Saturated and linear isometric transfer systems for cyclic groups of order $p^m q^n$

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A B S T R A C T

Transfer systems are combinatorial objects which classify N_∞ operads up to homotopy. By results of A. Blumberg and M. Hill [2], every transfer system associated to a linear isometries operad is also *saturated* (closed under a particular two-out-of-three property). We investigate saturated and linear isometric transfer systems with equivariance group $C_{p^m q^n}$, the cyclic group of order $p^m q^n$ for p, q distinct primes and $m, n \geq 0$. We give a complete enumeration of saturated transfer systems for $C_{p^m q^n}$. We also prove J. Rubin's *saturation conjecture* for C_{pq^n}; this says that every saturated transfer system is realized by a linear isometries operad for p, q sufficiently large (greater than 3 in this case).

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1. Introduction

Fix a finite group G. Highly structured commutative G-equivariant ring spectra can support multiplicative norm maps associated with a class of finite H-sets, H ranging through subgroups of G. The G-N_∞ operads of A. Blumberg and M. Hill [2] parametrize ring structures with such an admissible family of norms. Following the work of [2,4,7,10], J. Rubin [11] and S. Balchin, D. Barnes, and C. Roitzheim [1] independently prove that the homotopy category of G-N_∞ operads is equivalent to the combinatorially-defined category of G-transfer systems (see Theorem 2.6). The structure of the lattice of transfer systems on an Abelian group was recently explored in [5].

The transfer systems induced by certain natural families of G-N_∞ operads have additional special properties. In particular, any transfer system realized by an equivariant linear isometries operad (see Definition 2.3) is *saturated* (see Definition 2.5). It is not the case, though, that every saturated transfer system arises in this fashion, as shown in [11]. This raises two fundamental questions which we address in this paper:

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(1) For a given group G, how many saturated G-transfer systems exist?
(2) For a given group G, which saturated G-transfer systems can be realized by a linear isometries operad?

We say that a (necessarily saturated) transfer system realized by a linear isometries operad is *linear isometric*; thus the second question may be rephrased as asking "Which saturated G-transfer systems are linear isometric?"

Fix p, q distinct primes and let $C_{p^m q^n}$ denote the cyclic group of order $p^m q^n$, for $m, n \geq 0$. Let $s(m, n)$ denote the number of saturated $C_{p^m q^n}$-transfer systems.[1] We provide the following answers to the above questions for $G = C_{p^m q^n}$:

Theorem 1.1 *(see Theorem 4.9 and Theorem 4.13). For all $m, n \geq 0$,*

$$s(m,n) = \sum_{j=2}^{m+2} (-1)^{m-j} \begin{Bmatrix} m+1 \\ j-1 \end{Bmatrix} \frac{j!}{2} j^n,$$

where $\begin{Bmatrix} r \\ s \end{Bmatrix}$ denotes the Stirling number of the second kind enumerating s-block partitions of a set of cardinality r. Furthermore, the exponential generating function for $s(m, n)$ takes the form

$$\sum_{m,n \geq 0} \frac{s(m,n)}{m! n!} x^m y^n = \frac{e^{2x+2y}}{(e^x + e^y - e^{x+y})^3}.$$

Theorem 1.2 *(see Theorem 5.5). Let $n \geq 0$. For $G = C_{p q^n}$, and $p, q > 3$, all saturated transfer systems are linear isometric.*

The latter theorem verifies an instance of Rubin's saturation conjecture, which loosely says that every saturated transfer system for a cyclic group of order n is linear isometric as long as the prime divisors of n are sufficiently large; see Conjecture 2.8 for a precise statement.

Remark 1.3. E. Franchere, the third and fourth authors of this paper, W. Qin, and R. Waugh expose a surprising connection between transfer systems and weak factorization systems (in the sense of abstract homotopy theory) in [5]. Under this correspondence, it turns out that saturated transfer systems are in bijection with model structures on the poset category $\mathrm{Sub}(G)$ of subgroups of G for which all morphisms are fibrations. As such, Theorem 4.9 and Theorem 4.13 give a complete enumeration of such model structures as well. The details of the relation between transfer systems and model structures are explained in [3].

Organization

In Section 2, we recall the definitions of and fundamental theorems regarding N_∞ operads and (saturated) transfer systems. In Section 3, we begin our study of saturated transfer systems on $C_{p^m q^n}$ and prove some structural results about these objects. In particular, we reduce their study to a combinatorial game on the $m \times n$ grid. We use this description in Section 4 to complete our enumeration of saturated transfer systems on $C_{p^m q^n}$. This takes three forms: a recurrence, a closed formula, and an exponential generating function. Finally, in Section 5 we prove the saturation conjecture for $C_{p q^n}$.

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[1] After the definition is presented, it will be clear that this number is independent of p and q.

Acknowledgments

2. N_∞ operads and transfer systems

2.1. N_∞ operads

In order to frame our work on transfer systems, we need to recall Blumberg–Hill's notion of an N_∞ operad [2], paying special attention to the example of linear isometry operads. We assume that the reader is familiar with the basic theory of (symmetric) operads. For $n \geq 0$, we denote the symmetric group on n letters by $\mathfrak{S}_n$.

Definition 2.1. A G-N_∞ *operad* is a symmetric operad $\mathscr{O}$ in G-spaces satisfying the following three properties:

☞ for all $n \geq 0$, the $G \times \mathfrak{S}_n$-space $\mathscr{O}(n)$ is $\mathfrak{S}_n$-free,
☞ for every $\Gamma \leq G \times \mathfrak{S}_n$, the Γ-fixed point space $\mathscr{O}(n)^\Gamma$ is empty or contractible, and
☞ for all $n \geq 0$, $\mathscr{O}(n)^G$ is nonempty.

A *map of G-N_∞ operads* $\varphi \colon \mathscr{O}_1 \to \mathscr{O}_2$ is a morphism of operads in G-spaces. We denote the associated category of G-N_∞ operads by N_∞-$\mathbf{Op}^G$.

For a map $\varphi \colon \mathscr{O}_1 \to \mathscr{O}_2$ of G-N_∞ operads, the map at level n is in particular $G \times \mathfrak{S}_n$-equivariant. We say that φ is a *weak equivalence* if $\varphi \colon \mathscr{O}_1(n)^\Gamma \to \mathscr{O}_2(n)^\Gamma$ is a weak homotopy equivalence of topological spaces for all $n \geq 0$ and $\Gamma \leq G \times \mathfrak{S}_n$. The associated homotopy category (formed by inverting weak equivalences) is denoted $\mathrm{Ho}(N_\infty$-$\mathbf{Op}^G)$.

Note that N_∞ operads are, in particular, nonequivariant E_∞ operads, and thus parametrize operations that are associative and commutative up to higher homotopies. Additionally, N_∞ operads *admit norms* for particular finite H-sets, $H \leq G$ in the following sense. Given an H-set T, let $\Gamma(T) \leq G \times \mathfrak{S}_{|T|}$ denote the graph of a permutation representation of T.

Definition 2.2. A G-N_∞ operad $\mathscr{O}$ *admits norms* for a finite H-set T when $\mathscr{O}(|T|)^{\Gamma(T)}$ is nonempty.

Particular examples of N_∞ operads include linear isometries operads defined on G-universes [9]. We recall the definition.

Definition 2.3. A *G-universe U* is a countably infinite-dimensional real G-inner product space such that it contains each finite-dimensional subrepresentation infinitely often and contains the trivial representation. The *linear isometries operad* $\mathcal{L}(U)$ is given at level n by the space $\mathcal{L}(U^n, U)$ of all (not necessarily equivariant) linear isometries, with G acting by conjugation, and $\mathfrak{S}_n$ acting by permuting inputs. The operadic composition is given by composition of isometries.

2.2. Transfer systems

A G-transfer system is a combinatorial object defined as a particular sub-poset of $\operatorname{Sub}(G)$, the subgroup lattice of G. For $H \leq G$ and $g \in G$, let $^gH = gHg^{-1}$ denote the g-conjugate of H.

Definition 2.4. A *G-transfer system* is a relation $\to$ on $\operatorname{Sub}(G)$ that refines the inclusion relation[2] and satisfies the following properties:

- ☞ (reflexivity) $H \to H$ for all $H \leq G$,
- ☞ (transitivity) $K \to H$ and $L \to K$ implies $L \to H$,
- ☞ (closed under conjugation) $K \to H$ implies that $^gK \to {}^gH$ for all $g \in G$,
- ☞ (closed under restriction) $K \to H$ and $M \leq H$ implies $(K \cap M) \to M$.

We denote the collection of transfer systems by $\operatorname{Tr}(G)$ and view it as a poset under the refinement relation.

In other words, a transfer system is a sub-poset of $\operatorname{Sub}(G)$ where the relation is closed under conjugation and restriction. Note that conjugation is trivial when G is Abelian; in this case transfer systems only depend on the lattice structure of $\operatorname{Sub}(G)$. Saturated transfer systems have an additional two-out-of-three property:

Definition 2.5. A G-transfer system $\to$ is *saturated* if it additionally satisfies the following property:

- ☞ (two-out-of-three) if $L \leq K \leq H \leq G$ and two of the three relations $L \to K$, $L \to H$, $K \to H$ hold, then so does the third.

We denote the collection of saturated transfer systems by $\operatorname{STr}(G)$.

By transitivity and closure under restriction, the two-out-of-three property may be rephrased as follows:

$$\text{if } L \leq K \leq H \text{ and } L \to H, \text{ then } K \to H.$$

The link between N_∞ operads and transfer systems is provided by the following construction. Given a G-N_∞ operad $\mathscr{O}$, define the relation $\to_{\mathscr{O}}$ by the rule

$$K \to_{\mathscr{O}} H \text{ if and only if } K \leq H \text{ and } \mathscr{O}([H : K])^{\Gamma(H/K)} \neq \varnothing$$

where $\Gamma(H/K)$ is the graph of some permutation representation $H \to \mathfrak{S}_{[H:K]}$ of H/K.

Theorem 2.6. *The assignment*

$$N_\infty\text{-}\mathbf{Op}^G \longrightarrow \operatorname{Tr}(G)$$
$$\mathscr{O} \longmapsto (\to_{\mathscr{O}})$$

induces an equivalence

$$\operatorname{Ho}(N_\infty\text{-}\mathbf{Op}^G) \simeq \operatorname{Tr}(G)$$

(considering the poset $\operatorname{Tr}(G)$ as a category). Moreover, if $\mathscr{O}$ is a linear isometries operad, then $\to_{\mathscr{O}}$ is saturated.

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[2] This means that $K \to H$ implies $K \leq H$.

Remark 2.7. In [2], Blumberg and Hill defined G-indexing systems, which are collections of finite H-sets for varying subgroups H of G satisfying certain properties. These collections form a poset $\mathcal{I}(G)$ under inclusion. They proved that every N_∞ operad $\mathcal{O}$ gives rise to an indexing system, and that this assignment gives a functor that descends to the homotopy category, with the resulting functor being full and faithful. Blumberg and Hill further conjectured that the functor is surjective, which was established independently by P. Bonventre and L. Pereira [4], J. Gutiérrez and D. White [7], and Rubin [10]. Both Rubin [11] and Balchin, Barnes, and Roitzheim [1] proved that the poset $\mathcal{I}(G)$ of indexing systems is isomorphic to the poset $\mathrm{Tr}(G)$ of transfer systems. The result about linear isometries operads is the translation of the corresponding statement for indexing systems (cf. [2, p. 678]) into the language of transfer systems (cf. [11, Theorem 3.7]).

Theorem 2.6 gives the precise link between N_∞ operads and transfer systems, and explains the inclusion

$$\{\text{linear isometric } G\text{-transfer systems}\} \subseteq \mathrm{STr}(G).$$

As Rubin points out in [11, §5.1], it was initially expected that the reverse inclusion would hold as well. As examples in *op. cit.* show, this is not generally true. Nonetheless, Rubin conjectures that when G is cyclic and $|G|$ has large prime divisors, every saturated transfer system is linear isometric:

Conjecture 2.8 *(Rubin). Fix a sequence of positive integers $r_1, \ldots, r_k$. Then for distinct sufficiently large primes $p_1, \ldots, p_k$, every saturated transfer system on $C_{p_1^{r_1} \cdots p_k^{r_k}}$ can be realized by a linear isometries operad.*

Rubin verified the conjecture for C_{p^n} for all $n \geq 1$ and for C_{pq}. In Section 5, we verify the conjecture for C_{pq^n} for all $n \geq 1$.

2.3. Generating (saturated) transfer systems

Recall that for a poset P, we say that $x < y$ is a *cover relation* if there is no $z \in P$ such that $x < z < y$. The saturation property implies that saturated transfer systems can be described in terms of the cover relations it contains.

Definition 2.9. Let R be a binary relation on $\mathrm{Sub}(G)$ that refines inclusion. The transfer system *generated* by R, denoted by $\langle R \rangle$, is the minimal transfer system that contains R. An explicit description can be found in [11, Construction A.1].

Proposition 2.10 *([11, Proposition 5.8]). Let $\rightarrow$ be a saturated G-transfer system. Then $\rightarrow$ is generated by the relation*

$$\{(K, H) \mid K \rightarrow H \text{ and } (K, H) \text{ is a cover relation in } \mathrm{Sub}(G)\}.$$
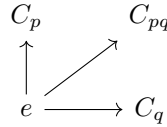
As a result, a saturated G-transfer system is uniquely determined by the cover relations in $\mathrm{Sub}(G)$ it contains.

Example 2.11. The saturated C_{pq}-transfer system

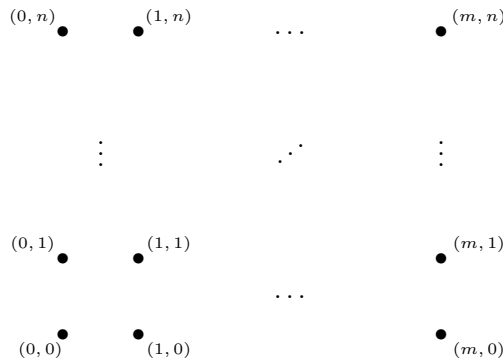is generated by its cover relations, *i.e.*, the edges of the square.

Note that a general G-transfer system is not necessarily generated by a set of cover relations, as the following example ("the chickenfoot") illustrates for $G = C_{pq}$.

$$
\begin{array}{ccc}
C_p & & C_{pq} \\
\uparrow & \nearrow & \\
e & \longrightarrow & C_q
\end{array}
$$

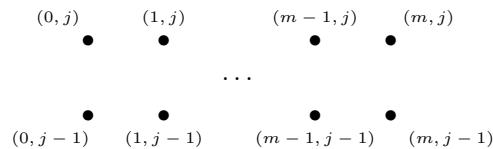## 3. Saturated transfer systems on $C_{p^m q^n}$

In this section we concentrate on studying transfer systems on the group $C_{p^m q^n}$. In what follows, for $k \geq 0$, we denote by $[k]$ the poset $\{0 < 1 < \cdots < k\}$.

The subgroup lattice $\mathrm{Sub}(C_{p^m q^n})$ is isomorphic to the grid $[m] \times [n]$, with the subgroup $C_{p^i q^j}$ corresponding to (i, j). For ease of notation, we will use this identification when referring to transfer systems on $C_{p^m q^n}$.

The following is meant to clarify how we will define and refer to rows and columns of the grid.

Notation 3.1. In a $m \times n$ grid, *row j* will refer to the edges between $(i, j-1)$ and (i, j) for $i = 0, \ldots, m$. As such, there are n *rows* in the grid, numbered from 1 to n (there is no row 0). Row j is shown below.
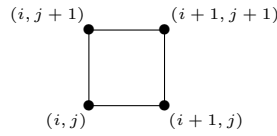

Similarly, column i will refer to the edges between $(i-1, j)$ and (i, j) for $j = 0, \ldots, n$. There are m columns, numbered from 1 to m.

3.1. Characterizing saturated transfer systems for $C_{p^m q^n}$

We now characterize the sets of cover relations within saturated transfer systems for $C_{p^m q^n}$. As above, we denote the subgroup $C_{p^i q^j}$ by the pair (i, j). The cover relations in this case are of the form $(i, j) \to (i+1, j)$ and $(i, j) \to (i, j+1)$, that is, the edges of the grid.

Theorem 3.2. *Let S be a set of cover relations within the lattice $[m] \times [n] \cong \mathrm{Sub}(C_{p^m q^n})$. Then S is the set of all cover relations within a saturated transfer system if and only if the following conditions are satisfied:*

(1) if $(i, j) \to (i+1, j)$ is in S, then $(i, k) \to (i+1, k)$ for all $k < j$;

(2) if $(i, j) \to (i, j+1)$ is in S, then $(k, j) \to (k, j+1)$ for all $k < i$;

(3) if three out of the four edges in the square

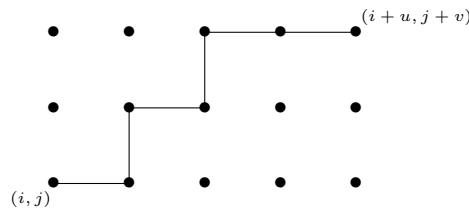*are in S, then so is the fourth.*

Moreover, if S satisfies the above conditions and $\mathscr{F} = \langle S \rangle$, then $\mathscr{F}$ is saturated and S is precisely the set of cover relations in $\mathscr{F}$.

Proof. We first prove the forward direction. Let $\mathscr{F} = \to$ be a saturated transfer system and let S be its set of cover relations. Conditions (1) and (2) follow from the restriction axiom on transfer systems. For condition (3), suppose that S contains three edges of the square. Then S either contains $(i, j) \to (i+1, j)$ and $(i+1, j) \to (i+1, j+1)$, or $(i, j) \to (i, j+1)$ and $(i, j+1) \to (i+1, j+1)$. In either case, by transitivity, $\mathscr{F}$ must contain $(i, j) \to (i+1, j+1)$. Thus, by restriction and saturation, $\mathscr{F}$, and hence S, must contain all the edges of the square.

To prove the backwards direction, let S be a set of cover relations satisfying the conditions, and consider the transfer system $\mathscr{F} = \langle S \rangle$ it generates. We will prove that $\mathscr{F}$ is saturated, and that S is precisely the set of cover relations within $\mathscr{F}$.

Recall that $\mathscr{F}$ is constructed in general by first closing S under restriction, and then closing under transitivity. Conditions (1) and (2) imply that ignoring identities, S itself is already closed under restriction. Indeed, suppose $(i, j) \to (i+1, j)$ is in S. Then restriction with respect to (a, b) for $a \leq i+1$ and $b \leq j$ gives $(i, b) \to (i+1, b)$ if $a = i+1$, and gives $(a, b) \to (a, b)$ otherwise. A similar consideration follows for vertical edges. Thus $\mathscr{F}$ is constructed by closing S under transitivity, and hence, the set of cover relations in $\mathscr{F}$ is precisely S.

To prove $\mathscr{F}$ is saturated, it suffices to prove that if the edge $(i, j) \to (i+u, j+v)$ is in $\mathscr{F}$, then all the horizontal and vertical edges of the corresponding $u \times v$ grid are in S. We proceed by induction on (u, v), with the base case $(0, 0)$ being trivially satisfied. Suppose the statement is true for $(u, v-1)$ and $(u-1, v)$, unless $v = 0$ or $u = 0$, in which case we only assume the one that makes sense. If $(i, j) \to (i+u, j+v)$ is in $\mathscr{F}$, then there is a path of cover relations in S that starts at (i, j) and ends at $(i+u, j+v)$. Assume without loss of generality that the last step of the path is the horizontal edge $(i+u-1, j+v) \to (i+u, j+v)$.
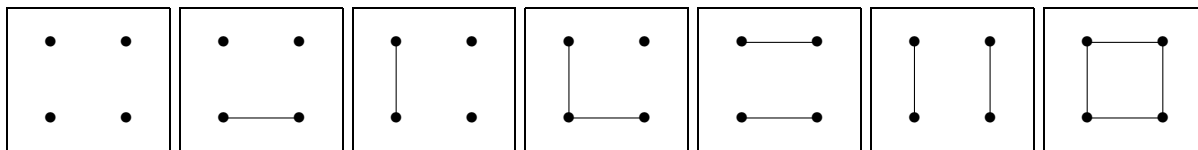
Then there is a path of cover relations from (i, j) to $(i+u-1, j+v)$, which implies $(i, j) \to (i+u-1, j+v)$ is in $\mathscr{F}$, and by the inductive hypothesis we have that all the edges of the $(u-1) \times v$ grid are in S. Furthermore, by condition (1), all edges $(i+u-1, k) \to (i+u, k)$ for $k \leq j+v$ are in S. Now, the squares in the last column of the grid have three of their edges in S, so by condition (3) the fourth edge must be in S as well, thus showing that all the edges of the $u \times v$ grid are in S. $\quad\square$

Definition 3.3. Let S be a subset of cover relations within the lattice $[m] \times [n]$. If S satisfies the conditions of Theorem 3.2, we say that S is a *saturated cover*. We denote the set of saturated covers for $[m] \times [n]$ by $\mathrm{SCov}(m, n)$.

Corollary 3.4. *There is a bijection between saturated transfer systems on $C_{p^m q^n}$ and $\mathrm{SCov}(m, n)$ given by $S \mapsto \langle S \rangle$.*

Proof. This follows from Proposition 2.10 and Theorem 3.2. $\square$

Remark 3.5. A set S of cover relations is a saturated cover if and only if the restriction of S to each of the 1×1 squares in the grid is a saturated transfer system, i.e., it is one of the following seven options.

Note that by (1) and (2) of Theorem 3.2 it is sufficient to record the highest horizontal edge in each column and the rightmost vertical edge in each row. Equivalently, we can record the number of horizontal edges in each column and the number of vertical edges in each row. This along with (3) of Theorem 3.2 allows us to demonstrate a bijection which leads us to encode $C_{p^m q^n}$-saturated transfer systems much more compactly in terms of *compatible codes* as follows.
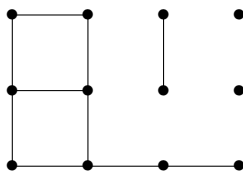
Definition 3.6. Let S be a saturated cover on $[m] \times [n]$. For $1 \le i \le m$, $1 \le j \le n$,

$$a_i = |\{k \colon (i-1, k) \to (i, k) \in S\}|$$
$$b_j = |\{k \colon (k, j-1) \to (k, j) \in S\}|$$

In other words, a_i is the number of horizontal edges in column i, which can range from 0 to $n + 1$, and b_j is the number of vertical edges in row j, which can range from 0 to $m + 1$. We call $a = (a_1, \ldots, a_m)$ the *horizontal code* of S, and $b = (b_1, \ldots, b_n)$ the *vertical code* of S.

Example 3.7. Consider the following saturated cover S on $[3] \times [2]$.

The horizontal code of S is $(3, 1, 1)$ and the vertical code of S is $(2, 3)$.

Definition 3.8. Let $m, n \ge 0$. A pair of *compatible codes* on $[m] \times [n]$ is a pair $(a_1, \ldots, a_m), (b_1, \ldots, b_n)$ of tuples of integers such that $0 \le a_i \le n + 1$, $0 \le b_j \le m + 1$, and
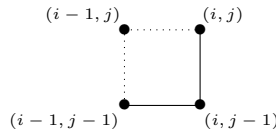
$$b_{a_i} \le i \quad \text{and} \quad a_{b_j} \le j,$$

whenever these are defined.

Example 3.9. Consider the same saturated cover S on $[3] \times [2]$ as in Example 3.7. In this case, the pair of codes $(3, 1, 1), (2, 3)$ are compatible since $a_{b_1} = a_2 = 1 \leq 1$ and $a_{b_2} = a_3 = 1 \leq 2$. At the same time, $b_{a_1} = b_3$ but b_3 is not defined. On the other hand, $b_{a_2} = b_1 = 2 \leq 2$ and $b_{a_3} = b_1 = 2 \leq 3$. The essential idea is to express Condition (3) of Theorem 3.2 for saturated transfer systems in terms of codes, as the following Proposition shows.

Proposition 3.10. *There is a bijection between the set of saturated covers and pairs of compatible codes on* $[m] \times [n]$.

Proof. Let $(a_1, \ldots, a_m), (b_1, \ldots, b_n)$ be the horizontal and vertical codes assigned to a saturated cover S, fix $i = 1, \ldots m$, and let $j = a_i$. If $j = 0$ or $n + 1$, b_j is undefined. Otherwise, consider the following square in the grid.

The fact that $a_i = j$ implies that $(i-1, j-1) \to (i, j-1)$ is in S, while $(i-1, j) \to (i, j)$ is not, as indicated in the picture. If $b_j > i$, then both vertical edges of the square must be in S, violating the 3-out-of-4 condition, thus $b_j \leq i$. The argument for the other inequality is symmetric, thus proving that horizontal and vertical codes are compatible.

Conversely, given compatible codes $(a_1, \ldots, a_m), (b_1, \ldots, b_m)$, consider the cover relation

$$S = \{(i-1, j) \to (i, j) \mid 1 \leq i \leq m \text{ and } j < a_i\} \cup \{(i, j-1) \to (i, j) \mid 1 \leq j \leq n \text{ and } i < b_j\}.$$

By construction S satisfies conditions (1) and (2), and just as above, the inequality constraints imply condition (3). These two constructions are inverses of each other, thus establishing the bijection. $\square$

4. Enumeration of saturated transfer systems on $C_{p^m q^n}$

The main result of this section is a closed formula for the number of saturated transfer systems on $C_{p^m q^n}$. Throughout we denote the number of saturated transfer systems for $C_{p^m q^n}$ as $s(m, n)$. We use the concept of saturated covers of Definition 3.3 and we denote the set of saturated covers for $C_{p^m q^n}$ as $\mathrm{SCov}(m, n)$ so that $s(m, n) = |\mathrm{SCov}(m, n)|$.

4.1. Recursive formula

We first prove a recursive formula for the number of saturated transfer systems. This recursive formula allows us to prove a closed formula for the number of saturated transfer systems that depends on m and n in Section 4.2.

Theorem 4.1. *Let* $m, n \geq 0$. *Then*

$$s(m, n+1) = s(m, n) + \sum_{k=0}^{m} \binom{m+1}{k} s(k, n).$$

In essence, we build saturated covers on $[m] \times [n+1]$ out of saturated covers on $[k] \times [n]$, where k ranges from 0 to m. In order to do so, we partition the set of saturated covers on $[m] \times [n+1]$ using the following
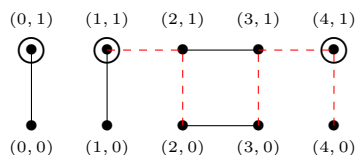
construction. Let $\mathcal{P}[m]$ denote the power set of $[m] = \{0, 1, \ldots, m\}$. For the remainder of the section, we fix $m, n \geq 0$.

Construction 4.2. We construct a function $c\colon \mathrm{SCov}(m, n+1) \to \mathcal{P}[m]$ as follows. Let S be a saturated cover, and suppose $(k, n) \to (k, n+1)$ is the rightmost vertical edge on the $(n+1)$-th row of S. That is, $(k, n) \to (k, n+1)$ is in S, but $(k+1, n) \to (k+1, n+1)$ is not. If S has no vertical edges on the $(n+1)$-th row, we set $k = -1$. Then we define

$$c(S) = \{0, 1, \ldots, k\} \cup \{i \mid k+1 < i \leq m \text{ and } (i-1, n+1) \to (i, n+1) \notin S\}.$$

We will prove Theorem 4.1 by enumerating the fibers of c. We first give an example to explain the information encoded by c.
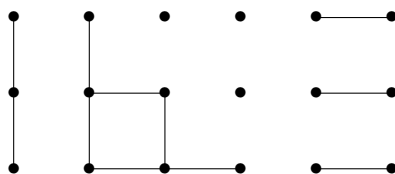
Remark 4.3. Let S be a saturated cover on $[4] \times [1]$ with $c(S) = \{0, 1, 4\}$. The figure below shows S, with edges that S must contain in solid black, and the edges S cannot contain in dashed red. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.) The vertices corresponding to the elements in $c(s)$ are marked with a circle.
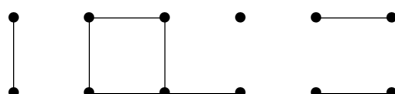

Note that the minimal element in the complement of $c(S)$, in this case 2, corresponds to $k+1$ in the construction above, and as such, it is the leftmost vertex without a vertical edge. This explains the dashed red edges $(i, 0) \to (i, 1)$ for $i = 2, 3, 4$. For $i > k+1$, $(i, n+1)$ is the target of a horizontal edge if and only if i is not in $c(S)$, which explains the solid black edge $(2, 1) \to (3, 1)$. By the saturated condition, this last edge implies that S must contain $(2, 0) \to (3, 0)$. Moreover, note that $(1, 1) \to (2, 1)$ cannot be in S; indeed, if it is, then $(1, 0) \to (2, 0)$ must be in S as well, violating 3-out-of-4.

The rest of the horizontal edges may or may not be there, with the caveat that the edge $(0, 1) \to (1, 1)$ is in S if and only if $(0, 0) \to (1, 0)$ is there too. Thus, the remaining bottom horizontal edges determine S.

Example 4.4. We will now illustrate a certain collapsing operation on saturated transfer systems to obtain saturated transfer systems for groups of smaller order. This will be key to the proof of Proposition 4.5. Let S be the saturated cover generating the following saturated transfer system on $[5] \times [2]$.

This saturated transfer cover S has $c(S) = \{0, 1\} \cup \{3, 4\}$ where $\{0, 1\}$ results from rightmost vertical $(1, 1) \to (1, 2)$ and $\{3, 4\}$ results since $(2, 2) \to (3, 2), (3, 2) \to (4, 2) \notin S$. To obtain a smaller saturated transfer system, we collapse the top row to obtain the following.

Then from the remaining column, we remove all columns which are indexed by i such that $i > k+1$ and in $[m] \setminus A = \{2, 5\}$ to give the below saturated transfer system.

Proposition 4.5. *Let $A \subseteq [m]$ and $c\colon \mathrm{SCov}(m, n+1) \to \mathcal{P}[m]$ as in Construction 4.2. Then*

$$|c^{-1}(A)| = \begin{cases} s(|A|, n) & \text{if } A \subsetneq [m], \\ s(m, n) & \text{if } A = [m]. \end{cases}$$

Proof. Fix $A \subsetneq [m]$. For notational convenience, let $\ell = |A|$, and let $k+1$ be the minimal element in $[m] \setminus A$. We construct a bijection between $c^{-1}(A)$ and saturated covers on the $[\ell] \times [n]$ grid.

Let T be a saturated cover on $[\ell] \times [n]$. We construct a saturated cover T^* on $[m] \times [n+1]$ as follows:

- ☞ For $i > k+1$ in $[m] \setminus A$ and all j, include the horizontal edges $(i-1, j) \to (i, j)$.
- ☞ Take T and expand it horizontally, so that the gaps coincide precisely with the columns filled in the previous step. When expanding, the vertical edges are repeated at both ends.
- ☞ For $i = 0, \ldots, k$, include the vertical edges $(i, n) \to (i, n+1)$.
- ☞ For $i = 1, \ldots, k$, include the horizontal edge $(i-1, n+1) \to (i, n+1)$ if and only if $(i-1, n) \to (i, n)$ is in T.

By construction, T^* is a saturated cover, $c(T^*) = A$, and $(T^*)' = T$. For a saturated cover $S \in c^{-1}(A)$, the considerations in Remark 4.3 show that $(S')^* = S$, thus proving that $(\ )'$ and $(\ )^*$ are inverse bijections.

Conversely, to $S \in c^{-1}(A)$ we assign the set of cover relations S' on $[\ell] \times [n]$ obtained by removing the top row and collapsing the columns indexed by all $i > k+1$ such that $(i-1, n+1) \to (i, n+1) \in S$. As noted in Remark 4.3, the horizontal $(i-1, n+1) \to (i, n+1) \in S$, and thus, $(i-1, j) \to (i, j) \in S$ for all $j \in [n]$. By the 3-out-of-4 condition, it follows that the left vertical boundary of the i-th column is identical to its right vertical boundary; thus this collapsing operation is well-defined. Note that we are collapsing precisely $(m+1) - |A| - 1$ columns, to obtain a grid with $|A| = \ell$ columns. Moreover, every 1×1 square in S' was a square in S, and hence is saturated. This implies S' is saturated.

In the case that $A = [m]$, if $S \in c^{-1}(A)$, we have that S contains all the vertical edges $(i, n) \to (i, n+1)$ for $i = 0, \ldots m$. Thus the horizontal edges $(i-1, n+1) \to (i, n+1)$ are determined by $(i-1, n) \to (i, n)$. Thus there is a bijection between $c^{-1}(A)$ and saturated covers on $[m] \times [n]$ obtained by removing the top row. □

We can now prove the recursion stated in Theorem 4.1:

$$s(m, n+1) = s(m, n) + \sum_{k=0}^{m} \binom{m+1}{k} s(k, n).$$

Proof of Theorem 4.1. We partition the set of saturated covers on $[m] \times [n+1]$ according to the fibers of the function c. Since there are $\binom{m+1}{k}$ subsets of $[m]$ of cardinality k, the formula above follows from Proposition 4.5. □

Example 4.6. We now illustrate the above theorem using the saturated cover as in Remark 4.3. By the collapsing operation described above, we can see that we obtain the following saturated transfer system,

•   •   •   •

where the blank edges indicate the possibility for an edge. In this case, there are 8 possible saturated transfer systems on the above graph. In this case, we have that $|c^{-1}(A)| = s(3,0)$.

4.2. Closed form

The recursion for the number of saturated transfer systems on $C_{p^m q^n}$ from the previous section allows us to prove a closed form for $s(m,n)$. We first recall the following definition.

Definition 4.7. For $\ell, k \geq 0$, the *Stirling number of the second kind* $\left\{ \begin{matrix} \ell \\ k \end{matrix} \right\}$ counts the number of partitions of a set with ℓ elements into k non-empty subsets.

Remark 4.8. Stirling numbers of the second kind satisfy the recurrence

$$\left\{ \begin{matrix} \ell + 1 \\ k \end{matrix} \right\} = k \left\{ \begin{matrix} \ell \\ k \end{matrix} \right\} + \left\{ \begin{matrix} \ell \\ k - 1 \end{matrix} \right\}$$

for $k > 0$ with

$$\left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\} = 1 \quad \text{and} \quad \left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} = \left\{ \begin{matrix} 0 \\ n \end{matrix} \right\} = 0$$

for $n > 0$. They are given by the closed formula

$$\left\{ \begin{matrix} \ell \\ k \end{matrix} \right\} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^{\ell}.$$

Theorem 4.9. *For all $m, n \geq 0$, the sequence $s(m,n)$ satisfies*

$$s(m,n) = \sum_{j=2}^{m+2} (-1)^{m-j} \left\{ \begin{matrix} m+1 \\ j-1 \end{matrix} \right\} \frac{j!}{2} j^n.$$

Proof. We prove this by induction. We begin by proving the base case, when $n = 0$. It can be seen that the saturated covers on $[m] \times [0]$ correspond to m-long bitstrings (because each column may have 0 or 1 horizontal edges). Therefore, $s(m,0) = 2^m$. We need to prove that

$$s(m,0) = 2^m = \sum_{j=2}^{m+2} (-1)^{m-j} \left\{ \begin{matrix} m+1 \\ j-1 \end{matrix} \right\} \frac{j!}{2}.$$

From [12, (1.94d)], we know that

$$x^{m+1} = \sum_{k=0}^{m+1} \left\{ \begin{matrix} m+1 \\ k \end{matrix} \right\} (x)_k = \sum_{k=1}^{m+1} \left\{ \begin{matrix} m+1 \\ k \end{matrix} \right\} (x)_k + \left\{ \begin{matrix} m+1 \\ 0 \end{matrix} \right\} (x)_0,$$

where $(x)_k := x(x-1)\ldots(x-(k-1))$. Recalling that $\left\{ \begin{matrix} m \\ 0 \end{matrix} \right\} = 0$ for all $m > 0$ we have that

$$x^{m+1} = \sum_{k=0}^{m+1} \left\{ \begin{matrix} m+1 \\ k \end{matrix} \right\} (x)_k = \sum_{k=1}^{m+1} \left\{ \begin{matrix} m+1 \\ k \end{matrix} \right\} (x)_k.$$

Dividing both sides by x and noting that $(x)_k/x = (x-1)_{k-1}$, we have that,

$$x^m = \sum_{k=1}^{m+1} \left\{ {m+1 \atop k} \right\} (x-1)_{k-1}.$$

In particular, when $x = -2$,

$$(-2)^m = \sum_{k=1}^{m+1} \left\{ {m+1 \atop k} \right\} (-3)(-4)\ldots(-k-1) = \sum_{k=1}^{m+1}(-1)^{k+1} \left\{ {m+1 \atop k} \right\} \frac{(k+1)!}{2},$$

from which the base case follows.

For the inductive step we fix $m, n \geq 0$, assume the statement for all (k, n) with $k \leq m$, and prove the case for $(m, n+1)$. Using the recursive formula from Theorem 4.1 and the inductive hypothesis, and changing the order of summation, we have that

$$s(m, n+1) = \sum_{j=2}^{m+2} \left[(-1)^{m-j} \left\{ {m+1 \atop j-1} \right\} + \sum_{k=j-2}^{m} (-1)^{k-j} \binom{m+1}{k} \left\{ {k+1 \atop j-1} \right\} \right] \frac{j!}{2} j^n.$$

Thus, to prove the inductive step, we need to show that,

$$\sum_{j=2}^{m+2} (-1)^{m-j} \left\{ {m+1 \atop j-1} \right\} \frac{j!}{2} j^n = \sum_{j=2}^{m+2} \left[(-1)^{m-j} \left\{ {m+1 \atop j-1} \right\} + \sum_{k=j-2}^{m} (-1)^{k-j} \binom{m+1}{k} \left\{ {k+1 \atop j-1} \right\} \right] \frac{j!}{2} j^{n-1},$$

where the right-hand side is obtained by computing $s(m, n)$ using the previous formula and the left-hand side is the desired formula for $s(m, n)$ to show. Thus, to prove the inductive step it suffices to prove that for all $2 \leq j \leq m+2$,

$$(-1)^{m-j} \left\{ {m+1 \atop j-1} \right\} \frac{j!}{2} j^n = (-1)^{m-j} \left\{ {m+1 \atop j-1} \right\} \frac{j!}{2} j^{n-1} + \sum_{k=j-2}^{m} (-1)^{k-j} \binom{m+1}{k} \left\{ {k+1 \atop j-1} \right\} \frac{j!}{2} j^{n-1}.$$

Collecting some terms together we have that,

$$\left\{ {m+1 \atop j-1} \right\} j \left((-1)^{m-j} \frac{j!}{2} j^{n-1} \right) = \left\{ {m+1 \atop j-1} \right\} \left((-1)^{m-j} \frac{j!}{2} j^{n-1} \right) + \sum_{k=j-2}^{m} (-1)^{k-j} \binom{m+1}{k} \left\{ {k+1 \atop j-1} \right\} \frac{j!}{2} j^{n-1}.$$

Dividing both sides by the quantity $(-1)^{m-j} j^{n-1} j!/2$, we obtain

$$\left\{ {m+1 \atop j-1} \right\} \cdot j = \left\{ {m+1 \atop j-1} \right\} + \sum_{k=j-2}^{m} (-1)^{m-k} \binom{m+1}{k} \left\{ {k+1 \atop j-1} \right\}.$$

This follows directly by applying the combinatorial identity in Lemma 4.10 (see below), with $\ell = m+1$ and $r = j-1$:

$$(\ell - r) \left\{ {\ell \atop r} \right\} = (m+1-j+1) \left\{ {m+1 \atop j-1} \right\}$$

$$= (m-j+2) \left\{ {m+1 \atop j-1} \right\}$$

$$= \sum_{t=1}^{m-j+2} (-1)^{t+1} \binom{m+1}{t+1} \left\{ {m+1-t \atop j-1} \right\} \qquad \square$$

Lemma 4.10. *Let* $\ell, r \in \mathbb{Z}$ *such that* $0 \leq r \leq \ell$. *Then,*

$$(\ell - r) \left\{ {\ell \atop r} \right\} = \sum_{t=1}^{\ell-r} (-1)^{t+1} \binom{\ell}{t+1} \left\{ {\ell - t \atop r} \right\}$$

Proof. We prove the above identity via a combinatorial proof. To do so, we consider *marked partitions*. Given two nonnegative integers ℓ and r with $r \leq \ell$ a *marked partition* is a pair $(\mathcal{P}, q)$ consisting of a partition $\mathcal{P}$ of $\{1, \ldots, \ell\}$ into r nonempty subsets together with a distinguished $q \in \{1, \cdots, \ell\}$ such that q is not the minimum of its subset. Note that the left hand side of the equation enumerates the number of marked partitions for given ℓ and r, as there are $\left\{ {\ell \atop r} \right\}$ ways to choose the partition $\mathcal{P}$ and $(\ell - r)$ possibilities for q.

To prove the identity, we count the set of marked partitions using the inclusion-exclusion principle with the following subsets. For $i < j \leq \ell$, we define the subset of the marked partitions

$$A_{ij} := \{(\mathcal{P}, j) \mid i, j \text{ are in the same subset in } \mathcal{P}\}.$$

The union of the A_{ij} is the set of all marked partitions. For $i < j$ and $k < l$, $A_{ij} \cap A_{kl}$ is empty unless $j = l$, as the elements marked will be distinct. Thus the total number of marked partitions is

$$\sum_{1 \leq j \leq \ell} \sum_{i_1, \ldots, i_t} (-1)^{t+1} |A_{i_1 j} \cap \cdots \cap A_{i_t j}| \qquad (4.1)$$

where the second sum is over distinct $i_1, \ldots, i_t < j$.

We now show that for any $j \leq \ell$ and distinct $i_1, \ldots, i_t < j$,

$$|A_{i_1 j} \cap \cdots \cap A_{i_t j}| = \left\{ {\ell - t \atop r} \right\}.$$

Indeed, suppose we have $(\mathcal{P}, j) \in A_{i_1 j} \cap \cdots \cap A_{i_t j}$. Then $i_1, \ldots, i_t$ are in the same subset as j in the marked partition. Thus, the number of such partitions is given by partitioning the remaining $\ell - t - 1$ numbers and the subset $\{i_1, \ldots, i_t, j\}$ into r parts, which can be done in $\left\{ {\ell - t \atop r} \right\}$ ways, proving the above identity.

Since the cardinality of the t-fold intersections is independent of j, we may group (4.1) according to the arity of the intersection. Each nonempty t-fold intersection is determined by the choice of $1 \leq i_1 < \cdots < i_t < j \leq \ell$, and there are exactly $\binom{\ell}{t+1}$ such choices. Since t-fold intersections are empty for $t > \ell - r$, (4.1) reduces to

$$\sum_{t=1}^{\ell-r} (-1)^{t+1} \binom{\ell}{t+1} \left\{ {\ell - t \atop r} \right\}.$$

This quantity and $(\ell - r) \left\{ {\ell \atop r} \right\}$ both count marked partitions of $\{1, \ldots, \ell\}$ with r parts, we are done. $\quad \square$

Remark 4.11. Notably, the closed formula for $s(m, n)$ in Theorem 4.9 is not symmetric in m and n, although by definition we know that $s(m, n) = s(n, m)$.

The authors conjectured Theorem 4.9 via the following process. We first observed that for small cases of m and n, we could write

$$s(m,n) = \sum_{j=2}^{m+2} b_{mj} j^n,$$

for some integers b_{mj} that were independent of n. To prove that is indeed the case, let $B_m(x)$ denote the generating function for $s(m,n)$ as a sequence in n. The recursion in Theorem 4.1 implies that

$$B_m(x) = \frac{2^m + x \sum_{k=0}^{m-1} \binom{m+1}{k} B_k(x)}{1 - (m+2)x}.$$

From this, one may prove that $B_m(x)$ is a rational function with denominator $\prod_{j=2}^{m+2}(1 - jx)$, whence

$$B_m(x) = \sum_{j=2}^{m+2} \frac{b_{mj}}{1 - jx}$$

for some $b_{mj} \in \mathbb{Q}$. It follows that

$$s(m,n) = \sum_{j=2}^{m+2} b_{mj} j^n,$$

but we did not succeed in producing the values of b_{mj} via this method. (The residue method for partial fractions would give the answer if we knew the numerator of $B_m(x)$.) Instead, we computed the values of b_{mj} in a range and guessed that

$$b_{mj} = (-1)^{m-j} \left\{ {m+1 \atop j-1} \right\} \frac{j!}{2}$$

via an act of OEIS-enabled perspicacity.

4.3. Exponential generating function

We now consider the two-variable exponential generating function corresponding to $\{s(m,n)\}$. The content of this section is mostly due to Igor Kriz [8].

Exponential generating functions are formal power series particularly well-suited to enumeration of labeled objects. We refer the interested reader to [6] for a recent textbook treatment of the subject that includes multivariate generating functions.

Definition 4.12. Let

$$f(x,y) = \sum_{m,n \geq 0} \frac{s(m,n)}{m!n!} x^m y^n$$

be the exponential generating function corresponding to $\{s(m,n)\}$.

The recursive formula in Theorem 4.1 allows us to get a closed formula for $f(x,y)$.

Theorem 4.13 ([8]). *The exponential generating function for $\{s(m,n)\}$ satisfies*

$$f(x,y) = \frac{e^{2x+2y}}{(e^x + e^y - e^{x+y})^3}.$$

Proof. Using standard techniques for exponential generating functions together with Theorem 4.1 shows that f is a solution to the PDE

$$\frac{\partial f}{\partial y} = (e^x + 1)f + (e^x - 1)\frac{\partial f}{\partial x},$$

subject to the initial conditions

$$f(x, 0) = e^{2x} \quad \text{and} \quad f(0, y) = e^{2y}.$$

These initial conditions follow from the fact that $s(m, 0) = 2^m$ and $s(0, n) = 2^n$. The general solution is of the form

$$f(x, y) = \frac{\phi(\frac{e^x - 1}{e^x - y})e^x}{(e^x - 1)^2},$$

with ϕ an arbitrary function. The initial conditions give the result. $\quad\square$

5. Saturation conjecture for C_{pq^n}

In this section we show that the saturation conjecture is true for $G = C_{pq^n}$ for all $n \geq 0$.

We first recall Rubin's characterization of linear isometric transfer systems in the case of finite cyclic groups. Let k be a positive integer, and let $G = C_k$ be the cyclic group of order k.

Definition 5.1. Call $I \subseteq \mathbb{Z}/k\mathbb{Z}$ that contains 0 and is closed under additive inverses an *index set*. Given an index set I, define the *I-modular* transfer system $\mathscr{F}_I$, by the following condition: given $d \mid e \mid k$,

$$(C_d \to C_e) \in \mathscr{F}_I \iff (I \bmod e) + d = (I \bmod e).$$

We say I is an *index set* for $\mathscr{F}_I$.

Example 5.2. Consider the cyclic group C_{pq} with $p = 5$ and $q = 7$ and the set

$$I = \{0, 5, 10, 15, 20, 25, 30\} \subseteq \mathbb{Z}/35\mathbb{Z}.$$

Since $0 \in I$ and I is closed under additive inverses mod 35 it is an index set. We can see that

$$\{0, 5, 10, 15, 20, 15, 20, 25, 30\} \bmod 7 = \{0, 5, 3, 1, 6, 4, 2\},$$

so it is invariant under addition by 1. By definition, it follows that $(C_e \to C_7) \in \mathscr{F}_I$. Similarly I is also invariant under addition by 5 modulo 35 as the below computation shows,

$$\{0, 5, 10, 15, 20, 15, 20, 25, 30\} \bmod 35 = \{0, 5, 3, 1, 6, 4, 2\}.$$

As a result, $(C_5 \to C_{35}) \in \mathscr{F}_I$ as well. On the other hand, $(I \bmod 5) + 1 \neq (I \bmod 5)$ and $(I \bmod 35) + 7 \neq (I \bmod 35)$ so altogether, the resulting transfer system is

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Proposition 5.3 *([11, Proposition 5.15]). A G-transfer system is I-modular for some I if and only if it is linear isometric.*

The proof is done in two main steps. First, every G-universe can be expressed as the direct sum of infinitely many copies of the 2-dimensional representations given by rotation by $2\pi m/k$ for $m \in I$, for some set I that contains 0 is closed under additive inverses. Second, one translates the general characterization of the transfer system associated to a linear isometries operad of [2, Theorem 4.18] in terms of this specific decomposition.
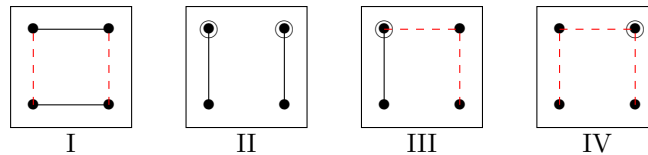
Proposition 5.3 implies that to verify the saturation conjecture it is sufficient to build an index set I for each saturated transfer system. Note that the following proposition follows directly from definitions and will allow us to recursively construct index sets via the $\ell = pq^n$, $k = pq^{n+1}$ case.

Proposition 5.4. *Let $\ell \mid k$, and suppose that J is an index set for C_k. Let $I = (J \mod \ell)$. Then I is an index set for C_ℓ, and*

$$\mathscr{F}_I = (\mathscr{F}_J)|_{C_\ell}. \quad \square$$

Our task now is to recursively generate index sets for saturated transfer systems on $C_{pq^{n+1}}$ from index sets for saturated transfer systems on C_{pq^n}. We use the language of saturated covers from Definition 3.3, and follow the strategy of Section 4.1 to split $\mathrm{SCov}(1, n+1)$ into four equivalence classes, based on the fibers of the map c of Construction 4.2.

The four equivalence classes have the following representative top rows. For a saturated cover S, the circles denote the vertices in $c(S)$, the edges in solid black are the edges that must be in S, and the edges in dashed red are the edges that S doesn't contain. The remaining edges may or may not be in S.

As explained in Section 4.1, within class I, a saturated cover is determined by its restriction to $[0] \times [n]$, while for the other three it is determined by its restriction to $[1] \times [n]$.

In order for the inductive step to work in all cases, we need to prove the following stronger statement.

Theorem 5.5. *Suppose p, q are primes greater than 3 and $n \geq 0$, and let $\mathscr{F}$ be a saturated transfer system on C_{pq^n}. Then there exists an index set $I \subseteq \mathbb{Z}/pq^n\mathbb{Z}$ such that $\mathscr{F} = \mathscr{F}_I$ and I contains a nonzero multiple of q^n.*

Remark 5.6. The nonzero multiple of q^n in I is necessary to define our index sets and perform our induction for type III and IV index sets.

Proof. We first prove the statement for type I directly, and we then prove the cases for types II, III and IV by induction. The strategy for all cases is to take a saturated transfer system $\mathscr{F}$, restrict it to a certain subgroup, take an index set for the restriction, and construct an index set for $\mathscr{F}$ based on the index set for the restriction. We will write (i, j) for the subgroup $C_{p^i q^j} \leq C_{pq^n}$ when it is convenient.

Type I: Let $\mathscr{F}$ be a saturated transfer system on C_{pq^n} of type I, and consider its restriction $\mathscr{F}|_{C_{q^n}}$. By [11, Theorem 5.18], this saturated transfer system on C_{q^n} is induced by an index set $I \subseteq \mathbb{Z}/q^n\mathbb{Z}$. Set

$$J := \{\alpha q^n + i \mid 0 \leq \alpha < p, \ i \in I\}.$$

By construction we have that $J \bmod q^n = I$, so Proposition 5.4 implies that the restrictions of $\mathscr{F}_J$ and $\mathscr{F}$ to C_{q^n} coincide. By construction again we have that $J + q^n = J$, thus showing that $(0,n) \to (1,n) \in \mathscr{F}_J$. By the conditions on saturation these two facts imply that $\mathscr{F}_J = \mathscr{F}$. Moreover, taking $\alpha = 1$ shows that $q^n \in J$.

For the remainder of the proof, we proceed by induction. For the base case, when $n = 0$, we are considering the two saturated systems for C_p: the trivial and the complete one (which is of type I). The trivial one is induced by $I = \{0, 1, p-1\}$ as long as $p > 3$ and the complete one is induced by $I = \{0, 1, \dots, p-1\}$. Note that the latter index set is the one we obtain from the direct proof above.

Our inductive hypothesis is that the statement of the theorem holds for some $n \geq 0$. Let $\mathscr{F}$ be a saturated transfer system of type II, III or IV on $C_{pq^{n+1}}$, and consider its restriction to C_{pq^n}. The inductive hypothesis implies that there exists and index set $I \subseteq \mathbb{Z}/pq^n\mathbb{Z}$ containing aq^n for some $0 < a < p$, and such that

$$\mathscr{F}_I = \mathscr{F}\mid_{C_{pq^n}}.$$

In all three cases, we will produce an index set $J \subseteq \mathbb{Z}/pq^{n+1}\mathbb{Z}$ such that $J \bmod pq^n = I$, so that by Proposition 5.4 we get that $\mathscr{F}_J$ and $\mathscr{F}$ coincide in the restriction to C_{pq^n}. We will then show that J does the right thing in the top square (depending on the type) and contains a nonzero multiple of q^{n+1}.

Type II: Suppose $\mathscr{F}$ has type II and take $I \subseteq \mathbb{Z}/pq^n\mathbb{Z}$ as described above. Set

$$J := \{\alpha pq^n + i \mid 0 \leq \alpha < q,\ i \in I\}.$$

By direct inspection, $J \bmod pq^n = I$, so $\mathscr{F}_J$ and $\mathscr{F}$ have the same restriction to C_{pq^n}. Since

$$J + pq^n = \{(\alpha+1)pq^n + i \mid 0 \leq \alpha < q,\ i \in I\} = J$$

we have that $(1,n) \to (1, n+1) \in \mathscr{F}_J$. By restriction, $(0,n) \to (0, n+1) \in \mathscr{F}_J$ as well. This proves that $\mathscr{F}_J$ has type II, whence $\mathscr{F}_J = \mathscr{F}$. To find a nonzero multiple of q^{n+1} in J, take α with residue class $-ap^{-1} \in \mathbb{Z}/q\mathbb{Z}^\times$ and $i = aq^n$. This completes the induction for type II.

Type III: Suppose $\mathscr{F}$ has type III and take $I \subseteq \mathbb{Z}/pq^n\mathbb{Z}$ and a such that $aq^n \in I$ as described above. We need to produce an index set $J \subseteq \mathbb{Z}/pq^{n+1}\mathbb{Z}$ such that $J \bmod pq^n = I$, some $0 \neq bq^{n+1} \in J$, and — since $\mathscr{F}$ has type III — such that $(1,n) \to (1, n+1) \notin \mathscr{F}_J$ and $(0,n) \to (0, n+1) \in \mathscr{F}_J$. By the saturation axioms, these conditions are enough to ensure $\mathscr{F}_J = \mathscr{F}$.

Set

$$J' := \{\alpha pq^n + i \mid 0 \leq \alpha < q,\ i \in I\}$$

and

$$J := J' \smallsetminus \{aq^n, pq^{n+1} - aq^n\}.$$

Then J is an index set and it is clear that $J \bmod pq^n \subseteq I$ with only aq^n and $-aq^n$ possibly in the set difference. For $q > 2$ and $\alpha = 1$, the element $pq^n + aq^n$ is in J and reduces to $aq^n \bmod pq^n$. Similarly, the mod pq^{n+1} negative of this element is in J and reduces to $-aq^n \bmod pq^n$.

We now check that $(0,n) \to (0, n+1) \in \mathscr{F}_J$, which amounts to $(J \bmod q^{n+1}) + q^n = J \bmod q^{n+1}$. We claim that $J \bmod q^{n+1} = J' \bmod q^{n+1}$, which suffices for this result. To verify the claim, take α that reduces to $ap^{-1} \bmod q$. Then

$$aq^n \equiv \alpha pq^n \bmod q^{n+1},$$

so $aq^n \in (J \bmod q^{n+1})$ as needed.

To show that $(1, n) \to (1, n+1) \notin \mathscr{F}_J$, we must verify that

$$J + pq^n \neq J.$$

Note that $aq^n \notin J$, but $(q-1)pq^n + aq^n \in J$, so $aq^n \in J + pq^n$.

Finally, we need to show that J contains some nonzero multiple bq^{n+1} of q^{n+1}. If $a \neq q$, then J contains

$$\alpha pq^n + aq^n = (\alpha p + a)q^n$$

which is divisible by q^{n+1} for some $0 < \alpha < q$. If $a = q$ (which is possible when $q < p$), then we actually must modify the definition of J, setting

$$J := J' \smallsetminus \{aq^n + pq^n, pq^{n+1} - (aq^n + pq^n)\}.$$

The above argument still goes through and we can then check that some $0 \neq bq^{n+1} \in J$.

Type IV: Suppose $\mathscr{F}$ has type IV and take $I \subseteq \mathbb{Z}/pq^n\mathbb{Z}$ as described above. We construct an index set $J \subseteq \mathbb{Z}/pq^{n+1}\mathbb{Z}$ containing a nonzero multiple of q^{n+1} such that $J \bmod pq^n = I$, $(0, n) \to (0, n+1) \notin \mathscr{F}_J$ and $(0, n+1) \to (1, n+1) \notin \mathscr{F}_J$. The fact that $\mathscr{F}$ is of type IV and the saturation axioms imply then that $\mathscr{F}_J = \mathscr{F}$.

Let $i \in I \smallsetminus 0$. Then by Lemma 5.7 (stated and proved immediately after this proof), there exists $0 \leq \alpha_i < q$ such that $\alpha_i pq^n + i \bmod q^{n+1}$ lies in the interval $[0, q^n)$. By Sunzi's theorem, there exists $0 < c < pq$ such that c is a multiple of q and $c \equiv a \bmod p$. We set

$$J := \{0, cq^n, pq^{n+1} - cq^n\} \cup \left\{\alpha_i pq^n + i, pq^{n+1} - (\alpha_i pq^n + i) \mid i \in I \smallsetminus \{0, aq^n, pq^n - aq^n\}\right\}.$$

Since c is a nonzero multiple of q, we know cq^n is a nonzero multiple of q^{n+1}. The other condition on c implies $cq^n \equiv aq^n \bmod pq^n$, and we thus have that $J \bmod pq^n = I$.

To prove that $(0, n) \to (0, n+1) \notin \mathscr{F}_J$ we need to check that

$$(J \bmod q^{n+1}) + q^n \neq (J \bmod q^{n+1}).$$

We have that

$$(J \bmod q^{n+1}) = \{0\} \cup \{\alpha_i pq^n + i, q^{n+1} - (\alpha_i pq^n + i) \mid i \in I \smallsetminus \{0, aq^n, pq^n - aq^n\}\}.$$

When considering this set in terms of representatives $\{0, 1, \ldots, q^{n+1} - 1\}$, its elements are concentrated in the intervals $[0, q^n - 1]$ and $[q^{n+1} - q^n + 1, q^{n+1} - 1]$. Basic arithmetic shows that if $q > 2$, the translation by q^n of any element in the first interval does not land in either of the two intervals, showing our result.

A similar argument using that $p, q > 3$ can be used to prove that $J + q^{n+1} \neq J$, thus showing that $(0, n+1) \to (1, n+1) \notin \mathscr{F}_J$, as needed. This finishes the proof. $\square$

Lemma 5.7. *Let i be an integer such that $0 < i < pq^n$. Then there exists $0 \leq \alpha < q$ such that the residue of $\alpha pq^n + i \bmod q^{n+1}$ lies in the interval $[0, q^n)$.*

Proof. Let r be the residue class of $i \bmod q^n$. Thus, $0 \leq r < q^n$, and there exists $0 \leq k < p$ such that $i = kq^n + r$. By Bézout's identity, there exist $c, d \in \mathbb{Z}$ such that $cp + dq = 1$. Let $\beta = -ck \in \mathbb{Z}$. Then basic arithmetic shows that

$$\beta pq^n + i \equiv r \bmod q^{n+1}.$$

Finally, letting α be the residue of β mod q achieves the result. $\quad\square$

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