Journal of Algebra 622 (2023) 197–219

Contents lists available at ScienceDirect

Journal of Algebra

journal homepage: www.elsevier.com/locate/jalgebra

Height zero characters in principal blocks ☆

Nguyen Ngoc Hung [a,*], A.A. Schaeffer Fry [b], Carolina Vallejo [c]

[a] *Department of Mathematics, The University of Akron, Akron, OH 44325, USA*
[b] *Dept. Mathematics and Statistics, MSU Denver, Denver, CO 80217, USA*
[c] *Dipartimento di Matematica e Informatica 'Ulisse Dini', 50134 Firenze, Italy*

A R T I C L E I N F O

Article history:
Received 20 December 2021
Available online 6 February 2023
Communicated by Gunter Malle

MSC:
primary 20C20, 20C15, 20C33

Keywords:
Height zero character
Principal block
Sylow subgroup
Alperin-McKay conjecture

A B S T R A C T

We show that the principal p-block of a finite group of order divisible by p has at least $2\sqrt{p-1}$ height-zero characters. Along the way, we describe the p-local structure of finite groups whose principal p-blocks have at most five height-zero ordinary irreducible characters.

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1. Introduction

In order to better understand the relationship between complex and modular representations of a finite group G, Brauer partitioned the set of ordinary and p-Brauer

☆ The first-named author thanks Álex Moretó for several stimulating discussions on the relationship between fields of character values and Sylow subgroups. The second-named author is grateful for support from the National Science Foundation, Award Nos. DMS-1801156 and DMS-2100912. The third-named author acknowledges support by grants PID2020-118193GA-I00 and PID2019-103854GB-I00 funded by MCIN/AEI/10.13039/501100011033. Finally, we are grateful to the referees for several helpful comments and suggestions.

* Corresponding author.
E-mail addresses: hungnguyen@uakron.edu (N.N. Hung), aschaef6@msudenver.edu (A.A. Schaeffer Fry), carolina.vallejorodriguez@unifi.it (C. Vallejo).

https://doi.org/10.1016/j.jalgebra.2023.01.021
0021-8693/Published by Elsevier Inc.

irreducible characters of G into naturally defined subsets called p-blocks of G, where p is a prime. Brauer's idea has developed into what is now known as block theory, a fundamental tool in the study of finite group representation theory. In a p-block B of a finite group, height-zero characters, which are irreducible ordinary characters in B whose degrees have minimal p-part, play an important role because of their direct involvement in several central problems in the area, notably Brauer's height zero conjecture [6, Problem 23] and the Alperin-McKay conjecture [1]. (We remark that the Brauer's height zero conjecture was announced complete [36] while the current paper was under review, as was the Alperin–McKay conjecture for the prime 2 [54].) In the following, we write $k(B)$ to denote the number of ordinary irreducible characters in a block B and $k_0(B)$ to denote the number of height-zero characters in B.

The principal p-block of a finite group G, which we will denote by $B_0(G)$, or sometimes just by B_0, is the one containing the trivial character $\mathbf{1}_G$ of G. Therefore, the height-zero characters in B_0 are simply those characters with degree prime to p. The numbers $k_0(B_0) \leqslant k(B_0)$ are conjecturally bounded from above by the order of a Sylow p-subgroup of G [6, Problem 20]. In our first main result we provide a lower bound for $k_0(B_0)$ in terms of the prime p.

Theorem 1.1. *Let G be a finite group of order divisible by a prime p and B_0 denote the principal p-block of G. Then*

$$k_0(B_0) \geqslant 2\sqrt{p-1}\,.$$

We remark that Theorem 1.1 improves the main result of [34] from a modular perspective. In [34], Malle and Maróti show that every finite group of order divisible by p has at least $2\sqrt{p-1}$ irreducible characters of degree prime to p. At the same time, Theorem 1.1 generalizes [22, Theorem 1.1] from the perspective of height-zero characters, where the first and second-named authors prove that $k(B_0) \geqslant 2\sqrt{p-1}$, confirming Héthelyi-Külshammer's conjecture [20] for principal blocks.

Notice that another way of reading the statement of Theorem 1.1 is as follows: if G is a finite group of order divisible by p and B_0 is its principal p-block, then $p \leqslant k_0(B_0)^2/4+1$. In particular, in order to prove Theorem 1.1 we need to show that the above bound on p holds for small values of $k_0(B_0)$. We derive this bound as a consequence of a more general result of independent interest. Indeed, we are able to completely determine the local structure of finite groups whose principal blocks have up to five height-zero characters. This is our second main result.

Theorem 1.2. *Let G a finite group and p a prime. Let P be a Sylow p-subgroup and B_0 denote the principal p-block of G. We have:*

(A) *For $k \in \{2,3\}$, $k_0(B_0) = k$ if, and only if, P has order k.*
(B) *$k_0(B_0) = 4$ if, and only if, exactly one of the following happens:*

(i) $[P : P'] = 4$,
(ii) $|P| = 5$ *and* $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] = 2$.

(C) $k_0(B_0) = 5$ *if, and only if, exactly one of the following happens:*
(i) $|P| = 5$ *and* $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] \in \{1, 4\}$,
(ii) $|P| = 7$ *and* $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] \in \{2, 3\}$.

Studying the local structure of finite groups whose principal blocks have a given number of irreducible characters is the modular analogue of a classical subject in finite group theory: classifying the finite groups with a given number of conjugacy classes (see [10], [58] and [57], for instance). This problem has recently attracted the interest of the community, and the structure of the Sylow p-subgroups of finite groups whose principal p-block has up to five irreducible characters has been determined in [26] and [53].

We care to mention that Theorem 1.2 generalizes the main results of [26] and [53] and, at the same time, significantly extends [45, Theorems A and C], which study the cases where $k_0(B_0) = 3$ for $p = 3$ and $k_0(B_0) = 4$ for $p = 2$.

Brauer's Problem 21 [6] predicts that, for every positive integer k, there are finitely many isomorphism classes of groups which can occur as defect groups of blocks with k ordinary irreducible characters. This was shown by to be a consequence of the Alperin-McKay conjecture and Zelmanov's solution of the restricted Burnside problem in [29]. In view of Theorems 1.1 and 1.2, we propose the following variation of Brauer's problem 21 for height zero characters.

Conjecture 1.3. *For every positive integer k_0, there are finitely many isomorphism classes of (abelian) groups (of prime power order) which can occur as abelianizations of defect groups of blocks (of finite groups) with precisely k_0 height-zero irreducible characters.*

In Lemma 6.1, we show that Conjecture 1.3 is another consequence of the Alperin-McKay conjecture and the (known) positive answer of Brauer's problem 21 for p-solvable groups.

For principal blocks, Conjecture 1.3 is equivalent to the statement that the index $[P : P']$ is bounded from above in terms of the number $k_0 := k_0(B_0(G))$, where $P \in \mathrm{Syl}_p(G)$. By Theorem 1.1, this is reduced to showing that $\mathrm{rk}(P/P')$ and $\log_p(\exp(P/P'))$ are both bounded in terms of k_0, where $\mathrm{rk}(P/P')$ and $\exp(P/P')$ are respectively the rank and the exponent of the abelian group P/P'. The problem of bounding $\log_p(\exp(P/P'))$ turns out to be related to recent advances on the study of fields of character values and Galois actions on characters, in the context of the Alperin-McKay-Navarro conjecture [44, Conjecture B]. We will exploit this relationship in Section 6. In particular, in Theorem 6.2 we prove that $\exp(P/P')$ is bounded in terms of k_0 when $p = 2$.

The structure of this paper is as follows. In Section 2, we collect some previous results on blocks and normal subgroups as well as some proven consequences of the Alperin-McKay conjecture. In Section 3, we obtain a lower bound for the number of irreducible height-zero characters in principal blocks of almost simple groups. The proof of Theorem 1.2 is contained in Section 4. In Section 5, and relying on all the previous sections, we present a proof of Theorem 1.1. We finish our work by discussing Conjecture 1.3 and proving Theorem 6.2 in Section 6.

2. Preliminaries

We start by collecting some results on the interplay between block theory and the normal structure of a group. We refer the reader to [42, Chapter 9] for first definitions and basic properties. Let G be a finite group and p a prime. Recall that if N is a normal subgroup of G and B and b are p-blocks of G and N respectively, then B covers b if there are $\chi \in \mathrm{Irr}(B)$ and $\theta \in \mathrm{Irr}(b)$ such that θ is an irreducible constituent of the restriction χ_N. For $\theta \in \mathrm{Irr}(N)$, we write $\mathrm{Irr}(G|\theta)$, respectively $\mathrm{Irr}(B|\theta)$, for the set of those irreducible characters of G, respectively B, containing θ as a constituent when restricted to N.

We denote by $B_0(G)$ the principal p-block of G whenever p is clear from, or irrelevant in, the context. It is clear that $B_0(G)$ covers $B_0(N)$. Recall that $\chi \in \mathrm{Irr}(G)$ belongs to $B_0(G)$ if, and only if,

$$\sum_{x \in G^0} \chi(x) \neq 0\,,$$

where G^0 is the set of p-regular elements in G. In particular, $\mathrm{Aut}(G)$ and $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ act on $\mathrm{Irr}(B_0(G))$, and also on the subset $\mathrm{Irr}_{p'}(B_0(G))$ of height-zero characters in $B_0(G)$. Here $\mathbb{Q}^{ab}$ is the smallest extension of $\mathbb{Q}$ containing all roots of unity.

Lemma 2.1. *Let G be a finite group and $N \trianglelefteq G$.*

(i) $\mathrm{Irr}(B_0(G/N)) \subseteq \mathrm{Irr}(B_0(G))$.
(ii) *For every $\theta \in \mathrm{Irr}(B_0(N))$, there exists $\chi \in \mathrm{Irr}(B_0(G)|\theta)$.*
(iii) *Suppose that $B \in \mathrm{Bl}(G)$ is the only block covering $b \in \mathrm{Bl}(N)$. Then for every $\theta \in \mathrm{Irr}(b)$, we have $\mathrm{Irr}(G|\theta) \subseteq \mathrm{Irr}(B)$.*

Proof. Part (i) follows as $B_0(G)$ dominates $B_0(G/N)$ in the sense of [42, p. 199]. Part (ii) is [42, Theorem 9.4]. Part (iii) follows from the definition of covering blocks since the block containing $\chi \in \mathrm{Irr}(G|\theta)$ covers the block b. $\square$

Note that if $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(B_0(G))$ satisfies that $N \subseteq \mathrm{Ker}(\chi)$, then it is not true in general that $\chi \in \mathrm{Irr}(B_0(G/N))$.

Lemma 2.2. *Let $N \trianglelefteq G$ and $P \in \mathrm{Syl}_p(G)$.*

(i) *If $\theta \in \mathrm{Irr}_{p'}(B_0(N))$ extends to PN, then there is some $\chi \in \mathrm{Irr}(B_0(G)|\theta)$ of degree not divisible by p.*

(ii) *If $\theta \in \mathrm{Irr}_{p'}(B_0(N))$ extends to some character in $B_0(G)$ and $B_0(G)$ is the only block of G covering $B_0(N)$, then*

$$|\mathrm{Irr}_{p'}(B_0(G)|\theta)| = |\mathrm{Irr}_{p'}(G/N)|\,,$$

where $\mathrm{Irr}_{p'}(B_0(G)|\theta) := \mathrm{Irr}(B_0(G)) \cap \mathrm{Irr}_{p'}(G|\theta)$.

Proof. Part (i) is due to Murai [41, Lemma 4.3]. We now prove part (ii). Let $\hat{\theta} \in \mathrm{Irr}(B_0(G))$ be an extension of θ. By Gallagher's theorem [23, Corollary 6.17],

$$\mathrm{Irr}_{p'}(G|\theta) = \{\beta\hat{\theta} \mid \beta \in \mathrm{Irr}_{p'}(G/N)\}\,.$$

By hypothesis and Lemma 2.1(iii), $\mathrm{Irr}_{p'}(G|\theta) \subseteq \mathrm{Irr}(B_0(G))$. Putting these facts together, we see that $|\mathrm{Irr}_{p'}(G|\theta) \cap \mathrm{Irr}(B_0(G))| = |\mathrm{Irr}_{p'}(G|\theta)| = |\mathrm{Irr}_{p'}(G/N)|$. □

Lemma 2.3. *Let $M \trianglelefteq G$ and $P \in \mathrm{Syl}_p(G)$. If $P\mathbf{C}_G(P) \subseteq M$, then $B_0(G)$ is the only block covering $B_0(M)$. In particular, $k(G/M) < k_0(B_0(G))$ as long as $P > 1$.*

Proof. The first statement is [53, Lemma 1.3]. Let $\chi \in \mathrm{Irr}(G/M)$. Viewing χ as a character of G, we see that χ lies over the trivial character of M, and therefore $\chi \in B_0(G)$. Since G/M has order coprime to p, we further have $\chi \in \mathrm{Irr}_{p'}(B_0(G))$. We have seen that $k(G/M) \leqslant k_0(B_0(G))$, and so it remains to argue that, when $P > 1$, there is a member in $\mathrm{Irr}_{p'}(B_0(G))$ lying over a nontrivial character in $B_0(M)$. Recall that, when $P > 1$, there does exists some nontrivial $\theta \in \mathrm{Irr}_{p'}(B_0(M))$ by [42, Problem 3.11]. The second statement now follows from Lemma 2.2(i). □

We will also make use of Alperin-Dade's theory of isomorphic principal blocks.

Theorem 2.4. *Suppose that N is a normal subgroup of G, with G/N a p'-group. Let $P \in \mathrm{Syl}_p(G)$ and assume that $G = N\mathbf{C}_G(P)$. Then restriction of characters defines a natural bijection between the irreducible characters of the principal blocks of G and N. In particular, $k_0(B_0(G)) = k_0(B_0(N))$.*

Proof. The case where G/N is solvable was proved in [2] and the general case in [15]. □

We end this section with some proven consequences of the Alperin-McKay conjecture, which posits that $k_0(B) = k_0(b)$, where for B a block of G, b is the Brauer first main correspondent of B [42, Theorems 4.12 and 4.17]. Note that if B has defect group D, then b is a block of $\mathbf{N}_G(D)$ with defect group D. By Brauer's third main theorem [42, Theorem 6.7], the Brauer first main correspondent of $B_0(G)$ is $B_0(\mathbf{N}_G(P))$.

Theorem 2.5. *If G is p-solvable and $P \in \mathrm{Syl}_p(G)$, then*

$$k_0(B_0(G)) = k_0(B_0(\mathbf{N}_G(P))) = k(\mathbf{N}_G(P)/\mathbf{O}_{p'}(\mathbf{N}_G(P))P')\,.$$

Proof. The first equality is the principal bock case of results by Dade [16] and Okuyama-Wajima [48] (see also [3]). The second equality follows from Fong's theorem [42, Theorem 10.20] and Itô's argument [23, Theorem 6.15]. $\square$

In this paper, we write C_n to denote the cyclic group of order n.

Lemma 2.6. *If the principal p-block $B_0(G)$ of a finite group G satisfies the Alperin-McKay conjecture, then $k_0(B_0(G)) \geqslant 2\sqrt{p-1}$ with equality if, and only if, $\sqrt{p-1} \in \mathbb{N}$ and $\mathbf{N}_G(P)/\mathbf{O}_{p'}(\mathbf{N}_G(P))$ is isomorphic to the Frobenius group $\mathsf{C}_p \rtimes \mathsf{C}_{\sqrt{p-1}}$.*

In particular, if G is p-solvable or all the non-abelian composition factors of G have cyclic Sylow p-subgroups, then $k_0(B_0(G)) \geqslant 2\sqrt{p-1}$.

Proof. The first part follows from [22, §2.1]. (Note that the 'if' part of the equality claim is clear. For the 'only if' part, assume that $k_0(B_0(G)) = 2\sqrt{p-1}$. Then, from [22], we have $k((\mathbf{N}_G(P)/P')/\mathbf{O}_{p'}(\mathbf{N}_G(P)/P')) = 2\sqrt{p-1}$. Arguing similarly as in the first paragraph of [22, p. 5], we have that $\mathbf{N}_G(P)/\mathbf{O}_{p'}(\mathbf{N}_G(P))$ is isomorphic to the Frobenius group $\mathsf{C}_p \rtimes \mathsf{C}_{\sqrt{p-1}}$.)

The Alperin-McKay conjecture is known to be true for p-solvable groups by Theorem 2.5. The so-called inductive Alperin-McKay conditions are satisfied for all blocks with cyclic defect groups by Koshitani and Späth [56,27], and thus the Alperin-McKay conjecture also holds true for finite groups in which all the non-abelian composition factors have cyclic Sylow p-subgroups. (Indeed, note that a simple group is involved in G if, and only if, it is involved in some composition factor of G, and hence any simple group involved in G has cyclic Sylow p-subgroups.) $\square$

Theorem 2.7. *Let G be a finite group with an abelian Sylow p-subgroup. Let $B_0(G)$ denote the principal p-block of G. Then $k_0(B_0(G)) \geqslant 2\sqrt{p-1}$ with equality if, and only if, $\sqrt{p-1} \in \mathbb{N}$ and $\mathbf{N}_G(P)/\mathbf{O}_{p'}(\mathbf{N}_G(P))$ is isomorphic to the Frobenius group $\mathsf{C}_p \rtimes \mathsf{C}_{\sqrt{p-1}}$.*

Proof. Note that, when P is abelian, $k_0(B_0(G)) = k(B_0(G))$, by the work of Kessar and Malle [25, Theorem 1.1] on the 'if part' of Brauer's height zero conjecture. The statement then follows by [22, Theorems 1.1 and 1.3]. $\square$

3. Bounding height-zero characters in (almost) simple groups

To prove Theorems 1.1 and 1.2, we need to bound from below the number of height-zero characters in (almost) simple groups. That is the purpose of this section. We begin with the case of alternating A_n and symmetric S_n groups.

Proposition 3.1. *Let $p \geqslant 3$ be a prime and n be a positive integer. Then*

(i) *If $n \geqslant p+2$ then $|\mathrm{Irr}_{p'}(\mathsf{A}_n)| \geqslant p$ and $|\mathrm{Irr}_{p'}(\mathsf{S}_n)| \geqslant 2p$.*
(ii) *If $n = p$ or $p+1$ then $|\mathrm{Irr}_{p'}(\mathsf{A}_n)| = (p+3)/2$ and $|\mathrm{Irr}_{p'}(\mathsf{S}_n)| = p$*

(iii) *If* $n \geqslant p^2$ *then* $|\mathrm{Irr}_{p'}(B_0(\mathsf{S}_n))| \geqslant p^2$*, and thus, there are at least* $p^2/2$ *orbits of characters in* $\mathrm{Irr}_{p'}(B_0(\mathsf{A}_n))$ *under the action of* S_n*.*

Proof. Basics on the representation theory of symmetric and alternating groups can be found in [24,52]. Let $\mathcal{P}(n)$ denote the set of all partitions of n. Irreducible ordinary characters of S_n are naturally labeled by partitions in $\mathcal{P}(n)$, and so for each such partition λ, we let $\chi^\lambda \in \mathrm{Irr}(\mathsf{S}_n)$ denote the corresponding character. For $q \in \mathbb{Z}^+$, the q-core of λ is the partition obtained from λ by successive removals of rim q-hooks until no q-hook is left.

A well-known result of Macdonald (see [50, §2]) asserts that, if $\lambda \in \mathcal{P}(n)$ and the p-adic expansion of n is

$$a_0 + a_1 p + \cdots + a_t p^t,$$

then the character χ^λ has p'-degree if, and only if, λ has precisely a_t hooks of length divisible by p^t and the character labeled by the p^t-core of λ has p'-degree. Moreover,

$$|\mathrm{Irr}_{p'}(\mathsf{S}_n)| = k(1, a_0)k(p, a_1) \cdots k(p^t, a_t),$$

where, for $m, a \in \mathbb{N}$, $k(m, a)$ is the number of m-tuples of partitions of a.

When $n \geqslant p + 2$ we have $|\mathrm{Irr}_{p'}(\mathsf{S}_n)| \geqslant 2k(p, 1) = 2p$, and therefore $|\mathrm{Irr}_{p'}(\mathsf{A}_n)| \geqslant p$, proving part (i). For part (ii) we note that the p'-degree irreducible characters of S_p are labeled by hook-shape partitions of the form $(x, 1^{p-x})$ with $0 \leqslant x \leqslant p$, and exactly one of them, namely the one with $x = (p+1)/2$, is self-conjugate; also, the p'-degree irreducible characters of S_{p+1} are labeled by $(p + 1)$, (1^{p+1}), and $(x, 2, 1^{p-x-1})$ with $2 \leqslant x \leqslant p - 1$, and again exactly one of them is self-conjugate.

For part (iii), the assumptions on p and n imply that $n \geqslant 9$, and so $\mathsf{S}_n = \mathrm{Aut}(\mathsf{A}_n)$. Let $n = mp + r$ for some integers $m \geqslant 1$ and $0 \leqslant r < p$. Then [40, Theorem 1.10] implies that the number of height-zero characters in the principal block of S_n is the same as $k_0(B_0(\mathsf{S}_{mp}))$. By [51, p. 44], this number is $\prod_{i \geqslant 0} k(p^{i+1}, b_i)$, where $m = \sum b_i p^i$ is the p-adic decomposition of m. Since $n \geqslant p^2$, we have $m \geqslant p$, and it follows that this number $\prod_{i \geqslant 0} k(p^{i+1}, b_i)$ is at least p^2, as desired. □

We next prove the key statement for (almost) simple groups needed for our main results.

Proposition 3.2. *Let* S *be a non-abelian finite simple group and* $p \geqslant 5$ *a prime dividing* $|S|$*. Assume that* $P \in \mathrm{Syl}_p(S)$ *is non-abelian. Then there are at least 6 characters in* $\mathrm{Irr}_{p'}(B_0(S))$*. Further, there are more than* $2\sqrt{p-1}$ *different* $\mathrm{Aut}(S)$*-orbits in* $\mathrm{Irr}_{p'}(B_0(S))$*.*

Proof. (I) First we note that the conclusion follows from Proposition 3.1(iii) for the alternating groups, since P is abelian for $n < p^2$ and $p^2/2 > \max\{6, 2\sqrt{p-1}\}$ for $p \geqslant 5$.

For sporadic groups and the Tits group, the assumptions on p and P imply that either $p \in \{5, 7\}$ or $(S, p) = (J_4, 11)$ or $(M, 13)$. The GAP character table library [17] contains the character table and block distributions for S for the prime p in these cases. From this information, we can see that the statement holds.

(II) We now assume that S is a simple group of Lie type defined over $\mathbb{F}_q$, where q is a power of some prime q_0. First assume that $q_0 = p$. Let $\mathbf{G}$ be a simple algebraic group of adjoint type and F a Steinberg endomorphism on $\mathbf{G}$ such that $S \cong [G, G]$ where $G := \mathbf{G}^F$. By [9, Lemma 5], the p'-degree irreducible characters of G are the same as semisimple characters, one for each conjugacy class of semisimple elements of ${\mathbf{G}^*}^{F^*}$, where $(\mathbf{G}^*, F^*)$ is the dual pair of $(\mathbf{G}, F)$. As the number of semisimple classes of ${\mathbf{G}^*}^{F^*}$ is at least q^r, where r is the rank of $\mathbf{G}$, by [13, Theorem 3.7.6], it follows that $|\mathrm{Irr}_{p'}(G)| \geqslant q^r$. Therefore, $|\mathrm{Irr}_{p'}(S)| \geqslant q^r/d$ where $d := |G/S|$ is the order of the group of diagonal automorphisms of S. By a result of Dagger and Humphreys (see [11, Theorem 3.3]), S has precisely two p-blocks: the principal block and the defect-zero block containing only the Steinberg character (of degree $|S|_p$). Therefore, we have $k_0(B_0(S)) \geqslant q^r/d$. It is now easy to check that $q^r/d > 2\sqrt{p-1}|\mathrm{Out}(S)|$ for all S of Lie type in characteristic $p \geqslant 11$, using available information of $\mathrm{Out}(S)$, in [14, p. xvi] for instance. Hence we are done unless $p \in \{5, 7\}$.

Now suppose $p \in \{5, 7\}$. In this case, we have $q^r/d \geqslant 6$ except if $S = \mathrm{PSL}_2(p)$, and we have $q^r/d \geqslant 5|\mathrm{Out}(S)|$ unless $S = \mathrm{PSL}_2(p)$; $\mathrm{PSL}_2(p^2)$; $\mathrm{PSL}_3^{\pm}(p)$; or $\mathrm{PSL}_4^{\pm}(5)$. However, if $S = \mathrm{PSL}_2(q)$, then $P \in \mathrm{Syl}_p(S)$ is abelian, and we are done in that case. So, assume $S = \mathrm{PSL}_n^{\pm}(p)$ with $(n, p) \in \{(3, 5), (3, 7), (4, 5)\}$. In these cases, we can see from the character table available in GAP that there are at least 5 distinct character values in $\mathrm{Irr}_{p'}(S)$, so that there are at least 5 $\mathrm{Aut}(S)$-orbits in $\mathrm{Irr}_{p'}(B_0(S))$, and we are again done.

(III) So, we may now assume that $p \nmid q$. Let $d := e_p(q)$ be the multiplicative order of q modulo p. If S is of exceptional type (including Suzuki and Ree groups and ${}^3\mathrm{D}_4(q)$), then the fact that P is non-abelian implies that d is a regular number.

So, we assume that S is not of Suzuki or Ree type and that d is a regular number. In [53, Lemma 3.7], it is shown in this case that there are at least 6 distinct $\mathrm{Aut}(S)$-orbits on $\mathrm{Irr}(B_0(S))$. In the proof of [53], it is in fact shown that there are at least 6 distinct p'-degree characters in $\mathrm{Irr}(B_0(S))$ lying in at least 5 $\mathrm{Aut}(S)$-orbits. (In fact, in most cases, there are at least 6 distinct such orbits.) Hence we are done in this case, since the assumption P is non-abelian also implies that $p < 11$.

(IV) We therefore assume for the remainder of the proof that S is of classical type. That is, S is of type A_n, ${}^2\mathrm{A}_n$, B_n, C_n, D_n, or ${}^2\mathrm{D}_n$. We may write $S = [G, G]$, where $G = \mathrm{PGL}_n(q)$, $\mathrm{PGU}_n(q)$, $\mathrm{SO}_{2n+1}(q)$, $\mathrm{PCSp}_{2n}(q)$, or $\mathrm{P}(\mathrm{CO}_{2n}^{\pm}(q))^0$, respectively.

Define e to be the smallest positive integer such that $p \mid (q^e - 1)$ when G is of type A, $p \mid (q^e - (-1)^e)$ when G is of type ${}^2\mathrm{A}$, or $p \mid (q^e \pm 1)$ when G is of type $\mathrm{B}, \mathrm{C}, \mathrm{D}$ or ${}^2\mathrm{D}$. Let $n = we + m$ where $0 \leqslant m < e$. The fact that P is non-abelian implies that $p \leqslant w$ (see for instance the proof of [34, Prop. 5.5]).

Let $\mathcal{W}$ denote the relative Weyl group of a Sylow d-torus of G. When $G = \mathrm{PGL}_n(q)$ or $\mathrm{PGU}_n(q)$, the group $\mathcal{W}$ is the wreath product $\mathsf{C}_e \wr \mathsf{S}_w$ and otherwise, it is a subgroup of index 1 or 2 of $\mathsf{C}_{2e} \wr \mathsf{S}_w$, see [8, §3A]. In all cases, $\mathcal{W}$ has a factor group isomorphic to S_w. Note that p is good for $\mathbf{G}$. By generalized d-Harish-Chandra theory [8, Theorems 3.2 and 5.24], there is a natural bijection between unipotent characters in the principal p-block of G and the irreducible characters of $\mathcal{W}$. Furthermore, by [31, Corollary 6.6], the number of unipotent characters in $\mathrm{Irr}_{p'}(B_0(G))$ is at least the number of p'-degree irreducible characters of $\mathcal{W}$. Note that each unipotent character in $\mathrm{Irr}(B_0(G))$ restricts irreducibly to one in $\mathrm{Irr}(B_0(S))$. Indeed, this follows by identifying S with $\mathbf{G}_{sc}^F/\mathbf{Z}(\mathbf{G}_{sc})^F$ under the simply connected covering $\pi\colon \mathbf{G}_{sc} \to \mathbf{G}$ (see [37, Proposition 24.21]) and the bijection between unipotent characters induced by π (see [18, Proposition 2.3.15]), along with the fact that $B_0(G)$ covers a unique block of S.

Recall that $w \geqslant p \geqslant 5$, and thus $n \geqslant 5$. Assume for a moment that G is not $\mathrm{P}(\mathrm{CO}_{2n}^+(q))^0$ with n even. Then, by a result of Lusztig [32, Theorem 2.5], every unipotent character of S is invariant under $\mathrm{Aut}(S)$. Therefore, the number of $\mathrm{Aut}(S)$-orbits on $\mathrm{Irr}_{p'}(B_0(S))$ is at least the number of p'-degree irreducible characters of $\mathcal{W}$, which in turn is at least $|\mathrm{Irr}_{p'}(\mathsf{S}_w)|$. Since $w \geqslant p$, and $p > 2\sqrt{p-1}$ for all $p \geqslant 5$, we are done by using Proposition 3.1(i) and (ii), except possibly if $w \in \{5, 6\}$ and $p = 5$.

(V) Now suppose $w \in \{5, 6\}$ and $p = 5$, and continue to assume G is not $\mathrm{P}(\mathrm{CO}_{2n}^+(q))^0$ with n even. Then part (IV) implies we have at least 5 $\mathrm{Aut}(S)$-orbits on $\mathrm{Irr}_{p'}(B_0(S))$ by considering unipotent characters. We claim that $\mathrm{Irr}_{p'}(B_0(S))$ must contain at least 6 characters.

In the cases of type A, ^{2}A, and B, we may naturally view G as a central quotient of $H := \mathrm{GL}_n(q)$, $\mathrm{GU}_n(q)$, and $\mathrm{SO}_{2n+1}(q)$. In the case of type C, S is a central quotient of $H := \mathrm{Sp}_{2n}(q)$. Therefore, in these cases by [42, Theorem 9.9], $\mathrm{Irr}_{p'}(B_0(G))$ (respectively $\mathrm{Irr}_{p'}(B_0(S))$) can be identified with the members of $\mathrm{Irr}_{p'}(B_0(H))$ that are trivial on $\mathbf{Z}(H)$. Further, the two sets $\mathrm{Irr}_{p'}(B_0(G))$ (respectively $\mathrm{Irr}_{p'}(B_0(S))$) and $\mathrm{Irr}_{p'}(B_0(H))$ can be identified except in the case $5 \mid |\mathbf{Z}(H)|$ (i.e., when $5 \mid (q-1)$ and $H = \mathrm{GL}_n(q)$ or $5 \mid (q+1)$ and $H = \mathrm{GU}_n(q)$). In the case of $\mathrm{D}_n(q)$, and ${}^2\mathrm{D}_n(q)$, let $\bar{H} := \mathrm{GO}_{2n}^\epsilon(q)$. In this case, S is a central quotient of the commutator subgroup Ω of $H := \mathrm{SO}_{2n}^\epsilon(q) \leqslant \bar{H}$ and $\mathrm{Irr}_{p'}(B_0(S))$ may be identified with $\mathrm{Irr}_{p'}(B_0(\Omega))$.

In the cases of type A, ^{2}A, B, and C, write $\bar{H} := H$. Now, by [40, Theorem (1.9)] and [33, Theorem 5.17], there is a bijection between $\mathrm{Irr}_{p'}(B_0(\bar{H}))$ and $\mathrm{Irr}_{p'}(B_0(H_{we}))$, where $H_{we} = \mathrm{GL}_{we}(q), \mathrm{GU}_{we}(q), \mathrm{SO}_{2we+1}(q)$, $\mathrm{Sp}_{2we}(q)$, or $\mathrm{GO}_{2we}^\epsilon(q)$. We further see from the formulas for $|\mathrm{Irr}_{p'}(B_0(H_{we}))|$ in [33, Theorem 5.17] and [40, Proposition (2.13)] that this number is at least 10 in the type A, ^{2}A cases and at least 20 in the other cases. Hence we are done in case B and C. Further, in the case $\bar{H} = \mathrm{GO}_{2n}^\epsilon(q)$, this yields at least 10 characters in $\mathrm{Irr}_{p'}(B_0(H))$ by restricting from $\bar{H}$.

Now consider the case $H = \mathrm{GL}_n(q)$, $\mathrm{GU}_n(q)$, or $\mathrm{SO}_{2n}^\epsilon(q)$. Write $H' := \mathrm{SL}_n(q)$, $\mathrm{SU}_n(q)$, respectively Ω, so that $S = H'/\mathbf{Z}(H')$. Characters of H are partitioned into so-called Lusztig series $\mathcal{E}(H, s)$, indexed by semisimple elements $s \in H^*$, where in this case the dual group H^* is isomorphic to H. In particular, $\mathrm{Irr}(B_0(H))$ lies in the union of $\mathcal{E}(H, s)$

where s has order a power of p, by [12, Theorem 9.12]. Recall from our discussion in (IV) that $B_0(S)$ contains at least 5 unipotent characters of p'-degree. First, suppose that $5 \nmid |\mathbf{Z}(H)|$. Then $5 \nmid |\mathbf{Z}(H')|$ as well, and we have $|\mathrm{Irr}_{p'}(B_0(H))| \geqslant 10$ from the discussion above and $|\mathrm{Irr}_{p'}(B_0(S))| = |\mathrm{Irr}_{p'}(B_0(H'))|$ by [42, Theorem 9.9]. Since restriction gives an injection from unipotent characters in $\mathrm{Irr}_{p'}(B_0(H))$ to $\mathrm{Irr}_{p'}(B_0(H'))$ (see, e.g. [18, Lemma 2.3.14]), we may assume that $\mathrm{Irr}_{p'}(B_0(H))$ contains at least one non-unipotent character. We claim that this character does not have the same restriction to H' as a unipotent character of H, forcing $|\mathrm{Irr}_{p'}(B_0(S))| \geqslant 6$ since $B_0(S)$ contains at least 5 unipotent characters of p'-degree. Indeed, if $\chi \in \mathrm{Irr}(B_0(H))$ is not unipotent but restricts the same as a unipotent character, then χ is the tensor product of a unipotent character with a linear character of H. But linear characters of H are in natural bijection with characters of $\mathbf{Z}(H^*)$, and it follows that $\chi \in \mathcal{E}(H, z)$, where $z \in \mathbf{Z}(H^*) \cong \mathbf{Z}(H)$ is nontrival with order a power of 5 (see, for example, [12, Proposition 8.26]), contradicting that $5 \nmid |\mathbf{Z}(H)|$.

We may now assume $5 \mid |\mathbf{Z}(H)|$. (Note, in particular, that then $H = \mathrm{GL}_n(q)$ or $\mathrm{GU}_n(q)$.) In this case, $e = 1$ and $w = n \in \{5, 6\}$. Here the principal block of H is the unique block containing unipotent characters. Then $\mathrm{Irr}(B_0(H))$ consists of all series $\mathcal{E}(H, s)$ where $s \in H^* \cong H$ has order a power of 5 by [12, Theorem 9.12]. First suppose that $n = 6$. Then there is a semisimple element s of $H \cong H^*$ that lies in H', has order a power of 5, and has $\mathbf{C}_{H^*}(s) \cong \mathrm{GL}_5(q) \times \mathsf{C}_{q-1}$, respectively $\mathrm{GU}_5(q) \times \mathsf{C}_{q+1}$. Then the members of $\mathcal{E}(H, s)$ are trivial on $\mathbf{Z}(H)$ (see, for example, [55, Proposition 2.6]) and restrict to non-unipotent characters of H', and hence S. Since $\mathbf{C}_{H^*}(s)$ is of index prime to 5 in H^*, there is a so-called semisimple character in this series of degree $[H : \mathbf{C}_{H^*}(s)]_{q_0'}$, and hence height-zero, and we are done. Now, consider the case $n = 5$. Then $|\mathbf{Z}(H')| = 5$. In this case, every member of $\mathrm{Irr}_{5'}(B_0(H))$ restricts to one of the five unipotent characters in $\mathrm{Irr}_{5'}(B_0(H'))$. However, consider the element $s \in H'$ of order 5 whose eigenvalues are $\{\zeta, \zeta^2, \zeta^3, \zeta^4, 1\}$, where $\zeta \in \mathbb{F}_{q^2}^\times$ has order 5. We have $\mathbf{C}_{H^*}(s) \cong \mathsf{C}_{q-1}^5$, respectively C_{q+1}^5, so that $[H^* : \mathbf{C}_{H^*}(s)] = 5$. Let $\chi \in \mathcal{E}(H, s)$ be the semisimple character, so that $\chi(1)_5 = 5$. Since $s \in H'$, we have χ is trivial on the center. Further, sz is $H = H^*$-conjugate to s, where $z = \zeta \cdot I_5 \in \mathbf{Z}(H')$. It follows that the restriction of χ to H' is not irreducible, and hence splits into 5 non-unipotent characters in $\mathrm{Irr}_{5'}(B_0(H'))$. Then $|\mathrm{Irr}_{5'}(B_0(S))| \geqslant 6$, as claimed.

(VI) So lastly, suppose $G = \mathrm{P}(\mathrm{CO}_{2n}^+(q))^0$ with $n \geqslant 6$ even. (Recall that $n \geqslant p \geqslant 5$.) Then every unipotent character of S is still invariant under the field automorphisms. The graph automorphism of order 2 fixes all unipotent characters labeled by non-degenerate symbols, but interchanges the two unipotent characters in all pairs labeled by the same degenerate symbol of defect 0 and rank n (see [32, Theorem 2.5] and also [13, p. 471] for the parametrization of unipotent characters of type D groups). Recall from our discussion in (IV) that the number of unipotent characters in $\mathrm{Irr}_{p'}(B_0(G))$ (and therefore the number of such characters in $\mathrm{Irr}_{p'}(B_0(S))$) is at least $|\mathrm{Irr}_{p'}(\mathcal{W})|$. Hence, since $\mathrm{Aut}(S)$ at worst permutes pairs of these characters, in this case it is sufficient to show that $|\mathrm{Irr}_{p'}(\mathcal{W})| > \max\{12, 4\sqrt{p-1}\}$.

Recall that $\mathcal{W}$ is a subgroup of index 1 or 2 in $X := \mathsf{C}_{2e} \wr \mathsf{S}_w$. Fix $\theta \in \mathrm{Irr}(\mathsf{C}_{2e})$. The character $\psi := \theta \times \cdots \times \theta \in \mathrm{Irr}(B)$ of the base subgroup B of X is X-invariant and hence extendible to X, by [39, Lemma 1.3]. It follows that the irreducible characters of X that lie over ψ are in bijective correspondence with irreducible characters of S_w by Gallagher's theorem (see [23, Corollary 6.17]), and therefore the number of those characters of p'-degree is exactly equal to $|\mathrm{Irr}_{p'}(\mathsf{S}_w)|$. According to [8, p. 51], irreducible characters of X are labeled by $2e$-tuples of partitions $(a_i \vdash w_i)$ with $\sum w_i = w$. When $\mathcal{W}$ is a subgroup of index 2 in X, those characters of X that split when restricted to $\mathcal{W}$ are described in [8]. In particular, the p'-degree characters of X lying over ψ discussed above all restrict irreducibly to $\mathcal{W}$. Letting θ be arbitrary in $\mathrm{Irr}(\mathsf{C}_{2e})$, we deduce that the number of irreducible p'-degree characters of $\mathcal{W}$ is at least $2e|\mathrm{Irr}_{p'}(\mathsf{S}_w)|$, which in turn is at least $2p$ by Proposition 3.1. Note again that $p > 2\sqrt{p-1}$ for all $p \geqslant 5$. We see then that we are done unless $p = 5$, $w \in \{5, 6\}$, and $e = 1$.

In the latter case, we have shown that $\mathrm{Irr}_{p'}(B_0(S))$ contains at least 5 $\mathrm{Aut}(S)$-orbits, so it again suffices to show that $\mathrm{Irr}_{p'}(B_0(S))$ contains 6 elements. The exact same argument in (V) in the case $\bar{H} = \mathrm{GO}^{\epsilon}_{2n}(q)$ applies here, and we are done. □

4. Principal blocks with at most 5 height-zero characters

The aim of this section is to prove Theorem 1.2. We begin by recording some divisibility results on $k_0(B)$ for small primes.

Lemma 4.1. *Let p be a prime and G a finite group. Let B be a p-block of positive defect of G.*

(i) *If $p = 2$ then $2 \mid k_0(B)$.*
(ii) *If $p = 3$ then $3 \mid k_0(B)$.*
(iii) *If $p = 2$ and the defect d of B is at least 2, then $4 \mid k_0(B)$. Furthermore, if B has no characters of height one, then $k_0(B) \equiv 2^d (\mathrm{mod}\, 8)$.*

Proof. This follows from [30, Corollaries 1.3 and 1.6] (see also [45, Lemma 2.2] and [53, Theorems 1.6 and 1.7]). □

Theorem 4.2. *Let p be a prime and G a finite group. Let P be a Sylow p-subgroup of G. Then the following are equivalent:*

(i) *$k_0(B_0(G)) = 2$,*
(ii) *$k(B_0(G)) = 2$,*
(iii) *P is cyclic of order 2.*

Proof. The fact that $k(B_0(G)) = 2$ is equivalent to $|P| = 2$ is [5, Theorem A]. Assume that $k_0(B_0(G)) = 2$. If $p = 2$ then $|P| = 2$ by Lemma 4.1(iii), as wanted, and $p = 3$ cannot happen by Lemma 4.1(ii). Now, if $p \geqslant 5$, [19, Theorem A] implies that G is p-solvable. Therefore, by Lemma 2.6, we have $k_0(B_0(G)) \geqslant 2\sqrt{p-1} \geqslant 4$, a contradiction. $\square$

Notice that a finite group G satisfying the equivalent conditions in Theorem 4.2 is always solvable. (Consider the homomorphism $T : G \to \{\pm 1\}$ sending $g \in G$ to the sign of the permutation $G \ni x \mapsto gx$ on G. If t is an involution of G, then $T(t)$ is a product of $|G|/2$ transpositions, and hence is an odd permutation, proving that T is surjective. Therefore G has a (normal and odd-order) subgroup, namely $\mathrm{Ker}(T)$, of index two. By Feit-Thompson's odd-order theorem, it follows that G is solvable.) While Theorem 4.2 on principal blocks with two height-zero characters easily follows from results already appearing in the literature, the following result on blocks with three height-zero characters is much more difficult to prove; in fact, the proof is already nontrivial when one considers just 3-blocks, see the remark before [45, Theorem C].

Theorem 4.3. *Let p be a prime and G a finite group. Let P be a Sylow p-subgroup of G. Then the following are equivalent:*

(i) $k_0(B_0(G)) = 3$,
(ii) $k(B_0(G)) = 3$,
(iii) *P is cyclic of order* 3.

Proof. The fact that $k(B_0(G)) = 3$ implies $|P| = 3$ follows from the main result of [4] (we refer the reader to [26, Theorem 3.1] for an independent proof of this result). Moreover, if $|P| = 3$ then [42, Theorem 11.1] implies that $k_0(B_0(G)) = k(B_0(G)) = 3$. Therefore, it remains to prove that (i) implies (iii). So assume that $k_0(B_0(G)) = 3$.

By Lemma 4.1(i), we may assume that $p \geqslant 3$, and as the statement we need to prove is precisely [45, Theorem C] when $p = 3$, we may assume furthermore that $p \geqslant 5$. Our aim is now to show that if $P > 1$ then $k_0(B_0) \geqslant 4$.

Notice that if G is p-solvable, then $k_0(B_0(\mathbf{N}_G(P))) = k(\mathbf{N}_G(P)/\mathbf{O}_{p'}(\mathbf{N}_G(P))P')$ by Theorem 2.5. That number can be seen to be greater than or equal to 4 by looking at [58, Table 1]. We may thus assume that G is not p-solvable.

We consider a chief series $1 = G_0 < G_1 < \cdots < G_n = G$ of G with $G_j \trianglelefteq G$ for every $0 \leqslant j \leqslant n$. Let k be maximal such that p divides $[G_{k+1} : G_k]$. Since $k_0(B_0) \geqslant k_0(B_0(G/G_k))$, in order to show that $k_0(B_0) \geqslant 4$ we may assume that $G_k = 1$, and thus $N := G_{k+1}$ is a minimal normal subgroup of G of order divisible by p with $[G : N]$ not divisible by p. If N is abelian, then G is p-solvable. Hence N is semisimple with, say t, simple chief factors isomorphic to the simple non-abelian group S (of order divisible by p).

Write $M = N\mathbf{C}_G(P)$. Since $P \in \mathrm{Syl}_p(N)$, by the Frattini argument, $G = N\mathbf{N}_G(P)$ so that $M \trianglelefteq G$. By Lemma 2.3 we have that $k(G/M) < k_0(B_0)$. If $k(G/M) \geqslant 3$, then we are done. Hence we may assume that $[G : M] \leqslant 2$. Again by Lemma 2.3, for every $\eta \in \mathrm{Irr}_{p'}(B_0(M))$ we have that $\mathrm{Irr}(G|\eta) = \mathrm{Irr}_{p'}(G|\eta) \subseteq \mathrm{Irr}_{p'}(B_0)$. Note that if

$k_0(B_0(M)) \geqslant 4$ and $[G : M] = 2$ then there would be at least two G-orbits of nontrivial members of $\mathrm{Irr}_{p'}(B_0(M))$, and so $k_0(B_0(G)) \geqslant 2 + k(G/M) = 4$. In particular, we would be done if $k_0(B_0(M)) \geqslant 4$, and thus we may assume $G = M$.

By Theorem 2.4 we have that $k_0(B_0) = k_0(B_0(N)) = k_0(B_0(S))^t$. By [42, Problem 3.11] a block of positive defect contains at least two height-zero characters. Therefore, if $t > 1$ then $k_0(B_0) \geqslant 4$. We may assume that $t = 1$ and that $G = S$ is a simple non-abelian group of order divisible by $p \geqslant 5$.

By Proposition 3.2, we may assume that P is abelian. Then $k_0(B_0) = k(B_0)$ by the main result of [25], and $k(B_0) \geqslant 2\sqrt{p-1} \geqslant 4$ by [22, Theorem 1.1]. □

We remark that Theorems 4.2 and 4.3 prove Theorem 1.2(A). In order to prove parts (B) and (C) of Theorem 1.2, we make use of the classification of Sylow p-subgroups of finite groups with precisely four or five ordinary irreducible characters in the principal p-block worked out in [26,53]. We record this classification in the following two results. In this note D_{2n} is the dihedral group of order $2n$ and Q_8 is the quaternion group.

Theorem 4.4. *Let G be a finite group and p a prime. Let B_0 denote the principal p-block of G. Then $k(B_0) = 4$ if, and only if, exactly one of the following happens:*

(i) $|P| = 4$,
(ii) $|P| = 5$ *and* $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] = 2$.

Proof. The 'if' implication is clear by Lemma 4.1 when $p = 2$ and [42, Theorem 11.1] when $p = 5$. Assume that $k(B_0(G)) = 4$. By [26], then $P \in \{\mathsf{C}_2 \times \mathsf{C}_2, \mathsf{C}_4, \mathsf{C}_5\}$. Moreover, if $|P| = 5$, then $k(B_0(G)) = 4$ forces $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] = 2$ by [42, Theorem 11.1]. □

Theorem 4.5. *Let G be a finite group and p a prime. Let B_0 denote the principal p-block of G. Then $k(B_0) = 5$ if, and only if, precisely one of the following happens:*

(i) $P = \mathsf{D}_8$,
(ii) $P = \mathsf{Q}_8$ *and* $\mathbf{N}_G(P) = P\mathbf{C}_G(P)$,
(iii) $|P| = 5$ *and* $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] \in \{1, 4\}$,
(iv) $|P| = 7$ *and* $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] \in \{2, 3\}$.

Proof. The 'if' implication follows from results of Brauer [42, Theorem 11.1] when $|P| = p$ and of Brauer [7, Theorem 7B] and Olsson [49, Theorem 3.13] when $P \in \{\mathsf{D}_8, \mathsf{Q}_8\}$. For the reverse implication, notice that, by the discussion above, it suffices to show that $P \in \{\mathsf{C}_5, \mathsf{C}_7, \mathsf{D}_8, \mathsf{Q}_8\}$. That is the main result of [53]. □

Next we prove part (B) of Theorem 1.2. Recall that if $\chi \in \mathrm{Irr}(G)$, then $\det(\chi)$ is a linear character of G uniquely determined by χ (see [23, Problem 2.3]). The determinantal order $o(\chi) = |G/\mathrm{Ker}(\det(\chi))|$ of χ is related to character extension properties.

Theorem 4.6. *Let p be a prime and G a finite group. Let P be a Sylow p-subgroup of G. Then $k_0(B_0(G)) = 4$ if, and only if, exactly one of the following happens:*

(i) $[P : P'] = 4$,
(ii) $|P| = 5$ *and* $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] = 2$.

Proof. By [45] the statement holds if $p = 2$, so we may assume p is odd.

If $|P| = 5$ and $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] = 2$ then $k_0(B_0) = 4$ by [42, Theorem 11.1], and the 'if' implication holds.

Suppose that $k_0(B_0) = 4$. We want to prove the 'only if' implication. We may further assume that $p \geqslant 5$ by Lemma 4.1(ii). By [42, Theorem 11.1] it is enough to show that if $k_0(B_0) = 4$ and $p \geqslant 5$, then $|P| = 5$. Let G be a counterexample of minimal order to such a statement.

Step 1. G is not p-solvable.

Write $K := \mathbf{O}_{p'}(\mathbf{N}_G(P))$. Assume, to the contrary, that G is p-solvable. Then by Theorem 2.5, we have that $k(\mathbf{N}_G(P)/KP') = 4$. Inspecting [58, Table 1], we see that $\mathbf{N}_G(P)/KP' \cong \mathsf{D}_{10}$. In particular, $[P : P'] = 5$, implying $|P| = 5$ and thus contradicting the choice of G as a counterexample.

Step 2. $\mathbf{O}_{p'}(G) = 1$.

Notice that $k_0(B_0(G/\mathbf{O}_{p'}(G))) = 4$ by [42, Theorem 9.9(c)], so $\mathbf{O}_{p'}(G) = 1$ by the minimality of G as a counterexample.

Step 3. Let $1 \neq N$ be a minimal normal subgroup of G. Then p does not divide $[G : N]$.

Assume otherwise, so that $1 < k_0(B_0(G/N)) \leqslant 4$. The fact that $p \geqslant 5$ implies $k_0(B_0(G/N)) = 4$, by Theorems 4.2 and 4.3. By the minimality of G as a counterexample, $p = 5$ and $[PN : N] = 5$.

The fact that $k_0(B_0(G/N)) = k_0(B_0)$ in particular means that every $\chi \in \mathrm{Irr}_{p'}(B_0)$ lies over $\mathbf{1}_N$. By Lemma 2.2(i) we conclude that no $\mathbf{1}_N \neq \theta \in \mathrm{Irr}_{p'}(B_0(N))$ extends to PN. By Step 2, the group N has order divisible by p and there are 2 cases.

Case (a). Suppose that N is an elementary abelian p-group, so $N \subseteq P$. Then P acts on N necessarily fixing some non-trivial element of N. Hence, there exists some $1_N \neq \theta \in \mathrm{Irr}_{p'}(B_0(N))$ that is P-invariant. By [23, Theorem 11.22], θ extends to P, and we get a contradiction.

Case (b). Suppose that N is semisimple with t chief factors isomorphic to S. By [19, Proposition 2.1] there is some $\mathbf{1}_S \neq \theta \in \mathrm{Irr}_{p'}(B_0(S))$ invariant under the action of a Sylow p-subgroup of $\mathrm{Aut}(S)$. Let $\mathbf{1}_N \neq \psi$ be equal to the direct product of t copies of θ in N. Then $\psi \in \mathrm{Irr}_{p'}(B_0(N))$ is P-invariant and ψ extends to PN by [23, Theorem 11.22], again yielding a contradiction. (Note that in this case $o(\psi) = 1$ because N is perfect, so another way of arguing that ψ extends to PN is by using [23, Corollary 8.16].)

Step 4. By Steps 1 and 3, we have that N is semisimple with t chief factors isomorphic to S, a simple non-abelian group of order divisible by p. Let $M = N\mathbf{C}_G(P)$. Then $M = G$.

By the Frattini argument, $G = N\mathbf{N}_G(P)$, and hence $M \trianglelefteq G$. Notice that the elements in $\mathrm{Irr}_{p'}(B_0(M))$ are the irreducible constituents of χ_M for every $\chi \in \mathrm{Irr}_{p'}(B_0)$.

Suppose that $M < G$. Then by Lemma 2.3 we have that $1 < k(G/M) < 4$. This leaves two possibilities.

First assume $k(G/M) = 2$, and so $[G : M] = 2$. Write $\mathrm{Irr}_{p'}(B_0) = \{\mathbf{1}_G, \alpha, \beta, \gamma\}$ where $M \subseteq \mathrm{Ker}(\alpha)$. If $\beta_M = \gamma_M$, then $k_0(B_0(M)) = 2$, which is absurd as $p \geqslant 5$. Otherwise $k_0(B_0(M)) = 5$. By Theorem 2.4, we have that $5 = k(B_0(S))^t$. This forces $t = 1$, $P \subseteq S$ and $k_0(B_0(S)) = 5$. By [19, Proposition 2.1] some $\mathbf{1}_S \neq \theta \in \mathrm{Irr}_{p'}(B_0(S))$ is $\mathrm{Aut}(S)$-invariant. By Theorem 2.4, let $\varphi \in \mathrm{Irr}_{p'}(B_0(M))$ be such that $\varphi_S = \theta$. For every $g \in G$, $\varphi^g \in \mathrm{Irr}_{p'}(B_0(M))$ extends $\theta^g = \theta$. By Theorem 2.4, φ is G-invariant. Consequently, φ has 2 extensions in $\mathrm{Irr}_{p'}(B_0)$, those must be β and γ by Lemma 2.1. Then $\beta_M = \gamma_M$, a contradiction.

Secondly assume that $k(G/M) = 3$. Then every nontrivial $\theta \in \mathrm{Irr}_{p'}(B_0(M))$ lies under the same member of $\mathrm{Irr}_{p'}(B_0)$. Hence $|\{\psi(1) \mid \psi \in \mathrm{Irr}_{p'}(B_0(M))\}| \leqslant 2$. By the main result of [19] we get that M is p-solvable, and hence so is G, contradicting Step 1.

Final step. We have $G = N\mathbf{C}_G(P)$, where N is semisimple with t chief factors isomorphic to S. By Theorem 2.4, $4 = k_0(B_0) = k_0(B_0(N)) = k_0(B_0(S))^t$. As $p \geqslant 5$, this forces $t = 1$, $P \subseteq S$, and $k_0(B_0(S)) = 4$. By Proposition 3.2, P is abelian. Then $k_0(B_0) = k(B_0) = 4$ by [25]. Then Theorem 4.4 implies that $|P| = 5$, the final contradiction. $\square$

Finally, we classify groups with 5 height-zero characters in the principal block, thus completing the proof of Theorem 1.2.

Theorem 4.7. *Let G be a finite group and p a prime. Let $P \in \mathrm{Syl}_p(G)$ and let B_0 denote the principal p-block of G. Then $k_0(B_0) = 5$ if, and only if, precisely one of the following happens:*

(i) $|P| = 5$ *and* $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] \in \{1, 4\}$.
(ii) $|P| = 7$ *and* $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] \in \{2, 3\}$.

Proof. First we remark that the 'if part' follows by [42, Theorem 11.1].

Assume that $k_0(B_0) = 5$. By Lemma 4.1, p cannot be 2 or 3, and hence $p \geqslant 5$. By [42, Theorem 11.1], it suffices to show that if $k_0(B_0) = 5$ and $p \geqslant 5$, then $|P| \in \{5, 7\}$. Assume that G is a counterexample of minimal order to such a statement. By the main result of [25] and Theorem 4.5, we have that P is not abelian. Also we can see that G is not p-solvable and $\mathbf{O}_{p'}(G) = 1$, proceeding as in the proof of the case $k_0(B_0) = 4$. (Some arguments will be similar to ones used in the proof of Theorem 4.6 so here we will just sketch those.)

Let N be a minimal normal subgroup of G, with $N \neq 1$. We first show that p does not divide the index $[G : N]$.

Assume otherwise, so that $1 < k_0(B_0(G/N)) \leqslant 5$. As $p \geqslant 5$, then $4 \leqslant k_0(B_0(G/N)) \leqslant 5$. In the case where $k_0(B_0(G/N)) = 5$, we obtain a contradiction from Lemma 2.2(i) as we can always find some $\theta \in \mathrm{Irr}_{p'}(B_0(N))$ that extends to PN (note that by minimality of G as a counterexample PN/N is cyclic and we can proceed as in the proof of the case $k_0(B_0) = 4$).

Hence $k_0(B_0(G/N)) = 4$. By Theorem 4.6, we have that $[PN : N] = 5$. Notice that in this case $\mathrm{Irr}_{p'}(B_0) = \{\mathbf{1}_G, \alpha, \beta, \gamma, \chi\}$, where χ is the only member of $\mathrm{Irr}_{p'}(B_0)$ not belonging to $\mathrm{Irr}_{p'}(B_0(G/N))$. We distinguish the cases where N is abelian and semisimple.

Case (a). Suppose that N is abelian, then N is an elementary abelian p-group and $N \leqslant P$. Since P is not abelian, and as P/N is cyclic of order 5, then $P \cap \mathbf{C}_G(N) = N$. Hence $N \in \mathrm{Syl}_p(\mathbf{C}_G(N))$. Since $\mathbf{O}_{p'}(G) = 1$, that implies $\mathbf{C}_G(N) = N$.

Let $1_N \neq \theta \in \mathrm{Irr}(N)$ be P-invariant. Since P/N is cyclic, θ extends to P by [23, Theorem 11.22]. Take $Q/N \in \mathrm{Syl}_q(G_\theta/N)$ with $q \neq p$. Then θ extends to Q by [23, Corollary 8.16] as $(|Q/N|, o(\theta)\theta(1)) = 1$. By [23, Corollary 11.31] θ extends to G_θ. By the Fong-Reynolds correspondence [42, Theorem 9.14],

$$|\mathrm{Irr}_{p'}(B_0|\theta)| = |\mathrm{Irr}_{p'}(B_0(G_\theta)|\theta)| \,.$$

By Lemma 2.2(i) some $\mathrm{Irr}_{p'}(B_0)$ lies over θ. Under our assumptions, χ is the only member of $\mathrm{Irr}_{p'}(B_0)$ possibly lying over a nontrivial character of N. Then

$$|\mathrm{Irr}_{p'}(B_0|\theta)| = 1.$$

Let $b_0 = B_0(N)$. By [42, Corollary 9.21], we have that $b_0^{G_\theta} = B_0(G_\theta)$ is the only block of G_θ covering b_0. Let $\eta \in \mathrm{Irr}(G_\theta)$ be an extension of θ. In particular, η lies in $B_0(G_\theta)$. By Lemma 2.2(ii)

$$|\mathrm{Irr}_{p'}(B_0(G_\theta)|\theta)| = |\mathrm{Irr}_{p'}(G_\theta/\mathbf{C}_G(N)))| \geqslant 2 \,,$$

a contradiction.

Case (b). Suppose that N is semisimple with t chief factors isomorphic to S. By [19, Proposition 2.1] there are $\mathbf{1}_S \neq \alpha, \beta \in \mathrm{Irr}_{p'}(B_0(S))$ invariant under the action of a Sylow p-subgroup of $\mathrm{Aut}(S)$ with $\alpha(1) \neq \beta(1)$. Let $\mathbf{1}_N \neq \psi$ be equal to the direct product of t copies of α in N and $\mathbf{1}_N \neq \varphi$ be equal to the direct product of t copies of β in N. Then $\psi, \varphi \in \mathrm{Irr}_{p'}(B_0(N))$ are P-invariant. Alos $o(\psi) = 1 = o(\varphi)$ because N is perfect. By [23, Corollary 8.16] both ψ and φ extend to PN, yielding a contradiction by Lemma 2.2(i).

We have shown that p does not divide $[G : N]$. In particular, N is semisimple with, say t, chief factors isomorphic to the non-abelian simple group S (of order divisible by p). Take $M = N\mathbf{C}_G(P) \trianglelefteq G$. Then $1 \leqslant k(G/M) < 5$ by Lemma 2.3. We show that $G = M$ by analyzing the different values $1 < k(G/M) < 5$. Before proceeding with the analysis, we make the following observation. By [19, Proposition 2.1] some $\mathbf{1}_S \neq \varphi \in \mathrm{Irr}_{p'}(B_0(S))$ is $\mathrm{Aut}(S)$-invariant. In particular, if θ is the direct product of t copies of φ, then $\theta \in \mathrm{Irr}_{p'}(B_0(N))$ is G-invariant. By Theorem 2.4, let $\psi \in \mathrm{Irr}_{p'}(B_0(M))$ be such that $\psi_S = \theta$. Then $\mathbf{1}_M \neq \psi$ is a G-invariant member of $\mathrm{Irr}_{p'}(B_0(M))$.

If $k(G/M) = 2$, then $[G : M] = 2$. Write $\mathrm{Irr}_{p'}(B_0) = \{\mathbf{1}_G, \alpha, \beta, \gamma, \chi\}$ where $M \subseteq \mathrm{Ker}(\alpha)$. Since ψ extends to G, we may assume that β and γ are the two extensions of ψ. In particular, χ_M must decompose as the sum of two distinct members of $\mathrm{Irr}_{p'}(B_0(M))$. In particular, $|\mathrm{Irr}_{p'}(B_0(M))| = k_0(B_0(M)) = 4$ and by Theorem 4.6 we obtain $|P| = 5$, a contradiction.

If $k(G/M) = 3$, then G/M is isomorphic to C_3 or S_3. Write $\mathrm{Irr}_{p'}(B_0) = \{\mathbf{1}_G, \alpha, \beta, \gamma, \chi\}$, where α and β contain M in their respective kernels. Recall that $\mathbf{1}_M \neq \psi \in \mathrm{Irr}_{p'}(B_0(M))$ is G-invariant. Notice that $|\mathrm{Irr}_{p'}(B_0|\psi)| = |\mathrm{Irr}(G|\psi)| \geqslant 3$, which is impossible.

If $k(G/M) = 4$, then every nontrivial $\eta \in \mathrm{Irr}_{p'}(B_0(M))$ lies under the same member of $\mathrm{Irr}_{p'}(B_0)$. Hence $|\{\eta(1) \mid \eta \in \mathrm{Irr}_{p'}(B_0(M))\}| \leqslant 2$. By the main result of [19] we conclude that M is p-solvable, then so is G, a contradiction.

Finally, if $G = M$, then by Theorem 2.4 we have that $k_0(B_0(S))^t = 5$. Hence $t = 1$ and $k_0(B_0) = 5$. By Proposition 3.2, P must be abelian, a contradiction. $\square$

5. Bounding height-zero characters in principal blocks

In this section we prove Theorem 1.1. We begin with a technical result due to G. Navarro.

Lemma 5.1 *(Navarro). Let $S_1 \times \cdots \times S_t = N \trianglelefteq G$, where $\{S_1, ..., S_t\}$ are transitively permuted by conjugation of G; $S_i = S_1^{x_i}$ for some $x_i \in G$, and have order divisible by a prime p. Let $\theta := \theta_1 \in \mathrm{Irr}(S_1)$ such that $\mathbf{Z}(S_1) \subseteq \mathrm{Ker}(\theta)$ and that there exists $\alpha \in \mathrm{Irr}_{p'}(B_0(\mathbf{N}_G(S_1)/\mathbf{C}_G(S_1)))$ with $\alpha_{S_1} = e\theta$ for some $e \in \mathbb{N}$. Set $\psi := \theta_1 \times \cdots \times \theta_t$ where $\theta_i := \theta_1^{x_i}$. Then there exists $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ such that $\chi_N = a\psi$ for some $e^t \geqslant a \in \mathbb{N}$.*

Proof. This is the content of [38, Lemma 4.4]. $\square$

Lemma 5.1 is useful when one wants to produce characters in $\mathrm{Irr}_{p'}(B_0(G))$ that lie above certain characters of a non-abelian minimal normal subgroup of G. In such a situation, the existence of θ and α satisfying the hypothesis of Lemma 5.1 is presented in the following, which is [19, Proposition 2.1].

Lemma 5.2. *Let S be a non-abelian simple group of order divisible by a prime $p \geqslant 5$. Then there exist $1_S \neq \theta \in \mathrm{Irr}_{p'}(S)$ and $\alpha \in \mathrm{Irr}_{p'}(B_0(\mathrm{Aut}(S)))$ such that $\alpha_S \in \{\theta, 2\theta\}$. Further, when S is not $P\Omega_8^+(q)$, one may choose α so that it extends θ.*

We can now prove Theorem 1.1 in the case of non-abelian Sylow subgroups.

Theorem 5.3. *Let G be a finite group and p a prime. Assume that the Sylow p-subgroups of G are non-abelian. Then $k_0(B_0(G)) > 2\sqrt{p-1}$.*

Proof. First, if $p \leqslant 7$ then it is sufficient to assume that $k_0(B_0(G)) \leqslant 4$. However, by Theorem 1.2, in such case, P is abelian or $k_0(B_0(G)) = 4$ and $p = 2$, and thus we are done by Theorem 2.7. Therefore, we may and will assume from now on that $p \geqslant 11$.

We adapt some arguments in the proof of [22, Theorem 1.1]. Let G be a counterexample with minimal order. In particular, $\mathbf{O}_{p'}(G)$ is trivial, $P \in \mathrm{Syl}_p(G)$ is non-abelian, and $k_0(B_0(G)) \leqslant 2\sqrt{p-1}$. Let $1 \neq N$ be a minimal normal subgroup of G. We claim that p does not divide $[G : N]$.

Assume, to the contrary, that $p \mid [G : N]$. Then $PN/N \in \mathrm{Syl}_p(G/N)$ must be abelian, by the fact $k_0(B_0(G)) \geqslant k_0(B_0(G/N))$ and the minimality of G. It then follows from Theorem 2.7 that $k_0(B_0(G/N)) \geqslant 2\sqrt{p-1}$. Altogether, we deduce that

$$k_0(B_0(G)) = k_0(B_0(G/N)) = 2\sqrt{p-1}.$$

Assume that N is abelian, which means that N is actually an elementary abelian p-group, because $\mathbf{O}_{p'}(G) = 1$. Let $\mathbf{1}_N \neq \theta \in \mathrm{Irr}(N)$ be P-invariant. Theorem 2.7 implies that $P/N \in \mathrm{Syl}_p(G/N)$ is of order p, and it follows that θ extends to P. By Lemma 2.2(i), we deduce that there exists some $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ that lies over θ. We now have $k_0(B_0(G)) > k_0(B_0(G/N))$, violating the conclusion of the previous paragraph.

We may assume that N is non-abelian. Suppose that S is a simple direct factor of N, and notice that p divides the order of S, because $\mathbf{O}_{p'}(G) = 1$. By Lemma 5.2, there exist $\theta \in \mathrm{Irr}_{p'}(S)$ and $\alpha \in \mathrm{Irr}_{p'}(B_0(\mathrm{Aut}(S)))$ such that $\alpha_S \in \{\theta, 2\theta\}$. Lemma 5.1 then implies that there exists $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ such that $N \nsubseteq \mathrm{Ker}(\chi)$, again violating the equality $k_0(B_0(G)) = k_0(B_0(G/N))$. The claim $p \nmid [G : N]$ is now fully proved.

Recall that $p \mid |N|$. By Lemma 2.6, we are done if N is abelian, so let us assume that N is not, and furthermore, as above let S be a (non-abelian) simple factor of N. By Proposition 3.2, there are more than $2\sqrt{p-1}$ different $\mathbf{N}_G(S)$-orbits on $\mathrm{Irr}_{p'}(B_0(S))$. If two characters $\eta, \theta \in \mathrm{Irr}_{p'}(B_0(S))$ are not conjugate under the action of $\mathbf{N}_G(S)$ then the characters $\eta \times \cdots \times \eta$ and $\theta \times \cdots \times \theta$ of N are not conjugate under the action of G. We deduce that there are more than $2\sqrt{p-1}$ different G-orbits on $\mathrm{Irr}_{p'}(B_0(N))$. It immediately follows that $k_0(B_0(G)) > 2\sqrt{p-1}$ since there is a character in $\mathrm{Irr}_{p'}(B_0(G))$ lying over characters in each such G-orbit, by Lemma 2.2(i). □

The following result covers Theorem 1.1 in the introduction, where we also analyze the local structure of a group with $k_0(B_0(G)) = 2\sqrt{p-1}$. The equivalence of (i) and (iv) was already shown in [22, Theorem 1.3].

Theorem 5.4. *Let G be a finite group and p a prime such that $p \mid |G|$. Then $k_0(B_0(G)) \geqslant 2\sqrt{p-1}$. Moreover, for $P \in \mathrm{Syl}_p(G)$, the following are equivalent:*

(i) $k(B_0(G)) = 2\sqrt{p-1}$.

(ii) $k_0(B_0(G)) = 2\sqrt{p-1}$.
(iii) $k_0(B_0(\mathbf{N}_G(P))) = 2\sqrt{p-1}$.
(iv) $\sqrt{p-1} \in \mathbb{N}$ *and* $\mathbf{N}_G(P)/\mathbf{O}_{p'}(\mathbf{N}_G(P))$ *is isomorphic to the Frobenius group* $\mathsf{C}_p \rtimes \mathsf{C}_{\sqrt{p-1}}$.

Proof. The first statement follows from Theorem 2.7 (which is a consequence of [22, Theorem 1.1] and [25, Theorem 1.1]) and Theorem 5.3. In fact, these results also imply the equivalence of (i) and (ii). The fact that (i) is equivalent to (iv) is precisely [22, Theorem 1.3], and the equivalence of (iii) and (iv) follows by Lemma 2.6. □

We remark that the second statement of Theorem 5.4 is consistent with both Brauer's height zero conjecture and the Alperin-McKay conjecture for principal blocks. We have learned that the unproven half of Brauer's height zero conjecture for principal blocks has been confirmed very recently by Malle and Navarro [35]. However, note that our proofs are independent of this result.

6. On Conjecture 1.3

We end the paper with some discussion on Conjecture 1.3. It asserts that, if one fixes the number of height-zero characters in a p-block of a finite group, then $[D : D']$ is bounded, where D is a defect group of the block. The conjecture therefore may be viewed as the analogue of Brauer's Problem 21 [6] for height-zero characters.

Lemma 6.1. *Conjecture 1.3 follows from the Alperin-McKay conjecture.*

Proof. Fix a positive integer k_0 and let B be a p-block of a finite group G with precisely k_0 height-zero characters. Let D be a defect group of B. Assume that the Alperin-McKay conjecture holds. Then $k_0(B) = k_0(b)$ where b, a block of $\mathbf{N}_G(D)$, is the Brauer correspondent of B. By a result of Reynolds (see [28, p. 399]), there is a finite group K with D as a normal Sylow p-subgroup and a block β of K such that $k_0(b) = k_0(\beta)$. Now β contains a block $\overline{\beta}$ of K/D' with defect group D/D' and $k_0(\beta) = k(\overline{\beta})$ (see [28, Theorem 6]). All together, we have

$$k(\overline{\beta}) = k_0.$$

As Brauer's Problem 21 has been known to have a positive answer for p-solvable groups by Külshammer and Robinson [29], it follows that $|D/D'|$ is bounded, as desired. □

We now turn to the principal block case of Conjecture 1.3. Recall that $p \leqslant k_0^2/4 + 1$ by Theorem 1.1, where $k_0 := k_0(B_0(G))$. Moreover,

$$[P : P'] \leqslant p^{\log_p(\exp(P/P'))\cdot \mathrm{rk}(P/P')}.$$

Conjecture 1.3 is therefore reduced to showing that $\log_p(\exp(P/P'))$ and $\mathrm{rk}(P/P')$ are both bounded in terms of k_0.

Note that $\mathrm{rk}(P/P') = \log_p([P : \Phi(P)])$, where $\Phi(P)$ is the Frattini subgroup of P. The problem of bounding $\mathrm{rk}(P/P')$ in terms of k_0 seems highly nontrivial to us at the moment. On the other hand, the problem of determining $\log_p(\exp(P/P'))$ appears to be related to the Alperin-McKay-Navarro conjecture. We take advantage of recent advances [46,47] on the study of fields of values of characters of degree not divisible by p to prove that $\exp(P/P')$ is bounded in terms of k_0 when $p = 2$ in Theorem 6.2 below.

We first need to introduce some notation. The field of values of $\chi \in \mathrm{Irr}(G)$ is $\mathbb{Q}(\chi) := \mathbb{Q}(\chi(g) \mid g \in G)$. Notice that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{\exp(G)}$, where for an integer m, we write $\mathbb{Q}_m := \mathbb{Q}(e^{2\pi i/m})$. We define $c(\chi)$ as the smallest positive integer c such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_c$. The number $c(\chi)$ has been referred to as the *Feit number* of χ in connection with a conjecture by W. Feit [43, §3.3] and as the *conductor* of χ [47]. We recall that χ is said to be p-rational if p does not divide $c(\chi)$. Moreover, in [21, §2], $c_p(\chi)$ the *p-rationality level* of χ is defined as the nonnegative integer $\log_p(c(\chi)_p)$, where n_p is the p-part of the integer n. The p-rationality level of χ measures how p-rational χ is. Indeed, χ is p-rational if, and only if, $c_p(\chi) = 0$.

The Galois group $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ acts on the set of irreducible characters of any finite group G preserving character degrees. It also acts on the set of height-zero characters of principal blocks of finite groups as discussed in Section 2. For a positive integer e, let σ_e denote the automorphism in $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ that fixes roots of unity of order not divisible by p and sends p-power roots of unity ξ to ξ^{1+p^e}. By [46, Theorem B], we know that if e is any positive integer such that all of the height-zero characters in the principal p-block of G are fixed by σ_e, then $\log_p(\exp(P/P'))$ is at most e.

Theorem 6.2. *Let $p = 2$ and $P \in \mathrm{Syl}_p(G)$. Then $\exp(P/P')$ is bounded in terms of $k_0 := k_0(B_0(G))$. In fact,*

$$\exp(P/P') \leqslant 2(k_0 - 1)$$

whenever P is nontrivial.

Proof. Let $B_0 = B_0(G)$ denote the principal p-block of G and set

$$e(G) := \max_{\chi \in \mathrm{Irr}_{p'}(B_0)} \{\log_p(c(\chi)_p)\}.$$

So this $e(G)$ is the largest p-rationality level of a character in $\mathrm{Irr}_{p'}(B_0)$. First suppose that $e(G) = 0$. Then all the characters in $\mathrm{Irr}_{p'}(B_0)$ are p-rational and therefore σ_1-invariant. It follows from [46, Theorem B] that $\exp(P/P') \leqslant p = 2$, and the theorem in turn follows since $k_0 \geqslant 2$ when $P > 1$ by Theorem 1.1.

So let $e(G) \geqslant 1$. Then all the characters in $\mathrm{Irr}_{p'}(B_0)$ are $\sigma_{e(G)}$-invariant, and therefore by [46, Theorem B] we have

$$\log_p(\exp(P/P')) \leqslant e(G).$$

Let $\psi \in \mathrm{Irr}_{p'}(B_0)$ be such that $c(\psi)_p = p^{e(G)}$; that is, choose $\psi \in \mathrm{Irr}_{p'}(B_0)$ with maximal p-rationality level. By [47, Theorem A1], we have $\mathbb{Q}_{p^{e(G)}} \subseteq \mathbb{Q}(\psi)$ and it follows that

$$[\mathbb{Q}(\psi) : \mathbb{Q}] \geqslant [\mathbb{Q}_{p^{e(G)}} : \mathbb{Q}] = (p-1)p^{e(G)-1} = p^{e(G)-1}.$$

On the other hand, any Galois conjugate of ψ belongs to $\mathrm{Irr}_{p'}(B_0(G))$. As the number of those conjugates is exactly $[\mathbb{Q}(\psi) : \mathbb{Q}]$, we deduce that

$$k_0 - 1 \geqslant [\mathbb{Q}(\psi) : \mathbb{Q}].$$

(Note that the 'minus 1' comes from the fact that the trivial character is not among the conjugates of ψ.) The last three displayed inequalities imply that

$$\exp(P/P') \leqslant 2(k_0 - 1),$$

and this concludes the proof. $\square$

The proof of Theorem 6.2 in fact shows that $\exp(P/P')/2 + 1$ is bounded above by the number of characters in $\mathrm{Irr}_{p'}(B_0)$ with maximal p-rationality level.

One might naturally ask what happens when p is odd. The p-odd analogue of [47, Theorem A1] is not true in general. Navarro and Tiep proposed in [47, Conjecture B3 and Theorem B1] that, if $\chi \in \mathrm{Irr}_{p'}(G)$ with $c(\chi)_p = p^a$, then $[\mathbb{Q}_{p^a} : (\mathbb{Q}(\chi) \cap \mathbb{Q}_{p^a})]$ is not divisible by p. If that turns out to be true, one may follow the same arguments as in the proof of Theorem 6.2 to show that

$$[\mathbb{Q}(\psi) : \mathbb{Q}] \geqslant p^{e(G)-1},$$

whenever ψ is a character in $\mathrm{Irr}_{p'}(B_0(G))$ with maximal p-rationality level. It would follow then that $e(G)$, and hence $\exp(P/P')$, is bounded in terms of the number k_0 of height-zero irreducible characters in $B_0(G)$. Note that the bound $[\mathbb{Q}(\psi) : \mathbb{Q}] \geqslant p^{e(G)-1}$ does not directly imply that p is bounded in terms of k_0 since $e(G)$ could be 1. Therefore we do need Theorem 1.1 for this argument to work.

Data availability

No data was used for the research described in the article.

References

[1] J.L. Alperin, The main problem of block theory, in: Proc. Conf. Finite Groups, Academic Press, New York, 1975, pp. 341–356.
[2] J.L. Alperin, Isomorphic blocks, J. Algebra 43 (1976) 694–698.

[3] L. Barker, On p-soluble groups and the number of simple modules associated with a given Brauer pair, Q. J. Math. Oxf. Ser. 48 (1997) 133–160.
[4] V.A. Belonogov, Finite groups with a small principal p-block, in: Group-Theoretic Investigations, Akad. Nauk SSSR Ural. Otdel., Sverdlovsk, 1990, pp. 8–30 (in Russian).
[5] J. Brandt, A lower bound for the number of irreducible characters in a block, J. Algebra 2 (1982) 509–515.
[6] R. Brauer, Representations of finite groups, in: Lectures on Modern Mathematics, Vol. I, 1963, pp. 133–175.
[7] R. Brauer, Some applications of the theory of blocks of characters of finite groups III, J. Algebra 3 (1966) 225–255.
[8] M. Broué, G. Malle, J. Michel, Generic blocks of finite reductive groups, Astérisque 212 (1993) 7–92.
[9] O. Brunat, On the inductive McKay condition in the defining characteristic, Math. Z. 263 (2009) 411–424.
[10] W. Burnside, Theory of Groups of Finite Order, 2nd ed., Dover, 1955.
[11] M. Cabanes, Local methods for blocks of finite simple groups, in: Local Representation Theory and Simple Groups, in: EMS Ser. Lect. Math., Eur. Math. Soc., Zürich, 2018, pp. 179–265.
[12] M. Cabanes, M. Enguehard, Representation Theory of Finite Reductive Groups, New Mathematical Monographs, vol. 1, Cambridge University Press, Cambridge, 2004.
[13] R.W. Carter, Finite Groups of Lie Type. Conjugacy Classes and Complex Characters, John Wiley and Sons, Inc., New York, 1985, 544 pp.
[14] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, Atlas of Finite Groups, Clarendon Press, Oxford, 1985.
[15] E.C. Dade, Remarks on isomorphic blocks, J. Algebra 45 (1977) 254–258.
[16] E.C. Dade, A correspondence of characters, in: The Santa Cruz Conference on Finite Groups, in: Proc. Sympos. Pure Math., vol. 37, Univ. California, Santa Cruz, CA, 1979, Amer. Math. Soc., Providence, RI, 1980, pp. 401–403.
[17] The GAP Group, GAP – groups, algorithms, and programming, version 4.11.0, http://www.gap-system.org, 2020.
[18] M. Geck, G. Malle, The Character Theory of Finite Groups of Lie Type: a Guided Tour, Cambridge Studies in Advanced Mathematics, vol. 187, Cambridge University Press, 2020.
[19] E. Giannelli, N. Rizo, B. Sambale, A.A. Schaeffer Fry, Groups with few p'-characters degrees in the principal block, Proc. Am. Math. Soc. 148 (2020) 4597–4614.
[20] L. Héthelyi, B. Külshammer, On the number of conjugacy classes of a finite solvable group, Bull. Lond. Math. Soc. 32 (2000) 668–672.
[21] N.N. Hung, G. Malle, A. Maróti, On almost p-rational characters of p'-degree, Forum Math. 34 (2022) 1475–1496.
[22] N.N. Hung, A.A. Schaeffer Fry, On Héthelyi-Külshammer's conjecture for principal blocks, Algebra Number Theory, in press, preprint available at arXiv:2102.09077.
[23] I.M. Isaacs, Character Theory of Finite Groups, AMS Chelsea Publishing, Providence, RI, 2006.
[24] G. James, A. Kerber, The Representation Theory of the Symmetric Group, Encyclopedia of Mathematics and Its Applications, vol. 16, Addison-Wesley Publishing Co., Reading, Mass, 1981.
[25] R. Kessar, G. Malle, Quasi-isolated blocks and Brauer's height zero conjecture, Ann. Math. 178 (2013) 321–384.
[26] S. Koshitani, T. Sakurai, The principal p-blocks with four irreducible characters, Bull. Lond. Math. Soc. 53 (2021) 1124–1138.
[27] S. Koshitani, B. Späth, The inductive Alperin-McKay and blockwise Alperin weight conditions for blocks with cyclic defect groups and odd primes, J. Group Theory 19 (2016) 777–813.
[28] B. Külshammer, A remark on conjectures in modular representation theory, Arch. Math. 49 (1987) 396–399.
[29] B. Külshammer, G.R. Robinson, Alperin-McKay implies Brauer's problem 21, J. Algebra 180 (1996) 208–210.
[30] P. Landrock, On the number of irreducible characters in a 2-block, J. Algebra 68 (1981) 426–442.
[31] G. Malle, Height 0 characters of finite groups of Lie type, Represent. Theory 11 (2007) 192–220.
[32] G. Malle, Extensions of unipotent characters and the inductive McKay condition, J. Algebra 320 (2008) 2963–2980.
[33] G. Malle, On the number of characters in blocks of quasi-simple groups, Algebr. Represent. Theory 23 (2020) 513–539.
[34] G. Malle, A. Maróti, On the number of p'-degree characters in a finite group, Int. Math. Res. Not. 20 (2016) 6118–6132.

[35] G. Malle, G. Navarro, Brauer's height zero conjecture for principal blocks, J. Reine Angew. Math. 778 (2021) 119–125.
[36] G. Malle, G. Navarro, A.A. Schaeffer Fry, P.H. Tiep, Brauer's height zero conjecture, preprint, arXiv:2209.04736, 2022.
[37] G. Malle, M. Testerman, Linear Algebraic Groups and Finite Groups of Lie Type, Cambridge Studies in Advanced Mathematics, vol. 133, Cambridge University Press, 2011.
[38] J.M. Martínez, Degrees of characters in the principal block, J. Algebra 579 (2021) 195–209.
[39] S. Mattarei, On character tables of wreath products, J. Algebra 175 (1995) 157–178.
[40] G.O. Michler, J.B. Olsson, Character correspondences in finite general linear, unitary and symmetric groups, Math. Z. 184 (1983) 203–233.
[41] M. Murai, Block induction, normal subgroups and characters of height zero, Osaka J. Math. 31 (1994) 9–25.
[42] G. Navarro, Characters and Blocks of Finite Groups, London Mathematical Society Lecture Note Series, vol. 50, Cambridge University Press, Cambridge, 1998.
[43] G. Navarro, Character Theory and the McKay Conjecture, Cambridge Studies in Advance Mathematics, vol. 175, Cambridge University Press, Cambridge, 2018.
[44] G. Navarro, The McKay conjecture and Galois automorphisms, Ann. Math. 160 (2004) 1129–1140.
[45] G. Navarro, B. Sambale, P.H. Tiep, Characters and Sylow 2-subgroups of maximal class revisited, J. Pure Appl. Algebra 222 (2018) 3721–3732.
[46] G. Navarro, P.H. Tiep, Sylow subgroups, exponents, and character values, Trans. Am. Math. Soc. 372 (2019) 4263–4291.
[47] G. Navarro, P.H. Tiep, The fields of values of characters of degree not divisible by p, Forum Math. Pi 9 (2021) 1–28.
[48] T. Okuyama, M. Wajima, Character correspondence and p-blocks of p-solvable groups, Osaka J. Math. 17 (1980) 801–806.
[49] J.B. Olsson, On 2-blocks with quaternion and quasidihedral defect groups, J. Algebra 36 (1975) 212–241.
[50] J.B. Olsson, McKay numbers and heights of characters, Math. Scand. 38 (1976) 25–42.
[51] J.B. Olsson, On the number of characters in blocks of finite general linear, unitary and symmetric groups, Math. Z. 186 (1984) 41–47.
[52] J.B. Olsson, Combinatorics and Representations of Finite Groups, Universität Essen, Fachbereich Mathematik, Essen, 1993.
[53] N. Rizo, A.A. Schaeffer Fry, C. Vallejo, Principal blocks with 5 irreducible characters, J. Algebra 585 (2021) 316–337.
[54] L. Ruhstorfer, The Alperin-McKay conjecture for the prime 2, preprint, arXiv:2204.06373, 2022.
[55] A.A. Schaeffer Fry, J. Taylor, Galois automorphisms and classical groups, in: Transformation Groups, 2022, Online first https://doi.org/10.1007/s00031-022-09754-4, 2022.
[56] B. Späth, A reduction theorem for the Alperin-McKay conjecture, J. Reine Angew. Math. 680 (2013) 153–189.
[57] A. Vera-López, J. Sangroniz, The finite groups with thirteen and fourteen conjugacy classes, Math. Nachr. 280 (2007) 676–694.
[58] A. Vera López, J. Vera López, Classification of finite groups according to the number of conjugacy classes, Isr. J. Math. 51 (1985) 305–338.