arXiv:2210.16876v2 [math.GR] 15 May 2023

ON SOME GENERATING SET OF THOMPSON'S GROUP F

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ABSTRACT. We prove that Thompson's group F has a generating set with two elements such that every two powers of them generate a finite index subgroup of F.

1. Introduction

Recall that Thompson's group F is the group of all piecewise linear homeomorphisms of the interval $[0, 1]$ where all breakpoints are dyadic fractions and all slopes are integer powers of 2.

Thompson's group F has many interesting properties. It is infinite and finitely presented, it does not have any free subgroups and it does not satisfy any law [1]. In 1984, Brown and Geogheghan [3] proved that Thompson's group F is of type FP_∞, making Thompson's group F the first example of a torsion-free infinite-dimensional FP_∞ group.

One of the most interesting and counter-intuitive results about Thompson's group F is that in a certain natural probabilistic model on the set of all finitely generated subgroups of F, every finitely generated nontrivial subgroup appears with positive probability [5]. In [9], the first author proved that in the natural probabilistic models studied in [5], a random pair of elements of F generates F with positive probability. In fact, one can prove that for every finite index subgroup H of F, a random pair of elements of F generates H with positive probability. This result shows that in some sense it is "easy" to generate F, or more generally, finite index subgroups of F. Several other results in the literature can be interpreted in a similar way. In [11], the first author proved that every element of F whose image in the abelianization $\mathbb{Z}^2$ is part of a generating pair of $\mathbb{Z}^2$ is part of a generating pair of F (and that a similar statement holds for all finitely generated subgroups of F).

Another result that demonstrates the abundance of generating pairs of F is Brin's result [2] that the free group of rank 2 is a limit of 2-markings of Thompson's group F in the space of all 2-marked groups. Lodha's new (and much shorter) proof [13] of Brin's theorem demonstrates even better the abundance of generating pairs of F.

In [6], Gelander, Juschenko and the first author proved that Thompson's group F is invariably generated. Recall that a subset S of a group G *invariably generates* G if $G = \langle s^{g(s)} | s \in S\rangle$ for every choice of $g(s) \in G, s \in S$. A group G is said to be *invariably generated* if such S exists, or equivalently if $S = G$ invariably generates G. Note that all virtually solvable groups are invariably generated, but Thompson's group F was one of the

The research of the first author was supported by ISF grant 2322/19. The research of the second author was supported by NSF grant DMS-1901976.

first examples of a non-virtually solvable group that is invariably generated. Note also that in [6] it is proved that Thompson's group F is invariably generated by a set of 3 elements. Using [10, Theorem 1.3], the proof from [6] implies that in fact, Thompson's group F is invariably generated by a set of 2 elements (see also Lemma 14 below).

In this paper, we prove a somewhat similar result.

Theorem 1. *Thompson group F has a 2-generating set $\{x, y\}$ such that for every $m, n \in \mathbb{N}$, the set $\{x^m, y^n\}$ generates a finite index subgroup of F.*

We will show that the generating set $\{x, y\}$ constructed in the proof of Theorem 1 below also invariably generates F. Note also that since the abelianization of Thompson's group F is $\mathbb{Z}^2$, we couldn't request the elements x^m and y^n from the theorem to generate the entire group F.

Theorem 1 does not hold for any non-elementary hyperbolic group. Indeed, if G is non-elementary hyperbolic, then there exists $n \in \mathbb{N}$ such that G/G^n is infinite, where G^n is the normal subgroup generated by all n^{th} powers of elements in G [12]. More generally, Theorem 1 does not hold for any group G which has an infinite periodic quotient (such as large groups (see [15]) and Golod Shafarevich-groups (see [16])).

Theorem 1 does hold for the Tarski monsters constructed by Ol'shanskii [14]. Recall that Tarski monsters are infinite finitely generated simple groups where every proper subgroup is infinite cyclic[1]. Let T be the Tarski monster constructed in [14], then 2 elements of T generate it if and only if they do not commute. Since powers of non-commuting elements in T do not commute (see [14, Theorem 28.3]), any generating pair of T satisfies the assertion in Theorem 1 (in fact, for every pair of generators of T, any pair of powers of the generators generates the entire group T). It is easy to see that there are virtually-abelian groups (such as $\mathbb{Z}^2$ and $\mathbb{Z} \wr \mathbb{Z}_2$) for which Theorem 1 holds. But to our knowledge, Thompson's group F is the first example of a finitely presented non virtually-abelian group which satisfies the assertion in Theorem 1.

2. Thompson's group F

2.1. **F as a group of homeomorphisms.** Recall that Thompson group F is the group of all piecewise linear homeomorphisms of the interval $[0, 1]$ with finitely many breakpoints where all breakpoints are dyadic fractions and all slopes are integer powers of 2. The group F is generated by two functions x_0 and x_1 defined as follows [4].

$$x_0(t) = \begin{cases} 2t & \text{if } 0 \le t \le \frac{1}{4} \\ t + \frac{1}{4} & \text{if } \frac{1}{4} \le t \le \frac{1}{2} \\ \frac{t}{2} + \frac{1}{2} & \text{if } \frac{1}{2} \le t \le 1 \end{cases} \qquad x_1(t) = \begin{cases} t & \text{if } 0 \le t \le \frac{1}{2} \\ 2t - \frac{1}{2} & \text{if } \frac{1}{2} \le t \le \frac{5}{8} \\ t + \frac{1}{8} & \text{if } \frac{5}{8} \le t \le \frac{3}{4} \\ \frac{t}{2} + \frac{1}{2} & \text{if } \frac{3}{4} \le t \le 1 \end{cases}$$

[1] There is another type of Tarski monsters, where every proper subgroup is cyclic of order p for some fixed prime p, but for them Theorem 1 clearly does not hold.

The composition in F is from left to right.

Every element of F is completely determined by how it acts on the set $\mathbb{Z}[\frac{1}{2}]$. Every number in $(0,1)$ can be described as $.s$ where s is an infinite word in $\{0,1\}$. For each element $g \in F$ there exists a finite collection of pairs of (finite) words (u_i, v_i) in the alphabet $\{0,1\}$ such that every infinite word in $\{0,1\}$ starts with exactly one of the u_i's and such that the action of g on a number $.s$ is the following: if s starts with u_i, we replace u_i by v_i. For example, x_0 and x_1 are the following functions:

$$x_0(t) = \begin{cases} .0\alpha & \text{if } t = .00\alpha \\ .10\alpha & \text{if } t = .01\alpha \\ .11\alpha & \text{if } t = .1\alpha \end{cases} \qquad x_1(t) = \begin{cases} .0\alpha & \text{if } t = .0\alpha \\ .10\alpha & \text{if } t = .100\alpha \\ .110\alpha & \text{if } t = .101\alpha \\ .111\alpha & \text{if } t = .11\alpha \end{cases}$$

where α is any infinite binary word.

The group F has the following finite presentation [4].

$$F = \langle x_0, x_1 \mid [x_0x_1^{-1}, x_1^{x_0}] = 1, [x_0x_1^{-1}, x_1^{x_0^2}] = 1\rangle,$$

where a^b denotes $b^{-1}ab$. Sometimes, it is more convenient to consider an infinite presentation of F. For $i \geq 1$, let $x_{i+1} = x_0^{-i}x_1x_0^i$. In these generators, the group F has the following presentation [4]

$$\langle x_i, i \geq 0 \mid x_i^{x_j} = x_{i+1} \text{ for every } j < i\rangle.$$

2.2. **Elements of F as pairs of binary trees.** Often, it is more convenient to describe elements of F using pairs of finite binary trees (see [4] for a detailed exposition). The considered binary trees are rooted *full* binary trees; that is, each vertex is either a leaf or has two outgoing edges: a left edge and a right edge. A *branch* in a binary tree is a simple path from the root to a leaf. If every left edge in the tree is labeled "0" and every right edge is labeled "1", then a branch in T has a natural binary label. We rarely distinguish between a branch and its label.

Let (T_+, T_-) be a pair of finite binary trees with the same number of leaves. The pair (T_+, T_-) is called a *tree-diagram*. Let $u_1, \dots, u_n$ be the (labels of) branches in T_+, listed from left to right. Let $v_1, \dots, v_n$ be the (labels of) branches in T_-, listed from left to right. For each $i = 1, \dots, n$, we say that the tree-diagram (T_+, T_-) has the *pair of branches* $u_i \to v_i$. We also say that the tree-diagram (T_+, T_-) *consists* of all the pairs of branches $u_1 \to v_1, \dots, u_n \to v_n$. The tree-diagram (T_+, T_-) *represents* the function $g \in F$ which takes binary fraction $.u_i\alpha$ to $.v_i\alpha$ for every i and every infinite binary word α. We also say that the element g takes the branch u_i to the branch v_i. For a finite binary word u, we denote by $[u]$ the dyadic interval $[.u, .u1^{\mathbb{N}}]$. If $u \to v$ is a pair of branches of (T_+, T_-), then g maps the interval $[u]$ linearly onto $[v]$.

A *caret* is a binary tree composed of a root with two children. If (T_+, T_-) is a tree-diagram and one attaches a caret to the i^{th} leaf of T_+ and the i^{th} leaf of T_- then the resulting tree diagram is *equivalent* to (T_+, T_-) and represents the same function in F.

The opposite operation is that of *reducing* common carets. A tree diagram (T_+, T_-) is called *reduced* if it has no common carets; i.e, if there is no i for which the i and $i+1$ leaves of both T_+ and T_- have a common father. Every tree-diagram is equivalent to a unique reduced tree-diagram. Thus elements of F can be represented uniquely by reduced tree-diagrams [4]. The reduced tree-diagrams of the generators x_0 and x_1 of F are depicted in Figure 1.

FIGURE 1. (A) The reduced tree-diagram of x_0. (B) The reduced tree-diagram of x_1. In both figures, T_+ is on the left and T_- is on the right.

When we say that a function $f \in F$ has a pair of branches $u_i \to v_i$, the meaning is that some tree-diagram representing f has this pair of branches. In other words, this is equivalent to saying that f maps the dyadic interval $[u_i]$ linearly onto $[v_i]$. Clearly, if $u \to v$ is a pair of branches of f, then for any finite binary word w, $uw \to vw$ is also a pair of branches of f. Similarly, if f has the pair of branches $u \to v$ and g has the pair of branches $v \to w$ then fg has the pair of branches $u \to w$.

2.3. **The derived subgroup of F.** The derived subgroup of F is an infinite simple group [4]. It can be characterized as the subgroup of F of all functions f with slope 1 both at 0^+ and at 1^- (see [4]). That is, a function $f \in F$ belongs to $[F, F]$ if and only if the reduced tree-diagram of f has pairs of branches of the form $0^m \to 0^m$ and $1^n \to 1^n$ for some $m, n \in \mathbb{N}$.

Since $[F, F]$ is infinite and simple, every finite index subgroup of F contains the derived subgroup of F. Hence, there is a one-to-one correspondence between finite index subgroups of F and finite index subgroups of the abelianization $F/[F, F]$.

Recall that the abelianization of F is isomorphic to $\mathbb{Z}^2$ and that the standard abelianization map $\pi_{ab} \colon F \to \mathbb{Z}^2$ maps an element $f \in F$ to $(\log_2(f'(0^+)), \log_2(f'(1^-)))$. Hence, a subgroup H of F has finite index in F if and only if H contains the derived subgroup of F and $\pi_{ab}(H)$ has finite index in $\mathbb{Z}^2$.

2.4. **Generating sets of F.** Let H be a subgroup of F. A function $f \in F$ is said to be a *piecewise-H* function if there is a finite subdivision of the interval $[0, 1]$ such that on each interval in the subdivision, f coincides with some function in H. Note that since all breakpoints of elements in F are dyadic fractions, a function $f \in F$ is a piecewise-H function if and only if there is a dyadic subdivision of the interval $[0, 1]$ into finitely many pieces such that on each dyadic interval in the subdivision, f coincides with some function in H.

Following [7, 8], we define the *closure* of a subgroup H of F, denoted $\mathrm{Cl}(H)$, to be the subgroup of F of all piecewise-H functions. A subgroup H of F is *closed* if $H = \mathrm{Cl}(H)$. In [8] (see also [10]), the first author proved that the generation problem in F is decidable. That is, there is an algorithm that decides given a finite subset X of F whether it generates the whole F.

Theorem 2. [10, Theorem 1.3] *Let H be a subgroup of F. Then $H = F$ if and only if the following conditions hold.*

1. $\mathrm{Cl}(H)$ *contains the derived subgroup of F.*
2. $H[F, F] = F$.

More generally, we have a criterion for when a subgroup H of F contains the derived subgroup of F.

Theorem 3. [8, Theorem 7.10] *Let H be a subgroup of F. Then H contains the derived subgroup $[F, F]$ if and only if the following conditions hold.*

1. $\mathrm{Cl}(H)$ *contains the derived subgroup $[F, F]$.*
2. *There is an element $h \in H$ and a dyadic fraction $\alpha \in (0, 1)$ such that h fixes α, $h'(\alpha^-) = 1$ and $h'(\alpha^+) = 2$.*

Below we apply Theorem 3 to prove that a given subset of F generates a finite index subgroup of F (by proving that it contains the derived subgroup of F and considering its image in the abelianization of F). The following two lemmas will be useful in proving that Condition (1) of Theorem 3 holds for a subgroup H of F.

Lemma 4. *Let H be a subgroup of F. Assume that for every pair of finite binary words u and v which both contain both digits "0" and "1" there is an element $h \in H$ with the pair of branches $u \to v$. Then $\mathrm{Cl}(H)$ contains the derived subgroup of F.*

Proof. Let $f \in [F, F]$. Then the reduced tree-diagram of f consists of the pairs of branches

$$f : \begin{cases} 0^m & \to 0^m \\ u_i & \to v_i \text{ for } i = 1, \ldots, k \\ 1^n & \to 1^n \end{cases}$$

where $k, m, n \in \mathbb{N}$ and where for each $i = 1, \ldots, k$, the binary words u_i and v_i contain both digits "0" and "1". By assumption, for each $i = 1, \ldots, k$ there is an element $h_i \in H$ with the pair of branches $u_i \to v_i$. Then h_i coincides with f on the interval $[u_i]$. we note also that f coincides with the identity function $\mathbf{1} \in H$ on $[0^m]$ and on $[1^n]$. Since $[0^m], [u_1], \ldots, [u_k], [1^n]$ is a subdivision of the interval $[0, 1]$ and on each of these intervals f coincides with a function in H, f is a piecewise-H function and as such $f \in \mathrm{Cl}(H)$. □

Given a subgroup $H \leq F$ we associate with H an equivalence relation on the set of finite binary words as follows. Let u and v be finite binary words. We write $u \sim_H v$ if there is an element $h \in H$ with the pair of branches $u \to v$. Note that $\sim_H$ is indeed an equivalence relation on the set of finite binary words. (Indeed, for every finite binary word u the identity function has the pair of branches $u \to u$; if $h \in H$ has the pair of branches $u \to v$ then h^{-1} has the pair of branches $v \to u$ and if $h, g \in H$ have the pairs of branches $u \to v$ and $v \to w$, respectively, then hg has the pair of branches $u \to w$). We note also that if $u \sim_H v$ then for any finite binary word w we have $uw \sim_H vw$. Indeed, if $h \in H$ has the pair of branches $u \to v$ then for each w (some non-reduced tree-diagram of) h has the pair of branches $uw \to vw$. By Lemma 4, to prove that $\mathrm{Cl}(H)$ contains the derived subgroup of F, it suffices to prove that all finite binary words which contain both digits "0" and "1" are $\sim_H$-equivalent.

Lemma 5. *Let H be a subgroup of F such that the following assertions hold.*

(1) *For every $r \in \mathbb{N}$, we have $1^r0 \sim_H 10$.*
(2) *For every $s \in \mathbb{N}$, we have $0^s1 \sim_H 01$.*
(3) $01 \sim_H 10 \sim_H 010 \sim_H 011$.

Then $\mathrm{Cl}(H)$ contains the derived subgroup of F.

Proof. First, note that since $10 \sim_H 01$, we have $100 \sim_H 010$ and $101 \sim_H 011$. Then (3) implies that

$$100 \sim_H 010 \sim_H 01 \sim_H 011 \sim_H 101. \tag{4}$$

Now, let u be a finite binary word which contains both digits "0" and "1". It suffices to prove that $u \sim_H 01$ (indeed, in that case, all finite binary words which contain both digits "0" and "1" are $\sim_H$-equivalent). If u is of length 2, this is true, since $10 \sim_H 01$. If u is of length ≥ 3, then it must have a prefix of the form 1^r0 (for some $r \geq 2$), 0^s1 (for some $s \geq 2$), 010, 011, 100 or 101. In all of these cases, u is $\sim_H$-equivalent to a shorter word (since it has a prefix that is $\sim_H$-equivalent to a shorter word by (1)-(4) above). Hence, we are done by induction. □

3. Proof of Theorem 1

For the rest of this section, let $x = x_0$ and $y = x_0^2 x_1$ (the element x appears in Figure 1a and the element y appears in Figure 2). Since $\{x_0, x_1\}$ is a generating set of F, the set $\{x, y\}$ is a generating set of F. We will prove that for every $m, n \in \mathbb{N}$ the set $\{x^m, y^n\}$ generates a finite index subgroup of F and that $\{x, y\}$ invariably generates F.

We begin with the following lemma.

Lemma 6. *Let $n \in \mathbb{N}$. Then the reduced tree diagrams of x^n and y^n consist of the following pairs of branches (that is, we list all the pairs of branches of x^n and y^n).*

$$x^n : \begin{cases} 0^{n+1} & \to 0 \\ 0^k1 & \to 1^{n+1-k}0, \ \text{for } 1 \leq k \leq n \\ 1 & \to 1^{n+1} \end{cases}$$

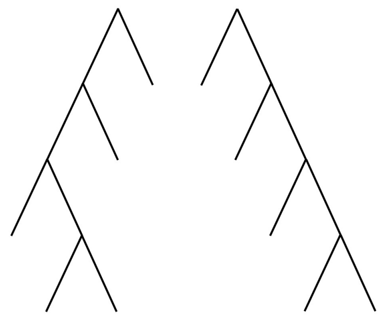

FIGURE 2. The reduced tree-diagram of y.

$$y^n : \begin{cases} 0^{2n+1} & \to 0 \\ 0^{2k}10 & \to 1^{1+3(n-k)}0, \ \textit{for } 1 \leq k \leq n \\ 0^{2k}11 & \to 1^{2+3(n-k)}0, \ \textit{for } 1 \leq k \leq n \\ 0^{2k-1}1 & \to 1^{3(n-k+1)}0, \ \textit{for } 1 \leq k \leq n \\ 1 & \to 1^{3n+1} \end{cases}$$

Proof. The lemma can be proved by induction. Note that for $n = 1$ the lemma follows from Figure 1a and from Figure 2. □

Now, for every $n \in \mathbb{N}$, we denote by H_n the subgroup of F generated by $\{x^n, y^n\}$. We claim that H_n contains the derived subgroup of F. To prove that, we will prove that H_n satisfies Conditions (1) and (2) from Theorem 3. First, we consider Condition (2).

Lemma 7. *Let $n \in \mathbb{N}$. Then there is an element $h \in H_n$ such that h fixes a dyadic fraction $\alpha \in (0,1)$ and such that $h'(\alpha^-) = 1$ and $h'(\alpha^+) = 2$.*

Proof. From the infinite presentation of F given above it follows that

$$y^n = (x_0^2 x_1)^n = x_0^{2n} x_1 x_4 x_7 \cdots x_{1+3(n-1)}.$$

Since $x^{2n} = x_0^{2n} \in H_n$ we have that

$$h = x_1 x_4 x_7 \cdots x_{1+3(n-1)} \in H_n.$$

Note that for $\alpha = \frac{1}{2}$ the function x_1 fixes $[0, \alpha]$ pointwise and satisfies $x_1'(\alpha^+) = 2$. For all $i > 1$, the function x_i fixes $[0, \frac{3}{4}]$ pointwise, hence for $\alpha = \frac{1}{2}$ we have $h(\alpha) = \alpha$, $h'(\alpha^-) = 1$ and $h'(\alpha^+) = 2$. □

To prove that Condition (1) from Theorem 3 holds for H_n, we let K_n be the minimal closed subgroup of F such that the following hold modulo $\sim_{K_n}$.

$$
\begin{array}{lll}
(a) & 0^k1\sim_{K_n}0^{k+n}1, & \text{for all } k\in\mathbb{N}\\
(b) & 1^k0\sim_{K_n}1^{k+n}0, & \text{for all } k\in\mathbb{N}\\
(c) & 0^k1\sim_{K_n}1^{n+1-k}0, & \text{for } 1\leq k\leq n\\
(d) & 0^{2k}10\sim_{K_n}1^{1+3(n-k)}0, & \text{for } 1\leq k\leq n\\
(e) & 0^{2k}11\sim_{K_n}1^{2+3(n-k)}0, & \text{for } 1\leq k\leq n\\
(f) & 0^{2k-1}1\sim_{K_n}1^{3(n-k+1)}0, & \text{for } 1\leq k\leq n.
\end{array}
$$

Note that the intersection of closed subgroups of F is a closed subgroup (see [8]) and that modulo $\sim_F$ relations $(a)-(f)$ hold. Hence, K_n is well defined.

Lemma 8. *Let $n\in\mathbb{N}$. Then $K_n\subseteq \mathrm{Cl}(H_n)$.*

Proof. It suffices to prove that equivalences $(a)-(f)$ hold when K_n is replaced by H_n. Indeed, in that case, the equivalences must also hold modulo $\sim_{\mathrm{Cl}(H_n)}$ and then the minimality of K_n implies that it is a subgroup of $\mathrm{Cl}(H_n)$.

Let us consider the relation $\sim_{H_n}$. Equivalences $(d),(e),(f)$ are true modulo $\sim_{H_n}$ since $y^n\in H_n$. Similarly, (c) holds modulo $\sim_{H_n}$ since $x^n\in H_n$. The branch $0^{n+1}\to 0$ of x^n implies that for all $k\in\mathbb{N}$, $0^k\sim_{H_n}0^{k+n}$. In particular, for all $k\in\mathbb{N}$, we have $0^k1\sim_{H_n}0^{k+n}1$, so (a) also holds modulo $\sim_{H_n}$. Finally, the branch $1\to 1^{n+1}$ of x^n implies that for all $k\in\mathbb{N}$, $1^k\sim_{H_n}1^{k+n}$. Hence, for all $k\in\mathbb{N}$, we have $1^k0\sim_{H_n}1^{k+n}0$, so (b) also holds modulo $\sim_{H_n}$. □

By Lemma 8, to prove that $[F,F]$ is contained in the closure of H_n for every $n\in\mathbb{N}$, it suffices to prove that $[F,F]\subseteq K_n$ for every $n\in\mathbb{N}$. To do so, we will make use of the following lemma.

Lemma 9. *Let $n\in\mathbb{N}$. If $2|n$ then $K_{\frac{n}{2}}\subseteq K_n$. If $3|n$ then $K_{\frac{n}{3}}\subseteq K_n$.*

Proof. Assume that $2|n$. The proof for the case $3|n$ is similar. $K_{\frac{n}{2}}$ is the minimal closed subgroup such that

$$
\begin{array}{lll}
(a') & 0^k1\sim_{K_{\frac{n}{2}}}0^{k+\frac{n}{2}}1, & \text{for all } k\in\mathbb{N}\\
(b') & 1^k0\sim_{K_{\frac{n}{2}}}1^{k+\frac{n}{2}}0, & \text{for all } k\in\mathbb{N}\\
(c') & 0^k1\sim_{K_{\frac{n}{2}}}1^{\frac{n}{2}+1-k}0, & \text{for } 1\leq k\leq \frac{n}{2}\\
(d') & 0^{2k}10\sim_{K_{\frac{n}{2}}}1^{1+3(\frac{n}{2}-k)}0, & \text{for } 1\leq k\leq \frac{n}{2}\\
(e') & 0^{2k}11\sim_{K_{\frac{n}{2}}}1^{2+3(\frac{n}{2}-k)}0, & \text{for } 1\leq k\leq \frac{n}{2}\\
(f') & 0^{2k-1}1\sim_{K_{\frac{n}{2}}}1^{3(\frac{n}{2}-k+1)}0, & \text{for } 1\leq k\leq \frac{n}{2}.
\end{array}
$$

It suffices to prove that $(a') - (f')$ hold with $K_{\frac{n}{2}}$ replaced by K_n. We would make use of equivalences $(a) - (f)$ above holding modulo $\sim_{K_n}$.

For every $k = 1, \dots, \frac{n}{2}$ we have by (a) and (d) that

$$0^{2k}10 \sim_{K_n} 0^{2k+n}10 = 0^{2(k+\frac{n}{2})}10 \sim_{K_n} 1^{1+3(n-k-\frac{n}{2})}0 = 1^{1+3(\frac{n}{2}-k)}0. \tag{1}$$

Hence (d') holds for K_n. Similarly, by (a) and (e), for every $k = 1, \dots, \frac{n}{2}$ we have

$$0^{2k}11 \sim_{K_n} 0^{2k+n}11 = 0^{2(k+\frac{n}{2})}11 \sim_{K_n} 1^{2+3(n-k-\frac{n}{2})}0 = 1^{2+3(\frac{n}{2}-k)}0. \tag{2}$$

Hence, (e') holds modulo $\sim_{K_n}$. Similarly, by (a) and (f), for every $k = 1, \dots, \frac{n}{2}$ we have

$$0^{2k-1}1 \sim_{K_n} 0^{2k+n-1}1 = 0^{2(k+\frac{n}{2})-1}1 \sim_{K_n} 1^{3(\frac{n}{2}-k+1)}0, \tag{3}$$

so (f') also holds with $K_{\frac{n}{2}}$ replaced by K_n.

To finish, it suffices to prove that equivalences $(a'), (b')$ and (c') hold modulo $\sim_{K_n}$. Since (b) holds modulo $\sim_{K_n}$, to prove (b'), it suffices to prove that for all $k \in \{1, \dots, \frac{n}{2}\}$ we have $1^k0 \sim_{K_n} 1^{k+\frac{n}{2}}0$. So let $k \in \{1, \dots, \frac{n}{2}\}$ and let $i \in \{1, 2, 3\}$ be such that $i \equiv k \pmod 3$. Let $r = \frac{n}{2} - \frac{k-i}{3}$ and note that $r \in \{1, \dots, \frac{n}{2}\}$. We will assume that $i = 3$, the proof for $i = 1, 2$ is similar. Note that if $i = 3$ then $k = 3(\frac{n}{2} - r + 1)$. Then, by (3), (f) and (b), we have

$$\begin{aligned} 1^k0 = 1^{3(\frac{n}{2}-r+1)}0 &\sim_{K_n} 0^{2r-1}1 \sim_{K_n} 1^{3(n-r+1)}0 \\ &\sim_{K_n} 1^{3+3n-3r-n}0 = 1^{3+2n-3r}0 = 1^{3(\frac{n}{2}-r+1)+\frac{n}{2}}0 = 1^{k+\frac{n}{2}}0. \end{aligned} \tag{4}$$

Thus (b') holds for K_n. To prove that (a') holds for K_n we note that for all $k = 1, \dots, \frac{n}{2}$, by applying (c) followed by (b') for $\sim_{K_n}$ followed by (c) again, we have

$$0^k1 \sim_{K_n} 1^{n+1-k}0 \sim_{K_n} 1^{n+1-k-\frac{n}{2}}0 = 1^{n+1-(k+\frac{n}{2})}0 \sim_{K_n} 0^{k+\frac{n}{2}}1. \tag{5}$$

Since (a) holds for K_n, (5) implies that (a') holds for K_n as well.

Finally, (5) shows that for all $k \in \{1, \dots \frac{n}{2}\}$ we have

$$0^k1 \sim_{K_n} 1^{n+1-(k+\frac{n}{2})}0 = 1^{\frac{n}{2}+1-k}0. \tag{6}$$

Hence, (c') also holds for K_n. □

Proposition 10. *Let $n \in \mathbb{N}$. Then K_n contains the derived subgroup of F.*

Proof. We prove the proposition by induction on n. If n is divisible by 2 or 3, then by Lemma 9, we are done by induction. Hence, we can assume that n is not divisible by 2 nor by 3. By Lemma 5, to prove that the closed subgroup K_n contains the derived subgroup of F, it suffices to prove that Conditions (1)-(3) of Lemma 5 hold for K_n.

By (a) and (c) we have

$$0^{2n}10 \sim_{K_n} 0^n10 \sim_{K_n} 1^{n+1-n}00 = 100. \tag{7}$$

On the other hand, by (d) we have

$$0^{2n}10 \sim_{K_n} 1^{1+3(n-n)}0 = 10. \tag{8}$$

Hence,

$$100 \sim_{K_n} 10. \tag{9}$$

Similarly, by (a) and (c) we have

$$0^{2n}11 \sim_{K_n} 0^n 11 \sim_{K_n} 10^{n-n+1}1 = 101. \tag{10}$$

By (e) we have

$$0^{2n}11 \sim_{K_n} 1^{2+3(n-n)}0 = 110. \tag{11}$$

Hence,

$$110 \sim_{K_n} 101. \tag{12}$$

Now, we make the observation that if Condition (1) of Lemma 5 holds for K_n, then Conditions (2) and (3) of Lemma 5 also hold for K_n. Indeed, assume that for all $r \in \mathbb{N}$ we have $1^r0 \sim_{K_n} 10$. Then in particular, $110 \sim_{K_n} 10$. Then, it follows from (12) and (9) that for all $r \in \mathbb{N}$,

$$1^r0 \sim_{K_n} 101 \sim_{K_n} 100 \sim_{K_n} 10. \tag{13}$$

In addition, (a) and (c) from the definition of K_n show that for every $s \in \mathbb{N}$ there is some $r \in \mathbb{N}$ such that $0^s1 \sim_{K_n} 1^r0$. Then it follows from (13) that for all $s \in \mathbb{N}$, $0^s1 \sim_{K_n} 10$. In particular, $01 \sim_{K_n} 10$. Hence, $0^s1 \sim_{K_n} 01$ for all $s \in \mathbb{N}$, so K_n satisfies Condition (2) of Lemma 5. In addition, since $01 \sim_{K_n} 10$, we have $010 \sim_{K_n} 100 \sim_{K_n} 10$ and $011 \sim_{K_n} 101 \sim_{K_n} 10$. Hence,

$$010 \sim_{K_n} 011 \sim_{K_n} 10 \sim_{K_n} 01. \tag{14}$$

Therefore, K_n satisfies Condition (3) of Lemma 5 as well.

Hence, it suffices to prove that Condition (1) of Lemma 5 holds for K_n, i.e., that for every $r \in \mathbb{N}$ we have $1^r0 \sim_{K_n} 10$.

Since n is co-prime to 2 and 3 there are $b, c \in \{1, \dots, n\}$ such that $2b \equiv 1 \pmod n$ and $3c \equiv 1 \pmod n$. Below, whenever an integer modulo n appears as an exponent of the digit "0" or "1" we assume that the chosen representative is in $\{1, \dots, n\}$. Recall that by (a) and (b) for K_n, for all $k \in \mathbb{N}$ we have that $0^k1 \sim_{K_n} 0^{k (\mathrm{mod}\ n)}1$ and $1^k0 \sim_{K_n} 1^{k (\mathrm{mod}\ n)}0$. We use this fact below, sometimes with no explicit reference.

We will need the following lemma.

Lemma 11. *Let $q \in \mathbb{N}$ be such that $1^q0 \sim_{K_n} 10$. Then $10 \sim_{K_n} 1^{q-c (\mathrm{mod}\ n)}0$.*

Proof. Let $p \in \mathbb{N}$ and let $s \in \{1, \dots, n\}$ be such that $s \equiv 1 - bp \pmod n$. Then $p \equiv 2 - 2s \pmod n$. Since $s \in \{1, \dots, n\}$, by (f) followed by (b) we have

$$0^{2s-1}1 \sim_{K_n} 1^{3(n+1-s)}0 \sim_{K_n} 1^{3-3s (\mathrm{mod}\ n)}0 = 1^{3-3(1-bp)(\mathrm{mod}\ n)}0 = 1^{3bp (\mathrm{mod}\ n)}0. \tag{15}$$

On the other hand, by $(a), (c)$ and (b)

$$0^{2s-1}1 \sim_{K_n} 0^{2s-1 (\mathrm{mod}\ n)}1 \sim_{K_n} 1^{1+n-(2s-1)(\mathrm{mod}\ n)}0 \sim_{K_n} 1^{2-2s (\mathrm{mod}\ n)}0 = 1^{p (\mathrm{mod}\ n)}0. \tag{16}$$

Hence,

$$1^{p(\text{mod } n)}0 \sim_{K_n} 1^{3bp(\text{mod } n)}0. \tag{17}$$

Since (17) holds for every $p \in \mathbb{N}$ and $(3b)(2c) \equiv 1 \pmod{n}$, we have that for all $p \in \mathbb{N}$,

$$1^{p(\text{mod } n)}0 = 1^{3b(2cp)(\text{mod } n)}0 \sim_{K_n} 1^{2cp(\text{mod } n)}0. \tag{18}$$

Now, let $t \in \{1, \ldots, n\}$ be such that $t \equiv b(1-q) \pmod{n}$ and note that $q \equiv 1 - 2t \pmod{n}$. Then by $(b), (c), (d)$ and the fact that $3b - 1 \equiv b \pmod{n}$ (indeed, $2b \equiv 1 \pmod{n}$), we have

$$\begin{aligned} 1^q00 &\sim_{K_n} 1^{n+1-2t(\text{mod } n)}00 \sim_{K_n} 0^{2t(\text{mod } n)}10 \sim_{K_n} 1^{1+3(n-t)(\text{mod } n)}0 \\ &= 1^{1-3t(\text{mod } n)}0 = 1^{1-3b(1-q)(\text{mod } n)}0 = 1^{3bq-3b+1(\text{mod } n)}0 = 1^{3bq-b(\text{mod } n)}0. \end{aligned} \tag{19}$$

Now, since by assumption $1^q0 \sim_{K_n} 10$ and by (9) we have $10 \sim_{K_n} 100$, it follows that $1^q00 \sim_{K_n} 100 \sim_{K_n} 10$. Then from equivalence (19) it follows that

$$10 \sim_{K_n} 1^{3bq-b(\text{mod } n)}0. \tag{20}$$

Then (20) and (18) imply that

$$10 \sim_{K_n} 1^{3bq-b(\text{mod } n)}0 \sim_{K_n} 1^{2c(3bq-b)(\text{mod } n)}0 \sim_{K_n} 1^{q-c(\text{mod } n)}0 \tag{21}$$

as required. □

Now we can finish proving the proposition. By lemma 11 applied to $q = 1$, we get that $10 \sim_{K_n} 1^{1-c(\text{mod } n)}0$. Another application of the lemma, now for $q \in \mathbb{N}$ such that $q \equiv 1 - c \pmod{n}$ shows that $10 \sim_{K_n} 1^{1-2c(\text{mod } n)}0$. Continuing inductively, we get that for all $\ell \in \mathbb{N}$, we have

$$10 \sim_{K_n} 1^{1-\ell c(\text{mod } n)}0. \tag{22}$$

Now, for each $r \in \mathbb{N}$, let $\ell \in \mathbb{N}$ be such that $\ell \equiv 3(1-r) \pmod{n}$. Then $r \equiv 1 - c\ell \pmod{n}$ and by (20) we have

$$1^r0 \sim_{K_n} 1^{1-\ell c(\text{mod } n)}0 \sim_{K_n} 10, \tag{23}$$

as required. Hence, the proposition holds. □

Corollary 12. *For every $n \in \mathbb{N}$, the subgroup H_n contains the derived subgroup of F.*

Proof. Let $n \in \mathbb{N}$. Proposition 10 and Lemma 8 imply that the derived subgroup of F is contained in $\text{Cl}(H_n)$. Hence, Condition (1) of Theorem 3 holds for H_n. Lemma 7 shows that Condition (2) of Theorem 3 also holds for H_n. Hence, by Theorem 3, H_n contains the derived subgroup of F. □

The following lemma completes the proof of Theorem 1.

Lemma 13. *Let $m, n \in \mathbb{N}$. Then $G = \langle x^m, y^n \rangle$ is a subgroup of F of index mn.*

Proof. Note that $x^{mn}, y^{mn} \in G$. Hence, H_{mn} is a subgroup of G. Hence, by Corollary 12, the derived subgroup $[F, F] \leq G$. Recall the map $\pi_{ab}\colon F \to \mathbb{Z}^2$ from Section 2.3. By Lemma 6, x^m has the pairs of branches $0^{m+1} \to 0$ and $1 \to 1^{m+1}$. Hence, $\pi_{ab}(x^m) = (m, -m)$. Similarly, $\pi_{ab}(y^n) = (2n, -3n)$. Hence, $\pi_{ab}(G) = \langle (m, -m), (2n, -3n) \rangle$. Since $\langle (m, -m), (2n, -3n) \rangle$ is a subgroup of $\mathbb{Z}^2$ of index $|-3mn + 2mn| = mn$, the subgroup G is a subgroup of F of index mn, as required. □

We finish with the following lemma.

Lemma 14. *The set $\{x, y\}$ invariably generates F.*

Proof. It suffices to prove that for any $g \in F$, the set $\{x, y^g\}$ is a generating set of F. Let $g \in F$ and let $m, n \in \mathbb{Z}$ be such that $\pi_{ab}(g) = (m, n)$. Since $\{\pi_{ab}(x), \pi_{ab}(y)\}$ generates $\mathbb{Z}^2$, there exist $i, j \in \mathbb{Z}$ such that $i\pi_{ab}(x) + j\pi_{ab}(y) = -(m, n)$. Let $h = y^j g x^i$ and note that $h \in [F, F]$. Now, $\{x, y^g\}$ generates F if and only if so does $\{x^{x^i}, y^{gx^i}\} = \{x, y^{y^{-j}h}\} = \{x, y^h\}$.

Let H be the subgroup of F generated by $X = \{x, y^h\}$. Then $H[F, F] = F$ (indeed, the image of X in the abelianization of F coincides with the image of the generating set $\{x, y\}$). Hence, by Theorem 2, to prove that $H = F$ it suffices to prove that $\mathrm{Cl}(H)$ contains the derived subgroup of F. For that, we will make use of Lemma 5. Since $x = x_0 \in H$ has the pairs of branches $00 \to 0$, $01 \to 10$ and $1 \to 11$, we have that for all $k \in \mathbb{N}$, $0^k \sim_H 0$, $1^k \sim_H 1$ and $10 \sim_H 01$. In particular, for every $k \in \mathbb{N}$, we have $0^k 1 \sim_H 01$ and $1^k 0 \sim_H 10$. Hence, Conditions (1) and (2) of Lemma 5 hold for H. To prove that Condition (3) from Lemma 5 holds as well, it suffices to prove that $010 \sim_H 011 \sim_H 10$.

Let us consider the element h. Since $h \in [F, F]$, there exist $a, b \in \mathbb{N}$ such that h has the pair of branches $0^a \to 0^a$ and $1^b \to 1^b$. Let $n = \max\{a, b\}$ and consider the element $f = h^{-1} y^{2n} h \in H$. We claim that f has the pairs of branches

(1) $0^{2n}10 \to 1^{1+3n}0$,
(2) $0^{2n}11 \to 1^{2+3n}0$.

Indeed, by Lemma 6, the element y^{2n} has the pairs of branches $0^{2n}10 \to 1^{1+3(2n-n)}0 = 1^{1+3n}0$ and $0^{2n}11 \to 1^{2+3(2n-n)}0 = 1^{2+3n}0$. Since h fixes the intervals $[0^{2n}] \subseteq [0^a]$ and $[1^{3n}] \subseteq [1^b]$ pointwise, the element f also has the pairs of branches $0^{2n}10 \to 1^{1+3n}0$ and $0^{2n}11 \to 1^{2+3n}0$, as claimed.

Now, from (1) and the fact that for all $k \in \mathbb{N}$, we have $0^k \sim_H 0$ and $1^k \sim_H 1$, we have that $010 \sim_H 10$. Similarly, using (2), we get that $011 \sim_H 10$. Hence, Condition (3) of Lemma 5 holds for H. Since H satisfies all the conditions of Lemma 5, $\mathrm{Cl}(H)$ contains the derived subgroup of F, as necessary. □

Acknowledgments: The authors would like to thank the referee for his/her careful reading of the text and for helpful comments and suggestions which helped simplify the text.

Conflict of Interest Statement: On behalf of all authors, the corresponding author states that there is no conflict of interest.

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