Z. Angew. Math. Phys. (2023) 74:126

https://doi.org/10.1007/s00033-023-02019-4

Zeitschrift für angewandte Mathematik und Physik ZAMP

Global existence of classical solutions to a regularized brine inclusion model

Yuan Chen and Keith Promislow

Abstract. The freezing of salt water leads to phase separation into ice and brine inclusions. The ejection of salt from the ice phase is akin to chemotaxis. We consider a regularization of a free energy for brine inclusions and address its gradient flow on a periodic domain in $\mathbb{R}^d(d=2,3)$. Uniqueness and global existence of classical solutions from initial data in an energy space are established for positive time.

Mathematics Subject Classification. 35A01.

Keywords. Chemotaxis, Global existence, Phase separation.

1. Introduction

The freezing of salt water requires the ejection of salt from the growing ice domain. This ejection is a natural form of chemotaxis. The brine inclusion model introduced in [5] incorporates the salt ejection by taking the entropy of the salt volume fraction to be relative to the liquid water volume fraction. This is a natural approach; however, the singularity implicit in the relative entropy formulation complicates rigorous analysis of the model. We present a regularization of the model, show that it fits within a broad system of Keller Segal equations, and establish that it is locally well-posed and supports global solutions. The regularized brine inclusion model addresses a phase function ϕ which distinguishes between liquid and frozen water and a salt density N. The flow for $u=(N,\phi)$ takes the form

$$\begin{cases} \partial_t\phi = \Delta\phi - \partial_\phi W(\phi,N) + \dfrac{N}{\phi+\sigma}, \\ \partial_t N = \Delta N - \nabla\cdot\left(g(N)\left(\dfrac{\nabla\phi}{\phi+\sigma} - \nabla\partial_N W(\phi,N)\right)\right), \end{cases} \tag{1.1}$$

on a periodic domain $\Omega\subset\mathbb{R}^d(d=2,3)$ subject to initial condition:

$$\phi(0,x)=\phi_0(x), \qquad N(0,x)=N_0(x), \qquad \text{for all} x\in\Omega. \tag{1.2}$$

Here $\sigma>0$ is a positive regularizing constant, $g(N)>0$ for $N>0$ is a mobility function depending on N, and W is a smooth potential. For spatially constant salt concentration N the function W is a double well potential in ϕ. The flow conserves the mass of N, viz.

$$\int_\Omega N\,dx = N_0. \tag{1.3}$$

Indeed, the flow arises as a gradient of the natural energy

$$\mathrm{E}[\phi,N] = \int_\Omega \left(\frac{1}{2}|\nabla\phi|^2 + W(\phi,N) + f(N) - N\ln(\phi+\sigma)\right)dx, \tag{1.4}$$

Published online: 06 June 2023

where $f = f(N)$ is a function of N satisfying

$$f''(N) = 1/g(N). \tag{1.5}$$

Particularly, if $g(N) = N$, then $f(N) = N \ln N$ is the classical entropy of the salt.

1.1. Model motivation

The flow (1.1) arises as a regularization of the isothermal version of the brine inclusion model introduced in [5]. Particularly, $N \geq 0$ denotes the salinity of the liquid water and $\phi \geq 0$ denotes the phase of the water with $\phi = 0$ corresponding to ice and $\phi = 1$ to liquid water. Following the GENERIC framework of [9], the prior work develops a thermodynamically consistent flow for the entropy S which is the negative of the energy E from (1.4). It considers the case $\sigma = 0$ and the choices

$$f(N) := N \ln N; \qquad W(\phi; N) := W_0(\phi) + \xi(N) W_1(\phi), \tag{1.6}$$

For fixed N, the potential W is a double well in ϕ with possibly unequal local minima at $\phi = 0$ and $\phi = 1$. The dominate phase potential W_0 is the classical double well $W_0(\phi) := \phi^2(1-\phi)^2$. The term W_1 breaks the equal-depth structure of W_0. It satisfies $W_1 \leq 0$, with compact support in $(\frac{1}{2}, \frac{3}{2})$ and a single minima $W_1(1) = -1$ at $\phi = 1$, The cryoscopic prefactor $\xi = \xi(N)$ incorporates the impact of salt on the freezing point of water, lowering the value of $W(1, N)$ with increasing salt concentration. The salt ejection is modeled by the $N \ln(\phi)$ term which incorporates the dependence of the salt entropy on the presence of *liquid* water. This is the origin of the chemotactic behavior in the system, as freezing removes liquid water molecules from the hydration sphere of a salt ion, the ion's entropy decreases.

Under time evolution, the entropy increases while the total salt content remains fixed. This motivates the gradient flow of (N, ϕ):

$$\begin{cases} \partial_t \phi = -\dfrac{\delta \mathrm{E}}{\delta \phi}, & \text{in}\, \Omega \times (0, \infty), \\ \partial_t N = \nabla \cdot \left(g(N) \nabla \dfrac{\delta \mathrm{E}}{\delta N} \right), & \text{in}\, \Omega \times (0, \infty). \end{cases} \tag{1.7}$$

Here, $\frac{\delta \mathrm{E}}{\delta \phi}, \frac{\delta \mathrm{E}}{\delta N}$ are the L^2 variational derivatives of the energy $\mathcal{E}$, (1.4), with respect to ϕ and N, respectively. Moreover, for the generic choice of mobility function $g(N) = N$ the flow reduces to

$$\begin{cases} \partial_t \phi = \Delta \phi - \partial_\phi W(\phi, N) + \dfrac{N}{\phi}; \\ \partial_t N = \Delta N - \nabla \cdot \left(N \dfrac{\nabla \phi}{\phi} - N \nabla \partial_N W(\phi, N) \right). \end{cases} \tag{1.8}$$

The flow (1.1) is a generalization of (1.8) subject to the regularization of the salt ejection mechanism at $\phi = 0$. The regularizing parameter $0 < \sigma \ll 1$ keeps the salt entropy $N \ln(\phi + \sigma)$ finite if $N > 0$ in the pure ice phase. The value $\sigma = 0$ corresponds to the unregularized model and is not considered in the analysis presented here. Macroscopic models of sea ice resolve effective transport properties based upon local averages of the ice/liquid phase, [11]. Degenerate diffusion of salt plays a crucial role in these models, as it allows the incorporation of a brine volume fraction threshold below which the salt phase is immobile. The model presented here resolves the brine microstructure explicitly and restricts salt diffusion in the ice by excluding the salt from the ice phase. When the brine regions become disconnected, the macroscopic salt flux is naturally extinguished, see [5] for details.

1.2. Main results and connection to Keller–Segel models

The system (1.1) can be compared to the general Keller–Segel model, see [3,4], who introduced the general framework:

$$\begin{cases} \partial_t \phi = D_\phi(N,\phi)\Delta\phi + K(N,\phi), \\ \partial_t N = \nabla \cdot \Big(D_N(N,\phi)\nabla N - \chi(N,\phi)N\nabla\phi \Big) + H(N,\phi). \end{cases} \tag{1.9}$$

In this system, N denotes a cell density and ϕ models the density of a chemo-attractant. The system (1.1) lies within the framework of (1.9) subject to the choices $D_\phi = 1, H = 0$ and

$$K = -\partial_\phi W + \frac{N}{\phi+\sigma},\ D_N = 1 + N\partial_N^2 W,\ \chi = \frac{1}{\phi+\sigma} - \partial_{N\phi}W.$$

There is a substantial literature work on the existence and nonexistence of solutions to the Keller–Segel model for a range of functional forms of the diffusivities D_N, D_ϕ, χ and potentials H, K. The survey paper [1] provides an excellent review of results on these models. Despite the breadth of the established work it does not accommodate the form (1.1). Specifically, the most relevant local existence result (see Lemma 3.1 in [1]) requires that the quantity $K - \phi$ be positive. This assumption affords a maximum principle which is not available to (1.1). Similarly relevant global existence results require a bound on the growth of K with respect to N of the form $|K(N,\phi)| \leq C|N+1|^\beta$ for some constant C and $\beta \in [0,1]$. This growth requirement is used to establish bounds on ϕ and close essential nonlinear estimates. These estimates do not seem to hold for the regularized brine inclusion model.

The classical Keller–Segel system corresponds to the choices $D_\phi \equiv 1, K = -\phi + N, D_N = \chi \equiv 1, H \equiv 0$. This form of (1.9) can be viewed as a gradient flow of the energy functional

$$\mathrm{F}_{KS}[\phi,N] = \int_\Omega \left(\frac{1}{2}|\nabla\phi|^2 + \phi^2 + N\ln N - N\phi \right) dx. \tag{1.10}$$

A key step to establish the global existence in the classical Keller–Segel system is to control the sign-indeterminate terms in the energy via the Moser–Trudinger inequality, [10]. This approach requires a particular structure in the nonlinearity that (1.1) does not possess. To surmount these difficulties, we impose assumptions on the form of f and the potential W to insure that the system is well-posed. Particularly we require

($\mathbf{H}_f$): f is a smooth function of N and satisfies
- (a) f'' is bounded away from zero and $|f'''|$ is uniformly bounded on $[1,\infty)$.
- (b) $f(N) = N\ln N$ for $N > 0$ sufficiently small.

($\mathbf{H}_W$): W is a smooth function of ϕ, N and satisfies
- (a) There exists $C > 0$ such that for all $N \in [0,\infty)$,

$$W(\phi,N) \geq -C\phi^2 - C; \quad |\partial_\phi^k W(\phi,N)| \leq C|\phi|^{4-k} + C(N+1), \tag{1.11}$$

  for $k = 1,2,3$.
- (b) The potential $W = W(\phi,N)$ is a linear function of N and satisfies the bounds

$$|\partial_{\phi N}W| + |\partial_{\phi\phi N}W| \leq C. \tag{1.12}$$

- (c) For $N > 0$ finite, there exists $\alpha_0, \beta_0 > 0$ such that $\partial_\phi W(\phi,N) \leq \alpha_0\phi$ for $\phi \in (0,\beta_0)$.

A motivating example of f satisfying ($\mathbf{H}_f$) is given by

$$f(N) = \begin{cases} N\ln N, & 0 \leq N < 1/2; \\ N^2, & N \geq 1, \end{cases}$$

with a smooth connection for $N \in [1/2, 1]$. It is useful to observe that the assumptions $(\mathbf{H}_f)$, imply that $g(N) := 1/f''(N)$ satisfies

$$|g(N)| + |g'(N)| \leq C. \tag{1.13}$$

Under these assumptions, we have a maximum principle for ϕ and the regularized energy bounds the L^2-norm of the salt density. We adopt the semigroup estimates and establish the global existence of solutions which are smooth for $t > 0$.

Theorem 1.1. *Let $\sigma > 0$ be a given positive constant. Consider the flow (1.1)–(1.2) on a periodic domain in $\mathbb{R}^d (d = 2, 3)$. Under the assumptions $(\mathbf{H}_f)$ and $(\mathbf{H}_W)$, if initial data $u_0 = (N_0, \phi_0)$ has finite energy, i.e., $\mathrm{E}(N_0, \phi_0) < \infty$, then there exists a global-in-time solution $u = (N, \phi)$ which is smooth after the initial time and recovers the initial data in the sense that*

$$\lim_{t \to 0^+} \left(\|\phi(t) - \phi_0\|_{W^{1,2}(\Omega)} + \|N(t) - N_0\|_{L^2(\Omega)} \right) = 0.$$

Moreover, the phase variable ϕ is uniformly bounded after an initial transient.

To prove the theorem, we first construct a local smooth solution via contraction mapping principle for sufficiently smooth initial data. An approximation method is then applied to extend the existence result to initial data in the energy space. The maximum principle cannot be used to establish an upper bound on N, instead we use semigroup and energy estimates to establish uniform bounds on $\|N\|_{L^4(\Omega)}$ and $\|\nabla\phi\|_{L^5(\Omega)}$. These allow the local solutions to be extended globally. These estimates and standard embedding theorems allow the phase variable to be uniformly bounded in time. Finally, a bootstrap argument shows that the solutions are smooth away from $t = 0$.

A maximum principle ensures the positivity $\phi > 0$. For the regularized model the term $1/(\phi + \sigma)$ is easy to bound – this fact plays an essential role in the proof. The regularization of ϕ at 0 is traditional in chemotaxis systems such as [4] and [13]. More recent work has addressed chemotaxis with singular sensitivity of the form $\chi(N, \phi) = \chi_0/\phi$. In [7], Lankeit established a global existence of classical solutions in dimension two. In higher dimension, Winkler, Fujie and Lankeit [2,8,13] proved the global existence under different assumptions on the size of χ_0 depending on dimension. In each of these cases the proof hinges on uniform estimates of energy in the form of $\int_\Omega N_\sigma^p \phi_\sigma^{-r}$ on any finite time interval. These depend heavily on their K term depends linearly on $N - \phi$. This approach does not apply to the regularized brine inclusion model, and we restrict our attention to the case $\sigma > 0$.

The assumptions $(\mathbf{H}_W)$ constrain the dependence of W upon N. We present the proof for the case W independent of N. The general case follows from a slight modification of the arguments. The remainder of the article is organized as follows. In Sect. 2, we introduce basic energy estimates and lower bounds on the energy. In Sect. 3, we apply the contraction mapping principle to establish the existence of local smooth positive solutions. In Sect. 4, we establish a uniform estimate of ϕ and N away from $t = 0$, which assures a uniform life span of local solutions. Finally, the global existence with initial data in the energy space is established by a classical approximation method.

2. Basic estimates

For on a bounded domain $\Omega \subset \mathbb{R}^2, \mathbb{R}^3$ with periodic boundary conditions, there exist constants C_1 and C_2 for which the embedding estimates hold

$$\|\phi\|_{L^2(\Omega)} \leq C_1 \|\phi\|_{L^6(\Omega)} \leq C_2 \|\nabla\phi\|_{L^2(\Omega)}.$$

These estimates will be used without comment in the sequel. From the gradient structure of the flow, we have the following basic energy identity regarding positive solutions. Particularly, this implies that the energy $\mathrm{E} = \mathrm{E}(N, \phi)$ introduced in (1.4) is dissipated by the flow.

Lemma 2.1. *Suppose (N,ϕ) is a positive classical solution to the system (1.1)–(1.2) on the periodic domain Ω, then the following energy identity holds:*

$$\frac{d\mathrm{E}}{dt}+\int\limits_{\Omega}\left(g(N)\left|\frac{\nabla N}{g(N)}-\frac{\nabla\phi}{\phi+\sigma}\right|^2+\left|\Delta\phi-W'(\phi)+\frac{N}{\phi+\sigma}\right|^2\right)dx=0. \tag{2.1}$$

Proof. The lemma is a direct result of the gradient structure of the flow (1.7). Taking the L^2-inner product of the first equation in (1.1) with $\frac{\delta\mathrm{E}}{\delta\phi}=-\left(\Delta\phi-W'(\phi)+\frac{N}{\phi+\sigma}\right)$ immediately yields

$$\int\limits_{\Omega}\partial_t\phi\,\frac{\delta\mathrm{E}}{\delta\phi}\,dx=-\int\limits_{\Omega}\left|\Delta\phi-W'(\phi)+\frac{N}{\phi+\sigma}\right|^2 dx. \tag{2.2}$$

Taking the L^2-inner product of the second equation in (1.1) with $\frac{\delta\mathrm{E}}{\delta N}=f'(N)-\ln(\phi+\sigma)$, integrating by parts, and using the fact $f''(N)=1/g(N)$ implies

$$\int\limits_{\Omega}\partial_t N\,\frac{\delta\mathrm{E}}{\delta N}\,dx=-\int\limits_{\Omega}g(N)\left|\frac{\nabla N}{g(N)}+\frac{\nabla\phi}{\phi+\sigma}\right|^2 dx. \tag{2.3}$$

The lemma follows by combining (2.2) and (2.3), with the identity

$$\frac{d\mathrm{E}}{dt}=\int\limits_{\Omega}\partial_t\phi\,\frac{\delta\mathrm{E}}{\delta\phi}\,dx+\int\limits_{\Omega}\partial_t N\,\frac{\delta\mathrm{E}}{\delta N}\,dx. \tag{2.4}$$

□

The energy $\mathrm{E}(N,\phi)$ admits a uniform lower bound, which controls the L^2-norm of N and $W^{1,2}(\Omega)$-norm of ϕ.

Lemma 2.2. *There exist positive constants C_1,C_2 depending on Ω and σ such that the energy enjoys the following lower bound:*

$$\mathrm{E}(N,\phi)\geq C_1\left(\|N\|^2_{L^2(\Omega)}+\|\phi\|^2_{W^{1,2}(\Omega)}\right)-C_2. \tag{2.5}$$

Proof. From the form of $\mathcal{E}$ and assumptions $(\mathbf{H}_f)$ we have

$$\mathrm{E}(N,\phi)\geq\frac{1}{2}\|\nabla\phi\|^2_{L^2(\Omega)}+C\|N\|^2_{L^2(\Omega)}+\int\limits_{\Omega}W(\phi)\,dx-\int\limits_{\Omega}N\ln|\phi+\sigma|\,dx. \tag{2.6}$$

By the assumption $(\mathbf{H}_W)_b$ on W, we have

$$\int\limits_{\Omega}W(\phi)\,dx\geq -C\int\limits_{\Omega}\phi^2\,dx-C. \tag{2.7}$$

Standard embedding estimates reduce (2.6) to the form

$$\mathrm{E}(N,\phi)\geq\frac{1}{4}\|\nabla\phi\|^2_{L^2(\Omega)}+C\|N\|^2_{L^2(\Omega)}-\int\limits_{\Omega}N\ln|\phi+\sigma|\,dx-C. \tag{2.8}$$

It remains to bound the sign-indeterminant integral on the right-hand side. For any positive constant $\varepsilon>0$, there exists a constant C for which the relation: $(\ln(x+\sigma))^2\leq\varepsilon^2x^2+C(\varepsilon)$ holds for all $x>0$.

Applying the Cauchy–Schwartz inequality and this pointwise identity yields

$$\begin{aligned}\int_\Omega N\ln(\phi+\sigma)\,dx &\le \varepsilon\int_\Omega N^2\,dx + \frac{C}{\varepsilon}\int_\Omega \ln^2|\phi+\sigma|\,dx\\ &\le \varepsilon\int_\Omega N^2\,dx + \varepsilon\int_\Omega \phi^2\,dx + \frac{C_\sigma}{\varepsilon}.\end{aligned}\tag{2.9}$$

The lemma follows by combining these two inequalities with embedding estimates and choosing ε small enough. □

3. Local well-posedness

In this section, we apply the contraction mapping principle to establish the local well-posedness of (1.1)–(1.2). To achieve this, we introduce an iteration space:

$$X_T := L^\infty\left(0,T;L^4(\Omega)\times W^{1,5}(\Omega)\right),$$

with the norm on $u=(u_1,u_2)$ given by

$$\|u\|_{X_T} := \sup_{t\in[0,T]}\left(\|u_1\|_{L^4(\Omega)} + \|u_2\|_{W^{1,5}(\Omega)}\right).\tag{3.1}$$

The associated space of initial data is

$$X_0 := L^4(\Omega)\times W^{1,5}(\Omega).\tag{3.2}$$

From the flow (1.1)–(1.2), given $u=(N,\phi)\in X_T$ and initial data $u_0=(N_0,\phi_0)\in X_0$, we introduce the iteration map $F(u)=(F_1,F_2)(u)$ defined by variation of parameters

$$\begin{aligned}F_1(u)(t) &:= e^{t\Delta}N_0 - \int_0^t e^{(t-s)\Delta}\nabla\cdot\left(\frac{g(N)}{\phi+\sigma}\nabla\phi\right)ds,\\ F_2(u)(t) &:= e^{t\Delta}\phi_0 - \int_0^t e^{(t-s)\Delta}\left(W'(\phi) - \frac{N}{\phi+\sigma}\right)ds,\end{aligned}\tag{3.3}$$

for $t\in[0,T]$. Here $e^{t\Delta}$ is the heat semigroup subject to periodic boundary conditions. We recall the classic $L^p - L^q$ estimates on the heat semigroup. Their proof can be found in many places, including [12].

Lemma 3.1. *Let $\Omega\subset\mathbb{R}^d$ be a bounded smooth domain. For $u\in L^q(\Omega)$ and $1<q\le p<\infty$, there exist positive constants $\lambda, C>0$ depending on Ω only such that*

$$\begin{aligned}\|e^{t\Delta}\nabla\cdot u\|_{L^p(\Omega)} &\le C\left(1+t^{-\frac{1}{2}-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\right)e^{-\lambda t}\|u\|_{L^q(\Omega)};\\ \|\nabla e^{t\Delta}u\|_{L^p(\Omega)} &\le C\left(1+t^{-\frac{1}{2}-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\right)e^{-\lambda t}\|u\|_{L^q(\Omega)}.\end{aligned}\tag{3.4}$$

Moreover, if u is mass-free, i.e., $\int_\Omega u = 0$, then

$$\|e^{t\Delta}u\|_{L^p(\Omega)} \le C\left(1+t^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\right)e^{-\lambda t}\|u\|_{L^q(\Omega)}.\tag{3.5}$$

In general for any $u\in L^q(\Omega)$, the inequality above holds with $\lambda=0$.

To prove that F is a contraction mapping on X_T, we first show that it is closed in a ball of X_T for suitable initial data and T small enough. The result requires an extra a priori lower bound condition on ϕ to avoid the singularity at $\phi = -\sigma$. This condition will be recovered in Lemma 3.5 via a maximum principle. The result is contained in the following lemma.

Lemma 3.2. *Let $R > 0, T \in (0, 1]$ be a given constant and B_{2R} be a ball in the functional space X_T with radius $2R$ and center at the origin. Suppose $u = (N, \phi)$ lies in the ball $B_{2R} \subset X_T$ and satisfies $\phi > -\sigma/2$, and has initial data u_0 with $\|e^{t\Delta}u_0\|_{X_0} \le R$ for all $t \in [0, T]$, then $F(u)$ also lies in B_{2R} provided that T is small enough in terms of R, σ, and the domain Ω.*

Proof. We first bound F_1 in $L^4(\Omega)$. From the definition of F_1 in (3.3), we have

$$\|F_1\|_{L^4(\Omega)} \le \|e^{t\Delta}N_0\|_{L^4(\Omega)} + \int_0^t \left\| e^{(t-s)\Delta}\nabla \cdot \left(\frac{g(N)}{\phi+\sigma}\nabla\phi\right)(s)\right\|_{L^4(\Omega)} ds. \tag{3.6}$$

Note that for $s, t \in [0, T]$ satisfying $s < t < T \le 1$.

$$1 + (t-s)^{-1/2} \le 2(t-s)^{-1/2}. \tag{3.7}$$

Applying the first inequality in Lemma 3.1 with $p = q = 4$ and using the inequality above implies

$$\|F_1\|_{L^4(\Omega)}(t) \le \|e^{t\Delta}N_0\|_{L^4(\Omega)} + C\int_0^t (t-s)^{-\frac{1}{2}} \left\|\frac{g(N)}{\phi+\sigma}\nabla\phi\right\|_{L^4(\Omega)}(s)\, ds. \tag{3.8}$$

Using the bound $\phi > -\sigma/2$ and $|g(N)| \le C$, and applying Young's inequality yields

$$\|F_1\|_{L^4(\Omega)}(t) \le \|e^{t\Delta}N_0\|_{L^4(\Omega)} + C\int_0^t (t-s)^{-\frac{1}{2}} \|\nabla\phi\|_{L^4(\Omega)}(s)\, ds. \tag{3.9}$$

Here and below, C is a constant possibly depending on σ and Ω, and its value may change from line to line. Taking the supremum of the time-dependent function $\|\nabla\phi\|_{L^4(\Omega)}(s)$ and integrating the remaining s-polynomial function with respect to time yields

$$\begin{aligned}\|F_1\|_{L^4(\Omega)}(t) &\le \|e^{t\Delta}N_0\|_{L^4(\Omega)} + CT^{\frac{1}{2}} \|\nabla\phi\|_{L^\infty([0,T];L^4(\Omega))} \\ &\le \|e^{t\Delta}N_0\|_{L^4(\Omega)} + CT^{\frac{1}{2}}\|u\|_{X_T}.\end{aligned} \tag{3.10}$$

To bound F_2 in $W^{1,5}(\Omega)$ we first estimate F_2 in $L^5(\Omega)$. By the definition of F_2 in (3.3), we have

$$\|F_2\|_{L^5(\Omega)} \le \|e^{t\Delta}\phi_0\|_{L^5(\Omega)} + \int_0^t \left\|e^{(t-s)\Delta}W'(\phi)\right\|_{L^5(\Omega)} ds + \int_0^t \left\|e^{(t-s)\Delta}\frac{N}{\phi+\sigma}\right\|_{L^5(\Omega)} ds. \tag{3.11}$$

Applying the last inequality in Lemma 3.1 and using the assumption $\phi > -\sigma/2$ yields

$$\|F_2\|_{L^5(\Omega)} \le \|e^{t\Delta}\phi_0\|_{L^5(\Omega)} + C\int_0^t (t-s)^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{5})} \|W'(\phi)\|_{L^2(\Omega)}(s)\, ds + C\int_0^t (t-s)^{-\frac{d}{2}(\frac{1}{4}-\frac{1}{5})}\|N\|_{L^4(\Omega)}\, ds. \tag{3.12}$$

From the assumption of W in ($\mathbf{H}_W$), it holds that $|W'(\phi)| \le C\phi^3 + 1$ and hence

$$\|W'(\phi)\|_{L^2(\Omega)}(s) \le C\|\phi\|^3_{L^6(\Omega)}(s) + C. \tag{3.13}$$

From the Gargliardo–Nirenberg inequality and $u = (N, \phi) \in X_T$ we have

$$\|\phi\|_{L^6(\Omega)}(s) \le \|\nabla\phi\|^{\frac{d}{30}}_{L^5(\Omega)}(s)\|\phi\|^{\frac{30-d}{30}}_{L^5(\Omega)}(s) \le \|u\|_{X_T}, \qquad \forall s \in (0, T). \tag{3.14}$$

Combining the previous two estimates with (3.12) yields the L^5 bound

$$\begin{aligned}\|F_2\|_{L^5(\Omega)} &\leq \|e^{t\Delta}\phi_0\|_{L^5(\Omega)} + \int_0^t (t-s)^{-\frac{3d}{20}} \left(\|u\|_{X_T}^3 + 1\right) ds + C\int_0^t (t-s)^{-\frac{d}{40}} \|N\|_{L^4(\Omega)}\, ds, \\ &\leq \|e^{t\Delta}\phi_0\|_{L^4(\Omega)} + CT^{\frac{20-3d}{20}} \left(\|u\|_{X_T}^3 + 1\right) + CT^{\frac{40-d}{40}} \|u\|_{X_T}.\end{aligned} \tag{3.15}$$

Finally, we bound ∇F_2 in $L^5(\Omega)$. We follow the argument for F_2 but apply the second inequality in Lemma 3.1 to obtain

$$\|\nabla F_2\|_{L^5(\Omega)} \leq \|\nabla e^{t\Delta}\phi_0\|_{L^5(\Omega)} + C\int_0^t (t-s)^{-\frac{1}{2}-\frac{3d}{20}} \|W'(\phi)\|_{L^2(\Omega)}\, ds + C\int_0^t (t-s)^{-\frac{1}{2}-\frac{40-d}{40}} \|N\|_{L^4(\Omega)}\, ds$$

Taking the supremum for $s \in [0, T]$ of the bounds (3.13)–(3.14) affords the estimate

$$\|\nabla F_2\|_{L^5(\Omega)} \leq \|\nabla e^{t\Delta}\phi_0\|_{L^5(\Omega)} + CT^{\frac{10-3d}{20}} \|u\|_{X_T}^3 + CT^{\frac{20-d}{40}} \|u\|_{X_T}. \tag{3.16}$$

Without loss of generality, we assume $T \leq 1$. Combining the estimates (3.10), (3.15) and (3.16) and taking the supremum of the left-hand side of the resulting inequality, the definition of X_T yields

$$\|F\|_{X_T} \leq \|e^{t\Delta}u_0\|_{X_0} + CT^{\frac{10-3d}{20}} \left(\|u\|_{X_T}^3 + \|u\|_{X_T} + 1\right). \tag{3.17}$$

By the assumption, $\|e^{t\Delta}u_0\|_{X_0} < R$ and $u = (N, \phi) \in B_{2R}$. We conclude that $F(u) \in B_{2R}$ for T small enough depending on Ω, σ and R. $\square$

We establish that the map $F : X_T \to X_T$ as defined by (3.3) is a contraction. Particularly, we establish the following lemma.

Lemma 3.3. *Suppose $u_1 := (N_1, \phi_1), u_2 := (N_2, \phi_2)$ lie in the ball $B_{2R} \subset X_T$ and satisfy $\phi_1, \phi_2 > -\sigma/2$. Then, there exists T_0 depending on σ, R and Ω only such that for all $T \in [0, T_0]$,*

$$\|F(u_1) - F(u_2)\|_{X_T} \leq \frac{1}{2}\|u_1 - u_2\|_{X_T}.$$

Proof. From (3.3), we first estimate

$$\|F_1(u_1) - F_1(u_2)\|_{L^4(\Omega)} \leq \int_0^t \left\| e^{(t-s)\Delta}\nabla\cdot\left(\frac{g(N_1)}{\phi_1+\sigma}\nabla\phi_1 - \frac{g(N_2)}{\phi_2+\sigma}\nabla\phi_2\right)\right\|_{L^4(\Omega)} ds. \tag{3.18}$$

It is natural to introduce the decomposition

$$\begin{aligned}\frac{g(N_1)}{\phi_1+\sigma}\nabla\phi_1 - \frac{g(N_2)}{\phi_2+\sigma}\nabla\phi_2 =& \frac{g(N_1)-g(N_2)}{\phi_1+\sigma}\nabla\phi_1 + \frac{g(N_2)}{\phi_2+\sigma}\nabla(\phi_1-\phi_2) \\ &+ \left(\frac{1}{\phi_1+\sigma} - \frac{1}{\phi_2+\sigma}\right) g(N_2)\nabla\phi_2.\end{aligned} \tag{3.19}$$

Recalling that $(\mathbf{H}_f)$ ensures $g = 1/f''$ and g' are uniformly bounded, and $\phi_1, \phi_2 > -\sigma/2$, we have

$$\left|\frac{g(N_1)}{\phi_1+\sigma}\nabla\phi_1 - \frac{g(N_2)}{\phi_2+\sigma}\nabla\phi_2\right| \leq C|N_1 - N_2||\nabla\phi_1| + C|\nabla(\phi_1-\phi_2)| + C|\nabla\phi_2||\phi_1-\phi_2|.$$

Here and below, the positive constant C may depend on σ, R, and domain Ω and may change from line to line. Applying the $L^p - L^q$ estimates from Lemma 3.1, we derive

$$\begin{aligned}\|F_1(u_1)-F_1(u_2)\|_{L^4(\Omega)} \le & C\int_0^t (t-s)^{-\frac{1}{2}-\frac{d}{2}(\frac{9}{20}-\frac{1}{4})}\|(N_1-N_2)\nabla\phi_1\|_{L^{20/9}(\Omega)}(s)\,ds \\ & + C\int_0^t (t-s)^{-\frac{1}{2}-\frac{d}{2}(\frac{2}{5}-\frac{1}{4})}\|(\phi_1-\phi_2)\nabla\phi_2\|_{L^{5/2}(\Omega)}(s)\,ds \\ & + C\int_0^t (t-s)^{-\frac{1}{2}-\frac{d}{2}(\frac{1}{5}-\frac{1}{4})}\|\nabla(\phi_1-\phi_2)\|_{L^5(\Omega)}(s)\,ds.\end{aligned} \tag{3.20}$$

Applying Young's inequality and using $u=(N,\phi)\in B_{2R}\subset X_T$ yields

$$\begin{aligned}\|(N_1-N_2)\nabla\phi_1\|_{L^{20/9}(\Omega)} &\le \|N_1-N_2\|_{L^4(\Omega)}\|\nabla\phi_1\|_{L^5(\Omega)} \le C\|N_1-N_2\|_{L^4(\Omega)}; \\ \|(\phi_1-\phi_2)\nabla\phi_2\|_{L^{5/2}(\Omega)} &\le \|\phi_1-\phi_2\|_{L^5(\Omega)}\|\nabla\phi_2\|_{L^5(\Omega)} \le C\|\phi_1-\phi_2\|_{L^5(\Omega)}.\end{aligned} \tag{3.21}$$

For $u\in X_T$, we have the Lipschitz estimate,

$$\|N_1-N_2\|_{L^4(\Omega)}+\|\phi_1-\phi_2\|_{L^5(\Omega)}+\|\nabla(\phi_1-\phi_2)\|_{L^5(\Omega)} \le \|u_1-u_2\|_{X_T}.$$

Therefore, for $t\le 1$ the difference of F_1 at $u_k (k=1,2)$ can be bounded by

$$\begin{aligned}\|F_1(u_1)-F_1(u_2)\|_{L^4(\Omega)} &\le C\|u_1-u_2\|_{X_T}\int_0^t (t-s)^{-\frac{5+d}{10}}\,ds \\ &\le CT^{\frac{5-d}{10}}\|u_1-u_2\|_{X_T}.\end{aligned} \tag{3.22}$$

Now we bound the difference $F_2(u_1)-F_2(u_2)$ in $W^{1,5}(\Omega)$. Particularly, from the definition of F_2 and Lemma 3.1 we have

$$\begin{aligned}\|F_2(u_1)-F_2(u_2)\|_{L^5(\Omega)} \le & C\int_0^t (t-s)^{-\frac{d}{2}(\frac{8}{15}-\frac{1}{5})}\,\|W'(\phi_1)-W'(\phi_2)\|_{L^{15/8}(\Omega)}\,ds \\ & +C\int_0^t (t-s)^{-\frac{d}{2}(\frac{9}{20}-\frac{1}{5})}\left\|\frac{N_1}{\phi_1+\sigma}-\frac{N_2}{\phi_2+\sigma}\right\|_{L^{20/9}(\Omega)}\,ds.\end{aligned} \tag{3.23}$$

Applying $(\mathbf{H}_W)_{(a)}$ to $W'(\phi)$ bounding $\phi_1,\phi_2\in L^6(\Omega)\subset W^{1,5}(\Omega)$, we have

$$\begin{aligned}\|W'(\phi_1)-W'(\phi_2)\|_{L^{15/8}(\Omega)} &\le C\|\phi_1-\phi_2\|_{L^5(\Omega)}\|W''(\phi_1+\mu\phi_2)\|_{L^3} \\ &\le C\|\phi_1-\phi_2\|_{L^5(\Omega)}.\end{aligned} \tag{3.24}$$

Moreover, from Young's inequality and the assumption $\phi_k > -\sigma/2$ we derive

$$\begin{aligned}\left\|\frac{N_1}{\phi_1+\sigma}-\frac{N_2}{\phi_2+\sigma}\right\|_{L^{20/9}(\Omega)} &\le C\|N_1-N_2\|_{L^4(\Omega)}+C\|(\phi_1-\phi_2)N_2\|_{L^{20/9}(\Omega)} \\ &\le C\|N_1-N_2\|_{L^4(\Omega)}+C\|(\phi_1-\phi_2)\|_{L^5(\Omega)}\|N_2\|_{L^4(\Omega)} \\ &\le C\|N_1-N_2\|_{L^4(\Omega)}+C\|(\phi_1-\phi_2)\|_{L^5(\Omega)}.\end{aligned} \tag{3.25}$$

Here, we used $N_2 \in L^4(\Omega)$ is uniformly bounded since $u \in B_{2R} \subset X_T$. Combining estimates (3.24)–(3.25) with (3.23), we obtain

$$\begin{aligned}\|F_2(u_1) - F_2(u_2)\|_{L^5(\Omega)} &\leq CT^{1-\frac{d}{6}} \sup_{s\in[0,T]} \left(\|N_1 - N_2\|_{L^4(\Omega)} + \|(\phi_1 - \phi_2)\|_{L^5(\Omega)}\right) \\ &\leq CT^{1-\frac{d}{6}} \|u_1 - u_2\|_{X_T}.\end{aligned} \tag{3.26}$$

Finally, we bound $\nabla(F(u_1) - F(u_2))$ in $L^5(\Omega)$. From (3.3) and Lemma 3.1, we derive

$$\begin{aligned}\|\nabla(F_2(u_1) - F_2(u_2))\|_{L^5(\Omega)} &\leq C\int_0^t (t-s)^{-\frac{1}{2}-\frac{d}{2}(\frac{1}{2}-\frac{1}{5})} \left\|W'(\phi_1) - W'(\phi_2)\right\|_{L^2(\Omega)} ds \\ &+C\int_0^t (t-s)^{-\frac{1}{2}-\frac{d}{2}(\frac{9}{20}-\frac{1}{5})} \left\|\frac{N_1}{\phi_1+\sigma} - \frac{N_2}{\phi_2+\sigma}\right\|_{L^{20/9}(\Omega)} ds.\end{aligned} \tag{3.27}$$

The second term can be bounded from (3.25). To bound the first term, we note that

$$\|W'(\phi_1) - W'(\phi_2)\|_{L^2} \leq \|W''(\mu\phi_1 + (1-\mu)\phi_2)\|_{L^{10/3}(\Omega)} \|\phi_1 - \phi_2\|_{L^5(\Omega)} \tag{3.28}$$

Using the assumptions on W in $(\mathbf{H}_W)$ we have

$$\|W''(\mu\phi_1 + (1-\mu)\phi_2)\|_{L^{10/3}(\Omega)} \leq C\|\phi_1\|^2_{L^{20/3}(\Omega)} + C\|\phi_2\|^2_{L^{20/3}(\Omega)} + C. \tag{3.29}$$

The right-hand side is uniformly bounded for $u_k \in B_{2R} \subset X_T$ from the embedding $L^{\frac{20}{3}}(\Omega) \subset W^{1,3}(\Omega)$. Combining these with (3.25) with (3.27) yields

$$\|\nabla(F_2(u_1) - F_2(u_2))\|_{L^5(\Omega)} \leq CT^{\frac{10-3d}{20}} \|u_1 - u_2\|_{X_T}. \tag{3.30}$$

Taking the supremum of the estimates (3.22), (3.26) and (3.30), for $t \in (0, T)$ yields

$$\|F(u_1) - F(u_2)\|_{X_T} \leq CT^{\frac{10-3d}{20}} \|u_1 - u_2\|_{X_T}.$$

The lemma follows provided for $T \in [0, T_0]$ if T_0 is small enough depending on σ, R, and the domain Ω.

□

We are in position to establish the existence of a fixed point to the mapping F in the iteration space X_T.

Proposition 3.4. *Suppose $u_0 = (N_0, \phi_0)$ lies in the functional space $X_0 = L^4(\Omega) \times W^{1,5}(\Omega)$ and $\phi_0 > 0$ in Ω. Then, there exists $T > 0$ depending only on the regularizing parameter σ, the initial data in X_0 and domain Ω, such that F possesses a fixed point $u \in X_T$.*

Proof. Since $u_0 = (N_0, \phi_0)$ lies in the functional space $X_0 := L^4(\Omega) \times W^{1,5}(\Omega)$ and $\phi_0 > 0$ in Ω, there exists $R > 0$ such that

$$\|e^{t\Delta}\phi_0\|_{X_T} \leq R, \qquad \forall T \in [0, 1].$$

With this R, it remains to show that there exists a uniform constant $T_0 > 0$ such that $F_2(u) > -\sigma/2$ for any $u \in X_T$ and any $T \in (0, T_0]$. This verifies the lower bound requirement of Lemmas 3.2–3.3, and the contraction mapping principle and the proposition follow.

From the maximum principle for the heat equation on a periodic domain, we know $e^{t\Delta}\phi_0 > 0$ for $\phi_0 > 0$. Instead of obtaining a lower bound of F_2 directly, it suffices to estimate the difference between $e^{t\Delta}\phi_0$ and F_2 in $L^\infty(\Omega)$. We claim there exists T_0 depending on σ, Ω and the radius R such that

$$\|F_2(u) - e^{t\Delta}\phi_0\|_{L^\infty(\Omega)} \leq \sigma/2, \qquad t \in [0, T_0]. \tag{3.31}$$

If so, then since $e^{t\Delta}\phi_0 > 0$ we may deduce from the triangle inequality that $F_2(u)$ is bigger than $-\sigma/2$. The proposition follows.

To prove the claim (3.31), it is sufficient to estimate the difference in $W^{1,5}(\Omega)$ due to the following Gagliardo–Nirenberg inequality:

$$\|f\|_{L^\infty(\Omega)} \le C\|\nabla f\|_{L^5(\Omega)}^{\frac{d}{5}}\|f\|_{L^5(\Omega)}^{1-\frac{d}{5}} + \|f\|_{L^5(\Omega)}. \tag{3.32}$$

Recalling that $F_2(0) = e^{t\Delta}\phi_0$, we write

$$F_2(u) - e^{t\Delta}\phi_0 = F_2(u) - F_2(0).$$

It follows directly from estimates (3.26) and (3.30) that

$$\|\nabla(F_2(u) - e^{t\Delta}\phi_0)\|_{L^5(\Omega)} + \|F_2(u) - e^{t\Delta}\phi_0\|_{L^5(\Omega)} \le CT^{\frac{10-3d}{20}}\|u\|_{X_T} \qquad \forall t \in [0,T]. \tag{3.33}$$

The claim (3.31) follows and the proof is complete. □

Lemma 3.5. (Maximum principle) *Suppose initial data ϕ_0, N_0 are strictly positive, (ϕ, N) is a classical solution of the flow (1.1)–(1.2) on $\Omega \times [0,T]$ and satisfies $\phi > -\sigma/2$ a priori, then ϕ, N preserve the positivity on $\Omega \times (0,T]$, that is,*

$$\phi(t,x) > 0, N(t,x) > 0, \qquad \forall (t,x) \in \Omega \times (0,T].$$

Proof. We first address N. Since N_0 is strictly positive on the compact domain $\Omega = \bar{\Omega}$, from the smoothness and a continuity argument there exists $t_0 > 0$ such that $N(t,x) > 0$ for all $(t,x) \in [0,t_0] \times \Omega$. We only need to show that $N(t,x)$ is strictly positive for any $(t,x) \in [t_0,T] \times \Omega$. This suffices to show $N(t,\cdot) > w(t) = \varepsilon e^{-Kt}$ on $[t_0,T]$ for some positive constants $K > 0$ and small $\varepsilon > 0$. The constant ε is chosen to be small enough so that $N(t_0,\cdot) \ge \varepsilon > w(t_0) = \varepsilon e^{-Kt_0}$ on Ω. If on the contrary $N(t,\cdot) > w(t)$ does not hold true on Ω, then there exists $t_1 \in (t_0,T]$ which is the first time that $N(t_1,x_1) = w(t_1)$ for some $x_1 \in \Omega$. Moreover, $x_1 \in \Omega$ is the minimum point of N in Ω at time t_1. Particularly at (t_1,x_1), we have

$$\begin{aligned} &-\varepsilon K e^{-Kt_1} = \partial_t w(t_1) \ge \partial_t N(t_1,x_1), \qquad N(t_1,x_1) = w(t_1) = \varepsilon e^{-Kt_1}, \\ &\nabla N(t_1,x_1) = 0, \qquad\qquad \Delta N(t_1,x_1) \ge 0. \end{aligned} \tag{3.34}$$

From the flow (1.1) for N and the facts above, we deduce that

$$\begin{aligned} -\varepsilon K e^{-Kt_1} \ge \partial_t N(t_1,x_1) &= \left(\Delta N - g'(N)\nabla N \cdot \frac{\nabla\phi}{\phi+\sigma} - g(N)\nabla\cdot\left(\frac{\nabla\phi}{\phi+\sigma}\right)\right)(t_1,x_1) \\ &\ge -g(N)\nabla\cdot\left(\frac{\nabla\phi}{\phi+\sigma}\right)(t_1,x_1) \end{aligned} \tag{3.35}$$

Since $g(N) = N$ for $N \ll 1$, we have $g(N)(t_1,x_1) = w(t_1) = \varepsilon e^{-Kt_1}$ and the inequality becomes

$$K \le \nabla\cdot\left(\frac{\nabla\phi}{\phi+\sigma}\right)(t_1,x_1). \tag{3.36}$$

Since ϕ is $C^2(\Omega)$ for $t > 0$ and hence bounded on any compact domain of $\Omega \times [t_0,T]$, we can choose K large enough depending on the values of ϕ and its derivatives at (t_1,x_1) so that a contradiction arises in the inequality above. This implies that $N(t,x)$ cannot attain $w(t) = \varepsilon e^{-Kt}$ at any time $t \in [t_0,T]$, and hence $N(t) > w(t) > 0$ for $t \in [t_0,T]$ which implies $N > 0$ on $[0,T]$.

Similarly, we show the positivity of ϕ. Since ϕ_0 is strictly positive on a compact domain Ω, by a continuity argument there exists some $t_0 > 0$ such that for all that $t \in [0,t_0]$ it holds $\phi(t,\cdot) > 0$ on Ω. It remains to show that $\phi(t,\cdot)(\in C^\infty)$ is strictly positive for $t \in [t_0,T]$ on Ω. This is sufficient to demonstrate that $\phi(t,\cdot) \ge v(t) = \varepsilon e^{-Kt}$ on $[t_0,T]$ for some $K > 0$ and for $\varepsilon > 0$ small enough that $\phi(t_0,\cdot) \ge \varepsilon > \varepsilon e^{-Kt_0}$ on Ω. Suppose to the contrary that $\phi(t,\dot{)} > v(t)$ does not hold on Ω, then there exists a first time $t_1 \in (t_0,T]$ such that $\phi(t_1,x_1) = v(t_1)$ at a minimizing point $x_1 \in \Omega$. Particularly at (t_1,x_1), we have

$$\phi(t_1,x_1) = v(t_1) = \varepsilon e^{-Kt_1}, \qquad -\varepsilon K e^{-Kt_1} = \partial_t v(t_1) \ge \partial_t\phi(t_1,x_1), \qquad \Delta\phi(t_1,x_1) \ge 0. \tag{3.37}$$

Hence, we derive

$$\begin{aligned} -\varepsilon K e^{-Kt_1} \geq \partial_t \phi(t_1, x_1) &= \left(\Delta\phi - W'(\phi) + \frac{N}{\phi + \sigma} \right)(t_1, x_1) \\ &\geq -W'(\phi) + \frac{N}{\phi + \sigma}. \end{aligned} \tag{3.38}$$

Since N is positive and $\phi(t_1, x_1) = \varepsilon e^{-Kt_1} > 0$, we have

$$-\varepsilon K e^{-Kt_1} \geq \partial_t \phi(t_1, x_1) \geq -W'(\phi). \tag{3.39}$$

From the assumption $(\mathbf{H}_W)_{(c)}$ $W'(\phi) \leq \alpha_0 \phi$ at $\phi = v(t_1) = \varepsilon e^{-Kt_1} > 0$ for ε small enough, a contradiction arises if we choose $K > \alpha_0$ and hence $\phi(t, x) \geq \varepsilon e^{-Kt} > 0$ for all $(t, x) \in [0, T] \times \Omega$. The lemma follows. □

The following two lemmas are required to establish the regularity of local solutions away from the initial time.

Lemma 3.6. *Let $T > 0, q \geq r > 1$ be some given positive constants, and A be the positive operator $-\Delta + 1$. Suppose $f = (f_1, f_2) \in L^\infty(0, T; L^q(\Omega)), v_0 \in L^r(\Omega)$ and v solves the initial value problem*

$$\partial_t v - \Delta v = f, \qquad v(0, x) = v_0(x) \tag{3.40}$$

on a bounded periodic domain, then for any $\tau > 0$ and $t \in [\tau, T]$ there exists a positive constant C depending on τ and T such that

$$\|A^\alpha v(\cdot, t)\|_{L^p(\Omega)} \leq C + C \sup_{s \in [0,T]} \|f(\cdot, \tau)\|_{L^q(\Omega)}, \tag{3.41}$$

for any $\alpha \in \left[0, 1 + \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)\right)$. In particular, the estimate holds if $\alpha = 0$ and $p \in [q, \infty)$; or if $\alpha = \frac{1}{2}$ and (p, q) satisfies

(a). $p \in [q, \infty)$ for $q \geq d$;

(b). $p \in \left[q, \frac{dq}{d - q}\right)$ for $q < d$.

Proof. Applying variation of parameters to the heat semigroup $e^{t\Delta}$ yields

$$v(t, x) = e^{t\Delta} v_0 + \int_0^t e^{(t-s)\Delta} f \, ds. \tag{3.42}$$

Applying the $L^p - L^q$ estimates we deduce that

$$\begin{aligned} \|A^\alpha v\|_{L^p(\Omega)} &\leq \|A^\alpha e^{t\Delta} v_0\|_{L^p(\Omega)} + \int_0^t \|A^\alpha e^{(t-s)\Delta} f\|_{L^p(\Omega)} \, ds \\ &\leq \frac{C\|v_0\|_{L^r(\Omega)}}{t^{\frac{d}{2}\left(\frac{1}{r} - \frac{1}{p}\right) + \alpha}} + C \sup_{s \in [0,T]} \|f\|_{L^q(\Omega)} \int_0^t (t - s)^{-\alpha - \frac{d}{2}\left(\frac{1}{q} - \frac{1}{p}\right)} \, ds. \end{aligned} \tag{3.43}$$

The last integral is bounded for T bounded if $\alpha < 1 + \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$, the Lemma follows. □

Lemma 3.7. *Let $T > 0$, and $r, q > 1$ be positive constants, and A be the positive operator $-\Delta + 1$. Suppose $f = (f_1, f_2) \in L^\infty(0, T; L^q(\Omega)), v_0 \in L^r(\Omega)$ and v solves the initial value problem of the heat equation with a divergence form inhomogeneity*

$$\partial_t v - \Delta v = \nabla \cdot f, \qquad v(0, x) = v_0(x) \tag{3.44}$$

on a bounded periodic domain. Then for any $\tau > 0$ and $t \in [\tau, T]$ there exists a positive constant C depending on τ and T such that

$$\|A^\alpha v(\cdot,t)\|_{L^p} \le C + C \sup_{s\in[0,T]} \|f(\cdot,\tau)\|_{L^q(\Omega)}, \tag{3.45}$$

for any (p,α) that satisfy $1 < q \le p < \infty$, $1 < r \le p$ and $\alpha \in \left[0, \frac{1}{2} + \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)\right)$. In particular, this inequality holds for any $p \in [q, \infty)$ if $q \ge d$ and $\alpha = 0$.

Proof. Applying variation of parameters to the heat semigroup $e^{t\Delta}$, we have

$$v(t,x) = e^{t\Delta} v_0 + \int_0^t e^{(t-s)\Delta} \nabla \cdot f \, ds. \tag{3.46}$$

The L^p-L^q estimates imply that

$$\begin{aligned} \|A^\alpha v\|_{L^p(\Omega)}(t) &\le \|A^\alpha e^{t\Delta} v_0\|_{L^p(\Omega)} + \int_0^t \|A^\alpha e^{(t-s)\Delta} \nabla \cdot f\|_{L^p(\Omega)} \, ds \\ &\le \frac{C\|v_0\|_{L^r(\Omega)}}{t^{\frac{d}{2}\left(\frac{1}{r}-\frac{1}{p}\right)+\alpha}} + C \sup_{s\in[0,T]} \|f\|_{L^p(\Omega)} \int_0^t (t-s)^{-\alpha - \frac{1}{2} - \frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \, ds. \end{aligned} \tag{3.47}$$

The last integral is bounded for T bounded if $\alpha < \frac{1}{2} + \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$, the Lemma follows. □

Theorem 3.8. *(Local existence) Given initial data $u_0 = (N_0, \phi_0) \in X_0$ satisfying $\phi_0, N_0 > 0$, then there exists a constant $T > 0$ such that on the domain $\Omega \times [0,T]$ there exists a unique positive solution $u = (N,\phi) \in X_T$ to the flow (1.1). Moreover, the solution is smooth away from initial time. That is, for any $\tau > 0$ there exists a positive constant C depending on $\tau, T, k, \Omega, \sigma$ such that*

$$\|u\|_{C^k(\Omega\times[\tau,T])} \le C(\tau, T, k, \Omega, \sigma). \tag{3.48}$$

Proof. The existence of solution $u = (N,\phi)$ in X_T is a direct result of Proposition 3.4. It remains to show the smoothness of $u = (N,\phi)$ on $\Omega \times (0,T]$ for which the positivity of N, ϕ follows from the Maximum Principle as established in Lemma 3.5.

Since $u = (N,\phi)$ lies in X_T, we have $N \in L^\infty(0,T; L^4(\Omega))$ and $\phi \in L^\infty(0,T; W^{1,5}(\Omega))$. For $d = 2,3$, Morrey's inequality implies $\phi(t) \in C^\alpha(\Omega)$ for some $\alpha > 0$. Therefore, from assumption $(\mathbf{H}_W)$ we have $W'(\phi) \in C^\alpha(\Omega)$. Taking the ϕ equation from (1.1), we apply Lemma 3.6 (b) with $\alpha = 1/2, q = 4$ and initial time at $t = \tau$. This yields that for any $\tau' > \tau$, $p \in [4,\infty)$ and all $t \in [\tau', T]$

$$\begin{aligned} \|\nabla\phi(t)\|_{L^p(\Omega)} &\le C\|A^{1/2}\phi(t)\|_{L^p(\Omega)} \\ &\le C_{\tau'} + C_{\tau'} \sup_{s\in[0,T]} \left\| -W'(\phi) + \frac{N}{\phi+\sigma} \right\|_{L^4(\Omega)}. \end{aligned} \tag{3.49}$$

We emphasize this holds for any $p \in [4,\infty)$ or any $p < \infty$ since Ω is bounded and hence $\nabla\phi \in L^\infty(\tau', T; L^p(\Omega))$ for any $\tau' > \tau$ and $p < \infty$.

We turn to the flow of N in (1.1). Since $N \in L^\infty(0,T; L^4(\Omega))$ and $\nabla\phi \in L^\infty(\tau', T; L^p(\Omega))$ for any $p < \infty$, from Young's inequality we have for any $q < 4$

$$\frac{N}{\phi+\sigma} \nabla\phi \in L^\infty(\tau', T; L^q(\Omega)). \tag{3.50}$$

On the other hand, taking the flow of N in (1.1) we apply Lemma 3.7 with $p \geq q, q < 4$ and derive

$$\|A^\alpha N\|_{L^p(\Omega)}(t) \leq C_{\tau'} + C_{\tau'} \sup_{s\in[\tau',T]} \left\| \frac{N}{\phi+\sigma}\nabla\phi \right\|_{L^q(\Omega)} \tag{3.51}$$

for all $t \in [\tau', T]$, $\alpha \in \left[0, \frac{1}{2} + \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)\right)$. We claim this inequality implies that $N(\cdot,t) \in C^\beta(\Omega)$ for some $\beta > 0$ and any $t \in [\tau', T]$. In fact, from the definition of the operator A the inequality above implies that $N \in W^{2\alpha,p}(\Omega)$ for $\alpha < \frac{1}{2} + \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$. In particular, these conditions are satisfied for $\alpha = \frac{5}{12}, p = 4, q = \frac{7}{2}$. Hence, $2\alpha p = \frac{10}{3} > d$ and by Morrey's inequality $N(\cdot,t) \in C^\beta(\Omega)$ for some $\beta > 0$ and $t \in [\tau', T]$. Since $\phi(\cdot,t) \in C^\beta(\Omega)$ for some $\beta > 0$ and all $t \in [0,T]$, we have $N(\cdot,t), \phi(\cdot,t) \in C^\beta(\Omega)$ for any $t \in [\tau', T]$. Furthermore this implies that $\frac{N}{\phi+\sigma}(\cdot,t)$ belongs to $C^\beta(\Omega)$ for any $t \in [\tau', T]$. Applying the parabolic Schauder estimates [6] to the ϕ equation, then a standard bootstrap argument establishes the C^k-smoothness of the solution away from initial time. The proof is complete. □

4. Global existence

We complete Theorem 1.1 by establishing the global existence of solutions to (1.1) for initial data $u_0 = (N_0, \phi_0)$ in energy space, that is, $N_0 \in L^2(\Omega), \phi_0 \in W^{1,2}(\Omega)$.

Proof of Theorem 1.1. For simplicity, we consider W to be independent of N. The general case, with W subject to assumption $(\mathbf{H}_W)$, follows by trivial modification of the proof. The construction is based upon extension of the local solutions constructed later in this section. First in view of Proposition 4.2, there exists a local solution $u = (N, \phi)$ which is smooth away from initial time, and both components are positive. The existence time of this solution depends on the energy space norm of the initial data $(N_0, \phi_0) \in L^2(\Omega) \times W^{1,2}(\Omega)$. A uniform bound on (N, ϕ) will allow the local solution to be extended globally in time. But this uniform bound is a direct result of the basic energy inequality in Lemma 2.1 and energy lower bound in 2.2. Indeed, we have the uniform a priori estimate:

$$\|\phi\|_{W^{1,2}(\Omega)}(t) + \|\phi\|_{L^6(\Omega)}(t) + \|N\|_{L^2(\Omega)}(t) \leq C, \qquad \forall t > 0.$$

Thus, the local solution can be extended inductively from $[0,T]$ to any $[nT, (n+1)T]$ for any natural number n. This establishes the global existence and regularity. It remains to bound ϕ uniformly on $[\tau,\infty) \times \Omega$ for any $\tau > 0$, this directly follows from the continuous embedding $L^\infty(\Omega) \subset W^{1,5}(\Omega)$ and uniform bounds of $\phi(t)$ in $W^{1,5}(\Omega)$ for $t \geq \tau$ afforded by Lemma 4.3. □

The main ingredient in the proof above is the local existence for initial data in energy space. This is established in Proposition 4.2. In preparation, we gather some needed estimates. For $u(t) = (N, \phi)(t)$, we introduce

$$\mathrm{B}(t) = \mathrm{B}[u](t) := \sup_{s\in[0,T]} \left(s^{\frac{d}{8}} \|N\|_{L^4(\Omega)}(s) + s^{\frac{3d}{20}} \|\nabla\phi\|_{L^5(\Omega)}(s) + \|\phi\|_{L^5(\Omega)}(s) \right). \tag{4.1}$$

Given initial data in energy space, we show that $\mathrm{B}(t)$ is bounded locally in time. The proof employs the following useful integral formula which results from a change of variable:

$$\int_0^t s^{-\alpha}(t-s)^{-\beta}\, ds = t^{1-(\alpha+\beta)} \int_0^1 s^{-\alpha}(1-s)^{-\beta}\, ds. \tag{4.2}$$

Crucially the integral on the right-hand side is bounded for $\alpha, \beta \in (0,1)$ and

$$\int_0^t s^{-\alpha}(t-s)^{-\beta}\, ds \leq C t^{1-(\alpha+\beta)}. \tag{4.3}$$

Lemma 4.1. *There exists a positive constant $t_0 > 0$ depending on Ω, σ and initial data $u_0 \in L^2(\Omega) \times W^{1,2}(\Omega)$ such that for all $t \in (0, t_0)$*

$$\mathrm{B}(t) \leq C \left(\|N_0\|_{L^2(\Omega)} + \|\phi_0\|_{W^{1,2}(\Omega)} + 1 \right).$$

Here, C is a positive constant depending on Ω and σ only.

Proof. From the identity (3.3) with $F_1 = N$ and $g(N) \lesssim 1$, we derive

$$\|N\|_{L^4(\Omega)}(t) \leq \|e^{t_0 \Delta} N_0\|_{L^4(\Omega)} + C \int_0^t (t-s)^{-\frac{1}{2}} \|\nabla \phi\|_{L^4(\Omega)}(s)\, ds. \tag{4.4}$$

The first term is estimated from the third inequality of Lemma 3.1 with $p = 4, q = 2$. From the definition of $\mathcal{A}(t)$ in (4.1) and embedding $L^4(\Omega) \subset L^5(\Omega)$, we see that $\|\nabla \phi\|_{L^4(\Omega)}(s) \leq \|\nabla \phi\|_{L^5(\Omega)}(s) \leq s^{-\frac{3d}{20}} \mathrm{A}(t)$, and hence for $t \leq 1$

$$\|N\|_{L^4(\Omega)}(t) \leq t^{-\frac{d}{8}} \|N_0\|_{L^2(\Omega)} + C\mathrm{B}(t) \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{3d}{20}}\, ds. \tag{4.5}$$

Applying the integral estimate (4.3) implies

$$\|N\|_{L^4(\Omega)}(t) \leq Ct^{-\frac{d}{8}} \|N_0\|_{L^2(\Omega)} + Ct^{\frac{10-3d}{20}} \mathrm{B}(t). \tag{4.6}$$

Similarly from the second identity in (3.3) with $F_2 = \phi$, we estimate

$$\|\phi\|_{L^5}(t) \leq C\|\phi_0\|_{L^5(\Omega)} + C \int_0^t (t-s)^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{5})} \left(\|W'(\phi)\|_{L^2(\Omega)} + \|N\|_{L^2(\Omega)} \right)(s)\, ds \tag{4.7}$$

From the basic energy estimate in Lemma 2.1 and the energy lower bound in Lemma 2.2, we find that (N, ϕ) is uniformly bounded in $L^2(\Omega) \times W^{1,2}(\Omega)$. From the embedding $L^6(\Omega) \subset W^{1,2}(\Omega)$, we deduce that $\|W'(\phi)\| \leq C\|\phi\|_{L^6} + C$ is bounded uniformly in time. A second application of the integral estimate (4.3) yields

$$\|\phi\|_{L^5(\Omega)}(t) \leq C\|\phi_0\|_{L^5(\Omega)} + Ct^{1-\frac{3d}{20}}. \tag{4.8}$$

The second identity in (3.3) with $F_2 = \phi$, the second $L^p - Lq$ inequality in Lemma 3.1, and the a priori estimate $\phi > 0$ collectively yield

$$\|\nabla\phi\|_{L^5(\Omega)}(t) \leq Ct^{-\frac{3d}{20}} \|\nabla \phi_0\|_{L^2(\Omega)} + C \int_0^t (t-s)^{-\frac{1}{2}-\frac{d}{2}(\frac{1}{2}-\frac{1}{5})} \left(\|W'(\phi)\|_{L^2(\Omega)} + \|N\|_{L^2(\Omega)} \right)(s)\, ds.$$

Following the derivation of (4.8), we may use the uniform bounded on $\|W'(\phi)\|_{L^2(\Omega)}$ and $\|N\|_{L^2(\Omega)}$, and the integral estimate (4.3) to obtain

$$\|\nabla\phi\|_{L^5(\Omega)}(t) \leq Ct^{-\frac{3d}{20}} \|\nabla\phi_0\|_{L^2(\Omega)} + Ct^{\frac{1}{2}-\frac{3d}{20}}. \tag{4.9}$$

Combining the estimates (4.6),(4.8) and (4.9) with the definition of $\mathcal{A}$ in (4.1) yields for $t \leq 1$

$$\mathrm{B}(t) \leq C \left(\|N_0\|_{L^2(\Omega)} + \|\phi_0\|_{L^5(\Omega)} + \|\nabla\phi_0\|_{L^2(\Omega)} \right) + Ct^{\frac{10-3d}{20}} \left(\mathrm{B}(t) + 1 \right). \tag{4.10}$$

The lemma follows from the embedding $L^5(\Omega) \subset W^{1,2}(\Omega)$ for $t \in (0, t_0)$ provided with t_0 is small enough. □

In the following proposition, we apply an approximation argument to establish local existence to solutions with initial data in the energy space only. This is based on the local existence of solutions established in Theorem 3.8 subject to initial data in smoother space X_0.

Proposition 4.2. *Let Ω be a periodic domain. Suppose $u_0 = (N_0, \phi_0)$ is positive and has finite energy, i.e., $\mathrm{E}(N_0, \phi_0) < \infty, \phi_0 > 0, N_0 > 0$. Then, there exists $T > 0$, depending on initial energy, such that on the domain $\Omega \times (0,T)$ the flow (1.1) has a solution $u = (N, \phi)$ satisfying the initial data in the sense of*

$$\lim_{t\to 0^+} \left(\|\phi(t) - \phi_0\|_{W^{1,2}(\Omega)} + \|N(t) - N_0\|_{L^2(\Omega)} \right) = 0.$$

Moreover, the solution is smooth on $\Omega \times (0,T)$.

Proof. For $(N_0, \phi_0) \in L^2(\Omega) \times W^{1,2}(\Omega)$ by Lemma 2.2, there exists an approximating sequence $\{(N_{k,0}, \phi_{k,0}), k = 1, 2 \cdots \} \subset X_0$ (for instance see (3.2)) such that

$$\lim_{k\to\infty} \|N_{k,0} - N_0\|_{L^2(\Omega)} = 0, \qquad \lim_{k\to\infty} \|\phi_{k,0} - \phi_0\|_{W^{1,2}(\Omega)} = 0 \tag{4.11}$$

For each initial data $(N_{k,0}, \phi_{k,0}) \in X_0 = L^4(\Omega) \times W^{1,5}(\Omega)$, by Theorem 3.8 there exists a $T_k > 0$ and a function $u_k = (N_k, \phi_k)$ which solves the flow (1.1) on $\Omega \times (0, T_k)$. Significantly, the value T_k depends on σ, Ω and the choice of the initial data in X_0. To iteratively extend these solutions we seek a lower bound on T_k in terms of norm of $u_k(t)$ in X_0.

The norm convergence in (4.11) to $(N_0, \phi_0) \in L^2(\Omega) \times W^{1,2}(\Omega)$ implies the sequence $\{(N_{k,0}, \phi_{k,0})\}_k$ is uniformly bounded in $L^2(\Omega) \times W^{1,2}(\Omega)$. Combined with the energy identity of Lemma 2.1 and energy lower bound in Lemma 2.2, we deduce the existence of a positive constant C independent of k such that

$$\|N_k\|_{L^2(\Omega)}(t) + \|\phi_k\|_{W^{1,2}(\Omega)}(t) \le C, \qquad \forall t \in [0, T_k). \tag{4.12}$$

Moreover, since (N_k, ϕ_k) is a classical solution to the flow (1.1) on $\Omega \times (0, T_k)$, Lemmas 4.3 and 4.4 apply, yielding a uniform bound on $(N_k, \phi_k)(t) \in X_0$ on any compact subset of $(0, T_k)$. That is, for each $\tau_k \in (0, T_k)$ there exists a constant C_{τ_k} such that

$$\|N_k\|_{L^4(\Omega)}(t) + \|\nabla\phi_k\|_{L^5(\Omega)}(t) \le C_{\tau_k}, \qquad \forall t \in [\tau_k, T_k). \tag{4.13}$$

From Theorem 3.8 we deduce the existence of a uniform lower bound, denoted by $T > 0$, of $\{T_k\}$.

To deduce the convergence we apply Arzelá–Ascoli theorem. There exists $u = (N, \phi)$ such that as k tends to infinity $u_k = (N_k, \phi_k)$ and its derivatives converge uniformly to u and corresponding derivatives on any compact set of $\Omega \times (0,T)$. Moreover, the limit $u = (N, \phi)$ solves the flow (1.1) and is smooth on $\Omega \times (0,T)$.

It remains to verify the initial condition. From Fatou's lemma, we have

$$\begin{aligned} &\lim_{t\to 0^+} \left(\|N(t) - N_0\|_{L^2(\Omega)} + \|\phi(t) - \phi_0\|_{W^{1,2}(\Omega)} \right) \\ &\le \lim_{t\to 0^+} \liminf_{k\to\infty} \left(\|N_k(t) - N_{k,0}\|_{L^2(\Omega)} + \|\phi_k(t) - \phi_{k,0}\|_{W^{1,2}(\Omega)} \right). \end{aligned} \tag{4.14}$$

The heat kernel has the property that $e^{t\Delta}\varphi \to \varphi$ in $L^p(\Omega)$ as $t \to 0^+$ if $\varphi \in L^p(\Omega)$. In light of the uniform bound (4.12), the following convergence is uniform in k:

$$\lim_{t\to 0^+} \left(\|e^{t\Delta} N_{k,0} - N_{k,0}\|_{L^2(\Omega)} + \|e^{t\Delta}\phi_{k,0} - \phi_{k,0}\|_{W^{1,2}(\Omega)} \right) = 0. \tag{4.15}$$

Combining (4.14) and (4.15), the triangle inequality implies that

$$\begin{aligned} &\lim_{t\to 0^+} \left(\|N(t) - N_0\|_{L^2(\Omega)} + \|\phi(t) - \phi_0\|_{W^{1,2}(\Omega)} \right) \\ &\le \lim_{t\to 0^+} \liminf_{k\to\infty} \left(\|N_k(t) - e^{t\Delta} N_{k,0}\|_{L^2(\Omega)} + \|\phi_k(t) - e^{t\Delta}\phi_{k,0}\|_{W^{1,2}(\Omega)} \right). \end{aligned} \tag{4.16}$$

From Lemma 4.1, we may deduce that the right-hand side of this inequality is zero. To this end, it is helpful to introduce $\mathrm{B}_k(t) := \mathrm{B}[u_k]$, where B is from (4.1), and $u_k = (N_k, \phi_k)$. The uniform bound from (4.12) and Lemma 4.1 imply for T small enough

$$\mathrm{B}_k(t) \le C, \qquad \forall t \in [0, T). \tag{4.17}$$

Here and below, C denotes a constant independent of k. Variation of parameters applied to the heat semigroup yields expression

$$N_k(t) = e^{t\Delta} N_{k,0} - \int_0^t e^{(t-s)\Delta} \nabla \cdot \left(\frac{g(N_k)}{\phi_k + \sigma} \nabla \phi_k \right) ds. \tag{4.18}$$

By Lemma 3.1 with $p = q = 2$, since $|g(N)| \leq 1$, we bound the difference as

$$\begin{aligned} \|N_k(t) - e^{t\Delta} N_{k,0}\|_{L^2(\Omega)} &\leq \int_0^t \left\| e^{(t-s)\Delta} \nabla \cdot \left(\frac{g(N_k)}{\phi_k + \sigma} \nabla \phi_k \right) \right\|_{L^2(\Omega)} ds \\ &\leq C \int_0^t (t-s)^{-\frac{1}{2}} \|g(N_k) \nabla \phi_k\|_{L^2(\Omega)}\, ds \\ &\leq C \int_0^t (t-s)^{-\frac{1}{2}} \|\nabla \phi_k\|_{L^5(\Omega)}\, ds. \end{aligned} \tag{4.19}$$

Applying (4.17) and definition of $\mathrm{B}_k(t)$ we have

$$\begin{aligned} \|N_k(t) - e^{t\Delta} N_{k,0}\|_{L^2(\Omega)} &\leq C \mathrm{B}_k(t) \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{d}{8}}\, ds, \\ &\leq C \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{d}{8}}\, ds. \end{aligned} \tag{4.20}$$

The integral estimate (4.3) with $\alpha = \frac{1}{2}, \beta = \frac{d}{8}$ yields

$$\|N_k(t) - e^{t\Delta} N_{k,0}\|_{L^2(\Omega)}(t) \leq C t^{\frac{4-d}{8}}. \tag{4.21}$$

To estimate ϕ_k we return to the variation of parameters expression

$$\phi_k = e^{t\Delta} \phi_{k,0} - \int_0^t e^{(t-s)\Delta} \left(W'(\phi_k) - \frac{N_k}{\phi_k + \sigma} \right) ds. \tag{4.22}$$

To bound the difference $\phi_k - e^{t\Delta} \phi_{k,0}$ in $W^{1,2}(\Omega)$ we apply Lemma 3.1, and use the estimates on $W'(\phi)$ and uniform bound on $\|N_k\|_{L^2(\Omega)}$ from (4.12). These estimates yield the bound

$$\begin{aligned} \|\phi_k - e^{t\Delta} \phi_{k,0}\|_{L^2(\Omega)} &\leq \int_0^t \left\| W'(\phi_k) - \frac{N_k}{\phi_k + \sigma} \right\|_{L^2(\Omega)} ds \\ &\leq C \int_0^t (1 + \|\phi_k\|_{L^6(\Omega)} + \|N_k\|_{L^2(\Omega)})\, ds. \end{aligned} \tag{4.23}$$

Since $L^6(\Omega) \subset W^{1,2}(\Omega)$ for $\Omega \subset \mathbb{R}^d (d = 2, 3)$ and $\|N_k\|_{L^2}, \|\phi\|_{W^{1,2}(\Omega)}$ are uniformly bounded by (4.12) we have

$$\|\phi_k - e^{t\Delta} \phi_{k,0}\|_{L^2(\Omega)} \leq Ct. \tag{4.24}$$

Finally, similar steps lead to the estimate

$$\|\nabla\left(\phi_k - e^{t\Delta}\phi_{k,0}\right)\|_{L^2(\Omega)} \le \int_0^t (t-s)^{-\frac{1}{2}} \left\|W'(\phi_k) - \frac{N_k}{\phi_k+\sigma}\right\|_{L^2(\Omega)} ds \le Ct^{\frac{1}{2}}. \tag{4.25}$$

Combining estimates (4.21), (4.24)–(4.25) yields

$$\lim_{t\to 0^+} \left(\|N_k(t) - e^{t\Delta}N_{k,0}\|_{L^2(\Omega)} + \|\phi_k(t) - e^{t\Delta}\phi_{k,0}\|_{W^{1,2}(\Omega)}\right) = 0$$

uniformly in k. The initial condition is satisfied from (4.16) and the Proposition follows. □

In the following two lemmas, we establish the uniform boundedness of $\nabla\phi$ in $L^5(\Omega)$ and N in $L^4(\Omega)$ away from initial time, which has been used to establish a lower bound of existence time interval in the proof of Proposition 4.2. We mention similar boundedness is also established in the chemotaxis case, see [10] for instance.

Lemma 4.3. *Let (N,ϕ) be a classical positive solution to (1.1) on $\Omega\times(0,T)$ for some $T>0$. Then the gradient $\nabla\phi$ is uniformly bounded in $L^5(\Omega)$ away from initial time. More precisely, for any $\tau\in(0,T)$ there exists some finite constant C_τ depending on τ such that*

$$\|\nabla\phi\|_{L^5(\Omega)}(t) \le C_\tau, \qquad \forall t\in[\tau,T).$$

Particularly, the constant C_τ is independent of T.

Proof. Applying variation of parameters to the evolution (1.1) for ϕ, we have

$$\phi = e^{t\Delta}\phi_0 - \int_0^t e^{(t-s)\Delta}\left(W'(\phi) - \frac{N}{\phi+\sigma}\right) ds. \tag{4.26}$$

Let $A = -\Delta + 1$, then for $\beta > 0$

$$\begin{aligned} A^\beta\phi &= A^\beta e^{t\Delta}\phi_0 - A^\beta \int_0^t e^{(t-s)\Delta}\left(W'(\phi) - \frac{N}{\phi+\sigma}\right) ds, \\ &= A^\beta e^{t\Delta}\phi_0 - \int_0^t A^\beta e^{(t-s)\Delta}\left(W'(\phi) - \frac{N}{\phi+\sigma}\right) ds, \end{aligned} \tag{4.27}$$

from which we deduce the bound

$$\|A^\beta\phi\|_{L^2} \le C\left(\frac{e^{-\lambda t}}{t^\beta}\|\phi_0\|_{L^2(\Omega)} + \int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^\beta}\|W'(\phi)\|_{L^2(\Omega)}\, ds + \int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^\beta}\|N\|_{L^2(\Omega)}\, ds\right),$$

for some C and $\lambda>0$. From the global energy identity and embedding estimate, we have uniform bounds on $\phi\in L^6(\Omega), N\in L^2(\Omega)$ for all time. In particular, from (3.13) we have that $W'(\phi)$ is bounded in $L^2(\Omega)$ uniformly in time. These bounds afford the estimate

$$\|A^\beta\phi\|_{L^2(\Omega)} \le C\left(\frac{e^{-\lambda t}}{t^\beta} + \int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^\beta}\, ds\right). \tag{4.28}$$

For $\beta\in[0,1)$, the right-hand side is uniformly bounded on any compact set of $(0,\infty]$. Moreover, by Gagliardo-Nirenberg inequality we have for $\beta > \frac{19}{20}$ that

$$\|\nabla\phi\|_{L^5(\Omega)} \le C\|A^\beta\phi\|_{L^2}^{\alpha}\|\nabla\phi\|_{L^2(\Omega)}^{1-\alpha} + C\|\nabla\phi\|_{L^2(\Omega)}, \qquad \alpha = \frac{3d}{10(2\beta-1)}.$$

The extra lower bound assumption on β ensures $\alpha \in (0,1)$ for spatial dimension $d = 2, 3$. The lemma follows from the two previous estimates and the uniform bound on $\nabla\phi$ in $L^2(\Omega)$. □

Lemma 4.4. *Let (N, ϕ) be a classical positive solution to (1.1) on $\Omega \times (0,T)$ for some $T > 0$. Then, N is uniformly bounded in $L^4(\Omega)$ away from initial time $t = 0$. More precisely, for any $\tau \in (0,T)$ there exists some finite constant C_τ depending on τ such that*

$$\|N(\cdot,t)\|_{L^4(\Omega)} \le C_\tau, \qquad \forall t \in [\tau, T). \tag{4.29}$$

Particularly, the constant C_τ is independent of T.

Proof. Let $w := N^2$, then

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2\,dx = 2\int_\Omega N^3 \partial_t N\,dx. \tag{4.30}$$

Replacing $\partial_t N$ by the equation of N yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2\,dx = 2\int_\Omega N^3 \Delta N\,dx - 2\int_\Omega N^3 \nabla\cdot\left(g(N)\frac{\nabla\phi}{\phi+\sigma}\right)dx.$$

Integrating by parts and using $\nabla w = 2N\nabla N$ yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega w^2\,dx &= -6\int_\Omega N^2|\nabla N|^2\,dx - 6\int_\Omega N^2 g(N)\frac{\nabla\phi}{\phi+\sigma}\cdot\nabla N\,dx\\ &= -\frac{3}{2}\int_\Omega |\nabla w|^2\,dx - 3\int_\Omega g(N)\sqrt{w}\frac{\nabla\phi}{\phi+\sigma}\cdot\nabla w\,dx.\end{aligned} \tag{4.31}$$

The hypothesis $(\mathbf{H}_f)$ implies (1.13) and hence $|g(N)| \le 1$. Since $\phi > 0$ and $\sigma > 0$ we may apply Young's inequality to the second term on the right-hand side, yielding

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2\,dx \le -\int_\Omega |\nabla w|^2\,dx + C\|\sqrt{w}\nabla\phi\|^2_{L^2(\Omega)} \tag{4.32}$$

where the constant C depends on σ. From Hölder's inequality, we have

$$\|\sqrt{w}\nabla\phi\|^2_{L^2(\Omega)} \le \|\sqrt{w}\|^2_{L^{10/3}(\Omega)}\|\nabla\phi\|^2_{L^5(\Omega)} \le \|w\|_{L^{5/3}(\Omega)}\|\nabla\phi\|^2_{L^5(\Omega)}. \tag{4.33}$$

In turn, the Gagliardo–Nirenberg inequality and uniform bound of $w = N^2 \in L^1(\Omega)$ imply

$$\begin{aligned}\|w\|_{L^{5/3}(\Omega)} &\le C\|\nabla w\|_{L^2}^{\frac{4d}{5(d+2)}}\|w\|_{L^1(\Omega)}^{\frac{d+10}{5(d+2)}} + C\|w\|^2_{L^1(\Omega)},\\ &\le C\|\nabla w\|_{L^2}^{\frac{4d}{5(d+2)}} + C,\\ &\le C\|\nabla w\|_{L^2} + C,\end{aligned} \tag{4.34}$$

where to obtain the last inequality we used Young's inequality and $\frac{4d}{5(d+2)} < 1$. Inserting the estimate above into the right-hand side of the previous inequality and applying Young's inequality, we derive

$$\begin{aligned}\|w\nabla\phi\|^2_{L^2(\Omega)} &\le C\|\nabla w\|_{L^2(\Omega)}\|\nabla\phi\|^2_{L^5(\Omega)} + C\|\nabla\phi\|^2_{L^5(\Omega)},\\ &\le \frac{1}{2}\|\nabla w\|^2_{L^2(\Omega)} + C\|\nabla\phi\|^4_{L^5(\Omega)} + C\|\nabla\phi\|^2_{L^5(\Omega)}.\end{aligned} \tag{4.35}$$

Inserting the estimate into the right-hand side of the inequality (4.32) yields

$$\frac{d}{dt}\int_\Omega w^2\,dx \le -\int_\Omega |\nabla w|^2\,dx + C\|\nabla\phi\|^4_{L^5(\Omega)} + C\|\nabla\phi\|^2_{L^5(\Omega)}. \tag{4.36}$$

Fix $\tau > 0$, we introduce τ-dependent constant

$$M_\tau := \sup_{t\geq\tau} \|\nabla\phi\|^2_{L^5(\Omega)} < \infty,$$

from which we obtain the bound

$$\frac{d}{dt}\int_\Omega w^2\,dx \leq -\int_\Omega |\nabla w|^2\,dx + CM_\tau^6 + CM_\tau. \tag{4.37}$$

Using the Gagliardo–Nirenberg inequality, Cauchy's inequality, and the uniform bound on $w = N^2 \in L^1(\Omega)$, we obtain

$$\begin{aligned}\|w\|^2_{L^2(\Omega)} &\leq C\|\nabla w\|^{\frac{2d}{d+2}}_{L^2(\Omega)}\|w\|^{\frac{4}{d+2}}_{L^1(\Omega)} + C\|w\|^2_{L^1(\Omega)},\\ &\leq \|\nabla w\|^2_{L^2(\Omega)} + C.\end{aligned} \tag{4.38}$$

We used $2d/(d+2) < 2$ to apply Young's inequality to arrive at the second inequality above. In conclusion,

$$\frac{d}{dt}\int_\Omega w^2\,dx + \int_\Omega w^2\,dx \leq K_\tau, \qquad K_\tau := CM_\tau^6 + CM_\tau + C \tag{4.39}$$

or equivalently,

$$\frac{d}{dt}\left(e^t\int_\Omega w^2\,dx\right) \leq K_\tau e^t. \tag{4.40}$$

Integrating this relation from τ to t yields

$$\begin{aligned}\int_\Omega w^2(x,t)\,dx &\leq e^{-t+\tau}\int_\Omega w^2(x,\tau)\,dx + K_\tau\\ &\leq e^{-t+\tau}\int_\Omega N^4(x,\tau)\,dx + K_\tau.\end{aligned} \tag{4.41}$$

The lemma follows from the $L^4(\Omega)$ bound of N at time $t = \tau$ by choosing $\tau > 0$ small enough. □

Acknowledgements

The first author is supported by the Chinese University of Hong Kong(Shenzhen) under the start-up funding UDF01002321 and by Shenzhen government under grant C10120230023. The second author was supported by the NSF under grant DMS 2205553.

References

[1] Bellomo, N., Bellouquid, A., Tao, Y., Winkler, M.: Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues. Math. Models Methods Appl. Sci. **25**, 1663–1763 (2015)
[2] Fujie, K.: Boundedness in a fully parabolic chemotaxis system with singular sensitivity. J. Math. Anal. Appl. **424**, 675–684 (2015)
[3] Keller, E.F., Segel, L.A.: Initiation of slime mold aggregation viewed as an instability. J. Theor. Biol. **26**, 399–415 (1970)
[4] Keller, E.F., Segel, L.A.: Traveling bands of chemotactic bacteria: a theoretical analysis. J. Theor. Biol. **30**, 235–248 (1971)
[5] Kraitzman, N., Promislow, K., Wetton, B.: Slow migration of brine inclusions in first-year sea ice. SIAM J. Appl. Math. **82**, 1470–1494 (2022)
[6] Ladyzhenskaia, O.A., Solonnikov, V.A., Ural'tseva, N.N.: Linear and Quasi-Linear Equations of Parabolic Type, vol. 23. AAmerican Mathematical Society, Providence (1988)
[7] Lankeit, J.: A new approach toward boundedness in a two-dimensional parabolic chemotaxis system with singular sensitivity. Math. Methods Appl. Sci. **3**, 394–404 (2016)
[8] Lankeit, J., Winkler, M.: A generalized solution concept for the Keller–Segel system with logarithmic sensitivity: global solvability for large nonradial data. Nonlinear Differ. Equ. Appl. **24**, 1–33 (2017)
[9] Mielke, A.: Formulation of thermoplastic dissipative material behavior using generic. Contin. Mech. Thermodyn. **23**, 233–256 (2011)
[10] Nagai, T., Senba, T., Yoshida, K.: Application of the trudinger-moser inequality to a parabolic system of chemotaxis. Funkc. Ekvacioj **40**, 411–433 (1997)
[11] Wells, A., Hitchen, J., Parkinson, J.: Mushy-layer growth and convection, with application to sea ice. Philos. Trans. R. Soc. A **377**, 20180165 (2019)
[12] Winkler, M.: Aggregation vs global diffusive behavior in the higher-dimensional Keller–Segel model. J. Differ. Equ. **248**, 2889–2905 (2010)
[13] Winkler, M.: Global solutions in a fully parabolic chemotaxis system with singular sensitivity. Math. Methods Appl. Sci. **34**, 176–190 (2011)

Yuan Chen
School of Science and Engineering, Chinese University of Hong Kong (Shenzhen)
Shenzhen
China
e-mail: chenyuan@cuhk.edu.cn

Keith Promislow
Michigan State University
East Lansing
USA
e-mail: kpromisl@math.msu.edu

(Received: December 12, 2022; revised: May 7, 2023; accepted: May 11, 2023)