Action-angle variables of a binary black hole with arbitrary eccentricity, spins, and masses at 1.5 post-Newtonian order

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Accurate and efficient modeling of the dynamics of binary black holes (BBHs) is crucial to their detection through gravitational waves (GWs), with LIGO/Virgo/KAGRA, and LISA in the future. Solving the dynamics of a BBH system with arbitrary parameters without simplifications (like orbit- or precession-averaging) in closed-form is one of the most challenging problems for the GW community. One potential approach is using canonical perturbation theory which constructs perturbed action-angle variables from the unperturbed ones of an integrable Hamiltonian system. Having action-angle variables of the integrable 1.5 post-Newtonian (PN) BBH system is therefore imperative. In this paper, we continue the work initiated by two of us in [Tanay *et al.*, Phys. Rev. D **103**, 064066 (2021)], where we presented four out of five actions of a BBH system with arbitrary eccentricity, masses, and spins, at 1.5PN order. Here we compute the remaining fifth action using a novel method of extending the phase space by introducing unmeasurable phase space coordinates. We detail how to compute all the frequencies, and sketch how to explicitly transform from the action-angle variables to the usual positions and momenta. This analytically solves the dynamics at 1.5PN. This lays the groundwork to analytically solve the conservative dynamics of the BBH system with arbitrary masses, spins, and eccentricity, at higher PN order, by using canonical perturbation theory.

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I. INTRODUCTION

Laser interferometer detectors have made numerous gravitational wave (GW) detections that have originated from compact binaries made up of black holes (BHs) or neutron stars [1–3]. Among these detections, the predominant sources of GWs are from binary black holes (BBHs), whose initial eccentricity is believed to be mostly radiated away by the time they enter the frequency band of the ground-based detectors such as LIGO, Virgo, and KAGRA. Since the upcoming LISA mission [4,5] will target compact binaries earlier in their inspiral phase compared to the ground based detectors, incorporating eccentricity becomes more relevant. Since the observation time for LISA sources will be much longer, it is imperative to find accurate closed-form solutions to the binary dynamics.

This brings us to the question of working out closed-form solutions of the dynamics of a *generic* BBH system, with arbitrary eccentricity, masses, and with both BHs spinning, without special alignment. Many such attempts have been made in the literature [6–14], but most (if not all) of them give the solution of the conservative sector of the dynamics under some simplifying conditions such as the quasi-circular limit, equal-mass case, only one or none of the BHs spinning, with orbit-averaging, etc. Only recently, one of us provided a method to find the closed-form solution to a BBH system with arbitrary eccentricity, spins, and masses at 1.5 post-Newtonian (PN) order for the first time [15] (with the 1PN part of the Hamiltonian being omitted, as it is not complicated to handle). The next natural question is: how can one construct the solutions at 2PN, or is it even feasible?

This line of questioning led two of us to probe the integrability, and therefore the existence of action-angle variables of the BBH system at 2PN in Ref. [16], wherein we found that a BBH system is indeed 2PN integrable when we applied the perturbative version of the Liouville-Arnold (LA) theorem, due to the existence of two new 2PN constants of motion that we discovered. Since integrability precludes chaos (which would obstruct finding closed-form solutions), establishing integrability at 2PN instills hope toward finding a closed-form solutions at this order. A straightforward extension of the methods of Ref. [15] from 1.5PN to 2PN appears too difficult to carry out, if not impossible. Our hope is to use nondegenerate canonical perturbation theory [17,18], which when supplied with 1.5PN action-angle variables, can yield 2PN action-angle variables. If this line of work is to be pursued, the 1.5PN action-angle variables are imperative. The calculation cannot start from a lower PN order because the lower order (1PN) system is degenerate in the action-angles context; this

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is discussed later. We initiated the action-angle calculation in Ref. [16], where we computed four (out of five) actions. In this paper, we compute the last action variable, and sketch how to explicitly transform from the action-angle variables to the usual positions and momenta. This basically comprises the closed-form solution to the 1.5PN spinning BBH dynamics.

The history of action-angle variables literature dates back centuries. The Kepler equation presented in 1609 gives the Newtonian angle variable [17], long before Newton proposed his laws of motion and gravitation. Important contributions were made by Delaunay to the action-angle formalism of the Newtonian two-body system [17] in the nineteenth century. More recently, on the post-Newtonian front, Damour and Deruelle gave the 1PN extension of the angle variable when they worked out the quasi-Keplerian solution to the nonspinning eccentric BBH system [19]. Damour, Schäfer and Jaranowski worked out action variables at 2PN and 3PN ignoring the spin effects. Such post-Newtonian calculations make use of the work of Sommerfeld for complex contour integration to evaluate the radial action variable [20]. Finally, Damour gave the requisite number (five) of 1.5PN constants of motion in Ref. [21], which is required for integrability as per the LA theorem.

This paper is a natural extension to our earlier work [16]. We compute the remaining fifth action variable using a novel method of extending the phase space by the introduction of new, unmeasurable (or fictitious) phase space variables. We then show how to PN expand the lengthy expression of this 1.5PN exact fifth action and retain the much shorter leading-order contribution. Next we discuss how to compute all the frequencies of the system. Then we give a clear roadmap on how to compute all angle variables of the system implicitly, by expressing the standard phase space variables of the system $(\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2)$ as explicit functions of the action-angle variables. Thereafter, we proceed toward constructing solution to the BBH system using action-angle variables at 1.5PN. This action-angle-based solution can be extended to higher PN orders via canonical perturbation theory. Finally, in one of the appendices, we point out a loophole in the definition of PN integrability that we presented in Ref. [16] and also provide a simple fix. We mention here that in a companion paper [22], we implemented our action-angle based solution using *Mathematica*, and compared it with the corresponding numerical solution.

The organization of this paper is as follows. In Sec. II, we lay the conceptual foundations, introducing the phase space (symplectic manifold) and the Hamiltonian of the system. This includes introducing important definitions like those of integrability and action-angle variables. In Sec. III, we discuss the idea of extending the phase space by introducing new, unmeasurable phase space variables; they make the computation of the fifth action possible. In the next section, we implement these ideas to actually compute the fifth action in explicit form. Then in Sec. V, we show how to PN expand this fifth action and present its shortened form. In Sec. VI, we finally show how to compute the five frequencies, the angle variables, and construct the action-angle-based solution to the system. Finally, we summarize our work and suggest its future extensions in Sec. VII. As far as appendices are concerned, some lengthy calculations have been pushed to Appendix A, which would have otherwise been a part of Sec. IV. In Appendix B, we prove that our fifth action calculated in the extended phase space is also an action in the standard phase space. Appendix C gives some commonly occurring derivatives that occur in the frequency calculations. Lastly, in Appendix D, we fix a loophole in the definition of PN integrability that we presented in Ref. [16].

II. THE SETUP

The paper is a continuation of the research initiated in Ref. [16] and uses the same conventions, which we now briefly describe. For an informal and pedagogical introduction to the mathematical machinery employed in this paper and Ref. [16], the reader is referred to the set of lecture notes at [23].

We will study the BBH system in the PN approximation within the Hamiltonian formalism. The system under consideration is schematically displayed in Fig. 1. We work in the center-of-mass frame with a relative separation vector $\vec{R} \equiv \vec{r}_1 - \vec{r}_2$ between the two black holes, and conjugate momentum $\vec{P} \equiv \vec{p}_1 = -\vec{p}_2$, where the labels 1 and 2 indicate the two black holes, with masses m_1 and m_2 respectively. In Ref. [16], $\vec{R}_{1,2}$ and $\vec{P}_{1,2}$ were used to denote the position and momentum vectors of the two BHs; but here we are reserving these symbols for to-be-introduced unmeasurable, fictitious variables (see Sec. III). The BHs

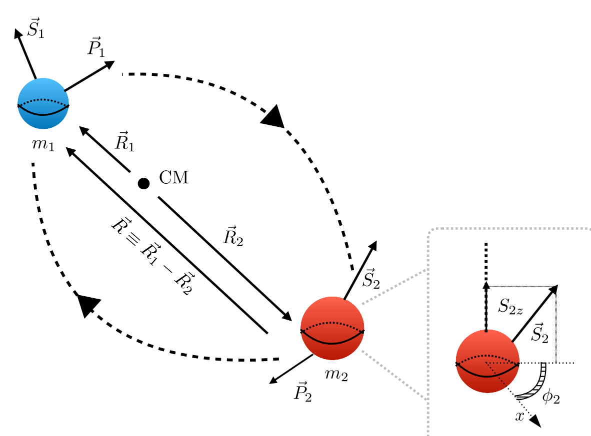

FIG. 1. Schematic setup of a precessing black hole binary. Positions, velocities and momenta are all defined as Newtonian vectors built from the center-of-mass.

possess spin angular momenta $\vec{S}_1$ and $\vec{S}_2$ which contribute to the total angular momentum $\vec{J} \equiv \vec{L} + \vec{S}_1 + \vec{S}_2$, where $\vec{L} \equiv \vec{R} \times \vec{P}$ is the orbital angular momentum of the binary. We will frequently use the effective spin

$$\vec{S}_{\text{eff}} \equiv \sigma_1 \vec{S}_1 + \sigma_2 \vec{S}_2, \tag{1}$$

$$\sigma_1 \equiv 1 + \frac{3m_2}{4m_1}, \tag{2}$$

$$\sigma_2 \equiv 1 + \frac{3m_1}{4m_2}. \tag{3}$$

The magnitude of any vector will be denoted by the same letter used to denote the vector, but without the arrow. Additionally, $S_{\text{eff}} \cdot L$, and $R_a \cdot P_a$ will stand for $\vec{S}_{\text{eff}} \cdot \vec{L}$, and $\vec{R}_a \cdot \vec{P}_a$, respectively. Einstein summation convention will be assumed unless stated otherwise.

The 1.5PN Hamiltonian that we will primarily be interested in is given by Eqs. (11), (12), (13), and (14) in Ref. [16], and will be denoted by H. Note that H in this paper is found by dropping the 2PN contribution in the H of Ref. [16]. The only nonvanishing Poisson brackets (PBs) between the phase space variables $\vec{R}, \vec{P}, \vec{S}_1$, and $\vec{S}_2$ are

$$\{R^i, P_j\} = \delta^i_j, \quad \text{and} \quad \left\{S^i_a, S^j_b\right\} = \delta_{ab}\epsilon^{ij}{}_k S^k_a, \tag{4}$$

and those related by antisymmetry

$$\{f, g\} = -\{g, f\}. \tag{5}$$

Here the letters a, b label the two black holes (a, $b = 1, 2$), and i, j, k are spatial vector indices. The PBs are derivations, so obey the chain rule (with ξ^i's standing for all phase-space variables)

$$\{f, g(\xi^i)\} = \{f, \xi^i\}\frac{\partial g}{\partial \xi^i}. \tag{6}$$

Equations (4), (5), and (6) enable us to compute the PB between any two functions of the phase-space variables. As usual, the evolution of any phase-space function f is given by $\dot{f} = \{f, H\}$. With this, it can be verified that both the spin magnitudes are constant, $\dot{S}_a = \{S_a, H\} = 0$. This means that we can specify each spin vector using only two variables: the z component and the azimuthal angle of the spin vector. This choice is particularly useful because these two variables act like canonical ones. This is so because Eqs. (4), (5), and (6) imply that

$$\{\phi_a, S^z_b\} = \delta_{ab}. \tag{7}$$

This means that there are five pairs of canonically conjugate variables, and a total of ten canonical phase space variables.

From a more mathematical point of view, Hamiltonian dynamics takes place on a symplectic manifold B which is a smooth manifold equipped with a closed, nondegenerate differential 2-form Ω, the symplectic form. The orbital variables R^i, P_j are canonical variables of the cotangent bundle $T^*\mathbb{R}^3$ (a symplectic manifold), while each spin vector S^i_a lives on the surface of a two-sphere (also a symplectic manifold, with symplectic form proportional to the area 2-form). The spin vectors $\vec{S}_a$ being on the above spherical symplectic manifolds is consistent with the constancy of the spin magnitudes. The symplectic manifold B which is the total phase space of the system is the Cartesian product of the above symplectic manifolds ($T^*\mathbb{R}^3$, and the two 2-spheres). The symplectic form on B is the sum of the symplectic forms from the three manifold factors [16]. In terms of canonically conjugate variables, Ω is[1]

$$\Omega = dP_i \wedge dR^i + dS^z_1 \wedge d\phi_1 + dS^z_2 \wedge d\phi_2. \tag{8}$$

This description of the phase space manifold using a symplectic geometry point of view makes it clear that each spin has only two degrees of freedom (S^z_a and ϕ_a), rather than three (S^x_a, S^y_a, and S^z_a). Although Ω itself is smooth, notice that this coordinate system is singular at the poles of each spin space.

Now we define integrable systems and action-angle variables at the same time, re-presenting the definition given in Ref. [24]. Two quantities f and g are called commuting or "in involution" if $\{f, g\} = 0$. Consider a system with Hamiltonian H in $2n$ canonical phase space variables $(\vec{\mathcal{P}}, \vec{\mathcal{Q}})$. This system is integrable if there exists a canonical transformation to coordinates $(\vec{\mathcal{J}}, \vec{\theta})$ such that all the actions $\mathcal{J}^i$ are mutually commuting, H is a function only of the actions, and all the $\vec{\mathcal{P}}$ and $\vec{\mathcal{Q}}$ variables are 2π-periodic functions of the angle variables $\vec{\theta}$.

The Liouville-Arnold (LA) theorem [16,18,24–26] which states that, on a $2n$-dimensional symplectic manifold, if $\partial_t H = 0$ and there are n independent, mutually commuting phase-space functions F_i, such that the level sets of these functions form compact and connected manifolds, then the system is integrable, and the above level sets are diffeomorphic to an n-torus. H being one of these F_i's implies that all the F_i's are also constants since $\dot{F}_i = \{F_i, H\} = 0$. Hence we call these F_i's the n commuting constants. When Ω is exact, there is a globally

[1]The relationship between symplectic form and PBs is encapsulated in Eq. (5.79) of Ref. [18]. The form Ω of Eq. (8) is consistent with the PBs of Eqs. (4) and (7).

well-defined potential one-form Θ (such that $\Omega = d\Theta$), then in canonical variables it will be

$$\Theta = \mathcal{P}_i d\mathcal{Q}^i, \tag{9}$$

and the action variables can be computed via [24,25]

$$\mathcal{J}_k = \frac{1}{2\pi}\oint_{\mathcal{C}_k} \Theta = \frac{1}{2\pi}\oint_{\mathcal{C}_k} \mathcal{P}_i d\mathcal{Q}^i, \tag{10}$$

where $\mathcal{C}_k$ is any loop in the kth homotopy class on the n-torus defined by the level sets $F_i = \text{const}$. The above integral is insensitive to the choice of loop in a certain homotopy class; see Proposition 11.2 of Ref. [24]. However, the 2-sphere (and therefore our symplectic manifold B) does not admit a global Θ, as mentioned earlier. In such cases, the actions are still well defined up to some global constants, but now using integrals over areas instead of loops; see Ref. [16] for details.

Before ending this section, we briefly introduce the concept of Hamiltonian flows and the associated Hamiltonian vector fields [18]. A quantity $f(\vec{\mathcal{P}}, \vec{\mathcal{Q}})$ defines a Hamiltonian vector field $\vec{X}_f$ via $\{\cdot, f\} = \partial/\partial\lambda$, such that it acts on another function $g(\vec{\mathcal{P}}, \vec{\mathcal{Q}})$ as $\partial g/\partial\lambda = \{g, f\}$. The collection of the integral curves of this vector field is referred to as the Hamiltonian flow of the field.

III. THE EXTENDED PHASE SPACE: A TOOL TO COMPUTE ACTIONS ON SPHERICAL MANIFOLDS

In this section, there are instances where we first explain some subtle concepts informally, before giving a more mathematically precise statement in the next paragraph. The reader may choose to skip the more advanced wording at the expense of some depth.

A. Motivation behind fictitious variables

In Ref. [16], we evaluated four of the five action integrals for the 1.5PN BBH system. The fifth action computation is a more complicated task and this leads us to invent certain "fictitious," "unmeasurable" variables, thereby extending the usual standard phase space (SPS) to the extended phase space (EPS). We now turn to explain the motivation behind them, which has two facets.

Actions are well defined on exact symplectic manifolds; an exact symplectic manifold admits a global potential one-form $\Theta = \vec{\mathcal{P}} \cdot d\vec{\mathcal{Q}}$). While $\vec{R}$ and $\vec{P}$ live in $T^*\mathbb{R}^3$, which is exact (with $\Theta = \vec{P} \cdot d\vec{R}$), the same is not true for the spin spherical symplectic manifolds, thereby making the SPS nonexact; see Problem 2 of Homework 2 in Ref. [27]. Although the SPS is not exact, the EPS will be. The two spaces will also be found to be equivalent (in a certain sense), which justifies the computation of action in the EPS, which we can then push forward to the SPS, since every EPS point would map to an SPS point by construction.

The other more practical problem the EPS cures is that of computation of the action integral in closed form. In the SPS (with variables R^i, P_i, ϕ_a, S_a^z with $a = 1, 2$), the action integral is broken down into the orbital and spin sector contributions,

$$\mathcal{J} = \mathcal{J}^{\text{orb}} + \mathcal{J}^{\text{spin}}, \tag{11}$$

$$\mathcal{J}^{\text{orb}} = \frac{1}{2\pi}\oint_{\mathcal{C}_k} P_i dR^i, \tag{12}$$

$$\mathcal{J}^{\text{spin}}_A = \frac{1}{2\pi}\oint_{\mathcal{C}_k} S_A^z d\phi_A. \tag{13}$$

Now under the flow of $S_{\text{eff}} \cdot L$, the above orbital sector integral of Eq. (12) is easy to compute. We state beforehand that the result comes out to be

$$\mathcal{J}^{\text{orb}} = \frac{(S_{\text{eff}} \cdot L)\Delta\lambda_{S_{\text{eff}}\cdot L}}{2\pi}, \tag{14}$$

where $\Delta\lambda_{S_{\text{eff}}\cdot L}$ is the flow amount under $S_{\text{eff}} \cdot L$ (to be determined). See Eqs. (30)–(33) for the intermediate steps.[2] Now although the orbital sector of the action integral under the $S_{\text{eff}} \cdot L$ flow is easy to compute, we do not know how to compute the spin sector integral of Eq. (13). We again state beforehand that writing the orbital angular momentum $\vec{L}$ as a cross product of a position $\vec{R}$ and a momentum $\vec{P}$ was critical to easily evaluating $\mathcal{J}^{\text{orb}}$ under the $S_{\text{eff}} \cdot L$ flow. This is something we cannot do with the spin angular momenta $\vec{S}_a$ because $\vec{S}_a$ are considered to be fundamental coordinates, not written as cross products of some positions and momenta. As we will see, the EPS gets rid of all these problems, thus making the action evaluation tractable.

B. Introducing fictitious phase-space variables

We refer to the phase space with coordinates $(\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2)$ as the SPS, the standard phase space. It is denoted by the letter B. We now invent a new 18-dimensional extended phase space (EPS) $E = (T^*\mathbb{R}^3)^3$ with canonical coordinates R^i, P_i, R_a^i, P_{ai} with $a = 1, 2$, with canonical Poisson bracket algebra

$$\{R^i, P_j\}_E = \delta^i_j, \qquad \{R^i_a, P_{bj}\}_E = \delta_{ab}\delta^i_j. \tag{15}$$

Here we use the subscript E to distinguish the Poisson brackets in E from those in B. We call the $\vec{R}_a, \vec{P}_a$ variables the unmeasurable, fictitious variables. For contrast, we

[2] Equations (30)–(33) are written for the EPS, but if we neglect the spin sector terms (like $\mathcal{J}^{\text{spin}}$ and $P_{ai} dR^i_a/d\lambda$ with $a = 1, 2$), then these are also valid for the SPS.

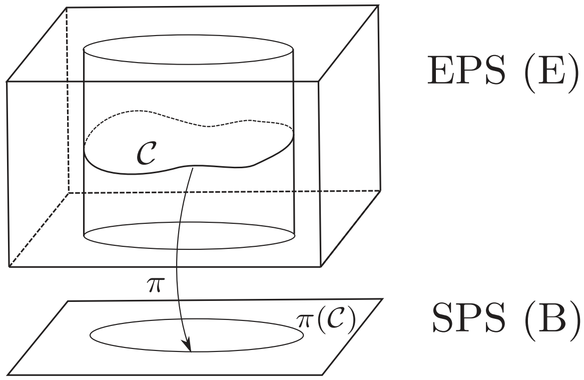

FIG. 2. The extended phase space E can be viewed as a fiber bundle with projection $\pi: E \to B$ down to the standard phase space B. Both are symplectic manifolds, but the symplectic form in the EPS is exact. In the context of this figure, a pure vertical motion in the EPS corresponds to keeping the SPS coordinates fixed and changing only the fictitious coordinates.

will sometimes refer to the SPS coordinates $\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2$ as the observable coordinates. We also demand that for an SPS point $(\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2)$, the corresponding EPS point must satisfy

$$\vec{S}_1 = \vec{R}_1 \times \vec{P}_1, \qquad \vec{S}_2 = \vec{R}_2 \times \vec{P}_2. \tag{16}$$

Of course, there are an infinity of EPS points which correspond to the same SPS point.

In more advanced language, this is a fiber bundle (with noncompact fibers) with projection

$$\pi\colon\ E \to B. \tag{17}$$

In coordinates, this projection takes a point in E and sends it to the point in B where its B coordinates are determined by[3]

$$\pi((\vec{R}, \vec{P}, \vec{R}_1, \vec{P}_1, \vec{R}_2, \vec{P}_2)) = (\vec{R}, \vec{P}, \vec{R}_1 \times \vec{P}_1, \vec{R}_2 \times \vec{P}_2). \tag{18}$$

This is depicted in Fig. 2. In this pictorial depiction of Fig. 2, a purely vertical displacement in the EPS space corresponds to changing the EPS coordinates in such a way that the observable coordinates do not change. To change the observable coordinates, a horizontal motion is needed, both in the SPS and the EPS. In addition to Fig. 2, we also follow these simple pictorial conventions in later figures like Figs. 3 and 4.

Any function on B can be pulled back with $\pi^\star$ to a function on E. So we can evolve the fictitious variables under the flow of the pulled-back version of H. While the fictitious variables can appear in intermediate calculations, they are a mathematical convenience for the purpose of computing $\mathcal{J}_5$. In the end, if physically observable quantities depend on $\vec{R}_a$ or $\vec{P}_a$, they must depend on them through $\vec{S}_a$. In other words, the observable quantities must be functions of only the observable coordinates.

In summary, this extended manifold E, the spins are now seen as cross products of fictitious positions and momenta. Also, now E is exact and admits a global potential one-form $\Theta_E = \vec{P} \cdot d\vec{R} + \vec{P}_1 \cdot d\vec{R}_1 + \vec{P}_2 \cdot d\vec{R}_2$. It is no longer a product of a Cartesian manifold and two-spheres. All three angular momenta $\vec{L}, \vec{S}_1$, and $\vec{S}_2$ stand on equal mathematical footing.

C. Comparing the EPS and SPS pictures

We can now sensibly talk about PBs on either the base SPS manifold B, or the extended EPS manifold E, denoted as $\{,\}_B$ and $\{,\}_E$, respectively. The former is computed using Eqs. (4) and (7), whereas the latter is computed using Eq. (15). Additional rules like Eqs. (5) and (6) apply universally to both $\{,\}_B$ and $\{,\}_E$. Now that we have rid ourselves of the problematic features of the SPS, the next natural question would be: are the two spaces (SPS and the EPS) equivalent in some sense so as to justify action computation in the EPS, instead of the SPS? It is easy to check that, when acting on any two functions f and g that only depend on the SPS coordinates, the two PBs agree, since Eqs. (15) imply Eqs. (4) and (7). Because of this crucial observation, we conclude that *the SPS picture and the EPS picture are equivalent* for the evolution of f under the flow of g. In other words,

$$\pi^\star\{f, g\}_B = \{\pi^\star f, \pi^\star g\}_E. \tag{19}$$

This means that either of the two pictures can be used to evolve the system under the H flow.

We can state the above compatibility relation of the PBs in B and E in more advanced language of differential geometry. Given some symplectic form Ω, its associated Poisson bracket $\{f, g\}$ is found from

$$\{f, g\} = \Omega^{-1}(df, dg), \tag{20}$$

where Ω^{-1} is the bivector that is the inverse of Ω, $[\Omega^{-1}]^{ij}\Omega_{jk} = \delta^i_k$. In our setting we have a symplectic form Ω_B in the SPS and Ω_E in the EPS. Eq. (19), the compatibility condition between the two PBs can be reexpressed as

[3]A more sophisticated way to think of this projection is as follows. Think of the three-dimensional spin manifold, with coordinates S^i_a, as $\mathfrak{so}(3)^*$, the vector space dual to the Lie algebra $\mathfrak{so}(3)$ of the rotation group SO(3). The dual of a Lie algebra naturally comes equipped with a Lie-Poisson structure (results developed by Kirillov, Kostant, and Souriau [28]). The usual action of SO(3) on $\mathbb{R}^3$ induces a Hamiltonian action on its cotangent bundle $T^*\mathbb{R}^3$ (a Poisson manifold), analogous to our fiber coordinates (R^i_a, P_{aj}). From here we can build the dual map $T^*\mathbb{R}^3 \to \mathfrak{so}(3)^*$, which is the *momentum map* [28]. Our projection π coincides with the momentum map.

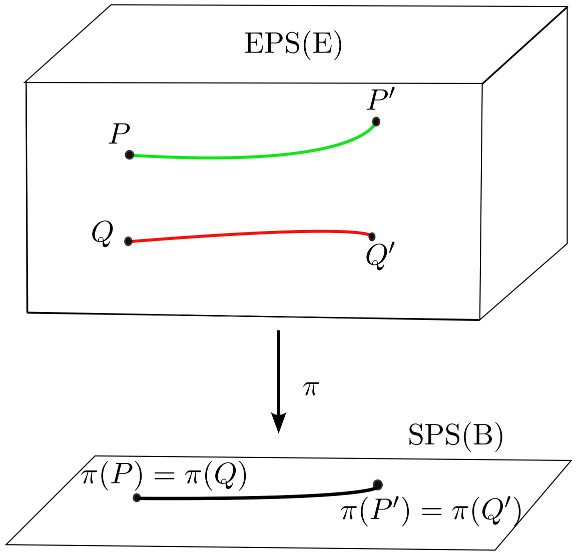

FIG. 3. Two EPS points (P and Q) with the same fictitious coordinates are mapped to new EPS points (P' and Q') again the same fictitious coordinates by a flow under a general $f(\vec{R}, \vec{P}, \vec{S}_1(\vec{R}_a, \vec{P}_a), \vec{S}_2(\vec{R}_a, \vec{P}_a))$ by any arbitrary amount.

$$\pi^\star(\Omega_B^{-1}(df, dg)) = \Omega_E^{-1}(\pi^\star df, \pi^\star dg), \tag{21}$$

where $\pi^\star$ is the pullback induced by the projection map π, and $f, g: B \to \mathbb{R}$. Since the left-hand side (lhs) is fiberwise constant,[4] so is the right-hand side (rhs); and so we can also consistently push forward this equality to B. Since f and g are arbitrary, this compatibility and the definition of pushforward implies that

$$\pi_\star(\Omega_E^{-1}) = \Omega_B^{-1}, \tag{22}$$

where $\pi_\star$ is the pushforward.

The equivalency needs to be pushed even to the integrability arena: the 1.5PN BBH system being integrable or chaotic must not depend on whether we choose to work in the SPS picture or the EPS one. Fortunately, the two pictures are also equivalent when we investigate the integrability of the system, following the LA theorem. In the base SPS manifold, we have the required $10/2 = 5$ mutually commuting constants to establish integrability: $H, J^2, J_z, L^2, S_{\mathrm{eff}} \cdot L$. In the EPS picture, we also have the requisite $18/2 = 9$ commuting constants required for integrability. Those are the five constants already listed above, plus S_a^2 and $R_a \cdot P_a$ for $a = 1$, 2. These nine constants are to be viewed as functions of the EPS coordinates. Because of the integrable nature of the system, there are five (nine) action variables in the SPS (EPS), and similarly for the angle variables.

An interesting question arises. Imagine two points P and Q in the EPS which have the same SPS coordinates $\vec{R}, \vec{P}, \vec{S}_1$, and $\vec{S}_2$, but some different fictitious coordinates (shown in Fig. 3). If we were to flow under $f(\vec{R}, \vec{P}, \vec{S}_1(\vec{R}_a, \vec{P}_a), \vec{S}_2(\vec{R}_a, \vec{P}_a))$, with P and Q as starting points for a fixed amount λ_0, then are the SPS coordinates of the two final points the same? In other words, is $\pi(P'(P, f, \lambda_0)) = \pi(Q'(Q, f, \lambda_0))$? The primes denote the final point reached at the end of the flow. It is easy to check that the answer to the above question is "yes," and this is due to the compatibility of the PBs [Eq. (19) or Eq. (22)]. In other words, when flowing under f in the EPS, the SPS coordinates of the final point reached by the flow depends only on f, λ_0 and the SPS coordinates (and not the fictitious coordinates) of the starting point. This is not just a desirable but a necessary feature because it assures us that among an infinity of EPS configurations (lying within a single fiber) that are compatible with a given SPS configuration, we can choose to work with any one of them.

We can state the same result in the language of Hamiltonian vector fields. We denote the Hamiltonian vector field generated by the flow under f (whether in the SPS or in the EPS) with

$$X_f^B \equiv \{\cdot, f\}_B = \Omega_B^{-1}(\cdot, df), \tag{23}$$

$$X_{\pi^\star f}^E \equiv \{\cdot, \pi^\star f\}_E = \Omega_E^{-1}(\cdot, d(\pi^\star f)) = \Omega_E^{-1}(\cdot, \pi^\star(df)), \tag{24}$$

where the Poisson bracket on the EPS $\{\cdot,\cdot\}_E$ acts on the pullback $\pi^\star f$ of the function $f = f(\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2)$. Now the compatibility of the brackets in the SPS and the EPS [Eqs. (19) and (22)] tells us that

$$\pi_\star(X_{(\pi^\star f)}^E) = X_f^B, \tag{25}$$

i.e., the SPS vector field is the pushforward of the EPS vector field. This is equivalent to the result arrived at in the previous paragraph.

D. Strategy to compute the action

Since the EPS and SPS are equivalent when acting on SPS functions, we can use either of them for our calculations. As already remarked in Sec. III A, we do not know how to compute the fifth action in the SPS. So we now turn to computing the fifth action in the EPS via

$$\mathcal{J}_k = \frac{1}{2\pi} \oint_{\mathcal{C}_k} \left(\vec{P} \cdot d\vec{R} + \vec{P}_1 \cdot d\vec{R}_1 + \vec{P}_2 \cdot d\vec{R}_2 \right), \tag{26}$$

which interestingly is tractable. We state in advance the necessary result that the fifth action [Eq. (35)] in the EPS is fiberwise constant (see Footnote 4), meaning it can be

[4] Fiberwise constancy means constancy when one moves along a fiber through points which map to the same base point. This means insensitivity to changing the fictitious variables (and moving through the EPS), so long as all these fictitious variables correspond to the same SPS point $(\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2)$.

written in terms of only the observable coordinates $(\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2)$. In other words, the dependence of this action on the unmeasurable variables occurs only through the combinations $\vec{S}_1 = \vec{R}_1 \times \vec{P}_1$ and $\vec{S}_2 = \vec{R}_2 \times \vec{P}_2$. This makes it possible to treat the fifth action as a function of only the SPS coordinates.

Another important question arises. Since we are computing the fifth action in the EPS, do we have a legitimate action in the SPS? The answer is "yes" in a certain sense (as explained below), although due to the SPS-EPS equivalency, we can in principle, totally disregard the SPS and work only in the EPS, through and through. Eq. (10) is the popular loop-integral definition of action. This action has an important property that under its flow by 2π (and not any smaller amount), we get a closed loop.[5] In fact, this is such an important feature that it can also serve as another definition of the action (call it the "loop-flow definition"). We will use the loop-integral definition to compute the action in the EPS, and we show in Appendix B that the pushforward of this action to SPS satisfies the loop-flow definition of action.

We make a few closing remarks before we turn to the evaluation of the fifth action integral of Eq. (26). We have numerically verified that flowing by 2π under the fifth action [to-be-computed from Eq. (26)] yields a closed loop (as required by the loop-flow definition), within numerical errors, whether the action is treated as an SPS function or an EPS function. Although, the first four action integrals computed in Ref. [16] were done in the SPS, we could have also computed them in the EPS, and then pushforwarded these integrals to the SPS. The results would be the same as the four action integrals already presented in Ref. [16], except for some irrelevant additive constants. In summary, the equivalence of the two pictures (in terms of integrability, action-angle variables, and most importantly, the evolution under a flow associated with any observable), the global exactness of the symplectic form Ω_E, and the ease of evaluation of the action variables, make us prefer the EPS over the SPS for the action computation.

IV. COMPUTING THE FIFTH ACTION

Four out of the five actions were already presented in Ref. [16]. Here we compute the fifth one. For the fifth action, we generate a closed loop on the invariant n-torus by flowing under $S_{\rm eff} \cdot L$, and other commuting constants. After flowing under $S_{\rm eff} \cdot L$ by a certain amount $\Delta\lambda_{S_{\rm eff}\cdot L}$ (to be computed), although the mutual angles between $(\vec{L}, \vec{S}_1, \vec{S}_2)$ return to their original values, these individual vectors have not. So we have not formed a closed loop yet. However, additional flows under J^2, L^2, S_1^2, and S_2^2 will close the loop (shown in Appendix A), and at the same time ensuring that this loop is in a different homotopy class than the four associated to the other actions. We will see that we do not need to flow along H or J_z for the fifth action computation. The fifth action integral can be computed piecewise as five integrals,

$$\mathcal{J}_5 = \mathcal{J}_{S_{\rm eff}\cdot L} + \mathcal{J}_{J^2} + \mathcal{J}_{L^2} + \mathcal{J}_{S_1^2} + \mathcal{J}_{S_2^2}, \tag{27}$$

where each part corresponds to the segment generated by flowing under the quantity in the subscript.

Focusing on $\mathcal{J}_{S_{\rm eff}\cdot L}$, we will need the evolution equations under the flow of $S_{\rm eff} \cdot L$ in the EPS, which read

$$\frac{d\vec{R}}{d\lambda} = \vec{S}_{\rm eff} \times \vec{R}, \tag{28a}$$

$$\frac{d\vec{P}}{d\lambda} = \vec{S}_{\rm eff} \times \vec{P}, \tag{28b}$$

$$\frac{d\vec{R}_a}{d\lambda} = \sigma_a\left(\vec{L} \times \vec{R}_a\right), \tag{28c}$$

$$\frac{d\vec{P}_a}{d\lambda} = \sigma_a\left(\vec{L} \times \vec{P}_a\right), \tag{28d}$$

and they imply

$$\frac{d\vec{L}}{d\lambda} = \vec{S}_{\rm eff} \times \vec{L}, \tag{29a}$$

$$\frac{d\vec{S}_a}{d\lambda} = \sigma_a\left(\vec{L} \times \vec{S}_a\right). \tag{29b}$$

From these evolution equations we have

$$2\pi\mathcal{J}_{S_{\rm eff}\cdot L} = 2\pi(\mathcal{J}^{\rm orb} + \mathcal{J}^{\rm spin}) \tag{30}$$

$$\begin{aligned} &= \int_{\lambda_i}^{\lambda_f}\left(P_i\frac{dR^i}{d\lambda} + P_{1i}\frac{dR_1^i}{d\lambda} + P_{2i}\frac{dR_2^i}{d\lambda}\right)d\lambda \\ &= \int_{\lambda_i}^{\lambda_f}\Big(\vec{P}\cdot(\vec{S}_{\rm eff}\times\vec{R}) + \vec{P}_1\cdot(\sigma_1\vec{L}\times\vec{R}_1) \\ &\quad + \vec{P}_2\cdot(\sigma_2\vec{L}\times\vec{R}_2)\Big)d\lambda \end{aligned} \tag{31}$$

$$= 2\int_{\lambda_i}^{\lambda_f}(S_{\rm eff}\cdot L)d\lambda = 2(S_{\rm eff}\cdot L)\Delta\lambda_{S_{\rm eff}\cdot L}, \tag{32}$$

$$\mathcal{J}_{S_{\rm eff}\cdot L} = \frac{(S_{\rm eff}\cdot L)\Delta\lambda_{S_{\rm eff}\cdot L}}{\pi}, \tag{33}$$

with $\Delta\lambda_{S_{\rm eff}\cdot L} = \lambda_f - \lambda_i$ being the required flow parameter amount (for the mutual angles of $(\vec{L}, \vec{S}_1, \vec{S}_2)$ to be restored). We could pull $S_{\rm eff} \cdot L$ out of the integral since it is a constant under the flow of $S_{\rm eff} \cdot L$. After performing similar calculations, we can also show that (see also Sec. III-A of Ref. [16])

[5]See the proof of Theorem 11.6 of Ref. [24] to arrive at this conclusion.

$$\mathcal{J}_{J^2} = \frac{J^2 \Delta\lambda_{J^2}}{\pi}, \tag{34a}$$

$$\mathcal{J}_{L^2} = \frac{L^2 \Delta\lambda_{L^2}}{\pi}, \tag{34b}$$

$$\mathcal{J}_{S_1^2} = \frac{S_1^2 \Delta\lambda_{S_1^2}}{\pi}, \tag{34c}$$

$$\mathcal{J}_{S_2^2} = \frac{S_2^2 \Delta\lambda_{S_2^2}}{\pi}, \tag{34d}$$

where the quantities $\Delta\lambda_i$'s are the flow amounts required to close the loop under the corresponding commuting constant in the subscript. This finally renders the fifth action to be

$$\mathcal{J}_5 = \frac{1}{\pi}\Big\{(S_{\mathrm{eff}} \cdot L)\Delta\lambda_{S_{\mathrm{eff}}\cdot L} + J^2\Delta\lambda_{J^2} + L^2\Delta\lambda_{L^2} + S_1^2\Delta\lambda_{S_1^2} + S_2^2\Delta\lambda_{S_2^2}\Big\}, \tag{35}$$

which means that the fifth action computation has now boiled down to computing the five flow amounts $\Delta\lambda_i$'s. Due to the tedious nature of the computation of these five parameter flow amounts, they have been relegated to Appendix A.

Summarizing, the fifth action is given by Eq. (35), where the $\Delta\lambda$'s are presented in Eqs. (A42), (A67), (A77), (A94), and (A95). Because our derivation assumed $m_1 > m_2$, this expression of the action is not manifestly symmetric under the exchange $1 \leftrightarrow 2$ (labels of the two black holes). However, as discussed in the text after Eq. (A28), the symmetry can be restored simply. Note that in Ref. [16], the flows under J^2, J_z, and L^2 individually form closed loops. This implies that the associated actions are functions of just these individual conserved quantities. Meanwhile, to get a closed loop for the fifth action evaluation, we need to flow under all of $J, L, S_{\mathrm{eff}} \cdot L, S_1$, and S_2, which makes the fifth action a function of all these five quantities.

A. Fifth action in the equal mass case

The above result for the fifth action in Eq. (35) is not manifestly finite in the equal mass limit: there are many factors of $(\sigma_1 - \sigma_2)$ which vanish in this limit, including some in denominators. We have checked numerically that the equal mass limit of $\mathcal{J}_5$ is finite, but trying to take this limit analytically is cumbersome. There is however a simpler way, and the solvability of the equal-mass case has been independently investigated in the literature, albeit in the orbit- and precession-averaged approach [29].

Working with only the SPS variables, when $\sigma_1 = \sigma_2$ (equal-mass case), it is easy to check that $\vec{S}_1 \cdot \vec{S}_2$, along with H, J^2, L^2, and J_z forms a set of five mutually commuting constants. In fact, $S_{\mathrm{eff}} \cdot L$ can then be seen as a function of these five constants, and is therefore no longer an independent constant. It can be checked that under the flow of $\vec{S}_1 \cdot \vec{S}_2$ we have the flow equations (with $\vec{S} \equiv \vec{S}_1 + \vec{S}_2$)

$$\Big\{\vec{S}_1, \vec{S}_1 \cdot \vec{S}_2\Big\} = \vec{S} \times \vec{S}_1 = \Big\{\vec{S}_1 \cdot \vec{S}_2, \vec{S}_2\Big\} = \vec{S}_2 \times \vec{S}, \tag{36}$$

which imply that both the spin vectors rotate around $\vec{S}$, which itself remains fixed under this flow. $\vec{R}$ and $\vec{P}$ do not move and hence only the spin sectors contribute to the action integral. At this point, we can simply use the result of Eq. (28) of Ref. [16] with $\hat{n} = \vec{S}/S$, which gives our fifth action variable for the equal mass case as

$$\tilde{\mathcal{J}}_{5(m_1=m_2)} = (\vec{S}_1 + \vec{S}_2) \cdot \vec{S}/S = S. \tag{37}$$

The reason we used a tilde in the above equation is because $\tilde{\mathcal{J}}_{5(m_1=m_2)}$ need not be the equal mass limit of $\mathcal{J}_5$, since action variables of a system are not unique; see Proposition 11.3 of Ref. [24].

Finally, using the equal mass relations

$$J^2 = L^2 + S_1^2 + S_2^2 + 2(\vec{L} \cdot \vec{S} + \vec{S}_1 \cdot \vec{S}_2), \tag{38}$$

$$S_{\mathrm{eff}} \cdot L = \frac{7}{4}\vec{L} \cdot \vec{S}, \tag{39}$$

$$S^2 = S_1^2 + S_2^2 + 2\vec{S}_1 \cdot \vec{S}_2, \tag{40}$$

in Eq. (38) of Ref. [16], it is possible to arrive at an equation connecting the Hamiltonian with the actions. Performing a PN series inversion thereafter, one can write an explicit expression for the Hamiltonian in terms of the actions, up to 1.5PN. This can be used to explicitly obtain the frequencies of the system via $\omega^i = \partial H/\partial \mathcal{J}_i$ for the equal-mass case.

V. FIFTH ACTION AT THE LEADING PN ORDER

The action variable given by Eq. (35) is in exact form with respect to the 1.5PN Hamiltonian H. It is a worthwhile exercise to write the leading order contribution of this action because it is a much shorter expression than the "exact" one. This is in the same spirit as the expression of the fourth action variable as a PN series which was presented in Eq. (38) of Ref. [16]. Another advantage is that we can then write $S_{\mathrm{eff}} \cdot L$ in terms of the actions, including the fifth one (discussed below), which when used with Eq. (38) of Ref. [16] can give an expression for Hamiltonian in terms of the actions.

Note that out of the five actions: $J, L, J_z, \mathcal{J}_4$, and $\mathcal{J}_5$ (see Ref. [16] for the first four), the first two coincide with each other at 1PN order due to the absence of spins. The next important action variable at 1PN is the 1PN version of $\mathcal{J}_4$ [20]. J_z is irrelevant when it comes to computing frequencies since the Hamiltonian is never a function of J_z.

This explains the presence of only two frequencies (resulting from effectively two actions) at 1PN. Now since $\mathcal{J}_5$ comes into play for the first time only at the 1.5PN order, it makes sense to expand it in a PN series and work with the leading order term only, if we are working at 1.5PN. We now turn our attention to extracting this leading order term.

We sketch the plan for how to obtain the leading PN contribution to $\mathcal{J}_5$. It comprises a couple of steps which were performed in *Mathematica*.

Step 1: To start with, instead of writing the various quantities which make up $\mathcal{J}_5$ in terms of the five commuting constants, write them only in terms of $\vec{L}, \vec{S}_1, \vec{S}_2, \sigma_1$, and σ_2 with the understanding that $\vec{S}_1$ and $\vec{S}_2$ are 0.5PN order higher than $\vec{L}$; see Ref. [16] for more details on this. Attach a formal PN order counting parameter ϵ to $\vec{S}_1$ and $\vec{S}_2$. This ϵ will be used as a PN perturbative expansion parameter: every power of ϵ stands for an extra 0.5PN order. At the end of the calculation, ϵ will be set equal to 1. Writing various quantities of interest in terms of $\vec{L}, \vec{S}_1$, and $\vec{S}_2$ is imperative since it serves to expose the PN powers explicitly. For example, $J^2 - L^2 = \mathcal{O}(\epsilon^1)$, though both J^2 and L^2 are $\mathcal{O}(\epsilon^0)$. This becomes manifestly clear when $J^2 - L^2$ is written in the above way.

Step 2: Instead of trying to series expand $\mathcal{J}_5$ directly in terms of ϵ in one go, we first series expand various quantities that make up $\mathcal{J}_5$, and then use these expanded versions to finally build up the series-expanded version of $\mathcal{J}_5$. As a first step, series expand the cubic expression of Eq. (A21), and its roots, keeping terms up to $\mathcal{O}(\epsilon^2)$. Expansion of the roots up to $\mathcal{O}(\epsilon^2)$ is necessary because the turning points f_1 and f_2 coincide at lower orders.

Step 3: Series expand various other quantities that make up $\mathcal{J}_5$, such as k^2, B_1, B_2, D_1, D_2, α_1 and α_2 in ϵ such that the resulting expansions have two nonzero post-Newtonian terms. We do not have to worry about series expanding certain other quantities which make up $\Delta\lambda_4$ and $\Delta\lambda_5$, since they do not contribute to the fifth action variable at the leading order.

Step 4: Using these series-expanded ingredients, build up $\mathcal{J}_5$ of Eq. (35). The PN orders of the five summands of $\mathcal{J}_5$ [as shown in Eq. (27)] are schematically shown here as

$$\mathcal{J}_{S_{\mathrm{eff}}\cdot L} = \mathcal{O}(\epsilon), \tag{41}$$

$$\mathcal{J}_{J^2} = \mathcal{J}_0\epsilon^0 + \mathcal{O}(\epsilon), \tag{42}$$

$$\mathcal{J}_{L^2} = -\mathcal{J}_0\epsilon^0 + \mathcal{O}(\epsilon), \tag{43}$$

$$\mathcal{J}_{S_1^2} = \mathcal{O}(\epsilon^2), \tag{44}$$

$$\mathcal{J}_{S_2^2} = \mathcal{O}(\epsilon^2), \tag{45}$$

where we have indicated that the leading order components of $\mathcal{J}_{J^2}$ and $\mathcal{J}_{L^2}$ cancel each other. Our leading order $\mathcal{J}_5$ is thus the sum of the first three contributions. The last two contributions being at subleading orders can be dropped. At this point we can set $\epsilon = 1$.

Step 5: At this point the resulting perturbative $\mathcal{J}_5$ is a function of $\vec{L}, \vec{S}_1, \vec{S}_2, \sigma_1, \sigma_2$ and dot products formed out of them. We still want to write this as a function of the commuting constants only, keeping in line with the tradition followed in the action-angle variables formalism. To do so, we eliminate $\vec{L}\cdot\vec{S}_1$ and $\vec{L}\cdot\vec{S}_2$ using the following results valid up to the leading PN order

$$\vec{L}\cdot\vec{S}_1 \sim \frac{2S_{\mathrm{eff}}\cdot L - (J^2 - L^2 - S_1^2 - S_2^2)\sigma_2}{2(\sigma_1 - \sigma_2)}, \tag{46a}$$

$$\vec{L}\cdot\vec{S}_2 \sim -\frac{2S_{\mathrm{eff}}\cdot L - (J^2 - L^2 - S_1^2 - S_2^2)\sigma_1}{2(\sigma_1 - \sigma_2)}, \tag{46b}$$

which finally yields the leading PN order contribution to $\mathcal{J}_5$ as

$$\mathcal{J}_5 \sim \frac{1}{4L(\sigma_1-\sigma_2)(\mathcal{C}_1^2 - 4L^2({S_1}^2+{S_2}^2))}\Big[\mathcal{C}_1^3\mathcal{C}_2(\sigma_1+\sigma_2) + 4\mathcal{C}_1L^2\{{S_1}^2(\mathcal{C}_2(\sigma_1-\sigma_2)+2\sigma_1) + {S_2}^2(\mathcal{C}_2(\sigma_2-\sigma_1)+2\sigma_2)\} - (S_{\mathrm{eff}}\cdot L)\{16L^2({S_1}^2+{S_2}^2) + 4\mathcal{C}_1^2\mathcal{C}_2\}\Big], \tag{47}$$

where we define the combinations

$$\mathcal{C}_1 = J^2 - L^2 - S_1^2 - S_2^2, \tag{48}$$

$$\mathcal{C}_2 = \left[1 - \frac{4(\mathcal{C}_1\sigma_1 - 2S_{\mathrm{eff}}\cdot L)(\mathcal{C}_1\sigma_2 - 2S_{\mathrm{eff}}\cdot L)}{(\mathcal{C}_1(\sigma_1+\sigma_2) - 4S_{\mathrm{eff}}\cdot L)^2 - 4L^2(\sigma_1-\sigma_2)^2({S_1}^2+{S_2}^2)}\right]^{1/2} - 1. \tag{49}$$

We could have chosen to eliminate $\vec{L}\cdot\vec{S}_1$ and $\vec{L}\cdot\vec{S}_2$ using slightly modified forms of Eqs. (46) by simply ignoring S_1^2 and S_2^2 terms in the numerator. These modified forms of Eqs. (46) and the resulting modified form of the leading order contribution to the fifth action would still agree with the original results [Eqs. (46) and Eq. (47)] up to the leading PN order. The above expression of linearized fifth action is not manifestly symmetric with respect to the label exchange $1 \leftrightarrow 2$. This is because from the beginning, we assumed $m_1 > m_2$ while deriving the 1.5PN exact fifth action; see the text after Eq. (A28). We can easily make this leading PN order version of fifth action symmetric by replacing $(\sigma_1 - \sigma_2)$ with $-|\sigma_1 - \sigma_2|$ only in the denominator of the rhs of Eq. (47). This is because $m_1 > m_2 \Rightarrow \sigma_1 < \sigma_2$.

We note that the expression of the leading PN order contribution to the fifth action in Eq. (47) is much shorter than that of the exact 1.5PN fifth action (when both are expressed in terms of the commuting constants). This could be used in an efficient implementation of the evaluation of the fifth action on a computer.

We also note that Eq. (47) can be used to arrive at a quartic equation in $S_{\rm eff} \cdot L$ with other action variables as parameters of this quartic equation. This means it is in principle possible to solve for $S_{\rm eff} \cdot L$ as a function of the actions. By inserting this into Eq. (38) of Ref. [16], we can explicitly find the 1.5PN $H(\vec{\mathcal{J}})$ as a function of all of the actions (after a PN series inversion). This gives an alternative approach for computing the frequencies $\omega^i = \partial H/\partial \mathcal{J}_i$ which can be compared with the approach in Sec. VI. We have also numerically verified that $\mathcal{J}_5$ as presented in Eq. (47) above converges to the exact 1.5PN version in the limit of small PN parameter $(S_1, S_2 \ll L)$.

VI. FREQUENCIES AND ANGLE VARIABLES

A. Computing the frequencies

Since we have an integrable Hamiltonian system, the Hamiltonian is a function of the actions and not the angles, though it may not be possible to write H explicitly in terms of the actions. In terms of the actions, the equations of motion for the respective angle variables are trivial,

$$\dot{\theta}^i = \frac{\partial H}{\partial \mathcal{J}_i} = \omega^i(\vec{\mathcal{J}}). \tag{50}$$

As a consequence, the usual phase space variables are all multiply-periodic functions of all of the angle variables. Concretely, this means a Fourier transform of some regular coordinate would consist of a forest of delta function peaks at integer-linear combinations of the fundamental frequencies ω^i [30]. Additionally, if we know the frequencies, we can locate resonances—where the ratio of two frequencies is a rational number—which are key to the KAM theorem and the onset of chaos.

With $\vec{C}$ standing for the vector of all five mutually commuting constants, H being one of these C_i's, H is automatically a function of $\vec{C}$. In principle, once can invert $\vec{\mathcal{J}}(\vec{C})$ (at least locally, via the inverse function theorem) for $\vec{C}(\vec{\mathcal{J}})$, and thus find an explicit expression for $H(\vec{\mathcal{J}})$ paving the road for the computation of the frequencies ω^i's. But this is not necessary.

Instead, we follow the approach given in Appendix A of Ref. [31] to find the frequencies as functions of the constants of motion, via the Jacobian matrix between the five C_i's and the five $\mathcal{J}_i$'s. For the purpose of frequency computations, we take our C_i's to be (in this specific order) $\vec{C} = \{J, J_z, L, H, S_{\rm eff} \cdot L\}$. As two of us showed in Ref. [16], the first three of these are already action variables. We take the order of the actions to be $\vec{\mathcal{J}} = \{J, J_z, L, \mathcal{J}_4, \mathcal{J}_5\}$. The expression for $\mathcal{J}_4$ was given as an explicit function of $(H, L, S_{\rm eff} \cdot L)$ in Ref. [16]. The Jacobian matrix $\partial \mathcal{J}^i/\partial C^j$ can be found explicitly, since we have analytical expressions for $\vec{\mathcal{J}}(\vec{C})$. This matrix is somewhat sparse, given by

$$\frac{\partial \mathcal{J}^i}{\partial C^j} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\partial \mathcal{J}_4}{\partial L} & \frac{\partial \mathcal{J}_4}{\partial H} & \frac{\partial \mathcal{J}_4}{\partial (S_{\rm eff} \cdot L)} \\ \frac{\partial \mathcal{J}_5}{\partial J} & 0 & \frac{\partial \mathcal{J}_5}{\partial L} & 0 & \frac{\partial \mathcal{J}_5}{\partial (S_{\rm eff} \cdot L)} \end{bmatrix}. \tag{51}$$

Now we use the simple fact that the Jacobian $\partial C^i/\partial \mathcal{J}^j$ is the inverse of this matrix (assuming it is full rank),

$$\frac{\partial \mathcal{J}^i}{\partial C^j} \frac{\partial C^j}{\partial \mathcal{J}^k} = \delta^i{}_k, \tag{52}$$

$$\frac{\partial \vec{C}}{\partial \vec{\mathcal{J}}} = \left[\frac{\partial \vec{\mathcal{J}}}{\partial \vec{C}}\right]^{-1}. \tag{53}$$

Because of the sparsity of the matrix in Eq. (51), we directly invert and find the only nonvanishing coefficients in the inverse are

$$\frac{\partial C^i}{\partial \mathcal{J}^j} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{\partial H}{\partial J} & 0 & \frac{\partial H}{\partial L} & \frac{\partial H}{\partial \mathcal{J}_4} & \frac{\partial H}{\partial \mathcal{J}_5} \\ \frac{\partial (S_{\rm eff} \cdot L)}{\partial J} & 0 & \frac{\partial (S_{\rm eff} \cdot L)}{\partial L} & 0 & \frac{\partial (S_{\rm eff} \cdot L)}{\partial \mathcal{J}_5} \end{bmatrix}. \tag{54}$$

The frequencies we seek are in the fourth row of this matrix. Matrix inversion yields the following expressions for the frequencies:

$$\frac{\partial H}{\partial J} = \omega^1 = \frac{(\partial \mathcal{J}_4/\partial (S_{\rm eff} \cdot L))(\partial \mathcal{J}_5/\partial J)}{(\partial \mathcal{J}_4/\partial H)(\partial \mathcal{J}_5/\partial (S_{\rm eff} \cdot L))}, \tag{55a}$$

$$\frac{\partial H}{\partial J_z} = \omega^2 = 0, \tag{55b}$$

$$\frac{\partial H}{\partial L} = \omega^3 = \left[(\partial \mathcal{J}_4/\partial (S_{\rm eff}\cdot L))(\partial \mathcal{J}_5/\partial L) - (\partial \mathcal{J}_4/\partial L)(\partial \mathcal{J}_5/\partial (S_{\rm eff}\cdot L)) \right] \times (\partial \mathcal{J}_4/\partial H)^{-1}(\partial \mathcal{J}_5/\partial (S_{\rm eff}\cdot L))^{-1}, \tag{55c}$$

$$\frac{\partial H}{\partial \mathcal{J}_4} = \omega^4 = (\partial \mathcal{J}_4/\partial H)^{-1}, \tag{55d}$$

$$\frac{\partial H}{\partial \mathcal{J}_5} = \omega^5 = -\frac{\partial \mathcal{J}_4/\partial (S_{\rm eff}\cdot L)}{(\partial \mathcal{J}_4/\partial H)(\partial \mathcal{J}_5/\partial (S_{\rm eff}\cdot L))}. \tag{55e}$$

The frequency $\omega^2 = \partial H/\partial J_z$ vanishes since H cannot depend on J_z, to preserve SO(3) symmetry. The derivatives of $\mathcal{J}_4$ with respect to $(H, L, S_{\rm eff}\cdot L)$ are easy to compute from the explicit expression given in Eq. (38) of Ref. [16]. Taking the derivatives of $\mathcal{J}_5$ in Eqs. (55) involves many intermediate quantities that arise from the chain rule, and are presented in Appendix C.

B. The angle variables

Canonical perturbation theory [17,18] has the potential to furnish 2PN action-angle variables when supplied with 1.5PN ones. To use this tool, we want to be able to express perturbations to the Hamiltonian (namely, higher PN order terms) as functions of the angle variables which are canonically conjugate to the actions. One of these angles —the mean anomaly, which is conjugate to our $\mathcal{J}_4$—has been presented previously in the literature, in pieces. We have explicitly checked that the Poisson bracket between $\mathcal{J}_4$ the 1.5PN mean anomaly (combining 1PN and 1.5PN pieces of the results from Refs. [19] and [32]) is 1, up to 1.5PN order.[6]

1. Constructing angle variables

We now lay out a roadmap on how to implicitly construct the rest of the angle variables on the invariant tori of constant $\vec{\mathcal{J}}$ (or constant $\vec{C}$). To be more precise, we show how to obtain the standard phase-space coordinates $(\vec{\mathcal{P}}, \vec{\mathcal{Q}})$ as explicit functions of action-angle variables $(\vec{\mathcal{J}}, \vec{\theta})$. This is in fact the more useful transformation [rather than $(\vec{\mathcal{J}}, \vec{\theta})$ as explicit functions of $(\vec{\mathcal{P}}, \vec{\mathcal{Q}})$] for canonical perturbation theory, since we will need to transform the 2PN and higher Hamiltonian [which is given as explicit function of $(\vec{\mathcal{P}}, \vec{\mathcal{Q}})$] into action-angle variables.

The method to assign angle variables on invariant tori is straightforward. Pick a fiducial point P_0 on an invariant torus, and give it angle coordinates $\vec{0} \equiv (0, \ldots, 0)$. Then every other point on this same torus, with angle coordinates θ^i, is reached by integrating a flow from P_0 by amounts θ^i under each of the actions $\mathcal{J}_i$. This is because the flow parameter is in fact the angle parameter: $d\theta^i/d\lambda^j = \{\theta^i, \mathcal{J}_j\} = \delta^i_j$. The Poisson brackets evaluating to Kronecker delta follows because θ^i and $\mathcal{J}_j$ are canonically conjugate coordinates; see Theorem 10.17 of Ref. [24]. Since the actions commute, we are free to flow under these actions in any order.

The construction explained above was only on an individual torus. The only requirement for extending these variables to being full phase space variables is that the choice of fiducial point $P_0(\vec{\mathcal{J}})$ is smooth in $\vec{\mathcal{J}}$. Given any choice of angle variables, we can always reparametrize them by adding a constant that is a smooth function of $\vec{\mathcal{J}}$. That is, if θ^i are angle variables, then so are $\bar{\theta}^i = \theta^i + \delta\theta^i(\vec{\mathcal{J}})$, with smooth $\delta\theta^i$, which can be verified by taking Poisson brackets: $\{\bar{\theta}^i, \mathcal{J}_j\} = \delta^i_j$. Some of these angle variables may be simpler than others, but here we are only interested in finding one such construction.

So now, the problem of assigning the angle coordinates on the torus has been transformed into that of flowing under all the actions, one by one, by amounts equal to the angle coordinates of the point whose angles are desired (assuming that the starting point had $\vec{\theta} = \vec{0}$). To integrate the equations under the flow associated with any of the five actions, we start with

$$\frac{d\xi}{d\lambda} = \left\{\xi, \mathcal{J}_i(\vec{C})\right\}, = \{\xi, C_j\}\frac{\partial \mathcal{J}_i}{\partial C_j}, \tag{56}$$

where ξ is any one of the phase space coordinates. This is the same sparse matrix $\partial \mathcal{J}_i/\partial C_j$ which appeared in the previous section in Eq. (51). The matrix $\partial \mathcal{J}_i/\partial C_j$ is a function of only the $\vec{C}$'s, and thus is constant on each torus and each of the flows we consider. Hence, integrating the above equation boils down to integrating under the flow of the C_i's. We will now briefly explain how to obtain the solution for the flow under each of the C_i's one by one.

2. Solutions to flow under the commuting constants

The solution for flow under H has been given in Ref. [22]; it has been termed as the "standard solution" there. It is found by filling in the gaps in the solution provided in Ref. [15].[7] The solution for the flow under $S_{\rm eff}\cdot L$ is constructed in Appendix A, with minor caveats. Eqs. (A39), (A66), and (A76) in Appendix A collectively

[6] The result in Refs. [19] [Eq. (7.1 a)] does not have the 1.5PN piece, whereas the result in Ref. [32] [Eq. (11b)] is missing the 1PN piece.

[7] Reference [15] ignored the 1PN Hamiltonian throughout for brevity since the authors deemed it straightforward. Equations (3.28-c, d) of this article have typos.

give solutions for $\vec{L}$ and $\vec{R}$, but the appendix does not give explicit solutions for $\vec{P}, \vec{S}_1$, and $\vec{S}_2$. However, this is not a major hurdle for the following reasons. The solution for $\vec{P}$ can be easily found from that for $\vec{R}$ by noting that P and the angular offset between $\vec{R}$ and $\vec{P}$ remain constant under the $S_{\rm eff} \cdot L$ flow. Also, the solutions for $\vec{S}_1$ can be had using similar calculations as for the solution for $\vec{L}$. Once we have $\vec{S}_1$, $\vec{S}_2$ can be found from $\vec{S}_2 = \vec{J} - \vec{L} - \vec{S}_1$, and the fact that $\vec{J}$ does not change under the $S_{\rm eff} \cdot L$ flow.

It now remains to show how to integrate under the flow of the remaining three C_i's, (J^2, J_z and L^2). Section III [specifically Eqs. (21)–(23)] of Ref. [16] showed that the equations for a flow under any of these quantities can be concisely written in a generalized form as

$$\frac{d\vec{V}}{d\lambda} = \left\{\vec{V}, \mathcal{J}_i\right\} = \vec{U} \times \vec{V}. \tag{57}$$

Here $\vec{U}$ is the constant vector (under the respective flow) $2\vec{J}, \hat{z}$, or $2\vec{L}$ when C_i is J^2, J_z, or L^2, respectively. In the above equation, $\vec{V}$ stands for any of $\vec{R}, \vec{P}, \vec{S}_1$, and $\vec{S}_2$, with the exception that under the flow of L^2, spin vectors do not move; so $\vec{V}$ stands for only $\vec{R}$ and $\vec{P}$ in this case. This basically means that $\vec{V}$ rotates around the fixed vector $\vec{U}$ with an angular velocity whose magnitude is simply U.

Constructing the solutions to the flows under J and L in terms of Cartesian components is cumbersome, so we will work with the magnitudes and the directions of the vectors instead. This paragraph assumes the reader is familiar with the definitions of the frames (ijk) and $(i'j'k')$ which have been introduced with the help of Fig. 5 in Appendix A. Now in light of Eq. (57), it is a simple matter to see that the equations for flow under J and L (or rather Eqs. (21) and (23) of Ref. [16]) imply that

(i) Under the flow of J by an amount $\Delta\lambda$, the azimuthal angles of $\vec{R}, \vec{P}, \vec{S}_1$, and $\vec{S}_2$ in the inertial (ijk) frame increase by $\Delta\lambda$. The magnitudes of the vectors do not change.

(ii) Under the flow of L by an amount $\Delta\lambda$, the azimuthal angles of $\vec{R}$, and $\vec{P}$ in the noninertial $(i'j'k')$ frame increase by $\Delta\lambda$, whereas the spin vectors do not move. The magnitudes of the vectors do not change.

The flow under J_z can be handled similarly. With all the individual pieces now identified, it is now straightforward, although lengthy to find each standard phase space variable as an explicit function of the angle variables θ^i, on any invariant torus.

C. Action-angle based solution at 1.5PN and higher PN orders

Now there are two approaches to solving the real-time dynamics of the system, i.e., a flow under H. The approach by one us in Ref. [15] was to directly integrate the differential equations[7], yielding a quasi-Keplerian parametrization. Although this method is direct, it seems quite difficult to extend this to higher PN orders. The second approach is the action-angle based one, the subject of this paper. All the angles have a trivial real time evolution, each one increasing linearly with time $\dot{\theta}^i = \omega^i(\vec{\mathcal{J}})$. After a certain time t, θ^i has changed by $\omega^i t$, which we can compute. So assuming that $\vec{\theta}(t=0) = \vec{0}$, we can compute the angles $\vec{\theta}(t)$ at any general time t, with the $\vec{\mathcal{J}}$ unchanged. Now the problem has become that of computing $(\vec{\mathcal{P}}, \vec{\mathcal{Q}})(t)$ given $(\vec{\mathcal{J}}, \vec{\theta})(t)$, whose road map has been clearly laid out in Sec. VI B. This concludes our brief description of the action-angle based method of computing the solution. This method has the advantage that evaluating the state of the system (or its derivatives, as needed for computing gravitational waveforms) can be trivially parallelized by evaluating each time independently. Both the above solution methods have been implemented by us in a public *Mathematica* package [22].

Moreover, our action-angle based solution allows for the possibility of using nondegenerate perturbation theory [17,18] to extend our solution to higher PN orders. The procedure of Sec. VI B will yield the standard phase-space variables $(\vec{\mathcal{P}}, \vec{\mathcal{Q}})$ as explicit functions of $(\vec{\mathcal{J}}, \vec{\theta})$. This is exactly what is required for computing perturbed action-angle variables at higher PN order with canonical perturbation theory. Higher-PN terms in the Hamiltonian are given in terms of $(\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2)$, and one must transform them to (unperturbed) action-angle variables to apply perturbation theory. If successful, our method can be seen as the foundation of closed-form solutions of BBHs with arbitrary masses, eccentricity, and spins to high PN orders under the conservative Hamiltonian (excluding radiation-reaction for now). This is in the same spirit as Damour and Deruelle's quasi-Keplerian solution method for nonspinning BBHs given in Ref. [19], which has been pushed to 4PN order recently [33]. We are also currently working to find the 2PN action-angle based solution via canonical perturbation theory.

Note that we could not have applied nondegenerate perturbation theory to a lower PN order (say 1PN) to arrive at 1.5PN or higher PN action-angle variables, because the lower PN systems are degenerate in the full phase space. This is because the spin variables are not dynamical until the 1.5PN order; so at lower orders, there are fewer than four action variables and frequencies.[8] At 1.5PN, the system becomes nondegenerate, and can be used as a starting point for perturbing to higher order. We therefore view our construction of the action-angle variables as

[8]There can be at most 4 different nonzero frequencies of this system irrespective of the PN order, since H must be independent of J_z to preserve SO(3) symmetry.

significant for finding closed-form solutions of the complicated spin-precession dynamics of BBHs with arbitrary eccentricity, masses, and spin.

VII. SUMMARY AND NEXT STEPS

In this paper, we continue the integrability and action-angle variables study of the most general BBH system (both components spinning in arbitrary directions, with arbitrary masses and eccentricity) initiated in Ref. [16]. There, two of us presented four (out of five) actions at 1.5PN and showed the integrable nature of the system at 2PN by constructing two new 2PN perturbative constants of motion. Here, we computed the remaining fifth action variable using a novel mathematical method of inventing unmeasurable phase space variables. We derived the leading order PN contribution to the fifth action, which is a much shorter expression than the "exact" one. We showed how to compute the fundamental frequencies of the system without needing to write the Hamiltonian explicitly in terms of the actions. Finally, we presented a recipe for computing the five angle variables implicitly, by finding $(\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2)$ as explicit functions of action-angle variables. We leave deriving the full expressions to future work. We also sketched how the 1.5PN action-angle variables can be used to construct solutions to the BBH system at higher PN orders via canonical perturbation theory.

Typically, action-angle variables are found by separating the Hamilton-Jacobi (HJ) equation [17], though we were able to work them out without effecting such a separation. Finally from this vantage point, we summarize the major ingredients that went into our action-angle based solution for the PN BBHs: (1) the classic Sommerfeld contour integration method for the Newtonian system, which gave the Newtonian radial action long ago [17]; (2) its PN extension by Damour and Schäfer [20]; (3) the integration techniques worked out in the context of the 1.5PN Hamiltonian flow by one of us in Ref. [15]; and finally, (4) the method of extending the phase space by inventing fictitious phase-space variables introduced in this paper.

A couple of extensions of the present work are possible in the near future. Currently we are working on presenting our 1.5 PN action-angle-based solution in a more concrete and consolidated form, as well as re-presenting the solution given in Ref. [34] with 1PN terms included that were ignored in the original work. We have developed a public *Mathematica* package that implements these two solutions [22], as well as the one from numerical integration. This will prepare a solid base for pushing our action-angle-based solution to 2PN.

Since the integrable nature (existence of action-angle variables) has already been shown in Ref. [16], constructing the 2PN action-angle variables (via canonical perturbation theory) and an action-angle based solution should be the next natural line of work. Our group has already initiated the efforts in that direction. With the motivation behind these action-angle variables study of the BBH systems being having closed-form solution to the system, it would be an interesting challenge to incorporate the radiation-reaction effects at 2.5PN into the to-be-constructed 2PN action-angle based solution. There is also hope that the action-angle variables at 1.5PN can be used to re-present the effective one-body (EOB) approach to the spinning binary of Ref. [21] (via a mapping of action variables between the one-body and the two-body pictures) as was originally done for nonspinning binaries in Ref. [35]. Also, it would be interesting to try to compare our action-angle and frequency results in the limit of extreme mass-ratios with similar work on Kerr extreme mass-ratio inspirals (EMRIs) [36] in some selected EMRI parameter space region where PN approximation is also valid. Comparison is also possible with the recently derived solution of EMRIs with spinning secondaries [37,38]. Another line of effort could be the task of building gravitational waveforms using the BBH solution presented in this paper; Ref. [14] may serve as one of the guides.

Lastly, there a possibility of a mathematically oriented study of our novel method of introducing the unmeasurable, fictitious variables to compute the fifth action. A few pertinent questions along this line could be (1) Is there a way to compute the fifth action without introducing the fictitious variables? (2) Are there other situations (with other topologically nontrivial symplectic manifolds) where an otherwise intractable action computation can be made possible using this new method? (3) What is the deeper geometrical reason that makes this method work?

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APPENDIX A: EVALUATING THE FLOW AMOUNTS $\Delta\lambda$'S

In this appendix, we will rely heavily on the methods of integration first presented in Ref. [15], which integrated the evolution equations for flow under H, with the 1PN Hamiltonian terms omitted. We first need to set up some vector bases before we can integrate the equations of motion. Figure 5 below displays two sets of bases. The one in which the components of a vector will be assumed to be written in this paper is the inertial triad (ijk), unless stated otherwise. Since derivatives of components of vectors depend on the basis, we mention here that this (ijk) triad is also the frame in which all the component derivatives of any general vector will be assumed to be taken, unless stated otherwise.[9]

1. Evaluating $\Delta\lambda_{S_{\rm eff}\cdot L}$

The evaluation of $\Delta\lambda_{S_{\rm eff}\cdot L}$ can happen only when we can compute the mutual angles between $\vec{L}, \vec{S}_1$ and $\vec{S}_2$ as a function of the flow parameter under the flow of $S_{\rm eff}\cdot L$. Therefore, most of Appendix A 1 deals with how to do this calculation and only toward the end we arrive at the expression of $\Delta\lambda_{S_{\rm eff}\cdot L}$.

Under the flow of $S_{\rm eff}\cdot L$, a generic quantity g evolves as $dg/d\lambda = \{g, S_{\rm eff}\cdot L\}$ which implies the three evolution equations for the dot products between the three angular momenta under the flow of $S_{\rm eff}\cdot L$,

$$\frac{1}{\sigma_2}\frac{d(\vec{L}\cdot\vec{S}_1)}{d\lambda} = -\frac{1}{\sigma_1}\frac{d(\vec{L}\cdot\vec{S}_2)}{d\lambda} = \frac{1}{(\sigma_1-\sigma_2)}\frac{d(\vec{S}_1\cdot\vec{S}_2)}{d\lambda} = \vec{L}\cdot(\vec{S}_1\times\vec{S}_2), \quad (A1)$$

which means that we can easily construct three constants of motion (dependent on the five mutually commuting constants as introduced before). These are the differences between the three quantities

$$\left\{\frac{\vec{L}\cdot\vec{S}_1}{\sigma_2}, -\frac{\vec{L}\cdot\vec{S}_2}{\sigma_1}, \frac{\vec{S}_1\cdot\vec{S}_2}{\sigma_1-\sigma_2}\right\}, \quad (A2)$$

whose λ derivatives all agree, the triple product $\vec{L}\cdot(\vec{S}_1\times\vec{S}_2)$. Namely, these constants of motion are

$$\Delta_1 = \frac{\vec{S}_1\cdot\vec{S}_2}{\sigma_1-\sigma_2} - \frac{\vec{L}\cdot\vec{S}_1}{\sigma_2} = \frac{1}{\sigma_1-\sigma_2}\left[\frac{1}{2}(J^2-L^2-S_1^2-S_2^2) - \frac{\vec{L}\cdot\vec{S}_{\rm eff}}{\sigma_2}\right], \quad (A3)$$

$$\Delta_2 = \frac{\vec{S}_1\cdot\vec{S}_2}{\sigma_1-\sigma_2} + \frac{\vec{L}\cdot\vec{S}_2}{\sigma_1} = \frac{1}{\sigma_1-\sigma_2}\left[\frac{1}{2}(J^2-L^2-S_1^2-S_2^2) - \frac{\vec{L}\cdot\vec{S}_{\rm eff}}{\sigma_1}\right], \quad (A4)$$

$$\Delta_{21} = \frac{\vec{L}\cdot\vec{S}_1}{\sigma_2} + \frac{\vec{L}\cdot\vec{S}_2}{\sigma_1} = \frac{\vec{L}\cdot\vec{S}_{\rm eff}}{\sigma_1\sigma_2}. \quad (A5)$$

[9]While the time derivative of a vector is a good geometric object, the time derivatives of basis components are not; see Ch. 4 of Ref. [17].

Stated differently, all this means that the three mutual angles between $\vec{L}$, $\vec{S}_1$, and $\vec{S}_2$ satisfy linear relationships. With the understanding that hatted letters denote unit vectors, if we define the mutual angles as $\cos\kappa_1 \equiv \hat{L}\cdot\hat{S}_1$, $\cos\kappa_2 \equiv \hat{L}\cdot\hat{S}_2$, and $\cos\gamma \equiv \hat{S}_1\cdot\hat{S}_2$, their relations are

$$\cos\gamma = \Sigma_1 + \frac{L}{S_2}\frac{\sigma_1-\sigma_2}{\sigma_2}\cos\kappa_1 \quad (A6)$$

$$\cos\kappa_2 = \Sigma_2 - \frac{\sigma_1 S_1}{\sigma_2 S_2}\cos\kappa_1, \quad (A7)$$

where

$$\Sigma_1 = \frac{(\sigma_1-\sigma_2)\Delta_1}{S_1S_2} = \frac{\sigma_2(J^2-L^2-S_1^2-S_2^2)-2S_{\rm eff}\cdot L}{2\sigma_2 S_1 S_2}, \quad (A8)$$

$$\Sigma_2 = \frac{S_{\rm eff}\cdot L}{\sigma_2 L S_2} = \frac{\Delta_{21}\sigma_1}{LS_2}. \quad (A9)$$

We will integrate the solution for

$$f \equiv \frac{\vec{S}_1\cdot\vec{S}_2}{\sigma_1-\sigma_2} = \frac{S_1S_2\cos\gamma}{\sigma_1-\sigma_2}, \quad (A10)$$

$$\frac{df}{d\lambda} = \vec{L}\cdot(\vec{S}_1\times\vec{S}_2), \quad (A11)$$

which is the most symmetrical of the three dot products given above. Thus if we have a solution for $f(\lambda)$, we automatically have solutions for the three dot products,

$$\vec{S}_1\cdot\vec{S}_2 = (\sigma_1-\sigma_2)f, \quad (A12)$$

$$\vec{L}\cdot\vec{S}_1 = \sigma_2(f-\Delta_1), \quad (A13)$$

$$\vec{L}\cdot\vec{S}_2 = -\sigma_1(f-\Delta_2). \quad (A14)$$

The triple product on the rhs of Eq. (A11) is the signed volume of the parallelepiped with ordered sides $\vec{L}, \vec{S}_1, \vec{S}_2$. In general, for a parallelepiped with sides $\vec{\mathcal{A}}, \vec{\mathcal{B}}, \vec{\mathcal{C}}$, and dot products

$$\vec{\mathcal{A}}\cdot\vec{\mathcal{B}} = \mathcal{A}\mathcal{B}\cos\gamma', \quad (A15)$$

$$\vec{\mathcal{A}}\cdot\vec{\mathcal{C}} = \mathcal{A}\mathcal{C}\cos\beta', \quad (A16)$$

$$\vec{\mathcal{B}}\cdot\vec{\mathcal{C}} = \mathcal{B}\mathcal{C}\cos\alpha', \quad (A17)$$

a standard result from analytical geometry is that the signed volume of this parallelepiped can be written as

$$V = \vec{\mathcal{A}} \cdot (\vec{\mathcal{B}} \times \vec{\mathcal{C}}) = \pm \mathcal{ABC}\Big[1 + 2(\cos\alpha')(\cos\beta')(\cos\gamma') - \cos^2\alpha' - \cos^2\beta' - \cos^2\gamma'\Big]^{1/2}, \tag{A18}$$

where the sign comes from the handedness of the $(\vec{\mathcal{A}}, \vec{\mathcal{B}}, \vec{\mathcal{C}})$ triad. The radicand is always non-negative. As above in Eqs. (A12)–(A14), we can rewrite all angles in terms of f. We can then use this volume equation to express the evolution for f as

$$\frac{df}{d\lambda} = \pm\sqrt{P(f)}, \tag{A19}$$

where the cubic $P(f) \geq 0$ and is given by

$$P(f) = L^2 S_1^2 S_2^2 + 2(\vec{L}\cdot\vec{S}_1)(\vec{L}\cdot\vec{S}_2)(\vec{S}_1\cdot\vec{S}_2) - L^2(\vec{S}_1\cdot\vec{S}_2)^2 - S_1^2(\vec{L}\cdot\vec{S}_2)^2 - S_2^2(\vec{L}\cdot\vec{S}_1)^2 \tag{A20}$$

$$= L^2 S_1^2 S_2^2 - 2\sigma_1\sigma_2(\sigma_1 - \sigma_2) f(f-\Delta_1)(f-\Delta_2) - L^2(\sigma_1-\sigma_2)^2 f^2 - S_2^2\sigma_2^2(f-\Delta_1)^2 - S_1^2\sigma_1^2(f-\Delta_2)^2. \tag{A21}$$

This is a general cubic, which we will write as

$$P(f) = a_3 f^3 + a_2 f^2 + a_1 f + a_0, \tag{A22}$$

with the coefficients

$$a_3 = 2\sigma_1\sigma_2(\sigma_2 - \sigma_1), \tag{A23a}$$

$$a_2 = 2(\Delta_1 + \Delta_2)(\sigma_1 - \sigma_2)\sigma_1\sigma_2 - L^2(\sigma_1 - \sigma_2)^2 - \sigma_1^2 S_1^2 - \sigma_2^2 S_2^2, \tag{A23b}$$

$$a_1 = 2[\sigma_1^2 S_1^2 \Delta_2 + \sigma_2^2 S_2^2 \Delta_1 + \sigma_1\sigma_2\Delta_1\Delta_2(\sigma_2 - \sigma_1)], \tag{A23c}$$

$$a_0 = L^2 S_1^2 S_2^2 - \sigma_1^2 S_1^2 \Delta_2^2 - \sigma_2^2 S_2^2 \Delta_1^2. \tag{A23d}$$

It is important here to note the sign of a_3,

$$\mathrm{sgn}(a_3) = \begin{cases} +1, & m_1 > m_2, \\ 0, & m_1 = m_2, \\ -1, & m_1 < m_2. \end{cases} \tag{A24}$$

The fact that the cubic becomes undefined when $m_1 = m_2$ is the reason we treated the equal-mass case separately toward the end of Sec. IV.

Now we rewrite the cubic in terms of its roots,

$$P(f) = A(f - f_1)(f - f_2)(f - f_3), \tag{A25}$$

where $A = a_3$ is the leading term, and when all three roots are real, we assume the ordering $f_1 < f_2 < f_3$. In other words, we assume the roots to be real and simple.

For completeness, we state the roots in the trigonometric form. The cubic can be depressed by defining $g \equiv f + a_2/(3a_3)$ in terms of which P becomes $P = a_3(g^3 + pg + q)$ with the coefficients

$$p = \frac{3a_1a_3 - a_2^2}{3a_3^2}, \qquad q = \frac{2a_2^3 - 9a_1a_2a_3 + 27a_0a_3^2}{27a_3^3}. \tag{A26}$$

When there are three real solutions, $p < 0$, and the argument to the arccos below will be in $[-1, +1]$. In terms of these depressed coefficients, the trigonometric solutions for the $k = 1, 2, 3$ roots are

$$f_k = -\frac{a_2}{3a_3} + 2\sqrt{\frac{-p}{3}}\cos\left[\frac{1}{3}\arccos\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right) + \frac{2\pi k}{3}\right]. \tag{A27}$$

This form yields the desired ordering $f_1 < f_2 < f_3$.

Whenever any two of the vectors $\{\vec{L}, \vec{S}_1, \vec{S}_2\}$ are collinear, the triple product on the rhs of Eq. (A11) vanishes. A less drastic degeneracy is if two roots coincide. Here we will restrict ourselves to the case of three simple roots. At the end of this subsection, we will argue that the cubic has three real roots for the cases of physical interest. Since $P(f) > 0$, we have

$$\begin{cases} f_1 \leq f \leq f_2, & m_1 > m_2, \\ f_2 \leq f \leq f_3, & m_1 < m_2. \end{cases} \tag{A28}$$

That is, f will lie between the two roots where $P(f) > 0$. Without loss of generality we will take $m_1 > m_2$ and handle only this case.

Since $P(f)$ is cubic, the ODE $df/d\lambda = \pm\sqrt{P(f)}$ can be integrated analytically in terms of elliptic integrals (and their inverses, elliptic functions). The behavior is typical: f oscillates between the two turning points f_1, f_2 (when $m_1 > m_2$). We cannot integrate through the turning point using the first-order form $df/d\lambda$ (it is not Lipschitz continuous there), but by taking a derivative of Eq. (A19) to find $d^2f/d\lambda^2$, we can see that the motion is regular at each turning point. At both turning points, the $\pm$ sign [the handedness of the triad $(\vec{L}, \vec{S}_1, \vec{S}_2)$] must flip, so that f oscillates between the two turning points.

Continuing further with Eq. (A19), we write

$$\frac{df}{\sqrt{(f-f_1)(f-f_2)(f-f_3)}} = \sqrt{A}d\lambda. \tag{A29}$$

Reparametrize this integral via

$$f = f_1 + (f_2 - f_1)\sin^2\phi_p \tag{A30}$$

$$df = 2(f_2 - f_1)\sin\phi_p\cos\phi_p d\phi_p. \tag{A31}$$

We define ϕ_p so it increases monotonically with λ as

$$\frac{2d\phi_p}{\sqrt{(f_3-f_1)-(f_2-f_1)\sin^2\phi_p}} = \sqrt{A}d\lambda. \tag{A32}$$

Now factor out $(f_3 - f_1)$ from the radicand in the denominator to give

$$\frac{d\phi_p}{\sqrt{1-k^2\sin^2\phi_p}} = \frac{1}{2}\sqrt{A(f_3-f_1)}d\lambda, \tag{A33}$$

where we have defined the elliptic modulus

$$k \equiv \sqrt{\frac{f_2-f_1}{f_3-f_1}}. \tag{A34}$$

Note that $0 < k < 1$, because of the ordering of the roots. Equation (A33) can be integrated to give

$$u - u_0 \equiv F(\phi_p, k) = \frac{1}{2}\sqrt{A(f_3-f_1)}(\lambda-\lambda_0), \tag{A35}$$

where $F(\phi_p, k)$ is the incomplete elliptic integral of the first kind defined as [39–42]

$$F(\phi, k) \equiv \int_0^{\phi} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}. \tag{A36}$$

In Eq. (A35), λ_0 is the initial value of the flow parameter and

$$u_0 \equiv u(\lambda_0) = F\left(\arcsin\sqrt{\frac{f(\lambda_0)-f_1}{f_2-f_1}}, k\right). \tag{A37}$$

We can now rewrite the parametrization in terms of sn and am, the Jacobi sine and amplitude functions [39],

$$sn(u, k) \equiv \sin(am(u, k)) \equiv \sin\phi_p. \tag{A38}$$

This turns our parametrization into

$$f(\lambda) = f_1 + (f_2 - f_1)sn^2(u(\lambda), k). \tag{A39}$$

The solution for f is thus given by Eq. (A39) accompanied by Eqs. (A35) and (A37).

It now remains to generalize the solution for f when f at $\lambda = \lambda_0$ may be in any arbitrary initial state (such as $df/d\lambda < 0$ or > 0) and it can oscillate between f_1 and f_2 any arbitrary number of times during the integration interval. In this most general scenario, the solution is still given by Eq. (A39), accompanied by Eq. (A35) and a variant of (A37), which reads

$$u_0 \equiv u(\lambda_0) = F\left(\arcsin \pm\sqrt{\frac{f(\lambda_0)-f_1}{f_2-f_1}}, k\right), \tag{A40}$$

where we use the $+$ sign if $(df/d\lambda)|_{\lambda_0} > 0$, and vice versa.

From this solution for $f(\lambda)$, we recover solutions for the three dot products $\vec{S}_1\cdot\vec{S}_2$, $\vec{L}\cdot\vec{S}_1$, and $\vec{L}\cdot\vec{S}_2$, by using Eqs. (A12)–(A14). We also immediately get the λ-period of the precession. One precession cycle occurs when ϕ_p goes from 0 to π, or when f starts from f_1, goes to f_2 and then returns back to f_1 [see parametrization in Eq. (A30)]. Integrating on this interval via Eq. (A35) gives the equation for the λ-period of precession, which we call Λ, in terms of the complete elliptic integral of the first kind $K(k) \equiv F(\pi/2, k) = F(\pi, k)/2$,

$$\Lambda\sqrt{A(f_3-f_1)} = 2F(\pi, k) = 4K(k). \tag{A41}$$

Recall that our goal is to close a loop in the EPS by successively flowing under $S_{\mathrm{eff}}\cdot L, J^2, L^2, S_1^2$, and S_2^2. A necessary condition for the phase-space loop to close is that the mutual angles between $\vec{L}$, $\vec{S}_1$, and $\vec{S}_2$ recur at the end of the flow. Since the flows under J^2, L^2, S_1^2, and S_2^2 do not change these mutual angles, we choose to flow under $S_{\mathrm{eff}}\cdot L$ by exactly the precession period,

$$\Delta\lambda_{S_{\mathrm{eff}}\cdot L} = \Lambda. \tag{A42}$$

This flow under $S_{\mathrm{eff}}\cdot L$ is pictorially represented by the red PQ curve in Fig. 4.

Now we try to address the issue of the nature of roots of the cubic $P(f)$ of Eq. (A22). It is predicated on the nature of the cubic discriminant D, with $D > 0$ implying three real roots, $D < 0$ implying one real and two distinct complex roots, and $D = 0$ implying repeated roots. The discriminant of the exact cubic $P(f)$ is too complicated for us to investigate its sign analytically. We rather choose to investigate the sign of its leading order PN contribution. It is in the same spirit as the calculation of the leading PN order contribution of $\mathcal{J}_5$ in Sec. V. We write D in terms of $\vec{L}$, $\vec{S}_1$, and $\vec{S}_2$ while attaching a formal power counting parameter ϵ to both $\vec{S}_1$ and $\vec{S}_2$, for every factor of ϵ signifies an extra 0.5PN order. Then series expand D in ϵ and keep only the leading order term, which comes out to be

$$D \sim 4L^4\left[L^2S_1^2 - \left(\vec{L}\cdot\vec{S}_1\right)^2\right]\left[L^2S_2^2 - \left(\vec{L}\cdot\vec{S}_2\right)^2\right] \times (\sigma_1-\sigma_2)^6\epsilon^4 + \mathcal{O}(\epsilon^5) > 0, \tag{A43}$$

and this implies three real roots. If both spins are aligned or antialigned with $\vec{L}$, we will have repeated roots, and the spins will remain aligned or antialigned with $\vec{L}$ as the system evolves under the flows of $S_{\mathrm{eff}}\cdot L$ or H. Aside from this special case, the above discussion suggests that the

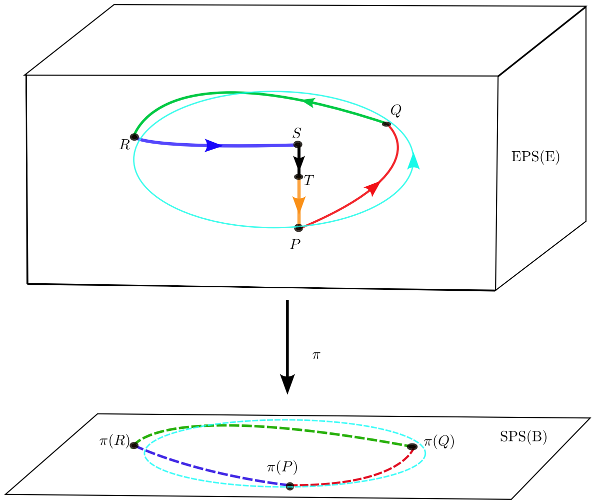

FIG. 4. Schematic depiction of closing the loop in the EPS over which the fifth action integral is computed. This is done by flowing under $S_{\rm eff}\cdot L$ (red), J^2 (green), L^2 (blue), S_1^2 (black), and S_2^2 (orange). The curve in cyan is the one found by flowing under $\mathcal{J}_5$ in the EPS. The corresponding π projections of the solid curves in the EPS is shown by broken curves in the SPS with the same color. The segments ST and TP are vertical because only the fictitious variables change along these.

$D<0$ case of only one real root is disallowed. This is also necessary on physical grounds, as there must be two turning points for the mutual angle variable f, otherwise f would be unbounded.

2. Evaluating $\Delta\lambda_{J^2}$

After flowing under $S_{\rm eff}\cdot L$ by parameter $\Delta\lambda_{S_{\rm eff}\cdot L}$, the mutual angles between $\vec{L}$, $\vec{S}_1$, and $\vec{S}_2$ have recurred, but $\vec{L}$, $\vec{S}_1$, and $\vec{S}_2$ have not. We now plan to flow under J^2 by $\Delta\lambda_{J^2}$ so that $\vec{L}$ is restored; this restoration is a necessary condition for closing the phase space loop. To find the required amount of flow under J^2 so that $\vec{L}$ is restored, we need to find the final state of $\vec{L}$ after flowing under $S_{\rm eff}\cdot L$ by $\Delta\lambda_{S_{\rm eff}\cdot L}$. Instead of working with Cartesian components, we find it more convenient to work with the polar and azimuthal angles of $\vec{L}$ in a new noninertial frame that we now introduce.

At this point we introduce a noninertial frame with $(i'j'k')$ axes whose basis vectors are unit vectors along $\vec{J}\times\vec{L}, \vec{L}\times(\vec{J}\times\vec{L})$, and $\vec{L}$ respectively, as depicted pictorially in Fig. 5. Without loss of generality, we choose the z-axis of the (ijk) frame to point along the $\vec{J}$ vector. Now there are two angles to find: the polar θ_{JL}, where $\cos\theta_{JL}=\vec{J}\cdot\vec{L}/(JL)$, and an azimuthal ϕ_L.

Since we have already solved for the angles between $\vec{L}, \vec{S}_1$, and $\vec{S}_2$ in Appendix A 1, we have the angle θ_{JL} from

$$\vec{J}\cdot\vec{L}=JL\cos\theta_{JL}=L^2+\vec{S}_1\cdot\vec{L}+\vec{S}_2\cdot\vec{L}. \quad (A44)$$

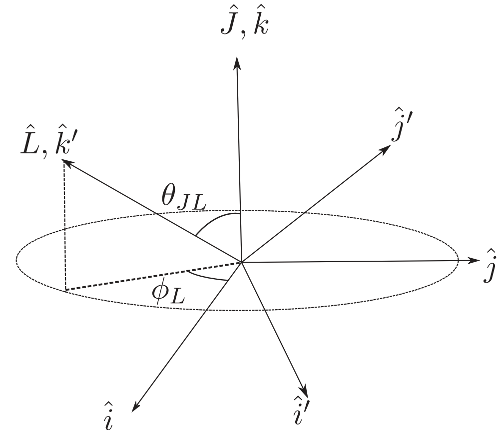

FIG. 5. The noninertial $(i'j'k')$ triad (centered around $\hat{L}\equiv\vec{L}/L$) is displayed along with the inertial (ijk) triad (centered around $\hat{J}\equiv\vec{J}/J$).

This shows that θ_{JL} has recurred after the $S_{\rm eff}\cdot L$ flow, because all the mutual angles between $\vec{L}, \vec{S}_1$, and $\vec{S}_2$ have. So, what remains to be tackled is the azimuthal angle ϕ_L. The inertial (ijk) components of $\vec{L}$ are

$$\vec{L}=L(\sin\theta_{JL}\cos\phi_L, \sin\theta_{JL}\sin\phi_L, \cos\theta_{JL}), \quad (A45)$$

and therefore it follows that

$$\frac{d\vec{L}}{d\lambda}=L\bigg(\cos\theta_{JL}\cos\phi_L\frac{d\theta_{JL}}{d\lambda}-\sin\theta_{JL}\sin\phi_L\frac{d\phi_L}{d\lambda}, \cos\theta_{JL}\sin\phi_L\frac{d\theta_{JL}}{d\lambda}+\sin\theta_{JL}\cos\phi_L\frac{d\phi_L}{d\lambda}, -\sin\theta_{JL}\frac{d\theta_{JL}}{d\lambda}\bigg). \quad (A46)$$

As mentioned in the beginning of Appendix A, all vector derivatives are assumed to be taken in the inertial (ijk) frame, unless stated otherwise. With the aid of the instantaneous azimuthal direction vector given by

$$\hat{\phi}=\frac{\vec{J}\times\vec{L}}{|\vec{J}\times\vec{L}|}=\frac{\vec{J}\times\vec{L}}{JL\sin\theta_{JL}}=\frac{\hat{z}\times\vec{L}}{L\sin\theta_{JL}}, \quad (A47)$$

we can extract $d\phi_L/d\lambda$ via an elementary result involving the dot product $\hat{\phi}\cdot(d\vec{L}/d\lambda)$

$$\hat{\phi}\cdot\frac{d\vec{L}}{d\lambda}=L\sin\theta_{JL}\frac{d\phi_L}{d\lambda}. \quad (A48)$$

This leads to

$$\frac{d\phi_L}{d\lambda}=\frac{\hat{\phi}\cdot(d\vec{L}/d\lambda)}{L\sin\theta_{JL}}=\frac{\vec{J}\times\vec{L}}{JL^2\sin^2\theta_{JL}}\cdot\frac{d\vec{L}}{d\lambda}. \quad (A49)$$

Now using $d\vec{L}/d\lambda = -d\vec{S}_1/d\lambda - d\vec{S}_2/d\lambda$, and inserting the precession equations for the two spins,

$$\frac{d\phi_L}{d\lambda} = \frac{1}{JL^2\sin^2\theta_{JL}}(\vec{J}\times\vec{L})\cdot(\vec{S}_{\rm eff}\times\vec{L}) = \frac{J}{J^2L^2-(\vec{J}\cdot\vec{L})^2}\left[(\vec{J}\cdot\vec{S}_{\rm eff})L^2-(\vec{J}\cdot\vec{L})(\vec{L}\cdot\vec{S}_{\rm eff})\right] \tag{A50}$$

$$= \frac{J\left[(\sigma_1 S_1^2+\sigma_2 S_2^2+(\sigma_1+\sigma_2)\vec{S}_1\cdot\vec{S}_2)L^2-(\vec{S}_1\cdot\vec{L}+\vec{S}_2\cdot\vec{L})(\vec{L}\cdot\vec{S}_{\rm eff})\right]}{J^2L^2-(L^2+\vec{S}_1\cdot\vec{L}+\vec{S}_2\cdot\vec{L})^2}. \tag{A51}$$

We see that everything on the rhs is given in terms of constants of motion $(J, L, \vec{L}\cdot\vec{S}_{\rm eff})$ and the inner products between the three angular momenta (which can be found from $f(\lambda)$ in the previous section). Put everything in terms of f using Eqs. (A12)–(A14) and separate into partial fractions,

$$\frac{d\phi_L}{d\lambda} = \frac{J\left[(\sigma_1 S_1^2+\sigma_2 S_2^2+(\sigma_1^2-\sigma_2^2)f)L^2-(\sigma_2(f-\Delta_1)-\sigma_1(f-\Delta_2))(\vec{L}\cdot\vec{S}_{\rm eff})\right]}{J^2L^2-(L^2+\sigma_2(f-\Delta_1)-\sigma_1(f-\Delta_2))^2} \tag{A52}$$

$$= \frac{B_1}{D_1-(\sigma_1-\sigma_2)f}+\frac{B_2}{D_2-(\sigma_1-\sigma_2)f}, \tag{A53}$$

where we have defined

$$B_1 = \frac{1}{2}\left[(\vec{L}\cdot\vec{S}_{\rm eff}+L^2(\sigma_1+\sigma_2))(J+L)+L(\sigma_1 S_1^2+\sigma_2 S_2^2+(\sigma_1+\sigma_2)(\Delta_2\sigma_1-\Delta_1\sigma_2))\right], \tag{A54}$$

$$B_2 = \frac{1}{2}\left[(\vec{L}\cdot\vec{S}_{\rm eff}+L^2(\sigma_1+\sigma_2))(J-L)-L(\sigma_1 S_1^2+\sigma_2 S_2^2+(\sigma_1+\sigma_2)(\Delta_2\sigma_1-\Delta_1\sigma_2))\right], \tag{A55}$$

$$D_1 = L(L+J)+\Delta_2\sigma_1-\Delta_1\sigma_2, \tag{A56}$$

$$D_2 = L(L-J)+\Delta_2\sigma_1-\Delta_1\sigma_2. \tag{A57}$$

So we need to be able to perform the two integrals (with $i = 1, 2$)

$$I_i \equiv \int\frac{B_i}{D_i-(\sigma_1-\sigma_2)f}d\lambda = \int\frac{B_i}{D_i-(\sigma_1-\sigma_2)f}\frac{d\lambda}{df}df = \int\frac{\pm B_i}{D_i-(\sigma_1-\sigma_2)f}\frac{df}{\sqrt{A(f-f_1)(f-f_2)(f-f_3)}}, \tag{A58}$$

where the last equality is due to Eq. (A19). With these integrals, we will have

$$\int\frac{d\phi_L}{d\lambda}d\lambda = \phi_L(f)-\phi_{L,0} = I_1+I_2. \tag{A59}$$

The integrals I_i are another type of incomplete elliptic integral (defined below). Using the parametrization of Eqs. (A30) and (A31), I_i becomes

$$I_i(\lambda) = \int^{\phi_p}\frac{B_i}{D_i-(\sigma_1-\sigma_2)(f_1+(f_2-f_1)\sin^2\phi_p)}\frac{2d\phi_p}{\sqrt{A(f_3-f_1)(1-k^2\sin^2\phi_p)}} \tag{A60}$$

$$= \frac{2B_i}{\sqrt{A(f_3-f_1)}}\frac{1}{D_i-f_1(\sigma_1-\sigma_2)}\int^{\phi_p}\frac{1}{1-\alpha_i^2\sin^2\phi_p}\frac{d\phi_p}{\sqrt{1-k^2\sin^2\phi_p}}, \tag{A61}$$

where we have defined

$$\alpha_i^2 \equiv \frac{(\sigma_1 - \sigma_2)(f_2 - f_1)}{D_i - f_1(\sigma_1 - \sigma_2)}. \tag{A62}$$

Thus we can identify the I_i's in terms of the incomplete elliptic integral of the third kind, which is defined as [39]

$$\Pi(a, b, c) \equiv \int_0^b \frac{1}{\sqrt{1 - c^2 \sin^2\theta}} \frac{d\theta}{1 - a\sin^2\theta}. \tag{A63}$$

I_i thus becomes

$$I_i(\lambda) = \frac{2B_i}{\sqrt{A(f_3 - f_1)}} \frac{\Pi(\alpha_i^2, am(u(\lambda), k), k)}{D_i - f_1(\sigma_1 - \sigma_2)}, \tag{A64}$$

and we get the solution for $\phi_L(\phi_p)$

$$\phi_L(\lambda) - \phi_{L,0} = \frac{2}{\sqrt{A(f_3 - f_1)}} \left[\frac{B_1 \Pi(\alpha_1^2, \phi_p, k)}{D_1 - f_1(\sigma_1 - \sigma_2)} + \frac{B_2 \Pi(\alpha_2^2, \phi_p, k)}{D_2 - f_1(\sigma_1 - \sigma_2)} \right]. \tag{A65}$$

Here $\phi_{L,0}$ is an integration constant to be determined by inserting $\lambda = \lambda_0$ and $\phi_L = \phi_L(\lambda_0)$ into the Eq. (A65).

To close the loop, we need to know the angle $\Delta\phi_L$ that ϕ_L goes through under one period of the precession cycle (when flowing under $\vec{L} \cdot \vec{S}_{\rm eff}$), that is, when ϕ_p advances by π. This is given in terms of the *complete* elliptic integral of the third kind, $\Pi(\alpha^2, k) \equiv \Pi(\alpha^2, \pi/2, k)$ yielding

$$\begin{aligned} \Delta\phi_L &\equiv \phi_L(\lambda_0 + \Lambda) - \phi_L(\lambda_0) \\ &= \frac{4}{\sqrt{A(f_3 - f_1)}} \left[\frac{B_1 \Pi(\alpha_1^2, k)}{D_1 - f_1(\sigma_1 - \sigma_2)} + \frac{B_2 \Pi(\alpha_2^2, k)}{D_2 - f_1(\sigma_1 - \sigma_2)} \right], \end{aligned} \tag{A66}$$

where we have used the fact that $\Pi(\alpha^2, \pi, k) = 2\Pi(\alpha^2, k)$.

To negate this angular offset caused by flowing under $S_{\rm eff} \cdot L$ and thereby closing the loop, we need to flow under J^2 by

$$\Delta\lambda_{J^2} = -\frac{\Delta\phi_L}{2J}. \tag{A67}$$

Note that this flow does not alter the mutual angles between $\vec{L}, \vec{S}_1$, and $\vec{S}_2$, as is necessary to close the loop in the phase space. Now that the mutual angles within the triad $(\vec{L}, \vec{S}_1, \vec{S}_2)$ have recurred and the full $\vec{L}$ vector has recurred, the concern is if the spin vectors have recurred or not. The spin vectors are constrained not only by their mutual angles with $\vec{L}$, but also $\vec{J}$. Their angles with $\vec{J}$ are algebraically related to the mutual angles that we have previously dealt with, e.g., $\vec{J} \cdot \vec{S}_2 = \vec{L} \cdot \vec{S}_1 + \vec{S}_1 \cdot \vec{S}_2 + S_2^2$.

After the respective flows under $S_{\rm eff} \cdot L$ and J^2 by amounts indicated in Eqs. (A42) and (A67), all of these angle cosines between $\vec{L}, \vec{S}_1$, and $\vec{S}_2$ have recurred, which narrows things down to two solutions: the original configuration for $(\vec{L}, \vec{S}_1, \vec{S}_2)$, and its reflection across the $J - L$ plane. We can rule out the reflected solution with the following observation. The original configuration and its reflection have opposite signs for the signed volume $\vec{L} \cdot (\vec{S}_1 \times \vec{S}_2)$, and thus opposite signs for the radical $\sqrt{P(f)}$ in Eq. (A19). Now once we return back to the same point on the f axis after flowing under $S_{\rm eff} \cdot L$, the handedness of the $(\vec{L}, \vec{S}_1, \vec{S}_2)$ triad is restored. This is because the handedness must have flipped twice: first when f touched f_1 and second when it touched f_2.[10] Therefore, after the flows by $S_{\rm eff} \cdot L$ and J^2 by the amounts specified in Eqs. (A42) and (A67), each of the three vectors $(\vec{L}, \vec{S}_1, \vec{S}_2)$ have recurred. This second flow under J^2 is pictorially represented by the green QR curve in Fig. 4.

3. Evaluating $\Delta\lambda_{L^2}$

After flowing under $S_{\rm eff} \cdot L$ and J^2, all the three angular momenta $\vec{L}, \vec{S}_1$, and $\vec{S}_2$ have recurred, but the orbital vectors ($\vec{R}$, $\vec{P}$) and fictitious vectors have not. We will now restore $\vec{R}$ and $\vec{P}$ by flowing under L^2 by $\Delta\lambda_{L^2}$, to be determined in this section.

Now, $\vec{R}$ has to be in the $i'j'$ plane because $\vec{R} \perp \vec{L}$. Denote by ϕ the angle made by $\vec{R}$ with the i' axis. The key point is that after successively flowing under $S_{\rm eff} \cdot L$ by $\lambda_{S_{\rm eff} \cdot L}$, J^2 by λ_{J^2}, and L^2 by a certain amount λ_{L^2} (to be calculated), if ϕ is restored, then so are $\vec{R}$ and $\vec{P}$. This is because under these three flows, R, P, and $\vec{R} \cdot \vec{P}$ do not change. Hence the restoration of ϕ after the above three flows by the stated amounts restores both $\vec{R}$ and $\vec{P}$.

Our strategy is to compute ϕ under the flow of $S_{\rm eff} \cdot L$. The flow under J^2 does not change the angle ϕ, since J^2 rigidly rotates all vectors together. And in the end, we will undo the change to ϕ (caused by the $S_{\rm eff} \cdot L$ flow) by flowing under L^2.

Under the flow of $S_{\rm eff} \cdot L$, we have

$$\dot{\vec{R}} = \{\vec{R}, S_{\rm eff} \cdot L\} = \vec{S}_{\rm eff} \times \vec{R}. \tag{A68}$$

[10] There is also a complex-analytic interpretation. The function $\sqrt{P(f)}$ is an analytic function on a Riemann surface of two sheets. The different signs of $\vec{L} \cdot (\vec{S}_1 \times \vec{S}_2)$ correspond to being on the two different sheets. The solution is periodic after completing a loop around both branch points, ending on the same sheet where we started.

To write the components of this equation in the $(i'j'k')$ frame, we need the components of all the individual vectors involved in the same frame which are given by

$$\vec{R} = \begin{bmatrix} R\cos\phi \\ R\sin\phi \\ 0 \end{bmatrix}_n, \qquad \vec{L} = \begin{bmatrix} 0 \\ 0 \\ L \end{bmatrix}_n,$$
$$\vec{J} = \begin{bmatrix} 0 \\ J\sin\theta_{JL} \\ J\cos\theta_{JL} \end{bmatrix}_n, \qquad \vec{S}_1 = S_1\begin{bmatrix} \sin\kappa_1\cos\xi_1 \\ \sin\kappa_1\sin\xi_1 \\ \cos\kappa_1 \end{bmatrix}_n,$$
$$\vec{S}_2 = S_2\begin{bmatrix} \sin\kappa_2\cos\xi_2 \\ \sin\kappa_2\sin\xi_2 \\ \cos\kappa_2 \end{bmatrix}_n, \tag{A69}$$

where ϕ is the azimuthal angle of $\vec{R}$ in the $(i'j'k')$ frame. Here the letter n beside these columns indicates that the components are in the $(i'j'k')$ frame, and ξ_i's are the azimuthal angles of $\vec{S}_i$ in this $(i'j'k')$ frame.

The Euler matrix $\tilde{\Lambda}$, which when multiplied with the column consisting of a vector's components in the inertial frame gives its components in the $(i'j'k')$ frame is

$$\tilde{\Lambda} = \begin{pmatrix} -\sin\phi_L & \cos\phi_L & 0 \\ -\cos\phi_L\cos\theta_{JL} & -\sin\phi_L\cos\theta_{JL} & \sin\theta_{JL} \\ \cos\phi_L\sin\theta_{JL} & \sin\phi_L\sin\theta_{JL} & \cos\theta_{JL} \end{pmatrix}. \tag{A70}$$

Now we take the $\vec{R}$ in Eq. (A69), evaluate its components in the inertial frame using $\tilde{\Lambda}^{-1}$. We then differentiate each of these components with respect to λ (the flow parameter under $S_{\mathrm{eff}}\cdot L$) and transform these components back to the $(i'j'k')$ frame using $\tilde{\Lambda}$, thus finally yielding the components (in the noninertial frame) of the derivative of $\vec{R}$. The result comes out to be (keeping in mind that $dR/d\lambda = 0$)

$$\dot{\vec{R}} = \begin{bmatrix} -R\sin\phi(\dot{\phi}_L\cos\theta_{JL} + \dot{\phi}) \\ R\cos\phi(\dot{\phi}_L\cos\theta_{JL} + \dot{\phi}) \\ R(-\dot{\phi}_L\sin\theta_{JL}\cos\phi + \dot{\theta}_{JL}\sin\phi) \end{bmatrix}_n. \tag{A71}$$

Plugging Eqs. (A69) and (A71) in Eq. (A68) and using the first two components of the resulting matrix equation gives us

$$\frac{d\phi}{d\lambda} = \sigma_1 S_1\cos\kappa_1 + \sigma_2 S_2\cos\kappa_2 - \cos\theta_{JL}\frac{d\phi_L}{d\lambda} \tag{A72}$$

Note that what we need for Eq. (A68) is the noninertial-frame components of the frame-independent vector $\dot{\vec{R}}$; not to be confused with the time derivatives of the noninertial-frame components of $\vec{R}$.

We digress a bit to write $\vec{J} = \vec{L} + \vec{S}_1 + \vec{S}_2$ in component form in the $(i'j'k')$ frame using Eqs. (A69). Only the third component is of interest to us, which reads

$$J\cos\theta_{JL} = L + S_1\cos\kappa_1 + S_2\cos\kappa_2. \tag{A73}$$

We use this equation for θ_{JL}, and Eqs. (A53) for $d\phi_L/d\lambda$, to write $d\phi/d\lambda$ in terms of κ_1, κ_2, and γ. Finally using Eqs. (A12)–(A14) to express everything in terms of f, we get

$$\frac{d\phi}{d\lambda} = \frac{B_1}{D_1 - (\sigma_1-\sigma_2)f} - \frac{B_2}{D_2 - (\sigma_1-\sigma_2)f} - \frac{S_{\mathrm{eff}}\cdot L + (\Delta_1-\Delta_2)\sigma_1\sigma_2 + L^2(\sigma_1+\sigma_2)}{L}. \tag{A74}$$

This is the equivalent of Eq. (A53) for $d\phi_L/d\lambda$, and therefore its solution can be found in a totally parallel way to what led us to $\phi_L(\lambda)$ in Eq. (A65). This gives us

$$\phi(\lambda) - \phi_0 = \frac{2}{\sqrt{A(f_3-f_1)}}\left[\frac{B_1\Pi(\alpha_1^2,\phi_p,k)}{D_1 - f_1(\sigma_1-\sigma_2)} - \frac{B_2\Pi(\alpha_2^2,\phi_p,k)}{D_2 - f_1(\sigma_1-\sigma_2)}\right] - (S_{\mathrm{eff}}\cdot L + (\Delta_1-\Delta_2)\sigma_1\sigma_2 + L^2(\sigma_1+\sigma_2))\frac{(\lambda-\lambda_0)}{L}, \tag{A75}$$

where again the integration constant ϕ_0 is determined by inserting $\lambda = \lambda_0$ and $\phi = \phi(\lambda_0)$ into this equation.

The angle $\Delta\phi$ that ϕ goes through under one period of the precession cycle when flowing under $\vec{L}\cdot\vec{S}_{\mathrm{eff}}$, is given in a similar manner as we arrived at Eq. (A66). We get

$$\Delta\phi \equiv \phi(\lambda_0+\Lambda) - \phi(\lambda_0) = \frac{4}{\sqrt{A(f_3-f_1)}}\left[\frac{B_1\Pi(\alpha_1^2,k)}{D_1 - f_1(\sigma_1-\sigma_2)} - \frac{B_2\Pi(\alpha_2^2,k)}{D_2 - f_1(\sigma_1-\sigma_2)}\right] - (S_{\mathrm{eff}}\cdot L + (\Delta_1-\Delta_2)\sigma_1\sigma_2 + L^2(\sigma_1+\sigma_2))\frac{\Lambda}{L}. \tag{A76}$$

To negate this angular offset caused by flowing under $S_{\rm eff}\cdot L$, we need to flow under L^2 by

$$\Delta\lambda_{L^2} = -\frac{\Delta\phi}{2L}. \tag{A77}$$

Note that this flow does not change any of the three angular momenta $\vec{L}, \vec{S}_1$, or $\vec{S}_2$, which is necessary for closing the loop in the phase space. This third flow under L^2 is pictorially represented by the blue RS curve in Fig. 4.

4. Evaluating $\Delta\lambda_{S_1^2}$ and $\Delta\lambda_{S_2^2}$

Once we have made sure that $\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2$ (and hence also $\vec{L}$) have been restored by successively flowing under $S_{\rm eff}\cdot L, J^2$, and L^2 by $\Delta\lambda_{S_{\rm eff}\cdot L}, \Delta\lambda_{J^2}$, and $\Delta\lambda_{L^2}$ respectively, now is the time to restore the fictitious vectors $\vec{R}_{1/2}$ and $\vec{P}_{1/2}$. The strategy and calculations are analogous to the ones for $\vec{R}$ and $\vec{P}$, so we will not explicate them in full detail. We will show the basic road map and the final results.

For the purposes of these calculations, the relevant figure is Fig. 6, which shows a second noninertial frame $(i''j''k'')$ adapted to $\vec{S}_1$. Its axes point along $\vec{J}\times\vec{S}_1, \vec{S}_1\times(\vec{J}\times\vec{S}_1)$ and $\vec{S}_1$, respectively. We also use this figure to introduce the definitions of the azimuthal angle ϕ_{S_1} and polar angle θ_{JS_1} pictorially. Also, just like ϕ was the angle between $\vec{R}$ and the i' axis in Appendix A 3, we define ϕ_1 to be the angle between $\vec{R}_1$ and the i'' axis, with the understanding that $\vec{R}_1$ lies in the $i''j''$ plane.

As far as the fictitious variables of the first black hole are concerned, just like in Appendices A 2 and A 3, all we have to worry about is to restore the change in ϕ_1 which the $S_{\rm eff}\cdot L$ flow (by $\lambda_{S_{\rm eff}\cdot L}$) brings about, for doing so would imply that both $\vec{R}_1$ and $\vec{P}_1$ have been restored. The justifications are analogous to those presented in Appendices A 2 and A 3 while dealing with the orbital sector. Now we proceed to compute the change in ϕ_1 brought about by the $S_{\rm eff}\cdot L$ flow.

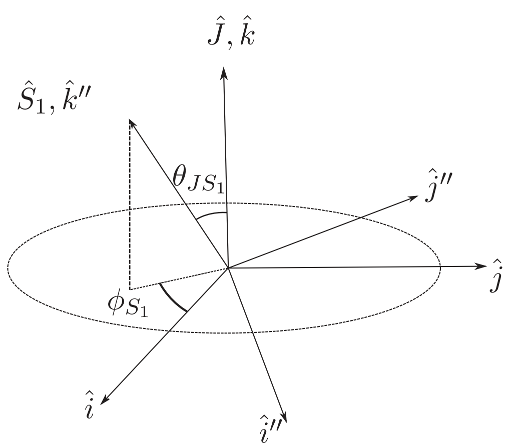

FIG. 6. The second noninertial $(i''j''k'')$ triad (centered around $\hat{S}_1\equiv\vec{S}_1/S_1$) is displayed along with the inertial (ijk) triad (centered around $\hat{J}\equiv\vec{J}/J$).

We denote components in the $(i''j''k'')$ frame by using the subscript $n2$. In this frame we have

$$\vec{J} = \begin{bmatrix} 0 \\ J\sin\theta_{JS_1} \\ J\cos\theta_{JS_1} \end{bmatrix}_{n2}, \qquad \vec{S}_1 = \begin{bmatrix} 0 \\ 0 \\ S_1 \end{bmatrix}_{n2}. \tag{A78}$$

We also have

$$\vec{L} = L\begin{bmatrix} \sin\kappa_1\cos\xi_3 \\ \sin\kappa_1\sin\xi_3 \\ \cos\kappa_1 \end{bmatrix}_{n2}, \quad \vec{S}_2 = S_2\begin{bmatrix} \sin\gamma\cos\xi_4 \\ \sin\gamma\sin\xi_4 \\ \cos\gamma \end{bmatrix}_{n2}. \tag{A79}$$

Here ξ_3 and ξ_4 are the azimuthal angles of $\vec{L}$ and $\vec{S}_2$, respectively, in the $(i''j''k'')$ frame. We now write the k'' component of $\vec{J}\equiv\vec{L}+\vec{S}_1+\vec{S}_2$ as

$$J\cos\theta_{JS_1} = S_1 + L\cos\kappa_1 + S_2\cos\gamma. \tag{A80}$$

The derivative of $\vec{S}_1$ along the flow of $S_{\rm eff}\cdot L$ is

$$\dot{\vec{S}}_1 \equiv \frac{d\vec{S}_1}{d\lambda} = \left\{\vec{S}_1, \vec{S}_{\rm eff}\cdot\vec{L}\right\} = \sigma_1\vec{L}\times\vec{S}_1. \tag{A81}$$

The analog of $d\phi/d\lambda$ given in Eq. (A49) becomes

$$\frac{d\phi_{S_1}}{d\lambda} = \frac{\vec{J}\times\vec{S}_1}{JS_1^2\sin^2\theta_{JS_1}}\cdot\frac{d\vec{S}_1}{d\lambda}. \tag{A82}$$

Using Eq. (A81), we can arrive at the analog of $d\phi/d\lambda$ as a function of f [Eq. (A53)],

$$\frac{d\phi_{S_1}}{d\lambda} = J\sigma_2 + \frac{B_{1S1}}{D_{1S1}+\sigma_1 f} + \frac{B_{2S1}}{D_{2S1}+\sigma_1 f}, \tag{A83}$$

where we have defined

$$B_{1S1} = \frac{1}{2}[-S_1\sigma_1(L^2 - JS_1 + S_1^2 + \Delta_2\sigma_1) + (J-S_1)^2 S_1\sigma_2 - (J-2S_1)\Delta_1\sigma_1\sigma_2 + (J-S_1)\Delta_1\sigma_2^2], \tag{A84}$$

$$B_{2S1} = \frac{1}{2}[S_1\sigma_1(L^2 + JS_1 + S_1^2 + \Delta_2\sigma_1) - (J+S_1)^2 S_1\sigma_2 - (J+2S_1)\Delta_1\sigma_1\sigma_2 + (J+S_1)\Delta_1\sigma_2^2], \tag{A85}$$

$$D_{1S1} = (S_1 - J)S_1 - \Delta_1\sigma_2, \tag{A86}$$

$$D_{2S1} = (S_1 + J)S_1 - \Delta_1\sigma_2. \tag{A87}$$

Analogous to matrix equations for $\vec{R}$ and $\dot{\vec{R}}$ in Eqs. (A69) and (A71), we can write $\vec{R}_1$ in component form as

$$\vec{R}_1 = \begin{pmatrix} R_1 \cos\phi_1 \\ R_1 \sin\phi_1 \\ 0 \end{pmatrix}_{n2}, \tag{A88}$$

and its derivative as (keeping in mind that $dR_1/d\lambda = 0$ along the flow under $S_{\rm eff} \cdot L$)

$$\dot{\vec{R}}_1 = \begin{pmatrix} -R_1 \sin\phi_1 (\dot{\phi}_{S1} \cos\theta_{JS_1} + \dot{\phi}_1) \\ R_1 \cos\phi_1 (\dot{\phi}_{S_1} \cos\theta_{JS_1} + \dot{\phi}_1) \\ R_1 (-\dot{\phi}_{S_1} \sin\theta_{JS_1} \cos\phi_1 + \dot{\theta}_{JS_1} \sin\phi_1) \end{pmatrix}_{n2} \tag{A89}$$

Also, along the flow under $S_{\rm eff} \cdot L$, $\vec{R}_1$ evolves as

$$\dot{\vec{R}}_1 = \sigma_1 \vec{L} \times \vec{R}_1. \tag{A90}$$

Using Eqs. (A79), (A88), and (A89) to express Eq. (A90) in component form and either the first or the second component of the equation when supplemented with Eqs. (A80) and (A83) to eliminate $\cos\theta_{JS_1}$ and $d\phi_{S_1}/d\lambda$ gives us $\dot{\phi}_1$. We again write the partial fraction form [analogous to Eq. (A74)]

$$\dot{\phi}_1 = S_1(\sigma_2 - \sigma_1) - \left(\frac{B_{1S1}}{D_{1S1} + \sigma_1 f} - \frac{B_{2S1}}{D_{2S1} + \sigma_1 f} \right). \tag{A91}$$

We have also used Eqs. (A6), (A7), and (A10) to write the cosines of κ_1, κ_2, and γ in terms of f.

Finally, in a way very similar to how $\Delta\phi$ in Eq. (A76) was found, we find the angle $\Delta\phi_1$ that ϕ_1 goes through under one period of the precession cycle when flowing under $S_{\rm eff} \cdot L$. We get

$$\Delta\phi_1 = \frac{-4}{\sqrt{A(f_3 - f_1)}} \left[\frac{B_{1S1} \Pi(\alpha_{1S1}^2, k)}{D_{1S1} + f_1 \sigma_1} - \frac{B_{2S1} \Pi(\alpha_{2S1}^2, k)}{D_{2S1} + f_1 \sigma_1} \right] + S_1(\sigma_2 - \sigma_1)\Lambda, \tag{A92}$$

where we have defined

$$\alpha_{iS1}^2 \equiv \frac{-\sigma_1(f_2 - f_1)}{D_{iS1} + f_1 \sigma_1}. \tag{A93}$$

To negate this angular offset brought about flowing under $S_{\rm eff} \cdot L$, we need to flow under S_1^2 by

$$\Delta\lambda_{S_1^2} = -\frac{\Delta\phi_1}{2S_1}, \tag{A94}$$

This fourth flow under S_1^2 is pictorially represented by the black ST curve in Fig. 4.

And finally, by performing similar calculations as above, we can see that $\Delta\lambda_{S_2^2}$ (the amount we need to flow under S_2^2) is given by the following set of equations

$$\Delta\lambda_{S_2^2} = -\frac{\Delta\phi_2}{2S_2}, \tag{A95}$$

$$\Delta\phi_2 = \frac{-4}{\sqrt{A(f_3 - f_1)}} \left[\frac{B_{1S2} \Pi(\alpha_{1S2}^2, k)}{D_{1S2} + f_1 \sigma_2} - \frac{B_{2S2} \Pi(\alpha_{2S2}^2, k)}{D_{2S2} + f_1 \sigma_2} \right] + S_2(\sigma_1 - \sigma_2)\Lambda, \tag{A96}$$

$$B_{1S2} = \frac{1}{2} [S_2 \sigma_2 (L^2 - J S_2 + S_2^2 - \Delta_1 \sigma_2) - (J - S_2)^2 S_2 \sigma_1 - (J - 2S_2)\Delta_2 \sigma_1 \sigma_2 + (J - S_2)\Delta_2 \sigma_1^2], \tag{A97}$$

$$B_{2S2} = \frac{1}{2} [S_2 \sigma_2 (-L^2 - J S_2 - S_2^2 + \Delta_1 \sigma_2) + (J + S_2)^2 S_2 \sigma_1 - (J + 2S_2)\Delta_2 \sigma_1 \sigma_2 + (J + S_2)\Delta_2 \sigma_1^2], \tag{A98}$$

$$D_{1S2} = (J - S_2) S_2 - \Delta_2 \sigma_1, \tag{A99}$$

$$D_{2S2} = -(J + S_2) S_2 - \Delta_2 \sigma_1, \tag{A100}$$

$$\alpha_{iS2}^2 \equiv \frac{-\sigma_2(f_2 - f_1)}{D_{iS2} + f_1 \sigma_2}. \tag{A101}$$

This final fifth flow under S_2^2 is pictorially represented by the orange TP curve in Fig. 4. Of course, this final set of flows under S_1^2 and S_2^2 do not disturb the already restored configurations of the other variables such as $\vec{R}, \vec{P}, \vec{S}_1$, and $\vec{S}_2$. We mention that it is not recommended to try to arrive at $\Delta\lambda_{S_2^2}$ from $\Delta\lambda_{S_1^2}$ by a mere label exchange $1 \leftrightarrow 2$ (signifying the exchange of the two black holes) because we have already introduced asymmetry in these labels when we assumed $m_1 > m_2$ in Appendix A 1.

Finally, although not required for the fifth action computation, we mention as an aside that the result of the integration of Eq. (A83) is

$$\phi_{S1}(\lambda) - \phi_{S10} = \frac{2}{\sqrt{A(f_3 - f_1)}} \left[\frac{B_{1S1} \Pi(\alpha_{1S1}^2, \phi_p, k)}{D_{1S1} + f_1 \sigma_1} + \frac{B_{2S1} \Pi(\alpha_{2S1}^2, \phi_p, k)}{D_{2S1} + f_1 \sigma_1} \right] + J\sigma_2(\lambda - \lambda_0), \tag{A102}$$

where again the integration constant ϕ_{S10} is determined by inserting $\lambda = \lambda_0$ and $\phi_{S1} = \phi_{S1}(\lambda_0)$ into this equation.

APPENDIX B: PROOF THAT $\pi_\star(\mathcal{J}_5)$ IS AN ACTION IN THE SPS

By construction, $\mathcal{J}_5$ is an action variable in the EPS (as per the loop-integral defintion), but we also need to show that its pushforward $\pi_\star(\mathcal{J}_5)$ is an action (as per the loop-flow definition) in the SPS; see Sec. III for these two definitions. The pushforward can be constructed since $\mathcal{J}_5$ is fiberwise constant. To show that $\pi_\star(\mathcal{J}_5)$ is an action, we need to show that (i) flowing under $\pi_\star(\mathcal{J}_5)$ forms a closed loop, and (ii) this flow by parameter 2π takes us around the loop exactly once.

Condition (i) can be shown to be satisfied automatically. Since the loop-integral definition of action implies the loop-flow definition, flowing under $\mathcal{J}_5$ in the EPS forms a loop. Call this loop γ (shown in solid cyan in Fig. 4). The image of this loop $\pi(\gamma)$ (shown in dashed cyan in Fig. 4) is a loop in the SPS. Meanwhile, because of the compatibility of the PBs (see Sec. III C), the pushforward of the Hamiltonian vector field $\pi_\star(\vec{X}_{\mathcal{J}_5})$ is the Hamiltonian vector field of the pushforward, $\vec{X}_{\pi_\star(\mathcal{J}_5)} = \pi_\star(\vec{X}_{\mathcal{J}_5})$. Therefore flowing under $\vec{X}_{\pi_\star(\mathcal{J}_5)}$ forms a loop, namely the image $\pi(\gamma)$.

The second part follows from homotopy equivalence. In Fig. 4, let γ_1 be the path $PQRS$ in the EPS, which is not a loop. However, its image $\pi(\gamma_1)$ (in dashed red-green-blue) is a loop in the SPS. Recall from Appendix A that we constructed γ_1 using three successive flows (under $S_{\mathrm{eff}} \cdot L, J^2$ and L^2) to bring the SPS coordinates back to their starting values, thereby making exactly one loop in the SPS. Let γ_2 be the segment STP, which is vertical in the EPS (it is contained in a single fiber); its image $\pi(STP)$ is a single point. Their composition is $\gamma_3 = \gamma_2 \cdot \gamma_1$, where $\cdot$ is composition of paths. Composing with the projection,

$$\pi(\gamma_3) = \pi(\gamma_2 \cdot \gamma_1) = \pi(\gamma_2) \cdot \pi(\gamma_1) = \pi(\gamma_1). \tag{B1}$$

Now, the Liouville-Arnold theorem is constructive, meaning that when we find the action $\mathcal{J}_5 = \oint_{\gamma_3} \mathcal{P}_i dQ^i/(2\pi)$, it generates a flow ($\gamma$) in the same homotopy class as the path we integrated over (γ_3). The two loops are homotopic, i.e., $[\gamma] = [\gamma_3]$, where the notation $[\gamma_i]$ denotes the homotopy class of a map γ_i. Since π is a continuous map, the two images are also homotopic. Therefore we also have the homotopy

$$[\pi(\gamma)] = [\pi(\gamma_3)] = [\pi(\gamma_1)]. \tag{B2}$$

Therefore we conclude that $\pi(\gamma)$ goes around exactly once, just like $\pi(\gamma_1)$, being in the same homotopy class.

APPENDIX C: FREQUENTLY OCCURRING DERIVATIVES IN FREQUENCY CALCULATIONS

Here we present some common derivatives that arise in the computation of frequencies in Eqs. (55). The most important ones are the derivatives of the roots f_i of the cubic P. These roots are implicit functions of the constants of motion, $f_i = f_i(\vec{C})$, and the coefficients of the cubic depend explicitly on the constants, $P = P(f; \vec{C})$. Since f_i is a root,

$$0 = P(f_i(\vec{C}); \vec{C}), \tag{C1}$$

and this identity is satisfied smoothly in $\vec{C}$, therefore

$$0 = \frac{\partial}{\partial C_j}\left[P(f_i(\vec{C}); \vec{C})\right], \tag{C2}$$

$$0 = P'(f_i)\frac{\partial f_i}{\partial C_j} + \left.\frac{\partial P}{\partial C_j}\right|_{f=f_i}, \tag{C3}$$

where we have expanded with the chain rule. We can now easily solve for the derivative of a root with respect to a constant of motion,

$$\frac{\partial f_i}{\partial C_j} = -\frac{1}{P'(f_i)}\left.\frac{\partial P}{\partial C_j}\right|_{f=f_i}. \tag{C4}$$

Here $P'(f) = \partial P/\partial f$ is the quadratic

$$P'(f) = 3a_3 f^2 + 2a_2 f + a_1, \tag{C5}$$

where the coefficients are given in Eq. (A23). The denominator $P'(f_i)$ only vanishes if f_i is a multiple root, which only happens if there is no precession. Notice that all the polynomials $\partial P/\partial C_j$ are also quadratics, since the leading coefficient a_3 in Eq. (A23a) does not depend on any constants of motion. We present these explicitly below.

Taking the derivative of $\mathcal{J}_5$ in Eq. (35) requires applying the product rule and chain rule many times. We need the derivatives of the $\Delta\lambda$'s from Eqs. (A42), (A67), (A77), (A94), and (A95), which involve the quantities $f_i, B_i, D_i, B_{iS}, D_{iS}$ and various elliptic integrals. Derivatives of f_i's have already been discussed above and those of B_i, D_i, B_{iS}, D_{iS} are not too hard to compute. Derivatives of the elliptic integrals via the application of the chain rule can be written in terms of derivatives of their arguments: α_i, α_{iS} and k. Derivatives of the first two can be written in terms of the derivatives of f_i, D_i and D_{iS}, whereas the derivative of k [Eq. (A34)] simplifies to

$$\frac{dk}{dC_i} = \frac{-(f_2-f_3)^2 \left.\frac{\partial P}{\partial C_i}\right|_{f=f_1} + (1\to 2\to 3) + (1\to 3\to 2)}{2kA(f_1-f_3)^2(f_1-f_2)(f_1-f_3)(f_2-f_3)}. \tag{C6}$$

The $\partial P/\partial C_i$ polynomials occur in both Eqs. (C4) and (C6). use of the expression of P as given in Eq. (A25) is to be made to compute it. The nonzero $\partial P/\partial C_i$'s are the quadratic polynomials

$$\frac{\partial P}{\partial L} = 2L\Big[-(\sigma_1^2+\sigma_2^2)f^2 + (2\delta\sigma\sigma_1\sigma_2\mathcal{G} - \delta\sigma^{-1}[(\sigma_1+\sigma_2)S_{\rm eff}\cdot L + S_1^2\sigma_1^2 + S_2^2\sigma_2^2])f + \mathcal{G}(S_1^2\sigma_1^2+S_2^2\sigma_2^2) - \delta\sigma^{-2}(S_1^2\sigma_1+S_2^2\sigma_2)S_{\rm eff}\cdot L + S_1^2S_2^2\Big], \tag{C7}$$

$$\frac{\partial P}{\partial J} = 2J\Big[2\sigma_1\sigma_2 f^2 - (2\delta\sigma\sigma_1\sigma_2\mathcal{G} - \delta\sigma^{-1}[(\sigma_1+\sigma_2)S_{\rm eff}\cdot L + S_1^2\sigma_1^2 + S_2^2\sigma_2^2])f - \mathcal{G}(S_1^2\sigma_1^2+S_2^2\sigma_2^2) + \delta\sigma^{-2}(S_1^2\sigma_1+S_2^2\sigma_2)S_{\rm eff}\cdot L\Big], \tag{C8}$$

$$\frac{\partial P}{\partial S_{\rm eff}\cdot L} = 2\Big[-(\sigma_1+\sigma_2)f^2 + [\delta\sigma(\sigma_1+\sigma_2)\mathcal{G} - \delta\sigma^{-1}(2S_{\rm eff}\cdot L + S_1^2\sigma_1 + S_2^2\sigma_2)]f + \mathcal{G}(S_1^2\sigma_1+S_2^2\sigma_2) - \delta\sigma^{-2}(S_1^2+S_2^2)S_{\rm eff}\cdot L\Big], \tag{C9}$$

where we have used the shorthands

$$\delta\sigma = \sigma_1 - \sigma_2, \tag{C10}$$

$$\mathcal{G} = \frac{J^2 - L^2 - S_1^2 - S_2^2}{2\delta\sigma^2}. \tag{C11}$$

The last piece are the derivatives of the complete elliptic integrals of the first and third kinds with respect to their arguments. By differentiating their integral definitions, the derivatives are expressible again as elliptic integrals [39],

$$\frac{\mathrm{d}}{\mathrm{d}k}K(k) = \frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k}, \tag{C12}$$

$$\frac{\partial\Pi(n,k)}{\partial n} = \frac{1}{2(k^2-n)(n-1)}\left(E(k) + \frac{1}{n}(k^2-n)K(k) + \frac{1}{n}(n^2-k^2)\Pi(n,k)\right), \tag{C13}$$

$$\frac{\partial\Pi(n,k)}{\partial k} = \frac{k}{n-k^2}\left(\frac{E(k)}{k^2-1} + \Pi(n,k)\right). \tag{C14}$$

APPENDIX D: REFINING THE DEFINITION OF PN INTEGRABILITY IN REF. [16]

The definition of PN integrability was first provided in Sec. IV-A of Ref. [16] which was later refined in Sec. IV-D, for it had some shortcomings. According to the refined definition, we have qPN perturbative integrability in a $2n$-dimensional phase space when we have n independent phase-space functions (including the $(q+1/2)$PN Hamiltonian) which are in mutual involution up to at least qPN order. One shortcoming in regard to even this refined definition has come to our notice which we attempt to point out and fix in this appendix.

In Ref. [16], we noted that $\widetilde{L^2} + c_1S_1^2h/c^2 + c_2S_2^2h/c^2$ was in mutual perturbative involution with all the other phase space constants, for arbitrary real coefficients c_1, c_2, where $\tilde{L^2}$ was given by Eqs. (50) and (53), and h was defined in Eq. (52) of Ref. [16]. Similarly, for any real coefficients c_3, c_4, c_5, the combination $\widetilde{S_{\rm eff}\cdot L} + c_3S_1^2h/c^2 + c_4S_1^2h/c^2 + c_5\vec{S}_1\cdot\vec{S}_2/c^2$ was in mutual perturbative involution with the other phase space constants. It is important to note that the free terms with coefficients c_i are at the same PN order that we are keeping, and that they are not simply built out of other constants of motion. With our previous definition of PN integrability, this seems to suggest far more than n independent functions in mutual perturbative involution. This is in stark contrast with exact integrability scenario where one cannot have more than n independent functions in mutual involution on a $2n$ dimensional phase space. Clearly, something is wrong.

Another way to look at this problem is to realize that for 2PN integrability, if we enumerate the required $n=5$ commuting constants by including the 2.5PN Hamiltonian, J^2, J_z, $\tilde{L^2}$ and $\widetilde{L^2} + c_1S_1^2h/c^2 + c_2S_2^2h/c^2$, the latter two quantities will coincide in the PN limit $1/c\to 0$, thereby leaving us with only four independent quantities in exact mutual involution, whereas the requisite number is 5 (both for PN perturbative and exact integrability). This means that the $1/c\to 0$ limit of the requisite number n of quantities in PN mutual involution (required for PN integrability) may not be enough for exact integrability (in the $1/c\to 0$ limit), which is bizarre. The definition of PN integrability clearly needs a fix.

To fix the definition, we add one more demand: the n independent phase-space functions (including the $(q+1/2)$PN Hamiltonian) must be such that in the limit $1/c\to 0$, they reduce to n independent phase-space functions in exact mutual involution. As per this new we cannot count $\widetilde{L^2}$ and $\widetilde{L^2} + c_1S_1^2h/c^2 + c_2S_2^2h/c^2$

simultaneously in our list of independent functions in mutual involution. This remedies the aforementioned problems with the definition of PN integrability. It's easy to see that the BBH system is still 2PN integrable per this revised definition of PN integrability since $\widetilde{L^2}$ and $\widetilde{S_{\mathrm{eff}} \cdot L}$ reduce to L^2 and $S_{\mathrm{eff}} \cdot L$ in the $1/c \to 0$ limit, which exactly mutually commute and are independent of each other.

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[1] B. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GWTC-1: A Gravitational-Wave Transient Catalog of Compact Binary Mergers Observed by LIGO and Virgo during the First and Second Observing Runs, Phys. Rev. X **9**, 031040 (2019).

[2] R. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GWTC-2: Compact Binary Coalescences Observed by LIGO and Virgo During the First Half of the Third Observing Run, Phys. Rev. X **11**, 021053 (2021).

[3] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral, Phys. Rev. Lett. **119**, 161101 (2017).

[4] C. Cutler, Angular resolution of the LISA gravitational wave detector, Phys. Rev. D **57**, 7089 (1998).

[5] P. Amaro-Seoane *et al.* (LISA Collaboration), Laser interferometer space antenna, arXiv:1702.00786.

[6] S. Tiwari, A. Gopakumar, M. Haney, and P. Hemantakumar, Ready-to-use Fourier domain templates for compact binaries inspiraling along moderately eccentric orbits, Phys. Rev. D **99**, 124008 (2019).

[7] M. Kesden, D. Gerosa, R. O'Shaughnessy, E. Berti, and U. Sperhake, Effective Potentials and Morphological Transitions for Binary Black Hole Spin Precession, Phys. Rev. Lett. **114**, 081103 (2015).

[8] D. Gerosa, M. Kesden, U. Sperhake, E. Berti, and R. O'Shaughnessy, Multi-timescale analysis of phase transitions in precessing black-hole binaries, Phys. Rev. D **92**, 064016 (2015).

[9] I. Hinder, L. E. Kidder, and H. P. Pfeiffer, Eccentric binary black hole inspiral-merger-ringdown gravitational waveform model from numerical relativity and post-Newtonian theory, Phys. Rev. D **98**, 044015 (2018).

[10] K. Chatziioannou, A. Klein, N. Yunes, and N. Cornish, Constructing gravitational waves from generic spin-precessing compact binary inspirals, Phys. Rev. D **95**, 104004 (2017).

[11] A. Klein and P. Jetzer, Spin effects in the phasing of gravitational waves from binaries on eccentric orbits, Phys. Rev. D **81**, 124001 (2010).

[12] A. Klein, Y. Boetzel, A. Gopakumar, P. Jetzer, and L. de Vittori, Fourier domain gravitational waveforms for precessing eccentric binaries, Phys. Rev. D **98**, 104043 (2018).

[13] C. Konigsdorffer and A. Gopakumar, Phasing of gravitational waves from inspiralling eccentric binaries at the third-and-a-half post-Newtonian order, Phys. Rev. D **73**, 124012 (2006).

[14] C. Konigsdorffer and A. Gopakumar, Post-Newtonian accurate parametric solution to the dynamics of spinning compact binaries in eccentric orbits: The Leading order spin-orbit interaction, Phys. Rev. D **71**, 024039 (2005).

[15] G. Cho and H. M. Lee, Analytic Keplerian-type parametrization for general spinning compact binaries with leading order spin-orbit interactions, Phys. Rev. D **100**, 044046 (2019).

[16] S. Tanay, L. C. Stein, and J. T. Gálvez Ghersi, Integrability of eccentric, spinning black hole binaries up to second post-Newtonian order, Phys. Rev. D **103**, 064066 (2021).

[17] H. Goldstein, C. Poole, and J. Safko, *Classical Mechanics* (Pearson, San Francisco, 2013).

[18] J. José and E. Saletan, *Classical Dynamics: A Contemporary Approach* (Cambridge University Press, Cambridge, England, 1998).

[19] T. Damour and N. Deruelle, General relativistic celestial mechanics of binary systems. I. The post-Newtonian motion., Ann. Inst. Henri Poincaré Phys. Théor. **43**, 107 (1985).

[20] T. Damour and G. Schafer, Higher-order relativistic periastron advances and binary pulsars., Nuovo Cimento Soc. Ital. Fis. **101B**, 127 (1988).

[21] T. Damour, Coalescence of two spinning black holes: An effective one-body approach, Phys. Rev. D **64**, 124013 (2001).

[22] R. Samanta, S. Tanay, and L. C. Stein, Closed-form solutions of spinning, eccentric binary black holes at 1.5 post-Newtonian order, arXiv:2210.01605.

[23] S. Tanay, Integrability and action-angle-based solution of the post-Newtonian BBH system (lecture notes), arXiv: 2206.05799.

[24] A. Fasano, S. Marmi, and B. Pelloni, *Analytical Mechanics: An Introduction*, Oxford Graduate Texts (Oxford University Press, Oxford, 2006).

[25] V. Arnold, K. Vogtmann, and A. Weinstein, *Mathematical Methods of Classical Mechanics*, Graduate Texts in Mathematics (Springer, New York, 2013).

[26] R. Abraham and J. Marsden, *Foundations of Mechanics*, AMS Chelsea Publishing (AMS Chelsea Pub./American Mathematical Society, Providence, 2008).

[27] A. da Silva, *Lectures on Symplectic Geometry*, Lecture Notes in Mathematics No. 1764 (Springer, New York, 2001).

[28] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry*, 2nd ed., Texts in Applied Mathematics Vol. 17 (Springer-Verlag, New York, 1999), pp. xviii +582, a basic exposition of classical mechanical systems.

[29] D. Gerosa, U. Sperhake, and J. Vošmera, On the equal-mass limit of precessing black-hole binaries, Classical Quantum Gravity **34**, 064004 (2017).

[30] L. Landau, E. Lifshitz, J. Sykes, and J. Bell, *Mechanics: Volume 1*, Course of Theoretical Physics (Elsevier Science, New York, 1976).

[31] W. Schmidt, Celestial mechanics in Kerr space-time, Classical Quantum Gravity **19**, 2743 (2002).

[32] A. Gopakumar and G. Schafer, Gravitational wave phasing for spinning compact binaries in inspiraling eccentric orbits, Phys. Rev. D **84**, 124007 (2011).

[33] G. Cho, S. Tanay, A. Gopakumar, and H. M. Lee, Generalized quasi-Keplerian solution for eccentric, nonspinning compact binaries at 4PN order and the associated IMR waveform, Phys. Rev. D **105**, 064010 (2022).

[34] G. Cho and H. M. Lee, Analytic Keplerian-type parametrization for general spinning compact binaries with leading order spin-orbit interactions, Phys. Rev. D **100**, 044046 (2019).

[35] A. Buonanno and T. Damour, Effective one-body approach to general relativistic two-body dynamics, Phys. Rev. D **59**, 084006 (1999).

[36] V. Witzany, Hamilton-Jacobi equation for spinning particles near black holes, Phys. Rev. D **100**, 104030 (2019).

[37] L. V. Drummond and S. A. Hughes, Precisely computing bound orbits of spinning bodies around black holes. I. General framework and results for nearly equatorial orbits, Phys. Rev. D **105**, 124040 (2022).

[38] L. V. Drummond and S. A. Hughes, Precisely computing bound orbits of spinning bodies around black holes. II. Generic orbits, Phys. Rev. D **105**, 124041 (2022).

[39] DLMF, *NIST Digital Library of Mathematical Functions*, http://dlmf.nist.gov/, Release 1.1.3 of 2021-09-15, edited by f. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain.

[40] P. Byrd and M. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, Grundlehren der Mathematischen Wissenschaften (Springer, Berlin, Heidelberg, 2012).

[41] E. Whittaker and G. Watson, *A Course of Modern Analysis*, A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions, with an Account of the Principal Transcendental Functions (Cambridge University Press, Cambridge, England, 1996).

[42] D. Lawden, *Elliptic Functions and Applications*, Applied Mathematical Sciences (Springer, New York, 2013).