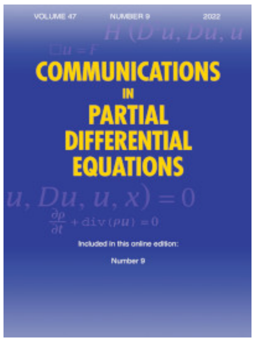

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Partial regularity of the heat flow of half-harmonic maps and applications to harmonic maps with free boundary

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ABSTRACT

We introduce a heat flow associated to half-harmonic maps, which have been introduced by Da Lio and Rivière. Those maps exhibit integrability by compensation in one space dimension and are related to harmonic maps with free boundary. We consider a new flow associated to these harmonic maps with free boundary which is actually motivated by a rather unusual heat flow for half-harmonic maps. We construct then weak solutions and prove their partial regularity in space and time via a Ginzburg-Landau approximation. The present paper complements the study initiated by Struwe and Chen-Lin.

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1. Introduction

In [1,2], Da Lio and Rivière introduced a new notion of harmonic map by considering critical points of a Gagliardo-type $\dot{H}^s(\mathbb{R}^d)$ semi-norm in the conformal case $s=\frac{1}{2}$ and $d=1$. Those maps have found a geometric application in the works of Fraser and Schoen about extremal metrics of Steklov eigenvalues (see e.g. [3] and references therein). These maps correspond to an extrinsinc version of harmonic maps with free boundary as proved by Millot and Sire in [4]. On the other hand, Moser [5] introduced an intrinsic version of those latter maps, and Roberts [6] investigated regularity of generalized versions of those maps, i.e. considering Gagliardo functionals for any $s\in(0,1)$. Whenever the extrinsic version of those maps is concerned, critical points of the functional introduced by Da Lio and Rivière satisfy the following equation in the distributional sense

$$(-\Delta)^{\frac{1}{2}}u\perp T_uN$$

whenever $u:\mathbb{S}^1\to N$. As pointed out in [4], the harmonic extension of those maps into the unit disk are so-called harmonic maps with free boundary. We now introduce such maps in a general setup: let (M, g) be an m-dimensional smooth Riemannian manifold with boundary ∂M and N be another smooth compact Riemannian manifold without boundary. Suppose Σ is a k-dimensional submanifold of N without boundary. Any continuous map $u_0: M\to N$ satisfying $u_0(\partial M)\subset\Sigma$ defines a relative homotopy class in maps from $(M, \partial M)$ to (N, Σ). A map $u : M \to N$ with $u(\partial M) \subset \Sigma$ is called homotopic to u_0 if there exists a continuous homotopy $h : [0,1] \times M \to N$ satisfying $h([0,1] \times \partial M) \subset \Sigma$, $h(0) = u_0$ and $h(1) = u$. An interesting problem is that whether or not each relative homotopy class of maps has a representation by harmonic maps, which is equivalent to the following problem:

$$\begin{cases} -\Delta u = \Gamma(u)(\nabla u, \nabla u), \\ u(\partial M) \subset \Sigma, \\ \dfrac{\partial u}{\partial \nu} \perp T_u \Sigma. \end{cases} \tag{1.1}$$

Here ν is the unit normal vector of M along the boundary ∂M, $\Delta \equiv \Delta_M$ is the Laplace-Beltrami operator of (M, g), Γ is the second fundamental form of N (viewed as a submanifold in $\mathbb{R}^\ell$ via Nash's isometric embedding), $T_p N$ is the tangent space in $\mathbb{R}^\ell$ of N at p and $\perp$ means orthogonal in $\mathbb{R}^\ell$. (1.1) is the Euler-Lagrange equation for critical points of the Dirichlet energy functional

$$E(u) = \int_M |\nabla u|^2 \, dv_g$$

defined over the space of maps

$$H^1_\Sigma(M, N) = \{u \in H^1(M, N) : u(x) \subset \Sigma \, \text{a.e.} \, x \in \partial M\}.$$

Here $H^1(M, N) = \{u \in H^1(M, \mathbb{R}^\ell) : u(x) \in N \, \text{a.e.} \, x \in M\}$. Both the existence and partial regularity of energy minimizing harmonic maps in $H^1_\Sigma(M, N)$ have been established (for example, in [7–11]). A classical approach to investigate (1.1) is to study the following parabolic problem

$$\begin{cases} \partial_t u - \Delta u = \Gamma(u)(\nabla u, \nabla u) & \text{on } M \times [0, \infty), \\ u(x, t) \in \Sigma & \text{on } \partial M \times [0, \infty), \\ \dfrac{\partial u}{\partial \nu}(x, t) \perp T_{u(x,t)} \Sigma & \text{on } \partial M \times [0, \infty) \\ u(\cdot, 0) = u_0 & \text{on } M. \end{cases} \tag{1.2}$$

This is the so-called harmonic map flow with free boundary. (1.2) was first studied by Ma [12] in the case $m = dim M = 2$, where a global existence and uniqueness result for finite energy weak solutions was obtained under suitable geometrical hypotheses on N and Σ. Global existence for weak solutions of (1.2) was established by Struwe in [13] for $m \geq 3$. In [14], Hamilton considered the case when $\partial N = \Sigma$ is totally geodesic and the sectional curvature $K_N \leq 0$. He proved the existence of a unique global smooth solution for (1.2). When N is an Euclidean space, the first equation in (1.2) is the standard heat equation

$$u_t - \Delta u = 0 \text{ on } M \times [0, \infty). \tag{1.3}$$

As pointed out in [15] and [13], estimates near the boundary for (1.2) are difficult because of the highly nonlinear boundary conditions. Struwe in [13] introduced the heat flow for the intrinsic version of harmonic maps with free boundary. In particular, he used a Ginzburg-Landau approximation in the interior, hence keeping the boundary condition highly nonlinear.

In the present paper we revisit the Struwe approximation argument by considering a natural, though unusual, heat flow associated to the equation derived by Da Lio and Rivière, that we called half-harmonic maps. Wettstein [16,17] considered the natural L^2-gradient flow of the $\dot{H}^{\frac{1}{2}}$-energy of half-harmonic map defined distributionally by

$$\partial_t u + (-\Delta)^{\frac{1}{2}} u \perp T_u N \text{ in } \mathbb{R} \times [0, \infty), \tag{1.4}$$

where $\partial_t + (-\Delta)^{\frac{1}{2}}$ is the so-called Poisson operator whose expression is explicit. Some weak solutions for this flow have been constructed in [18]. Infinite-time blow up has been considered in [19].

As far as the (partial) regularity of the heat flow of harmonic maps is concerned, a way to construct weak solutions is to have a suitable monotonicity formula for a Ginzburg-Landau approximation of the system (see the monograph [20] for an up to date account). At the moment such a monotonicity formula is not available for the latter system (1.4), despite this flow being the natural one analytically.

Therefore, we replace the previous flow by

$$\begin{cases} (\partial_t - \Delta)^{\frac{1}{2}} u \perp T_u N & \text{in } \mathbb{R}^m \times (0, +\infty), \\ u(x,t) = u_0(x,t) & \text{in } \mathbb{R}^m \times (-\infty, 0]. \end{cases} \tag{1.5}$$

Clearly, these two flows admit the same stationary solutions, which are (weak) half-harmonic maps into N. However, it is known (see [21]) that, suitably formulated, the flow (1.5) does enjoy a monotonicity formula. This is due to the existence of a suitable (caloric) extension to the upper-half space (see [22] and [23]). As we will see below, though the operator $(\partial_t - \Delta)^{\frac{1}{2}}$ defined as a Fourier-Laplace multiplier seems unnatural, its caloric extension to the upper half-space is naturally associated to extrinsic harmonic maps with free boundary. Considering a Ginzburg-Landau approximation at the boundary, which is more in the spirit of the approach by Da Lio and Rivière and motivated by the Ginzburg-Landau approximation of extrinsic harmonic maps with free boundary proved in [4], we construct weak solutions which are partially regular.

We will always assume in the following that $(M, g) = (\mathbb{R}^m, dx^2)$. To keep the technicalities as simple as possible we will present the detailed proof for the case that the target manifold is a sphere, and provide necessary modifications of proof for general target manifolds N in Appendix B. Let $(\mathbb{S}^{\ell-1}, g_{can})$ be the $(\ell-1)$ dimensional unit sphere in $\mathbb{R}^\ell$ equipped with the standard metric. Given $u_0 : \mathbb{R}^m \times (-\infty, 0] \to \mathbb{S}^{\ell-1}$ with $u_0(\cdot, t) \in \dot{H}^s(\mathbb{R}^m)$ for $t \le 0$, we introduce the following evolution: for $(X, t) = (x, y, t) \in \mathbb{R}^{m+1}_+ \times \mathbb{R}$,

$$\begin{cases} \dfrac{\partial u_\varepsilon(X,t)}{\partial t} = \Delta_X u_\varepsilon(X,t) & \text{in } \mathbb{R}^{m+1}_+ \times (0, \infty), \\ u_\varepsilon(x, 0, t) = u_0(x,t) & \text{in } \mathbb{R}^m \times (-\infty, 0], \\ \lim_{y \to 0^+} \dfrac{\partial u_\varepsilon(X,t)}{\partial y} = -\dfrac{1}{\varepsilon^2}(1 - |u_\varepsilon|^2) u_\varepsilon & \text{in } \mathbb{R}^m \times (0, +\infty). \end{cases} \tag{1.6}$$

The following result is our main theorem.

Theorem 1.1. *For any given $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^m, N)$, the following statements hold:*

A. *There exists a global solution $u \in L^\infty(\mathbb{R}_+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^m, N))$ of the equation of $\frac{1}{2}$-harmonic map heat flow:*

$$\begin{cases} (\partial_t - \Delta)^{\frac{1}{2}} u \perp T_u N & \text{in}\, \mathbb{R}^m \times (0,\infty), \\ u|_{t\le 0} = u_0 & \text{in}\, \mathbb{R}^m. \end{cases} \tag{1.7}$$

Furthermore, there exists a closed subset $\Sigma \subset \mathbb{R}^m \times (0,\infty)$, with locally finite m-dimensional parabolic Hausdorff measure, such that $u \in C^\infty(\mathbb{R}^m \times (0,\infty) \setminus \Sigma)$, and

B. *there exists $T_0 > 0$, depending on $\|u_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^m)}$, such that $\Sigma \cap (\mathbb{R}^m \times [T_0,\infty)) = \emptyset$ and*

$$\|\nabla u(\cdot,t)\|_{L^\infty(\mathbb{R}^m)} \le \frac{C}{\sqrt{t}}, \qquad \forall t \ge T_0.$$

Hence there exists a point $p \in N$ such that $u(\cdot,t) \to p$ in $C^2_{\mathrm{loc}}(\mathbb{R}^m)$ as $t \to \infty$, and

C. *for any $0 < t < T_0$, $\Sigma_t = \Sigma \cap (\mathbb{R}^m \times \{t\})$ has finite $(m-1)$-dimensional Hausdorff measure.*

At the end of this section, we would like to remark that when $\frac{1}{2} \neq s \in (0,1)$, while Lemma 3.1 for the energy monotonicity inequality remains true, the arguments presented in Lemma 4.3 (for the ϵ_0-regularity) and in Proposition 5.1 (for uniform boundary $C^{1,\alpha}$-estimates) do not seem to be valid because of the degeneracy of coefficient function y^{1-2s} in the extended equation. Thus Theorem 1.1 remains open for $s \in (0,1) \setminus \{\frac{1}{2}\}$.

2. Existence of weak solutions

In this section we prove the existence of a weak solution of

$$\begin{cases} (\partial_t - \Delta)^s u \perp T_u \mathbb{S}^{\ell-1} & \text{in } \mathbb{R}^m \times (0,\infty), \\ \\ u(x,t) = u_0(x) & \text{in } \mathbb{R}^m \times (-\infty, 0], \end{cases} \tag{2.1}$$

for any $s \in (0,1)$, here $u_0 \in \dot{H}^s(\mathbb{R}^m, \mathbb{S}^{\ell-1})$. This equation is a mere generalization of (1.5), and thanks to [22,23] fits well in our framework (see also [24] for a similar setup and related results). It is important to remark that the case $s = 1/2$ and $m = 1$ corresponds to a geometric problem since the image by those maps are minimal surfaces with free boundary. See [4]. We will then consider only the case $s = 1/2$ in any dimension in the subsequent sections. However, we provide here the existence of weak solutions (but not their partial regularity) for the general system (1.5) for all $0 < s < 1$ when the initial datum u_0 is a function of x only.

Here $(\partial_t - \Delta)^s u$ is defined by the Poisson representation formula (found independently by Nyström-Sande [22] and by Stinga-Torrea [23]): For any u belonging to a suitable class of functions (see [22,23])

$$(\partial_t - \Delta)^s u(x,t) = \int_0^\infty \int_{\mathbb{R}^m} (u(x,t) - u(x-z,t-\tau))K_s(z,\tau)\ dzd\tau, \tag{2.2}$$

where the kernel K_s is given by

$$K_s(z,\tau) = \frac{1}{(4\pi)^{\frac{m}{2}}|\Gamma(-s)|}\frac{e^{-\frac{|z|^2}{4\tau}}}{\tau^{\frac{m}{2}+1+s}},\ \forall z \in \mathbb{R}^m, \tau > 0,$$

where Γ denotes the Gamma function.

As in [25] and in [26], we relax the constraint $u \in \mathbb{S}^{\ell-1}$ and introduce the Ginzburg-Landau type approximation. For any $\varepsilon > 0$, we consider the problem (c_s is a normalization constant that will be defined later)

$$\begin{cases} (\partial_t - \Delta)^s u_\varepsilon = \frac{c_s}{\varepsilon^2}(1-|u_\varepsilon|^2)u_\varepsilon & \text{in } \mathbb{R}^m \times (0,+\infty), \\ u_\varepsilon(x,t) = u_0(x) & \text{in } \mathbb{R}^m \times (-\infty, 0]. \end{cases} \tag{2.3}$$

Here $c_s = \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}$.

The proof of the existence of a solution to the approximate problem (2.3) and of its convergence to a solution of (2.1) heavily relies on the possibility of reformulating the nonlocal problems (2.1) and (2.3) as local problems but in an extended variable setting (see [22] and [23]).

First we recall the extension method for the nonlocal operator $(\partial_t - \Delta)^s$, then we prove the existence of a solution of the Ginzburg-Landau approximation (2.3). Finally, we address the problem of the convergence when $\varepsilon \to 0$.

2.1. Extension method

In this subsection we briefly recall the extension method of [22] and [23]. If $u = u(x,t)$ is a function belonging[1] to

$$D(H^s) := \left\{ v \in \mathcal{S}'(\mathbb{R}^{m+1}) : \ \widehat{v} \in L^1_{\text{loc}}(\mathbb{R}^{m+1}), \quad (\xi,\sigma) \mapsto \left((2\pi|\xi|)^2 + 2\pi i\sigma\right)^s \widehat{v}(\xi,\sigma) \in L^2(\mathbb{R}^{m+1}) \right\}, \tag{2.4}$$

where $\mathcal{S}'(\mathbb{R}^{m+1})$ is the space of tempered distributions and $\widehat{v}$ is the Fourier transform with respect to $(x,\ t)$, then we can consider the degenerate parabolic problem in the extended variables $(X,t) := (x,y,t) \in \mathbb{R}^m \times (0,+\infty) \times \mathbb{R}$:

$$\begin{cases} y^{1-2s}\dfrac{\partial U(X,t)}{\partial t} = \text{div}_X(y^{1-2s}\nabla_X U(X,t)) & \text{in } \mathbb{R}^{m+1}_+ \times \mathbb{R}, \\ U(x,0,t) = u(x,t), & \text{in } \mathbb{R}^m \times \mathbb{R}. \end{cases} \tag{2.5}$$

Given the boundary datum u in the regularity class $D(H^s)$ above, there exists a smooth solution U of the parabolic problem above. Moreover, there holds (see [22] and [23])

[1]Note that in the papers [29] and [33] it is actually considered a slightly different definition for $D(H^s)$that prescribes that its elements belong to $L^2(\mathbb{R}^{m+1})$. The reason for considering the "homogeneous" version (2.4) lies in the fact that we have to deal with maps satisfying the constraint $|v| = 1$ in the whole $\mathbb{R}^{m+1}$.

$$-\frac{1}{c_s}\lim_{y\to 0^+} y^{1-2s}\frac{\partial U(X,t)}{\partial y} = (\partial_t - \Delta)^s u. \tag{2.6}$$

The limit in (2.6) is understood in the $L^2(\mathbb{R}^m \times \mathbb{R})$ sense. See also [27].

With this discussion in mind we rewrite the nonlocal and nonlinear system (2.1) as the following local and degenerate parabolic problem with nonlinear boundary conditions in the extended variables $(X,t) \in \mathbb{R}^{m+1}_+ \times \mathbb{R}$:

$$\begin{cases} y^{1-2s}\dfrac{\partial U(X,t)}{\partial t} = \mathrm{div}_X(y^{1-2s}\nabla_X U(X,t)), & \text{in } \mathbb{R}^{m+1}_+ \times \mathbb{R}, \\ U(x,0,t) = u_0(x), & \text{in } \mathbb{R}^m \times (-\infty, 0], \\ \lim_{y\to 0^+} y^{1-2s}\dfrac{\partial U(X,t)}{\partial y} \perp T_u\mathbb{S}^{\ell-1}, & \text{on } \mathbb{R}^m \times (0,+\infty), \end{cases} \tag{2.7}$$

where the limit in the last condition is understood in the L^2 sense. We note that the previous system for the case $s = 1/2$ arises as the harmonic map flow with a free boundary and has been investigated in [15].

Notice that our solution u to (2.1) is $\mathbb{S}^{\ell-1}$ valued, and therefore it is not in $L^2(\mathbb{R}^m)$. Nevertheless, one can interpret distributional solutions of (2.1) directly through traces of weak solutions of (2.7), which are defined below. In particular, in [22,23] the domain $D(H^s)$ is designed so that the R.H.S. of (2.2) makes sense. As previously mentioned, we slightly modify this domain to take into account the constraint. In any case, we always interpret solutions of (2.1) via its extension.

Remark 2.1. We also want to point out that this is the flow of harmonic maps with free boundary from a manifold with edge-singularities into the sphere. Indeed, for $a > -1$, the operator $y^{2-a}\mathrm{div}(y^a\nabla)$ is an edge-operator in the sense of [28]. Therefore, the flow (2.7) is the Ginzburg-Landau approximation of the heat flow of harmonic maps of a manifold with edge-singularities into the round sphere. See also [29] for related results. We postpone a deeper investigation of such flows on singular manifolds to future work.

Remark 2.2. We would like to point out that the approach used in [30] would be an alternative way to build weak solutions for our system too.

Now we discuss the weak formulation of (2.7). First of all, we introduce some functional spaces. Given an open set $A \subset \mathbb{R}^{m+1}_+$, we introduce the Lebesgue and the Sobolev spaces with weights

$$L^2(A; y^{1-2s}\mathrm{d}X) := \left\{ V : A \to \mathbb{R}^\ell : \int_A |V|^2 y^{1-2s}\mathrm{d}X < +\infty \right\}, \tag{2.8}$$

and

$$H^1(A; y^{1-2s}\mathrm{d}X) := \left\{ V : A \to \mathbb{R}^\ell : V \text{ and } \nabla_X V \in L^2(A, y^{1-2s}\mathrm{d}X) \right\}, \tag{2.9}$$

endowed with the norm

$$\|V\|_{H^1(A;y^{1-2s}\mathrm{d}X)} := \left(\int_A |V|^2 y^{1-2s}\mathrm{d}X + \int_A |\nabla_X V|^2 y^{1-2s}\mathrm{d}X\right)^{\frac{1}{2}}. \tag{2.10}$$

Moreover, we let

$$\mathbb{X}^{2s}(A) := \{V : A \to \mathbb{R}^\ell : \nabla_X V \in L^2(A, y^{1-2s}\mathrm{d}X)\}, \tag{2.11}$$

endowed with the semi-norm

$$\|V\|_{\mathbb{X}^{2s}(A)} := \left(\int_A y^{1-2s}|\nabla_X V|^2 \mathrm{d}X\right)^{1/2}. \tag{2.12}$$

Thanks to [31, Theorem 2.8], there exists a unique bounded linear operator (the trace operator)

$$\mathrm{Tr} : \mathbb{X}^{2s}(\mathbb{R}^{m+1}_+) \to \dot{H}^s(\mathbb{R}^m), \tag{2.13}$$

such that $\mathrm{Tr}V := V|_{\mathbb{R}^m\times\{0\}}$ for any $V \in C^1_c(\mathbb{R}^{m+1})$

Finally, given a Banach space $\mathcal{X}$ with norm $\|\cdot\|_{\mathcal{X}}$, we let $L^p(a, b; \mathcal{X})$ ($p \in [1, +\infty]$) denote the space of classes of functions which are strongly measurable on $[a, b]$ and with values in $\mathcal{X}$ and such that

$$\|v\|_{L^p(a,\,b;\mathcal{X})} < +\infty,$$

where

$$\|v\|_{L^p(a,\,b;\mathcal{X})} := \begin{cases} \left(\int_a^b \|v(t)\|^p_X \mathrm{d}t\right)^{1/p} & \text{if} \quad p \in [1, +\infty) \\ \operatorname{ess\,sup}_{t\in(a,\,b)}\|v(t)\|_{\mathcal{X}} & \text{if} \quad p = +\infty. \end{cases}$$

Moreover, we let

$$H^1(a, b; \mathcal{X}) := \left\{v \in L^2(a, b; \mathcal{X}) : \quad \frac{\mathrm{d}}{\mathrm{d}t} v \in L^2(a, b; \mathcal{X})\right\},$$

where the derivative is understood in the sense of distributions (see, e.g., [32, Chapter 1])

Definition 2.1. Given a $u_0 \in \dot{H}^s(\mathbb{R}^m, \mathbb{S}^{\ell-1})$, a map $U : \mathbb{R}^{m+1}_+ \times \mathbb{R} \to \mathbb{R}^\ell$, with $|U(x, 0, t)| = 1$ for almost every $(x, t) \in \mathbb{R}^m \times \mathbb{R}$, is weak solution of (2.7) if

$$\partial_t U \in L^2(\mathbb{R}_+; L^2(\mathbb{R}^{m+1}_+, y^{1-2s}\mathrm{d}X)), \tag{2.14}$$

$$U \in L^\infty(\mathbb{R}_+; \mathbb{X}^{2s}(\mathbb{R}^{m+1}_+)), \tag{2.15}$$

$$U(x, 0, t) = u_0(x) \qquad \text{a.e.}(\mathrm{x}, t) \in \mathbb{R}^{\mathrm{m}} \times (-\infty, 0], \tag{2.16}$$

and

$$\int_0^\infty \int_{\mathbb{R}^{m+1}_+} (\langle \partial_t U, \Phi\rangle + \langle \nabla_X U, \nabla_X \Phi\rangle) y^{1-2s}\mathrm{d}X\mathrm{d}t = 0, \tag{2.17}$$

for any $\Phi \in L^\infty(\mathbb{R}_+; \mathbb{X}^{2s}(\mathbb{R}^{m+1}_+)) \cap L^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^{m+1}_+))$ with $\Phi(x, 0, t) \in T_{U(x,\,0,\,t)}\mathbb{S}^{\ell-1}$ for almost every $(x, t) \in \mathbb{R}^m \times (0, +\infty)$.

Note that if U is a weak solution according to the above definition taking Φ with $\Phi(x,0,t)=0$ for almost any $(x,t)\in\mathbb{R}^m\times(0,+\infty)$ we get that U verifies

$$y^{1-2s}\frac{\partial U(X,t)}{\partial t}=\operatorname{div}_X\big(y^{1-2s}\nabla_X U(X,t)\big),\qquad \text{in } \mathbb{R}^{m+1}_+\times\mathbb{R}. \tag{2.18}$$

Owing to the previous definition, we now define what we mean by a weak solution of the original system (2.1):

Definition 2.2. Given $u_0\in\dot{H}^s(\mathbb{R}^m,\mathbb{S}^{\ell-1})$, we say that $u:\mathbb{R}^m\times\mathbb{R}\to\mathbb{S}^{\ell-1}$ is a weak solution of (2.1) if the pair $(U,\,u)$ with $u=\operatorname{Tr}(U)$ is a weak solution of the extended equation according to Definition 2.1.

Remark 2.3. It would be possible of course to have a more straightforward definition of weak solutions for (2.1) by defining suitable function spaces so that the Fourier-Laplace multiplier $(\partial_t-\Delta)^s$ is well defined. This would actually introduce some additional technicalities which are unnecessary for our purposes and we do not pursue along this line. We refer the reader to [27] for a related construction.

Following [20, 25, 26], in the next Lemma we exploit the symmetry of the constraint $\mathbb{S}^{\ell-1}$ to write (2.17) in an equivalent way that is more suited for the treatment of the nonlinear boundary condition in the limit procedure. The reformulation of (2.17) makes use of test functions defined in $\mathbb{R}^{m+1}_+\times\mathbb{R}$ with values in $\bigwedge_k(\mathbb{R}^\ell)$. Therefore we have to introduce some notation. The exterior algebra of $\mathbb{R}^\ell$ is denoted by $\bigwedge(\mathbb{R}^\ell)$ and the exterior (or wedge) product by $\wedge$. If $e_1,\dots,e_\ell$ is the canonical orthonormal basis of $\mathbb{R}^\ell$, we let $\bigwedge_k(\mathbb{R}^\ell)$ $(k\le\ell)$ be the space of k-vectors, namely the subspace of $\bigwedge(\mathbb{R}^\ell)$ spanned by $e_{i_1}\wedge\dots\wedge e_{i_k}$ with $(1\le i_1\le\dots,i_k\le\ell)$. We let $\langle\cdot,\cdot\rangle$ denote the scalar product in $\mathbb{R}^\ell$. We denote with the same symbol the induced scalar product in $\bigwedge_k(\mathbb{R}^\ell)$

$$\langle v_1\wedge\dots\wedge v_k, w_1\wedge\dots\wedge w_k\rangle := \det(\langle v_i,w_i\rangle), \tag{2.19}$$

where $v_i,w_i\in\mathbb{R}^\ell$ for $i=1,\dots,k$.

We finally introduce the Hodge star operator

$$\star:\textstyle\bigwedge_k(\mathbb{R}^\ell)\to\bigwedge_{\ell-k}(\mathbb{R}^\ell)\qquad 0\le k\le\ell,$$

by

$$\star(e_{i_1}\wedge\dots\wedge e_{i_k}) := e_{j_1}\wedge\dots\wedge e_{j_{\ell-k}},$$

where $j_1,\dots,j_{\ell-k}$ is chosen in such a way that $e_{i_1},\dots,e_{i_p},e_{j_1},\dots,e_{j_{\ell-k}}$ is a (positive) basis of $\mathbb{R}^\ell$. The following hold

$$\begin{gathered}\star(1)=e_1\wedge\dots\wedge e_\ell,\\ \star(e_1\wedge\dots\wedge e_\ell)=1,\\ \star\star v=(-1)^{k(\ell-k)}v,\qquad \forall v\in\textstyle\bigwedge_k(\mathbb{R}^\ell),\end{gathered}$$

and

$$u\wedge\star v=\langle u,v\rangle e_1\wedge\dots\wedge e_\ell,\qquad \text{for any } u,v\in\textstyle\bigwedge_k(\mathbb{R}^\ell), \tag{2.20}$$

or, equivalently,

$$\star(u\wedge\star v)=\langle u,v\rangle \qquad \text{for any } u,v\in \textstyle\bigwedge_k(\mathbb{R}^\ell). \tag{2.21}$$

In the familiar case in which u, v are vectors in $\mathbb{R}^3$, then the relation above with $\ell=3$ and $k=1$ gives

$$\star(u\wedge v)=u\times v.$$

Then, we introduce some new function space. We set

$$\mathbb{X}^{2s}\Big(\mathbb{R}^{m+1}_+;\textstyle\bigwedge_{\ell-2}(\mathbb{R}^\ell)\Big):=\Big\{V:\mathbb{R}^{m+1}_+\to\textstyle\bigwedge_k(\mathbb{R}^\ell):\quad \nabla_X V\in L^2\Big(\mathbb{R}^{m+1}_+,y^{1-2s}\mathrm{d}X\Big)\Big\}.$$

We have the following

Lemma 2.4. *U is a weak solution in the sense of Definition 2.1 if and only if U verifies (2.14), (2.15), (2.16), (2.18) and*

$$\int_0^\infty\int_{\mathbb{R}^{m+1}_+}\big(\langle\partial_t U,\star(U\wedge\Psi)\rangle+\langle\nabla_X U,\star(U\wedge\nabla_X\Psi)\rangle\big)y^{1-2s}\mathrm{d}X\mathrm{d}t=0, \tag{2.22}$$

for any $\Psi\in L^\infty(\mathbb{R}_+;\mathbb{X}^{2s}(\mathbb{R}^{m+1}_+;\bigwedge_{\ell-2}(\mathbb{R}^\ell)))\cap L^\infty(\mathbb{R}_+;L^\infty(\mathbb{R}^{m+1}_+;\bigwedge_{\ell-2}(\mathbb{R}^\ell)))$.

Proof. If U is a weak solution in the sense of Definition 2.1, then we take $\Phi=\star(U\wedge\Psi)$ where $\Psi\in L^\infty(\mathbb{R}_+;\mathbb{X}^{2s}(\mathbb{R}^{m+1}_+;\bigwedge_{\ell-2}(\mathbb{R}^\ell)))$. Thanks to the properties of the wedge product and of the Hodge-star operator, it is immediate to check that Φ is indeed a vector field. The fact that $\Phi\in\mathbb{X}^{2s}(\mathbb{R}^{m+1}_+)$ for a.e. t is a consequence of the the fact that its components are product of functions which lie in $\mathbb{X}^{2s}(\mathbb{R}^{m+1}_+)$ and in L^∞ for almost every t. We have to check that $\Phi(x,0,t)\in T_{U(x,0,t)}\mathbb{S}^{\ell-1}$, namely that, denoting with $u(\cdot,\cdot):=U(\cdot,0,\cdot)$ (in the sense of traces),

$$\langle u,\star(u\wedge\Psi)\rangle=0 \qquad \text{a.e. in } \mathbb{R}^m\times\mathbb{R}.$$

This is a consequence of (2.21). In fact,

$$\langle u,\star(u\wedge\Psi)\rangle=\star(u\wedge(\star\star(u\wedge\Psi)))=(-1)^{\ell-1}\star((u\wedge(u\wedge\Psi)))=0. \tag{2.23}$$

Finally, since the Hodge star operator commutes with the covariant differentiation (here derivation in $\mathbb{R}^{m+1}_+$) we have that

$$\begin{aligned}\langle\nabla_X U,\nabla_X(\star(U\wedge\Psi))\rangle&=\langle\nabla_X U,\star(\nabla_X U\wedge\Psi)\rangle+\langle\nabla_X U,\star(U\wedge\nabla_X\Psi)\rangle\\&=\langle\nabla_X U,\star(U\wedge\nabla_X\Psi)\rangle,\end{aligned}$$

where the first addendum is treated as in (2.23). As a result we have that U verifies also (2.22).

On the other hand, let U be a function verifyng (2.14), (2.15), (2.16), (2.17) and (2.18). For any given vector field Φ as in the Definition 2.1 we set

$$\Psi:=\star(U\wedge\Phi).$$

We have that $\Psi\in L^\infty(\mathbb{R}_+;\mathbb{X}^{2s}(\mathbb{R}^{m+1}_+;\bigwedge_{\ell-2}(\mathbb{R}^\ell)))$. Moreover for almost any $(x,t)\in\mathbb{R}^m\times(0,+\infty)$ there holds

$$\star(U\wedge\Psi)=\Phi.$$

Thus (2.18) and (2.22) give that U verifies also (2.17) and thus it is a weak solution in the sense of Definition (2.1). □

According to [23], given a solution U of the above problem, its trace on $\mathbb{R}^m\times\{0\}$

$$u(x,t):=\mathrm{Tr}U(x,y,t),$$

is indeed a (weak) solution of (2.1). Weak solutions to (2.7) are constructed as limits of solution of the the (local) extension of the Ginzburg Landau approximation (2.3) of (2.1). Therefore, for any $\varepsilon>0$ we consider the following system

$$\begin{cases} y^{1-2s}\dfrac{\partial U_\varepsilon(X,t)}{\partial t}=\mathrm{div}_X(y^{1-2s}\nabla_X U_\varepsilon(X,t)) & \text{in } \mathbb{R}^{m+1}_+\times(0,\infty),\\ U_\varepsilon(x,0,t)=u_0(x), & \text{in } \mathbb{R}^m\times(-\infty,0],\\ \lim_{y\to 0^+} y^{1-2s}\dfrac{\partial U_\varepsilon(X,t)}{\partial y}=-\dfrac{c_s}{\varepsilon^2}(1-|U_\varepsilon|^2)U_\varepsilon, & \text{in } \mathbb{R}^m\times(0,+\infty). \end{cases} \tag{2.24}$$

2.2. Existence for the approximate problem and a priori estimates

In this subsection we discuss the existence of the approximate problem (2.24).

First of all, we introduce some notation. For $\varepsilon>0$ and $V\in\mathbb{X}^{2s}(\mathbb{R}^{m+1}_+)$, we introduce the following energy functional

$$\mathscr{E}_\varepsilon(V,v):=\frac{1}{2}\int_{\mathbb{R}^{m+1}_+}y^{1-2s}|\nabla_X V|^2\mathrm{d}X+\frac{c_s}{4\varepsilon^2}\int_{\mathbb{R}^m}(1-|v|^2)^2\mathrm{d}x,\ \text{for } V\in\mathcal{V}, \tag{2.25}$$

where $v=\mathrm{Tr}\ V$. We seek for minimizers in the space

$$\mathcal{V}:=\left\{V\in\mathbb{X}^{2s}(\mathbb{R}^{m+1}_+):v:=\mathrm{Tr}(V)|(|v|^2-1)^2\in L^1(\mathbb{R}^m)\right\}.$$

We let $U_0:\mathbb{R}^{m+1}_+\times(-\infty,0]\to\mathbb{R}^\ell$ be the Caffarelli-Silvestre extension of u_0, namely, for any $t\in(-\infty,0]$,

$$\begin{cases} -\mathrm{div}\big(y^{1-2s}\nabla U_0(X,t)\big)=0 & \text{in } \mathbb{R}^{m+1}_+,\\ U_0(x,0,t)=u_0(x) & \text{on } \mathbb{R}^m. \end{cases} \tag{2.26}$$

Thanks to [33] we have that the extension operator

$$E:v\mapsto V,\ \text{with } V \text{ the unique solution of (2.26) with boundary datum } v, \tag{2.27}$$

is an isometry from $\dot{H}^s(\mathbb{R}^m)$ to $\mathbb{X}^{2s}(\mathbb{R}^{m+1}_+)$ and we have

$$\|\mathrm{Tr}V\|_{\dot{H}^s(\mathbb{R}^m)}=\|E(\mathrm{Tr}(V))\|_{\mathbb{X}^{2s}(\mathbb{R}^{m+1}_+)}\le\|V\|_{\mathbb{X}^{2s}(\mathbb{R}^{m+1}_+)}, \tag{2.28}$$

for any $V\in\mathbb{X}^{2s}(\mathbb{R}^{m+1}_+)$.

Toward the construction of a solution to (2.24) we observe that, since U_0 is constant with respect to time, the function U_0 verifies

$$\begin{cases} y^{1-2s}\dfrac{\partial U_0(X,t)}{\partial t} = \mathrm{div}_X(y^{1-2s}\nabla_X U_0(X,t)), & \text{in } \mathbb{R}^{m+1}_+ \times (-\infty, 0], \\ U_0(x,0,t) = u_0(x), & \text{in } \mathbb{R}^m \times (-\infty, 0]. \end{cases} \tag{2.29}$$

Therefore, we study existence of a solution of the following initial and boundary value problem:

$$\begin{cases} y^{1-2s}\dfrac{\partial U_\varepsilon(X,t)}{\partial t} = \mathrm{div}_X(y^{1-2s}\nabla_X U_\varepsilon(X,t)) & \text{in } \mathbb{R}^{m+1}_+ \times (0, +\infty), \\ U_\varepsilon(x,y,0) = U_0(x,y), & \text{in } \mathbb{R}^{m+1}_+ \times \{0\}, \\ \lim_{y\to 0^+} y^{1-2s}\dfrac{\partial U_\varepsilon(X,t)}{\partial y} = -\dfrac{c_s}{\varepsilon^2}\left(1-|U_\varepsilon|^2\right)U_\varepsilon, & \text{in } \mathbb{R}^m \times (0, +\infty). \end{cases} \tag{2.30}$$

As a result, if we let $\tilde{U}_\varepsilon$ be a solution of the above problem, then

$$U_\varepsilon(X,t) := \begin{cases} \tilde{U}_\varepsilon(X,t) & \text{for } (X,t) \in \mathbb{R}^{m+1}_+ \times (0,\infty), \\ U_0(X,t) & \text{for } (X,t) \in \mathbb{R}^{m+1}_+ \times (-\infty, 0], \end{cases} \tag{2.31}$$

is a solution of (2.24). As the behavior of $t \le 0$ of U_ε is ruled by U_0 which only depends on the known "initial" condition u_0, with some abuse of notation we will use the same symbol U_ε to denote both a solution of (2.24) and a solution of (2.30).

We concentrate on (2.30). Since for the moment we work at fixed ε, we do not indicate the dependence on ε in the notation. Existence of a solution can be proven, for instance, by using a time discretization scheme. More precisely, for $n \in \mathbb{N}$ we set $\tau := \frac{T}{n}$ and $t^k := \tau k$ for $k = 0, ..., n$. We set $U^0 := U_0$ and we (iteratively) let U^k (with $k = 1, ..., n$) be the solution of

$$\begin{cases} U^k - \tau y^{-(1-2s)}\mathrm{div}\left(y^{1-2s}\nabla_X U^k\right) = U^{k-1}, & \text{in } \mathbb{R}^{m+1}_+, \\ \lim_{y\to 0^+} y^{1-2s}\dfrac{\partial U^k}{\partial y} = -\dfrac{c_s}{\varepsilon^2}\left(1-|U^k|^2\right)U^k & \text{in } \mathbb{R}^m \times \{0\}. \end{cases} \tag{2.32}$$

Equation (2.32) is the Euler-Lagrange equation for the minimizer of the energy (as in (2.25) we indicate with u the trace of U on $\mathbb{R}^m \times \{0\}$)

$$F(U,u) := \frac{1}{2}\int_{\mathbb{R}^{m+1}_+} \frac{|U - U^{k-1}|^2}{\tau} y^{1-2s}\mathrm{d}X + \mathscr{E}_\varepsilon(U,u).$$

Existence of a minimizer in the space $\mathcal{V}$ is a consequence of the Direct method of Calculus of Variations. Once we have constructed the discrete solutions U^k for $k = 1, ..., n$, we can standardly introduce the piecewise constant and piecewise affine (in time) interpolants of the discrete solutions and pass to limit when τ (the time step) tends to 0. This limit procedure gives (we restore the ε-dependence) a solution U_ε of (2.30). We let $u_\varepsilon(\cdot,\cdot) := U_\varepsilon(\cdot, 0, \cdot)$ (in the sense of traces). The function U_ε satisfies by construction the following a priori estimate (that correspond with testing (2.30) with $\frac{\partial U_\varepsilon}{\partial t}$ and integrating on $\mathbb{R}^{m+1}_+$)

$$\int_{\mathbb{R}^{m+1}_+} y^{1-2s}|\frac{\partial U_\varepsilon}{\partial t}|^2 \mathrm{d}X + \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_\varepsilon(U_\varepsilon, u_\varepsilon)(t) = 0. \tag{2.33}$$

Thus, integrating with respect to time in $(0, T)$ for $0 < T < \infty$, we get (recall that $\mathrm{Tr}(U_0(\cdot, \cdot, 0)) = u_0(\cdot)$ and that $u_0 \in \mathbb{S}^{l-1}$ a.e. in $\mathbb{R}^m$)

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^{m+1}_+} y^{1-2s}\left|\frac{\partial U_\varepsilon(X,t)}{\partial t}\right|^2 \mathrm{d}X\mathrm{d}t + \int_{\mathbb{R}^{m+1}_+} y^{1-2s}|\nabla_X U_\varepsilon(X,t)|^2 \mathrm{d}X \\ &\quad + \frac{c_s}{4\varepsilon^2}\int_{\mathbb{R}^m}(1-|u_\varepsilon|^2)^2 \mathrm{d}x = \int_{\mathbb{R}^{m+1}_+} y^{1-2s}|\nabla_X U_\varepsilon(X,0)|^2 \mathrm{d}X \le \|u_0\|^2_{\dot{H}^s(\mathbb{R}^m)}. \end{aligned} \tag{2.34}$$

Thus, we obtain

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^{m+1}_+} y^{1-2s}\left|\frac{\partial U_\varepsilon(X,t)}{\partial t}\right|^2 \mathrm{d}X\mathrm{d}t + \int_{\mathbb{R}^{m+1}_+} y^{1-2s}|\nabla_X U_\varepsilon(X,t)|^2 \mathrm{d}X \\ &\quad + \frac{c_s}{4\varepsilon^2}\int_{\mathbb{R}^m}(1-|u_\varepsilon|^2)^2 \mathrm{d}x \le C, \end{aligned} \tag{2.35}$$

where the constant C does not depend on ε. Thus, we conclude that $\partial_t U_\varepsilon$ and U_ε are uniformly bounded with respect to ε in the spaces

$$L^2(\mathbb{R}_+; L^2(\mathbb{R}^{m+1}_+, y^{1-2s}\mathrm{d}X)) \quad \text{and} \quad L^\infty(\mathbb{R}_+; \mathbb{X}^{2s}(\mathbb{R}^{m+1}_+)), \tag{2.36}$$

respectively. Moreover, recalling (2.31) we have indeed constructed a solution (still denoted with U_ε) of (2.30) that satisfies

$$\|U_\varepsilon\|_{H^1(\mathbb{R}_+;L^2(\mathbb{R}^{m+1}_+, y^{1-2s}\mathrm{d}X))} + \|U_\varepsilon\|_{L^\infty(\mathbb{R}_+;\mathbb{X}^{2s}(\mathbb{R}^{m+1}_+))} \le C. \tag{2.37}$$

Note that u_ε is a solution of (2.3).

2.3. Limit procedure and existence of a weak solution

The energy estimate (2.35) and weak compactness results guarantee the existence of a map $U : \mathbb{R}^{m+1}_+ \times \mathbb{R}_+ \to \mathbb{R}^\ell$ with

$$\partial_t U \in L^2\Big(\mathbb{R}_+; L^2(\mathbb{R}^{m+1}_+, y^{1-2s}\mathrm{d}X)\Big) \quad \text{and} \quad U \in L^\infty\Big(\mathbb{R}_+; \mathbb{X}^{2s}(\mathbb{R}^{m+1}_+)\Big)$$

and of a subsequence of ε (not relabeled) such that

$$\partial_t U_\varepsilon \overset{\varepsilon\to 0}{\rightarrow} \partial_t U \qquad \text{weakly in } L^2\Big(\mathbb{R}_+; L^2(\mathbb{R}^{m+1}_+, y^{1-2s}\mathrm{d}X)\Big), \tag{2.38}$$

$$\nabla_X U_\varepsilon \overset{\varepsilon\to 0}{\rightarrow} \nabla_X U \qquad \text{weakly star in } L^\infty\Big(\mathbb{R}_+; L^2(\mathbb{R}^{m+1}_+, y^{1-2s}\mathrm{d}X)\Big). \tag{2.39}$$

Moreover, the Aubin-Lions compactness Lemma gives that

$$U_\varepsilon \overset{\varepsilon\to 0}{\rightarrow} U \text{ strongly in } L^2_{\mathrm{loc}}\Big(\mathbb{R}_+; L^2_{\mathrm{loc}}(\mathbb{R}^{m+1}_+, y^{1-2s}\mathrm{d}X)\Big). \tag{2.40}$$

Now, denoting with u and with u_ε the traces of U and of U_ε on $\mathbb{R}^m \times \{0\}$, respectively, we have that

$$u_\varepsilon \overset{\varepsilon\to 0}{\to} u \qquad \text{in } L^2_{\text{loc}}(\mathbb{R}_+; L^2_{\text{loc}}(\mathbb{R}^m)), \tag{2.41}$$

and thus $u_\varepsilon \to u$ almost everywhere in $\mathbb{R}^m \times \mathbb{R}_+$, up to the extraction of a further subsequence. The convergence almost everywhere above combined with the fact that, thanks to estimate (2.35),

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^m} \left(1 - |u_\varepsilon|^2\right)^2 dx = 0,$$

allows us to reach that $|u(x,t)| = 1$ for almost any $(x,t) \in \mathbb{R}^m \times \mathbb{R}_+$. To conclude that U is a weak solution of (2.7) in the sense of Definition 2.1 we have to prove that U verifies (2.22). We consider $\Psi \in L^\infty(\mathbb{R}_+; \mathbb{X}^{2s}(\mathbb{R}^{m+1}_+; \bigwedge_{\ell-2}(\mathbb{R}^\ell))) \cap L^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^{m+1}_+; \bigwedge_{\ell-2}(\mathbb{R}^\ell)))$ and we test (2.30) with $\star(U_\varepsilon \wedge \Psi)$. For almost any $(x,t) \in \mathbb{R}^m \times (0, +\infty)$

$$\begin{aligned}\left\langle \frac{1}{\varepsilon^2}(1 - |u_\varepsilon|^2)u_\varepsilon, \star(u_\varepsilon \wedge \Psi) \right\rangle &= \star\left(\frac{1}{\varepsilon^2}(1 - |u_\varepsilon|^2)u_\varepsilon \wedge \star\star(u_\varepsilon \wedge \Psi)\right) \\ &= (-1)^{\ell-1} \star \left(\frac{1}{\varepsilon^2}(1 - |u_\varepsilon|^2)u_\varepsilon \wedge (u_\varepsilon \wedge \Psi)\right) = 0,\end{aligned}$$

thanks to (2.21) (recall (2.23)). For $t \le 0$, we have that $U_\varepsilon(x, 0, t) = u_\varepsilon(x,t) = u_0(x)$ and therefore, since $|u_0| = 1$ by hypothesis, we conclude that

$$\frac{1}{\varepsilon^2}\left(1 - |u_\varepsilon(x,t)|^2\right)u_\varepsilon(x,t) = 0, \qquad \text{for a.e. } \mathrm{x} \in \mathbb{R}^{\mathrm{m}} \qquad \text{and } t \le 0.$$

Thus, after integration by parts in space we conclude that U_ε verifies

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}^{m+1}_+} \left(\langle \partial_t U_\varepsilon, \star(U_\varepsilon \wedge \Psi)\rangle + \langle \nabla_X U_\varepsilon, \star(U_\varepsilon \wedge \nabla_X \Psi)\rangle\right) y^{1-2s} dX dt = 0. \tag{2.42}$$

Convergences (2.38)-(2.40) are enough to pass to the limit in Equation (2.42) and to obtain that U verifies (2.22). Thus, thanks to Lemma 2.4 we conclude that U is indeed a weak solution of (2.7). Therefore, the trace of U on $\mathbb{R}^m \times \{0\}$ is a weak solution of (2.1).

3. Monotonicity formula for the approximate problem

This section is devoted to the derivation of monotonicity formula for (2.30). For the later purpose, we will provide both global and local versions of such formulas.

For $t_0 \ge 0$ and $0 \le R \le \frac{t_0}{2}$, we set

$$\begin{aligned}&T^+_R(t_0) := \{(X,t) \in \mathbb{R}^{m+1}_+ \times \mathbb{R}_+ : t_0 - 4R^2 < t < t_0 - R^2\}, \quad T^+_1 := T^+_1(0),\\ &\partial^+ T^+_R(t_0) := \{(x, 0, t) \in \mathbb{R}^m \times \{0\} \times \mathbb{R}_+ : t_0 - 4R^2 < t < t_0 - R^2\}, \quad \partial^+ T^+_1 := \partial T^+_1(0).\end{aligned}$$

For $X_0 = (x_0, 0) \in \mathbb{R}^m \times \{0\}$ and $0 < s < 1$, let

$$\mathcal{G}^s_{X_0, t_0}(X,t) := \frac{1}{\Gamma(s)(4\pi)^{\frac{m}{2}}|t - t_0|^{\frac{m}{2}+1-s}} e^{-\frac{|X-X_0|^2}{4|t-t_0|}}, \ t < t_0$$

be the backward fundamental solution of (2.30). For $X_0 = 0$ and $t_0 = 0$, we write $\mathcal{G}^s = \mathcal{G}^s_{X_0, t_0}$. Note that

$$\nabla \mathcal{G}^s(X,t) = -\frac{X}{2|t|}\mathcal{G}^s(X,t),\ \mathcal{G}^s(RX, R^2t) = R^{-m-2+2s}\mathcal{G}^s(X,t),\ \forall (X,t) \in \mathbb{R}^{m+1}_+ \times \mathbb{R}_-,\ R > 0.$$

Lemma 3.1. *For every $Z_0 = (X_0, t_0)$ with $X_0 \in \partial\mathbb{R}^{m+1}_+$ and $t_0 > 0$, if U_ε solves (2.30) then the following two renormalized energies*

$$\mathcal{D}(U_\varepsilon, Z_0, R) := R^2\Bigg(\frac{1}{2}\int_{\mathbb{R}^{m+1}_+\times\{t_0-R^2\}} \mathcal{G}^s_{X_0,t_0}(X,t)y^{1-2s}|\nabla U_\varepsilon|^2 dX + \frac{c_s}{4\varepsilon^2}\int_{\mathbb{R}^m\times\{t_0-R^2\}} \mathcal{G}^s_{X_0,t_0}(X,t)(1-|u_\varepsilon|^2)^2 dx\Bigg)$$

and

$$\mathcal{E}(U_\varepsilon, Z_0, R) := \frac{1}{2}\int_{T^+_R(Z_0)} \mathcal{G}^s_{X_0,t_0}(X,t)y^{1-2s}|\nabla U_\varepsilon|^2 dXdt + \frac{c_s}{4\varepsilon^2}\int_{\partial^+T^+_R(Z_0)} \mathcal{G}^s_{X_0,t_0}(X,t)(1-|u_\varepsilon|^2)^2 dxdt$$

are monotone nondecreasing with respect to R. Namely,

$$\begin{cases} \mathcal{D}(U_\varepsilon, Z_0, r) \le \mathcal{D}(U_\varepsilon, Z_0, R),\ 0 < r \le R < \sqrt{t_0}, \\ \mathcal{E}(U_\varepsilon, Z_0, r) \le \mathcal{E}(U_\varepsilon, Z_0, R), \quad 0 < r \le R < \frac{1}{2}\sqrt{t_0}. \end{cases} \tag{3.1}$$

Proof. Here we just sketch a proof for $\mathcal{E}(U_\varepsilon, Z_0, R)$. Let us set

$$U_{\varepsilon,R}(X,t) := U_\varepsilon(RX + X_0, R^2t + t_0), \quad u_{\varepsilon,R}(x,t) := u_\varepsilon(Rx + x_0, R^2t + t_0)$$

for $X \in \mathbb{R}^{m+1}_+$ and $t > -R^{-2}t_0$. Then $U_{\varepsilon,R}$ satisfies

$$\begin{cases} y^{1-2s}\partial_t U_{\varepsilon,R}(X,t) = \mathrm{div}_X(y^{1-2s}\nabla_X U_{\varepsilon,R}(X,t)), & \text{in } \mathbb{R}^{m+1}_+ \times (-R^{-2}t_0, \infty), \\ U_{\varepsilon,R}(x,0,t) = u_{\varepsilon,R}(x,t), & \text{in } \mathbb{R}^m \times (-R^{-2}t_0, \infty), \\ U_{\varepsilon,R}(x,0,t) = u_0(Rx + X_0), & \text{in } \mathbb{R}^m \times (-\infty, -R^{-2}t_0], \\ \lim_{y\to 0^+} y^{1-2s}\partial_y U_{\varepsilon,R}(X,t) = -R^{2s}\frac{c_s}{\varepsilon^2}(1-|u_{\varepsilon,R}|^2)u_{\varepsilon,R}, & \text{in } \mathbb{R}^m \times (-R^{-2}t_0, \infty). \end{cases} \tag{3.2}$$

By the change of variables $X \mapsto RX + X_0,\ t \mapsto R^2t + t_0,$ we get

$$\mathcal{E}(U_\varepsilon, Z_0, R) := \frac{1}{2}\int_{T^+_1} \mathcal{G}^s y^{1-2s}|\nabla_X U_{\varepsilon,R}|^2 dXdt + \frac{c_s}{4\varepsilon^2}R^{2s}\int_{\partial^+T^+_1} \mathcal{G}^s(1-|u_{\varepsilon,R}|^2)^2 dxdt.$$

Therefore, integrating by parts we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dR}\int_{T^+_1} \mathcal{G}^s y^{1-2s}|\nabla_X U_{\varepsilon,R}|^2 dXdt \\ &= \textstyle\int_{T^+_1} \mathcal{G}^s y^{1-2s}\nabla_X U_{\varepsilon,R}\cdot\nabla_X\partial_R U_{\varepsilon,R} dXdt \\ &= -\textstyle\int_{T^+_1} \mathrm{div}_X\left[\mathcal{G}^s y^{1-2s}\nabla_X U_{\varepsilon,R}\right]\cdot\partial_R U_{\varepsilon,R} dXdt - \int_{\partial^+T^+_1} \lim_{y\to 0^+}\left[\mathcal{G}^s y^{1-2s}\partial_y U_{\varepsilon,R}\cdot\partial_R U_{\varepsilon,R}\right] dxdt \\ &= -\textstyle\int_{T^+_1} \mathcal{G}^s y^{1-2s}\left[\frac{X}{2t}\cdot\nabla_X U_{\varepsilon,R} + \partial_t U_{\varepsilon,R}\right]\cdot\partial_R U_{\varepsilon,R} dXdt \\ &\quad + R^{2s}\frac{c_s}{\varepsilon^2}\int_{\partial^+T^+_1} \mathcal{G}^s(1-|u_{\varepsilon,R}|^2)u_{\varepsilon,R}\cdot\partial_R u_{\varepsilon,R} dxdt. \end{aligned}$$

Here we have used the fact $\frac{X}{2t} = -\frac{X}{2|t|}$, and that $\partial_R U_{\varepsilon,R}(x,0,t) = \partial_R u_{\varepsilon,R}(x,t)$ for $t > -R^{-2}t_0$, which is a consequence of

$$\lim_{y\to 0^+} y\partial_y U_\varepsilon(RX, R^2t) = \lim_{y\to 0^+} y^{2s}\left[y^{1-2s}\partial_y U_\varepsilon(RX, R^2t)\right] = 0.$$

While

$$\begin{aligned}
&\frac{d}{dR}\left\{\frac{c_s}{4\varepsilon^2}R^{2s}\int_{\partial^+T_1^+}\mathcal{G}^s(1-|u_{\varepsilon,R}|^2)^2dxdt\right\}\\
&=\frac{sc_s}{2\varepsilon^2}R^{2s-1}\int_{\partial^+T_1^+}\mathcal{G}^s(1-|u_{\varepsilon,R}|^2)^2dxdt\\
&\quad - R^{2s}\frac{c_s}{\varepsilon^2}\int_{\partial^+T_1^+}\mathcal{G}^s(1-|u_{\varepsilon,R}|^2)u_{\varepsilon,R}\cdot\partial_R u_{\varepsilon,R}dxdt.
\end{aligned}$$

Since

$$\partial_R U_{\varepsilon,R} = \frac{1}{R}(X\cdot\nabla_X U_{\varepsilon,R} + 2t\partial_t U_{\varepsilon,R}),$$

we obtain

$$\begin{aligned}
\frac{d}{dR}\mathcal{E}(U_\varepsilon, Z_0, R) &= \frac{1}{2R}\int_{T_1^+}\frac{\mathcal{G}^s}{|t|}y^{1-2s}|X\cdot\nabla_X U_{\varepsilon,R}(X,t) + 2t\partial_t U_{\varepsilon,R}(X,t)|^2 dXdt\\
&\quad + \frac{sc_s}{2\varepsilon^2}R^{2s-1}\int_{\partial^+T_1^+}\mathcal{G}^s(1-|u_{\varepsilon,R}|^2)^2dxdt\\
&\ge 0.
\end{aligned} \tag{3.3}$$

This yields the monotonicity of $\mathcal{E}(U_\varepsilon, Z_0, R)$ with respect to $R > 0$ and hence completes the proof. □

We will also need the following local energy inequality. Here we denote

$$P_R^+(Z_0) = B_R^+(X_0)\times\left[t_0 - R^2, t_0 + R^2\right],\ \partial^+P_R^+(Z_0) = P_R^+(Z_0)\cap(\partial\mathbb{R}_+^{m+1}\times(0,\infty))$$

for $R > 0$ and $Z_0 = (X_0, t_0) \in \overline{\mathbb{R}_+^{m+1}}\times\mathbb{R}$.

Lemma 3.2. *If U_ε solves (2.30) then for any $\eta \in C_0^\infty(\mathbb{R}^{m+1})$ it holds that*

$$\begin{aligned}
\frac{d}{dt}\left\{\int_{\mathbb{R}_+^{m+1}}\frac{1}{2}|\nabla U_\varepsilon|^2\eta^2 + \int_{\mathbb{R}^m}\frac{c_{\frac{1}{2}}}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2\eta^2\right\} + \frac{1}{2}\int_{\mathbb{R}_+^{m+1}}|\partial_t U_\varepsilon|^2\eta^2\\
\le 4\int_{\mathbb{R}_+^{m+1}}|\nabla U_\varepsilon|^2|\nabla\eta|^2.
\end{aligned} \tag{3.4}$$

In particular, for any $Z_0 = (X_0, t_0) \in \overline{\mathbb{R}_+^{m+1}}\times(0,\infty)$ and $0 < R < \frac{\sqrt{t_0}}{2}$, we have that

$$\int_{P_R^+(Z_0)}|\partial_t U_\varepsilon|^2 \le CR^{-2}\left(\int_{P_{2R}^+(Z_0)}|\nabla U_\varepsilon|^2 + \int_{\partial^+P_{2R}^+(Z_0)}\frac{c_{\frac{1}{2}}}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2\right). \tag{3.5}$$

Proof. Multiplying the first equation of (2.30) by $\partial_t U_\varepsilon\eta^2$ and integrating the resulting equation over $\mathbb{R}_+^{m+1}$, and applying the third equation of (2.30) in integration by parts, we obtain

$$\frac{d}{dt}\left\{\int_{\mathbb{R}^{m+1}_+}\frac{1}{2}|\nabla U_\varepsilon|^2\eta^2+\int_{\mathbb{R}^m}\frac{c_{\frac{1}{2}}}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2\eta^2\right\}+\int_{\mathbb{R}^{m+1}_+}|\partial_t U_\varepsilon|^2\eta^2$$
$$=-2\int_{\mathbb{R}^{m+1}_+}\langle\eta\partial_t U_\varepsilon,\nabla U_\varepsilon\nabla\eta\rangle\le\frac{1}{2}\int_{\mathbb{R}^{m+1}_+}|\partial_t U_\varepsilon|^2\eta^2+2\int_{\mathbb{R}^{m+1}_+}|\nabla U_\varepsilon|^2|\nabla\eta|^2.$$

This yields (3.4). To see (3.5), let $\eta\in C_0^\infty(\mathbb{R}^{m+1})$ be a cutoff function of $B_R(X_0)$, i.e. $0\le\eta\le1$, $\eta\equiv1$ in $B_R(X_0)$, $\eta\equiv0$ outside $B_{2R}(X_0)$, and $|\nabla\eta|\le4R^{-1}$. By Fubini's theorem, there exists $t_*\in(t_0-4R^2,t_0-R^2)$ such that

$$\begin{aligned}&\int_{B^+_{2R}(X_0)\times\{t_*\}}\frac{1}{2}|\nabla U_\varepsilon|^2+\int_{(B_{2R}(X_0)\cap\partial\mathbb{R}^{m+1}_+)\times\{t_*\}}\frac{c_{\frac{1}{2}}}{4\varepsilon^2}\left(1-|u_\varepsilon|^2\right)^2\\&\le\frac{16}{R^2}\left(\int_{P^+_{2R}(Z_0)}\frac{1}{2}|\nabla U_\varepsilon|^2+\int_{\partial^+P^+_{2R}(Z_0)}\frac{c_{\frac{1}{2}}}{4\varepsilon^2}\left(1-|u_\varepsilon|^2\right)^2\right).\end{aligned}\tag{3.6}$$

Now if we integrate (3.4) for $t_*\le t\le t_0+R^2$ and apply (3.6), we obtain that

$$\begin{aligned}\int_{P^+_R(Z_0)}|\partial_t U_\varepsilon|^2&\le\int_{B^+_{2R}(X_0)\times\{t_*\}}\frac{1}{2}|\nabla U_\varepsilon|^2+\int_{(B_{2R}(X_0)\cap\partial\mathbb{R}^{m+1}_+)\times\{t_*\}}\frac{c_{\frac{1}{2}}}{4\varepsilon^2}\left(1-|u_\varepsilon|^2\right)^2\\&\quad+CR^{-2}\int_{P^+_{2R}(Z_0)}|\nabla U_\varepsilon|^2\\&\le CR^{-2}\left(\int_{P^+_{2R}(Z_0)}|\nabla U_\varepsilon|^2+\int_{\partial^+P^+_{2R}(Z_0)}\frac{c_{\frac{1}{2}}}{4\varepsilon^2}\left(1-|u_\varepsilon|^2\right)^2\right).\end{aligned}$$

This implies (3.5) and completes the proof. □

4. ε−Regularity result

From now on, we will always assume $s=\frac{1}{2}$. Therefore, according to

$$c_s:=\frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)},$$

we have that $c_{1/2}=1$.

As previously mentioned, we now focus only on the system (2.30). We will derive a priori estimates of its solutions U_ε under a smallness condition on the renormalized energy.

For $X_0=(x_0,0)$, $t_0>0$ and $0<R\le\sqrt{t_0}$, we set

$$P^+_R(X_0,t_0):=\left\{(X,t)\in\overline{\mathbb{R}^{m+1}_+}\times(0,\infty):|X-X_0|\le R,\ \ t_0-R^2\le t\le t_0+R^2\right\}.$$

Lemma 4.1. *Assume U_ε is a bounded, smooth solution of (2.30). Then $|U_\epsilon|\le1$ in $\mathbb{R}^{m+1}_+\times(0,\infty)$.*

Proof. We argue by contradiction. Suppose the conclusion were false. Then by the maximum principle there exists $Z_0=(x_0,0,t_0)\in\partial\mathbb{R}^{m+1}_+\times(0,\infty)$ such that

$$(|U_\varepsilon|^2-1)(Z_0)=\max_{Z\in\overline{\mathbb{R}^{m+1}_+}\times[0,\infty)}(|U_\varepsilon|^2-1)(Z)>0.$$

Set $\Phi_\varepsilon=|U_\varepsilon|^2-1$. Then it satisfies

$$\begin{cases} \partial_t \Phi_\varepsilon - \Delta\Phi_\varepsilon = -2|\nabla U_\varepsilon|^2 \le 0 & \text{in} \quad \mathbb{R}^{m+1}_+ \times [0,\infty), \\ \lim_{y\to 0^+} \dfrac{\partial \Phi_\varepsilon}{\partial y}(X,t) = \dfrac{2c_s}{\varepsilon^2}\Phi_\varepsilon(x,0,t)|u_\varepsilon|^2(x,t) & \text{on} \quad \partial\mathbb{R}^{m+1}_+ \times [0,\infty). \end{cases}$$

It follows from the Hopf boundary Lemma that

$$\frac{\partial \Phi_\varepsilon}{\partial y}(Z_0) = \lim_{y\to 0^+} \frac{\partial \Phi_\varepsilon}{\partial y}(x_0,y,t_0) < 0.$$

On the other hand, the boundary condition of Φ_ε yields that

$$\frac{\partial \Phi_\varepsilon}{\partial y}(Z_0) = \frac{2c_s}{\varepsilon^2}\Phi_\varepsilon(Z_0))|u_\varepsilon|^2(Z_0) > 0.$$

We get the desired contradiction. □

The next Lemma is a clearing-out result, which plays a crucial role in the small-energy regularity result.

Lemma 4.2. *There exists $\varepsilon_0 > 0$ such that if U_ε is a smooth solution of (2.30) with $|U_\varepsilon| \le 1$, that satisfies*

$$\mathcal{E}(U_\varepsilon,(X_0,t_0),1) \le \varepsilon_0$$

for some $X_0 \in \partial\mathbb{R}^{m+1}_+$ and $t_0 > 4$, then

$$|U_\varepsilon| \ge \frac{1}{2} \quad \text{on } P^+_\delta(X_0,t_0)$$

for some $\delta > 0$ independent of U_ε, X_0 and t_0.

Proof. We divide the proof into two cases:

Case 1: $\varepsilon \ge \frac{1}{2}$. Set

$$V_\varepsilon(x,y,t) = \int_0^y U_\varepsilon(x,s,t)ds, \quad y > 0.$$

Then

$$\partial_y(\partial_t - \Delta)V_\varepsilon(x,y,t) = 0,$$

that is, $(\partial_t - \Delta)V_\varepsilon$ is independent of $y \in (0,\infty)$. In particular, we get

$$(\partial_t - \Delta)V_\varepsilon(x,y,t) = (\partial_t - \Delta)V_\varepsilon(x,0,t) = -\partial_y U_\varepsilon(x,0,t) = \frac{c_{\frac{1}{2}}}{\varepsilon^2}(1-|u_\varepsilon|^2)u_\varepsilon, \quad y > 0.$$

Note that

$$\frac{c_{\frac{1}{2}}}{\varepsilon^2}|1-|u_\varepsilon|^2||u_\varepsilon| \le 4c_{\frac{1}{2}} \,\text{in}\, \mathbb{R}^{m+1}_+ \times (0,\infty), \text{ and } V_\varepsilon = 0 \text{ for } y = 0.$$

Hence, by the standard parabolic theory [21, Theorem 2.13] we conclude that V_ε is bounded in $C^{2,1}(P^+_{\frac{1}{2}}(X_0,t_0))$. In other words, U_ε is bounded in $C^{1,1}(P^+_{\frac{1}{2}}(X_0,t_0))$, which gives

$$|U_\varepsilon(X,t) - U_\varepsilon(\tilde{X},\tilde{t})| \le C_1\left(|X-\tilde{X}| + |t-\tilde{t}|^{\frac{1}{2}}\right) \quad \text{for } (X,t) \text{ and } (\tilde{X},\tilde{t}) \in P^+_{\frac{1}{2}}(X_0,t_0), \tag{4.1}$$

for some $C_1 > 0$. Now we choose $\delta_1 \in \left(0, \frac{1}{2}\right)$ such that $\delta_1 C_1 \le \frac{1}{8}$. By the monotonicity inequality (3.1) we get that

$$\begin{aligned}
&\frac{c_{\frac{1}{2}}}{4\varepsilon^2}\int_{\partial\mathbb{R}^{m+1}_+\cap B_{\delta_1}(X_0)}\int_{t_0-4\delta_1^2}^{t_0-\delta_1^2}(1-|u_\varepsilon|^2)^2dxdt\\
&\le C\delta_1^{-(m+1)}\frac{c_{\frac{1}{2}}}{4\varepsilon^2}\int_{\partial\mathbb{R}^{m+1}_+}\int_{t_0-4\delta_1^2}^{t_0-\delta_1^2}\mathcal{G}^{\frac{1}{2}}_{X_0,t_0}(X,t)(1-|u_\varepsilon|^2)^2dxdt\\
&\le C\delta_1^{-(m+1)}\mathcal{E}(U_\varepsilon,\delta_1,X_0,t_0)\\
&\le C\delta_1^{-(m+1)}\mathcal{E}(U_\varepsilon,1,X_0,t_0)\\
&\le C\delta_1^{-(m+1)}\varepsilon_0.
\end{aligned}$$

Therefore, by choosing $\varepsilon_0 > 0$ sufficiently small we obtain that

$$|u_\varepsilon| \ge \frac{4}{5} \quad \text{for } |X-X_0| \le \delta_1,\ X\in\partial\mathbb{R}^{m+1}_+,\ t_0-4\delta_1^2 \le t \le t_0-\delta_1^2.$$

From the choice of $\delta_1 > 0$ we conclude that $|U_\varepsilon| \ge \frac{1}{2}$ on $P^+_{\delta_1}(X_0,t_0)$, thanks to (4.1).

Case 2: $\varepsilon \le \frac{1}{2}$. Let $X_1 = (x_1,y_1) \in B_\delta(X_0)$ with $y_1 \ge 0$ and $t_1 \in (t_0-\delta^2, t_0+\delta^2)$ being fixed. Set $\tilde{X}_1 = (x_1,0)$, and

$$\tilde{U}_\varepsilon(X,t) = U_\varepsilon(\tilde{X}_1+\varepsilon^2X, t_1+\varepsilon^4t).$$

Then $\tilde{U}_\varepsilon$ satisfies

$$\begin{cases}
\partial_t\tilde{U}_\varepsilon(X,t) = \Delta\tilde{U}_\varepsilon(X,t) & \text{in } \mathbb{R}^{m+1}_+\times\mathbb{R},\\
\tilde{U}_\varepsilon(x,0,t) = \tilde{u}_\varepsilon(x,t) & \text{in } \mathbb{R}^m\times(-\varepsilon^{-4}t_1,\infty),\\
\tilde{U}_\varepsilon(x,0,t) = \tilde{u}_0(x,t) & \text{in } \mathbb{R}^m\times(-\infty,-\varepsilon^{-4}t_1],\\
\lim_{y\to0^+}\dfrac{\partial\tilde{U}_\varepsilon}{\partial y}(X,t) = -c_{\frac{1}{2}}(1-|\tilde{u}_\varepsilon|^2)\tilde{u}_\varepsilon & \text{in } \mathbb{R}^m\times(-\varepsilon^{-4}t_1,\infty).
\end{cases} \tag{4.2}$$

By the monotonicity inequality (3.1), we obtain

$$\mathcal{E}(\tilde{U}_\varepsilon,(0,0),1) = \mathcal{E}(U_\varepsilon,(\tilde{X}_1,t_1),\varepsilon^2) \le \mathcal{E}\left(U_\varepsilon,(\tilde{X}_1,t_1),\frac{1}{2}\right) \le C(\varepsilon_0+\varepsilon_1),$$

where the last inequality follows from Lemma 4.4. Now one can proceed as in Case 1 to show that

$$|\tilde{U}_\varepsilon| \ge \frac{1}{2} \quad \text{for } (X,t)\in P^+_{\delta_1}(0,0),$$

for some $\delta_1 > 0$ independent of $\varepsilon > 0$. In particular, we obtain

$$|U_\varepsilon(X,t)| \ge \frac{1}{2} \quad \text{for } (X,t)\in P^+_{\delta_1}(X_0,t_0),\ 0\le y\le\delta_1\varepsilon^2. \tag{4.3}$$

Next we find a small $\delta_2 > 0$ such that $|U_\varepsilon(X,t)| \geq \frac{1}{2}$ on $P^+_{\delta_2}(X_0, t_0)$, provided $\varepsilon_0 > 0$ is sufficiently small. Note that it suffices to consider points $(X,t) \in P^+_{\delta_2}(X_0,t_0)$, $X = (x,y)$ for which $\varepsilon^2\delta_1 \leq y \leq \delta_2$, and $\varepsilon > 0$ satisfies $\varepsilon^2 \leq \frac{\delta_2}{\delta_1}$. To this end we fix an arbitrary point $(X_1,t_1) \in P^+_\delta(X_0,t_0)$ with $X_1 = (x_1,y_1)$, $y_1 > 0$, and set $R := \frac{1}{4}y_1$, $\tilde{X}_1 = (x_1, 0)$.

We claim that for $\delta > 0$ small (depending on ε_0) it holds that

$$\int_{t_1-10R^2}^{t_1}\int_{B^+_{10R}(\tilde{X}_1)} |\nabla U_\varepsilon|^2 dXdt \leq C(\varepsilon_0+\varepsilon_1)R^{m+1}, \tag{4.4}$$

and

$$\frac{1}{\varepsilon^2}\int_{t_1-10R^2}^{t_1}\int_{|x-x_1|<10R} (1-|u_\varepsilon|^2)^2 dxdt \leq C(\varepsilon_0+\varepsilon_1)R^{m+1}. \tag{4.5}$$

To prove the above claims, let us first choose $\delta > 0$ small so that we can apply Lemma 4.4 with $\varepsilon_1 = \varepsilon_0$, $X_1 = \tilde{X}_1$ and $t_1 = t_1 + 4R^2$. Then by the monotonicity inequality (3.1) and Lemma 4.4,

$$\mathcal{E}(U_\varepsilon, (\tilde{X}_1, t_1+4R^2), 2R) \leq \mathcal{E}(U_\varepsilon, (\tilde{X}_1, t_1+4R^2), \frac{1}{2}) \leq C(\varepsilon_0+\varepsilon_1).$$

Since

$$G_{\tilde{X}_1, t_1+4R^2}(X,t) \sim R^{-(m+1)} \quad \text{for} |X-\tilde{X}_1| \leq 10R \quad \text{and } t_1 - 10R^2 \leq t \leq t_1,$$

(4.4) and (4.5) follow immediately.

As $B_{4R}(X_1) \subset B^+_{10R}(\tilde{X}_1)$, and $\nabla_X U_\varepsilon$ satisfies the heat equation on $\mathbb{R}^{m+1}_+ \times (0,\infty)$, by (4.4) we obtain (see [34, page 61])

$$|\nabla_X U_\varepsilon(X,t)| \leq \frac{C\sqrt{\varepsilon_0+\varepsilon_1}}{R} \quad \text{for } |X-X_1| \leq 4R \quad \text{and} \quad t_1 - 9R^2 \leq t \leq t_1, \tag{4.6}$$

and consequently, by the standard parabolic estimates,

$$|\partial_t U_\varepsilon(X,t)| \leq \frac{C\sqrt{\varepsilon_0+\varepsilon_1}}{R^2} \quad \text{for } |X-X_1| \leq 3R \quad \text{and } t_1 - 8R^2 \leq t \leq t_1. \tag{4.7}$$

Setting

$$\bar{U}_\varepsilon(t) := \fint_{B_R(X_1)} U_\varepsilon(X,t)dX,$$

we see that

$$|U_\varepsilon(X,t) - \bar{U}_\varepsilon(t_1)| \leq C\sqrt{\varepsilon_0+\varepsilon_1} \quad \text{for } |X-X_1| \leq R \text{ and } t_1 - 8R^2 \leq t \leq t_1, \tag{4.8}$$

thanks to (4.6)-(4.7).

For $\xi \in \mathbb{R}^\ell$ we set $d(\xi) = |1 - |\xi||$. Then d is 1-Lipschitz. Since

$$d(\bar{U}_\varepsilon(t_1)) \leq d(U_\varepsilon(X,t)) + |U_\varepsilon(X,t) - \bar{U}_\varepsilon(t_1)|, \forall X \in B_R(X_1),\ t_1 - R^2 \leq t \leq t_1,$$

taking an average integral one gets

$$\begin{aligned}
&d(\bar{U}_\varepsilon(t_1))\\
&\leq \fint_{t_1-R^2}^{t_1}\fint_{B_R(X_1)} d(U_\varepsilon(X,t))dXdt + \fint_{t_1-R^2}^{t_1}\fint_{B_R(X_1)} |U_\varepsilon(X,t)-\bar{U}_\varepsilon(t_1)|dXdt\\
&\leq \fint_{t_1-R^2}^{t_1}\fint_{B_R(X_1)} d(U_\varepsilon(X,t))dXdt + C\sqrt{\varepsilon_0+\varepsilon_1},
\end{aligned}$$

thanks to (4.8). By Jensen's inequality, and using that $B_R(X_1)\subset B^+_{5R}(\tilde{X}_1)$, we get

$$\begin{aligned}
&\left(\fint_{t_1-R^2}^{t_1}\fint_{B_R(X_1)} d(U_\varepsilon(X,t))dXdt\right)^2\\
&\leq \frac{C}{R^{m+1}}\fint_{t_1-R^2}^{t_1}\int_{B^+_{5R}(\tilde{X}_1)} d(U_\varepsilon(X,t))^2dXdt\\
&\leq \frac{C}{R^{m+1}}\fint_{t_1-R^2}^{t_1}\left(R\int_{|x-x_1|<5R} d(u_\varepsilon(x,t))^2dx + R^2\int_{B^+_{5R}(\tilde{X}_1)}|\nabla d(U_\varepsilon(X,t))|^2dX\right)dt\\
&\leq \frac{C}{R^{m+2}}\int_{t_1-R^2}^{t_1}\int_{|x-x_1|<5R}(1-|u_\varepsilon|^2)^2dxdt + \frac{C}{R^{m+1}}\int_{t_1-R^2}^{t_1}\int_{B^+_{5R}(\tilde{X}_1)}|\nabla U_\varepsilon|^2dXdt\\
&\leq C(\varepsilon_0+\varepsilon_1)\left(\frac{\varepsilon^2}{R}+1\right).
\end{aligned}$$

The second inequality above follows from the Poincarè inequality, and we have used that d is 1-Lipschitz in the third inequality, and the last inequality follows from the estimates (4.4)-(4.5).

As we have mentioned before, we only need to consider $4R = y_1 \geq \delta_1\varepsilon^2$ and $\varepsilon^2 < \frac{\delta}{\delta_1}$. Thus we obtain

$$d(\bar{U}_\varepsilon(t_1)) \leq C\sqrt{\varepsilon_0+\varepsilon_1}\left(1+\frac{1}{\sqrt{\delta_1}}\right).$$

Hence, if $\varepsilon_0>0$ is sufficiently small, from (4.8) we have

$$|U_\varepsilon(X_1,t_1)| \geq \frac{1}{2}.$$

Consequently, corresponding to this ε_0 we obtain $\delta=\delta_2>0$ as determined by Lemma 4.4 for the choice of $\varepsilon_1=\varepsilon_0$. □

Next we show that under a smallness condition on the renormalized energy, U_ε enjoys a gradient estimate. More precisely, we have

Lemma 4.3. *There exists $\varepsilon_0>0$, depending only on m, such that if U_ε is a smooth solution of (2.30), with $|U_\varepsilon|\leq 1$, which satisfies, for $Z_0=(X_0,t_0)\in\partial\mathbb{R}^{m+1}_+\times(0,\infty)$ and some $0<R<\frac{\sqrt{t_0}}{2}$,*

$$\mathcal{E}(U_\varepsilon, Z_0, R) < \varepsilon_0^2, \tag{4.9}$$

then

$$\sup_{P^+_{\delta_0 R}(Z_0)} R^2|\nabla U_\varepsilon|^2 \le C\delta_0^{-2}, \qquad \sup_{P^+_{\delta_0 R}(Z_0)} R^4|\partial_t U_\varepsilon|^2 \le C\delta_0^{-4}, \tag{4.10}$$

where $0 < \delta_0 < 1$ *and* $C > 0$ *are independent of* ε.

Proof. By scaling, we may assume that $t_0 > 4$ and $R = 1$. Let $\delta > 0$ be as determined by Lemma 4.2. Since U_ε is smooth in $\overline{\mathbb{R}^{m+1}_+} \times (0, \infty)$, there exists $\sigma_\varepsilon \in (0, \delta)$ such that

$$(\delta - \sigma_\varepsilon)^2 \max_{P^+_{\sigma_\varepsilon}(Z_0)} (|\nabla U_\varepsilon|^2 + |\partial_t U_\varepsilon|) = \max_{0 \le \sigma \le \delta} (\delta - \sigma)^2 \max_{P^+_\sigma(Z_0)} (|\nabla U_\varepsilon|^2 + |\partial_t U_\varepsilon|).$$

Let $Z_1^\varepsilon = (X_1^\varepsilon, t_1^\varepsilon) \in P^+_{\sigma_\varepsilon}(Z_0)$ be such that

$$\max_{P^+_{\sigma_\varepsilon}(Z_0)} (|\nabla U_\varepsilon|^2 + |\partial_t U_\varepsilon|) = (|\nabla U_\varepsilon|^2 + |\partial_t U_\varepsilon|)(Z_1^\varepsilon) := e_\varepsilon^2.$$

Set $\rho_\varepsilon = \frac{1}{2}(\delta - \sigma_\varepsilon)$. Since $P^+_{\rho_\varepsilon}(Z_1^\varepsilon) \subset P^+_{\rho_\varepsilon + \sigma_\varepsilon}(Z_0)$, we have that

$$\begin{aligned} &\max_{P^+_{\rho_\varepsilon}(Z_1^\varepsilon)} (|\nabla U_\varepsilon|^2 + |\partial_t U_\varepsilon|) \\ &\le \max_{P^+_{\rho_\varepsilon + \sigma_\varepsilon}(Z_0)} (|\nabla U_\varepsilon|^2 + |\partial_t U_\varepsilon|) \le 4e_\varepsilon^2. \end{aligned} \tag{4.11}$$

Write $X_1^\varepsilon = (x_1^\varepsilon, y_1^\varepsilon) \in \mathbb{R}^{m+1}_+$ with $x_1^\varepsilon \in \mathbb{R}^m$ and $y_1^\varepsilon \ge 0$. Set $\tilde{X}_1^\varepsilon = (x_1^\varepsilon, 0) \in \partial\mathbb{R}^{m+1}_+$ and define

$$\tilde{U}_\varepsilon(X, t) = U_\varepsilon\left(\tilde{X}_1^\varepsilon + \frac{X}{e_\varepsilon}, t_1^\varepsilon + \frac{t}{e_\varepsilon^2}\right), \; (X, t) \in P^+_{r_\varepsilon}(Y_1^\varepsilon, 0),$$

where $r_\varepsilon = \rho_\varepsilon e_\varepsilon$ and $Y_1^\varepsilon = (0, y_1^\varepsilon e_\varepsilon) \in \mathbb{R}^{m+1}_+$. Setting

$$\frac{1}{\tilde{\varepsilon}^2} := \frac{c_{\frac{1}{2}}}{\varepsilon^2 e_\varepsilon},$$

one gets

$$\begin{cases} (\partial_t - \Delta)\tilde{U}_\varepsilon = 0 & \text{in } P^+_{r_\varepsilon}(Y_1^\varepsilon, 0), \\ \partial_\nu \tilde{U}_\varepsilon = \frac{1}{\tilde{\varepsilon}^2}(1 - |\tilde{u}_\varepsilon|^2)\tilde{u}_\varepsilon & \text{on } \partial P^+_{r_\varepsilon}(Y_1^\varepsilon, 0) \cap (\partial\mathbb{R}^{m+1}_+ \times \mathbb{R}), \\ (|\nabla \tilde{U}_\varepsilon|^2 + |\partial_t \tilde{U}_\varepsilon|)(X, t) \le 4 & \text{in } P^+_{r_\varepsilon}(Y_1^\varepsilon, 0), \\ (|\nabla \tilde{U}_\varepsilon|^2 + |\partial_t \tilde{U}_\varepsilon|)(Y_1^\varepsilon, 0)| = 1. \end{cases} \tag{4.12}$$

Note that if $r_\varepsilon \le 2$, then from the definition of σ_ε we obtain

$$\frac{\delta^2}{4} \sup_{P_{\frac{\delta}{2}^+}(Z_0)} (|\nabla U_\varepsilon|^2 + |\partial_t U_\varepsilon|) \le (\delta - \sigma_\varepsilon)^2 \sup_{P^+_{\sigma_\varepsilon}(Z_0)} (|\nabla U_\varepsilon|^2 + |\partial_t U_\varepsilon|) = 4\rho_\varepsilon^2 e_\varepsilon^2 = 4r_\varepsilon^2 \le 16.$$

This yields (4.10). Thus we may assume $r_\varepsilon > 2$ and proceed as follows. In case that $y_1^\varepsilon e_\varepsilon \ge \frac{1}{8}$, one can use the interior regularity of heat equations to conclude that

$$
\begin{aligned}
1 &= (|\nabla \tilde{U}_\varepsilon|^2 + |\partial_t \tilde{U}_\varepsilon|)(Y_1^\varepsilon, 0) \\
&\le C(\|\nabla \tilde{U}_\varepsilon\|^2_{L^2(P_{\frac{1}{9}}(Y_1^\varepsilon,0))} + \|\partial_t \tilde{U}_\varepsilon\|_{L^1(P_{\frac{1}{18}}(Y_1^\varepsilon,0))}) \\
&\le C\bigg((9e_\varepsilon)^{-(m+1)}\int_{P^+_{\frac{1}{9e_\varepsilon}}(X_1^\varepsilon,t_1^\varepsilon)} |\nabla U_\varepsilon|^2 + (18e_\varepsilon)^{-(m+1)}\int_{P^+_{\frac{1}{18e_\varepsilon}}(X_1^\varepsilon,t_1^\varepsilon)} |\partial_t U_\varepsilon|\bigg).
\end{aligned}
\tag{4.13}
$$

Next we need

Claim. For $0 < r, \sigma < \delta$ with $2r + \sigma < \delta,$ it holds

$$
\begin{aligned}
&\left(\frac{r}{2}\right)^{2-(m+1)}\int_{P^+_{\frac{r}{2}}(Z_1)} |\partial_t U_\varepsilon|^2 + r^{-(m+1)}\left(\int_{P^+_r(Z_1)} \frac{1}{2}|\nabla U_\varepsilon|^2 + \int_{\partial^+ P^+_r(Z_1)} \frac{c_{\frac{1}{2}}}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2\right) \\
&\le C(\varepsilon_0^2 + \varepsilon_1 E_0),\ \forall Z_1 \in P^+_\sigma(Z_0).
\end{aligned}
\tag{4.14}
$$

Here $E_0 = \int_{\mathbb{R}^{m+1}_+} |\nabla U_0|^2.$

Assume (4.14) for the moment. Since $\frac{1}{9e_\varepsilon} \le \frac{8y_1^\varepsilon}{9} \le \sigma_\varepsilon,$ it follows from (4.14) and (4.13) that

$$
1 \le C(\varepsilon_0^2 + \varepsilon_1 E_0) + C\sqrt{\varepsilon_0^2 + \varepsilon_1 E_0},
$$

which is impossible if we choose a sufficiently small ε_0. Therefore we must have $y_1^\varepsilon e_\varepsilon \le \frac{1}{8}$ and $r_\varepsilon > 2.$ In this case, we see that $\tilde{U}_\varepsilon$ satisfies (5.1), (5.2) and (5.3) with $\varepsilon = \tilde{\varepsilon}.$ Hence, by Proposition 5.1, we have that for any $0 < \alpha < 1,$

$$
\|\tilde{U}_\varepsilon\|_{C^{1+\alpha}(P^+_{\frac{1}{7}}(Y_1^\varepsilon,0))} \le C(\alpha).
$$

In particular, for any $0 < r < \frac{1}{60}$ it holds that

$$
|(|\nabla \tilde{U}_\varepsilon|^2 + |\partial_t \tilde{U}_\varepsilon|)(X,t) - (|\nabla \tilde{U}_\varepsilon|^2 + |\partial_t \tilde{U}_\varepsilon|)(Y_1^\varepsilon, 0)| \le Cr^{\frac{1}{4}}, \quad \forall (X,t) \in P^+_r(Y_1^\varepsilon, 0).
$$

Since $(|\nabla \tilde{U}_\varepsilon|^2 + |\partial_t \tilde{U}_\varepsilon|)(Y_1^\varepsilon, 0) = 1,$ it follows that

$$
(|\nabla \tilde{U}_\varepsilon|^2 + |\partial_t \tilde{U}_\varepsilon|)(X,t) \ge 1 - Cr^{\frac{1}{4}}, \quad \forall (X,t) \in P^+_r(Y_1^\varepsilon, 0).
$$

Thus we can find a $0 < r_0 < \frac{1}{60},$ independent of $\varepsilon,$ such that

$$
(|\nabla \tilde{U}_\varepsilon|^2 + |\partial_t \tilde{U}_\varepsilon|)(X,t) \ge \frac{1}{2}, \quad \forall (X,t) \in P^+_{r_0}(Y_1^\varepsilon, 0).
$$

This, combined with (4.14), yields that

$$
\begin{aligned}
r_0^2 &\le C r_0^{-(m+1)}\int_{P^+_{r_0}(Y_1^\varepsilon,0)} (|\nabla \tilde{U}_\varepsilon|^2 + |\partial_t \tilde{U}_\varepsilon|) \\
&= C(\frac{r_0}{2}\rho_\varepsilon)^{-(m+1)}\int_{P^+_{\frac{r_0}{2}\rho_\varepsilon}(Z_1^\varepsilon)} (|\nabla U_\varepsilon|^2 + |\partial_t U_\varepsilon|) \\
&\le C(\varepsilon_0^2 + \varepsilon_1 E_0) + C\sqrt{\varepsilon_0^2 + \varepsilon_1 E_0}.
\end{aligned}
$$

This is again impossible if we choose a sufficiently small ε_0. Therefore we show that $r_\varepsilon = \rho_\varepsilon e_\varepsilon \le 2$ so that (4.10) holds.

Now we return to the proof of (4.14). For simplicity, assume $Z_0 = (0,0)$ and write

$$\mathcal{G}_{X_*,t_*}(X,t) = \mathcal{G}^{\frac{1}{2}}_{X_*,t_*}(X,t).$$

For any $Z_1 = (X_1,t_1) \in P^+_\sigma(0,0)$, the monotonicity formula (3.1) implies

$$\begin{aligned}
&r^{-(m+1)}\left(\int_{P^+_r(Z_1)} \frac{1}{2}|\nabla U_\varepsilon|^2 dXdt + \int_{\partial^+ P^+_r(Z_1)} \frac{c_{\frac{1}{2}}}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2 dxdt\right)\\
&\le c\left(\int_{T^+_r(t_1+2r^2)} \frac{1}{2}|\nabla U_\varepsilon|^2 \mathcal{G}_{X_1,t_1+2r^2} dXdt + \int_{\partial^+ T^+_r(t_1+2r^2)} \frac{c_{\frac{1}{2}}}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2 \mathcal{G}_{X_1,t_1+2r^2} dxdt\right)\\
&\le c\left(\int_{T^+_{\frac{1}{2}}(t_1+2r^2)} \frac{1}{2}|\nabla U_\varepsilon|^2 \mathcal{G}_{X_1,t_1+2r^2} dXdt + \int_{\partial^+ T^+_{\frac{1}{2}}(t_1+2r^2)} \frac{c_{\frac{1}{2}}}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2 \mathcal{G}_{X_1,t_1+2r^2} dxdt\right)\\
&= c\mathcal{E}\left(U_\varepsilon,(X_1,t_1+2r^2),\frac{1}{2}\right)\\
&\le C(\mathcal{E}(U_\varepsilon,(0,0),1) + \varepsilon_1 E_0) \le C(\varepsilon_0^2 + \varepsilon_1 E_0),
\end{aligned} \tag{4.15}$$

where we have used (4.16) from Lemma 4.4 below, since $(X_1, t_1+2r^2) \in P^+_\delta(0,0)$. Now one can see that (4.14) follows from (4.15) and (3.5). □

Lemma 4.4. *Let $\varepsilon_1 > 0$ be given. Then there exists $\delta = \delta(\varepsilon_1) > 0$ such that for every $(X_0,t_0) \in \partial\mathbb{R}^{m+1}_+ \times (4,\infty)$, we have*

$$\mathcal{E}\left(U_\varepsilon,(X_1,t_1),\frac{1}{2}\right) \le C(\mathcal{E}(U_\varepsilon,(X_0,t_0),1) + \varepsilon_1 E_0) \quad \text{for every } (X_1,t_1) \in P^+_\delta(X_0,t_0), \tag{4.16}$$

where $C > 0$ is independent of ε, δ and ε_1, and $E_0 = \int_{\mathbb{R}^{m+1}_+} |\nabla U_0|^2$.

Proof. Proof of this Lemma is essentially contained in [26] Lemma 2.4. Here we give a sketch of it. For $(X_1,t_1) \in P^+_\delta(X_0,t_0)$, we see that

$$\begin{aligned}
G_{X_1,t_1}(X,t) &\le \left(\frac{|t-t_0|}{|t-t_1|}\right)^{\frac{m+1}{2}} \exp\left(\frac{|X-X_0|^2}{4|t-t_0|} - \frac{|X-X_1|^2}{4|t-t_1|}\right) G_{X_0,t_0}(X,t)\\
&\le C\exp\left(\frac{|X-X_0|^2}{4|t-t_0|} - \frac{|X-X_1|^2}{4|t-t_1|}\right) G_{X_0,t_0}(X,t)\\
&\le C\exp\left(c\delta^2 \frac{|X-X_0|^2}{4|t-t_0|}\right) G_{X_0,t_0}(X,t)\\
&\le \begin{cases} CG_{X_0,t_0}(X,t), & |X-X_0| \le \delta^{-1}\\ C\exp(-c\delta^{-2}), & |X-X_0| > \delta^{-1}\end{cases}
\end{aligned}$$

for any $(X,t) \in T^+_{\frac{1}{2}}(t_1)$, where $C, c > 0$ are independent of δ and ε.

Therefore, for a given $\varepsilon_1 > 0$ small one can find $\delta > 0$ small so that for $(X, t) \in T^+_{\frac{1}{2}}(t_1)$,

$$G_{X_1,t_1}(X,t) \le C \begin{cases} G_{X_0,t_0}(X,t) & \text{if } |X - X_0| \le \delta^{-1}, \\ \varepsilon_1 & \text{if } |X - X_0| \ge \delta^{-1}. \end{cases}$$

Hence

$$\begin{aligned} &\mathcal{E}\left(U_\varepsilon, (X_1, t_1), \frac{1}{2}\right) \\ &\le C\mathcal{E}(U_\varepsilon, (X_0, t_0), 1) + C\varepsilon_1 \left(\int_{t_1-1}^{t_1} \int_{\mathbb{R}^{m+1}_+} |\nabla \tilde{U}_\varepsilon|^2 dX dt + \int_{t_1-1}^{t_1} \int_{\partial \mathbb{R}^{m+1}_+} (1 - |\tilde{u}_\varepsilon|^2)^2 dx dt \right) \\ &\le C(\varepsilon_0 + \varepsilon_1 E_0), \end{aligned}$$

thanks to (2.35). □

5. Boundary $C^{1+\alpha}$-estimate

This section is devoted to establishing an uniform boundary $C^{1+\alpha}$-estimates for solutions U_ε to (2.30), under the smallness condition of the renormalized energy of U_ε.

Notations: for $r > 0$, set

$$P_r := B_r \times (-r^2, 0],\ P_r^+ := P_r \cap (\mathbb{R}^{m+1}_+ \times \mathbb{R}),$$

and

$$\begin{aligned} \Gamma_r &:= \{(x, 0, t) : |x| < r, \quad -r^2 < t \le 0\},\ \Gamma_r^+ : \\ &= \{(x, y, t) : |x|^2 + y^2 = r^2,\ \ y > 0, \quad -r^2 < t \le 0\}. \end{aligned}$$

Let $\{U_\varepsilon\}_{\varepsilon>0} : P_1^+ \to \mathbb{R}^l$ be a family of solutions to

$$\partial_t U_\varepsilon - \Delta U_\varepsilon = 0 \quad \text{in } P_1^+, \tag{5.1}$$

subject to the Neumann boundary condition

$$\frac{\partial U_\varepsilon}{\partial \nu} = \frac{1}{\varepsilon^2}(1 - |u_\varepsilon|^2)u_\varepsilon \quad \text{on } \Gamma_1. \tag{5.2}$$

Then we have

Proposition 5.1. *Let $\{U_\varepsilon\}$ be a family of solutions to (5.1)-(5.2). Assume that*

$$\frac{1}{2} \le |U_\varepsilon| \le 1, \quad |\partial_t U_\varepsilon| + |\nabla U_\varepsilon| \le 4 \quad in\ P_1^+. \tag{5.3}$$

Then $\|U_\varepsilon\|_{C^{1+\alpha}(P^+_{\frac{1}{4}})} \le C(\alpha)$ for every $0 < \alpha < 1$ and $\varepsilon > 0$.

Proof. Write U_ε in the polar form. Namely, $U_\varepsilon = \rho_\varepsilon \omega_\varepsilon$, with $\rho_\varepsilon = |U_\varepsilon|$ and $\omega_\varepsilon = \frac{U_\varepsilon}{\rho_\varepsilon}$. Then ρ_ε and ω_ε solve

$$\begin{cases} \partial_t \rho_\varepsilon - \Delta \rho_\varepsilon = -|\nabla \omega_\varepsilon|^2 \rho_\varepsilon & \text{in } P_1^+ \\ \dfrac{\partial \rho_\varepsilon}{\partial \nu} = \dfrac{c_{\frac{1}{2}}}{\varepsilon^2}(1 - |\rho_\varepsilon|^2)\rho_\varepsilon & \text{on } \Gamma_1, \end{cases} \tag{5.4}$$

and

$$\begin{cases} \partial_t \omega_\varepsilon - \Delta\omega_\varepsilon = f_\varepsilon := 2\dfrac{\nabla\rho_\varepsilon}{\rho_\varepsilon}\cdot\nabla\omega_\varepsilon + |\nabla\omega_\varepsilon|^2\omega_\varepsilon & \text{in } P_1^+ \\ \dfrac{\partial\omega_\varepsilon}{\partial\nu} = 0 & \text{on } \Gamma_1. \end{cases} \tag{5.5}$$

It follows from (5.3) that

$$\frac{1}{2} \le \rho_\varepsilon \le 1,\ |\nabla\rho_\varepsilon| \le |\nabla U_\varepsilon| \le 8, \wedge |\nabla\omega_\varepsilon| \le 2\frac{|\nabla U_\varepsilon|}{|U_\varepsilon|} \le 16, \quad \text{in} P_1^+, \tag{5.6}$$

and hence

$$|f_\varepsilon| \le 1000 \quad \text{in} \quad P_1^+.$$

From (5.5), we can apply the regularity theory for parabolic equations to conclude that

$$\|\omega_\varepsilon\|_{C^{1+\alpha}(P^+_{\frac{7}{8}})} \le C(\alpha) \quad \text{for every } \alpha \in (0,1), \tag{5.7}$$

uniformly with respect to ε.

Next we show that ρ_ε is uniformly bounded in $C^{1+\alpha}\left(P^+_{\frac{1}{4}}\right)$. From (5.4), it suffices to prove C^α-regularity of $\partial_\nu\rho_\varepsilon$ on $\Gamma_{\frac{1}{2}}$. To this end, we set

$$h_\varepsilon(X,t) = 1 - \rho_\varepsilon(X,t), \quad (X,t) \in P_1^+.$$

The boundary condition in (5.4), and (5.3) imply that

$$0 \le h_\varepsilon \le c\varepsilon^2 \quad \text{on} \quad \Gamma_1. \tag{5.8}$$

For any fixed $(Y,0) \in \Gamma_{\frac{7}{16}}$, set

$$\mathbf{h}_Y^\varepsilon(X,t) := h_\varepsilon(X+Y,t) - h_\varepsilon(X,t) \quad \text{for } (X,t) \in P^+_{\frac{7}{16}}.$$

By direct calculations, $\mathbf{h}_Y^\varepsilon$ satisfies

$$\begin{cases} (\partial_t - \Delta)\mathbf{h}_Y^\varepsilon = \mathbf{f}_Y^\varepsilon & \text{in } P^+_{\frac{7}{16}}, \\ \dfrac{\varepsilon^2}{\rho_\varepsilon(1+\rho_\varepsilon)}\partial_\nu\mathbf{h}_Y^\varepsilon + \mathbf{h}_Y^\varepsilon = \mathbf{g}_Y^\varepsilon & \text{on } \Gamma_{\frac{7}{16}}, \end{cases} \tag{5.9}$$

where

$$\mathbf{f}_Y^\varepsilon(X,t) = |\nabla\omega_\varepsilon(X+Y,t)|^2\rho_\varepsilon(X+Y,t) - |\nabla\omega_\varepsilon(X,t)|^2\rho_\varepsilon(X,t),$$

and

$$\mathbf{g}_Y^\varepsilon(X,t) = \left(1 - \frac{\rho_\varepsilon(X+Y,t)(1+\rho_\varepsilon(X+Y,t))}{\rho_\varepsilon(X,t)(1+\rho_\varepsilon(X,t))}\right)h_\varepsilon(X+Y,t).$$

From the estimates (5.3), (5.7) and (5.8) we infer

$$|\mathbf{f}_Y^\varepsilon(X,t)| \le C|Y|^\alpha \quad \text{in } P^+_{\frac{7}{16}}, \tag{5.10}$$

and

$$|\mathbf{g}_Y^\varepsilon(X,t)| \le C\varepsilon^2|Y|^\alpha \quad \text{on } \Gamma_{\frac{7}{16}}. \tag{5.11}$$

From (5.6) we also have

$$|\mathbf{h}_Y^\varepsilon(X,t)| \le C|Y|^\alpha, \quad \text{in } P^+_{\frac{7}{16}}. \tag{5.12}$$

Denote by

$$m_Y^\varepsilon = \|\mathbf{f}_Y^\varepsilon\|_{L^\infty(P^+_{\frac{7}{16}})},\ n_Y^\varepsilon = \|\mathbf{g}_Y^\varepsilon\|_{L^\infty(P^+_{\frac{7}{16}})},\ p_Y^\varepsilon = \|\mathbf{h}_Y^\varepsilon\|_{L^\infty(P^+_{\frac{7}{16}})}.$$

Now we need

Claim. *There exists a function* $\phi_Y^\varepsilon \in C^\infty\left(\overline{B^+_{\frac{1}{3}}}\right)$ *satisfying*

$$\begin{cases} \mathbf{h}_Y^\varepsilon \le \phi_Y^\varepsilon \le 10p_Y^\varepsilon \le C|Y|^\alpha & \text{in } B^+_{\frac{1}{3}}, \\ \dfrac{4\varepsilon^2}{3}\partial_\nu\phi_Y^\varepsilon + \phi_Y^\varepsilon = 0 & \text{on } \partial B^+_{\frac{1}{3}} \cap \{x \in \mathbb{R}^{m+1}_+ : x_{m+1} = 0\}. \end{cases} \tag{5.13}$$

To verify this claim, first choose a $\frac{1}{3} < r_0 < \frac{7}{16}$ such that $B^+_{\frac{1}{3}} \subset B^m_{r_0} \times [0, r_0]$. Next define $\phi_Y^\varepsilon(x) = \psi_Y^\varepsilon(x_{m+1}) : B^m_{r_0} \times [0, r_0] \to \mathbb{R}$, where $\psi_Y^\varepsilon \in C^\infty([0, r_0])$ satisfies

$$\psi_Y^\varepsilon(\varepsilon^2) = 10p_Y^\varepsilon;\ 10p_Y^\varepsilon \le \psi_Y^\varepsilon(t) \le 20p_Y^\varepsilon,\ \varepsilon^2 < t \le r_0,$$

and

$$\frac{4\varepsilon^2}{3}(\psi_Y^\varepsilon)'(t) + \psi_Y^\varepsilon(t) = 0,\ 0 \le t \le \varepsilon^2. \tag{5.14}$$

Solving (5.14) yields

$$\psi_Y^\varepsilon(t) = 10e^{\frac{3}{4}\left(1-\frac{t}{\varepsilon^2}\right)}p_Y^\varepsilon,\ 0 \le t \le \varepsilon^2.$$

It is clear that (5.13) holds by restricting ϕ_Y^ε on $B^+_{\frac{1}{3}}$.

Finally we let $\widehat{\mathbf{h}}_Y^\varepsilon : B^+_{\frac{1}{3}} \to \mathbb{R}$ be the unique solution of the initial and boundary value problem:

$$\begin{cases} (\partial_t - \Delta)\widehat{\mathbf{h}}_Y^\varepsilon = m_Y^\varepsilon & \text{in } P^+_{\frac{1}{3}}, \\ \widehat{\mathbf{h}}_Y^\varepsilon = \phi_Y^\varepsilon & \text{on } B^+_{\frac{1}{3}} \times \{-(\frac{1}{3})^2\}, \\ \widehat{\mathbf{h}}_Y^\varepsilon = \phi_Y^\varepsilon & \text{on } \Gamma^+_{\frac{1}{3}}, \\ \dfrac{4\varepsilon^2}{3}\partial_\nu\widehat{\mathbf{h}}_Y^\varepsilon + \widehat{\mathbf{h}}_Y^\varepsilon = 0 & \text{on } \Gamma_{\frac{1}{3}}. \end{cases} \tag{5.15}$$

For $\widehat{\mathbf{h}}_Y^\varepsilon$, it follows from both the maximum principle and the C^0-estimate (see [35]) that

$$0 \le \widehat{\mathbf{h}}_Y^\varepsilon \le C\left(m_Y^\varepsilon + \|\phi_Y^\varepsilon\|_{L^\infty(B^+_{\frac{1}{3}})}\right) \le C|Y|^\alpha \quad \text{in } P^+_{\frac{1}{3}}. \tag{5.16}$$

Furthermore, we have the following uniform gradient estimate, whose proof will be given in the Appendix A.

$$\|\widehat{\mathbf{h}}^{\varepsilon}_Y\|_{C^1(P^+_{\frac{7}{24}})} \le C\Big(m^{\varepsilon}_Y + \|\phi^{\varepsilon}_Y\|_{L^\infty(B^+_{\frac{1}{3}})}\Big) \le C|Y|^{\alpha}, \quad \text{for every } \varepsilon > 0. \tag{5.17}$$

This, combined with the boundary condition on $\Gamma_{\frac{1}{3}}$ in (5.15), implies that there exists a constant $C > 0$, independent of ε, such that

$$0 \le \widehat{\mathbf{h}}^{\varepsilon}_Y \le C\varepsilon^2|Y|^{\alpha} \quad \text{on } \Gamma_{\frac{7}{24}}. \tag{5.18}$$

Define functions $H^+_{Y,\varepsilon}$ and $H^-_{Y,\varepsilon}$ on $P^+_{\frac{1}{3}}$ by letting

$$H^{\pm}_{Y,\varepsilon}(X,t) = \widehat{\mathbf{h}}^{\varepsilon}_Y(X,t) \pm \mathbf{h}^{\varepsilon}_Y(X,t) + n^{\varepsilon}_Y \quad \text{in } P^+_{\frac{1}{3}}.$$

Then it is easy to verify that

$$\begin{cases} (\partial_t - \Delta)H^{\pm}_{Y,\varepsilon} \ge 0 & \text{in } P^+_{\frac{1}{3}}, \\ H^{\pm}_{Y,\varepsilon} \ge 0 & \text{on } B^+_{\frac{1}{3}} \times \{-(\frac{1}{3})^2\}, \\ H^{\pm}_{Y,\varepsilon} \ge 0 & \text{on } \Gamma^+_{\frac{1}{3}}, \\ \frac{4\varepsilon^2}{3}\partial_\nu H^{\pm}_{Y,\varepsilon} + H^{\pm}_{Y,\varepsilon} \ge 0 & \text{on } \Gamma_{\frac{1}{3}}. \end{cases} \tag{5.19}$$

Applying the maximum principle to (5.19) (see [35]), we conclude that

$$H^{\pm}_{Y,\varepsilon} \ge 0 \text{ in } P^+_{\frac{1}{3}},$$

or, equivalently,

$$-\widehat{\mathbf{h}}^{\varepsilon}_Y(X,t) - n^{\varepsilon}_Y \le \mathbf{h}^{\varepsilon}_Y(X,t) \le \widehat{\mathbf{h}}^{\varepsilon}_Y(X,t) + n^{\varepsilon}_Y,\ (X,t) \in P^+_{\frac{1}{3}}.$$

Hence we obtain that

$$|\mathbf{h}^{\varepsilon}_Y(X,t)| \le \widehat{\mathbf{h}}^{\varepsilon}_Y(X,t) + n^{\varepsilon}_Y \le C\varepsilon^2|Y|^{\alpha},\ (X,t) \in \Gamma_{\frac{7}{24}},$$

thanks to (5.10), (5.11) and (5.18). In other words, we have

$$\frac{1}{\varepsilon^2}|\rho_\varepsilon(X+Y,t) - \rho_\varepsilon(X,t)| \le C|Y|^{\alpha} \quad \text{for } (X,t) \in \Gamma_{\frac{7}{24}},\ |Y| \le \frac{7}{16}. \tag{5.20}$$

For every fixed $T \in (-\frac{7}{16}, 0]$, set

$$\mathbf{h}^{\varepsilon}_T(X,t) = h_\varepsilon(X,t+T) - h_\varepsilon(X,t),\ (X,t) \in P^+_{\frac{7}{16}}.$$

Then by an argument similar to that for (5.20) we can show that

$$\frac{1}{\varepsilon^2}|\rho_\varepsilon(X,t+T) - \rho_\varepsilon(X,t)| \le C(r)|T|^{\frac{\alpha}{2}} \quad \text{for } (X,t) \in \Gamma_{\frac{7}{24}},\ -\frac{7}{16} \le T \le 0. \tag{5.21}$$

Combining (5.20) and (5.21), and applying the boundary condition of ρ_ε in equation (5.4), we conclude that

$$\|\partial_\nu \rho_\varepsilon\|_{C^{\alpha}(\Gamma_{\frac{7}{24}})} \le C$$

holds uniformly with respect to ε.

Now Proposition 5.1 follows immediately from the standard parabolic regularity theory for equation (5.4) (see [35]). □

6. Passing to the limit and partial regularity

This section is devoted to the proof of our main theorem on the existence of partially smooth solutions of the heat flow of 1/2-harmonic maps.

Completion of proof of Theorem 1.1:

Proof. For $s = \frac{1}{2}$, let $\{U_\varepsilon\}_{\varepsilon>0}$ be a family of solutions of (2.30), satisfying the bound (2.34), and $U : \mathbb{R}^{m+1}_+ \times [0, \infty) \to \mathbb{R}^l$ be the weak limit of U_ε as $\varepsilon \to 0$. It it readily seen that $U \in C^\infty(\mathbb{R}^{m+1}_+ \times (0, \infty))$ solves

$$\partial_t U - \Delta U = 0 \text{in}\, \mathbb{R}^{m+1}_+ \times (0, \infty); \quad U|_{t=0} = U_0 \quad \text{on}\, \mathbb{R}^{m+1}_+, \tag{6.1}$$

and $u := U|_{\partial\mathbb{R}^{m+1}_+ \times (0,\infty)}$ is a weak solution of the equation of 1/2-harmonic map heat flow:

$$(\partial_t - \Delta)^{\frac{1}{2}} u \perp T_u \mathbb{S}^{l-1} \quad \text{on } \mathbb{R}^m \times (0, \infty); \quad u|_{t=0} = u_0 \quad \text{on } \mathbb{R}^m.$$

We are left with showing u enjoys the partial regularity as stated in Theorem 1.1. To show this, let $\varepsilon_0 > 0$ be the constant determined by Lemma 4.3 and define the singular set $\Sigma \subset \partial\mathbb{R}^{m+1}_+ \times (0, \infty)$ by

$$\Sigma = \bigcap_{R>0} \{Z_0 = (X_0, t_0) \in \partial\mathbb{R}^{m+1}_+ \times (0, \infty) : \liminf_{\varepsilon\to 0} \mathcal{E}(U_\varepsilon, Z_0, R) \geq \varepsilon_0^2\}. \tag{6.2}$$

It is well-known that the monotonicity inequality (3.1) implies that Σ is a closed set in $\partial\mathbb{R}^{m+1}_+ \times (0, \infty)$. Furthermore, similar to the proof of Lemma 4.3 and Lemma 4.4, we have that for any $Z_0 = (X_0, t_0) \in \Sigma$, there exists a $0 < r_0 < \sqrt{t_0}$ such that for all $0 < r < r_0$,

$$r^{-m} \left(\int_{P_r^+(Z_0)} |\nabla U_\varepsilon|^2 + \int_{\partial^+ P_r^+(Z_0)} \frac{1}{\varepsilon^2} (1 - |u_\varepsilon|^2)^2 \right) \geq c\varepsilon_0^2.$$

Now we can apply Vitali's covering Lemma to show that for any compact set $K \subset \overline{\mathbb{R}^{m+1}_+} \times (0, \infty)$, the m-dimensional Hausdorff measure of $\Sigma \cap K$ is finite, i.e.,

$$\mathcal{P}^m(\Sigma \cap K) \leq C(E_0, K) < \infty.$$

It follows from the definition of Σ that for any $Z_1 = (X_1, t_1) \in \partial\mathbb{R}^{m+1}_+ \times (0, \infty) \setminus \Sigma$, we can find a radius $0 < R_1 < \frac{\sqrt{t_1}}{2}$ so that

$$\liminf_{\varepsilon\to 0} \mathcal{E}(U_\varepsilon, Z_1, R_1) < \varepsilon_0^2.$$

Hence by Lemma Lemma 4.3 and Lemma 4.4 we can conclude that there exists a $\delta_1 > 0$, independent of ε, such that for any $\alpha \in (0, 1)$,

$$\|U_\varepsilon\|_{C^{1+\alpha}(P^+_{2\delta_1 R_1}(Z_1))} \leq C(\varepsilon_0, \alpha). \tag{6.3}$$

Thus $U_\varepsilon \to U$ in $C^{1+\alpha}(P^+_{\delta_1 R_1}(Z_1))$. In particular, $U \in C^{1+\alpha}_{\text{loc}}(\overline{\mathbb{R}^{m+1}_+} \times (0, \infty) \setminus \Sigma)$. Applying higher order boundary regularity theory of (6.1), we conclude that $U \in C^\infty_{\text{loc}}(\overline{\mathbb{R}^{m+1}_+} \times (0, \infty) \setminus \Sigma)$. This yields part A) of Theorem 1.1.

Observe that for any sufficiently large $t_0 > 0$, and $X_0 \in \partial\mathbb{R}^{m+1}_+$, if choose $R = \frac{\sqrt{t_0}}{2}$, then

$$\begin{aligned}\mathcal{E}(U_\varepsilon, (X_0, t_0), R) &= \int_0^{\frac{3t_0}{4}} \int_{\mathbb{R}^{m+1}_+} \frac{1}{2}|\nabla U_\varepsilon|^2 \mathcal{G}_{X_0,t_0} dXdt + \int_0^{\frac{3t_0}{4}} \int_{\partial\mathbb{R}^{m+1}_+} \frac{c_{\frac{1}{2}}}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2 \mathcal{G}_{X_0,t_0} dxdt \\ &\le Ct_0^{-\frac{m+1}{2}} \int_0^{\frac{3t_0}{4}} \left(\int_{\mathbb{R}^{m+1}_+} \frac{1}{2}|\nabla U_\varepsilon|^2 dX + \int_{\partial\mathbb{R}^{m+1}_+} \frac{c_{\frac{1}{2}}}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2 dx \right) dt \\ &\le Ct_0^{-\frac{m-1}{2}} \int_{\mathbb{R}^{m+1}_+} \frac{1}{2}|\nabla U_0|^2 dXdt = Ct_0^{-\frac{m-1}{2}} E_0 < \varepsilon_0^2,\end{aligned}$$

uniformly in ε, provided $t_0 > (\frac{CE_0}{\varepsilon_0^2})^{\frac{2}{m-1}}$. Here we have used (2.34).

Hence by Lemma 4.3 and Lemma 4.4, we can conclude that $\Sigma \cap (\partial\mathbb{R}^{m+1}_+ \times [t_0, \infty)) = \emptyset$, and $U_\varepsilon \to U$ in $C^2_{loc}(\overline{\mathbb{R}^{m+1}_+} \times [t_0, \infty))$. Furthermore, it holds that

$$|\nabla U(X, t)| \le \frac{c}{\sqrt{t}},$$

for all $X \in \overline{\mathbb{R}^{m+1}_+}$ and t sufficiently large. There exists a point $p \in \mathbb{S}^{l-1}$ such that $U(\cdot, t) \to p$ in $C^2_{loc}(\overline{\mathbb{R}^{m+1}_+})$ as $t \to \infty$. Hence $u(\cdot, t)$ also converges to p in $C^2_{loc}(\partial\mathbb{R}^{m+1}_+)$ as $t \to \infty$. This yields part B) of Theorem 1.1.

The proof of part C) can be done in the same way as in Cheng [36]. We sketch it as follows. First recall that for any $\delta > 0$, there exists a sufficiently large $K(\delta) > 0$ such that for any $t_0 > 0$ and $0 < R < \frac{\sqrt{t_0}}{2}$, it holds for $t_0 - 4R^2 \le t \le t_0 - R^2$,

$$\mathcal{G}_{(X_0,t_0)}(X, t) \le \begin{cases} R^{-(m+1)} & \forall X \in \mathbb{R}^{m+1}_+, \\ \delta\mathcal{G}_{(X_0,t_0)+(0,R^2)}(X, t) & \text{if } X \in \mathbb{R}^{m+1}_+ \text{ and } |X - X_0| \ge K(\delta)R. \end{cases}$$

Hence

$$\begin{aligned}&\mathcal{E}(U_\epsilon, (X_0, t_0), R) \\ &\le R^{-(m+1)} \int_{t_0-4R^2}^{t_0-R^2} \left(\int_{B^+_{K(\delta)R}(X_0)} |\nabla U_\varepsilon|^2 + \int_{B^+_{K(\delta)R}(X_0)\cap\partial\mathbb{R}^{m+1}_+} \frac{c_{\frac{1}{2}}}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2 \right) dt \\ &+ \delta \int_{t_0-4R^2}^{t_0-R^2} \left(\int_{\mathbb{R}^{m+1}_+} |\nabla U_\varepsilon|^2 \mathcal{G}_{(X_0,t_0+R^2)} + \int_{\partial\mathbb{R}^{m+1}_+} \frac{c_{\frac{1}{2}}}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2 \mathcal{G}_{(X_0,t_0+R^2)} \right) dt.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\delta \int_{t_0-4R^2}^{t_0-R^2} \left(\int_{\mathbb{R}^{m+1}_+} |\nabla U_\varepsilon|^2 \mathcal{G}_{(X_0,t_0+R^2)} + \int_{\partial\mathbb{R}^{m+1}_+} \frac{c_{\frac{1}{2}}}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2 \mathcal{G}_{(X_0,t_0+R^2)} \right) dt \\ &\le 2\delta \int_{t_0-4R^2}^{t_0-R^2} (R^2 + t_0 - t)^{-1} \mathcal{D}(U_\varepsilon, (X_0, t_0 + R^2), \sqrt{R^2 + t_0 - t})\ dt \\ &\le 2\delta \left(\int_{t_0-4R^2}^{t_0-R^2} (R^2 + t_0 - t)^{-1}\ dt \right) \mathcal{D}(U_\varepsilon, (X_0, t_0 + R^2), \sqrt{R^2 + t_0}) \\ &\le C\delta(t_0 + R^2)^{\frac{1-m}{2}} E_0 \le \frac{1}{2}\varepsilon_0^2,\end{aligned}$$

provided $\delta > 0$ is chosen to be sufficiently small. Here we have used the monotonicity inequality for $\mathcal{D}(U_\varepsilon, (X_0, t_0 + R^2), r)$ in the proof.

Note that $\Sigma_{t_0} = \cap_{0<R<\sqrt{t_0}} \Sigma_{t_0}^R$, where

$$\Sigma_{t_0}^R = \{X_0 \in \partial\mathbb{R}_+^{m+1} : \liminf_{\varepsilon\to 0} \mathcal{E}(U_\varepsilon, (X_0, t_0), R) \geq \varepsilon_0^2\}.$$

Thus we obtain that for any $X_0 \in \Sigma_{t_0}^R$, it holds

$$R^{m+1} \leq \frac{2}{\varepsilon_0^2} \lim_{\varepsilon\to 0} \int_{t_0-4R^2}^{t_0-R^2} \left(\int_{B^+_{K(\delta)R}(X_0)} |\nabla U_\varepsilon|^2 + \int_{B^+_{K(\delta)R}(X_0)\cap\partial\mathbb{R}_+^{m+1}} \frac{c_{\frac{1}{2}}}{4\varepsilon^2} (1 - |u_\varepsilon|^2)^2 \right) dt$$

so that by Vitali's covering Lemma we can show that

$$H^{m-1}_{K(\delta)R}(\Sigma_{t_0}^R) \leq C(K(\delta), E_0).$$

This implies $H^{m-1}(\Sigma_{t_0}) < \infty$, after sending $R \to 0$. □

7. Appendix A: uniform estimate of heat kernels

In this section, we will sketch a proof of the gradient estimate (5.17) for the solution $\widehat{\mathbf{h}}_Y^\varepsilon$ of the auxiliary Equation (5.15), which holds uniformly with respect to ε. We refer the reader to [35] Theorem 4.31, in which an estimate similar to (5.17) is established but with a constant possibly depending on ε. Here we will provide a proof based on an explicit Green function representation of the heat equation under an oblique boundary condition.

First recall the heat kernel in $\mathbb{R}^{m+1}$ given by

$$\Gamma(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{m+1}{2}}} \exp\left(-\frac{|x|^2}{4t}\right), & (x,t) \in \mathbb{R}^{m+1} \times (0,\infty), \\ 0, & (x,t) \in \mathbb{R}^{m+1} \times (-\infty, 0]. \end{cases}$$

For $y = (y_1, ..., y_m, y_{m+1}) \in \mathbb{R}_+^{m+1}$, denote $y^* = (y_1, ..., y_m, -y_{m+1})$. Define $G^\varepsilon(x,y,t) : \mathbb{R}_+^{m+1} \times \mathbb{R}_+^{m+1} \times \mathbb{R} \to \mathbb{R}$ by

$$G^\varepsilon(x,y,t) = \Gamma(x-y,t) - \Gamma(x-y^*,t) - 2\int_0^\infty e^{-\frac{3}{4\varepsilon^2}\tau} D_{m+1}\Gamma(x - y^* + \tau e_{m+1}, t)\, d\tau, \tag{7.1}$$

where $D_{m+1}\Gamma(z,t) = \frac{\partial\Gamma}{\partial x_{m+1}}(z,t)$ and $e_{m+1} = (0', 1) \in \mathbb{R}^{m+1}$. Then we have

Lemma 7.1. *G^ε is the Green function of the heat equation in $\mathbb{R}_+^{m+1}$ with an oblique boundary condition: for any fixed $y \in \mathbb{R}_+^{m+1}$,*

$$\begin{cases} (\partial_t - \Delta)G^\varepsilon(x,y,t) = \delta(x-y)\delta(t), & (x,t) \in \mathbb{R}_+^{m+1} \times \mathbb{R}_+, \\ \frac{\partial G^\varepsilon}{\partial x_{m+1}}(x,y,t) - \frac{3}{4\varepsilon^2} G^\varepsilon(x,y,t) = 0, & x \in \partial\mathbb{R}_+^{m+1} \times [0,\infty). \end{cases} \tag{7.2}$$

Proof. Since $y^* \in \mathbb{R}_-^{m+1}$ for $y \in \mathbb{R}_+^{m+1}$, it follows that $x - y^* \neq 0$ and $x - y^* + \tau e_{m+1} \neq 0$ for any $x \in \mathbb{R}_+^{m+1}$ and $\tau > 0$. Hence we have

$$(\partial_t - \Delta)G^\varepsilon(x,y,t) = (\partial_t - \Delta)\Gamma(x-y,t) = \delta(x-y)\delta(t).$$

To check the boundary condition, let $x \in \partial\mathbb{R}_+^{m+1}$. Then we have that $x_{m+1} = 0$ and $|x - y| = |x - y^*|$ so that $\Gamma(x-y,t) = \Gamma(x-y^*,t)$ and $D_{m+1}\Gamma(x-y^*,t) = -D_{m+1}\Gamma(x-y,t)$. Hence

$$
\begin{aligned}
&\frac{\partial G^\varepsilon}{\partial x_{m+1}}(x,y,t) - \frac{3}{4\varepsilon^2} G^\varepsilon(x,y,t) \\
&= -2D_{m+1}\Gamma(x-y^*,t) - 2\int_0^\infty e^{-\frac{3}{4\varepsilon^2}\tau} \frac{\partial}{\partial x_{m+1}} \left[D_{m+1}\Gamma(x-y^*+\tau e_{m+1},t)\right]\, d\tau \\
&\quad + \frac{6}{4\varepsilon^2}\int_0^\infty e^{-\frac{3}{4\varepsilon^2}\tau} D_{m+1}\Gamma(x-y^*+\tau e_{m+1},t)\, d\tau \\
&= -2D_{m+1}\Gamma(x-y^*,t) - 2\int_0^\infty \frac{\partial}{\partial \tau}(e^{-\frac{3}{4\varepsilon^2}\tau} D_{m+1}\Gamma(x-y^*+\tau e_{m+1},t))\, d\tau \\
&= 0
\end{aligned}
$$

holds for $x \in \partial\mathbb{R}^{m+1}_+$. □

For any bounded $f \in C^\infty(\mathbb{R}^{m+1}_+ \times [0,\infty))$, it is well-known that the unique smooth solution of

$$
\begin{cases}
(\partial_t - \Delta)u = f & \text{in } \mathbb{R}^{m+1}_+ \times [0,\infty), \\
\dfrac{\partial u}{\partial x_{m+1}} - \dfrac{3}{4\varepsilon^2}u = 0 & \text{on } \partial\mathbb{R}^{m+1}_+ \times [0,\infty), \\
u = 0 & \text{on } \mathbb{R}^{m+1}_+ \times \{0\},
\end{cases}
\tag{7.3}
$$

is given by the Duhamel formula

$$
u_\varepsilon(x,t) = \int_0^t \int_{\mathbb{R}^{m+1}_+} G^\varepsilon(x,y,t-s) f(y,s)\, dyds,\ (x,t) \in \mathbb{R}^{m+1}_+ \times [0,\infty). \tag{7.4}
$$

Now we are ready with the proof of the following theorem.

Theorem 7.2. *For any $f \in C^\infty(\overline{\mathbb{R}^{m+1}_+ \times [0,\infty)})$ and $\varepsilon > 0$, let $u^\varepsilon : \mathbb{R}^{m+1}_+ \times [0,\infty) \to \mathbb{R}$ be given by (7.4). Then for any $0 < \alpha < 1$ there exists a constant $C = C(m,\alpha) > 0$ such that*

$$
\|u^\varepsilon\|_{C^{2+\alpha}(\mathbb{R}^{m+1}_+ \times [0,\infty))} \le C\|f\|_{C^\alpha(\mathbb{R}^{m+1}_+ \times [0,\infty))},\ \forall \varepsilon > 0. \tag{7.5}
$$

Proof. Decompose G^ε by $G^\varepsilon = G^\varepsilon_1 + G^\varepsilon_2$, where

$$
G^\varepsilon_1(x,y,t) = \Gamma(x-y,t) - \Gamma(x-y^*,t);\ G^\varepsilon_2 = G^\varepsilon - G^\varepsilon_1,
$$

and write $u^\varepsilon = u^\varepsilon_1 + u^\varepsilon_2$, where

$$
u^\varepsilon_1(x,t) = \int_0^t \int_{\mathbb{R}^{m+1}_+} G^\varepsilon_1(x,y,t-s) f(y,s)\, dyds;\ u^\varepsilon_2 = u^\varepsilon - u^\varepsilon_1.
$$

Since $G^\varepsilon_1(x,y,t)$ is the Green function of the heat equation on $\mathbb{R}^{m+1}_+$ with zero Dirichlet boundary condition, by the standard Schauder theory (see [35]) we have that $u^\varepsilon_1 \in C^\infty(\overline{\mathbb{R}^{m+1}_+} \times [0,\infty))$ and

$$
\|u^\varepsilon_1\|_{C^{2+\alpha}(\overline{\mathbb{R}^{m+1}_+} \times [0,\infty))} \le C(m,\alpha)\|f\|_{C^\alpha(\overline{\mathbb{R}^{m+1}_+} \times [0,\infty))}.
$$

To prove a similar estimate for u^ε_2 we first note that (7.1) gives

$$
G^\varepsilon_2(x,y,t) = -2\int_0^\infty e^{-\frac{3}{4\varepsilon^2}\tau} D_{m+1}\Gamma(x-y^*+\tau e_{m+1},t)\, d\tau. \tag{7.6}
$$

By direct computation we have that

$$
D_{m+1}\Gamma(x-y^*+\tau e_{m+1},t) = \frac{-1}{2}\frac{1}{(4\pi t)^{\frac{m+1}{2}}}\frac{(x_{m+1}+y_{m+1}+\tau)}{t}\exp\left(-\frac{|x-y^*+\tau e_{m+1}|^2}{4t}\right).
$$

Moreover, by the very definition of y^*

$$|x-y^*+\tau e_{m+1}|^2 = |x'-y'|^2 + (x_{m+1}+y_{m+1}+\tau)^2,$$

where we recall

$$x' := (x_1, \dots, x_m, 0) \qquad y' = (y_1, \dots, y_m, 0).$$

Therefore (7.6) becomes

$$G_2^\varepsilon(x,y,t) = \frac{1}{(4\pi t)^{\frac{m+1}{2}}} e^{-|x'-y'|^2} \int_0^{+\infty} \frac{(x_{m+1}+y_{m+1}+\tau)}{t} \exp\left(-\frac{3\tau}{4\varepsilon^2} - \frac{|x_{m+1}+y_{m+1}+\tau|^2}{4t}\right) d\tau.$$

We change variables in the integral according to

$$r := \frac{x_{m+1}+y_{m+1}+\tau}{\sqrt{t}}.$$

Moreover, we write

$$-|x'-y'|^2 = -|x-y^*|^2 + (x_{m+1}+y_{m+1})^2.$$

Then,

$$G_2^\varepsilon(x,y,t) = \frac{1}{(4\pi t)^{\frac{m+1}{2}}} e^{-|x-y^*|^2} e^{(x_{m+1}+y_{m+1})^2 + \frac{3}{4\varepsilon^2}(x_{m+1}+y_{m+1})} \int_{\frac{x_{m+1}+y_{m+1}}{\sqrt{t}}}^{+\infty} r \exp\left(-\frac{3r\sqrt{t}}{4\varepsilon^2} - \frac{r^2}{4}\right) dr.$$

We introduce the function $\Theta_\varepsilon : [0, +\infty) \to \mathbb{R}$ given by

$$\Theta_\varepsilon(\lambda) := e^{(x_{m+1}+y_{m+1})^2 + \frac{3}{4\varepsilon^2}(x_{m+1}+y_{m+1})} \int_\lambda^{+\infty} r \exp\left(-\frac{3r\sqrt{t}}{4\varepsilon^2} - \frac{r^2}{4}\right) dr,$$

and thus G_2^ε is represented as

$$G_2^\varepsilon(x,y,t) = \frac{1}{(4\pi t)^{\frac{m+1}{2}}} e^{-|x-y^*|^2} \Theta_\varepsilon\left(\frac{x_{m+1}+y_{m+1}}{\sqrt{t}}\right).$$

We have that for any $\varepsilon > 0$, $\Theta_\varepsilon \in C^\infty([0,\infty))$. Moreover, since

$$\Theta_\varepsilon'(\lambda) = -e^{(x_{m+1}+y_{m+1})^2 + \frac{3}{4\varepsilon^2}(x_{m+1}+y_{m+1})} \lambda \exp\left(-\frac{3\lambda\sqrt{t}}{4\varepsilon^2} - \frac{\lambda^2}{4}\right),$$

we get that

$$\Theta_\varepsilon'\left(\frac{x_{m+1}+y_{m+1}}{\sqrt{t}}\right) = -\left(\frac{x_{m+1}+y_{m+1}}{\sqrt{t}}\right) e^{-\frac{(x_{m+1}+y_{m+1})^2+4t}{4t}},$$

which is bounded, uniformly with respect to ε and with respect to $t>0$. Therefore, we conclude thanks to Schauder theory that $u_2^\varepsilon \in C^\infty(\mathbb{R}_+^{m+1} \times [0,\infty))$ and

$$\|u_2^\varepsilon\|_{C^{2+\alpha}(\overline{\mathbb{R}_+^{m+1}} \times [0,\infty))} \le C(m,\alpha)\|f\|_{C^\alpha(\overline{\mathbb{R}_+^{m+1}} \times [0,\infty))}.$$

Combining the estimates for u_1^ε and u_2^ε yields (7.5). □

Now we will give a proof of (5.17). To do it, let $\eta_1 \in C_0^\infty\left(B_{\frac{1}{3}}^m \times \left(-(\frac{1}{3})^2, 0\right)\right)$ be such that $\eta_1 = 1$ in $B_{\frac{7}{24}}^m \times \left(-(\frac{7}{24})^2, 0\right)$, and $\eta_2 \in C_0^\infty([0,\infty)$ be such that $\eta_2 = 1$ in $[0, \frac{1}{3}]$ and $\eta_2 = 0$ in $[\frac{2}{3}, \infty)$. Define $\eta(x,t) = \eta_1(x',t)\eta_2(x_{m+1})$ for $(x,t) \in \mathbb{R}_+^{m+1} \times \mathbb{R}$. Then by direct calculations we obtain that

$$\begin{aligned}
&\left(\frac{\partial}{\partial x_{m+1}}(\widehat{\mathbf{h}}_Y^\varepsilon\eta) - \frac{3}{4\varepsilon^2}\widehat{\mathbf{h}}_Y^\varepsilon\eta\right)(x,t) \\
&\quad = \widehat{\mathbf{h}}_Y^\varepsilon(x,t)\eta_1(x',t)\eta_2'(x_{m+1}) + \left(\frac{\partial}{\partial x_{m+1}}\widehat{\mathbf{h}}_Y^\varepsilon - \frac{3}{4\varepsilon^2}\widehat{\mathbf{h}}_Y^\varepsilon\right)(x,t)\eta(x,t) \\
&\quad = 0 + 0 = 0
\end{aligned}$$

holds for any $(x,t) \in \partial\mathbb{R}_+^{m+1} \times (0,\infty) \cap \Gamma_{\frac{1}{3}}^+$. Hence by Duhamel's formula, we conclude that for any $(x,t) \in P_{\frac{7}{24}}^+$, it holds

$$\begin{aligned}
(\widehat{\mathbf{h}}_Y^\varepsilon\eta)(x,t) &= \int_{\mathbb{R}_+^{m+1}\times(0,\infty)} G^\varepsilon(x,y,t-s)(\partial_t - \Delta)(\widehat{\mathbf{h}}_Y^\varepsilon\eta)(y,s)\ dyds \\
&= m_Y^\varepsilon \int_{\mathbb{R}_+^{m+1}\times(0,\infty)} G^\varepsilon(x,y,t-s)\eta(y,s)\ dyds \\
&\quad + \int_{\mathbb{R}_+^{m+1}\times(0,\infty)} G^\varepsilon(x,y,t-s)\widehat{\mathbf{h}}_Y^\varepsilon(y,s)(\partial_t\eta - \Delta\eta)(y,s)\ dyds \\
&\quad + 2\int_{\mathbb{R}_+^{m+1}\times(0,\infty)} (\nabla_y G(x,y,t-s)\nabla\eta(y,s) + G(x,y,t-s)\Delta\eta(y,s))\widehat{\mathbf{h}}_Y^\varepsilon(y,s)\ dyds \\
&=: A^\varepsilon(x,t) + B^\varepsilon(x,t) + C^\varepsilon(x,t).
\end{aligned} \tag{7.7}$$

Applying Theorem 7.2, there exists a constant $C > 0$ independent of ε such that

$$\|A^\varepsilon\|_{C^{2+\alpha}(\mathbb{R}_+^{m+1}\times(0,\infty))} \le Cm_Y^\varepsilon \le C|Y|^\alpha.$$

For B^ε and C^ε, it is not hard to see that

$$\|\nabla B^\varepsilon\|_{C^\alpha(\mathbb{R}_+^{m+1}\times(0,\infty))} + \|\nabla C^\varepsilon\|_{C^\alpha(\mathbb{R}_+^{m+1}\times(0,\infty))} \le C\|\widehat{\mathbf{h}}_Y^\varepsilon\|_{C^0(P_{\frac{1}{3}}^+)} \le Cp_Y^\varepsilon \le C|Y|^\alpha.$$

Putting these estimates together, we conclude that $\widehat{\mathbf{h}}_Y^\varepsilon$ satisfies the gradient estimate (5.17).

8. Appendix B: Proof of Theorem 1.1 for general targets

In this section, we will sketch the modifications that are necessary in order to show Theorem 1.1 for any compact Riemannian manifold $N \hookrightarrow \mathbb{R}^l$.

To do it, first recall that there exists a constant $\delta_N > 0$ such that both the nearest point projection map

$$\Pi_N : N_{\delta_N} \equiv \{y \in \mathbb{R}^l : d(y,N) < \delta_N\} \to N$$

and the square of distance function to N, $d^2(p,N) = |p - \Pi_N(p)|^2$, are smooth in the δ_N-neighborhood of N.

Now let $\chi \in C_0^\infty([0,\infty))$ be such that

$$\chi(t) = t \text{ for } 0 \le t \le \delta_N^2; \quad \chi(t) = 2\delta_0 \text{ for } t \ge (2\delta_N)^2.$$

Then we replace the potential function $\frac{1}{4\varepsilon^2}(1-|u|^2)^2$ by $\frac{1}{\varepsilon^2}\chi(d^2(u,N))$. More precisely, we consider the following approximated system:

$$\begin{cases} (\partial_t - \Delta)U_\varepsilon = 0 & \text{in } \mathbb{R}^{m+1}_+ \times (0,\infty), \\ U_\varepsilon|_{t=0} = U_0 & \text{on } \mathbb{R}^{m+1}_+, \\ \lim_{y\to 0^+} \frac{\partial U_\varepsilon}{\partial y} = \frac{c_{\frac{1}{2}}}{\varepsilon^2}\chi'(d^2(U_\varepsilon,N))D_{U_\varepsilon}d^2(U_\varepsilon,N) & \text{on } \mathbb{R}^m \times (0,\infty). \end{cases} \tag{8.1}$$

As in (2.34), it is readily seen that any solution U_ε of (8.1) satisfies the following energy inequality:

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^{m+1}_+} \left|\frac{\partial U_\varepsilon(X,r)}{\partial t}\right|^2 \mathrm{d}X\mathrm{d}r + \int_{\mathbb{R}^{m+1}_+} |\nabla_X U_\varepsilon(X,t)|^2 \mathrm{d}X \\ &\quad + \frac{c_{\frac{1}{2}}}{\varepsilon^2}\int_{\mathbb{R}^m} \chi(d^2(u_\varepsilon,N))\mathrm{d}x = \int_{\mathbb{R}^{m+1}_+} |\nabla_X U_0(X)|^2 \mathrm{d}X \le \|u_0\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^m)}. \end{aligned} \tag{8.2}$$

As in Section 3, we can similarly define the renormalized energies $\mathcal{D}(U_\varepsilon, Z_0, R)$ and $\mathcal{E}(U_\varepsilon, Z_0, R)$ for U_ε by simply replacing the term $(1-|u_\varepsilon|^2)^2$ by $\chi(d^2(u_\varepsilon,N))$. For example,

$$\begin{aligned} \mathcal{E}(U_\varepsilon, Z_0, R) \quad &:= \frac{1}{2}\int_{T_R^+(Z_0)} \mathcal{G}_{X_0,t_0}(X,t)|\nabla U_\varepsilon|^2 dXdt \\ &\quad + \frac{c_{\frac{1}{2}}}{\varepsilon^2}\int_{\partial^+ T_R^+(Z_0)} \mathcal{G}_{X_0,t_0}(X,t)\chi(d^2(u_\varepsilon,N))dxdt. \end{aligned}$$

Then by the same argument as in Lemma 3.1, we have

Lemma 8.1. *For $Z_0 = (X_0, t_0) \in \partial\mathbb{R}^{m+1}_+ \times (0,\infty)$, if U_ε solves (8.1) then it holds that*

$$\mathcal{D}(U_\varepsilon, Z_0, r) \le \mathcal{D}(U_\varepsilon, Z_0, R),\ \forall 0 < r \le R < \sqrt{t_0},$$
$$\mathcal{E}(U_\varepsilon, Z_0, r) \le \mathcal{E}(U_\varepsilon, Z_0, R),\ \forall 0 < r \le R < \frac{\sqrt{t_0}}{2}.$$

As in Lemma 3.2, we also have the local energy inequality.

Lemma 8.2. *For any $\eta \in C_0^\infty(\mathbb{R}^{m+1})$, if U_ε solves (8.1) then it holds that*

$$\frac{d}{dt}\left\{\int_{\mathbb{R}^{m+1}_+} \frac{1}{2}|\nabla U_\varepsilon|^2\eta^2 + \int_{\mathbb{R}^m} \frac{c_{\frac{1}{2}}}{\varepsilon^2}\chi(d^2(u_\varepsilon,N))\eta^2\right\} + \frac{1}{2}\int_{\mathbb{R}^{m+1}_+} |\partial_t U_\varepsilon|^2\eta^2 \le 4\int_{\mathbb{R}^{m+1}_+} |\nabla U_\varepsilon|^2|\nabla\eta|^2. \tag{8.3}$$

In particular, for any $Z_0 = (X_0, t_0) \in \overline{\mathbb{R}^{m+1}_+} \times (0,\infty)$ and $0 < R < \frac{\sqrt{t_0}}{2}$, we have that

$$\int_{P_R^+(Z_0)} |\partial_t U_\varepsilon|^2 \le CR^{-2}\left(\int_{P_{2R}^+(Z_0)} |\nabla U_\varepsilon|^2 + \int_{\partial^+ P_{2R}^+(Z_0)} \frac{c_{\frac{1}{2}}}{\varepsilon^2}\chi(d^2(u_\varepsilon,N))\right). \tag{8.4}$$

We also have the following clearing out result for any solution U_ε of (8.1).

Lemma 8.3. *There exists $\varepsilon_0 > 0$ such that if U_ε solves (8.1) and satisfies*

$$\mathcal{E}(U_\varepsilon, (X_0, t_0), 1) \le \varepsilon_0^2,$$

for some $X_0 \in \partial\mathbb{R}^{m+1}_+$ and $t_0 > 4$, then $d(U_\varepsilon, N) \le \delta_N$ and $\chi(d^2(U_\varepsilon,N)) = d^2(U_\varepsilon,N)$ hold on $P_\beta^+(X_0,t_0)$ for some $\beta > 0$ that is independent of $U_\varepsilon, X_0,$ and t_0.

The next Lemma, analogous to Proposition 5.1, plays a crucial role in the proof.

Lemma 8.4. *Let $\{U_\varepsilon\}_{\varepsilon>0}$ be a family of solutions to (8.1). Assume that*

$$d(U_\varepsilon, N) \le \delta_N,\ |\partial_t U_\varepsilon| + |\nabla U_\varepsilon| \le 4 \text{ in } P_1^+. \tag{8.5}$$

Then $\|U_\varepsilon\|_{C^{1+\alpha}(P^+_{\frac{1}{4}})} \le C(\alpha)$ for any $\alpha \in (0,1)$ and $\varepsilon > 0$.

Proof. The proof is similar to that of Proposition 5.1 (see also [6, pages 342-346]). Since $U_\varepsilon(P_1^+) \subset N_{\delta_N}$, we can decompose

$$U_\varepsilon = V_\varepsilon + \nu_N(V_\varepsilon)\rho_\varepsilon \text{ in } P_1^+.$$

Here $V_\varepsilon = \Pi_N(U_\varepsilon)$, $\rho_\varepsilon = d(U_\varepsilon, N) = |U_\varepsilon - V_\varepsilon|$, and $\nu_N(V_\varepsilon) \in (T_{V_\varepsilon}N)^\perp$ is a smooth unit vector field in the normal space $(T_{V_\varepsilon}N)^\perp$. By direct calculations, we obtain that

$$\begin{aligned}
0 &= \partial_t U_\varepsilon - \Delta U_\varepsilon \\
&= (\mathbb{I}_l + \rho_\varepsilon \nabla_{V_\varepsilon}\nu_N(V_\varepsilon))(\partial_t V_\varepsilon - \Delta V_\varepsilon) + \nu_N(V_\varepsilon)(\partial_t\rho_\varepsilon - \Delta\rho_\varepsilon) \\
&\quad - 2\nabla(\nu_N(V_\varepsilon))\nabla\rho_\varepsilon - \rho_\varepsilon \nabla^2_{V_\varepsilon}\nu_N(V_\varepsilon)(\nabla V_\varepsilon, \nabla V_\varepsilon)
\end{aligned}$$

hold in P_1^+. If we multiply the equation above by $\nu_N(V_\varepsilon)$ and observe that

$$\langle(\mathbb{I}_l + \rho_\varepsilon\nabla_{V_\varepsilon}\nu_N(V_\varepsilon))(\partial_t V_\varepsilon - \Delta V_\varepsilon), \nu_N(V_\varepsilon)\rangle = \langle \Delta V_\varepsilon, \nu_N(V_\varepsilon)\rangle = -\nabla_{V_\varepsilon}\nu_N(V_\varepsilon)(\nabla V_\varepsilon, \nabla V_\varepsilon),$$

we can show that V_ε and ρ_ε solve

$$\begin{cases}
(\mathbb{I}_l + \rho_\varepsilon\nabla_{V_\varepsilon}\nu_N(V_\varepsilon))(\partial_t V_\varepsilon - \Delta V_\varepsilon) = \rho_\varepsilon \Pi_N(V_\varepsilon)(\nabla^2_{V_\varepsilon}\nu_N(V_\varepsilon)(\nabla V_\varepsilon, \nabla V_\varepsilon)) & \\
\qquad - 2\nabla(\nu_N(V_\varepsilon))\nabla\rho_\varepsilon + \rho_\varepsilon\nabla_{V_\varepsilon}\nu_N(V_\varepsilon)(\nabla V_\varepsilon, \nabla V_\varepsilon)\nu_N(V_\varepsilon) & \text{in } P_1^+, \\
\dfrac{\partial V_\varepsilon}{\partial y} = 0 & \text{in } \Gamma_1.
\end{cases} \tag{8.6}$$

and

$$\begin{cases}
\partial_t\rho_\varepsilon - \Delta\rho_\varepsilon = \rho_\varepsilon\langle\nabla^2_{V_\varepsilon}\nu_N(V_\varepsilon)(\nabla V_\varepsilon, \nabla V_\varepsilon), \nu_N(V_\varepsilon)\rangle & \\
\qquad - \rho_\varepsilon\nabla_{V_\varepsilon}\nu_N(V_\varepsilon)(\nabla V_\varepsilon, \nabla V_\varepsilon) & \text{in } P_1^+, \\
\dfrac{\partial\rho_\varepsilon}{\partial y} = \dfrac{2c_{\frac{1}{2}}}{\varepsilon^2}\rho_\varepsilon & \text{in } \Gamma_1.
\end{cases} \tag{8.7}$$

Here we have used the fact that $\nabla_p\rho_\varepsilon(p) = \nu_N(\Pi_N(p))$ for $p \in N_{\delta_N}$, so that the boundary condition for U_ε implies that on Γ_1,

$$\begin{aligned}
0 &= \frac{\partial U_\varepsilon}{\partial y} - \frac{c_{\frac{1}{2}}}{\varepsilon^2}\chi'(d^2(U_\varepsilon, N))D_{U_\varepsilon}d^2(U_\varepsilon, N) \\
&= \frac{\partial V_\varepsilon}{\partial y} + \frac{\partial\nu_N(V_\varepsilon)}{\partial y}\rho_\varepsilon + \left(\frac{\partial\rho_\varepsilon}{\partial y} - \frac{2c_{\frac{1}{2}}}{\varepsilon^2}\rho_\varepsilon\right)\nu_N(V_\varepsilon).
\end{aligned}$$

If we multiply this equation by $\nu_N(V_\varepsilon)$ and observe that $\langle\frac{\partial V_\varepsilon}{\partial y}, \nu_N(V_\varepsilon)\rangle = \langle\frac{\partial\nu_N(V_\varepsilon)}{\partial y}, \nu_N(V_\varepsilon)\rangle = 0$, we would obtain the above boundary condition for ρ_ε. On the other hand, the boundary condition for V_ε follows from the following identity

$$0 = \frac{\partial V_\varepsilon}{\partial y} + \frac{\partial\nu_N(V_\varepsilon)}{\partial y}\rho_\varepsilon = (\mathbb{I}_l + \rho_\varepsilon\nabla_{V_\varepsilon}\nu_N(V_\varepsilon))\frac{\partial V_\varepsilon}{\partial y},$$

and the invertibility of the map $(\mathbb{I}_l + \rho_\varepsilon\nabla_{V_\varepsilon}\nu_N(V_\varepsilon)) : \mathbb{R}^l \to \mathbb{R}^l$.

Note that (8.5) implies that

$$(|\partial_t V_\varepsilon| + |\nabla V_\varepsilon|) + (|\partial_t\rho_\varepsilon| + |\nabla\rho_\varepsilon|) \le 8 \text{ in } P_1^+.$$

This implies

$$\|(\mathbb{I}_l + \rho_\varepsilon\nabla_{V_\varepsilon}\nu_N(V_\varepsilon)) - \mathbb{I}_l\|_{L^\infty(P_1^+)} \le C\delta_N,$$

and

$$\|\rho_\varepsilon\langle\nabla^2_{V_\varepsilon}\nu_N(V_\varepsilon)(\nabla V_\varepsilon,\nabla V_\varepsilon),\nu_N(V_\varepsilon)\rangle-\rho_\varepsilon\nabla_{V_\varepsilon}\nu_N(V_\varepsilon)(\nabla V_\varepsilon,\nabla V_\varepsilon)\|_{L^\infty(P_1^+)}\le C.$$

Hence by the $W_p^{2,1}$-estimate for linear parabolic equations, we obtain that

$$\|V_\varepsilon\|_{C^{1+\alpha}(P^+_{\frac{7}{8}})}\le C(\alpha),\ \forall\,\alpha\in(0,1),$$

uniformly with respect to ε.

The boundary $C^{1+\alpha}$-estimate of ρ_ε can be done exactly as in Proposition 5.1. This completes the proof of Lemma 8.4. □

Finally with Lemma 8.4 at hand, we can show that U_ε also satisfies the gradient estimate as in Lemma 4.3. More precisely, we have that

Lemma 8.5. *There exists $\varepsilon_0>0$, depending only on m, such that if U_ε solves (8.1) and satisfies, for $Z_0=(X_0,t_0)\in\partial\mathbb{R}^{m+1}_+\times(0,\infty)$ and some $0<R<\frac{\sqrt{t_0}}{2}$,*

$$\mathcal{E}(U_\varepsilon,Z_0,R)<\varepsilon_0^2, \tag{8.8}$$

then

$$\sup_{P^+_{\delta_0R}(Z_0)}R^2|\nabla U_\varepsilon|^2\le C\delta_0^{-2},\quad \sup_{P^+_{\delta_0R}(Z_0)}R^4|\partial_tU_\varepsilon|^2\le C\delta_0^{-4}, \tag{8.9}$$

where $0<\delta_0<1$ and $C>0$ are independent of ε.

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