



# Enhanced dissipation by circularly symmetric and parallel pipe flows

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## ABSTRACT

We study enhanced dissipation due to the combined effect of diffusion or hyperdiffusion and advection by an incompressible flow with circular or cylindrical symmetry in 2 and 3 space dimensions, respectively. By using resolvent estimates for  $m$ -accretive operators (Wei, 2021), under a suitable condition on the velocity adapted from Gallay and Coti Zelati (2021), we establish enhanced dissipation for the advection-(hyper)diffusion equation and quantify it in terms of rates of decay in time for the solution, suitably projected, with an improved explicit dependence on the diffusivity. Our results extend prior results in Coti Zelati and Dolce (2020).

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## 1. Introduction

This article concerns enhanced dissipation due to the combined effect of diffusion or hyperdiffusion and advection by an incompressible flow with circular or cylindrical symmetry in 2 and 3 space dimensions, respectively. By *enhanced dissipation*, we mean that the solution operator of the advection–diffusion equation acts on time scales smaller than the diffusion time scale, which in turn implies a faster rate of decay in time for solutions in comparison to (hyper) diffusion alone. We will quantify this effect in terms of bounds on the decay rates with an improved, explicit dependence on the diffusivity coefficient (see (1.12)–(1.13)). Of course, improved decay can only be achieved after projecting out the kernel of the advection operator. We restrict our attention to *incompressible flows*.

Enhanced dissipation by advection is a well-known mechanism in fluid mechanics, observed in turbulent mixing and leading to anomalous dissipation in scalar turbulence (see e.g. [1] and reference therein, see also [2–4] and the preprint [5]) and other important phenomena. A related phenomenon is Taylor dispersion due to shearing (see e.g. [6] and references therein, see also [7] and the preprint [8]), where the dispersion leads to a large effective diffusivity provided the time scale on which advection acts is large enough. Significant progress has been made recently on the mathematical justification and rigorous quantification of enhanced dissipation, starting from the seminal work of Constantin et al. on so-called *relaxation enhancing flows* [9] (we refer, among several works, to [10–16]), due in part to recent advances in the analysis and construction of flows with optimal mixing rates [17–26] (see also the recent preprint [27]) and in the analysis of instabilities of viscous flows around steady profiles (we mention [28–39]). Enhanced dissipation has been used as a mechanism to prevent blow up and separation, and prove global existence in dissipative systems, such as for aggregation–diffusion models [40–42], for the Cahn–Hilliard phase field model [43], and for the Kuramoto–Sivashinsky equation in 2 and 3 space dimensions [43–45].

Most of the results concerning enhanced dissipation are obtained in the torus, a setting that, while more geometrically constrained, allows more explicit calculations. A few consider bounded domains with no slip or no penetration boundary conditions for the flow and compatible boundary conditions for the advected scalar. Steady relaxation enhancing flows are characterized by a condition on their spectrum, and it is known that mixing flows with enough regularity are dissipation enhancing. Certain shear and cellular flows

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can also be dissipation enhancing (we refer the reader to the references cited above for more details). In all cases, enhancement can only happen on the complement of the kernel of the advection operator. In this work, we study dissipation enhancement for certain flows with circular or cylindrical symmetry, which can be considered as a generalization of shear flows.

Let  $f : [0, \infty) \times \Omega \rightarrow \mathbb{R}$  solve the initial value problem (IVP for short) for a linear advection–diffusion equation

$$\begin{aligned} \partial_t f + \mathbf{u} \cdot \nabla f + \nu(-\Delta)^p f &= 0, \\ f|_{t=0} &= f_0, \end{aligned} \quad (1.1)$$

where  $\Omega$  is either  $\mathbb{R}^n$ ,  $n = 2, 3$ , or a domain with smooth boundary in  $\mathbb{R}^n$ ,  $p = 1$  or  $2$ ,  $0 < \nu \ll 1$  is the diffusivity coefficient,  $\mathbf{u}$  is a given divergence-free vector field, and  $f_0$  is the initial data. This IVP is complemented by suitable boundary conditions specified later.

We denote a point in  $\mathbb{R}^2$  by  $\mathbf{x} = (x, y)$  and let  $(r, \theta)$  be the polar coordinates in the plane with associated o.n. frame  $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ . Similarly, we denote a point in  $\mathbb{R}^3$  by  $\mathbf{x} = (x, y, z)$  and let  $(r, \theta, z)$  be cylindrical coordinates in space with associated o.n. frame  $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ . In the two-dimensional case, we take  $\mathbf{u}$  to be the velocity associated with a *steady circularly symmetric* flow

$$\mathbf{u}(x, y) = v(r) \mathbf{e}_\theta, \quad (1.2)$$

in the full plane  $\Omega = \mathbb{R}^2$  (we could also consider a disk centered at the origin). In the three-dimensional case, we take  $\mathbf{u}$  to be the velocity associated with a *steady pipe parallel* flow in :

$$\mathbf{u}(x, y, z) = v(r) \mathbf{e}_\theta + w(r, \theta, z) \mathbf{e}_z, \quad (1.3)$$

in the cylinder parallel to the  $z$ -axis  $\Omega = D(0, 1) \times \mathbb{T} \subset \mathbb{R}^3$ , where  $D(0, 1)$  is the disk centered at the origin with unit radius in the  $xy$  plane, (any radii and any axis can be chosen). With slight abuse of notation, we will identify  $\mathbf{u}$  with the flow it generates.

We then rigorously establish enhanced dissipation for (1.1) and quantify it in terms of rates of decay in time for the solution operator in terms of the viscosity. Our results extend the work of Coti-Zelati and Dolce [46], who studied enhanced dissipation in the case  $p = 1$  for certain circularly symmetric flows in  $\mathbb{R}^2$ . The example in [46] is, to our knowledge, the only rigorous example of dissipation enhancing flow in the whole plane. It is a circularly symmetric flow with velocity

$$\mathbf{u}(x, y) = (x^2 + y^2)^{\frac{q}{2}} \begin{pmatrix} -y \\ x \end{pmatrix}, \quad (1.4)$$

where  $q$  is an integer. Informally, dissipation enhancement in shear flows occurs by mixing across streamlines. Here, the streamlines are circles concentric to the origin of increasing radius, hence the velocity must grow sufficiently fast at infinity and its critical points cannot be too many or too degenerate for enhancement to occur. More specifically, the authors in [46] prove a separation of time scales: for time-scales between  $\frac{1}{\lambda_\nu} = (1 + \frac{2(q-1)}{q+2} |\ln \nu|) / (\nu^{\frac{q}{q+2}})$  and  $\sim \frac{1}{\nu}$  the dominant mechanism is mixing along streamlines. At times-scale of order  $\frac{1}{\nu}$  diffusion takes over and at later times mixing happens across streamlines. The result, proven by employing hypocoercivity methods [47], presented difficulties given by the absence of the Poincaré inequality (which is generally employed for deriving exponential decays) and the unboundedness of the velocity field, which potentially causes the solution to grow at infinity. In this work, we establish this result by employing stability estimates for the semigroup associated to the operator  $\mathbf{u} \cdot \nabla - \nu \Delta$ . The method we employed is inspired by a recent result of Wei ([48, Theorem 1.3]), which in turn uses certain spectral estimates for  $m$ -accretive operators on Hilbert spaces. Spectral estimates can be more naturally adapted to different boundary conditions and to hyper- and hypo-diffusion operators, in particular fractional powers of the Laplace operator [49], although we do not pursue this extension here.

Wei's result in [48] is based on the following Gearhart–Prüss-type theorem. Let  $(\mathcal{H}, \|\cdot\|)$  be a complex Hilbert space and let  $H$  be a closed, densely defined operator on  $\mathcal{H}$  with domain  $\mathcal{D}(H)$ . If  $H$  is an  $m$ -accretive operator on  $\mathcal{H}$ , then

$$\|e^{-tH}\|_{\text{op}} \leq e^{-t\Psi(H) + \pi/2}, \quad \forall t \geq 0, \quad (1.5)$$

where  $\|\cdot\|_{\text{op}}$  denotes the operator norm,  $e^{-tH}$ ,  $t \geq 0$ , denotes the semigroup generated by the operator  $H$  on  $\mathcal{H}$ , and

$$\Psi(H) = \inf \{ \|(H - i\lambda)g\| : g \in \mathcal{D}(H), \lambda \in \mathbb{R}, \|g\| = 1 \}, \quad (1.6)$$

Our goal is to derive semigroup estimates from bounds on the resolvent of the operator  $H = H_\nu := \nu(-\Delta)^p + \mathbf{u} \cdot \nabla$  on  $\mathcal{H} = L^2(\Omega)$ .

To achieve dissipation enhancement, we impose certain conditions on the velocity profiles  $v$  and  $w$  in (1.2) and (1.3) (see Assumptions 1.1 and 1.2). These conditions are adapted from the work of Coti Zelati and Gallay in the case of higher-dimensional shear flows [8], which some of the authors of this paper have employed to prove global existence for the 2D advective Kuramoto–Sivashinsky equation [44]. More precisely, in  $\mathbb{R}^2$  we consider the circularly symmetric flow,

$$\mathbf{u}(x, y) = u(\sqrt{x^2 + y^2}) \begin{pmatrix} -y \\ x \end{pmatrix} \quad (1.7)$$

with  $u : [0, \infty) \rightarrow \mathbb{R}$  a smooth profile satisfying the following assumption.

**Assumption 1.1.** There exist  $m, N \in \mathbb{N}$ ,  $c_1 > 0$  and  $\delta_0 \in (0, \infty)$  with the property that, for any  $\lambda \in \mathbb{R}$  and any  $\delta \in (0, \delta_0)$ , there exist  $n \leq N$  and points  $r_1, \dots, r_n \in [0, \infty)$  such that

$$|u(r) - \lambda| \geq c_1 \delta^m, \quad \forall |r - r_j| \geq \delta, \quad \forall j \in \{1, \dots, n\}. \quad (1.8)$$

Observe that this condition quantifies the *degeneracy of the critical points* as one can easily see by choosing  $\lambda = u(r_j)$ . The condition of finitely many critical points with finite order arises naturally in studying the solution of the advection equation by a shear flow by the method of stationary phase. Such method allows to show that there is mixing along the streamlines, quantified by the decay of the  $H^{-1}$  negative Sobolev norm of the solution in the direction orthogonal to shear (we refer to [50], Appendix A and references therein for a more in-depth discussion). This decay in turn allows to prove hypocoercivity of the advection–diffusion operator and enhanced dissipation (see e.g. [47] and references therein).

We note that  $\mathbf{u}$  is divergence free and vanishes at the origin. It is of the form (1.2) with  $v(r) = r u(r)$ . The flow (1.4) considered in [46] is an example of a circularly symmetric flow satisfying Assumption 1.1 with  $m = q$  (see Remark 3.1, but this assumption is satisfied by more general flows).

To estimate  $e^{-tH}$ , it is convenient to pass to polar coordinates  $(r, \theta)$ . To this end, we recall that the gradient and the Laplacian in polar coordinates are given by

$$\nabla = \left( \frac{\partial_r}{r}, \frac{1}{r} \partial_\theta \right), \quad \Delta = \partial_{rr} + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_{\theta\theta}.$$

For notational convenience, we write  $\tilde{f}(r, \theta, t) = f(r \cos \theta, r \sin \theta, t)$ . Then Eq. (1.1) transforms into

$$\partial_t f + u(r) \partial_\theta f + v(-\partial_{rr} - \frac{1}{r} \partial_r - \frac{1}{r^2} \partial_{\theta\theta})^p f = 0 \quad (1.9)$$

on  $(0, \infty) \times \mathbb{T}$ , for  $t > 0$ , where  $\mathbb{T}$  is the 1D torus.

Given  $f \in L^2(\mathbb{R}^2)$ , we denote the average of  $f$  over circles centered at the origin by

$$\langle f \rangle_\theta(x, y, t) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(t, \sqrt{x^2 + y^2}, \theta) d\theta, \quad (1.10)$$

which is well defined a.e., and we denote the difference by

$$f_\neq(x, y, t) = f(x, y, t) - \langle f \rangle_\theta. \quad (1.11)$$

We observe that  $\langle f \rangle_\theta$  gives the projection of  $f$  onto the kernel of  $\mathbf{u} \cdot \nabla$  and  $f_\neq$  the projection onto its complement. The initial data of  $f_\neq$  is

$$(f_\neq)_0(x, y, t) = f_0(x, y, t) - \langle f_0 \rangle_\theta,$$

where  $f_0(x, y, t) = f|_{t=0}$  and  $\langle f_0 \rangle_\theta = \langle f \rangle_\theta|_{t=0}$ . We can now state our main result in the two-dimensional case.

**Theorem 1.1.** *Let  $\mathbf{u}$  satisfy (1.7)–(1.8), and let  $f_0 \in L^2(\mathbb{R}^2)$ . Let  $f$  satisfy the IVP (1.1) with velocity  $\mathbf{u}$  and initial data  $f_0$ . Then there exists constants  $C_1, C_2 > 0$ , independent of  $v$  and  $f$ , such that*

$$\|f_\neq(\cdot, t)\|_{L^2(\mathbb{R}^2)} \leq C_1 e^{-C_2 \lambda_v t} \|(f_0)_\neq\|_{L^2(\mathbb{R}^2)}, \quad (1.12)$$

where

$$\lambda_v = v^{\frac{m}{m+2p}}. \quad (1.13)$$

The above result can also be interpreted as a bound in time on the oscillation in direction orthogonal to streamlines of the solution. There are several bounds of this type available in the literature. Young and Rhines studied the effective diffusivity due to the interplay between dispersion along the streamlines due to advection and diffusivity orthogonal to the streamlines for some circularly and elliptically polarized shear flows by a formal moment analysis, motivated by the study of internal waves in the ocean [51,52]. Their estimate corresponds to the case  $p = 1$  and  $m = 1$  in (1.12). We also mention the work of Novikov, Papanicolau, and Ryzhik, who derive an estimate for the oscillation along streamlines of solutions to the stationary advection–diffusion equation for cellular flows in the high Peclet number regime [53].

The spectral approach allows us to improve on the results in [46] in two ways. First, there is no  $\log v^{-1}$  correction in (1.13). Second, we can take a more general circularly symmetric velocity  $\mathbf{u}$ . The proof is also more streamlined than that using hypocoercivity estimates. It is an interesting question, to which we do not know the answer, whether it is possible to exploit the Hamiltonian structure of the flow to generalize our results to flows with closed streamlines if the travel time along each streamline is controlled (cf. the work of Freidlin and Wentzell in the context of large deviation theory [54]).

To prove Theorem 1.1, we apply the Fourier Transform in the angular variable  $\theta$ , and bound the resolvent of the transformed operator

$$H_{v,k} := iku(r) + v(-\partial_{rr} - \frac{1}{r} \partial_r + \frac{k^2}{r^2})^p, \quad (1.14)$$

for fixed  $k \in \mathbb{Z}$ . By Wei's result, such bounds in turn give estimates on the semigroup generated by  $H_v$ , provided the  $m$ -accretivity of  $H_{v,k}$  is established first. This is perhaps the most delicate part of our proof as this operator is singular at  $r = 0$  and  $u(r)$  is unbounded at infinity, so we have to work with weighted spaces.

In the three dimensional case, for technical reasons we limit ourselves to flows in domains bounded in the radial direction, i.e., a cylinder to preserve circular symmetry, and the case  $p = 1$ . We take  $\mathbf{u}$  to be a parallel pipe flow in the periodic cylinder  $\Omega = D(0, 1) \times \mathbb{T}^1$ :

$$\mathbf{u}(x, y, z) = u(r) \left( \sin(2\pi r) \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} + \cos(2\pi r) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right), \quad (1.15)$$

with  $u : [0, 1] \rightarrow \mathbb{R}$  a smooth profile satisfying the following assumption.

**Assumption 1.2.** *There exist  $m, N \in \mathbb{N}$ ,  $c_1 > 0$  and  $\delta_0 \in (0, \infty)$  with the property that, for any  $\alpha, \lambda \in \mathbb{R}$  and any  $\delta \in (0, \delta_0)$ , there exist  $n \leq N$  and points  $r_1, \dots, r_n \in [0, 1]$  such that*

$$|u(r) \sin(2\pi r + \alpha) - \lambda| \geq c_1 \delta^m, \quad \forall |r - r_j| \geq \delta, \quad \forall j \in \{1, \dots, n\}. \quad (1.16)$$

We observe that this velocity field is divergence free, constant in  $z$ , tangent to the boundary of the cylinder, and vanishes along the axis. It is of the form (1.3) with  $v(r) = r u(r) \sin(2\pi r)$  and  $w(r, \theta) = u(r) \cos(2\pi r)$ . We then impose periodicity in  $z$  and homogeneous Neumann boundary conditions on the solution to (1.1). Any period  $L > 0$  can be taken along the axis, but for simplicity of exposition we assume  $L = 1$ . We then identify  $\Omega$  with  $D(0, 1) \times \mathbb{T}$ .

The planar equivalent of this flow was studied by one of the author on the 3D torus, proving enhanced dissipation and using it to establish global existence for the Kuramoto–Sivashinsky and the Keller–Segel equations with advection [55].

To estimate  $e^{-tH_v}$ , it is convenient to pass to cylindrical coordinates  $(r, \theta, z)$ . We recall that the gradient and Laplacian in these coordinates are given by

$$\nabla = \left( \frac{\partial_r}{r}, \frac{1}{r} \frac{\partial_\theta}{\partial_z} \right), \quad \text{and} \quad \Delta = \partial_{rr} + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_{\theta\theta} + \partial_{zz}.$$

Again, we write  $\tilde{f}(r, \theta, z, t) = f(r \cos \theta, r \sin \theta, z, t)$ . Then the advection–diffusion in (1.1) becomes

$$\partial_t f + u(r) \sin(2\pi r) \partial_\theta f + u(r) \cos(2\pi r) \partial_z f - \nu \Delta f = 0$$

on  $(0, 1) \times \mathbb{T}^2$ , for  $t > 0$ .

Similarly to the 2-dimensional case, given  $f \in L^2(\Omega)$ , we denote the average in the angular and axial variables  $\theta, z$  by

$$\langle f \rangle_{\theta, z}(t, r) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(t, \sqrt{x^2 + y^2}, \theta, z) d\theta dz, \quad (1.17)$$

which exists a.e., and the difference by

$$f_\neq = f(x, y, z, t) - \langle f \rangle_{\theta, z}. \quad (1.18)$$

We note that  $\langle f \rangle_{\theta, z}$  is the projection of  $f$  onto the kernel of the operator  $\mathbf{u} \cdot \nabla$  and  $f_\neq$  the projection onto its complement. The initial data of  $f_\neq$  is

$$(f_\neq)_0(x, y, z, t) = f_0(x, y, z, t) - \langle f_0 \rangle_{\theta, z},$$

where  $f_0(x, y, z, t) = f|_{t=0}$  and  $\langle f_0 \rangle_\theta = \langle f \rangle_{\theta, z}|_{t=0}$ .

Our main results in 3 dimensions is the following theorem.

**Theorem 1.2.** *Let  $\Omega = D(0, 1) \times \mathbb{T}$ . Let  $\mathbf{u}$  satisfy (1.15)–(1.16), and let  $f_0 \in L^2(\Omega)$  be mean free. Let  $f$  solve (1.1) with velocity  $\mathbf{u}$ , initial data  $f_0$ , and Neumann boundary conditions  $\frac{\partial f}{\partial n} = 0$  at  $\partial D(0, 1)$ . Then there exists constants  $C_1, C_2 > 0$ , independent of  $\nu$  and  $f$ , such that*

$$\|f_\neq(\cdot, t)\|_{L^2(\Omega)} \leq C_1 e^{-C_2 \lambda_\nu t} \|(f_0)_\neq\|_{L^2(\Omega)}, \quad (1.19)$$

where

$$\lambda_\nu = \nu^{\frac{m}{m+2}}. \quad (1.20)$$

As in the two-dimensional case, to prove Theorem 1.2 we apply the 2D Fourier Transform in the angular and axial variables  $\theta, z$ , and bound the resolvent of the transformed operator

$$H_{v,k} := ik_1 u(r) \sin(2\pi r) + ik_2 u(r) \cos(2\pi r) + \nu(-\partial_{rr} - \frac{1}{r} \partial_r + \frac{k_1^2}{r^2} + k_2^2), \quad (1.21)$$

for fixed  $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$ . (Instead of writing  $H_{v,\mathbf{k}}$ , we employ the same notation in (1.14) and (1.21), as no confusion can arise, since the two cases are dealt with separately). By Wei's result, such bounds in turn give estimates on the semigroup generated by  $H_v$ , provided  $H_{v,k}$  is  $m$ -accretive on a weighted  $L^2$  space.

Finally we remark that in [56] S. Pottel and the third author of this paper proved decay estimates in  $\mathbb{R}^d$  ( $d = 2, 3$ ) of the filamentation length  $\lambda$  defined as the ratio between the (homogeneous)  $H^{-1}$  norm and the  $L^2$  norm of the passive scalar. The estimates indicate either dispersion or mixing depending of the decay properties of the time-dependent velocity field. The velocity fields considered in this paper (unbounded and time independent) are not admissible and whether or not the class of velocity fields considered in [56] is dissipation enhancing is not known and object of investigation.

We close with a brief outline of the paper. In Section 2, we recall the notion of  $m$ -accretivity and give a proof of the  $m$ -accretivity of the operators introduced in (1.14) and (1.21), using some abstract functional analysis and the  $m$ -accretivity of  $H_v$  on  $L^2(\Omega)$ . Then in Section 3 we prove Theorem 1.1 and the needed resolvent estimates, while a proof of Theorem 1.2 and the needed resolvent estimates are contained in Section 4.

## 2. $m$ -accretivity

Our goal in this section is to show the  $m$ -accretivity of the operators defined in (1.14) and (1.21) on the space  $L^2(\mathbb{R}_+, r dr)$  and  $L^2((0, 1), r dr)$  respectively. For brevity we only discuss the operator given in (1.14). A similar argument (in fact simpler) applies to the operator given in (1.21). We exploit the fact that the operator  $H_{v,k}$  is obtained by conjugating  $H_v$  with an isometric isomorphism between Hilbert spaces and by applying an orthogonal projections, and  $m$ -accretivity is preserved under both operations.

Let  $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$  be a complex Hilbert space and let  $H$  be an unbounded linear operator on  $\mathcal{H}$  with dense domain  $\mathcal{D}(H)$  and with range  $\mathcal{R}(H)$ . We endow  $\mathcal{D}(H)$  with the graph norm  $\|f\|_{\mathcal{D}(H)} = \|f\|_{\mathcal{H}} + \|Hf\|_{\mathcal{H}}$ . If  $\Phi : \mathcal{H} \rightarrow \tilde{\mathcal{H}}$  is an invertible isometry between Hilbert spaces, we denote

$$\tilde{H} = \Phi(H) =: \Psi H \Psi^{-1},$$

which is an unbounded linear operator on  $\overline{\mathcal{H}}$  with dense domain and range given by:

$$\begin{aligned}\mathcal{D}(\tilde{H}) &= \Phi(\mathcal{D}(H)) := \{\overline{H} \ni g = \Psi(f) : f \in \mathcal{D}(H)\}, \\ \mathcal{R}(\tilde{H}) &= \Phi(\mathcal{R}(H)) := \{\overline{H} \ni g = \Psi(f) : f \in \mathcal{R}(H)\}.\end{aligned}$$

We begin by recalling the notion of  $m$ -accretivity.

**Definition 2.1.** The operator  $H : \mathcal{D}(H) \subset \mathcal{H} \rightarrow \mathcal{H}$  is called *maximally accretive* or  *$m$ -accretive* if

- (1)  $\operatorname{Re}\langle Hf, f \rangle \geq 0$  for all  $f \in \mathcal{D}(H)$ ,
- (2)  $\mathcal{R}(H + \xi Id) = \mathcal{H}$  for some  $\xi > 0$ ,

where  $Id$  denotes the identity map.

Property (1) gives the accretivity of  $H$ , while property (2) shows maximality.

The  $m$ -accretivity of an operator is a property that is preserved under isometric isomorphisms, as specified in the following lemma. We include a proof of this simple fact for completeness.

**Lemma 2.1.** Let  $H$  be an  $m$ -accretive operator on  $\mathcal{H}$  and let  $\Phi : \mathcal{H} \rightarrow \overline{\mathcal{H}}$  be an isometric isomorphism. Then  $\tilde{H} = \Phi(H)$  is an  $m$ -accretive operator on  $\overline{\mathcal{H}}$ .

**Proof.** Since  $\mathcal{D}(\tilde{H}) = \Phi\mathcal{D}(H)$ , for all  $f \in \mathcal{D}(\tilde{H})$

$$\operatorname{Re}\langle \tilde{H}f, f \rangle_{\overline{\mathcal{H}}} = \operatorname{Re}\langle \Phi H \Phi^{-1}f, f \rangle_{\overline{\mathcal{H}}} = \operatorname{Re}\langle \Phi H \Phi^{-1}f, \Phi \Phi^{-1}f \rangle_{\overline{\mathcal{H}}} = \operatorname{Re}\langle H \Phi^{-1}f, \Phi^{-1}f \rangle_{\mathcal{H}} \geq 0$$

where, in the last two identities, we have used that  $\Phi$  is an isometry and that  $H$  is  $m$ -accretive. Moreover,

$$\mathcal{R}(\tilde{H} + \xi Id) = \Phi(H + \xi Id) = \Phi(\mathcal{H}) = \overline{\mathcal{H}},$$

since  $\Phi$  is invertible.  $\square$

We will also need the following fact. Again, we include a proof for completeness.

**Lemma 2.2.** Let  $H$  be an  $m$ -accretive operator on a Hilbert space  $\mathcal{H}$ . Let  $P$  be an orthogonal projection onto a closed subspace  $V$  of  $\mathcal{H}$ . Let  $H_V := PH$ . Then  $H_V$  is  $m$ -accretive on  $V$  with the induced inner product.

**Proof.** We observe that  $H_V : V \rightarrow V$  is a linear operator with domain  $\mathcal{D}(H_V) = \mathcal{D}(H) \cap V$ . Since  $V$  is closed and  $\mathcal{D}(H)$  is dense in  $\mathcal{H}$ ,  $\mathcal{D}(H_V)$  is dense in  $V$ . Accretivity follows from a direct computation. In fact

$$\operatorname{Re}\langle H_V f, f \rangle_V = \operatorname{Re}\langle PHf, f \rangle_{\mathcal{H}} = \operatorname{Re}\langle Hf, Pf \rangle_{\mathcal{H}} = \operatorname{Re}\langle Hf, f \rangle_{\mathcal{H}} \geq 0$$

for all  $f \in \mathcal{D}(H_V)$ , since  $Pf = f$  if  $f \in V$ . Now notice that  $\mathcal{R}(H + \xi Id) = \mathcal{H}$  for some  $\xi > 0$ , by the  $m$ -accretivity of  $H$ . This implies

$$\mathcal{R}(H_V + \xi Id_V) = \mathcal{R}(PH + \xi PId) = V,$$

which shows the  $m$ -accretivity of  $H_V$ .  $\square$

We now apply these results to the operator  $H_{v,k}$  given in (1.14). We first set  $p = 1$  and let  $H = H_v = -v\Delta + \mathbf{u} \cdot \nabla$ , where  $\mathbf{u}$  is a circularly symmetric flow with smooth, possibly unbounded profile  $v(r)$  (see (1.2)), defined on  $\mathcal{H} = L^2(\mathbb{R}^2)$ .

**Lemma 2.3.**  $H_v$  is an  $m$ -accretive operator on  $L^2(\mathbb{R}^2)$ .

**Proof.** We write  $H_v = A_v + L$ , where  $A_v = -v\Delta$  and  $L = \mathbf{u} \cdot \nabla$  with domains  $\mathcal{D}(A_v) = H^2(\mathbb{R}^2)$  and  $\mathcal{D}(L) = \{f \in H^1(\mathbb{R}^2) : \mathbf{u} \cdot \nabla f \in L^2(\mathbb{R}^2)\}$ . Since  $A_v$  is self-adjoint and  $L$  is anti-selfadjoint,<sup>1</sup> it follows that the adjoint is  $H_v^* = A_v - L$ . We want to prove that  $H_v$  is densely defined, closable, accretive and that  $H_v^*$  is accretive. Then, from the characterization in Theorem I-4.4 in [57], it follows that  $H_v$  is  $m$ -accretive.

- **Density of  $H_v$ :** We take  $\mathcal{D}(H_v) = \mathcal{D}(A_v) \cap \mathcal{D}(L)$  and want to show that  $\mathcal{D}(H_v)$  is dense in  $\mathcal{H} = L^2(\mathbb{R}^2)$ . We first observe that  $C_c^\infty(\mathbb{R}^2)$  is dense in both  $\mathcal{D}(A_v)$  and  $\mathcal{D}(L)$ , and in  $\mathcal{H}$  in the respective norms. The density of  $C_c^\infty(\mathbb{R}^2)$  in  $\mathcal{D}(A_v)$  is clear. To see the density in  $\mathcal{D}(L)$ , let  $f \in \mathcal{D}(L)$  and  $\chi$  be a radially symmetric smooth bump function on the unit disk (again for notational convenience we identify  $\chi(x, y)$  with  $\chi(r)$ ). For  $n \in \mathbb{N}$ , let  $\chi_n(r) = \chi(\frac{r}{n})$ . Then  $f\chi_n \rightarrow f$  in  $\mathcal{D}(L)$ , since  $\|\mathbf{u} \cdot \nabla(f\chi_n) - \mathbf{u} \cdot \nabla f\|_{L^2} \leq \|(\mathbf{u} \cdot \nabla f)\chi_n - \mathbf{u} \cdot \nabla f\|_{L^2} + \|(\mathbf{u} \cdot \nabla \chi_n)f\|_{L^2}$  and  $\mathbf{u} \cdot \nabla \chi_n = 0$  due to the fact that  $\chi_n$  is radial and  $\mathbf{u}$  is circularly symmetric. The mollification of  $f\chi_n$  converges to  $f\chi_n$  in  $\mathcal{D}(L)$ , as  $f\chi_n$  is compactly supported. Hence  $C_c^\infty(\mathbb{R}^2)$  is dense in  $\mathcal{D}(L)$ . In particular, we showed that  $C_c^\infty(\mathbb{R}^2)$  is dense in  $\mathcal{D}(H_v)$  and from this it follows that  $\mathcal{D}(H_v)$  is dense in  $\mathcal{H}$ .
- **$H_v$  is closable:** Since  $\mathcal{D}(H_v) = \mathcal{D}(H_v^*)$  is dense in  $\mathcal{H}$ , the result follows from Theorem VIII.1 (page 252) in [58].
- **Accretivity of  $H_v$ :** For the ease of notation, we continue to denote the closure of  $H_v$  by  $H_v$  and its domain by  $\mathcal{D}(H_v)$ . We note that  $\mathcal{D}(H_v)$  contains  $\mathcal{D}(A_v) \cap \mathcal{D}(L)$  and that  $C_c^\infty$  is dense in  $\mathcal{D}(H_v)$ . Then the accretivity of  $H_v$ , property (1) in Definition 2.1, follows from a direct computation:

$$\operatorname{Re}\langle A_v f, f \rangle + \operatorname{Re}\langle Lf, f \rangle = v \operatorname{Re}\left(\int_{\mathbb{R}^2} \nabla \bar{f} \cdot \nabla f \, dx \, dy\right) + \frac{1}{2}\langle Lf, f \rangle - \frac{1}{2}\langle f, L^* f \rangle$$

<sup>1</sup> In fact,  $\langle Lf, g \rangle_{\mathcal{H}} = \int_{\mathbb{R}^2} \nabla \cdot (\mathbf{u} \bar{f}) g \, dx \, dy = - \int_{\mathbb{R}^2} \bar{f} \mathbf{u} \cdot \nabla g \, dx \, dy$ , since  $\mathbf{u}$  is divergence free.

$$\begin{aligned}
&= \nu \int_{\mathbb{R}^2} |\nabla f|^2 dx dy + \frac{1}{2} \operatorname{Re} [(\mathbf{v} \cdot \nabla \tilde{f})f - (\mathbf{v} \cdot \nabla f)\tilde{f}] dx dy \\
&= \nu \int_{\mathbb{R}^2} |\nabla f|^2 dx dy \geq 0.
\end{aligned} \tag{2.1}$$

• **Accretivity of  $H_v^*$ :** It follows by the same computation as in (2.1)  $\square$

Next, we let  $\Phi$  be the transformation

$$(\Phi f)(r, k) = \tilde{f}_k(r),$$

where  $\tilde{f}(r, \theta) = f(r \cos \theta, r \sin \theta)$  and  $\tilde{f}_k(r)$  is the  $k$ -th Fourier coefficient of  $\tilde{f}(r, \cdot)$ :

$$\tilde{f}_k(r) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(r, \theta) e^{-ik\theta} d\theta, \quad k \in \mathbb{Z}.$$

By the change of variable formula and Plancherel's identity,  $\Phi$  is an isometric isomorphism:

$$\Phi : \mathcal{H} = L^2(\mathbb{R}^2) \rightarrow \ell^2(\mathbb{Z}; L_r^2(\mathbb{R}^+)) =: \overline{\mathcal{H}}.$$

Above  $L_r^2(\mathbb{R}^+) := L^2(\mathbb{R}^+, r dr)$  and  $\ell^2(\mathbb{Z}; L_r^2(\mathbb{R}^+)) = \{\{\tilde{f}_k\}_{k \in \mathbb{Z}} : \|\{\tilde{f}_k\}\|_{\overline{\mathcal{H}}} := \sum_k \int_0^\infty |\tilde{f}_k(r)|^2 r dr < \infty\}$  with the induced inner product  $(\{\tilde{f}\}, \{\tilde{g}\})_{\overline{\mathcal{H}}} = \sum_k \int_0^\infty \tilde{f}_k(r) \overline{\tilde{g}_k(r)} r dr$ . Lemmas 2.1 and 2.3 then give the following result.

**Corollary 2.4.**  $\tilde{H}_v = \Phi(H_v)$  is  $m$ -accretive on  $\overline{\mathcal{H}} = \ell^2(\mathbb{Z}; L_r^2(\mathbb{R}^+))$ .

Let  $P_k$  denote the projection of  $f$  onto the  $k$ th Fourier mode. Then  $P_k$  is an  $L^2$ -orthogonal projection and  $\Phi(P_k H f) = \tilde{f}_k$ , so that

$$H_{v,k} := \Phi(P_k H_v) = \nu(\partial_r^2 + \frac{1}{r} \partial_r - \frac{k^2}{r^2}) + iu(r)k,$$

defined as an unbounded operator on  $V_k := \{\tilde{f}_k : f \in L^2(\mathbb{R}^2)\}$ .

Finally, by Lemma 2.2 and Corollary 2.4, we obtain the needed  $m$ -accretivity of the operator in (1.14) for  $p = 1$ .

**Corollary 2.5.**  $H_{v,k}$  is an  $m$ -accretive operator on  $V_k \simeq L_r^2(\mathbb{R}^+)$  for all  $k \in \mathbb{Z}$ ,  $\nu > 0$ .

The proof of  $m$ -accretivity for  $p = 2$  follows the same approach, using the  $m$ -accretivity of  $H_v = \nu \Delta^2 + \mathbf{u} \cdot \nabla$ , which can be proved in a manner similar to the case  $p = 1$ . In fact,  $H_v = A_v + L$ , where  $L$  is as before and  $A_v = \Delta^2$ , a self-adjoint, non-negative operator on  $L^2(\mathbb{R}^2)$  with dense domain  $\mathcal{D}(A_v) = H^2(\mathbb{R}^2)$ .

### 3. Dissipation enhancement in $\mathbb{R}^2$ by a circularly symmetric flow

In this section, we focus on the proof of Theorem 1.1. We follow the convention set in the Introduction to denote  $f$  in polar coordinates by  $\tilde{f}$ , and its  $k$ th Fourier coefficient by  $\tilde{f}_k$ . Then  $\tilde{f}_k$  belongs to  $L_r^2(\mathbb{R}^+) := L^2(\mathbb{R}^+, r dr)$  with the induced inner product  $\langle \cdot, \cdot \rangle_r$ . We note that

$$\|\nabla f\|_{L^2(\mathbb{R}^2)}^2 = \sum_{k \in \mathbb{Z}} \|\partial_r \tilde{f}_k\|_{L_r^2(\mathbb{R}^+)}^2 + k^2 \left\| \frac{\tilde{f}_k}{r} \right\|_{L_r^2(\mathbb{R}^+)}^2.$$

Furthermore, for each  $k \in \mathbb{Z}$  if  $f$  solves (1.1), then  $\tilde{f}_k$  satisfies the equation:

$$\partial_t \tilde{f}_k + H_{v,k} \tilde{f}_k = 0, \quad r > 0, t > 0, \tag{3.1}$$

and the initial condition  $\tilde{f}_k(r, 0) = (\tilde{f}_0)_k(r)$ , where  $H_{v,k}$  is given in (1.14). We recall that  $H_{v,k} = P_k \Phi(H_v)$  is an unbounded operator on  $V_k$ , canonically identified with  $L_r^2(\mathbb{R}^+)$ . Since  $\mathcal{D}(H_v) \subset \mathcal{D}(\Delta)$  and  $\Delta_k := P_k \Phi(\Delta) = \partial_{rr} + \frac{1}{r} \partial_r - \frac{k^2}{r^2}$ , if  $h, h' \in \mathcal{D}(H_{v,k})$  then

$$\langle \Delta_k h, h' \rangle_r = \langle h, \Delta_k h' \rangle_r = -\langle \partial_r h, \partial_r h' \rangle_r - k^2 \left\langle \frac{h}{r}, \frac{h'}{r} \right\rangle_r.$$

We also observe that  $(\tilde{f}_{\neq})_0 = 0$  and  $(\tilde{f}_{\neq})_k = \tilde{f}_k$  for  $k \neq 0$ . By Plancherel's identity, we then have that

$$\|f_{\neq}(\cdot, t)\|_{L^2(\mathbb{R}^2)}^2 = \|e^{-tH_v}(f_0)_{\neq}\|_{L^2(\mathbb{R}^2)}^2 = \sum_{k \in \mathbb{Z}, k \neq 0} \left\| e^{-tH_{v,k}}(\tilde{f}_0)_k \right\|_{L_r^2(\mathbb{R}^+)}^2.$$

Consequently, it is enough to bound  $e^{-tH_{v,k}}$ .

Since the operator  $H_{v,k}$  is  $m$ -accretive by Corollary 2.5, we can apply the result in (1.5). To conclude the proof of Theorem 1.1, it is then sufficient to establish a lower bound for the spectral function  $\Psi(H_{v,k})$  (see (1.5)–(1.6)).

Given  $\lambda \in \mathbb{R}$ , for ease of notation we set

$$H_p := H_{v,k} - i\lambda = \nu(-\Delta_k)^p + ik(u(r) - \tilde{\lambda}),$$

where  $\tilde{\lambda} = \frac{\lambda}{k}$ ,  $p = 1, 2$ . Also, to streamline the proof of the next result, we will write  $f$  instead of  $\tilde{f}_k$ , as no confusion may arise.



**Proposition 3.1.** Let the velocity profile  $u$  in (1.7) satisfy Assumption 1.1, and let  $k \neq 0$  and  $v$  satisfy  $v|k|^{-1} \leq 1$ . Then there exists a positive constant  $\epsilon_0$ , independent of  $v$ , such that

$$\Psi(H_{v,k}) \geq \epsilon_0 v^{\frac{m}{m+2p}} |k|^{\frac{2p}{m+2p}}$$

for  $p = 1$  or  $2$ .

**Proof.** We fix  $\lambda \in \mathbb{R}$  and pick  $g \in \mathcal{D}(H_{v,k})$  with unit  $L_r^2$  norm. By density of  $C_c^\infty(\mathbb{R}^2)$  in the domain of  $H_v$ , we can assume that  $g \in C^\infty(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$ . We let  $\chi : [0, +\infty) \rightarrow [-1, 1]$  be a smooth approximation of the signum function  $\text{sign}(r^q - \tilde{\lambda})$  such that  $\|\chi'\|_{L^\infty} \leq c_2 \delta^{-1}$ ,  $\|\chi''\|_{L^\infty} \leq c_2 \delta^{-2}$ ,  $\chi(u - \tilde{\lambda}) \geq 0$  and

$$\chi(r)(u(r) - \tilde{\lambda}) = |u(r) - \tilde{\lambda}|, \quad \text{whenever } |r - r_j| \geq \delta, \quad \forall j \in \{1, \dots, n\},$$

We distinguish two cases.

**Case 1** ( $p = 1$ ): We note that

$$\text{Re}(H_1 g, g)_r = v \|\partial_r g\|_{L_r^2(\mathbb{R}^+)}^2 + v k^2 \left\| \frac{g}{r} \right\|_{L_r^2(\mathbb{R}^+)}^2, \quad (3.2)$$

which implies

$$\|\partial_r g\|_{L_r^2(\mathbb{R}^+)}^2 \leq \frac{1}{v} \|H_1 g\|_{L_r^2(\mathbb{R}^+)} \|g\|_{L_r^2(\mathbb{R}^+)}.$$

On the other hand,

$$\langle H_1 g, \chi g \rangle_r = v \langle \partial_r g, \chi' g \rangle_r + v \langle \partial_r g, \chi \partial_r g \rangle_r + v k^2 \left\langle \frac{g}{r^2}, \chi g \right\rangle_r + i k \langle (u(r) - \tilde{\lambda}) g, \chi g \rangle_r,$$

so that

$$\text{Im} \langle H_1 g, \chi g \rangle_r = v \text{Im} \langle \partial_r g, \chi' g \rangle_r + k \langle (u(r) - \tilde{\lambda}) g, \chi g \rangle_r.$$

By applying the properties of the function  $\chi$  and using (3.2), we have

$$\begin{aligned} |k \langle (u(r) - \tilde{\lambda}) g, \chi g \rangle_r| &\leq \frac{c_2 v}{\delta} \|\partial_r g\|_{L_r^2(\mathbb{R}^+)} \|g\|_{L_r^2(\mathbb{R}^+)} + \|H_1 g\|_{L_r^2(\mathbb{R}^+)} \|g\|_{L_r^2(\mathbb{R}^+)} \\ &\leq \frac{c_2 v^{1/2}}{\delta} \|H_1 g\|_{L_r^2(\mathbb{R}^+)}^{1/2} \|g\|_{L_r^2(\mathbb{R}^+)}^{3/2} + \|H_1 g\|_{L_r^2(\mathbb{R}^+)} \|g\|_{L_r^2(\mathbb{R}^+)}. \end{aligned} \quad (3.3)$$

Next, we define

$$E := \{r \in [0, \infty) : |r - r_j| \geq \delta, \quad \text{for } j = 1, \dots, n\},$$

where  $r_j$ ,  $n \leq N$ , and  $\delta$  are as in Assumption 1.1, and observe that

$$|\langle (u(r) - \tilde{\lambda}) g, \chi g \rangle_r| \geq \int_E |u(r) - \tilde{\lambda}| |g|^2 r dr \geq c_1 \delta^m \int_E |g|^2 r dr. \quad (3.4)$$

Combining (3.3) and (3.4) gives

$$\begin{aligned} \int_E |g|^2 r dr &\leq \frac{c_2 v^{1/2}}{c_1 |k| \delta^{m+1}} \|H_1 g\|_{L_r^2(\mathbb{R}^+)}^{1/2} \|g\|_{L_r^2(\mathbb{R}^+)}^{3/2} + \frac{1}{c_1 |k| \delta^m} \|H_1 g\|_{L_r^2(\mathbb{R}^+)} \|g\|_{L_r^2(\mathbb{R}^+)} \\ &\leq \frac{c_2^2 v}{c_1^2 |k|^2 \delta^{2m+2}} \|H_1 g\|_{L_r^2(\mathbb{R}^+)} \|g\|_{L_r^2(\mathbb{R}^+)} + \frac{1}{4} \|g\|_{L_r^2(\mathbb{R}^+)}^2 + \frac{1}{c_1 |k| \delta^m} \|H_1 g\|_{L_r^2(\mathbb{R}^+)} \|g\|_{L_r^2(\mathbb{R}^+)}. \end{aligned} \quad (3.5)$$

We now estimate the  $L_r^2$  norm of  $g$  on the complement of  $E$ . Using the fact that  $|E^c| \leq N\delta$ , we obtain

$$\int_{E^c} |g|^2 r dr \leq N\delta \|g^2 r\|_{L^\infty(\mathbb{R}^+)}. \quad (3.6)$$

To bound the right-hand side, we use that  $g$  is smooth and bounded on  $\mathbb{R}^+$  to compute

$$\begin{aligned} g^2(r_0)r_0 &= \int_0^{r_0} \partial_r(g^2r) dr = 2 \int_0^{r_0} \partial_r g \cdot g \cdot r dr + \int_0^{r_0} g^2 dr \\ &\leq 2 \|\partial_r g\|_{L^2_r(\mathbb{R}^+)} \|g\|_{L^2_r(\mathbb{R}^+)} + \left\| \frac{g}{r} \right\|_{L^2_r(\mathbb{R}^+)} \|g\|_{L^2_r(\mathbb{R}^+)} , \end{aligned}$$

from which it follows that

$$\|g^2(r)r\|_{L^\infty(\mathbb{R}^+)} \leq 2 \|\partial_r g\|_{L^2_r(\mathbb{R}^+)} \|g\|_{L^2_r(\mathbb{R}^+)} + \left\| \frac{g}{r} \right\|_{L^2_r(\mathbb{R}^+)} \|g\|_{L^2_r(\mathbb{R}^+)} .$$

Hence (3.6) becomes

$$\begin{aligned} \int_{E^c} |g|^2 r dr &\leq N\delta \left( 2 \|\partial_r g\|_{L^2_r(\mathbb{R}^+)} \|g\|_{L^2_r(\mathbb{R}^+)} + \left\| \frac{g}{r} \right\|_{L^2_r(\mathbb{R}^+)} \|g\|_{L^2_r(\mathbb{R}^+)} \right) \\ &\leq N\delta \left( \frac{2}{\nu^{1/2}} \|Hg\|_{L^2_r(\mathbb{R}^+)}^{1/2} \|g\|_{L^2_r(\mathbb{R}^+)}^{3/2} + \frac{1}{\nu^{1/2}|k|} \|Hg\|_{L^2_r(\mathbb{R}^+)}^{1/2} \|g\|_{L^2_r(\mathbb{R}^+)}^{3/2} \right) \\ &\leq \frac{3N\delta}{\nu^{1/2}} \|Hg\|_{L^2_r(\mathbb{R}^+)}^{1/2} \|g\|_{L^2_r(\mathbb{R}^+)}^{3/2} \\ &\leq \frac{9N^2\delta^2}{\nu} \|Hg\|_{L^2_r(\mathbb{R}^+)} \|g\|_{L^2_r(\mathbb{R}^+)} + \frac{1}{4} \|g\|_{L^2_r(\mathbb{R}^+)}^2 , \end{aligned} \quad (3.7)$$

where we applied (3.2) in the second inequality and used the fact that  $|k| \geq 1$  in the third one. Adding (3.5) and (3.7) together gives

$$\|g\|_{L^2_r(\mathbb{R}^+)} \leq \left( \frac{2c_2^2\nu}{c_1^2|k|^2\delta^{2m+2}} + \frac{2}{c_1|k|\delta^m} + \frac{18N^2\delta^2}{\nu} \right) \|H_1g\|_{L^2_r(\mathbb{R}^+)} .$$

Since by hypothesis  $\nu \leq |k|$ , we can choose

$$\delta = \delta_0 \left( \frac{\nu}{|k|} \right)^{\frac{1}{m+2}} ,$$

ensuring  $\delta \geq \delta_0$ . Then finally

$$\|H_1g\|_{L^2_r(\mathbb{R}^+)} \geq \epsilon_0 \nu^{\frac{m}{m+2}} |k|^{\frac{2}{m+2}} \|g\|_{L^2_r(\mathbb{R}^+)} .$$

**Case 2** ( $p = 2$ ): Since for  $g$  and  $f$  in  $\mathcal{D}(H_{\nu,k})$   $\langle -\Delta_k f, g \rangle_r = \langle f, -\Delta_k g \rangle_r$ , by setting  $f = -\Delta_k g$  we have

$$\langle (-\Delta_k)^2 f, g \rangle = \|\Delta_k g\|_{L^2_r(\mathbb{R}^+)}^2 ,$$

and therefore,

$$\operatorname{Re} \langle H_2 g, g \rangle_r = \nu \|\Delta_k g\|_{L^2_r(\mathbb{R}^+)}^2 . \quad (3.8)$$

Moreover, with  $\chi$  as in Case 1,

$$\begin{aligned} \operatorname{Im} \langle H_2 g, \chi g \rangle_r &= \nu \operatorname{Im} \langle \Delta_k g, \chi'' g + 2\chi' \partial_r g + \frac{\chi'}{r} g + \chi \Delta_k g \rangle_r + k \langle (u(r) - \tilde{\lambda})g, \chi g \rangle_r \\ &= \nu \operatorname{Im} \langle \Delta_k g, \chi'' g + 2\chi' \partial_r g + \frac{\chi'}{r} g \rangle_r + k \langle (u(r) - \tilde{\lambda})g, \chi g \rangle_r , \end{aligned}$$

which further implies

$$|k \langle (u(r) - \tilde{\lambda})g, \chi g \rangle_r| \leq \frac{c_2\nu}{\delta^2} \|\Delta_k g\|_{L^2_r(\mathbb{R}^+)} \|g\|_{L^2_r(\mathbb{R}^+)} + \frac{2c_2\nu}{\delta} \|\Delta_k g\|_{L^2_r(\mathbb{R}^+)} \|\partial_r g\|_{L^2_r(\mathbb{R}^+)} \quad (3.9)$$

$$+ \frac{c_2\nu}{\delta} \|\Delta_k g\|_{L^2_r(\mathbb{R}^+)} \left\| \frac{g}{r} \right\|_{L^2_r(\mathbb{R}^+)} + \|H_2 g\|_{L^2_r(\mathbb{R}^+)} \|g\|_{L^2_r(\mathbb{R}^+)} . \quad (3.10)$$

Next, we recall that for  $g \in \mathcal{D}(H_{\nu,k})$ , the following identity holds

$$\langle -\Delta_k g, g \rangle_r = \|\partial_r g\|_{L^2_r(\mathbb{R}^+)}^2 + k^2 \left\| \frac{g}{r} \right\|_{L^2_r(\mathbb{R}^+)}^2 ,$$

from which it follows, on one hand, that

$$\|\partial_r g\|_{L^2_r(\mathbb{R}^+)}^2 \leq \|\Delta_k g\|_{L^2_r(\mathbb{R}^+)} \|g\|_{L^2_r(\mathbb{R}^+)} ,$$

and, on the other hand, that

$$\left\| \frac{g}{r} \right\|_{L^2_r(\mathbb{R}^+)}^2 \leq \|\Delta_k g\|_{L^2_r(\mathbb{R}^+)} \|g\|_{L^2_r(\mathbb{R}^+)} ,$$

where we used that  $|k| \geq 1$ . With these estimates at hand, (3.9) becomes

$$\begin{aligned} |k \langle (u(r) - \tilde{\lambda})g, \chi g \rangle_r| &\leq \frac{c_2\nu^{1/2}}{\delta^2} \|H_2 g\|_{L^2_r(\mathbb{R}^+)}^{1/2} \|g\|_{L^2_r(\mathbb{R}^+)}^{3/2} + \frac{3c_2\nu^{1/4}}{\delta} \|H_2 g\|_{L^2_r(\mathbb{R}^+)}^{3/4} \|g\|_{L^2_r(\mathbb{R}^+)}^{5/4} \\ &\quad + \|H_2 g\|_{L^2_r(\mathbb{R}^+)} \|g\|_{L^2_r(\mathbb{R}^+)} . \end{aligned}$$



In view of [Assumption 1.1](#), estimate (3.4) holds, which further gives

$$\begin{aligned} \int_E |g|^2 r \, dr &\leq \frac{c_2 v^{1/2}}{c_1 |k| \delta^{2+m}} \|H_2 g\|_{L^2_r(\mathbb{R}^+)}^{1/2} \|g\|_{L^2_r(\mathbb{R}^+)}^{3/2} + \frac{3c_2 v^{1/4}}{c_1 |k| \delta^{m+1}} \|H_2 g\|_{L^2_r(\mathbb{R}^+)}^{3/4} \|g\|_{L^2_r(\mathbb{R}^+)}^{5/4} \\ &\quad + \frac{1}{c_1 |k| \delta^m} \|H_2 g\|_{L^2_r(\mathbb{R}^+)} \|g\|_{L^2_r(\mathbb{R}^+)} \end{aligned} \quad (3.11)$$

$$\leq \frac{1}{4} \|g\|_{L^2_r(\mathbb{R}^+)}^2 + \left( \frac{c v}{|k|^2 \delta^{2(m+2)}} + \frac{c v^{1/3}}{|k|^{4/3} \delta^{4(m+1)/3}} + \frac{1}{c_1 |k| \delta^m} \right) \|H_2 g\|_{L^2_r(\mathbb{R}^+)} \|g\|_{L^2_r(\mathbb{R}^+)} . \quad (3.12)$$

On the complement on the set  $E$ , we have instead

$$\begin{aligned} \int_{E^c} |g|^2 r \, dr &\leq N \delta \left( 2 \|\partial_r g\|_{L^2_r(\mathbb{R}^+)} \|g\|_{L^2_r(\mathbb{R}^+)} + \left\| \frac{g}{r} \right\|_{L^2_r(\mathbb{R}^+)} \|g\|_{L^2_r(\mathbb{R}^+)} \right) \\ &\leq 3N \delta \|\Delta_k g\|_{L^2_r(\mathbb{R}^+)}^{1/2} \|g\|_{L^2_r(\mathbb{R}^+)}^{3/2} \\ &\leq \frac{3N \delta}{v^{1/4}} \|H_2 g\|_{L^2_r(\mathbb{R}^+)}^{1/4} \|g\|_{L^2_r(\mathbb{R}^+)}^{7/4} \\ &\leq \frac{c N^4 \delta^4}{v} \|H_2 g\|_{L^2_r(\mathbb{R}^+)} \|g\|_{L^2_r(\mathbb{R}^+)} + \frac{1}{4} \|g\|_{L^2_r(\mathbb{R}^+)}^2 . \end{aligned} \quad (3.13)$$

Adding (3.11) and (3.13) together yields that

$$\|g\|_{L^2_r(\mathbb{R}^+)}^2 \leq 2 \left( \frac{c v}{|k|^2 \delta^{2(m+2)}} + \frac{c v^{1/3}}{|k|^{4/3} \delta^{4(m+1)/3}} + \frac{1}{c_1 |k| \delta^m} + \frac{c N^4 \delta^4}{v} \right) \|H_2 g\|_{L^2_r(\mathbb{R}^+)} \|g\|_{L^2_r(\mathbb{R}^+)} .$$

Arguing as in Case 1, since by hypothesis  $v \leq |k|$ , we can choose  $\delta = \delta_0 \left( \frac{v}{|k|} \right)^{\frac{1}{m+4}}$ . Then we obtain

$$\|H_2 g\|_{L^2(rdr)} \geq \epsilon_0 v^{\frac{m}{m+4}} |k|^{\frac{4}{m+4}} \|g\|_{L^2(rdr)} ,$$

where  $\epsilon_0$  is independent of  $v$ .  $\square$

**Remark 3.1.** If  $u(r) = r^q$  with  $q \geq 1$  as in example (1.4), then [Assumption 1.1](#) holds with  $m = q$ . In fact, denote  $S := \{u^{-1}(\lambda), 0\}$  and note that  $\#S \leq 2$ . Fix any  $\delta > 0$ . For  $\lambda \leq 0$ ,  $S := \{0\}$  and  $|r^q - \lambda| > \delta^q$  for any  $r > \delta$ .

For  $0 < \lambda < \left(\frac{\delta}{2}\right)^q$ , we have

$$|r^q - \lambda| > \delta^q - \left(\frac{\delta}{2}\right)^q \geq \frac{\delta^q}{2}, \quad \forall |r - r_i| > \delta, \text{ with } r_i \in S.$$

For  $\lambda > \left(\frac{\delta}{2}\right)^q$ , we have instead  $r_\lambda := u^{-1}(\lambda) > \frac{\delta}{2}$  and there exists  $\tilde{r}$  in between  $r$  and  $r_\lambda$  such that

$$|r^q - \lambda| = q \tilde{r}^{q-1} |r - r_\lambda| > q (\delta/2)^{q-1} \delta \geq (q/2^{q-1}) \delta^q, \quad \forall |r - r_i| > \delta, \text{ with } r_i \in S.$$

**Remark 3.2.** The validity of [Assumption 1.8](#) is crucial in our argument to prove enhanced dissipation and restrict the class of flows that we can treat with this approach. In particular,  $\mathbf{u}$  can only have a finite amount of critical points. For example, the field

$$\mathbf{u}(x, y) = u(r) \sin(2\pi r)(-y, x),$$

with  $u$  unbounded on  $\mathbb{R}^+$  does not satisfy [Assumption 1.8](#).

[Theorem 1.1](#) readily follows by combining [Corollary 2.5](#) in Section 2 with [Proposition 3.1](#).

#### 4. Dissipation enhancement in $\mathbb{R}^3$ by a pipe parallel flow

In this section, we aim to prove [Theorem 1.2](#). We proceed in a manner analogous to that for circularly symmetric flows in Section 3, passing to cylindrical coordinates  $(r, \theta, z)$  in  $\Omega = D(0, 1) \times \mathbb{T}^2$ .

We apply the Fourier Transform in both the angular variable  $\theta$  and the axial variable  $z$ , and follow the notation discussed in the Introduction, denoting  $f(r \cos \theta, r \sin \theta, z, t) = \tilde{f}_k(r, t)$  and its  $\mathbf{k}$ -th Fourier coefficient by  $\tilde{f}_k(r, t)$ ,  $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$ , which belongs to  $L^2_r((0, 1)) = L^2((0, 1), rdr)$  with the induced inner product  $\langle \cdot, \cdot \rangle_r$  and norm. We observe that

$$\|\nabla f\|_{L^2(\mathbb{R}^2)}^2 = \sum_{\mathbf{k} \in \mathbb{Z}^2} \|\partial_r \tilde{g}_k\|_{L^2_r((0, 1))}^2 + k_1^2 \left\| \frac{\tilde{g}_k}{r} \right\|_{L^2_r((0, 1))}^2 + k_2^2 \|\tilde{g}_k\|_{L^2_r((0, 1))}^2 .$$

Again, if  $f$  satisfies (1.1) in  $\Omega$  for  $t > 0$  with homogeneous Neumann boundary conditions,  $\tilde{f}_k$  satisfies Eq. (3.1) with  $H_{v,k}$  given in (1.21) for  $0 < r < 1$  and  $t > 0$ , with initial condition  $\tilde{f}_k(r, 0) = (\tilde{f}_0)_k(r)$  and boundary condition

$$\partial_r \tilde{f}_k = 0 \text{ for } r = 1, \quad (4.1)$$

but we notice that all the arguments will hold also for Dirichlet boundary conditions  $f = 0$  for  $r = 1$ . To ensure uniqueness of the solution, we recall that we take  $f_0$  to be mean free, a condition preserved under the time evolution of  $f$  due to the divergence-free condition on  $\mathbf{u}$ , so that we can assume throughout  $\mathbf{k} \neq \mathbf{0}$ . As in the two-dimensional case, we view  $H_{v,k}$  as an unbounded operator on

$V_k = P_k \Phi(L^2(\Omega))$ , which we identify with  $L_r^2((0, 1))$ , where  $P_k$  is the projection onto the  $\mathbf{k}$ -th Fourier coefficient and  $\Phi$  is the isometry induced by the change from Cartesian to cylindrical coordinates. The domain of  $H_{v,k}$  is given by  $P_k \Phi(\dot{H}^2 \cap \mathcal{D}(L))(\Omega)$ ,  $p = 1, 2$ , where  $L$  is the transport operator  $\mathbf{u} \cdot \nabla$  and  $\dot{H}^2$  consists of mean-free functions in the Sobolev space  $H^2$ . Since  $\mathcal{D}(H_{v,k}) \subset \mathcal{D}(\Delta_N)$  with  $\Delta_N$  the 3D Laplacian with Neumann boundary conditions in  $\Omega$ , denoting again  $\Delta_k = P_k(\Phi(\Delta))$ , if  $h, h' \in \mathcal{D}(H_{v,k})$  then

$$\langle \Delta_k h, h' \rangle_r = \langle h, \Delta_k h' \rangle_r = -\langle \partial_r h, \partial_r h' \rangle_r - k_1^2 \left\langle \frac{h}{r}, \frac{h'}{r} \right\rangle_r - k_2^2 \langle h, h' \rangle_r.$$

By the change of variables formula and Plancherel's identity, we have

$$\|f_{\neq}(\cdot, t)\|_{L_r^2(\Omega)}^2 = \|e^{-tH_v}(f_0)_{\neq}\|_{L_r^2(\Omega)}^2 = \sum_{\mathbf{k} \in \mathbb{Z}^2, \mathbf{k} \neq \mathbf{0}} \|e^{-tH_{v,k}}(\tilde{f}_0)_k\|_{L_r^2((0,1))}^2.$$

Consequently, it is enough to bound  $e^{-tH_{v,k}}$ . Given that  $H_{v,k}$  is  $m$ -accretive, following the arguments in Section 2, we can again apply the result in [48], and it is sufficient to obtain a lower bound on the spectral function  $\Psi(H_{v,k})$  (see (1.5)–(1.6)).

For notational convenience, given  $\lambda \in \mathbb{R}$  we set

$$H_* := H_{v,\mathbf{k}} - i\lambda = v(-\partial_{rr} - \frac{1}{r}\partial_r + \frac{k_1^2}{r^2} + k_2^2) + ik_1 u(r) \sin(2\pi r) + ik_2 u(r) \cos(2\pi r) - i\lambda,$$

which we can rewrite as

$$\begin{aligned} H_* &= v(-\partial_{rr} - \frac{1}{r}\partial_r + \frac{k_1^2}{r^2} + k_2^2) + i|\mathbf{k}| \left( \frac{k_1}{|\mathbf{k}|} u(r) \sin(2\pi r) + \frac{k_2}{|\mathbf{k}|} u(r) \cos(2\pi r) - \frac{\lambda}{|\mathbf{k}|} \right) \\ &= v(-\partial_{rr} - \frac{1}{r}\partial_r + \frac{k_1^2}{r^2} + k_2^2) + i|\mathbf{k}| (\cos \alpha_{\mathbf{k}} \sin(2\pi r) u(r) + \sin \alpha_{\mathbf{k}} \cos(2\pi r) u(r) - \Lambda_{\mathbf{k}}), \end{aligned}$$

where we set

$$\cos \alpha_{\mathbf{k}} = \frac{k_1}{|\mathbf{k}|}, \quad \sin \alpha_{\mathbf{k}} = \frac{k_2}{|\mathbf{k}|}, \quad \Lambda_{\mathbf{k}} = \frac{\lambda}{|\mathbf{k}|}.$$

Also, to streamline the proof of the next result, we write  $g$  for  $\tilde{g}_k$ , as no confusion may arise.

**Proposition 4.1.** *Let the velocity profile  $u$  in (1.15) satisfy Assumption 1.2. Let  $\mathbf{k} \neq \mathbf{0}$  and  $v$  satisfy  $v|\mathbf{k}|^{-1} \leq 1$ . Then there exists a positive constant  $\epsilon_0$  independent of  $v$  such that*

$$\Psi(H_{v,k}) \geq \epsilon_0 v^{\frac{m}{m+2}} |\mathbf{k}|^{\frac{2}{m+2}},$$

**Proof.** We fix  $\lambda \in \mathbb{R}$  and pick  $g \in D(H_{v,k})$  with  $\|g\|_{L_r^2} = 1$ . Since  $C^\infty(\overline{\Omega})$  is dense in  $\mathcal{D}(H_v)$ , we can assume that  $g \in C^\infty((0, 1)) \cap L^\infty((0, 1))$ .

Let  $\chi : [0, 1] \rightarrow [-1, 1]$  be a smooth approximation of  $\text{sgn}(u(r)(\sin(2\pi r + \alpha_k) - \Lambda_{\mathbf{k}}))$  such that  $\|\chi'\|_{L^\infty} \leq c_2 \delta^{-1}$ ,  $\|\chi''\|_{L^\infty} \leq c_2 \delta^{-2}$ ,  $\chi(u(r)(\sin(2\pi r + \alpha_k) - \Lambda_{\mathbf{k}})) \geq 0$  and

$$\chi(r)(u(r) \sin(2\pi r + \alpha_k) - \Lambda_{\mathbf{k}}) = |u(r) \sin(2\pi r + \alpha_k) - \Lambda_{\mathbf{k}}|, \quad \text{whenever } |r - r_j| \geq \delta, \quad \forall j \in \{1, \dots, n\},$$

We note that

$$\text{Re}\langle H_* g, g \rangle_r = v \|\partial_r g\|_{L_r^2((0,1))}^2 + v k_1^2 \left\| \frac{g}{r} \right\|_{L_r^2((0,1))}^2 + v k_2^2 \|g\|_{L_r^2((0,1))}^2, \quad (4.2)$$

using the boundary conditions on  $g$ , which implies

$$\|\partial_r g\|_{L_r^2((0,1))}^2 \leq \frac{1}{v} \|H_* g\|_{L_r^2((0,1))}^2 \|g\|_{L_r^2((0,1))}^2.$$

On the other hand, since  $g \in \mathcal{D}(H_{v,k})$ , we have

$$\langle H_* g, \chi g \rangle_r = v \langle \partial_r g, \chi' g \rangle_r + v \langle \partial_r g, \chi \partial_r g \rangle_r + v k_1^2 \left\langle \frac{g}{r^2}, \chi g \right\rangle_r + v k_2^2 \langle g, \chi g \rangle_r + i|\mathbf{k}| \langle (u(r) \sin(2\pi r + \alpha_k) - \Lambda_{\mathbf{k}}) g, \chi g \rangle_r.$$

from which it follows that

$$\text{Im}\langle H_* g, \chi g \rangle_r = v \text{Im}\langle \partial_r g, \chi' g \rangle_r + |\mathbf{k}| \langle (u(r) \sin(2\pi r + \alpha_k) - \Lambda_{\mathbf{k}}) g, \chi g \rangle_r.$$

From the choice of the function  $\chi$  and by using (4.2), we also have

$$\begin{aligned} ||\mathbf{k}| \langle (u(r) \sin(2\pi r + \alpha_k) - \Lambda_{\mathbf{k}}) g, \chi g \rangle_r| &\leq \frac{c_2 v}{\delta} \|\partial_r g\|_{L_r^2((0,1))} \|g\|_{L_r^2((0,1))} + \|H g\|_{L_r^2((0,1))} \|g\|_{L_r^2((0,1))} \\ &\leq \frac{c_2 v^{1/2}}{\delta} \|H g\|_{L_r^2((0,1))}^{1/2} \|g\|_{L_r^2((0,1))}^{3/2} + \|H_* g\|_{L_r^2((0,1))} \|g\|_{L_r^2((0,1))}. \end{aligned} \quad (4.3)$$

Next, we denote

$$E := \{r \in [0, \infty) : |r - r_j| \geq \delta, \quad \text{for } j = 1, \dots, n\},$$

where  $r_j, \delta$ , and  $n \leq N$  are as in Assumption 1.2, and observe that

$$|\langle (u(r) \sin(2\pi r + \alpha_k) - \Lambda_{\mathbf{k}}) g, \chi g \rangle_r| \geq \int_E |u(r) \sin(2\pi r + \alpha_k) - \Lambda_{\mathbf{k}}| |g|^2 r dr \geq c_1 \delta^m \int_E |g|^2 r dr. \quad (4.4)$$

Combining (4.3) and (4.4), one has

$$\begin{aligned} \int_E |g|^2 r \, dr &\leq \frac{c_2 \nu^{1/2}}{c_1 |\mathbf{k}| \delta^{m+1}} \|H_* g\|_{L^2_r((0,1))}^{1/2} \|g\|_{L^2_r((0,1))}^{3/2} + \frac{1}{c_1 |\mathbf{k}| \delta^m} \|H_* g\|_{L^2_r((0,1))} \|g\|_{L^2_r((0,1))} \\ &\leq \frac{c_2^2 \nu}{c_1^2 |\mathbf{k}|^2 \delta^{2m+2}} \|H_* g\|_{L^2_r((0,1))} \|g\|_{L^2_r((0,1))} + \frac{1}{4} \|g\|_{L^2_r((0,1))}^2 + \frac{1}{c_1 |\mathbf{k}| \delta^m} \|H_* g\|_{L^2_r((0,1))} \|g\|_{L^2_r((0,1))}. \end{aligned} \quad (4.5)$$

On the complement  $E^c$ , since  $|E^c| \leq N\delta$ , it follows that

$$\int_{E^c} |g|^2 r \, dr \leq N\delta \|g^2 r\|_{L^\infty} \quad (4.6)$$

Using the smoothness and boundedness of  $g$ , for all  $r_0 \in [0, 1]$ ,

$$\begin{aligned} g^2(r_0)r_0 &= \int_0^{r_0} \partial_r(g^2 r) \, dr = 2 \int_0^{r_0} \partial_r g \cdot g \cdot r \, dr + \int_0^{r_0} g^2 \, dr \\ &\leq 2 \|\partial_r g\|_{L^2_r((0,1))} \|g\|_{L^2_r((0,1))} + \left\| \frac{g}{r} \right\|_{L^2_r((0,1))} \|g\|_{L^2_r((0,1))}, \end{aligned}$$

which implies

$$\|g^2(r)r\|_{L^\infty} \leq 2 \|\partial_r g\|_{L^2_r((0,1))} \|g\|_{L^2_r((0,1))} + \left\| \frac{g}{r} \right\|_{L^2_r((0,1))} \|g\|_{L^2_r((0,1))}.$$

Hence (4.6) becomes

$$\begin{aligned} \int_{E^c} |g|^2 r \, dr &\leq N\delta \left( 2 \|\partial_r g\|_{L^2_r((0,1))} \|g\|_{L^2_r((0,1))} + \left\| \frac{g}{r} \right\|_{L^2_r((0,1))} \|g\|_{L^2_r((0,1))} \right) \\ &\leq N\delta \left( \frac{2}{\nu^{1/2}} \|H_* g\|_{L^2_r((0,1))}^{1/2} \|g\|_{L^2_r((0,1))}^{3/2} + \frac{1}{\nu^{1/2} |\mathbf{k}|} \|H_* g\|_{L^2_r((0,1))}^{1/2} \|g\|_{L^2_r((0,1))}^{3/2} \right) \\ &\leq \frac{3N\delta}{\nu^{1/2}} \|H_* g\|_{L^2_r((0,1))}^{1/2} \|g\|_{L^2_r((0,1))}^{3/2} \\ &\leq \frac{9N^2\delta^2}{\nu} \|H_* g\|_{L^2_r((0,1))} \|g\|_{L^2_r((0,1))} + \frac{1}{4} \|g\|_{L^2_r((0,1))}^2, \end{aligned} \quad (4.7)$$

where we applied (4.2) in the second inequality and we used the fact that  $|\mathbf{k}| \geq 1$  in the third one. Adding up (4.5) and (4.7), we get

$$\|g\|_{L^2_r((0,1))} \leq \left( \frac{2c_2^2 \nu}{c_1^2 |\mathbf{k}|^2 \delta^{2m+2}} + \frac{2}{c_1 |\mathbf{k}| \delta^m} + \frac{18N^2\delta^2}{\nu} \right) \|H_* g\|_{L^2_r((0,1))}. \quad (4.8)$$

In the regime  $\nu \leq |\mathbf{k}|$ , choosing

$$\delta_0 = c_3 \left( \frac{\nu}{|\mathbf{k}|} \right)^{\frac{1}{m+2}},$$

with  $c_3$  small enough, we get

$$\|H_* g\|_{L^2_r((0,1))} \geq \epsilon_0 \nu^{\frac{m}{m+2}} |\mathbf{k}|^{\frac{2}{m+2}} \|g\|_{L^2_r((0,1))}.$$

This estimate, in turn, gives the desired bound

$$\Psi(H_{\nu,k}) \geq \epsilon_0 \nu^{\frac{m}{m+2}} |\mathbf{k}|^{\frac{2}{m+2}},$$

for  $\nu \leq |\mathbf{k}|$ .  $\square$

The proof of Theorem 1.2 now follows directly from Proposition 4.1.

**Remark 4.1.** If  $u$  is chosen as  $u(r) = 1$  or  $u(r) = \cos(2\pi r)$ , then Assumption 1.2 is fulfilled with  $m = 2$ . For a proof of this fact, we refer to [55] Example 2.1.

## Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Camilla Nobili reports financial support was provided by German Research Foundation.

## Data availability

No data was used for the research described in the article.

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