the following sequence is exact:

$$0 \to \Psi_{\mathrm{b}}^{m-1,\ell}(M) \to \Psi_{\mathrm{b}}^{m,\ell}(M) \to \rho^{-\ell}C^\infty({}^{\mathrm{b}}S^*M) \to 0.$$

(In the case of $\Psi_{\mathrm{b},\infty}^{m}$, the symbol space must be replaced by the quotient S^m/S^{m-1}.)

ii. There is a (non-canonical) quantization map $\mathrm{Op} : \rho^{-\ell}S^m({}^{\mathrm{b}}T^*M) \to \Psi_{\mathrm{b}}^{m,\ell}(M)$ so that

$$\sigma_{\mathrm{b},m,\ell}(\mathrm{Op}(a)) = a$$

as an element of $\rho^{-\ell}S^m({}^{\mathrm{b}}T^*M)/\rho^{-\ell}S^{m-1}({}^{\mathrm{b}}T^*M)$.

iii. The algebras $\Psi_{\mathrm{b}}^{m,\ell}(M)$ and $\Psi_{\mathrm{b},\infty}^{m}(M)$ are closed under adjoints, and

$$\sigma(A^*) = \overline{\sigma(A)}.$$

iv. If $A \in \Psi_{\mathrm{b}}^{m,\ell}(M)$ and $B \in \Psi_{\mathrm{b}}^{m',\ell'}(M)$, then $[A,B] = AB - BA \in \Psi_{\mathrm{b}}^{m+m'-1,\ell+\ell'}(M)$, and

$$\sigma_{\mathrm{b},m+m'-1,\ell+\ell'}\left(i[A,B]\right) = \left\{\sigma(A),\sigma(B)\right\},$$

where the right hand side denotes the Poisson bracket induced by the symplectic structure on ${}^{\mathrm{b}}T^*M$ as in Section 3.

v. Elements of $\Psi_{\mathrm{b}}^{0}(M)$ are bounded on L^2. In particular, given $A \in \Psi_{\mathrm{b}}^{0}(M)$, there is an $A' \in \Psi_{\mathrm{b}}^{-1}(M)$ so that

$$\|Au\|_{L^2} \leq 2\sup|\sigma(A)|\|u\|_{L^2} + \|A'u\|_{L^2}.$$

vi. If $A \in \Psi_{\mathrm{b}}^{m,\ell}(M)$ (or $\Psi_{\mathrm{b},\infty}^{m}(M)$), the *microsupport* (or operator wavefront set) $\mathrm{WF}'_{\mathrm{b}}(A) \subset {}^{\mathrm{b}}S^*M$ of A is the set of points and directions in which the total symbol of A fails to be rapidly decaying, and obeys the usual microlocality property:

$$\mathrm{WF}'_{\mathrm{b}}(AB) \subset \mathrm{WF}'_{\mathrm{b}}(A) \cap \mathrm{WF}'_{\mathrm{b}}(B).$$

The analysis below requires commuting b-pseudodifferential operators with the components $D_x, \frac{1}{x}D_z$, and $\frac{1}{x}$ of the operators on the cone. As commutators with $\frac{1}{x}D_z$ are not necessarily lower order, we are careful to select commutants that commute with derivatives in z to top order. In other words, we require the notion of a *basic operator* introduced by Melrose–Vasy–Wunsch [MVW08, Section 9].

Definition 5.1. We say a symbol $a \in C^\infty({}^{\mathrm{b}}T^*M)$ is basic if $\partial_z a = 0$ at $\{x = 0, \underline{\xi} = 0, \underline{\zeta} = 0\}$. The quantization of such a symbol is called a basic operator.

We now recall from Melrose–Vasy–Wunsch [MVW08, Lemma 8.6] how the b-calculus interacts with $\frac{1}{x}, D_x$, and $\frac{1}{x}D_{z_j}$.

Lemma 5.2. *If $A \in \Psi_{\mathrm{b}}^m(M)$, then there are $B \in \Psi_{\mathrm{b}}^m(M)$ and $C \in \Psi_{\mathrm{b}}^{m-1}(M)$ depending continuously on A so that*

$$i\,[D_x, A] = B + CD_x,$$

with $\sigma(B) = \partial_x \sigma(A)$ and $\sigma(C) = \partial_{\underline{\xi}}\sigma(A)$.

Similarly, there are $C_L, C_R \in \Psi_{\mathrm{b}}^{m-1}(M)$ with $\sigma(C) = \partial_{\underline{\xi}}\sigma(A)$ so that

$$i\left[\frac{1}{x}, A\right] = C_L\frac{1}{x} = \frac{1}{x}C_R.$$

If, in addition, A is a basic operator, then

$$i\left[\frac{1}{x}D_{z_j}, A\right] = B_j + C_jD_x + \sum_k E_{jk}\frac{1}{x}D_{z_k} + \frac{1}{x}F_j,$$

with $B_j \in \Psi_{\mathrm{b}}^m(M)$, $C_j, E_{jk}, F_j \in \Psi_{\mathrm{b}}^{m-1}(M)$, and

$$\partial_{z_j}\sigma(A) + \underline{\zeta}_j\partial_{\underline{\xi}}\sigma(A) = x\sigma(B_j) + \underline{\xi}\sigma(C_j) + \sum_k \underline{\zeta}_k\sigma(E_{jk}).$$

As in the work of Vasy [Vas08] and Melrose–Vasy–Wunsch [MVW08], we define

$$x^{-k}\,\mathrm{Diff}_{\mathrm{e}}^k\,\Psi_{\mathrm{b}}^m \subset x^{-k}\Psi_{\mathrm{b}}^{k+m}$$

to be the span of the products QA with $Q \in x^{-k}\,\mathrm{Diff}_{\mathrm{e}}^k$ and $A \in \Psi_{\mathrm{b}}^m$. By Lemma 5.2, it is also generated by the products AQ and so the union

$$\bigcup_{k,m} x^{-k}\,\mathrm{Diff}_{\mathrm{e}}^k\,\Psi_{\mathrm{b}}^m$$

is a bigraded ring closed under adjoints. This observation allows us to freely commute elements of $x^{-k}\,\mathrm{Diff}_{\mathrm{e}}^k$ through b-pseudodifferential operators at the cost of lower order terms.

5.2. **The semiclassical b-calculus.** On the boundary hypersurface $X = \mathrm{mf}$, we further employ the b-calculus as well as its semiclassical variant $\Psi^*_{\mathrm{b},h}$. In this section we briefly describe properties satisfied by the semiclassical b-calculus $\Psi^*_{\mathrm{b},h}(X)$. We refer the reader to Gannot–Wunsch [GW18, Section 3] for details. We remind the reader that X is an n-dimensional compact manifold with boundary.[10]

We can obtain an explicit quantization procedure on X near its boundary in terms of coordinates (x, z) by fixing $\phi \in C_c^\infty((-1/2, 1/2))$ so that $\phi(s) \equiv 1$ near $s = 0$ and $\psi \in C_c^\infty$ localizing to a fixed coordinate chart in z. Given a semiclassical symbol $a \in S^m_h({}^{\mathrm{b}}T^*X)$, define $\mathrm{Op}_{\mathrm{b},h}(a) \in \Psi^m_{\mathrm{b},h}(X)$ by

$$\mathrm{Op}_{\mathrm{b},h}(a)u(x,z) = \\ \frac{1}{(2\pi h)^n} \int\int e^{i\left((x-x')\underline{\xi}+(z-z')\cdot\underline{\zeta}\right)} \phi\left(\frac{x-x'}{x}\right) \psi(z) a(x, z, x\underline{\xi}, \underline{\zeta}) u(x', z')\, d\underline{\xi}\, d\underline{\zeta}\, dx'\, dz'.$$

As in the homogeneous setting, the space of semiclassical b-pseudodifferential operators on X satisfies the following properties:

i. There is a principal symbol map $\sigma_{\mathrm{b},h} : \Psi^m_{\mathrm{b},h}(X) \to S^m({}^{\mathrm{b}}T^*X)/hS^{m-1}({}^{\mathrm{b}}T^*X)$ so that the sequence
$$0 \to h\Psi^{m-1}_{\mathrm{b},h}(X) \to \Psi^m_{\mathrm{b},h}(X) \to S^m({}^{\mathrm{b}}T^*X)/hS^{m-1}({}^{\mathrm{b}}T^*X) \to 0$$
is exact. This map is multiplicative.

ii. There is a (non-canonical) quantization map $\mathrm{Op}_{\mathrm{b},h} : S^m({}^{\mathrm{b}}T^*X) \to \Psi^m_{\mathrm{b},h}(X)$ so that if $a \in S^m({}^{\mathrm{b}}T^*X)$, then
$$\sigma_{\mathrm{b},h}(\mathrm{Op}_{\mathrm{b},h}(a)) = a$$
as an element of $S^m({}^{\mathrm{b}}T^*X)/hS^{m-1}({}^{\mathrm{b}}T^*X)$.

iii. The algebra $\Psi^*_{\mathrm{b},h}(X)$ is closed under adjoints and
$$\sigma_{\mathrm{b},h}(A^*) = \overline{\sigma_{\mathrm{b},h}(A)}.$$

iv. If $A \in \Psi^m_{\mathrm{b},h}(X)$ and $B \in \Psi^{m'}_{\mathrm{b},h}(X)$, then $[A, B] \in h\Psi^{m+m'-1}_{\mathrm{b},h}(X)$ and has principal symbol
$$\sigma_{\mathrm{b},h}\left(\frac{i}{h}[A, B]\right) = \{\sigma_{\mathrm{b},h}(A), \sigma_{\mathrm{b},h}(B)\},$$
where the Poisson bracket is taken with respect to the symplectic structure on ${}^{\mathrm{b}}T^*X$.

v. Each $A \in \Psi^0_{\mathrm{b},h}(X)$ extends to a bounded operator on L^2 and there exists $A' \in \Psi^{-\infty}_{\mathrm{b},h}(X)$ so that
$$\|Au\|_{L^2} \leq 2\sup|\sigma_{\mathrm{b},h}(A)|\|u\|_{L^2} + \mathcal{O}(h^\infty)\|A'u\|_{L^2}.$$

vi. If $A \in \Psi^*_{\mathrm{b},h}(X)$, the microsupport (or operator wavefront set) $\mathrm{WF}'_{\mathrm{b},h}(A) \subset {}^{\mathrm{b}}T^*X$ is the set of points in the b-cotangent bundle at which A fails to lie in $h^\infty\Psi^{-\infty}_{\mathrm{b},h}$. It obeys the standard microlocality property:
$$\mathrm{WF}'_{\mathrm{b},h}(AB) \subset \mathrm{WF}'_{\mathrm{b},h}(A) \cap \mathrm{WF}'_{\mathrm{b},h}(B).$$

As in the homogeneous setting, we say that a *basic operator* is the quantization of a symbol a with $\partial_z a = 0$ at $\{x = 0, \underline{\xi} = 0, \underline{\zeta} = 0\}$. We also require the semiclassical analogue of Lemma 5.2, with proof essentially identical to the one in the homogeneous setting.

[10]The shift in dimension arises because we employ the semiclassical calculus only on $X = \mathrm{mf}$ rather than the bulk M.

Lemma 5.3 (cf. [MVW08, Lemma 8.6] and [GW18, Lemma 3.6]). *If $A \in \Psi^m_{\mathrm{b},h}(X)$, there are $B \in \Psi^m_{\mathrm{b},h}(X)$ and $C \in \Psi^{m-1}_{\mathrm{b},h}(X)$ so that*

$$\frac{i}{h}[hD_x, A] = B + C(hD_x),$$

with $\sigma_{\mathrm{b},h}(B) = \partial_x \sigma_{\mathrm{b},h}(A)$ and $\sigma_{\mathrm{b},h}(C) = \partial_{\underline{\xi}} \sigma_{\mathrm{b},h}(A)$.

Moreover, there are $C_L, C_R \in \Psi^{m-1}_{\mathrm{b},h}(X)$ with $\sigma_{\mathrm{b},h}(C_\bullet) = \partial_{\underline{\xi}} \sigma_{\mathrm{b},h}(A)$ and

$$i\left[\frac{1}{x}, A\right] = \frac{h}{x} C_R = C_L \frac{h}{x}.$$

If, in addition, A is a basic operator, then

$$\frac{i}{h}\left[\frac{h}{x} D_{z_j}, A\right] = B_j + C_j(hD_x) + \sum_k E_{jk} \frac{h}{x} D_{z_k} + \frac{h}{x} F_j,$$

where $B_j \in \Psi^m_{\mathrm{b},h}(X)$, $C_j, E_{jk}, F_j \in \Psi^{m-1}_{\mathrm{b},h}(X)$, and

$$\partial_{z_j} \sigma_{\mathrm{b},h}(A) + \underline{\zeta}_j \partial_{\underline{\xi}} \sigma_{\mathrm{b},h}(A) = x \sigma_{\mathrm{b},h}(B_j) + \underline{\xi} \sigma_{\mathrm{b},h}(C_j) + \sum_k \underline{\zeta}_k \sigma_{\mathrm{b},h}(E_{jk}).$$

Just as in the homogeneous setting, Lemma 5.3 allows us to freely commute factors of hD_x, $\frac{h}{x} D_{z_j}$, and $\frac{1}{x}$ through semiclassical b-pseudodifferential operators at the cost of lower order terms.

6. Function spaces

As described above, our analysis is based on mixed differential-pseudodifferential structures on both M and mf. The associated analytic objects we employ are therefore adapted to the Friedrichs form domain of the conic Laplacian.

We denote by $\mathcal{D}$ the Friedrichs form domain of the Laplacian on the cone $C(Z)$, i.e., the domain of $\Delta^{1/2}$, where Δ is the Friedrichs extension of the Laplacian. It is equipped with a natural norm given by

$$\|u\|^2_{\mathcal{D}} = \|u\|^2 + \langle \Delta u, u \rangle,$$

where the norm and inner product are taken with respect to the L^2 space induced by the conic metric on $C(Z)$. Writing the Laplacian in coordinates, the norm on $\mathcal{D}$ is controlled by

$$\|v\| + \|\partial_r v\| + \left\|r^{-1} \nabla_z v\right\|,$$

where the pointwise magnitude of the last term is measured by the metric k on the cross-section.

Just as in Euclidean space in three dimensions and higher, $\mathcal{D}$ enjoys an analogue of the Hardy inequality:

Lemma 6.1 ([MVW08, Lemma 5.2]). *If $\dim Z > 1$, then there is some C so that for all $v \in C^\infty_c((0,\infty) \times Z)$,*

$$\left\|r^{-1} v\right\|^2 + \|v\|^2 + \left\|r^{-1} \nabla_z v\right\|^2 + \|\partial_r v\|^2 \leq C \|v\|^2_{\mathcal{D}}.$$

We often appeal to Lemma 6.1 and its analogues in order to estimate terms of the form $r^{-1} v$ arising in commutator estimates; the lemma asserts that they can be controlled by one "conic derivative". Just as the Hardy inequality fails in dimension two, Lemma 6.1 is false when $\dim Z = 1$. On the other hand, when $\dim Z = 1$, the manifold Z must be a circle,

hence modifications similar to those used by Melrose–Vasy–Wunsch [MVW08, Section 10] allow us to recover the propagation results of Sections 7 and 8 in this case. For the purpose of exposition, we omit these arguments in this paper.

6.1. **b-Sobolev spaces on the bulk M.** Although it is standard in the b-calculus literature to define b-Sobolev spaces with respect to a fixed b-density, the proofs in Sections 7 and 8 below more naturally employ a cone-type density, i.e., a rescaling of the density reflecting the conic structure of the problems.

In other words, on M we consider the density associated to the Lorentzian metric $\rho^2 g$, which in local coordinates has the form

$$\frac{x^{n-1}\sqrt{k}}{\rho}\, d\rho\, dx\, dz.$$

All L^2 norms on M are taken with respect to this density.

We let $H^m_{\mathrm{b}}(M)$ denote the b-Sobolev space of order m relative to the function space $L^2(M)$ and the algebras $\mathrm{Diff}^m_{\mathrm{b}}(M)$ and $\Psi^m_{\mathrm{b}}(M)$. In particular, for $m \geq 0$, if $A \in \Psi^m_{\mathrm{b}}(M)$ is a fixed invertible elliptic operator, then $u \in H^m_{\mathrm{b}}(M)$ if and only if $u, Au \in L^2(M)$.[11] For $m < 0$, the space $H^m_{\mathrm{b}}(M)$ is defined as the dual space of $H^{-m}_{\mathrm{b}}(M)$ with respect to the $L^2(M)$ pairing. We further require an additional filtration of the Sobolev spaces. For $\ell \in \mathbb{R}$, we set $H^{m,\ell}_{\mathrm{b}}(M) = \rho^\ell H^m_{\mathrm{b}}(M)$ as defined in for instance Section 2 of [MW04].

In an abuse of notation, we use $\widetilde{\mathcal{D}}$ to denote a differential Sobolev space of order 1 on the spacetime M:

Definition 6.2. We let $\widetilde{\mathcal{D}}$ denote the set of functions $u \in H^1_{\mathrm{b}}(M)$ for which the norms $\|\partial_x u\|$ and $\|x^{-1}\nabla_z u\|$ are both finite.

Just as it is well-known (see, e.g., [MW04, Section 3][12]) that $\mathcal{D} = rH^1_{\mathrm{b}}(C(Z))$, we could instead define $\widetilde{\mathcal{D}}$ as a type of weighted b-Sobolev space with only partial regularity. It is convenient, however, for our purposes, to ensure that $\rho\partial_\rho$ and ∂_x are on nearly equal footing.

Away from cf, $\widetilde{\mathcal{D}}$ is a standard b-Sobolev space (defined with respect to the density above). Near cf, it inherits the norm

$$\|u\|^2_{\widetilde{\mathcal{D}}} = \|u\|^2 + \|\rho\partial_\rho u\|^2 + \|\partial_x u\|^2 + \left\|x^{-1}\nabla_z u\right\|^2,$$

and is closed with respect to this norm. Just as in Lemma 6.1, we have the additional Hardy-type inequality

$$\left\|x^{-1}u\right\| \leq C\|u\|_{\widetilde{\mathcal{D}}}.$$

As solutions of the wave equation are not typically L^2 in time, we require a weighted variant of $\widetilde{\mathcal{D}}$: for $\ell \in \mathbb{R}$, we let $\rho^{-\ell}\widetilde{\mathcal{D}}$ denote the space of those u for which $\rho^\ell u \in \widetilde{\mathcal{D}}$. The $\rho^{-\ell}\widetilde{\mathcal{D}}$ norm of a distribution u is the $\widetilde{\mathcal{D}}$ norm of $\rho^\ell u$.

Integrating energy estimates shows that solutions of the wave equation with compactly supported finite energy initial data[13] lie in $\rho^{-\ell}\widetilde{\mathcal{D}}$ for some ℓ.

[11] If m is a positive integer, H^m_{b} can be characterized in terms of $\mathrm{Diff}^m_{\mathrm{b}}(M)$. A characterization for other values of m then follows by interpolation and duality.

[12] Because we adopt the convention that L^2 and b-Sobolev are measured with respect to the metric density on $C(Z)$ rather than a b-density, this identification of $\mathcal{D}$ with a b-Sobolev space differs from the one in that paper by a factor of $r^{-n/2}$.

[13] We state and prove the result for the forward problem with smooth compactly supported initial data, but an inspection of the proof reveals that it needs only finite energy and compact support.

Lemma 6.3. *If u is the forward solution of $Lu = f$, where $f \in C_c^\infty(M^\circ)$, then there is some $\ell \in \mathbb{R}$ so that $u \in \rho^{-\ell}\widetilde{\mathcal{D}}$.*

Proof. The lemma follows by energy bounds and changing variables. Indeed, for each fixed t, standard energy estimates show that

$$\int_{C(Z)} \left(|\partial_t u|^2 + |\partial_r u|^2 + |\frac{1}{r}\nabla_z u|^2 \right) \sqrt{k} r^{n-1}\, dr\, dz < C(f),$$

and so the Hardy-type inequality also shows that

$$\int_{C(Z)} |r^{-1}u|^2 \sqrt{k} r^{n-1}\, dr\, dz < C(f)$$

is uniformly bounded. Integrating these estimates in time shows that for any $\alpha > 1/2$, we have

$$\int_t \int_{C(Z)} \left(|r^{-1}u|^2 + |\partial_t u|^2 + |\partial_r u|^2 + |\frac{1}{r}\nabla_z u|^2 \right) \sqrt{k} r^{n-1} \langle t \rangle^{-2\alpha}\, dr\, dz\, dt < C_\alpha(f),$$

where this new constant differs from the previous one by a factor of $\int_{\mathbb{R}} \langle t \rangle^{-2\alpha}\, dt$.

By the finite speed of propagation and possibly translating the coordinate system in t, it suffices to estimate the $\rho^{-\ell}\widetilde{\mathcal{D}}$ norm of u in the region where $r \leq At$ for some $A > 1$. We may use the boundary defining function $\rho = 1/t$ in this region and $x = r/t$ as a replacement for the radial coordinate and then the region corresponds to $x \leq A$. We then aim to show that there is some ℓ for which

$$\int_0^{\rho_0} \int_0^A \int_Z \left(|u|^2 + |\rho \partial_\rho u|^2 + |\partial_x u|^2 + |x^{-1}\nabla_z u|^2 \right) \sqrt{k}\, dz\, x^{n-1}\, dx\, \rho^{2\ell}\, \frac{d\rho}{\rho} < \infty.$$

Changing coordinates back to (t, r, z), this is equivalent to estimating

$$\int_{t_0^{-1}}^\infty \int_0^{At} \int_Z \left(|u|^2 + |t\partial_t u + r\partial_r u|^2 + |t\partial_r u|^2 + |\frac{t}{r}\nabla_z u|^2 \right) \sqrt{k}\, dz r^{n-1}\, dr\, t^{-n+1-2\ell}\, \frac{dt}{t}.$$

As $r \leq At$ in this region, this integral is bounded by $C_{\ell + \frac{n}{2} - 1}(f)$, provided that $\ell - 1 + n/2 > 1/2$. □

The main use of the space $\widetilde{\mathcal{D}}$ is to act as the base level against which we measure regularity of distributions on M. To that end, we let $H^1_{\mathrm{b},\widetilde{\mathcal{D}}}(M) = \widetilde{\mathcal{D}}$ and define, for $m \geq 1$ and $\ell \in \mathbb{R}$, the finite order conormal spaces $H^{m,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(M)$:

Definition 6.4. Let $A \in \Psi^{m-1}_{\mathrm{b}}(M)$ be an invertible basic (in the sense of Section 5) elliptic operator. For $m \geq 1$, the space $H^{m,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(M)$ consists of those $u \in \rho^\ell \widetilde{\mathcal{D}}$ for which $Au \in \rho^\ell \widetilde{\mathcal{D}}$.

In other words, $H^{m,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(M)$ consists of those distributions conormal to mf and cf of finite order $m-1$ relative to $\rho^\ell \widetilde{\mathcal{D}}$. Away from cf, they agree with the weighted b-Sobolev spaces $\rho^\ell H^m_{\mathrm{b}}(M)$ and indeed we have the inclusion $H^m_{\mathrm{b},\widetilde{\mathcal{D}}} \hookrightarrow H^m_{\mathrm{b}}(M)$.

The following lemma shows that these spaces do not depend on the choice of basic A (as in the work of Vasy [Vas08, Remark 3.6]), as basic operators of order 0 preserve $\widetilde{\mathcal{D}}$.

Lemma 6.5. *If $A \in \Psi_{\mathrm{b}}^0(M)$ is a basic operator, then*

$$A : \rho^\ell \widetilde{\mathcal{D}} \to \rho^\ell \widetilde{\mathcal{D}}, \quad A : \rho^\ell \widetilde{\mathcal{D}}' \to \rho^\ell \widetilde{\mathcal{D}}'$$

are bounded.

Proof. As conjugation by ρ^ℓ yields another basic element of $\Psi_{\mathrm{b}}^0(M)$, we must prove the lemma only for $\ell = 0$.

The result follows from the commutator expressions of Lemma 5.3. Indeed, to estimate $\|Au\|_{\widetilde{\mathcal{D}}}$, it suffices to estimate the quantities

$$\|\rho\partial_\rho Au\|, \|\partial_x Au\|, \left\|\frac{1}{x}\nabla_z Au\right\|, \|Au\|,$$

where all norms taken are with respect to L^2.

We show explicitly this bound only for the term $\frac{1}{x}\nabla_z Au$; the $\partial_x Au$ term is treated similarly while the other two terms amount to the boundedness of Ψ_{b} on b-Sobolev spaces. Appealing to Lemma 5.3, we write

$$\frac{1}{x}\partial_{z_j} Au = A\frac{1}{x}\partial_{z_j} u + B_j u + C_j \partial_x u + \sum_k E_{jk}\frac{1}{x} D_{z_k} u + F_j \frac{1}{x} u,$$

where $B_j \in \Psi_{\mathrm{b}}^0$, and $C_j, E_{jk}, F_j \in \Psi_{\mathrm{b}}^{-1}$. As elements of Ψ_{b}^s are bounded on L^2 for $s \leq 0$, we may then estimate

$$\left\|\frac{1}{x}\partial_{z_j} Au\right\|_{L^2} \leq C\left(\sum_k \left\|\frac{1}{x}\partial_{z_k} u\right\|_{L^2} + \|\partial_x u\|_{L^2} + \left\|\frac{1}{x}u\right\| + \|u\|_{L^2}\right) \leq C\|u\|_{\widetilde{\mathcal{D}}}.$$

□

We finally describe a microlocal characterization of regularity, the wavefront set.

Definition 6.6. Let $u \in H_{\mathrm{b},\widetilde{\mathcal{D}}}^{s,\ell}$ for some $s \geq 0$ and $\ell \in \mathbb{R}$ and suppose that $m \geq 0$. We say $q \in {}^{\mathrm{b}}T^*M \setminus 0$ is not in $\mathrm{WF}_{\mathrm{b},\widetilde{\mathcal{D}}}^{m,\ell}(u)$ if there is some $A \in \Psi_{\mathrm{b}}^{m,\ell}(M)$ elliptic at q so that $Au \in \widetilde{\mathcal{D}}$.

For $m = \infty$, q is not in $\mathrm{WF}_{\mathrm{b},\widetilde{\mathcal{D}}}^{\infty,\ell}(u)$ if there is some $A \in \Psi_{\mathrm{b}}^{m,\ell}(M)$ elliptic at q with $Au \in H_{\mathrm{b},\widetilde{\mathcal{D}}}^{\infty,\ell}$.

Note that if $\mathrm{WF}_{\mathrm{b},\widetilde{\mathcal{D}}}^{\infty,\ell}(u) = \emptyset$, then u is fully conormal to mf and cf relative to the space $\rho^\ell\widetilde{\mathcal{D}}$.

6.2. **Variable-order Sobolev spaces on the boundary** mf**.** We now turn our attention to the function spaces on the boundary mf. We fix a density on mf against which we integrate functions; away from the boundary of mf we ask only that it be smooth and nondegenerate, while at the boundary $\mathrm{mf} \cap \mathrm{cf}$ of mf, we demand that it take the following form in local coordinates (x, z):

$$x^{n-1}\sqrt{k}\, dx\, dz.$$

Near $\mathrm{mf} \cap \mathrm{cf}$ (i.e., near the boundary of $X = \mathrm{mf}$), the operator $\widetilde{P}_\sigma$ is a conjugate of the Laplacian on a hyperbolic cone (see Section 9). In fact, near the boundary $\widetilde{P}_\sigma$ differs from the Laplacian on $C(Z)$ by an element of $\mathrm{Diff}_{\mathrm{b}}^2(\mathrm{mf})$. Lemma 6.1 applies on mf as well, motivating the following abuse of notation:

Definition 6.7. We let $\mathcal{D}$ denote the space of functions on mf that:

(1) lie in H^1 away from $\mathrm{mf} \cap \mathrm{cf}$, and
(2) lie in the Friedrichs form domain of $\Delta_{C(Z)}$ near $\mathrm{mf} \cap \mathrm{cf}$.

We let $\mathcal{D}'$ denote the dual of $\mathcal{D}$ with respect to the L^2 pairing.

A more patently invariant way to define $\mathcal{D}$ involves fixing an invertible elliptic operator (in, e.g., Hintz's semiclassical cone calculus [Hin20]) agreeing with $(1 + \Delta_{C(Z)})^{1/2}$ near the boundary. As our function spaces depend on $\mathcal{D}$ only near the boundary, however, we need not take this approach.

As the main propagation result in Section 8 is semiclassical, we introduce a rescaled version of the domain norm, denoted $\mathcal{D}_h$. For u supported near the boundary of mf, this norm is given by

$$\|u\|_{\mathcal{D}_h}^2 = \|u\|^2 + \|h\partial_x u\|^2 + \left\| \frac{h}{x} \nabla_z u \right\|^2.$$

Lemma 6.1 then shows that $\|u\|_{\mathcal{D}_h}$ also controls $h\|x^{-1}u\|$. As above, we use $\mathcal{D}_h'$ to denote the dual of $\mathcal{D}_h$.

We point out that the characterization of $\mathcal{D}$ stemming from Lemma 6.1 shows that the inclusions $\mathcal{D} \hookrightarrow L^2$ and $L^2 \hookrightarrow \mathcal{D}'$ are compact. This observation is crucial to the Fredholm statement proved in Section 8.1. When $\dim Z = 1$, the characterization of the Friedrichs form domain given by Melrose–Wunsch [MW04, Equation 3.11] also shows the compactness of these inclusions.

Just as in the bulk spacetime, pseudodifferential operators of order 0 (and their semiclassical counterparts) preserve these spaces. The following lemma is proved in the same way as its classical analogue (Lemma 6.5):

Lemma 6.8. *If $A \in \Psi_{\mathrm{b},h}^0(X)$ is a basic operator, then*

$$A : \mathcal{D}_h \to \mathcal{D}_h, \quad A : \mathcal{D}_h' \to \mathcal{D}_h'$$

are bounded.

As we aim to reduce problems on the bulk spacetime M to problems on its main boundary hypersurface $X = \mathrm{mf}$, we record the following lemma relating the spaces $\mathcal{D}$ and $\widetilde{\mathcal{D}}$. The proof of the lemma with $\widetilde{\mathcal{D}}$ replaced by a Sobolev space H^k is standard; the proof for $\widetilde{\mathcal{D}}$ proceeds identically.

Lemma 6.9 (cf. [BVW15, Lemma 2.3])**.** *Suppose $u \in \rho^{-\ell}\widetilde{\mathcal{D}}$ and that $\chi_1, \chi_2 \in C_c^\infty([0,\infty))$ with χ_2 supported in $\{x < 1/4\}$. The Mellin transform (in ρ) of $\chi_1(\rho)\chi_2(x)u$ is a holomorphic function for $\operatorname{Im}\sigma > \ell$ taking values in $L^\infty_{\operatorname{Im}\sigma} L^2_{\operatorname{Re}\sigma}(\mathbb{R}; \mathcal{D})$.*

We now describe the Sobolev spaces on which $\widetilde{P}_\sigma$ is a Fredholm operator. As in prior work, these have variable orders; see [BVW15, Appendix A] for details.

We fix a future regularity function $s_{\mathrm{ftr}} : {}^{\mathrm{b}}S^*\,\mathrm{mf} \to \mathbb{R}$ satisfying the following:

(1) s_{ftr} is constant near $\Lambda^\pm$ and $s_{\mathrm{ftr}} \equiv 1$ in a neighborhood of the conic singularity ∂X,
(2) Along the flow in the classical characteristic set of $\widetilde{P}_\sigma$ (oriented so as to flow from Λ^- to Λ^+), s_{ftr} is monotonically decreasing, and
(3) s_{ftr} is less than the threshold exponent at Λ^+ and greater than the threshold at Λ^-.

As the classical characteristic set of $\widetilde{P}_\sigma$ lies solely over the closure of C_0, the first condition is always compatible with the second and third.

Note that the thresholds at $\Lambda^\pm$ are σ-dependent, so the spaces we consider necessarily depend on which operators in the family $\widetilde{P}_\sigma$ are under consideration. Indeed, as in the previous paper [BVW15, Section 5], the thresholds are given by

$$\begin{aligned} &\frac{1}{2} + \operatorname{Im}\sigma \quad \text{for } \widetilde{P}_\sigma, \\ &\frac{1}{2} - \operatorname{Im}\sigma \quad \text{for } \widetilde{P}_\sigma^*. \end{aligned}$$

We further define $s_{\mathrm{ftr}}^* = -s_{\mathrm{ftr}} + 1$. With these functions in hand, we define (as in [BVW15, Appendix A]) the variable order Sobolev spaces $H^{s_{\mathrm{ftr}}}$ and $H^{s_{\mathrm{ftr}}^*}$ away from the conic singularity $\partial\,\mathrm{mf}$. Recall that standard elliptic regularity estimates still hold in these spaces; hyperbolic propagation estimates also remain valid provided that the order function is decreasing along the flow.

We now fix a partition of unity $\phi, 1-\phi \in C^\infty(\mathrm{mf})$ so that ϕ is supported near the conic singularity where $s_{\mathrm{ftr}} \equiv 1$ and $1-\phi \equiv 0$ in a neighborhood of $\partial\,\mathrm{mf}$. We now define the spaces[14]

$$\begin{aligned} \mathcal{Y}^{s_{\mathrm{ftr}}-1} &= \left\{ u = (1-\phi)u_1 + \phi u_2 \mid u_1 \in H^{s_{\mathrm{ftr}}-1},\ u_2 \in L^2 \right\}, \\ \mathcal{Y}^{s_{\mathrm{ftr}}^*-1} &= \left\{ u = (1-\phi)u_1 + \phi u_2 \mid u_1 \in H^{s_{\mathrm{ftr}}^*-1},\ u_2 \in \mathcal{D}' \right\}, \end{aligned}$$

where we have abused notation slightly: the spaces $\mathcal{Y}^{s_{\mathrm{ftr}}}$ and $\mathcal{Y}^{s_{\mathrm{ftr}}^*}$ differ by how they look near the conic singularity. As $s_{\mathrm{ftr}} = 1$ near the cone points, $\mathcal{Y}^{s_{\mathrm{ftr}}-1}$ agrees with L^2 there, while $s_{\mathrm{ftr}}^* = 0$ near these points, so $\mathcal{Y}^{s_{\mathrm{ftr}}^*-1}$ is a stand-in for H^{-1} there. We equip these two spaces with the norms

$$\begin{aligned} \|u\|^2_{\mathcal{Y}^{s_{\mathrm{ftr}}-1}} &= \|(1-\phi)u\|^2_{H^{s_{\mathrm{ftr}}-1}} + \|\phi u\|^2_{L^2}, \\ \|u\|^2_{\mathcal{Y}^{s_{\mathrm{ftr}}^*-1}} &= \|(1-\phi)u\|^2_{H^{s_{\mathrm{ftr}}^*-1}} + \|\phi u\|^2_{\mathcal{D}'}. \end{aligned}$$

The semiclassical versions of these norms are defined by replacing the Sobolev part of the norm with a semiclassical Sobolev norm and replacing the $\mathcal{D}'$ part of the norm with the $\mathcal{D}'_h$ norm.

We again rely on the localizer ϕ to define the $\mathcal{X}^{s_{\mathrm{ftr}}}$ spaces:

$$\begin{aligned} \mathcal{X}^{s_{\mathrm{ftr}}} &= \left\{ u = (1-\phi)u_1 + \phi u_2 \mid u_1 \in H^{s_{\mathrm{ftr}}},\ u_2 \in \mathcal{D},\ \widetilde{P}_\sigma u \in \mathcal{Y}^{s_{\mathrm{ftr}}-1} \right\}, \\ \mathcal{X}^{s_{\mathrm{ftr}}^*} &= \left\{ u = (1-\phi)u_1 + \phi u_2 \mid u_1 \in H^{s_{\mathrm{ftr}}^*},\ u_2 \in L^2,\ \widetilde{P}_\sigma u \in \mathcal{Y}^{s_{\mathrm{ftr}}^*-1} \right\}. \end{aligned}$$

We have abused notation in the same way as in the definitions of the $\mathcal{Y}$ spaces.[15] Observe also that the condition on $\widetilde{P}_\sigma u$ in the definition of the $\mathcal{X}$ spaces is independent of σ as σ only appears in subprincipal (both classically and semiclassically) terms in $\widetilde{P}_\sigma$. The norms

[14] In analogy with the definition of the $\widetilde{\mathcal{D}}$-based function spaces on the full spacetime, we could have defined the $\mathcal{Y}^{s_{\mathrm{ftr}}}$ space more directly using $\mathcal{D}$. We take the approach above to avoid translating the variable-order Sobolev spaces into the b-setting.

[15] Just as we built the Sobolev spaces in the full spacetime on top of $\widetilde{\mathcal{D}}$, we have built $\mathcal{X}^{s_{\mathrm{ftr}}}$ on $\mathcal{D}$, $\mathcal{Y}^{s_{\mathrm{ftr}}^*-1}$ on $\mathcal{D}'$, and the other two spaces on L^2.

on the $\mathcal{X}$ spaces are given by

$$\|u\|^2_{\mathcal{X}^{s_{\mathrm{ftr}}}} = \|(1-\phi)u_1\|^2_{H^{s_{\mathrm{ftr}}}} + \|\phi u_2\|^2_{\mathcal{D}} + \left\|\widetilde{P}_\sigma u\right\|^2_{\mathcal{Y}^{s_{\mathrm{ftr}}-1}},$$
$$\|u\|^2_{\mathcal{X}^{s^*_{\mathrm{ftr}}}} = \|(1-\phi)u_1\|^2_{H_{s^*_{\mathrm{ftr}}}} + \|\phi u_2\|^2_{L^2} + \left\|\widetilde{P}_\sigma u\right\|^2_{\mathcal{Y}^{s^*_{\mathrm{ftr}}-1}},$$

with the semiclassical analogues obtained in the same way as for the $\mathcal{Y}$ spaces.

One of the main reasons for this setup is that the dual of $\mathcal{Y}^{s_{\mathrm{ftr}}-1}$ consists of those distributions of the form $(1-\phi)u_1 + \phi u_2$, where $u_1 \in H^{s^*_{\mathrm{ftr}}}$ and $u_2 \in L^2$. Similarly, the dual of $\mathcal{Y}^{s^*-1}$ consists of those distributions $(1-\phi)u_1 + \phi u_2$ with $u_1 \in H^{s_{\mathrm{ftr}}}$ and $u_2 \in \mathcal{D}$. Moreover, because the inclusions $\mathcal{D} \hookrightarrow L^2$ and $L^2 \hookrightarrow \mathcal{D}'$ are compact, the inclusions $\mathcal{X}^{s_{\mathrm{ftr}}} \hookrightarrow \mathcal{Y}^{s_{\mathrm{ftr}}-1}$ and $\mathcal{X}^{s^*_{\mathrm{ftr}}} \hookrightarrow \mathcal{Y}^{s^*_{\mathrm{ftr}}-1}$ are also compact.

As our results in Section 8 are stated entirely in terms of estimates, it is unnecessary to define the wavefront set associated to these spaces.

In Section 9 below, we also use variable-order b-Sobolev spaces $H_{\mathrm{b}}^{s_{\mathrm{ftr}}}$ *not* based on $\mathcal{D}$. As s_{ftr} is constant near ∂X, these spaces can be defined in the standard way (see, e.g., [BVW15, Appendix A]). We note that, with our definitions of s_{ftr} and s^*_{ftr}, we have the inclusions

$$\mathcal{X}^{s_{\mathrm{ftr}}} \hookrightarrow H_{\mathrm{b}}^{s_{\mathrm{ftr}}}(X), \qquad \mathcal{Y}^{s_{\mathrm{ftr}}-1} \hookrightarrow H_{\mathrm{b}}^{s_{\mathrm{ftr}}-1}(X),$$
$$\mathcal{X}^{s^*_{\mathrm{ftr}}} \hookrightarrow H_{\mathrm{b}}^{1-s_{\mathrm{ftr}}}(X), \qquad \mathcal{Y}^{s^*_{\mathrm{ftr}}} \hookrightarrow H_{\mathrm{b}}^{-s_{\mathrm{ftr}}}(X).$$

7. Propagation of singularities in the bulk

The aim of this section is to prove a regularity result for forward solutions u of $Lu \in C_c^\infty(M^\circ)$. In particular, we establish that u lies in a weighted $H_{\mathrm{b},\widetilde{\mathcal{D}}}$ space and enjoys additional regularity with respect to the $\Psi_{\mathrm{b}}^0(M)$-module

$$\mathcal{M} = \{A \in \Psi_{\mathrm{b}}^1(M) \mid \sigma(A)|_{\mathcal{R}^+} = 0\}.$$

The main result of this section is the following proposition:

Proposition 7.1. *If $u \in \rho^\ell \widetilde{\mathcal{D}}$ satisfies $Lu \in C_c^\infty(M^\circ)$ and $u \equiv 0$ for $t \ll 0$, then there are $s, \gamma \in \mathbb{R}$ so that $s + \gamma < 1/2$ and $u \in H^{s,\gamma}_{\mathrm{b},\widetilde{\mathcal{D}}}$. Moreover, u possesses module regularity with respect to this space, i.e., if $A_1, \dots, A_N \in \mathcal{M}$, then $A_1 \dots A_N u \in H^{s,\gamma}_{\mathrm{b},\widetilde{\mathcal{D}}}$.*

Away from the cone points and the future radial set, standard elliptic regularity and hyperbolic propagation arguments apply to establish $H^{\bullet,\bullet}_{\mathrm{b},\widetilde{\mathcal{D}}}$ regularity of any order. Our aim therefore is to establish the proposition microlocally in these regions. In Section 7.1 we recall the propagation estimates at the radial sets $\mathcal{R}^\pm$, while in Section 7.2 we establish the necessary estimates near the singularities.

7.1. The radial set.

At $\mathcal{R}^+$ (i.e., at N^*S_+), the Hamilton vector field of L is radial and so we appeal to the radial point propagation estimates of Vasy [Vas13].[16] Though we state the estimates with reference to the domain $\widetilde{\mathcal{D}}$, this is immaterial as the estimates localize and the radial sets are disjoint from the conic singularities.

[16] As we are working with the forward solution in the bulk, we have no need for the estimates at $\mathcal{R}^-$, though these estimates would of course be necessary to show that $\Box$ is Fredholm on appropriate spaces.

Proposition 7.2 (cf. [BVW18, Proposition 5.4])**.** *If $u \in H^{-\infty,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(M)$ for some l, $Lu \in H^{m-1,l}_{\mathrm{b},\widetilde{\mathcal{D}}}$, and $u \in H^{m,l}_{\mathrm{b},\widetilde{\mathcal{D}}}$ on a punctured neighborhood $U \setminus \partial\mathcal{R}^+$ of $\partial\mathcal{R}^+$ in ${}^{\mathrm{b}}S^*M$, then for $m' \leq m$ with $m' + l < 1/2$, we have $w \in H^{m',l}_{\mathrm{b},\widetilde{\mathcal{D}}}(M)$ at $\partial\mathcal{R}^+$ and for $N \in \mathbb{N}$ with $m' + N \leq M$ and $A \in \mathcal{M}^N$, Aw is in $H^{m',l}_{\mathrm{b},\widetilde{\mathcal{D}}}$ at $\partial\mathcal{R}^+$.*

In particular, if $Lu \in H^{\infty,l}_{\mathrm{b},\widetilde{\mathcal{D}}}$ and $u \in H^{\infty,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}$ on a punctured neighborhood of $\partial\mathcal{R}^+$, then as long as $m' + l < 1/2$, $Au \in H^{m',l}_{\mathrm{b},\widetilde{\mathcal{D}}}$ at $\partial\mathcal{R}^+$ for $A \in \mathcal{M}^N$. We remark that as $\partial\mathcal{R}^+$ is disjoint from cf, $H^{m',l}_{\mathrm{b},\widetilde{\mathcal{D}}}$ regularity agrees with $H^{m',l}_{\mathrm{b}}$ regularity.

7.2. **Near the singular points.** For finite times, the work of Melrose–Wunsch [MW04] establishes the needed propagation results. We therefore prove the analogous statement near the intersection $\mathrm{mf} \cap \mathrm{cf}$. Recall that the compressed characteristic set is defined in Section 4.2.

Proposition 7.3. *If $u \in \rho^\ell \widetilde{\mathcal{D}}$ is the forward solution of $Lu = f$ for $f \in C^\infty_c(M^\circ)$, then $\mathrm{WF}^{m,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}} u \subset \dot{\Sigma}$. For*

$$q_0 = \{(\rho = 0, x = 0, z \in Z, \underline{\tau}_0 = \pm 1, \underline{\xi} = 0, \underline{\zeta} = 0)\} \subset \dot{\Sigma} \cap \{\rho = 0\}$$

and let U denote a neighborhood of $q_0 \in \dot{\Sigma}$. If

$$U \cap \{\underline{\xi}/\underline{\tau} > 0\} \cap \mathrm{WF}^{s,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(u) = \emptyset,$$

then

$$q_0 \cap \mathrm{WF}^{s,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(u) = \emptyset.$$

As the wavefront set is closed, this proposition yields regularity at the outgoing points $(\underline{\xi}/\underline{\tau} < 0)$ sufficiently near q_0.

The first statement (that the wavefront set lies in the characteristic set) is the main result of Section 7.2.1, while the diffractive theorem (the absence of "incoming" wavefront set implies the absence of "outgoing" wavefront set) is proved in Section 7.2.2.

Throughout the rest of this section we use Q_j to denote those first-order conic differential operators not lying in $\mathrm{Diff}^1_{\mathrm{b}}$. We set $Q_0 = 1/x$, $Q_1 = D_x$, and $Q_j = \frac{1}{x} D_{z_j}$ for local coordinates $z_2, \ldots, z_n$ on Z. We further *assume all pseudodifferential operators and distributions are localized to a region with $x \leq 1/4$.* As mentioned above, we continue to abuse notation by using the symbol $\mathrm{Diff}_{\mathrm{e}}$ to denote differential operators that are edge-like at cf (i.e., in x) and otherwise b-like at mf (i.e., in ρ). We measure L^2 with respect to the density for the conic-b-metric $\rho^2 g$; in local coordinates this has the form

$$\frac{x^{n-1}\sqrt{k}}{\rho}\, d\rho\, dx\, dz.$$

With respect to this density, we observe that L has the following form:

$$L = (\rho D_\rho + x D_x)^* (\rho D_\rho + x D_x) - D_x^* D_x - \left(\frac{1}{x}\nabla_z\right)^* \left(\frac{1}{x}\nabla_z\right) - \frac{n^2 - 1}{4}.$$

7.2.1. *Elliptic regularity.* The elliptic part of Proposition 7.1 follows from a main lemma and the ellipticity of the operator away from $\dot\Sigma$. Before stating the main lemma, we introduce for brevity the shorthand notation

$$|d_{x,z}f|^2 = |\partial_x f|^2 + |x^{-1}\nabla_z f|^2,$$

where the latter norm is measured with respect to the metric k on Z.

The main estimate follows by pairing Lv with v for a family of v and then integrating by parts; its proof is essentially identical to the one given by Melrose–Vasy–Wunsch [MVW08, Lemma 8.9] with a minor modification we will describe below.

Lemma 7.4 (cf. [MVW08, Lemma 8.9])**.** *Suppose that $K \subset U \subset {}^{\mathrm{b}}S^*M$ with K compact and U open, and suppose further that A_r constitute a bounded family of basic elements of $\Psi_{\mathrm{b},\infty}$ with $\mathrm{WF}'_{\mathrm{b}}(A_r) \subset K$ in the sense of uniform wavefront sets of families, and $A_r \in \Psi_{\mathrm{b}}^{s-1}$ for all $r \in (0,1)$. There exist $G \in \Psi_{\mathrm{b}}^{s-1/2}$ and $\widetilde G \in \Psi_{\mathrm{b}}^{s}$ with $\mathrm{WF}'_{\mathrm{b}}(G), \mathrm{WF}'_{\mathrm{b}}(\widetilde G) \subset U$ and $C_0 > 0$ so that for all $\epsilon > 0$, $r \in (0,1)$, and $u \in \widetilde{\mathcal{D}}$ with $\mathrm{WF}'^{s-1/2}_{\mathrm{b},\widetilde{\mathcal{D}}}(u) \cap U = \emptyset$ and $\mathrm{WF}^{s}_{\mathrm{b},\widetilde{\mathcal{D}}'}(Lu) \cap U = \emptyset$, we have*

$$\left| \int \left(|d_{x,z}A_r u|^2 + \frac{n^2-1}{4}|A_r u|^2 - |(\rho\partial_\rho + x\partial_x)A_r u|^2 \right) \frac{x^{n-1}\sqrt{k}}{\rho}\, d\rho\, dx\, dz \right|$$
$$\leq \epsilon \left(\|d_{x,z}A_r u\|^2_{L^2} + \|\rho\partial_\rho A_r\|^2_{L^2} \right) + C_0 \left(\|u\|^2_{\widetilde{\mathcal{D}}} + \|Gu\|^2_{\widetilde{\mathcal{D}}} + \epsilon^{-1}\|Lu\|^2_{\widetilde{\mathcal{D}}'} + \epsilon^{-1}\left\|\widetilde G Lu\right\|^2_{\widetilde{\mathcal{D}}'} \right).$$

After observing that for $v \in \widetilde{\mathcal{D}}$,

$$\langle Lv, v\rangle = \|(\rho\partial_\rho + x\partial_x)v\|^2 - \|\partial_x v\|^2 - \left\|\frac{1}{x}\nabla_z v\right\|^2 - \frac{n^2-1}{4}\|v\|^2,$$

the proof of Lemma 7.4 is identical to its counterpart in the work of Melrose–Vasy–Wunsch in [MVW08] with ∂_t replaced by $\rho\partial_\rho + x\partial_x$.

At this stage, we record a corollary useful in the next subsection:

Corollary 7.5. *Under the hypotheses of Lemma 7.4, we can estimate the domain norm of $A_r u$ by*

$$\|A_r u\|_{\widetilde{\mathcal{D}}} \leq C \left(\|u\|_{\widetilde{\mathcal{D}}} + \|Gu\|_{\widetilde{\mathcal{D}}} + \|Lu\|_{\widetilde{\mathcal{D}}'} + \left\|\widetilde G Lu\right\|_{\widetilde{\mathcal{D}}'} + \|(\rho\partial_\rho + x\partial_x)A_r u\|_{L^2} \right).$$

Corollary 7.5 allows us to replace factors of Q_j with the b-differential operator $\rho\partial_\rho + x\partial_x$ at the cost of terms already on the right side of Lemma 7.4. In other words, we can control the $\widetilde{\mathcal{D}}$ norm of Au by the H^1_{b} norm of Au and the other terms on the right.

We conclude this section with the proof of the first part of Proposition 7.3, namely that $\mathrm{WF}^{m,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}} u \subset \dot\Sigma$. We employ a simpler version of the argument used by Melrose–Vasy–Wunsch [MVW08, Proposition 8.10].

Suppose $q \in {}^{\mathrm{b}}S^*M \setminus \dot\Sigma$. For finite times (i.e., $\rho > 0$, the theorem of Melrose–Wunsch in [MW04] applies and so we may assume q projects to $\rho = 0$. Likewise, standard elliptic arguments apply away from $x = 0$ and so we may assume q projects to $x = 0$, so that

$$q = (\rho = 0, x = 0, z \in Z, \underline{\tau}, \underline{\xi}, \underline{\zeta}),$$

where $\underline{\xi}^2 + |\underline{\zeta}|^2 > 0$. We assume inductively that $q \notin \mathrm{WF}^{s-1/2,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(u)$ and aim to show that $q \notin \mathrm{WF}^{s,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(u)$. Let $A \in \Psi_{\mathrm{b}}^{s,\ell}$ be a basic operator so that

(1) $\mathrm{WF}'_{\mathrm{b}}(A) \cap \mathrm{WF}^{s-1/2,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(u) = \emptyset$, and

(2) $\mathrm{WF}'_{\mathrm{b}}(A)$ is a subset of a small neighborhood U of q on which $\hat{\underline{\xi}}^2 + |\hat{\underline{\zeta}}|^2 > c > 0$.

We now introduce $\Lambda_r \in \Psi_{\mathrm{b}}^{-2}$ for $r > 0$ with symbol $(1 + r(\underline{\tau}^2 + \underline{\xi}^2 + |\underline{\zeta}|^2))^{-1}$ so that $\Lambda_r \in \Psi^0_{\mathrm{b},\infty}$ uniformly and $\Lambda_r \to \mathrm{Id}$ as $r \to 0$. We set $A_r = \Lambda_r A$ so that for $r > 0$, we have

$$\sigma(A_r) = \frac{a}{1 + r(\underline{\tau}^2 + \underline{\xi}^2 + |\underline{\zeta}|^2)},$$

where a is the symbol of A and $A_r\rho^\ell$ and $\rho^{-\ell}u$ satisfy the hypotheses of Lemma 7.4.

By the Lemma 7.4, the difference

$$\|\partial_x A_r u\|^2 + \left\|\frac{1}{x}\nabla_z A_r u\right\|^2 + \frac{n^2-1}{4}\|A_r u\|^2 - \|(\rho\partial_\rho + x\partial_x)A_r u\|^2 - \epsilon\left\|d_{(x,z)}A_r u\right\|^2$$

is uniformly bounded in r. Writing this quantity as

$$\frac{1-\epsilon}{2}\left(\|\partial_x A_r u\|^2 + \left\|\frac{1}{x}\nabla_z A_r u\right\|\right) + I,$$

we now show $I \geq 0$.

Indeed, we observe that if $\delta > 0$ is sufficiently small, then the operator $B \in \Psi^1_{\mathrm{b}}$ with principal symbol given by

$$\sigma_{\mathrm{b}}(B) = \left(\frac{1-\epsilon}{\delta^2}\left(\underline{\xi}^2 + \underline{\zeta}^2\right) - (1+\epsilon)\left(\underline{\tau} + \underline{\xi}\right)^2\right)^{1/2}$$

is elliptic on U.

Moreover, if A is supported in $\{x < \delta\}$, then

$$\|\partial_x A_r u\|^2 \geq \frac{1}{\delta^2}\|x\partial_x A_r u\|^2,$$
$$\left\|\frac{1}{x}\nabla_z A_r u\right\|^2 \geq \frac{1}{\delta^2}\|\nabla_z A_r u\|^2,$$

and so by shrinking the support of A, I is bounded below by

$$\begin{aligned} I &\geq \frac{1-\epsilon}{\delta^2}\left(\|x\partial_x A_r u\|^2 + \left\|\frac{1}{x}\nabla_z A_r u\right\|^2\right) - (1+\epsilon)\|(\rho\partial_\rho + x\partial_x)A_r u\|^2 \\ &= \|BA_r u\|^2 + \langle FA_r u, A_r u\rangle, \end{aligned}$$

where $B, F \in \Psi^1_{\mathrm{b}}$ and B has principal symbol given above. As F is order 1 and $Au \in \widetilde{\mathcal{D}}$, the second term is uniformly bounded in r.

As I is bounded below, we deduce that

$$\frac{1-\epsilon}{2}\left(\|\partial_x A_r u\|^2 + \left\|\frac{1}{x}\nabla_z A_r u\right\|^2\right)$$

is uniformly bounded in r. Extracting weak limits shows that $Au \in \widetilde{\mathcal{D}}$ and proves the first part of the proposition.

7.2.2. *Hyperbolic propagation.* The aim of this subsection is to complete the proof of Proposition 7.3. We proceed by a positive commutator estimate; the positivity essentially stems from the commutator of L with $x\partial_x$. We treat the case of $\underline{\tau}_0 > 0$ here; the other case follows by flipping the sign of $\underline{\xi}$. Indeed, for

$$\hat{\underline{\xi}} = \frac{1}{\underline{\tau}}\underline{\xi}, \quad p_0 = \sigma_{\mathrm{b}}(L),$$

the Hamilton vector field of p_0 satisfies

$$\frac{1}{2}H_{p_0}(-\hat{\underline{\xi}}) = \frac{1}{x^2}\left(\underline{\xi}^2 + |\underline{\zeta}|^2\right).$$

As in Vasy [Vas08], we define two auxiliary functions

$$\omega = x^2 + \rho^2,$$

and

$$\phi = -\hat{\underline{\xi}} + \frac{1}{\beta^2}\delta\omega,$$

where $\beta > 0$ is a parameter to be chosen. The first function acts as a localizer near the corner, while the second function provides the positivity in the estimate. As long as $\omega < \delta$, we can bound

$$\frac{1}{\tau}H_{p_0}\omega = \mathcal{O}\left(\sqrt{\omega}\left(\frac{\hat{\underline{\xi}}^2}{x^2} + \frac{|\hat{\underline{\zeta}}|^2}{x^2} + 1\right)^{1/2}\right).$$

We now fix three smooth functions of one variable χ_0, χ_1, and χ_2. We demand that $\chi(s) = \exp(-1/s)$ for $s > 0$ so that $\chi_0'(s) = s^{-2}\chi_0(s)$. We take χ_1 supported in $[0,\infty)$ to be equal to 1 on $[1,\infty)$ and so that $\chi_1' \geq 0$ is compactly supported in $(0,1)$. Finally, for a given parameter c_1, we take $\chi_2 \in C_c^\infty(\mathbb{R})$ supported in $[-2c_1, 2c_1]$ and identically 1 on $[-c_1, c_1]$. We insist that all cut-off functions and their derivatives have smooth square roots up to sign.

With $\chi_\bullet$ in hand, we finally define the basic test symbol a by

$$a = \chi_0\left(1 - \frac{\phi}{\delta}\right)\chi_1\left(\frac{-\hat{\underline{\xi}}}{\delta} + 1\right)\chi_2\left(\hat{\underline{\xi}}^2 + |\hat{\underline{\zeta}}|^2\right),$$

where $\delta > 0$ is another parameter to be chosen.

As in Melrose–Vasy–Wunsch [MVW08] and Gannot–Wunsch [GW18], we can arrange that a is well-localized near $q_0 = \{(\rho = 0, x = 0, z \in Z, \underline{\tau} = \pm 1, \underline{\xi} = 0, \underline{\zeta} = 0)\}$.

Lemma 7.6. *Given any neighborhood U of q_0 and any $\beta > 0$, there are $\delta_0 > 0$ and $c_1 > 0$ so that a is supported in U for all $0 < \delta < \delta_0$.*

We now choose a basic operator $B \in \Psi_{\mathrm{b}}^{1/2}$ with

$$b = \sigma_{\mathrm{b}}(B) = \tau^{1/2}\delta^{-1/2}(\chi_0\chi_0')^{1/2}\chi_1\chi_2,$$

so that, when taking derivatives of a, those falling on χ_0 yield factors of b^2. We further choose $C \in \Psi_{\mathrm{b}}^0$ with principal symbol

$$\sigma_{\mathrm{b}}(C) = \frac{\sqrt{2}}{\tau}|\tau + \underline{\xi}|\psi,$$

where $\psi \in S^0({}^{\mathrm{b}}T^*M)$ is identically 1 on the support of the symbol of B.

We can now compute the commutator of A^*A and L:

Lemma 7.7 (cf. [MVW08, Lemma 9.6 and Theorem 9.7])**.** *There is a $\delta_0 > 0$ so that for all $0 < \delta < \delta_0$, the commutator of L and A^*A is given by*

$$i\,[A^*A, L] = R'L + B^*\left(C^*C + R_0 + \sum_j R_jQ_j + \sum_{j,k} Q_j^*R_{jk}Q_k\right)B + R'' + E' + E'',$$

where the terms enjoy the following properties:

- *all factors are microlocalized near q_0,*
- *$R_0 \in \Psi_{\mathrm{b}}^0$, $R', R_j \in \Psi_{\mathrm{b}}^{-1}$, $R_{jk} \in \Psi_{\mathrm{b}}^{-2}$,*
- *E', $E'' \in x^{-2}\operatorname{Diff}_{\mathrm{e}}^2\Psi_{\mathrm{b}}^{-1}$, $R'' \in x^{-2}\operatorname{Diff}_{\mathrm{e}}^2\Psi_{\mathrm{b}}^{-2}$,*
- *the symbols r_0, r_j, and r_{jk} of R_0, R_j, and R_{jk} are supported in $\{\omega \leq 9\delta^2\beta\}$,*
- *the symbols r_0, $\underline{\tau}r_j$, and $\underline{\tau}^2 r_{jk}$ are bounded by both*

$$c\left(1 + \frac{1}{\beta^2\delta}\right), \quad \text{and} \quad c\left(\delta\beta + \beta^{-1}\right),$$

- *$\mathrm{WF}_{\mathrm{b}}'(E') \subset \underline{\xi}^{-1}((0,\infty)) \cap U$, and*
- *$\mathrm{WF}_{\mathrm{b}}'(E'') \cap \dot{\Sigma} = \emptyset$.*

Proof. The principal symbol of the commutator is given by the action of the Hamilton vector field of p_0 on a^2; the choice of the function χ_0 ensures that when derivatives fall on this term, we obtain the contributions sandwiched between B^* and B. The positive term arises from the near homogeneity (in x) of L. Indeed, we exchange the leading term in $a\partial_\xi a$ with L, leaving the symbol $|\underline{\tau} + \underline{\xi}|^2/\underline{\tau}^2$ and obtaining the C^*C term as well as the $R'L$ term.

Derivatives falling on χ_1 give contributions to the E' term; those falling on χ_2 provide contributions to E''. Commuting the Q_j through B also leads to contributions to E' and E''.

The R'' term arises as the computation occurs only at the principal symbol level; this term is also used to further absorb other lower order commutation terms. □

We also observe that we can estimate the remainder terms via the symbol calculus:

Lemma 7.8. *Given $\epsilon > 0$, there is a $\delta_1 \in (0, \delta_0)$ so that for all $0 < \delta < \delta_1$, and all $v \in \widetilde{\mathcal{D}}$,*

$$\begin{aligned}|\langle R_0Bv, Bv\rangle| + \sum_j |\langle R_jQ_jBv, Bv\rangle| + \sum_{j,k} |\langle Q_j^*R_{jk}Q_kBv, Bv\rangle| \\ \leq \epsilon\|Bv\|^2 + C\|R'Bv\|^2 + C\left(\|u\|_{\widetilde{\mathcal{D}}}^2 + \|Gu\|_{\widetilde{\mathcal{D}}}^2 + \|Lu\|_{\widetilde{\mathcal{D}}'}^2 + \left\|\widetilde{G}Lu\right\|_{\widetilde{\mathcal{D}}'}^2\right)\end{aligned}$$

for some $R' \in \Psi_{\mathrm{b}}^{-1}$.

Proof. The lemma follows from the symbol estimates of Lemma 7.7 and Corollary 7.5, together with the observation that for $A \in \Psi_{\mathrm{b}}^0$, there is an $A' \in \Psi_{\mathrm{b}}^{-1}$ so that for all $u \in L^2$

$$\|Au\| \leq \sup|\sigma_{\mathrm{b}}(A)|\|v\| + C\|A'v\|.$$

□

We now finish the proof of Proposition 7.3.

Proof of Proposition 7.3. We first consider the case of $\ell = 0$. Suppose $s < \sup\{s' : q_0 \notin \mathrm{WF}^{s'}_{\mathrm{b},\widetilde{\mathcal{D}}} u\}$; shrinking U if necessary we may assume $U \cap \mathrm{WF}^{s}_{\mathrm{b},\widetilde{\mathcal{D}}}(u) = \emptyset$. Our aim is to show $q \notin \mathrm{WF}^{s+1/2}_{\mathrm{b},\widetilde{\mathcal{D}}}(u)$.

As we measure regularity with respect to $\widetilde{\mathcal{D}}$, we know that if $B \in \Psi^s_{\mathrm{b}}$ localizes to U, then Bu, $Q_i Bu$, and $\rho\partial_\rho Bu$ all lie in L^2. By the hypothesis and Corollary 7.5, it suffices to control $\rho\partial_\rho Bu$ at q_0. In particular, it suffices to find a b-pseudodifferential operator of order $s+3/2$ that is elliptic at q_0 and for which $Bu \in L^2$. (This explains the apparent shift in order by one below.)

Let A, B, and C be as in the discussion preceding Lemma 7.7 and let Λ_r be a quantization of

$$\underline{\tau}^{s+1}\left(1 + r\underline{\tau}^2\right)^{-(s+1)/2}, \quad r \in [0,1],$$

and set $A_r = A\Lambda_r \in \Psi^0_{\mathrm{b}}$ for $r > 0$ and A_r is uniformly bounded in $\Psi^{s+1}_{\mathrm{b},\infty}$. We may further arrange that $[L, \Lambda_r] = 0$.

By the calculation in Lemma 7.7, we may write

$$\begin{aligned} i\left\langle [A_r^* A^*, L]u, u\right\rangle &= \left\|CB\Lambda_r u\right\|^2 + \left\langle R' L\Lambda_r u, \Lambda_r u\right\rangle + \left\langle R_0 B\Lambda_r u, \Lambda_r u\right\rangle \\ &\quad + \sum_j \left\langle R_j Q_j B\Lambda_r u, B\Lambda_r u\right\rangle + \sum_{j,k} \left\langle R_{jk} Q_j B\Lambda_r u, Q_k B\Lambda_r u\right\rangle \\ &\quad + \left\langle R''\Lambda_r u, \Lambda_r u\right\rangle + \left\langle (E' + E'')\Lambda_r u, \Lambda_r u\right\rangle. \end{aligned} \tag{12}$$

As $u \in \widetilde{\mathcal{D}}$, the pairing on the left is well-defined:

$$\left\langle [A_r^* A, L]u, u\right\rangle = \left\langle A_r Lu, A_r u\right\rangle - \left\langle A_r u, A_r Lu\right\rangle.$$

As Lu is residual, these terms are uniformly bounded in r and so we may estimate $\|CB\Lambda_r u\|^2$ by the other terms in equation (12). The second term is uniformly bounded because Lu is residual, while the next three terms are estimated by Lemma 7.8. The R'' term is bounded by the regularity hypothesis of u on U, while the E'' term is bounded by elliptic regularity. Finally, the E' term is bounded by the hypothesis of the theorem. We can therefore find a constant $\mathcal{C}$ independent of r so that

$$\left\|CB\Lambda_r u\right\|^2 \leq \mathcal{C} + \epsilon\|B\Lambda_r u\|^2 + \mathcal{C}\left(\left\|R'Bv\right\|^2 + \|u\|^2_{\widetilde{\mathcal{D}}} + \|Gu\|^2_{\widetilde{\mathcal{D}}} + \|Lu\|^2_{\widetilde{\mathcal{D}}'} + \left\|\widetilde{G}Lu\right\|^2_{\widetilde{\mathcal{D}}'}\right),$$

where $G \in \Psi^{s+1/2}_{\mathrm{b}}$, $\widetilde{G} \in \Psi^{s+1}_{\mathrm{b}}$ are supported in U. An application of the symbol calculus shows that $\|CB\Lambda_r u\|$ (and the rest of the right side) controls $\|B\Lambda_r u\|$. The other terms on the right are uniformly bounded by the assumed regularity of u, so we can extract a subsequence and conclude that $B\Lambda_0 u \in L^2$, so that $q_0 \notin \mathrm{WF}^{s+1/2}_{\mathrm{b},\widetilde{\mathcal{D}}}(u)$. By iteratively shrinking the neighborhoods U, one can then show that in fact $q_0 \notin \mathrm{WF}^{\infty}_{\mathrm{b},\widetilde{\mathcal{D}}}(u)$.

Finally, we now suppose that $\ell \neq 0$. As $Lu \in C^\infty_c(M^\circ)$, we can apply the above argument to $v = \rho^\ell u$ and $\tilde{L} = \rho^\ell L\rho^{-\ell}$. As L and $\tilde{L}$ differ only by an element of $\mathrm{Diff}^1_{\mathrm{b}}$, the same proof applies to v. □

8. The boundary operator $\widetilde{P}_\sigma$

The aim of this section is to establish the mapping properties of $\widetilde{P}_\sigma$ (recall that $\Lambda^\pm$ are the radial sets for $\widetilde{P}_\sigma$ and are the fiber infinities of $N^*S_\pm$):

Proposition 8.1. *The family $\widetilde{P}_\sigma$ has the following mapping properties:*

(1) $\widetilde{P}_\sigma : \mathcal{X}^{s_{\mathrm{ftr}}} \to \mathcal{Y}^{s_{\mathrm{ftr}}-1}$ and $\widetilde{P}_\sigma^* : \mathcal{X}^{s^*_{\mathrm{past}}} \to \mathcal{Y}^{s^*_{\mathrm{past}}-1}$ are Fredholm.

(2) The operators $\widetilde{P}_\sigma$ form a holomorphic Fredholm family on these spaces in

$$\mathbb{C}_{s_+,s_-} = \left\{ \sigma \in \mathbb{C} \mid s_+ < \frac{1}{2} + \operatorname{Im}\sigma < s_- \right\},$$

with $s_{\mathrm{ftr}}|_{\Lambda^\pm} = s_\pm$. The formal adjoint $\widetilde{P}_\sigma^*$ is antiholomorphic in the same region.

(3) The inverse $\widetilde{P}_\sigma^{-1}$ has only finitely many poles in each strip $a < \operatorname{Im}\sigma < b$.

(4) For all a and b, there is a constant C so that

$$\left\| \widetilde{P}_\sigma^{-1} \right\|_{\mathcal{Y}^{s_{\mathrm{ftr}}-1}_{|\sigma|^{-1}} \to \mathcal{X}^{s_{\mathrm{ftr}}}_{|\sigma|}} \leq C \langle \operatorname{Re}\sigma \rangle^{-1} \tag{13}$$

on $a < \operatorname{Im}\sigma < b$, $|\operatorname{Re}\sigma| > C$, with a similar estimate holding for $(\widetilde{P}_\sigma^*)^{-1}$.

(5) The set of poles of $\widetilde{P}_\sigma^{-1}$ (and $(\widetilde{P}_\sigma^*)^{-1}$) is independent of the choice of s_{ftr}.

The first two parts of Proposition 8.1 follow from a sequence of propagation estimates; the second two parts follow from semiclassical analogues of the same sorts of estimates. For the Fredholm statement, we propagate regularity *out of* S_- via radial point estimates (as the $\mathcal{X}$ spaces are more regular than the threshold there), then rely on standard hyperbolic propagation estimates to carry this regularity to a neighborhood of S_+, where we then finish the propagation argument with below-threshold radial point estimates. The regularity in $C_\pm$ is treated by means of the elliptic theory on cones, as $\widetilde{P}_\sigma$ is classically elliptic there. In the semiclassical case, however, the semiclassical characteristic set of $\widetilde{P}_\sigma$ extends into $C_\pm$ and we establish a semiclassical diffractive estimate to carry the regularity of the solution through the singularity of the operator.

Many of the microlocal estimates employed to establish Proposition 8.1 are already in the literature; the main missing components are the Fredholm statement (Section 8.1) and the semiclassical propagation estimate for $\widetilde{P}_\sigma$ near the cone point (Sections 8.2.2 and 8.2.3). The last part of Proposition 8.1 follows from standard arguments in the resonances literature.

8.1. **The Fredholm property.** We first show that $\widetilde{P}_\sigma$ is Fredholm on the desired spaces (parts 1 and 2 of Proposition 8.1). In particular, we prove the following proposition:

Proposition 8.2. *Given $s_\pm$ and $s_{\mathrm{ftr/past}}|_{\Lambda^\pm} = s_\pm$, there is*

$$\|u\|_{\mathcal{X}^{s_{\mathrm{ftr}}}} \leq C \left(\left\| \widetilde{P}_\sigma u \right\|_{\mathcal{Y}^{s_{\mathrm{ftr}}-1}} + \|u\|_{H^{-N}_{\mathrm{b},\widetilde{\mathcal{D}}}} \right).$$

Away from the radial sets and the conic singularity $\mathrm{mf} \cap \mathrm{cf}$, standard elliptic regularity and hyperbolic propagation arguments can be pieced together. Near the singularity at the poles, we appeal to the following elementary lemma, which follows essentially immediately after integrating by parts:

Lemma 8.3. *Fix $\chi \in C^\infty(\mathrm{mf})$ supported in $\{x < 1/4\}$. For any N, there is a constant C so that*

$$\|\chi u\|_{\mathcal{D}} \leq C \left(\left\| \widetilde{P}_\sigma(\chi u) \right\|_{L^2} + \|\chi u\|_{L^2} \right).$$

In particular, for all s and all N, we may estimate

$$\|\chi u\|_{\mathcal{X}^{s_{\mathrm{ftr}}}} \leq C\left(\left\|\widetilde{P}_\sigma(\chi u)\right\|_{\mathcal{Y}^{s_{\mathrm{ftr}}-1}} + \|\chi u\|_{\mathcal{Y}^{s_{\mathrm{ftr}}-1}}\right).$$

The same argument provides a similar estimate for $\widetilde{P}_\sigma^*$ in the appropriate (dual) spaces.

Near the radial sets $\Lambda^\pm$, the radial point estimates of Vasy [Vas13] (building on work of Melrose [Mel94]) apply without change:

Lemma 8.4 ([Vas13, Propositions 2.3 and 2.4])**.** *For all N and for $s_0 > m > \frac{1}{2} + \operatorname{Im}\sigma$, and for all $A, B, G \in \Psi_{\mathrm{b}}^0(\mathrm{mf})$ supported near Λ^- with A, G elliptic at Λ^- and so that all bicharacteristics from the microsupport of B tend to Λ^- in one direction while remaining in the elliptic set of G, we have*

$$\text{If } Au \in H^m \text{ then } \|Bu\|_{H^{s_0}} \leq C\left(\left\|G\widetilde{P}_\sigma u\right\|_{H^{s_0-1}} + \|u\|_{H^{-N}_{\mathrm{b},\mathcal{D}}}\right).$$

For $s_0 < \frac{1}{2} + \operatorname{Im}\sigma$ and $A, B, G \in \Psi_{\mathrm{b}}^0(\mathrm{mf})$ supported near Λ^+ with B, G elliptic at Λ^+ so that all bicharacteristics from $\mathrm{WF}'(B) \setminus \Lambda^+$ reach the microsupport of A in one direction while remaining in the elliptic set of G, we have

$$\|Bu\|_{H^{s_0}} \leq C\left(\left\|G\widetilde{P}_\sigma u\right\|_{H^{s_0-1}} + \|Au\|_{H^{s_0}} + \|u\|_{H^{-N}}\right).$$

An analogous theorem holds for $\widetilde{P}_\sigma^*$ with σ replaced by its complex conjugate and the direction of propagation reversed (so that the roles of $\Lambda^\pm$ are exchanged).

Taking microlocal partitions of unity as appropriate, we therefore have the two estimates

$$\|u\|_{\mathcal{X}^{s_{\mathrm{ftr}}}} \leq C\left(\left\|\widetilde{P}_\sigma u\right\|_{\mathcal{Y}^{s_{\mathrm{ftr}}-1}} + \|u\|_{\mathcal{Y}^{s_{\mathrm{ftr}}-1}}\right),$$
$$\|u\|_{\mathcal{X}^{s^*_{\mathrm{ftr}}}} \leq C\left(\left\|\widetilde{P}^*_\sigma u\right\|_{\mathcal{Y}^{s^*_{\mathrm{ftr}}-1}} + \|u\|_{\mathcal{Y}^{s^*_{\mathrm{ftr}}-1}}\right).$$

As the inclusions $\mathcal{X}^{s_{\mathrm{ftr}}} \hookrightarrow \mathcal{Y}^{s_{\mathrm{ftr}}-1}$ and $\mathcal{X}^{s^*_{\mathrm{ftr}}} \hookrightarrow \mathcal{Y}^{s^*_{\mathrm{ftr}}-1}$ are compact, the operators $\widetilde{P}_\sigma$ and $\widetilde{P}_\sigma^*$ are Fredholm in the stated spaces, proving the first part of the theorem. The second part of the theorem follows from the facts that the coefficients of $\widetilde{P}_\sigma$ are holomorphic and that s_{ftr} may be chosen to satisfy the desired properties for all σ in such a strip.

8.2. **Semiclassical estimates.** The third part of Proposition 8.2 follows from the fourth; this rest of this section is devoted to proving the estimate there.

As the estimate (13) is a semiclassical hyperbolic estimate, we work semiclassically with $h = |\sigma|^{-1}$ as our semiclassical parameter and $\widetilde{P}_h = h^2\widetilde{P}_\sigma$. In these terms, the estimate (13) is immediately implied by an estimate of the form

$$\|u\|_{\mathcal{X}_h^{s_{\mathrm{ftr}}}} \leq \frac{C}{h}\left\|\widetilde{P}_h u\right\|_{\mathcal{Y}_h^{s_{\mathrm{ftr}}-1}} + Ch\|u\|_{\mathcal{X}_h^{s_{\mathrm{ftr}}}} \tag{14}$$

for some N, together with an analogous estimate for $\widetilde{P}_h^*$ on the appropriate spaces. Away from $\Lambda^\pm$ and from $\{x = 0\}$, the microlocal version of the estimate follows from standard elliptic regularity and hyperbolic propagation estimates.

8.2.1. *The radial set.* Near $\Lambda^{\pm}$, the estimate follows from a semiclassical version of radial propagation estimates as in earlier work [BVW15, BVW18, Vas13].

Proposition 8.5 (cf. [Vas13, Propositions 2.8 and 2.9])**.** *For $s|_{\Lambda^-} > m > \frac{1}{2} + \operatorname{Im}\sigma$ and $A, B, G \in \Psi^0_{\mathrm{b},h}$ supported near Λ^- with A, G elliptic at Λ^- and so that semiclassical bicharacteristics from the microsupport of B tend to Λ^- in one direction with closure in the elliptic set of G, we have*

$$\text{If } Au \in H^m, \text{ then } \|Bu\|_{\mathcal{X}_h^{s_{\mathrm{ftr}}}} \leq \frac{C}{h}\left\|G\widetilde{P}_h u\right\|_{\mathcal{Y}_h^{s_{\mathrm{ftr}}-1}} + Ch\|u\|_{\mathcal{Y}^{s_{\mathrm{ftr}}-1}}.$$

For $s|_{\Lambda^+} < \frac{1}{2} + \operatorname{Im}\sigma$, and for all $A, B, G \in \Psi^0_{\mathrm{b},h}$ supported near Λ^+ with B, G elliptic at Λ^+ and so that semiclassical bicharacteristics from $\mathrm{WF}'_{\mathrm{b},h}(B) \setminus \Lambda^+$ reach the microsupport of A in one direction while remaining in the elliptic set of G, we have

$$\|Bu\|_{\mathcal{X}_h^{s_{\mathrm{ftr}}}} \leq \frac{C}{h}\left\|G\widetilde{P}_h u\right\|_{\mathcal{Y}_h^{s_{\mathrm{ftr}}-1}} + C\|Au\|_{\mathcal{X}_h^{s_{\mathrm{ftr}}}} + Ch\|u\|_{\mathcal{Y}^{s_{\mathrm{ftr}}-1}}.$$

Analogous estimates hold for $\widetilde{P}_h^*$ on the dual spaces as well (with the roles of $\Lambda^{\pm}$ interchanged).

8.2.2. *Elliptic regularity near the singularity.* We now consider the problem in the region $\{x < c_0\}$ where $s_{\mathrm{ftr}} = 1$; we assume all pseudodifferential operators and distributions are supported in this region. Here we have an explicit expression for $\widetilde{P}_\sigma$:

$$\widetilde{P}_\sigma = (\sigma + xD_x)^2 - ni\,(\sigma + xD_x) - D_x^2 + \frac{(n-1)i}{x}D_x - \frac{1}{x^2}\Delta_k - \frac{n^2-1}{4}.$$

After rescaling and letting $\lambda = \sigma/|\sigma|$, we have

$$\widetilde{P}_h = h^2\widetilde{P}_\sigma = (\lambda + hxD_x)^2 - nih\,(\lambda + hxD_x) - h^2D_x^2 + \frac{(n-1)ih}{x}hD_x - \frac{h^2}{x^2}\Delta_k - \frac{n^2-1}{4}h^2.$$

As we are only ever concerned with $\operatorname{Im}\sigma \in [a, b]$ for some fixed a, b, we observe that $\lambda = \pm 1 + \mathcal{O}(h)$.

We prove the estimate near $x = 0$ in two main steps; we first consider the microlocally elliptic region (i.e., away from the characteristic set) and then the hyperbolic region (near the characteristic set). In this section and the next, we consider only the forward problem (for $\widetilde{P}_h$); the adjoint problem (for $\widetilde{P}_h^*$) proceeds nearly identically, though with a shift downward in the norms considered (i.e., $\mathcal{D}_h$ replaced by L^2 and L^2 replaced by $\mathcal{D}'_h$).

The main elliptic estimate near the singularity is the following proposition:

Proposition 8.6. *Suppose $A \in \Psi^0_{\mathrm{b},h}$ is basic and satisfies $\mathrm{WF}'_{\mathrm{b},h}(A) \cap \Sigma_h = \emptyset$. For any $G \in \Psi^0_{\mathrm{b},h}$ with $\mathrm{WF}'_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b},h}(G)$, there is a constant C so that*

$$\|Au\|_{\mathcal{X}_h^{s_{\mathrm{ftr}}}} \leq C\left\|G\widetilde{P}_h u\right\|_{\mathcal{Y}_h^{s_{\mathrm{ftr}}-1}} + Ch^{1/2}\|Gu\|_{\mathcal{X}_h^{s_{\mathrm{ftr}}}} + \mathcal{O}(h^\infty)\|u\|_{\mathcal{X}_h^{s_{\mathrm{ftr}}}}.$$

By enlarging the microsupport of G, one can improve the factor of $h^{1/2}$ to h^N for any fixed N.

Integration by parts allows us to prove the following lemma, which reduces the problem of controlling the domain norm of solutions to controlling a b-norm.

Lemma 8.7. *Suppose $A, G \in \Psi^0_{\mathrm{b},h}$ with A a basic operator satisfying $\mathrm{WF}'_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b}}(G)$. There is a constant C so that*

$$\int \left(|hD_x Au|^2 + |\frac{1}{x}\nabla_z Au|^2 - |(hxD_x + \lambda)Au|^2 \right) x^{n-1}\, dx \ \mathrm{dvol}_k$$
$$\leq \epsilon \|Au\|^2_{\mathcal{D}_h} + \frac{C}{\epsilon}\left\| G\widetilde{P}_\sigma u \right\|^2_{\mathcal{D}'_h} + Ch\|Gu\|^2_{\mathcal{D}_h} + \mathcal{O}(h^\infty)\|u\|^2_{\mathcal{D}_h}$$

for all $u \in \mathcal{D}_h$.

Proof. Integration by parts shows that if $v \in \mathcal{D}_h$, then

$$\left\langle \widetilde{P}_h v, v \right\rangle =$$
$$\int \left(|hD_x v|^2 + |\frac{1}{x}\nabla_z v|^2 - |(hxD_x + \lambda)v|^2 + \frac{n^2-1}{4}h^2|v|^2 \right) 2i(\operatorname{Im}\lambda)\, \langle (hxD_x + \lambda)v, \overline{v} \rangle ,$$

where the pairing on the left side is of $\mathcal{D}_h$ with $\mathcal{D}'_h$.

We apply this identity to $v = Au \in \mathcal{D}_h$ and then first estimate

$$\left\langle A\widetilde{P}_h, Au \right\rangle + \left\langle [\widetilde{P}_h, A]u, u \right\rangle - 2i(\operatorname{Im}\lambda)\, \langle (hxD_x + \lambda)Au, Au \rangle .$$

The first term is estimated by Cauchy–Schwarz:

$$|\left\langle A\widetilde{P}_h u, Au \right\rangle| \leq \frac{1}{4\epsilon}\left\| A\widetilde{P}_h u \right\|^2_{\mathcal{D}'_h} + \epsilon \|Au\|^2_{\mathcal{D}_h}.$$

Microlocal elliptic regularity lets us estimate $A\widetilde{P}_h u$ in terms of $G\widetilde{P}_h u$. As $\operatorname{Im}\lambda = \mathcal{O}(h)$, the final term is bounded by

$$Ch\left(\|hxD_x Au\|^2 + \|Au\|^2 \right).$$

The additional factor of h allows these terms to be absorbed into the $h\|Gu\|^2_{\mathcal{D}_h}$ term.

We now turn to the term involving $[P, A]$. After applying Lemma 5.3 and keeping track of the factors lying in $\frac{1}{x}\operatorname{Diff}^1_{\mathrm{b},h}$ but not $\operatorname{Diff}^1_{\mathrm{b},h}$, we can estimate this term by $h\|Gu\|^2_{\mathcal{D}_h}$. □

As we have assumed that the operators in Lemma 8.7 are supported in $\{x < c_0\}$, we obtain the following corollary, which we record for use in the hyperbolic section below:

Corollary 8.8. *If A and G are as in Lemma 8.7, there are constants C_0 (independent of A) and C so that*

$$\|Au\|_{\mathcal{D}_h} \leq C_0\|Au\|_{L^2} + C\left(\left\| G\widetilde{P}_h u \right\|_{\mathcal{D}'_h} + \|Gu\|_{\mathcal{D}_h} \right) + \mathcal{O}(h^\infty)\|u\|_{\mathcal{D}_h}.$$

Proof. As $x < c_0$, we can bound $-|(hxD_x + \lambda)Au|^2$ below by

$$-2c_0^2|hD_x Au|^2 - 2|\lambda|^2|Au|^2.$$

The first of these terms can be absorbed (together with $\epsilon\|Au\|^2_{\mathcal{D}_h}$) into the first term on the left in Lemma 8.7, while the second term is moved to the right side. □

Proposition 8.6 then follows immediately by applying the following lemma and bounding $\left\| G\widetilde{P}_h u \right\|_{\mathcal{D}'_h} \leq \left\| G\widetilde{P}_h u \right\|_{L^2}$:

Lemma 8.9. *Suppose A and G are as in the statement of Proposition 8.6. If, in addition, A is supported in $\{x < \delta/\sqrt{2}\}$ and $\{(\underline{\xi}+\lambda)^2 < \frac{1}{2}\delta^{-2}(\underline{\xi}^2 + |\underline{\zeta}|^2)\}$, then*

$$\|Au\|_{\mathcal{D}_h} \leq C\left\|G\widetilde{P}_h u\right\|_{\mathcal{D}'_h} + Ch^{1/2}\|Gu\|_{\mathcal{D}_h} + \mathcal{O}(h^\infty)\|u\|_{\mathcal{D}_h}.$$

Proof. As A is supported in $\{x < \delta/\sqrt{2}\}$, we know

$$\delta^{-2}\int \left(|hxD_xAu|^2 + |h\nabla_z u|^2\right) - \int |(hxD_x+\lambda)Au|^2 \leq \\ \frac{1}{2}\int \left(|hD_xAu|^2 + |\frac{1}{x}\nabla_z Au|^2 - |(hxD_x+\lambda)Au|^2\right).$$

Our other hypothesis on the support of A shows that we can find operators $B, F \in \Psi^1_{\mathrm{b},h}$ with $\mathrm{WF}'_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b},h}(B)$ so that

$$Z = \delta^{-2}\left((hxD_x)^*(hxD_x) + \frac{h^2}{x^2}\Delta_k - (hxD_x+\lambda)^*(hxD_x+\lambda)\right) - (B^*B + hF)$$

satisfies

$$\mathrm{WF}_{\mathrm{b},h}(Zu) \cap \mathrm{WF}'_{\mathrm{b},h}(A) = \emptyset.$$

Integrating by parts and applying Lemma 8.7 shows that

$$\|BAu\|^2_{L^2} + \frac{1}{2}\int \left(|hD_xAu|^2 + |\frac{h}{x}\nabla_z Au|^2\right) \leq \epsilon\|Au\|^2_{\mathcal{D}_h} + \frac{C}{\epsilon}\left\|G\widetilde{P}_h u\right\|^2_{\mathcal{D}'_h} \\ + Ch\|Gu\|^2_{\mathcal{D}_h} + Ch\|FAu\|\|Au\| + \mathcal{O}(h^\infty)\|u\|^2_{\mathcal{D}_h}.$$

As B is elliptic on $\mathrm{WF}'_{\mathrm{b}}(A)$, the left side controls $\|Au\|^2_{\mathcal{D}'_h}$, while the right side is controlled by

$$\epsilon\|Au\|^2_{\mathcal{D}_h} + C\left\|G\widetilde{P}_h u\right\|^2_{\mathcal{D}'_h} + Ch\|Gu\|^2_{\mathcal{D}_h} + \mathcal{O}(h^\infty)\|u\|^2_{\mathcal{D}_h}.$$

Absorbing the first term into the left side finishes the proof. □

8.2.3. *Hyperbolic propagation near the singularity.* In this subsection we complete the proof of the third and fourth parts of Proposition 8.1. In particular, we establish the following proposition:

Proposition 8.10 (cf. [GW18, Proposition 5.8])**.** *If $G \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ is elliptic at $\{(0,z,0,0) \mid z \in Z\}$, then there are $\mathcal{Q}, \mathcal{Q}_1 \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ with $\mathcal{Q}$ elliptic at $\{(0,z,0,0) \mid z \in Z\}$ and*

$$\mathrm{WF}'_{\mathrm{b},h}\,\mathcal{Q} \subset \mathrm{ell}_{\mathrm{b},h}(G),$$
$$\mathrm{WF}'_{\mathrm{b},h}\,\mathcal{Q}_1 \subset \mathrm{ell}_{\mathrm{b},h}(G) \cap \{-\underline{\xi} > 0\},$$

so that for all $u \in \mathcal{D}_h$,

$$\|\mathcal{Q}u\|_{\mathcal{D}_h} \leq \frac{C}{h}\left\|G\widetilde{P}_h u\right\|_{\mathcal{D}'_h} + C\|\mathcal{Q}_1 u\|_{\mathcal{D}_h} + Ch\|Gu\|_{\mathcal{D}_h} + \mathcal{O}(h^\infty)\|u\|_{\mathcal{D}_h}.$$

We note that the estimate in Proposition 8.10 immediately implies the estimate

$$\|\mathcal{Q}u\|_{\mathcal{X}_h^{s_{\mathrm{ftr}}}} \leq \frac{C}{h}\left\|G\widetilde{P}_h u\right\|_{\mathcal{Y}_h^{s_{\mathrm{ftr}}-1}} + C\|\mathcal{Q}_1 u\|_{\mathcal{X}_h^{s_{\mathrm{ftr}}}} + Ch\|Gu\|_{\mathcal{X}_h^{s_{\mathrm{ftr}}}} + \mathcal{O}(h^\infty)\|u\|_{\mathcal{X}_h^{s_{\mathrm{ftr}}}},$$

finishing the proof of the fourth part of Proposition 8.1.

As in Section 7.2.2, we introduce a basic operator $A \in \Psi_{\mathrm{b},h}^{\mathrm{comp}}$ with symbol given by

$$a = \chi_0(2 - \phi/\delta)\chi_1(2 - \underline{\xi}/\delta)\chi_2(\underline{\xi}^2 + |\underline{\zeta}|^2),$$

where χ_i are the same functions as in that section and $\phi = -\underline{\xi} + \frac{1}{\beta^2\delta}x^2$. Recall that χ_2 is supported in $[-2c_1, 2c_1]$ and identically one on $[-c_1, c_1]$, so that a is essentially determined by the three parameters c_1, β, and δ.

We also choose a basic operator $B \in \Psi_{\mathrm{b},h}^{\mathrm{comp}}$ with symbol

$$b = \frac{2}{\sqrt{\delta}}(\chi_0\chi_0')^{1/2}\chi_1\chi_2,$$

so that factors of B arise when derivatives land on χ_0 in A.

As in that section (and Melrose–Vasy–Wunsch [MVW08] or Gannot–Wunsch [GW18]), the symbol a is well-localized:

Lemma 8.11. *Given any neighborhood U of $\{(0, z, 0, 0) \mid z \in Z\}$ in ${}^{\mathrm{b}}T^*\,\mathrm{mf}$ and any $\beta > 0$, there are $\delta_0 > 0$ and $c_1 > 0$ so that a is supported in U for all $0 < \delta < \delta_0$.*

We now compute the commutator of $\widetilde{P}_h$ with A^*A:

Lemma 8.12. *With Q_0, Q_1, and Q_j denoting the conic vector fields as in Section 5.2, the commutator of $\widetilde{P}_h$ and A^*A is given by*

$$\frac{i}{h}\left(\widetilde{P}_h^* A^*A - A^*A\widetilde{P}_h\right) = \\ - B_0\widetilde{P}_h + B^*\left(C^*C + R_0 + \sum_j R_jQ_j + \sum_{j,k} Q_j^*R_{jk}Q_k\right)B + E' + E'' + hR',$$

where the terms enjoy the following properties:

- $C = hxD_x + \lambda$,
- $\sigma_{\mathrm{b},h}(B_0) = 2\partial_{\underline{\xi}}(a^2)$,
- $R_0, R_j, R_{jk} \in \Psi_{\mathrm{b},h}^{\mathrm{comp}}$ *satisfy*
$$|\sigma_{\mathrm{b},h}(R_\bullet)| \leq C_1(\delta\beta + \beta^{-1}),$$
- $R'' \in x^{-2}\,\mathrm{Diff}_{\mathrm{b},h}^2\,\Psi_{\mathrm{b},h}^{\mathrm{comp}}$,
- $E', E'' \in x^{-2}\,\mathrm{Diff}_{\mathrm{b},h}^2\,\Psi_{\mathrm{b},h}^{\mathrm{comp}}$ *satisfy*
$$\mathrm{WF}'_{\mathrm{b},h}(E') \subset \{-\underline{\xi} > 0\}, \quad \mathrm{WF}'_{\mathrm{b},h}(E'') \cap \Sigma_h = \emptyset.$$

Proof. We use Lemma 5.3 to commute A^*A through $\widetilde{P}_h$, using that A is basic. The main term arising from the commutator reproduces the main terms in $\widetilde{P}_h$; indeed, it is of the form

$$B^*\left((hD_x)^*(hD_x) + \frac{h^2}{x^2}\Delta_k\right)B.$$

We use the form of the operator to exchange this term for $B_0\widetilde{P}_h$ and B^*C^*CB. The other terms in the expression arise in a similar way as those in Melrose–Vasy–Wunsch [MVW08] (explained above in the proof of Lemma 5.2). The term arising from $\widetilde{P}_h^* - \widetilde{P}_h$ can be absorbed into the R_0 term as the symbol of A is estimated by $\sqrt{\delta}b$. □

We also require that the remainder terms are sufficiently small as to be estimable:

Lemma 8.13. *For any $\epsilon > 0$, there are $\beta > 0$ and $\delta_1 \in (0, \delta_0)$ so that for all $0 < \delta < \delta_1$,*

$$|\langle R_0 Bu, Bu\rangle| + \sum_j |\langle R_j Q_j Bu, Bu\rangle| + \sum_{j,k} |\langle Q_j^* R_{jk} Q_k Bu, Bu\rangle| \leq \epsilon \|Bu\|^2_{\mathcal{D}_h} + \mathcal{O}(h^\infty)\|u\|^2_{\mathcal{D}_h}.$$

Proof. As in the proof of Lemma 7.8, we rely on the symbol estimates in Lemma 8.12. Indeed, we bound

$$\begin{aligned}\|R_\bullet v\|_{L^2} &\leq 2\sup|\sigma_{\mathrm{b},h}(R_\bullet)|\|v\|_{L^2} + \mathcal{O}(h^\infty)\|v\|_{L^2} \\ &\leq 2C_1\left(\delta\beta + \beta^{-1}\right)\|v\|_{L^2} + \mathcal{O}(h^\infty)\|v\|_{L^2}.\end{aligned}$$

We now fix $\beta > 0$ sufficiently large and then take $\delta_1 \in (0, \delta_0)$ sufficiently small to make $2C_1(\delta_1\beta + \beta^{-1}) < \epsilon/3$.

We now consider the individual terms. For the R_0 term, we apply the above inequality with $v = Bu$ and appeal to Cauchy–Schwarz. The R_j and R_{jk} terms are nearly identical, e.g.,

$$\begin{aligned}|\langle Q_j R_{jk} Q_k Bu, Bu\rangle| &= |\langle R_{jk} Q_k Bu, Q_j Bu\rangle| \\ &\leq 2C_1(\delta\beta + \beta^{-1})\|Bu\|^2_{\mathcal{D}_h} \leq \epsilon\|Bu\|^2_{\mathcal{D}_h}.\end{aligned}$$

□

We now finish the proof of Proposition 8.10.

Proof of Proposition 8.10. Given $u \in \mathcal{D}_h$, we apply Lemma 8.12 to write

$$\begin{aligned}\frac{2}{h}\operatorname{Im}\left\langle A\widetilde{P}_h u, Au\right\rangle &= \frac{i}{h}\left\langle \left(\widetilde{P}_h^* A^* A - A^* A\widetilde{P}_h\right)u, u\right\rangle \\ &= \|CBu\|^2_{L^2} + \langle R_0 Bu, Bu\rangle + \sum_j \langle R_j Q_j Bu, Bu\rangle \\ &\quad + \sum_{j,k}\langle R_{jk} Q_k Bu, Q_j Bu\rangle + \langle E' u, u\rangle + \langle E'' u, u\rangle + h\langle R' u, u\rangle - \langle B_0 Pu, u\rangle.\end{aligned}$$

As shown above in Lemma 6.8, A, B, and CB preserve $\mathcal{D}_h$, while B_0 preserves $\mathcal{D}'_h$.

By Corollary 8.8 and the ellipticity of C on $\mathrm{WF}'_{\mathrm{b},h}(B)$, there is a constant $c > 0$ so that

$$c\|Bu\|^2_{\mathcal{D}_h} \leq \|CBu\|^2_{L^2} + C\left\|G\widetilde{P}_h u\right\|^2_{\mathcal{D}'_h} + Ch\|Gu\|^2_{\mathcal{D}_h} + \mathcal{O}(h^\infty)\|u\|^2_{\mathcal{D}'_h},$$

where $c > 0$ is independent of β and δ and G is elliptic on $\mathrm{WF}'_{\mathrm{b},h}(B)$.

We now take $G \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ to be elliptic on $\mathrm{WF}'_{\mathrm{b},h}(B)$ and $\mathcal{Q}_1 \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ to be elliptic on $\mathrm{WF}'_{\mathrm{b},h}(E')$ with $\mathrm{WF}'_{\mathrm{b},h}(\mathcal{Q}_1) \subset \mathrm{ell}_{\mathrm{b},h}(G) \cap \{-\underline{\xi} > 0\}$. Applying Lemma 8.13 yields an estimate of the form

$$\begin{aligned}\frac{c}{2}\|Bu\|^2_{\mathcal{D}_h} \leq \frac{2}{h}|\left\langle A\widetilde{P}_h u, Au\right\rangle| + C\left\|G\widetilde{P}_h u\right\|^2_{\mathcal{D}'_h} + Ch\|Gu\|^2_{\mathcal{D}_h} \\ + |\langle (E' + E'')u, u\rangle| + h|\langle R' u, u\rangle| + |\left\langle B_0\widetilde{P}_h u, u\right\rangle| + \mathcal{O}(h^\infty)\|u\|^2_{\mathcal{D}_h}.\end{aligned}$$

We estimate the E' term by $\mathcal{Q}_1$ via microlocal elliptic regularity and the E'' term by Proposition 8.6. The second line is therefore bounded by

$$\frac{C}{h}\left\|G\widetilde{P}_h u\right\|^2_{\mathcal{D}'_h} + Ch\|Gu\|^2_{\mathcal{D}_h} + C\|\mathcal{Q}_1 u\|^2_{\mathcal{D}_h} + \mathcal{O}(h^\infty)\|u\|^2_{\mathcal{D}_h}.$$

Because $\mathrm{WF}'_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b},h}(G)$, we can further estimate

$$\frac{2}{h}|\left\langle A\widetilde{P}_h u, Au\right\rangle| \leq \frac{C}{h^2\epsilon}\left\|G\widetilde{P}_h u\right\|^2_{\mathcal{D}'_h} + C\epsilon\|Au\|^2_{\mathcal{D}_h} + \mathcal{O}(h^\infty)\|u\|^2_{\mathcal{D}_h}.$$

By construction, $\chi_0(s) = s^2\chi_0'(s)$ for $s > 0$, and so

$$a = \frac{1}{2}\delta^{1/2}(2 - \phi/\delta)b.$$

We may therefore write $A = FB + hF'$ for some $F, F' \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ in order to estimate Au by Bu. Putting the above together yields the estimate

$$\|Bu\|_{\mathcal{D}_h} \leq \frac{C}{h}\left\|G\widetilde{P}_h u\right\|_{\mathcal{D}'_h} + C\|\mathcal{Q}_1 u\|_{\mathcal{D}_h} + Ch^{1/2}\|Gu\|_{\mathcal{D}_h} + \mathcal{O}(h^\infty)\|u\|_{\mathcal{D}_h}.$$

Taking $\mathcal{Q} = B$ finishes the proof. □

9. Proof of Theorem 1.1

This section is devoted to a sketch of the proof of the main theorem, which is implied by the more refined theorem below:

Theorem 9.1. *Suppose w is a solution of the wave equation on a cone. If the initial data of w are smooth and compactly supported away from the conic singularity, i.e.,*

$$\Box w = 0 \text{ on } \mathbb{R} \times C(Z),$$
$$(w, \partial_t w)|_{t=0} \in C_c^\infty(C(Z)) \times C_c^\infty(C(Z)),$$

then, viewed as a distribution on $[M; S_+ \cup S_-]$,

1. *w is conormal to all six boundary hypersurfaces, and*
2. *w is partially polyhomogeneous (i.e., $w \in \mathcal{A}^{\mathcal{E}}_{\mathrm{pphg}}([M; S_+ \cup S_-])$) at all boundary hypersurfaces other than* cf *with index sets*

$$\mathcal{E} = \begin{cases} \emptyset & \text{at } C_0 \\ \left\{\left(-i\left(\frac{n-1}{2} + j\right), 0\right) \mid j = 0, 1, 2, \dots\right\} & \text{at } \mathcal{I}^+, \mathcal{I}^- \\ \left\{\left(-i\left(\frac{n}{2} + k + \sqrt{\left(\frac{n-2}{2}\right)^2 + \mu_j^2}\right), 0\right) \mid j, k = 0, 1, 2, \dots\right\} & \text{at } C_+, C_- \end{cases},$$

where μ_j^2 are those eigenvalues of Δ_k on Z so that

$$\sqrt{\left(\frac{n-2}{2}\right)^2 + \mu_j^2} \notin \frac{1}{2} + \mathbb{Z}.$$

In terms of the radiation field $\mathcal{R}_+[w]$, the expansion at C_+ implies the expansion in Theorem 1.1. Theorem 9.1 is stronger than Theorem 1.1, as it implies a joint asymptotic expansion at $C_+ \cap \mathcal{I}^+$.

The proof follows the same outline as in the setting of asymptotically Minkowski spaces to obtain the existence of the asymptotic expansion; the key missing steps require extending the propagation and Fredholm statements near the conic singularities and are formalized in Propositions 7.1 and 8.1. As the same approach works here, we provide only an abbreviated sketch.

Our strategy is to show first that the solution is partially polyhomogeneous. As the initial data are compactly supported, finite speed of propagation implies that the solution is trivial

near C_0. The finite speed of propagation also allows us to replace w with χw, where χ is a smooth cutoff function to a neighborhood of $\overline{C_+}$ in M; χw is then the forward solution of an *inhomogeneous* wave equation on $\mathbb{R}\times C(Z)$. We show that χw is partially polyhomogeneous on the blown-up space $[M;S_+]$ and an identical argument near $\overline{C_-}$ then establishes the claim for w. Establishing the partial polyhomogeneity of w has as its byproduct a proof that the index sets at $\mathcal{I}^\pm$ are as stated. Finally, we establish that the exponents seen in the expansion at $C_\pm$ can be characterized as resonances associated to the hyperbolic cone with the same link. It suffices to show this for the forward solution as the backward solution has the same form near $\mathcal{I}^-\cup C_-$.

We therefore begin by considering the equation

$$\Box_g w = f'$$

on M°, where $f'\in C_c^\infty$ and suppose w is the forward solution. By translating in time and replacing w with χw, we may assume that f' (and therefore w) is supported in the forward light cone $\{t>r\}$ and in $\{t>1\}$. With ρ denoting a defining function[17] for mf and x a defining function for cf, we consider the conjugated equation

$$Lu = f,$$

where

$$\begin{aligned} L &\equiv \rho^{-\frac{n-1}{2}-2}\Box_g\rho^{\frac{n-1}{2}},\\ u &= \rho^{-\frac{n-1}{2}}w\in C^{-\infty}(M),\\ f &= \rho^{-\frac{n-1}{2}-2}f'\in C_c^\infty(M^\circ). \end{aligned}$$

This conjugation and rescaling transform $\Box_g$ into L, a "wedge-b-differential-operator", i.e., a b-differential operator at mf and a wedge-type operator at cf. Note that the partial polyhomogeneity of u implies that of w with index sets shifted by $(n-1)/2$.

Due to the scaling invariance (in the variable ρ) of the metric, L agrees with its normal operator, so $N(L)\equiv L$. This observation greatly simplifies the analysis of the problem by eliminating remainder terms and thus allows us to avoid an additional iterative argument; the lack of remainder terms accounts for the absence of logarithmic terms in the expansions of Theorem 9.1.[18]

For convenience, we recall from Section 4 the form of the operator L in a neighborhood of $\overline{C_+}$ in the coordinate system given by $(\rho=1/t, x=r/t, z)$. As we eventually pass to the blown-up space $[M;S_+]$, it is often convenient to include a coordinate defining S_+. We therefore also include the coordinate systems $(\rho=1/t, v=(t-r)/t, z)$, which are valid in a neighborhood of S_+.

Near S_+, in the coordinate system given by $(\rho=1/t, x=r/t, z)$, L has the following form:

$$L = (\rho D_\rho + xD_x)^2 - ni\,(\rho D_\rho + xD_x) - D_x^2 + \frac{(n-1)i}{x}D_x - \frac{1}{x^2}\Delta_k - \frac{n^2-1}{4}.$$

Similarly, in terms of $(\rho=1/t, v=(t-r)/t, z)$, L takes the form

$$L = (\rho D_\rho - (1-v)D_v)^2 - ni\,(\rho D_\rho - (1-v)D_v) - D_v^2 - \frac{(n-1)i}{1-v}D_v - \frac{1}{(1-v)^2}\Delta_k - \frac{n^2-1}{4}.$$

[17]Near S_+, the primary region of interest, we recall that $\rho = t^{-1}$.

[18]If we instead perturb the spacetime metric, the remainder terms can be handled as in the asymptotically Minkowski setting [BVW18].

After applying the Mellin transform to the identity $Lu = f$, we obtain a family of equations[19]

$$\widetilde{P}_\sigma \widetilde{u}_\sigma = \widetilde{f}_\sigma,$$

where $\widetilde{P}_\sigma = \widehat{N}(L)$ is the reduced normal operator of L. As w vanishes near $\overline{C_-}$ in M, we may arrange that $\widetilde{u}_\sigma$ also vanishes in a neighborhood of $\overline{C_-}$ in $X = \mathrm{mf}$. In fact, as we are able to assume that f' and w are supported in the interior of the forward light cone $\{t > r\}$, we may further assume that $\widetilde{f}_\sigma$ and $\widetilde{u}_\sigma$ are supported in $\overline{C_+}$.

We start by showing that $\widetilde{u}_\sigma$ lies in the following space of conormal distributions:

Definition 9.2. Suppose $\widetilde{u}$ is a distribution on $X = \mathrm{mf}$. We say that $\widetilde{u} \in I^{(s)}(S_+)$ if

(1) $\widetilde{u} \in H^s_{\mathrm{b}}(X)$,
(2) away from S_+, $\widetilde{u} \in H^\infty_{\mathrm{b}}(X)$, and
(3) if $V_1, \dots, V_r$ are b-vector fields on X with principal symbols vanishing on N^*S_+, then $V_1 \dots V_r \widetilde{u} \in H^s_{\mathrm{b}}(X)$.

In other words, $\widetilde{u} \in I^{(s)}(S_+)$ if it lies in H^s_{b} and lies in $H^\infty_{\mathrm{b}}(X)$ away from S_+.

One consequence of Proposition 7.1 and mapping properties of the Mellin transform is the following proposition:

Proposition 9.3. *There are ς_0, s so that $\widetilde{u}_\sigma$ is holomorphic on the upper half-plane $\operatorname{Im}\sigma > -\varsigma$ taking values in $I^{(s-0)}(S_+)$ and obeys the following estimate for each N and each seminorm $\|\bullet\|$ on $I^{(s)}(S_+)$:*

$$\sup_{\operatorname{Im}\sigma > -\varsigma_0} \int_{\operatorname{Im}\sigma = C} \|\widetilde{u}_\sigma\|^2 \langle \sigma \rangle^N \, d(\operatorname{Re}\sigma) < \infty.$$

In order to aid in bookkeeping, we introduce a compact name for these spaces. In what follows, $\mathcal{H}(\Omega)$ refers to the space of holomorphic functions on the domain $\Omega \subset \mathbb{C}$.

Definition 9.4. For $\varsigma, s, \in \mathbb{R}$, we let $\mathbb{C}_\varsigma$ denote the upper half-plane $\operatorname{Im}\sigma > -\varsigma$ and then define

$$\mathcal{B}(\varsigma, s) = \mathcal{H}(\mathbb{C}_\varsigma) \cap \langle \sigma \rangle^{-\infty} L^\infty_{\operatorname{Im}\sigma} L^2(\mathbb{R}_{\operatorname{Re}\sigma}; I^{(s)}(S_+)).$$

In other words, $\mathcal{B}(\varsigma, s)$ consists of those g_σ holomorphic in $\sigma \in \mathbb{C}_\varsigma$ taking values in $I^{(s)}(S_+)$ so that for each seminorm on $I^{(s)}(S_+)$,

$$\int_{-\infty}^{\infty} \|g_{\mu+i\nu}\|^2_\bullet \langle \mu \rangle^{2k} \, d\mu$$

is uniformly bounded in $\nu > -\varsigma$.

Observe that because $f \in C^\infty_c(M^\circ)$, we have

$$\widetilde{f}_\sigma \in \mathcal{B}(C, s') \text{ for all } C, s'.$$

Proposition 9.3 can be restated as saying that there are ς_0, s so that $\widetilde{u}_\sigma \in \mathcal{B}(\varsigma_0, s - 0)$. We now turn our attention to its proof.

[19]Recall that u has already been localized to $\{\rho < 1\}$, so it is unnecessary to include an additional cut-off function here.

Proof of Proposition 9.3. Because $\rho^{(n-1)/2}u$ lies in some $H^{s,\gamma}_{\mathrm{b},\widetilde{\mathcal{D}}}(M)$, we have

$$\widetilde{u}_\sigma \in \mathcal{H}(\mathbb{C}_{\varsigma_0}) \cap \langle\sigma\rangle^{\max(0,-s)} L^\infty L^2(\mathbb{R}, H^s(\mathrm{mf})), \tag{15}$$

where $\varsigma_0 = \gamma - (n-1)/2$. By reducing γ, we may assume that $s + \gamma < 1/2$ so as to be able to apply the regularity results of Proposition 7.1. We may also arrange that $\widetilde{u}_\sigma$ vanishes in a neighborhood of $\overline{C_-}$ in mf because u vanishes near $\overline{C_-}$ in M.

Proposition 7.1 implies that w is jointly conormal to S_+ and cf and so by the mapping properties of the Mellin transform (see, e.g., an earlier work in this series [BVW15, Lemma 2.3]),

$$\widetilde{u}_\sigma \in \mathcal{B}(\varsigma_0, -\infty).$$

Interpolating with equation (15) yields the result. □

Having placed $\widetilde{u}_\sigma$ in the holomorphic conormal space $\mathcal{B}(\varsigma_0, s-0)$, we may begin the inversion procedure. Because

$$\widetilde{P}_\sigma \widetilde{u}_\sigma = \widetilde{f}_\sigma,$$

our aim is to invert $\widetilde{P}_\sigma$ and employ a contour-shifting argument to enlarge the domain of meromorphy for $\widetilde{u}_\sigma$.

By Proposition 8.1, $\widetilde{P}_\sigma^{-1}$ forms a meromorphic family in any strip in the complex plane (though its domain and range are dependent on the location of the strip). As $\widetilde{f}_\sigma$ is entire, writing $\widetilde{u}_\sigma = \widetilde{P}_\sigma^{-1}\widetilde{f}_\sigma$, we see that $\widetilde{u}_\sigma$ is meromorphic in any upper half-plane taking values in the $\mathcal{X}$ spaces. More precisely, we shift the contour N units to see that $\widetilde{u}_\sigma$ is meromorphic in the half plane $\operatorname{Im}\sigma > -\varsigma_0 - N$ with values in $\mathcal{X}^{s_{\mathrm{ftr}}}$, where $s_{\mathrm{ftr}}|_{\Lambda^+} < \frac{1}{2} - \varsigma_0 - N$. In particular, $\widetilde{u}_\sigma$ is meromorphic with values in $\langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; H_{\mathrm{b}}^{\min(s-0,\frac{1}{2}-\varsigma_0-N)})$. On the other hand, since $\widetilde{P}_\sigma$ maps the expression to a conormal space, it must in fact take values in the conormal space

$$\langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(\min(s-0,\frac{1}{2}-\varsigma_0-N-0))}),$$

by propagation of the propagation results of Section 8.2 as well as the first case of Theorem 6.3 of Haber–Vasy [HV15], which concerns the propagation of Lagrangian regularity into conic Lagrangian submanifolds of radial points.

We have therefore shown that for any N,

$$\widetilde{u}_\sigma \in \mathcal{B}(\varsigma_0 + N, \min(s-0, 1/2 - \varsigma_0 - N - 0)) + \sum_{\substack{(\sigma_j,m_j)\in\mathcal{E}_0 \\ -\varsigma_0 > \operatorname{Im}\sigma_j > -\varsigma_0 - N}} (\sigma - \sigma_j)^{-m_j} a_j,$$

where $\mathcal{E}_0$ is the set of poles of P_σ^{-1} and

$$a_j \in \mathcal{B}(\varsigma_0 + N, \operatorname{Im}\sigma_j + 1/2 - 0).$$

After inverting the Mellin transform, we conclude that u enjoys a partial asymptotic expansion. In fact, on M, we have

$$u = \sum_{\substack{(\sigma_j,k)\in\mathcal{E}_0 \\ -\varsigma_0 > \operatorname{Im}\sigma_j > -\ell}} \rho^{i\sigma_j} (\log\rho)^k b_{jk} + u',,$$

where, for some $C = s + \varsigma_0$ (with s as in Proposition 9.3),

$$u' \in \rho^\ell H_{\mathrm{b}}^{\min(C-\ell-0, 1/2-\varsigma_0-\ell-0)}(M).$$

The coefficients b_{jk} are smooth functions of ρ taking values in $I^{(1/2+\operatorname{Im}\sigma_j-0)}$. Looking further into the asymptotic expansion of u, one finds that the coefficients and the remainder term are growing more singular owing to the radiation field "hiding" at S_+.

In fact, after blowing up S_+, Proposition 3.2 implies that the same arguments in the preceding discussion provide one step toward the joint partial polyhomogeneity of u. Indeed, u enjoys an asymptotic expansion at C_+ uniformly up to the corner $\mathrm{mf}\cap\mathcal{I}^+$ in $[M;S_+]$.

The other needed step involves estimates at $\mathcal{I}^+$. This argument relies on the observation that on M, the operators L and $2D_v\left(\rho D_\rho + vD_v\right)$ differ only by terms with additional vanishing at N^*S_+. The vector field $\rho D_\rho + vD_v$ lifts to the b-normal vector field for $\mathcal{I}^+$ in $[M;S_+]$. Writing

$$R = \rho D_\rho + vD_v,$$

the other step establishing the polyhomogeneity of u requires that u enjoy additional vanishing after the application of $(R+ik)\dots(R+i)R$.

We ignore for now the additional terms in L[20] and suppose $L = 2D_v(\rho D_\rho + vD_v) = 4D_vR$. As these statements are local to S_+, a simple argument with cut-off functions shows we are free to ignore the differentiation near the conic singularity ($x=0$).

As Lu is smooth and compactly supported, and $u\in H_{\mathrm{b}}^{s,\gamma}$, we know that $D_vRu\in H_{\mathrm{b}}^{s,\gamma}$. Because D_v is elliptic on $\mathrm{WF}_{\mathrm{b}}(u)$, it is microlocally invertible and so $Ru\in H_{\mathrm{b}}^{s+1,\gamma}$, i.e., Ru is one order better than u.

To continue this iterative process, observe that $RD_v = D_v(R+i)$, so that

$$\left(\prod_{j=0}^{k-1}(R+ij)\right)L = \left(\prod_{j=0}^{k-1}(R+ij)\right)D_vR = D_v\left(\prod_{j=0}^{k}(R+ij)\right).$$

An inductive argument then shows that

$$\left(\prod_{j=0}^{k}(R+ij)\right)u\in H_{\mathrm{b}}^{s+k+1,\gamma}(M),$$

so that $(R+ik)\dots(R+i)Ru$ enjoys $k+1$ additional orders of regularity at S_+.

As u is already conormal to S_+, measure of regularity there are essentially based on applications of D_v. The vector field vD_v is tangent to S_+ (and so can be applied to u as many times as we like), so we may interpret additional regularity at S_+ as additional vanishing at S_+.[21] This extra vanishing is precisely what is needed for the application of Proposition 3.2 and completes the bulk of the proof of Theorem 1.1.

We finally characterize the exponents seen in Theorems 1.1 and 9.1. As noted above, these exponents are the poles of $\widetilde{P}_\sigma^{-1}$ acting as an operator $\mathcal{X}^{s_{\mathrm{ftr}}}\to\mathcal{Y}^{s_{\mathrm{past}}}$.

As $\widetilde{P}_\sigma$ is the Mellin conjugate of L, we may write

$$\widetilde{P}_\sigma = -(1-x^2)D_x^2 - i(n+1+2i\sigma)xD_x + \frac{n-1}{x}iD_x - \frac{1}{x^2}\Delta_k + \sigma^2 - ni\sigma + \frac{n^2-1}{4}.$$

[20]Of course, these additional terms are always there. Managing these terms forms a sizable part of Section 9.2 of the previous paper [BVW18] and we refer the reader there for a thorough discussion.

[21]This interpretation can be formalized by an integration argument and requires keeping track of the factors of the module for which w already possesses regularity; see [BVW18, Section 9.2] for details.

In particular, in C_+ we have the following identity:

$$\left(1-x^2\right)^{\frac{n-1}{4}+i\frac{\sigma}{2}+1}\widetilde{P}_\sigma\left(1-x^2\right)^{-\frac{n-1}{4}-i\frac{\sigma}{2}} = \\ -\left((1-x^2)D_x\right)^2+i\frac{n-1}{x}(1-x^2)D_x-\frac{1-x^2}{x^2}\Delta_k+\left(\frac{n-1}{2}\right)^2+\sigma^2.$$

Taking $x=\tanh r$ identifies C_+ with the hyperbolic cone $C_{\mathrm{hyp}}(Z)$ over (Z,h); the conjugation above yields

$$\left(1-x^2\right)^{\frac{n-1}{4}+i\frac{\sigma}{2}+1}\widetilde{P}_\sigma\left(1-x^2\right)^{-\frac{n-1}{4}-i\frac{\sigma}{2}} = -\left(\Delta_{C_{\mathrm{hyp}}(Z)}-\left(\frac{n-1}{2}\right)^2-\sigma^2\right).$$

Using this identification, for $\widetilde{f}_\sigma$ compactly supported in C_+, a straightforward adaptation of the arguments[22] in previous work [BVW15, Section 7] shows that

$$\widetilde{P}_\sigma^{-1}f|_{C_+} = -\left(1-x^2\right)^{-\frac{n-1}{4}-i\frac{\sigma}{2}}R_{C_{\mathrm{hyp}}(Z)}(\sigma)\left((1-x^2)^{1+\frac{n-1}{4}+i\frac{\sigma}{2}}f\right).$$

Here $R_{C_{\mathrm{hyp}}(Z)}(\sigma)=(\Delta_{C_{\mathrm{hyp}}(Z)}-\frac{(n-1)^2}{4}-\sigma^2)^{-1}$ is the resolvent of the Laplacian on the hyperbolic cone that is invertible for $\operatorname{Im}\sigma\gg 0$. The exponents appearing in the expansion of u are therefore the poles of the resolvent on the hyperbolic cone; these poles were found explicitly in a previous paper of the authors [BM19].

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[22] In fact, this adaptation is not necessary; in this structured setting the operator $\widetilde{P}_\sigma^{-1}$ can be found explicitly in terms of hypergeometric functions, though we do not include it here.

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