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Reaching a consensus with limited information[☆]

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ABSTRACT

In its simplest form the well known consensus problem for a networked family of autonomous agents is to devise a set of protocols or update rules, one for each agent, which can enable all of the agents to adjust or tune their "agreement variables" to the same value by utilizing real-time information obtained from their "neighbors" within the network. The aim of this paper is to study the problem of achieving a consensus in the face of limited information transfer between agents. By this it is meant that instead of each agent receiving an agreement variable or real-valued state vector from each of its neighbors, it receives a linear function of each state instead. The specific problem of interest is formulated and provably correct algorithms are developed for a number of special cases of the problem.

1. Introduction

In its simplest form the well known consensus problem [1] for a networked family of autonomous agents is to devise a set of protocols or update rules, one for each agent, which can enable all of the agents to adjust or tune their "agreement variable" to the same value by utilizing real-time information obtained from their "neighbors" within the network. The consensus problem is one of the most fundamental problems in the area of distributed computation and control. Consensus algorithms can be found as components of a large variety of more specialized algorithms in the area of distributed computation and control such as distributed algorithms for solving linear algebraic equations [2], distributed optimization problems [3], distributed estimation problems [4], and even some distributed control problems [5].

There are a great many variations of the consensus problem. For example, the agreement variables could be restricted to real-valued vectors or alternatively integer-valued vectors [6]. The updating of agreement variables could be executed either synchronously or asynchronously [7]. The topology of the network could be fixed or changing with time [8]. There could be malicious agents attempting to prevent consensus [9]. There could be communication delays [10] or bit-rate constraints [11]. The target value of the agreement variables could be unconstrained or it could be some specified function of the initial values of the agents' agreement variables as for example in distributed averaging [12] or gossiping [13]. Some versions of the problem such as when agreement variables take values in a finite set, defy deterministic solutions [6] whereas other versions of the problem do not.

The aim of this paper is to study the problem of achieving a consensus in the face of limited information transfer between agents. The problem setup is as follows. We consider a group of $m > 1$ autonomous agents labeled 1 to m. Each agent i has a set of neighbors from whom agent i can receive information; the set of labels of agent i's neighbors (excluding itself), denoted by[1] $\mathcal{N}_i \subset \mathbf{m} \stackrel{\Delta}{=} \{1, 2, \ldots, m\}$, is part of the problem formulation. The neighbor sets $\mathcal{N}_i$, $i \in \mathbf{m}$, determine an m-vertex directed graph $\mathbb{N}$ defined so that there is an arc (or a directed edge) from vertex j to vertex i just in case agent j is a neighbor of agent i. Each agent i has an agreement variable or state $x_i \in \mathbb{R}^n$ which it can adjust synchronously at times $t \in \{0, 1, 2, \ldots\}$. At time t, agent i receives from each neighbor $j \in \mathcal{N}_i$ a signal $s_{ji}(t) = C_{ji}x_j(t)$ where C_{ji} is a fixed real-valued matrix. Associating each arc (j, i) in $\mathbb{N}$ with matrix C_{ji} leads to a matrix-valued weighted neighbor graph $\bar{\mathbb{N}}$. It is assumed that for each $j \in \mathcal{N}_i$, $i \in \mathbf{m}$, both agents i and j know C_{ji}. There are no priori constraints on C_{ji}. Some could, for example, be matrices with less rows then columns in which cases the information transferred by each such corresponding signal $s_{ji}(t) = C_{ji}x_j(t)$ would be insufficient to determine $x_j(t)$. In this

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1 We use $\mathcal{A} \subset \mathcal{B}$ to denote that $\mathcal{A}$ is a subset of $\mathcal{B}$.

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sense the information agent i receives from neighbor j at time t is limited to only a "part of" $x_j(t)$. Given this setup, the consensus problem of interest is to devise update rules using the $s_{ji}(t)$, one for each agent, which if possible will cause all m agents' states x_i, $i \in \mathbf{m}$, to converge to the same value in the limit as $t \to \infty$.

2. Well-configured systems

Consider the multi-agent system just described. We say that the m agents are in *local agreement* with specific states x_i, $i \in \mathbf{m}$, if $C_{ji}x_i = C_{ji}x_j$ for all $i \in \mathbf{m}$ and $j \in \mathcal{N}_i$. We say that the m agents have reached a *consensus* with specific states x_i, $i \in \mathbf{m}$, if $x_i = x_j$ for all $i, j \in \mathbf{m}$. A weighted neighbor graph $\bar{\mathbb{N}}$ is called *well-configured* if local agreement implies consensus.

A well-configured weighted neighbor graph $\bar{\mathbb{N}}$ has the following equivalent mathematical description. For each vertex i in $\mathbb{N}$, let d_i denote the number of neighbors of agent i. Then $d = \sum_{i=1}^{m} d_i$ equals the total number of directed edges in $\mathcal{E}$. Let $k_{i1}, \dots, k_{id_i}$ be an arbitrary ordering of the labels in $\mathcal{N}_i$. Label all the d arcs from 1 to d according to the sequence $k_{11}, \dots, k_{1d_1}, \dots, k_{m1}, \dots, k_{md_m}$. Define the corresponding *incidence matrix* J as an $m \times d$ matrix in which column k has exactly one 1 in row i and exactly one -1 is row j if the kth arc in $\mathbb{N}$ is (j, i). For any finite set of matrices $\{M_1, M_2, \dots, M_k\}$, we use blockdiag$\{M_1, M_2, \dots, M_k\}$ to denote the block diagonal matrix whose ith diagonal block is M_i. Define

$$C = \text{blockdiag}\left\{C_{k_{11},1}, \dots, C_{k_{1d_1},1}, \dots, C_{k_{m1},m}, \dots, C_{k_{md_m},m}\right\}$$

Let $\bar{J} = J \otimes I_n$ and $\bar{I} = \mathbf{1}_m \otimes I_n$, where $\otimes$ denotes the Kronecker product, I_n denotes the $n \times n$ identity matrix, and $\mathbf{1}_m$ denotes the m-dimensional column vector whose entries all equal 1. Then it is not hard to verify that a weighted neighbor graph $\bar{\mathbb{N}}$ is well-configured if and only if

$$\text{kernel } C\bar{J}' = \text{span } \bar{I} \tag{1}$$

In the case when $\mathbb{N}$ is weakly connected,[2] kernel $\bar{J}' = \text{span } \bar{I}$ [14, Theorem 8.3.1]; then (1) will be true if and only if

$$\text{span } \bar{J}' \cap \text{kernel } C = 0 \tag{2}$$

It is worth emphasizing that C and J are defined according to the same ordering of the arcs in $\bar{\mathbb{N}}$, and the necessary and sufficient condition (1) or (2) is independent of the ordering.

With the above in mind, the following two questions arise. First, what are the necessary and/or sufficient conditions on $\mathbb{N}$ for which there exist C_{ji} matrices so that $\bar{\mathbb{N}}$ is well-configured? Second, if $\bar{\mathbb{N}}$ is well-configured, how one can construct a recursive distributed algorithm for each agent which will drive the system from arbitrary start states to local agreement and thus to a consensus? These are precisely what we consider in this paper.

3. System design

The goal of this section is to derive graph-theoretic conditions on which a multi-agent system can be well-configured.

As described, for any pair of neighboring agents, say agent i and its neighbor j, agent j only sends $C_{ji}x_j$ to agent i so that the transmitted vector size may be reduced and x_j may not be identified. Thus it is sometimes desirable that $\mathcal{K}_{ji} \neq 0$, where $\mathcal{K}_{ji}$ denotes the kernel of C_{ji}; otherwise, x_j can be uniquely determined from $C_{ji}x_j$. Also, if $\mathcal{K}_{ji} \neq 0$, the size of $C_{ji}x_j$ will be no smaller than that of x_j.

A directed graph $\mathbb{G}$ is called rooted if it contains a directed spanning tree of $\mathbb{G}$, and called strongly connected if there is a directed path between each pair of distinct vertices. Every strongly connected graph is rooted, but not vice versa.

First, it is easy to see that if $\mathbb{N}$ is not rooted, a consensus cannot be guaranteed for arbitrary initial values. We next consider some examples of rooted graphs.

3.1. Rooted graphs

If $\mathbb{N}$ is rooted, $\bar{\mathbb{N}}$ cannot be always well-configured with all $\mathcal{K}_{ji} \neq 0$, as shown in the following lemma for path graphs.

Lemma 1. *If $\mathbb{N}$ is a directed path, then $\bar{\mathbb{N}}$ can be well-configured only if all $\mathcal{K}_{ji} = 0$.*

Proof of Lemma 1. For a directed path with m vertices $1 \to 2 \to \dots \to m$, local agreements are $C_i(x_i - x_{i+1}) = 0, i \in \{1, \dots, m-1\}$. Suppose to the contrary that there exists an i such that $\mathcal{K}_i \neq 0$, then there exists a nonzero x such that $C_i x = 0$. Let

$$x_j = x_1, \quad j \in \{1, \dots, i\}$$
$$x_j = x_1 + x, \quad j \in \{i+1, \dots, m\}$$

Then $C_i(x_i - x_{i+1}) = 0$ for all $i \in \{1, \dots, m-1\}$, while all x_i, $i \in \mathbf{m}$ do not reach a consensus. ■

The following example shows that there exists a rooted graph which can be well-configured with all $\mathcal{K}_{ji} \neq 0$.

Example 1. Consider a three-agent network with arcs $1 \to 2, 2 \to 1, 3 \to 1, 3 \to 2$. Then local agreement equations are

$$\begin{aligned} C_{12}(x_1 - x_2) &= 0 \\ C_{21}(x_2 - x_1) &= 0 \\ C_{31}(x_3 - x_1) &= 0 \\ C_{32}(x_3 - x_2) &= 0 \end{aligned} \tag{3}$$

Note that the existence x_i satisfying the four equalities above imply that there are vectors p_1, p_2, p_3, namely $p_1 = x_1 - x_2$, $p_2 = x_3 - x_1$, $p_3 = x_2 - x_3$, such that

$$\begin{aligned} &p_1 + p_2 + p_3 = 0 \\ &\quad p_1 \in \mathcal{K}_{12} \cap \mathcal{K}_{21} \\ &\quad p_2 \in \mathcal{K}_{31} \\ &\quad p_3 \in \mathcal{K}_{32} \end{aligned} \tag{4}$$

Conversely, for any set of vectors p_1, p_2, p_3 satisfying (4), there are vectors x_i, namely $x_1 = p_1$, $x_2 = 0$, $x_3 = p_3$ for which the four equalities in (3) hold. Note that there will exist p_1, p_2, p_3 for which (4) holds if and only if

$$\begin{aligned} p_1 &\in \mathcal{K}_{12} \cap \mathcal{K}_{21} \cap (\mathcal{K}_{31} + \mathcal{K}_{32}) \\ p_2 &\in \mathcal{K}_{31} \cap (\mathcal{K}_{12} \cap \mathcal{K}_{21} + \mathcal{K}_{32}) \\ p_3 &\in \mathcal{K}_{32} \cap (\mathcal{K}_{12} \cap \mathcal{K}_{21} \cap \mathcal{K}_{31}) \end{aligned}$$

Thus the conditions for the p_i to all equal zero are

$$\begin{aligned} &\mathcal{K}_{12} \cap \mathcal{K}_{21} \cap (\mathcal{K}_{31} + \mathcal{K}_{32}) = 0 \\ &\mathcal{K}_{31} \cap (\mathcal{K}_{12} \cap \mathcal{K}_{21} + \mathcal{K}_{32}) = 0 \\ &\mathcal{K}_{32} \cap (\mathcal{K}_{12} \cap \mathcal{K}_{21} \cap \mathcal{K}_{31}) = 0 \end{aligned}$$

which are the conditions for the three subspaces $\mathcal{K}_{12} \cap \mathcal{K}_{21}$, $\mathcal{K}_{31}$ and $\mathcal{K}_{32}$ to be independent. Thus the weighted neighbor graph of interest is well-configured just in case the three subspaces are independent, and do not necessarily have to equal 0. □

It turns out that well-configuration characterization of rooted graphs is quite complicated. We thus leave it as a future direction and focus on strongly connected graphs in the next subsection.

[2] A directed graph is weakly connected if there is an undirected path between each pair of distinct vertices.

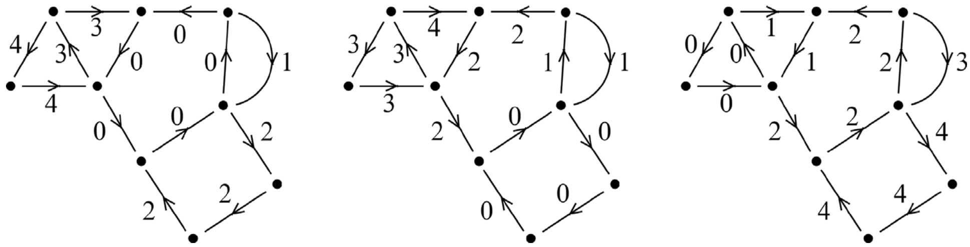

Fig. 1. An example of different ear decompositions of a strongly connected graph. The number associated to each arc represents the index of the ear which the arc belongs to in an ear decomposition.

3.2. Strongly connected graphs

Strong connectedness itself cannot guarantee well-configuration. To state our sufficient condition for well-configuration, we need the following concept from graph theory [15].

An *ear decomposition* of a directed graph without self-arcs[3] $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ with at least two vertices is a sequence of subgraphs of $\mathbb{G}$, denoted $\{\mathbb{E}_0, \mathbb{E}_1, \ldots, \mathbb{E}_p\}$, in which $\mathbb{E}_0$ is a directed cycle, and each $\mathbb{E}_i$, $i \in \mathbf{p}$, is a directed path or a directed cycle with the following properties:

1. $\{\mathbb{E}_0, \mathbb{E}_1, \ldots, \mathbb{E}_p\}$ form an arc partition of $\mathbb{G}$, i.e., $\mathbb{E}_i$ and $\mathbb{E}_j$ are arc disjoint if $i \neq j$, and $\bigcup_{k=0}^{p} \mathbb{E}_k = \mathbb{G}$;
2. For each $i \in \mathbf{p}$, if $\mathbb{E}_i$ is a directed cycle, then it has precisely one vertex in common with $\bigcup_{k=0}^{i-1} \mathbb{E}_k$; if $\mathbb{E}_i$ is a directed path, then its two end-vertices are the only two vertices in common with $\bigcup_{k=0}^{i-1} \mathbb{E}_k$.

Each of $\mathbb{E}_0, \mathbb{E}_1, \ldots, \mathbb{E}_p$ is called an *ear* of the decomposition. Not all directed graphs admit an ear decomposition. It has been proved that a directed graph has an ear decomposition if and only if it is strongly connected [15, Theorem 7.2.2]. It is also known that there exists a linear algorithm to find one ear decomposition of a strongly connected graph [15, Corollary 7.2.5]. A strongly connected graph may admit multiple ear decompositions, and apparently, the number of all possible different ear decompositions of a strongly connected graph is finite. It turns out that every ear decomposition of a strongly connected graph with m vertices and e arcs has $e - m + 1$ ears [15, Corollary 7.2.3]. To help understand the concept, an illustrative example is provided in Fig. 1.

Two subspaces $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathbb{R}^n$ are independent if their intersection is the zero subspace, i.e., if $\mathcal{S}_1 \bigcap \mathcal{S}_2 = 0$. A finite family of subspaces $\{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_p\}$ is independent if

$$\mathcal{S}_i \bigcap \Big(\sum_{j \neq i} \mathcal{S}_j\Big) = 0, \qquad i \in \mathbf{p}$$

Theorem 1. *Suppose that $\mathbb{N}$ is strongly connected and let* D *be an ear decomposition of $\mathbb{N}$. If for each ear $\mathbb{E} \in$ D, $\{\mathcal{K}_{ji} : (j, i) \in \mathbb{E}\}$ is an independent family, then $\bar{\mathbb{N}}$ is well-configured.*

To prove the theorem, we first study directed cycles and paths since they are basic components in ear decompositions.

To simplify notation, we label the vertices of an m-vertex directed cycle as $1 \to 2 \to \cdots \to m \to 1$. Suppose that $C_1, C_2, \ldots, C_m$ are given matrices, each with n columns. Suppose that for each $i \in \mathbf{m}$, agent i receives $C_i x_{i-1}$ from agent $i-1$, where it is understood that agent 0 and agent m are one and the same, and that $x_0 \stackrel{\Delta}{=} x_m$. Thus for this $\bar{\mathbb{N}}$ to be well-configured means that the relations

$$C_i x_i = C_i x_{i-1}, \qquad i \in \mathbf{m} \tag{5}$$

must imply that $x_i = x_{i-1}$, $i \in \mathbf{m}$. Let $\mathcal{K}_i$ denote the kernel of C_i for all $i \in \mathbf{m}$.

Lemma 2. *If $\mathbb{N}$ is an m-vertex directed cycle, then $\bar{\mathbb{N}}$ is well-configured by matrices C_i, $i \in \mathbf{m}$, if and only if $\{\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_m\}$ is an independent family.*

Proof of Lemma 2. Since $\mathbb{N}$ is a cycle, $m \geq 2$. We first prove the sufficiency. Let $\{\mathcal{K}_i,\ i \in \mathbf{m}\}$ be an independent family. Suppose to the contrary that $\bar{\mathbb{N}}$ is not well-configured. Then there must exist a non-consensus set $\{x_1, \ldots, x_m\}$ which satisfies (5). Let $y_i = x_i - x_{i+1}$ for each $i \in \{1, \ldots, m-1\}$ and $y_m = x_m - x_1$. Then at least one of $y_1, \ldots, y_m$ is nonzero. Since $\sum_{i=1}^{m} y_i = 0$, at least two of $y_1, \ldots, y_m$ are nonzero. Let $\mathcal{A} = \{i \in \mathbf{m} : y_i \neq 0\}$. Then $|\mathcal{A}| \geq 2$ and $\sum_{i \in \mathcal{A}} y_i = 0$. Since each $y_i \in \mathcal{K}_i$ and $\{\mathcal{K}_i,\ i \in \mathcal{A}\}$ is an independent family, y_i, $i \in \mathcal{A}$, are linear independent, which is contradictory to $\sum_{i \in \mathcal{A}} y_i = 0$.

We next prove the necessity. Let $\{C_1, \ldots, C_m\}$ be any set of matrices which make $\bar{\mathbb{N}}$ well-configured. Suppose to the contrary that $\{\mathcal{K}_1, \ldots, \mathcal{K}_m\}$ is not an independent family, which implies that there exists an index p such that $\mathcal{K}_p \cap (\sum_{i \neq p} \mathcal{K}_i) \neq 0$. Then there exist $k_i \in \mathcal{K}_i$, $i \in \mathbf{m}$, such that $k_p = \sum_{i \neq p} k_i$, which is nonzero. For any x_1, let $x_{i+1} = x_i + k_i$ for each $i \in \{1, \ldots, p-1, p+1, \ldots, m-1\}$ and $x_{p+1} = x_p - k_p$. It is easy to see check that such a set of non-consensus vectors $x_1, \ldots, x_m$ satisfy (5). But this is impossible as $\bar{\mathbb{N}}$ is well-configured. ∎

It is easy to see that n is the maximum possible number of subspaces in an independent family of nonzero subspaces of $\mathbb{R}^n$. We thus have the following immediate consequence of Lemma 2.

Corollary 1. *If $\mathbb{N}$ is an m-vertex directed cycle, then $\bar{\mathbb{N}}$ can be well-configured with all $\mathcal{K}_i \neq 0$, $i \in \mathbf{m}$, if and only if $m \leq n$.*

More can be said.

Lemma 3. *Let $\mathbb{N}$ be an m-vertex directed cycle with $\mathcal{E}_{\mathbb{N}}$ being the edge set and x_i being the state of vertex i. Let $\mathcal{E}$ be a subset of $\mathcal{E}_{\mathbb{N}}$ defined as $\mathcal{E} = \{(i, j) \in \mathcal{E}_{\mathbb{N}} : x_i = x_j\}$. Then $\bar{\mathbb{N}}$ is well-configured by matrices C_i, $i \in \mathbf{m}$, if and only if $\{\mathcal{K}_i : i \in \mathbf{m}, (i-1, i) \notin \mathcal{E}\}$ is an independent family.*

Proof of Lemma 3. The case of $m - |\mathcal{E}| = 0$ is trivial. We claim that $m - |\mathcal{E}| \neq 1$. To see this, suppose to the contrary that $m - |\mathcal{E}| = 1$. Then the edges in $\mathcal{E}$ forms a directed spanning path of $\mathbb{N}$, which guarantees that all m agents reach a consensus. This implies that all the edges of $\mathbb{N}$ belong to $\mathcal{E}$, which is impossible. Thus we focus on $m - |\mathcal{E}| \geq 2$ in the remaining proof.

Each vertex i in directed cycle $\mathbb{N}$ has a unique outgoing neighbor, denoted as $v[i]$. Let $\mathcal{V}$ be the vertex subset defined as $\mathcal{V} = \{i \in \mathbf{m} : (i, v[i]) \notin \mathcal{E}\}$. Then $|\mathcal{V}| = m - |\mathcal{E}|$. Relabel the vertices in $\mathcal{V}$ as $v_1, \ldots, v_p$, $p = m - |\mathcal{E}| \geq 2$, along with the same direction

[3] The definition can be extended to more general directed multigraphs with self-arcs [15].

as the directed cycle. It is not hard to verify that

$$C_{v_1}(x_{v_1} - x_{v_2}) = 0$$
$$\vdots$$
$$C_{v_{p-1}}(x_{v_{p-1}} - x_{v_p}) = 0$$
$$C_{v_p}(x_{v_p} - x_{v_1}) = 0$$

which are mathematically equivalent to (5) with m being replaced by p. Thus the above equations are equivalent to local agreements of an p-vertex directed cycle. From Lemma 2, $\bar{\mathbb{N}}$ can be well-configured with all $\mathcal{K}_{v_i} \neq 0$, $i \in \mathbf{p}$, if and only if $p \leq n$. ■

Lemma 3 immediately implies the following result.

Corollary 2. *Let $\mathbb{N}$ be an m-vertex directed cycle with edge set $\mathcal{E}_{\mathbb{N}}$. Let $\mathcal{E}$ be a subset of $\mathcal{E}_{\mathbb{N}}$ defined as $\mathcal{E} = \{(i,j) \in \mathcal{E}_{\mathbb{N}} : x_i = x_j\}$. Then $\bar{\mathbb{N}}$ can be well-configured with all $\mathcal{K}_i \neq 0$ if and only if $m - |\mathcal{E}| \leq n$.*

The above results can be directly applied to the following special case of path graphs.

To simplify notation, we label the vertices of an m-vertex directed path as $1 \to 2 \to \cdots \to m$. Suppose that $C_2, \ldots, C_m$ are given matrices, each with n columns. Suppose that for each $i \in \mathbf{m}$, agent i receives $C_i x_{i-1}$ from agent $i-1$. Thus for this $\bar{\mathbb{N}}$ to be well-configured means that the relations $C_i x_i = C_i x_{i-1}$, $i \in \{2, \ldots, m\}$, must imply that $x_i = x_{i-1}$, $i \in \{2, \ldots, m\}$. Adding the arc $(m, 1)$ to the above path and imposing $x_1 = x_m$ will lead to a special case satisfying the condition in Lemma 3 and Corollary 2, which immediately implies the following result.

Corollary 3. *If $\mathbb{N}$ is an m-vertex directed path with $x_1 = x_m$, then $\bar{\mathbb{N}}$ is well-configured by matrices C_i, $i \in \{2, \ldots, m\}$, if and only if $\{\mathcal{K}_2, \ldots, \mathcal{K}_m\}$ is an independent family, and thus $\bar{\mathbb{N}}$ can be well-configured with all $\mathcal{K}_i \neq 0$, $i \in \{2, \ldots, m\}$, if and only if $m - 1 \leq n$.*

Compared with Lemma 1, it is worth emphasizing that assuming $x_1 = x_m$ significantly changes the condition for well-configuration of path graphs.

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Let $\mathrm{D} = \{\mathbb{E}_0, \mathbb{E}_1, \ldots, \mathbb{E}_p\}$ be the given ear decomposition of $\mathbb{N}$. We claim that for each $i \in \{0, 1, \ldots, p\}$, $\bigcup_{k=0}^{i} \mathbb{E}_k$ is well-configured. The claim will be proved by induction on the index i.

By definition, $\mathbb{E}_0$ is a directed cycle. From Lemma 2, $\mathbb{E}_0$ is well-configured. Now suppose that the claim holds for all i in the range $0 \leq i \leq j$, where j is a nonnegative integer smaller than p. Consider ear $\mathbb{E}_{i+1}$, which is either a directed cycle or a directed path. We treat these two cases separately. If $\mathbb{E}_{i+1}$ is a directed cycle, using the preceding argument, it is well-configured. Since ear $\mathbb{E}_{i+1}$ shares one common vertex with $\bigcup_{k=0}^{i} \mathbb{E}_k$, $\bigcup_{k=0}^{i+1} \mathbb{E}_k$ is well-configured. If $\mathbb{E}_{i+1}$ is a directed path, its two end-vertices belong to $\bigcup_{k=0}^{i} \mathbb{E}_k$. Since the well-configuration of $\bigcup_{k=0}^{i} \mathbb{E}_k$ guarantees that the two end-vertices have the same value, from Corollary 3, $\mathbb{E}_{i+1}$ is well-configured, and so is $\bigcup_{k=0}^{i+1} \mathbb{E}_k$. By induction, the claim is established. Since $\bigcup_{k=0}^{p} \mathbb{E}_k = \mathbb{N}$, the proof is complete. ■

Theorem 1 is not a necessary condition. To see this, consider the strongly connected graph given in Fig. 2 which has 4 agents and 6 arcs. If $\{\mathcal{K}_{12}, \mathcal{K}_{21}\}$, $\{\mathcal{K}_{34}, \mathcal{K}_{43}\}$, and $\{\mathcal{K}_{23}, \mathcal{K}_{41}\}$ are independent families, then from Lemmas 2 and 3 the weighted neighbor graph is well-configured. However, if in addition none of $\{\mathcal{K}_{23}, \mathcal{K}_{34}, \mathcal{K}_{41}\}$, $\{\mathcal{K}_{41}, \mathcal{K}_{12}, \mathcal{K}_{23}\}$, and $\{\mathcal{K}_{41}, \mathcal{K}_{12}, \mathcal{K}_{23}, \mathcal{K}_{34}\}$ is an independent family, then none of the ear decompositions of the strongly connected graph satisfies the condition in Theorem 1.

Given a strongly connected weighted neighbor graph, if an ear decomposition satisfies the condition in Theorem 1, the condition

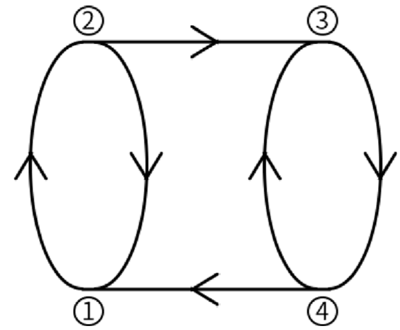

Fig. 2. A four-agent strongly connected graph.

may not hold for the other ear decompositions. To see this, still consider the graph in Fig. 2. If $\{\mathcal{K}_{12}, \mathcal{K}_{21}\}$ and $\{\mathcal{K}_{23}, \mathcal{K}_{34}, \mathcal{K}_{41}\}$ are independent families, then the ear decomposition $\mathbb{E}_0 = (1,2)(2,1)$, $\mathbb{E}_1 = (2,3)(3,4)(4,1)$, $\mathbb{E}_2 = (4,3)$ satisfies the condition in Theorem 1 provided $\mathcal{K}_{43} = 0$. Meanwhile, if neither $\{\mathcal{K}_{41}, \mathcal{K}_{12}, \mathcal{K}_{23}\}$ or $\{\mathcal{K}_{41}, \mathcal{K}_{12}, \mathcal{K}_{23}, \mathcal{K}_{34}\}$ is an independent family, it is easy to verify that all the other ear decompositions do not satisfy the condition.

The proof of Theorem 1 provides a constructive approach that systematically designs C_{ji} matrices for a strongly connected multi-agent system to be well-configured.

For each ear decomposition, say $\mathrm{D} = \{\mathbb{E}_0, \mathbb{E}_1, \ldots, \mathbb{E}_p\}$, let $l(\mathbb{E}_i)$ denote the length of ear $\mathbb{E}_i$, i.e., the number of arcs in $\mathbb{E}_i$. Theorem 1 immediately implies the following sufficient conditions for well-configuration.

Corollary 4. *Suppose that $\mathbb{N}$ is strongly connected and let D be an ear decomposition of $\mathbb{N}$. If*

$$\max_{\mathbb{E} \in \mathrm{D}} l(\mathbb{E}) \leq n$$

then $\bar{\mathbb{N}}$ can be well-configured with all $\mathcal{K}_{ji} \neq 0$, $i \in \mathbf{m}$, $j \in \mathcal{N}_i$.

More can be said. For a strongly connected graph $\mathbb{G}$, write $\mathcal{D}$ for the set of all possible ear decompositions of $\mathbb{G}$. Define

$$\chi(\mathbb{G}) = \min_{\mathrm{D} \in \mathcal{D}} \max_{\mathbb{E} \in \mathrm{D}} l(\mathbb{E})$$

Since each ear decomposition begins with a directed cycle and the shortest possible length of a cycle is two, e.g., a pair of agents which are neighbors of each other, $\chi(\mathbb{G}) \geq 2$.

Corollary 5. *If $\mathbb{N}$ is strongly connected and $\chi(\mathbb{N}) \leq n$, then $\bar{\mathbb{N}}$ can be well-configured with all $\mathcal{K}_{ij} \neq 0$, $i \in \mathbf{m}$, $j \in \mathcal{N}_i$.*

Although Corollary 5 provides a weaker condition, to our knowledge, it is still an open problem to construct an efficient algorithm to find all ear decompositions of a strongly connected graph.

3.3. Symmetric directed graphs

A directed graph is called *symmetric* if whenever (i,j) is an arc in the graph, so is (j,i). A symmetric directed graph is often called undirected in the literature, which simplifies each pair of directed edges, say (i,j) and (j,i), to one undirected edge between vertices i and j. We stick to the term "symmetric directed graphs" because of definition of the incidence matrix given in Section 2. Consider a symmetric directed graph with m vertices and d directed edges. Then d must be an even number. Our definition of an incidence matrix is of size $m \times d$, while the standard definition of an incidence matrix of the corresponding undirected graph is of size

$m \times (d/2)$. Thus using the term "undirected" may cause confusion. It is worth noting that rooted and strong connectedness boil down to the same connectivity for symmetric directed graphs.

For any symmetric directed graph $\mathbb{G}$, since each pair of arcs between any pair of neighboring agents in a symmetric directed graph is a cycle with length 2, all these cycles form an ear decomposition, which leads to $\chi(\mathbb{G}) = 2$. The following necessary and sufficient condition on well-configuration for symmetric directed graphs is easy to derive from Corollary 5.

Theorem 2. *If $\mathbb{N}$ is a symmetric directed graph, then $\bar{\mathbb{N}}$ can be well-configured with all $\mathcal{K}_{ji} \neq 0$, $i \in \mathbf{m}$, $j \in \mathcal{N}_i$, if and only if $\mathbb{N}$ is strongly connected and $n \geq 2$.*

As will be seen in the next section, there is a motivation, for the purpose of algorithm design, to figure out a condition under which a symmetric directed graph can be well-configured with the additional constraint that $C_{ij} = C_{ji}$ for all $i \in \mathbf{m}$ and $j \in \mathcal{N}_i$. Note that any weighted symmetric directed graph with $C_{ij} = C_{ji}$ can be equivalently simplified as an undirected graph with each edge (i, j) associated with matrix C_{ij}, whenever vertices i and j are a pair of neighbors. We thus need the following concept of ear decompositions for undirected graphs.

An *ear decomposition* of an undirected graph without self-loops $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ with at least two vertices is a sequence of undirected subgraphs of $\mathbb{G}$, denoted $\{\mathbb{E}_0, \mathbb{E}_1, \ldots, \mathbb{E}_p\}$, in which $\mathbb{E}_0$ is an undirected cycle, and each $\mathbb{E}_i$, $i \in \mathbf{p}$, is an undirected path or cycle with the following properties:

1. $\{\mathbb{E}_0, \mathbb{E}_1, \ldots, \mathbb{E}_p\}$ form an edge partition of $\mathbb{G}$, i.e., $\mathbb{E}_i$ and $\mathbb{E}_j$ are arc disjoint if $i \neq j$, and $\bigcup_{k=0}^{p} \mathbb{E}_k = \mathbb{G}$;
2. For each $i \in \mathbf{p}$, if $\mathbb{E}_i$ is an undirected cycle, then it has precisely one vertex in common with $\bigcup_{k=0}^{i-1} \mathbb{E}_k$; if $\mathbb{E}_i$ is an undirected path, then its two end-vertices are the only two vertices in common with $\bigcup_{k=0}^{i-1} \mathbb{E}_k$.

Each of $\mathbb{E}_0, \mathbb{E}_1, \ldots, \mathbb{E}_p$ is called an *ear* of the decomposition. Not all undirected graphs admit an ear decomposition. An undirected graph is called *k-edge-connected* if, upon removal of any $k-1$ edges, the resulting graph is still connected. It has been proved that an undirected graph has an ear decomposition if and only if it is 2-edge-connected [16]. A 2-edge-connected undirected graph may admit multiple ear decompositions, and apparently, the number of all possible different ear decompositions is finite. For each ear decomposition, say $\mathbb{D} = \{\mathbb{E}_0, \mathbb{E}_1, \ldots, \mathbb{E}_p\}$, let $l(\mathbb{E}_i)$ denote the length of ear $\mathbb{E}_i$, i.e., the number of edges in $\mathbb{E}_i$.

Let us agree to say that $\mathbb{D}$ is an *undirected ear decomposition* of a symmetric directed graph $\mathbb{G}$ if $\mathbb{D}$ is an ear decomposition of the undirected graph generated by replacing each pair of arcs (i, j) and (j, i) with an edge between neighboring vertices i and j in $\mathbb{G}$. We call $\mathbb{G}$ *2-connected* if the undirected graph is 2-edge-connected. From the preceding a symmetric directed graph has an undirected ear decomposition if and only if it is 2-connected.

Using the same arguments as in the proof of Theorem 1, we have the following result.

Theorem 3. *Suppose that $\mathbb{N}$ is 2-connected symmetric directed graph and let $\mathbb{D}$ be an undirected ear decomposition of $\mathbb{N}$. If for each ear $\mathbb{E} \in \mathbb{D}$, $\{\mathcal{K}_{ij} \vee \mathcal{K}_{ji} : (i, j) \in \mathbb{E}\}$[4] is an independent family, then $\bar{\mathbb{N}}$ is well-configured by matrices $C_{ij} = C_{ji}$, $i \in \mathbf{m}$, $j \in \mathcal{N}_i$. If, in addition, $\max_{\mathbb{E} \in \mathbb{D}} l(\mathbb{E}) \leq n$, then $\bar{\mathbb{N}}$ can be well-configured with all $\mathcal{K}_{ij} = \mathcal{K}_{ji} \neq 0$, $i \in \mathbf{m}$, $j \in \mathcal{N}_i$.*

In the sequel, we will propose and analyze a few distributed algorithms for well-configured systems under different scenarios.

[4] We use $\{a \vee b\}$ to denote that either a or b is an element in the set.

4. Algorithms for symmetric directed graphs

In this section, we assume that the neighbor graph is symmetric and $C_{ij} = C_{ji}$ whenever agents i and j are a pair of neighbors. The following algorithms (6)–(8) all involve a term $(C_{ij}'C_{ij} + C_{ji}'C_{ji})(x_i(t) - x_j(t))$ in each agent i's update, where j is any neighbor of agent i; with $C_{ij} = C_{ji}$, only one signal, $C_{ij}x_j(t)$, is transferred from agent j to agent i. It is worth noting that in this case the underlying symmetric directed graph will need to be 2-connected to guarantee well-configuration. If $C_{ij} \neq C_{ji}$, it will implicitly require that each agent i receives two signals, $C_{ji}x_j(t)$ and $C_{ij}x_j(t)$, from each of its neighbors at each time step. Although allowing $C_{ij} \neq C_{ji}$ in a symmetric directed graph does not affect the proofs and makes well-configuration easier in light of Theorem 2, transmitting two signals could be an issue in communication.

We begin with the simplest case in which the neighbor graph $\mathbb{N}$ is fixed.

4.1. Fixed symmetric directed graphs

Consider any strongly connected, symmetric directed graph $\mathbb{N}$ with m agents. Our first algorithm appeals to the idea of gradient descent in convex optimization, which is for each agent i,

$$x_i(t+1) = x_i(t) - \alpha(t) \sum_{j \in \mathcal{N}_i} (C_{ij}'C_{ij} + C_{ji}'C_{ji})(x_i(t) - x_j(t)) \tag{6}$$

where $\alpha(t)$ is a positive time-varying stepsize satisfying $\sum_t \alpha(t) = \infty$ and $\sum_t \alpha^2(t) < \infty$.

Theorem 4. *If $\mathbb{N}$ is a strongly connected symmetric directed graph and $\bar{\mathbb{N}}$ is well-configured, then algorithm (6) will lead all the agents to reach a consensus.*

Proof of Theorem 4. The m update equations in (6) can be combined into one state form as

$$x(t+1) = x(t) - \alpha(t)\bar{J}C'C\bar{J}'x(t)$$

where $x = \text{column}\{x_1, x_2, \ldots, x_m\}$, which is exactly the gradient descent of minimizing the convex function $\|C\bar{J}'x\|_2^2$. Thus with appropriate time-varying stepsize $\alpha(t)$ (i.e., $\sum_t \alpha(t) = \infty$ and $\sum_t \alpha^2(t) < \infty$), $x(t)$ will asymptotically converge to an optimal point of $\|C\bar{J}'x\|_2^2$, which must be a consensus vector as kernel $C\bar{J}' = \text{span}\,\bar{I}$. ■

The above algorithm requires all m agents share the same sequence of diminishing stepsizes. Our second algorithm gets around this limitation and is thus fully distributed, which is described as follows.

Since well-configuration only depends on $\mathcal{K}_{ij}$, the kernel of C_{ij}, $i \in \mathbf{m}$, $j \in \mathcal{N}_i$, without loss of generality, we assume each C_{ij} has full row rank and its rows are orthonormal, which implies that $C_{ij}C_{ij}' = I$ and $P_{ij} \triangleq C_{ij}'(C_{ij}C_{ij}')^{-1}C_{ij} = C_{ij}'C_{ij}$ is an orthogonal projection matrix. For each agent $i \in \mathbf{m}$,

$$x_i(t+1) = x_i(t) - \frac{1}{2(d_i+1)} \sum_{j \in \mathcal{N}_i} (C_{ij}'C_{ij} + C_{ji}'C_{ji})(x_i(t) - x_j(t)) \tag{7}$$

Theorem 5. *If $\mathbb{N}$ is symmetric, strongly connected and $\bar{\mathbb{N}}$ is well-configured, then algorithm (7) will lead all the agents to reach a consensus exponentially fast.*

To prove the theorem, we need the following lemmas.

Lemma 4. *If $\bar{\mathbb{N}}$ is well-configured, then $\bar{J}C'C\bar{J}'$ is positive semidefinite with exactly m eigenvalues at zero.*

Proof of Lemma 4. It is clear that $\bar{J}C'C\bar{J}'$ is positive semidefinite. Since kernel $C\bar{J}' =$ span $\bar{I}$, $\bar{J}C'C\bar{J}'$ has exactly rank$(\bar{I}) = n$ eigenvalues at zero. ■

Lemma 5. *Let $\bar{W} = W \otimes I$, where W is a positive diagonal matrix. If $\bar{\mathbb{N}}$ is well-configured, then $\bar{W}\bar{J}C'C\bar{J}'$ has exactly n eigenvalues at zero, and all the remaining eigenvalues are positive.*

Proof of Lemma 5. Note that $\bar{W}\bar{J}C'C\bar{J}'$ has the same spectrum as $\bar{W}^{-\frac{1}{2}}\bar{W}\bar{J}C'C\bar{J}'\bar{W}^{\frac{1}{2}} = \bar{W}^{\frac{1}{2}}\bar{J}C'C\bar{J}'\bar{W}^{\frac{1}{2}}$. It is clear that $\bar{W}^{\frac{1}{2}}\bar{J}C'C\bar{J}'\bar{W}^{\frac{1}{2}}$ is positive semidefinite. From Lemma 4, $C\bar{J}'\bar{W}^{\frac{1}{2}}x = 0$ if and only if $\bar{W}^{\frac{1}{2}}x \in$ span $\bar{I}$, which implies that kernel $C\bar{J}'\bar{W}^{\frac{1}{2}} =$ span $\bar{W}^{-\frac{1}{2}}\bar{I}$ and thus $\bar{W}\bar{J}C'C\bar{J}'$ has exactly n eigenvalues at zero. ■

We also need the following "mixed matrix norm" concept introduced in [2]. Let $\|\cdot\|_\infty$ denote the induced infinity norm and write $\mathbb{R}^{mn\times mn}$ for the vector space of all $m \times m$ block matrices $Q = [Q_{ij}]$ whose ijth entry is an $n \times n$ matrix $Q_{ij} \in \mathbb{R}^{n\times n}$. Define the $(2, \infty)$ norm of $Q \in \mathbb{R}^{mn\times mn}$, written $\|Q\|_{2,\infty}$, to be

$$\|Q\|_{2,\infty} = \|\langle Q\rangle\|_\infty$$

where $\langle Q\rangle$ is the $m\times m$ matrix in $\mathbb{R}^{m\times m}$ whose ijth entry is $\|Q_{ij}\|_2$, where $\|\cdot\|_2$ denotes the induced 2-norm. It has been shown in [2, Lemma 3] that $\|\cdot\|_{2,\infty}$ is a sub-multiplicative matrix norm.

In the sequel, for a matrix $Q \in \mathbb{R}^{mn\times mn}$, we use $[Q]_{ij}$, $i, j \in \mathbf{m}$ to denote the ijth block of Q, which is an $n \times n$ matrix.

We are now in a position to prove Theorem 5.

Proof of Theorem 5. The m update equations in (7) can be combined into one state form as

$$x(t+1) = x(t) - \bar{D}\bar{J}C'C\bar{J}'x(t)$$

where $\bar{D} = \text{blockdiag}\{\frac{1}{2(d_1+1)}I_m, \dots, \frac{1}{2(d_m+1)}I_m\}$. Consider each block of $\bar{D}\bar{J}C'C\bar{J}'$. For any $i \in \mathbf{m}$,

$$[\bar{D}\bar{J}C'C\bar{J}']_{ii} = \frac{1}{2(d_i+1)}\sum_{j\in\mathcal{N}_i}(C_{ij}'C_{ij} + C_{ji}'C_{ji}) = \frac{1}{2(d_i+1)}\sum_{j\in\mathcal{N}_i}(P_{ij} + P_{ji})$$

and for any $i \in \mathbf{m}$, $j \in \mathcal{N}_i$,

$$[\bar{D}\bar{J}C'C\bar{J}']_{ij} = -\frac{1}{2(d_i+1)}(C_{ij}'C_{ij} + C_{ji}'C_{ji}) = -\frac{1}{2(d_i+1)}(P_{ij} + P_{ji})$$

Since each P_{ij} is an orthogonal projection matrix, $\|P_{ij}\|_2 = 1$. Consider the 2-norm for each block. For any $i \in \mathbf{m}$,

$$\|[\bar{D}\bar{J}C'C\bar{J}']_{ii}\|_2 \le \frac{1}{2(d_i+1)}\sum_{j\in\mathcal{N}_i}(\|P_{ij}\|_2 + \|P_{ji}\|_2) \le \frac{d_i}{d_i+1}$$

and for any $i \in \mathbf{m}$, $j \in \mathcal{N}_i$,

$$\|[\bar{D}\bar{J}C'C\bar{J}']_{ij}\|_2 \le \frac{\|P_{ij}\|_2 + \|P_{ji}\|_2}{2(d_i+1)} \le \frac{1}{d_i+1}$$

Next consider the $(2, \infty)$-norm of $\bar{D}\bar{J}C'C\bar{J}'$:

$$\begin{aligned}\|\bar{D}\bar{J}C'C\bar{J}'\|_{2,\infty} &= \min_{i\in\mathbf{m}} \|[\bar{D}\bar{J}C'C\bar{J}']_{ii}\|_2 + \sum_{j\in\mathcal{N}_i}\|[\bar{D}\bar{J}C'C\bar{J}']_{ij}\|_2 \\ &\le \min_{i\in\mathbf{m}}\Big(\frac{d_i}{d_i+1} + \sum_{j\in\mathcal{N}_i}\frac{1}{d_i+1}\Big) \le \min_{i\in\mathbf{m}}\frac{2d_i}{d_i+1} < 2\end{aligned}$$

which implies that the spectral radius of $\bar{D}\bar{J}C'C\bar{J}'$ is less than 2. It follows that $I - \bar{D}\bar{J}C'C\bar{J}'$ has n eigenvalues at one and all the other eigenvalues lie in $(-1, 1)$, which implies that $x(t)$ will reach a consensus exponentially fast. ■

4.2. Time-varying symmetric directed graphs

In this subsection, we consider the following scenario of time-varying symmetric directed graphs. Let an m-vertex symmetric directed graph $\mathbb{N}$ represent all allowable communication among the m agents. In other words, agents i and j are allowed to communicate with each other if and only if (i, j) is an arc in $\mathbb{N}$. For each time t, we use a time-dependent m-vertex symmetric directed graph $\mathbb{N}(t)$ to describe the neighbor relations among the m agents at time t. That is, if agents i and j communicate at time t, then (i, j) is an arc in $\mathbb{N}(t)$. It is easy to see that $\mathbb{N}(t)$ is a spanning subgraph of $\mathbb{N}$, and all such possible spanning subgraphs is a finite set. We assume that $\bar{\mathbb{N}}$ is well-configured, i.e., each arc (i, j) in $\mathbb{N}$ is associated with a matrix C_{ij} such that kernel $C\bar{J}' =$ span $\bar{I}$, with J being the incidence matrix of $\mathbb{N}$.

For any time-varying symmetric directed graph sequence just described, we propose the following algorithm using the Metropolis weights:

$$x_i(t+1) = x_i(t) - \frac{1}{2}\sum_{j\in\mathcal{N}_i(t)} w_{ij}(t)(C_{ij}'C_{ij} + C_{ji}'C_{ji})(x_i(t) - x_j(t)) \tag{8}$$

where $\mathcal{N}_i(t)$ is the neighbor set of agent i at time t and $w_{ij}(t)$ are the Metropolis weights corresponding to $\mathbb{N}(t)$, which are proposed in [17] for solving the distributed averaging problem over symmetric directed graphs and defined as

$$w_{ij}(t) = \frac{1}{1 + \max\{d_i(t), d_j(t)\}}, \qquad j \in \mathcal{N}_i(t)$$

where $d_i(t) = |\mathcal{N}_i(t)|$ denotes the number of neighbors of agent i at time t.

Theorem 6. *Suppose that $\bar{\mathbb{N}}$ is well-configured. If $\mathbb{N}$ is symmetric, strongly connected and each edge of $\mathbb{N}$ appears infinitely often in the infinite sequence of neighbor graphs $\mathbb{N}(1), \mathbb{N}(2), \mathbb{N}(3), \dots$, then algorithm (8) will guarantee all m agents to reach a consensus.*

To prove the theorem, we first combine the m update equations in (8) into one state form. To this end, we tailor the definition of an incidence matrix for spanning subgraphs as follows. Consider a directed graph $\mathbb{G}$ with m vertices and d directed edges. Let $\mathcal{E}$ denote the arc set of $\mathbb{G}$ and J denote the $m \times d$ incidence matrix of $\mathbb{G}$ according to some ordering of the arcs in $\mathcal{E}$. Let $\mathbb{H}$ be a spanning subgraph of $\mathbb{G}$. We define the *spanning incidence matrix* of $\mathbb{H}$ as an $m \times d$ matrix in which column k has exactly one 1 in row i and exactly one -1 is row j if the kth arc in $\mathbb{G}$ is (j, i) and (j, i) is also an arc in $\mathbb{H}$. It is clear that the spanning incidence matrix of any spanning subgraph of $\mathbb{G}$ has the same size as the incidence matrix of $\mathbb{G}$. If the kth arc in $\mathbb{G}$ is not in a spanning subgraph, then the kth column of the incidence matrix of $\mathbb{G}$ is replaced by a zero vector in the spanning incidence matrix.

We also need the following definition. Consider a symmetric directed graph $\mathbb{G}$ with d arcs. Let $\bar{\mathbb{G}}$ be a spanning subgraph of $\mathbb{G}$ which is also symmetric. Since $\bar{\mathbb{G}}$ is symmetric, its Metropolis weights $\bar{w}_{ij}$ are well-defined; specifically, $\bar{w}_{ij} = 1/(1+\max\{\bar{d}_i, \bar{d}_j\})$, where $\bar{d}_k$ denotes the number of neighbors of vertex k in $\bar{\mathbb{G}}$. Given an ordering of all the arcs in $\mathbb{G}$, the *spanning weight matrix* of $\bar{\mathbb{G}}$ is the $d \times d$ diagonal matrix whose kth diagonal entry equals $\bar{w}_{ij}$ if the kth arc in $\mathbb{G}$ is (j, i) and (j, i) is also an arc in $\bar{\mathbb{G}}$, or 0 if the kth arc in $\mathbb{G}$ is not in $\bar{\mathbb{G}}$.

With the above definitions, it is not hard to verify that the m update equations in (8) can be written as

$$x(t+1) = x(t) - \frac{1}{2}\bar{J}(t)C'\bar{W}(t)C\bar{J}'(t)x(t) \tag{9}$$

where $\bar{J}(t) = J(t) \otimes I_n$ with $J(t)$ being the spanning incidence matrix of $\mathbb{N}(t)$, and $\bar{W}(t) = W(t) \otimes I_n$ with $W(t)$ being the spanning weight matrix of $\mathbb{N}(t)$. It is worth noting that all $W(t)$ are nonnegative diagonal matrices with the same size. It is also worth emphasizing that the definitions of C, $J(t)$, and $W(t)$ are based on the same ordering of the arcs in $\mathbb{N}$, and the equality (9) is independent of the ordering.

To proceed, we need the following concept and result.

A square matrix M is called *paracontracting* with respect to a vector norm $\|\cdot\|$ if $\|Mx\| \leq \|x\|$ and the strict inequality holds whenever $Mx \neq x$. It is easy to see that any symmetric matrix is paracontracting with respect to the 2-norm if all its eigenvalues lie in the interval $(-1, 1]$.

For a square matrix M, we define its fixed point set as

$$\mathcal{F}(M) = \{x : Mx = x\}$$

Paracontracting matrices have the following properties.

Lemma 6. *Suppose that a finite set of square matrices $\{M_1, M_2, \dots, M_p\}$ are paracontracting with respect to the same vector norm. Let $\sigma(1), \sigma(2), \dots$ be an infinite sequence of integers taking values in $\{1, 2, \dots, p\}$ and $\mathcal{I}$ be the set of all integers that appears infinitely often in the sequence. Then for any initial vector $z(0)$, the sequence of vectors generated by $z(t+1) = M_{\sigma(t)} z(t)$ has a limit $z^* \in \bigcap_{i\in\mathcal{I}} \mathcal{F}(M_i)$.*

The lemma is a special case of Theorem 1 in [18].

We also need the following lemmas.

Lemma 7. *Let $\bar{W} = W \otimes I$, where W is a positive diagonal matrix. If $\bar{\mathbb{N}}$ is well-configured, then $\bar{J}C'\bar{W}C\bar{J}'$ has exactly n eigenvalues at zero, and all the remaining eigenvalues are positive.*

Proof of Lemma 7. It is clear that $\bar{J}C'\bar{W}C\bar{J}'$ is positive semidefinite. Note that for any x and $y = \bar{W}^{1/2}C\bar{J}'x$, there holds $C\bar{J}'x = W^{-1/2}y$. From this fact and Lemma 4, $C\bar{J}'x = W^{-1/2}y = 0$ if and only if $x \in \operatorname{span} \bar{I}$, which implies that $W^{1/2}C\bar{J}'x = y = 0$, i.e., kernel $W^{1/2}C\bar{J}' = \operatorname{span} \bar{I}$, and thus $\bar{J}C'\bar{W}C\bar{J}'$ has exactly n eigenvalues at zero. ■

Lemma 8. *Let $\mathbb{G}$ be a symmetric, spanning subgraph of $\mathbb{N}$, W be the spanning weight matrix of $\mathbb{G}$, and J be the spanning incidence matrix of $\mathbb{G}$. Then all the eigenvalues of $I - \frac{1}{2}\bar{J}C'\bar{W}C\bar{J}'$ lie in $(-1, 1]$. If furthermore $\mathbb{G} = \mathbb{N}$, $I - \frac{1}{2}\bar{J}C'\bar{W}C\bar{J}'$ has exactly n eigenvalues at one and all the remaining eigenvalues lie in $(-1, 1)$.*

Proof of Lemma 8. Consider each block of $\frac{1}{2}\bar{J}C'\bar{W}C\bar{J}'$. For any $i \in \mathbf{m}$, each diagonal block is

$$\Big[\frac{1}{2}\bar{J}C'\bar{W}C\bar{J}'\Big]_{ii} = \frac{1}{2}\sum_{j\in\mathcal{N}_i}(w_{ij}C_{ij}'C_{ij} + w_{ji}C_{ji}'C_{ji}) = \frac{1}{2}\sum_{j\in\mathcal{N}_i}(w_{ij}P_{ij} + w_{ji}P_{ji})$$

and for any $i \in \mathbf{m}$, $j \in \mathcal{N}_i$, each off-diagonal block is

$$\Big[\frac{1}{2}\bar{J}C'\bar{W}C\bar{J}'\Big]_{ij} = -\frac{w_{ij}}{2}(C_{ij}'C_{ij} + C_{ji}'C_{ji}) = -\frac{w_{ij}}{2}(P_{ij} + P_{ji})$$

Note that $\|P_{ij}\|_2 = 1$, $w_{ij} = w_{ji} \leq 1$, and $\sum_{j\in\mathcal{N}_i} w_{ij} < 1$. Consider 2-norm for each block. For any $i \in \mathbf{m}$,

$$\|\Big[\frac{1}{2}\bar{J}C'\bar{W}C\bar{J}'\Big]_{ii}\|_2 \leq \frac{1}{2}\sum_{j\in\mathcal{N}_i} w_{ij}(\|P_{ij}\|_2 + \|P_{ji}\|_2) \leq \sum_{j\in\mathcal{N}_i} w_{ij}$$

and for any $i \in \mathbf{m}$, $j \in \mathcal{N}_i$,

$$\|\Big[\frac{1}{2}\bar{J}C'\bar{W}C\bar{J}'\Big]_{ij}\|_2 \leq \frac{1}{2} w_{ij}(\|P_{ij}\|_2 + \|P_{ji}\|_2) \leq w_{ij}$$

Next consider the $(2, \infty)$-norm of $\frac{1}{2}\bar{J}C'\bar{W}C\bar{J}'$:

$$\|\frac{1}{2}\bar{J}C'\bar{W}C\bar{J}'\|_{2,\infty} = \min_{i\in\mathbf{m}} \|\Big[\frac{1}{2}\bar{J}C'\bar{W}C\bar{J}'\Big]_{ii}\|_2 + \sum_{j\in\mathcal{N}_i} \|\Big[\frac{1}{2}\bar{J}C'\bar{W}C\bar{J}'\Big]_{ij}\|_2$$

$$\leq \min_{i\in\mathbf{m}}\Big(\sum_{j\in\mathcal{N}_i} w_{ij} + \sum_{j\in\mathcal{N}_i} w_{ij}\Big)$$
$$= 2\min_{i\in\mathbf{m}} \sum_{j\in\mathcal{N}_i} \frac{1}{1+\max(d_i, d_j)}$$
$$\leq 2\min_{i\in\mathbf{m}} \frac{d_i}{1+\min_{k\in\mathbf{m}} d_k} = \frac{2\min_{i\in\mathbf{m}} d_i}{1+\min_{k\in\mathbf{m}} d_k} < 2$$

which implies that the spectral radius of $\frac{1}{2}\bar{J}C'\bar{W}C\bar{J}'$ is less than 2. It follows that $I - \frac{1}{2}\bar{J}C'\bar{W}C\bar{J}'$ has n eigenvalues at one and all the other eigenvalues lie in $(-1, 1)$.

In the case when $\mathbb{G} = \mathbb{N}$, W is positive diagonal matrix. Then the lemma is true by Lemma 7. ■

The above lemma implies that each update matrix $(I - \frac{1}{2}\bar{J}(t)C'\bar{W}(t)C\bar{J}'(t))$ in (9) is paracontracting with respect to the 2-norm.

Lemma 9. *Let $\mathbb{G}_1, \mathbb{G}_2, \dots, \mathbb{G}_p$ be a finite set of symmetric, spanning subgraphs of $\mathbb{G}$. If the union of $\mathbb{G}_1, \mathbb{G}_2, \dots, \mathbb{G}_p$ is $\mathbb{G}$, then* kernel $C\bar{J}'$ = kernel $C(\sum_{i=1}^p \bar{W}_i^{1/2}\bar{J}_i')$, *where J is the incidence matrix of $\mathbb{G}$, J_i is the spanning incidence matrix of $\mathbb{G}_i$, and W_i is the spanning weight matrix of $\mathbb{G}_i$.*

Proof of Lemma 9. If (i, j) is an edge in $\mathbb{G}$ but not in $\mathbb{G}_k$, then the corresponding Metropolis weight $w_{ij} = 0$ for $\mathbb{G}_k$, which implies that $\bar{W}_k^{1/2}\bar{J}_k' = \bar{W}_k^{1/2}\bar{J}'$. Then $C(\sum_{i=1}^p \bar{W}_i^{1/2}\bar{J}_i') = C(\sum_{i=1}^p \bar{W}_i^{1/2})\bar{J}' = (\sum_{i=1}^p \bar{W}_i^{1/2})C\bar{J}'$. Since the union of $\mathbb{G}_1, \mathbb{G}_2, \dots, \mathbb{G}_p$ is $\mathbb{G}$, $\sum_{i=1}^p \bar{W}_i^{1/2}$ is a positive diagonal matrix and thus nonsingular. Then kernel $C(\sum_{i=1}^p \bar{W}_i^{1/2}\bar{J}_i')$ = kernel $(\sum_{i=1}^p \bar{W}_i^{1/2})C\bar{J}'$ = kernel $C\bar{J}'$. ■

Now we are in a position to prove Theorem 6.

Proof of Theorem 6. Let $\mathcal{S}$ denote the set of all possible spanning subgraphs of $\mathbb{N}$, which apparently is a finite set. Let $\mathcal{I} \subset \mathcal{S}$ denote the set of those spanning subgraphs which appears infinitely often in the infinite sequence $\mathbb{N}(1), \mathbb{N}(2), \mathbb{N}(3), \dots$. Denote all spanning graphs in $\mathcal{I}$ as $\mathbb{N}_1, \mathbb{N}_2, \dots, \mathbb{N}_p$. Since each edge of $\mathbb{N}$ appears infinitely often, the union of $\mathbb{N}_1, \mathbb{N}_2, \dots, \mathbb{N}_p$ is $\mathbb{N}$.

Let J be the incidence matrix of $\mathbb{N}$ and J_i be the spanning incidence matrix of $\mathbb{N}_i$ for all $i \in \mathbf{p}$. Let W and W_i be the spanning weight matrices of $\mathbb{N}$ and $\mathbb{N}_i$, $i \in \mathbf{p}$, respectively. From Lemma 8, each update matrix $(I - \frac{1}{2}\bar{J}(t)C'\bar{W}(t)C\bar{J}'(t))$ in (9) is paracontracting with respect to 2-norm for all t. From Lemma 6, $x(t)$ will asymptotically converge to a common fixed point all $(I - \frac{1}{2}\bar{J}_iC'\bar{W}_iC\bar{J}_i')$, $i \in \mathbf{p}$. It is easy to see that $\mathcal{F}(I - \frac{1}{2}\bar{J}_iC'\bar{W}_iC\bar{J}_i') =$ kernel $\bar{W}_i^{\frac{1}{2}}C\bar{J}_i'$ = kernel $C\bar{W}_i^{\frac{1}{2}}\bar{J}_i'$. Thus $x(t)$ will converge to a point in the intersection of kernel $C\bar{W}_i^{\frac{1}{2}}\bar{J}_i'$, $i \in \mathbf{p}$.

It is clear that the intersection of kernel $C\bar{W}_i^{\frac{1}{2}}\bar{J}_i'$, $i \in \mathbf{p}$ is a subset of kernel $C(\sum_{i=1}^p \bar{W}_i^{1/2}\bar{J}_i')$. From Lemma 9, kernel $C(\sum_{i=1}^p \bar{W}_i^{1/2}\bar{J}_i')$ = kernel $C\bar{J}'$. Since $\bar{\mathbb{N}}$ is well-configured, kernel $C\bar{J}'$ = span $\bar{I}$, which implies that the intersection of kernel $\bar{C}W_i^{\frac{1}{2}}\bar{J}_i'$, $i \in \mathbf{p}$, is a subset of span $\bar{I}$. ■

5. Algorithms for directed graphs

In this section, we discuss some special types of strongly connected graphs. We begin with directed cycles, the simplest strongly connected graphs.

5.1. Directed cycles with specific initial states

Consider an m-vertex directed cycle $1 \to 2 \to \cdots \to m \to 1$ whose local agreement equations are given in (5). The agents update their states as follows:

$$x_i(t+1) = x_i(t) - \frac{1}{2}P_i(x_i(t) - x_{i-1}(t)), \quad i \in \mathbf{m} \tag{10}$$

where $P_i = C_i'(C_iC_i')^{-1}C_i$ is a projection on $\mathcal{K}_i^{\perp}$. In this subsection, we assume that each agent i initializes its state $x_i(0) \in \text{image } P_i$, which can be implemented in a distributed manner.

Proposition 1. *If $\mathbb{N}$ is an m-vertex directed cycle, then algorithm (10) with $x_i(0) \in \text{image } P_i$, $i \in \mathbf{m}$, will lead all m agents to reach a consensus exponentially fast.*

Proof of Proposition 1. It is easy to see that with the specific initialization $x_i(t) \in \text{image } P_i$ for all time t. Then the individual update can be written as

$$x_i(t+1) = P_ix_i(t) - \frac{1}{2}P_i(x_i(t) - x_{i-1}(t)) = P_i\Big(\frac{1}{2}x_i(t) + \frac{1}{2}x_{i-1}(t)\Big)$$

for all $i \in \mathbf{m}$, which leads to the system update as

$$x(t+1) = P(F \otimes I)x(t)$$

where P is the block diagonal matrix of all P_i and F is the flocking matrix[5] of the cycle. The update has the same form as the distributed linear equation solver in [2], which guarantees exponentially fast consensus. ■

5.2. Directed cycles with arbitrary initial states

In this subsection, we consider directed cycles and algorithm (10) without any specific initialization.

Theorem 7. *If $\mathbb{N}$ is an m-vertex directed cycle and $\bar{\mathbb{N}}$ is well-configured, then algorithm (10) will lead all m agents to reach a consensus exponentially fast for any initial states.*

To prove the theorem, we first rewrite the m equations in (10) as one state form as

$$x(t+1) = Mx(t) \tag{11}$$

where M is an $mn \times mn$ matrix whose blocks can be easily figured out via (10). The update matrix M has the following properties.

Lemma 10. *If $\lambda \neq 1$ is an eigenvalue of M, then $|\lambda| < 1$.*

Proof of Lemma 12. Let $v = [v_1' \ \cdots \ v_m']' \neq 0$ be an eigenvector of M for eigenvalue $\lambda \neq 1$, with each $v_i \in \mathbb{R}^n$, $i \in \mathbf{m}$. Then

$$\Big(I - \frac{1}{2}P_i\Big)v_i + \frac{1}{2}P_iv_l = \lambda v_i$$

where $l = i+1$ when $i \in \{1, \ldots, m-1\}$ and $l = 1$ when $i = m$. Re-arranging the above equation yields

$$\Big(\frac{1}{2}P_i - (1-\lambda)I\Big)v_i = \frac{1}{2}P_iv_l \tag{12}$$

For any $i, j \in \mathbf{m}$, v_i can be decomposed into $v_i = \alpha_{i,j} + \beta_{i,j}$, where $\alpha_{i,j} \in \text{image } P_j$ and $\beta_{i,j} \in \text{kernel } P_j$. Since P_j is symmetric, such decomposition is unique for any $j \in \mathbf{m}$ because image $P_j \oplus$ kernel $P_j = \mathbb{R}^n$. Thus $\alpha_{i,k} \perp \beta_{j,k}$ for any i, j, k. Substituting $v_i = \alpha_{i,i} + \beta_{i,i}$ and $v_l = \alpha_{l,i} + \beta_{l,i}$ into (12) leads to

$$(\lambda - \frac{1}{2})\alpha_{i,i} - (1-\lambda)\beta_{i,i} = \frac{1}{2}\alpha_{l,i}$$

Left multiplying $\beta_{i,i}'$ on both sides of the above equation, $(1-\lambda)\beta_{i,i}'\beta_{i,i} = 0$. If $\lambda \neq 1$, then $\beta_{i,i} = 0$, which implies that $v_i \in \text{image } P_i$ when $\lambda \neq 1$. Applying this fact to (12) with $\lambda \neq 1$,

$$\frac{1}{2}(v_i + \alpha_{l,i}) = \lambda v_i \tag{13}$$

Note that index l is dependent on index i and $l \neq i$. We claim that there exists an index $i \in \mathbf{m}$ such that $v_i \neq v_l$ and $\|v_i\|_2 \geq \|v_l\|_2$. To prove the claim, suppose to the contrary that for each $i \in \mathbf{m}$ there holds $v_i = v_l$ or $\|v_i\|_2 < \|v_l\|_2$, which implies that $\|v_i\|_2 \leq \|v_l\|_2$ for all $i \in \mathbf{m}$. Since the mapping from index i to index l is bijective from $\mathbf{m}$ to $\mathbf{m}$, there cannot exist any index i for which $\|v_i\|_2 < \|v_l\|_2$, or $\sum_{i\in\mathbf{m}} \|v_i\|_2 < \sum_{l\in\mathbf{m}} \|v_l\|_2$ which is impossible. It then follows that $v_i = v_l$ for all $i \in \mathbf{m}$, which with (12) implies that either $\lambda = 1$ or $v = 0$. But this conflicts with the facts that $\lambda \neq 1$ or $v \neq 0$. Thus the claim is true. Pick any index $i \in \mathbf{m}$ such that $v_i \neq v_l$ and $\|v_i\|_2 \geq \|v_l\|_2$. Note that $\|v_l\|_2^2 = \|\alpha_{l,i}\|_2^2 + \|\beta_{l,i}\|_2^2$. It follows that $\|\alpha_{l,i}\|_2 \leq \|v_i\|_2$. If $v_i = \alpha_{l,i}$, then $\beta_{li} = 0$, and thus $v_l = \alpha_{l,i} = v_i$, which contradicts with $v_i \neq v_l$. This observation leads to $v_i \neq \alpha_{l,i}$. This inequality, together with $\|\alpha_{l,i}\|_2 \leq \|v_i\|_2$, implies that $v_i'\alpha_{l,i} < \|v_i\|_2^2$. Then taking squared 2-norm on both sides of (13) leads to

$$|\lambda|^2\|v_i\|^2 = \frac{1}{4}\Big(\|v_i\|^2 + \|\alpha_{l,i}\|^2 + 2v_i'\alpha_{l,i}\Big) < \|v_i\|^2$$

which implies that $|\lambda| < 1$. ■

Lemma 11. *If $\bar{\mathbb{N}}$ is well-configured, $\{x : Mx = x\} = \text{span } \bar{I}$.*

Proof of Lemma 11. It is easy to verify that $\{x : Mx = x\} \supset \text{span } \bar{I}$, and we thus focus on $\{x : Mx = x\} \subset \text{span } \bar{I}$ in the remaining proof. Let $x = [x_1' \ \cdots \ x_m']' \neq 0$ be an eigenvector of M for eigenvalue 1, with each $x_i \in \mathbb{R}^n$, $i \in \mathbf{m}$. Then for each $i \in \mathbf{m}$,

$$\Big(I - \frac{1}{2}P_i\Big)x_i + \frac{1}{2}P_ix_l = x_i$$

where $l = i+1$ when $i \in \{1, \ldots, m-1\}$ and $l = 1$ when $i = m$. Re-arranging the above equation yields $P_i(x_i - x_l) = 0$. Let $y_i = x_i - x_l$ for all $i \in \mathbf{m}$. Then $y_i \in \text{kernel } P_i = \mathcal{K}_i$ and

$$\sum_{i\in\mathbf{m}} y_i = \sum_{i\in\mathbf{m}} x_i - \sum_{l\in\mathbf{m}} x_l = 0$$

Note that if $\mathcal{K}_i = 0$, then $y_i = 0$. Then

$$\sum_{i\in\mathbf{m},\ \mathcal{K}_i\neq 0} y_i = 0 \tag{14}$$

Since $\bar{\mathbb{N}}$ is well-configured, from Lemma 2, $\{\mathcal{K}_i : i \in \mathbf{m}, \mathcal{K}_i \neq 0\}$ is an independent family. Since $y_i \in \mathcal{K}_i$ for each $i \in \mathbf{m}$, (14) implies that $y_i = 0$ for all $i \in \mathbf{m}$ such that $\mathcal{K}_i \neq 0$. Thus $y_i = 0$ for all $i \in \mathbf{m}$, namely, x_i are all equal, which implies that $x \in \text{span } \bar{I}$ and thus $\{x : Mx = x\} \subset \text{span } \bar{I}$. ■

Lemma 12. *If $\bar{\mathbb{N}}$ is well-configured, then M has exactly n eigenvalues at one.*

Proof of Lemma 12. To prove the lemma, it is sufficient to show that the algebraic and geometric multiplicity of eigenvalue one are equal, namely, all the Jordan blocks of eigenvalue one are of size 1. This and Lemma 11 will imply that M has exactly n eigenvalues at one.

To this end, it is sufficient to prove that kernel $(I - M) =$ kernel $(I - M)^2$. To see this, consider the Jordan canonical form of M. From the proof of Theorem 3.1.11 in [20], dim(kernel $(I - M)^2$) − dim(kernel $(I - M)$) equals the number of those Jordan blocks of eigenvalue one whose sizes are larger than 1, where dim(·) denotes the dimension of a subspace. Since kernel $(I -$

[5] The flocking matrix of a directed graph $\mathbb{G}$ is defined as $D_{\mathbb{G}}^{-1}A_{\mathbb{G}}'$, where $D_{\mathbb{G}}$ is the diagonal matrix whose ith diagonal entry is the in-degree of vertex i in $\mathbb{G}$ and $A_{\mathbb{G}}$ is the adjacency matrix of $\mathbb{G}$. A flocking matrix is a stochastic matrix [19].

$M) \subset$ kernel $(I - M)^2$, all the Jordan blocks of eigenvalue one being of size 1 is equivalent to the condition kernel $(I - M) =$ kernel $(I - M)^2$.

To prove kernel $(I - M) =$ kernel $(I - M)^2$, suppose to the contrary that there exists a nonzero vector $y \in$ kernel $(I - M)^2$ but $y \notin$ kernel $(I - M)$, which implies that $(I - M)y \in$ kernel $(I - M)$. From Lemma Lemma 11, kernel $(I - M) =$ span $\bar{I}$, and thus there exists a nonzero vector $z \in \mathbb{R}^n$ for which $(I - M)y = \mathbf{1}_m \otimes z$. Write $y = [y_1' \ \cdots \ y_m']'$ with each $y_i \in \mathbb{R}^n$, $i \in \mathbf{m}$. Then it is straightforward to verify that $\frac{1}{2}P_i(y_i - y_l) = z$, where $l = i + 1$ when $i \in \{1, \dots, m - 1\}$ and $l = 1$ when $i = m$. These equations imply that $z \in \bigcap_{i\in\mathbf{m}}$ image P_i and thus

$$\frac{1}{2}P_i(y_i - y_l) - z = \frac{1}{2}P_i(y_i - y_l - 2z) = 0$$

further implying that

$$y_i - y_l - 2z \in \text{kernel } P_i$$

Note that index l is dependent on index i and that the mapping from index i to index l is bijective from $\mathbf{m}$ to $\mathbf{m}$. Taking summation over index i on both sides of the above relation,

$$\sum_{i\in\mathbf{m}}(y_i - y_l - 2z) = \sum_{i\in\mathbf{m}} y_i - \sum_{l\in\mathbf{m}} y_l - 2mz = -2mz \in \sum_{i\in\mathbf{m}} \text{kernel } P_i$$

and thus $z \in \sum_{i\in\mathbf{m}}$ kernel P_i. Since each P_i is symmetric, its kernel is the orthogonal complement to its image. Recall that $z \in \bigcap_{i\in\mathbf{m}}$ image P_i. It follows that $z \perp$ kernel P_i for all $i \in \mathbf{m}$ and thus $z \perp \sum_{i\in\mathbf{m}}$ kernel P_i. Then it must be true that $z = 0$. But it conflicts the fact that z is a nonzero vector. This completes the proof. ■

We are now in a position to prove Theorem 7.

Proof of Theorem 7. From Lemmas 10–12, the linear system (11) will converge to the eigenspace of eigenvalue one as $t \to \infty$. From Lemma 11, the eigenspace of eigenvalue one is span $\bar{I}$, which implies that all $x_i(t)$, $i \in \mathbf{m}$, will reach a consensus. Since the linear system is time-invariant, the consensus will be reached exponentially fast. ■

5.3. A counterexample

One may conjecture that following algorithm

$$x_i(t+1) = x_i(t) - \frac{1}{d_i + 1}\sum_{j\in\mathcal{N}_i} P_{ji}(x_i(t) - x_j(t)) \tag{15}$$

where $P_{ji} = C_{ji}'(C_{ji}C_{ji}')^{-1}C_{ji}$ is a projection on $\mathcal{K}_{ji}^{\perp}$, will lead to a consensus for any strongly connected graphs, considering algorithm (10) is a special case of (15). It turns out that it is not the case, as shown in the following counterexample.

Consider a strongly connected graph with 3 vertices and 4 directed edges $(1, 2)(2, 3)(3, 1)(2, 1)$. For simplicity, we write 4 matrices as C_1, C_2, C_3, C_4 for the 4 directed edges whose kernels are $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4$. The corresponding local agreements are

$$\begin{aligned}
&C_1(x_2 - x_1) = 0\\
&C_2(x_3 - x_2) = 0\\
&C_3(x_1 - x_3) = 0\\
&C_4(x_1 - x_2) = 0
\end{aligned}$$

We claim that the weighted neighbor graph $\bar{\mathbb{N}}$ is well-configured with all $\mathcal{K}_i \neq 0$ if and only if $\{\mathcal{K}_1 \cap \mathcal{K}_4, \mathcal{K}_2, \mathcal{K}_3\}$ is an independent family, i.e.,

$$\begin{aligned}
&\mathcal{K}_1 \cap \mathcal{K}_4 \cap (\mathcal{K}_2 + \mathcal{K}_3) = 0\\
&\mathcal{K}_2 \cap (\mathcal{K}_1 \cap \mathcal{K}_4 + \mathcal{K}_3) = 0 \qquad (16)\\
&\mathcal{K}_3 \cap (\mathcal{K}_1 \cap \mathcal{K}_4 + \mathcal{K}_2) = 0
\end{aligned}$$

To prove the claim, let $p_1 = x_1 - x_2$, $p_2 = x_2 - x_3$ and $p_3 = x_3 - x_1$. From local agreements

$$\begin{aligned}
&C_1(x_1 - x_2) = C_1 p_1 = 0\\
&C_2(x_2 - x_3) = C_2 p_2 = 0\\
&C_3(x_3 - x_1) = C_3 p_3 = 0\\
&C_4(x_1 - x_2) = C_4 p_1 = 0\\
&p_1 + p_2 + p_3 = 0
\end{aligned}$$

from which

$$\begin{aligned}
&p_1 \in \mathcal{K}_1 \cap \mathcal{K}_4 \cap (\mathcal{K}_2 + \mathcal{K}_3)\\
&p_2 \in \mathcal{K}_2 \cap (\mathcal{K}_1 \cap \mathcal{K}_4 + \mathcal{K}_3)\\
&p_3 \in \mathcal{K}_3 \cap (\mathcal{K}_1 \cap \mathcal{K}_4 + \mathcal{K}_2)
\end{aligned}$$

We first prove the sufficiency. Suppose (16) holds, that is, $\{\mathcal{K}_1 \cap \mathcal{K}_4, \mathcal{K}_2, \mathcal{K}_3\}$ is an independent family, which implies that $p_1 = p_2 = p_3 = 0$, i.e., $x_1 = x_2 = x_3$.

We next prove the necessity. Suppose to the contrary that $\bar{\mathbb{N}}$ is well-configured but (16) does not hold. Then there must exist a nonzero vector $y_1 \in \mathcal{K}_1 \cap \mathcal{K}_4 \cap (\mathcal{K}_2 + \mathcal{K}_3)$, which implies that there exist $y_2 \in \mathcal{K}_2$ and $y_3 \in \mathcal{K}_3$ such that $y_1 = y_2 + y_3$ and $y_1 \in \mathcal{K}_1 \cap \mathcal{K}_4$. Since y_1 is nonzero, so is either y_2 or y_3. Letting $x_1 = y_1$, $x_2 = 0$, and $x_3 = y_2$, it is easy to check that all the local agreement equations hold:

$$\begin{aligned}
&C_1(x_1 - x_2) = C_1 y_1 = 0\\
&C_2(x_2 - x_3) = -C_2 y_2 = 0\\
&C_3(x_3 - x_1) = -C_3 y_3 = 0\\
&C_4(x_1 - x_2) = C_4 y_1 = 0\\
&p_1 + p_2 + p_3 = 0
\end{aligned}$$

while, since $x_1 \neq x_2$, $\bar{\mathbb{N}}$ is not well configured. This completes the proof of the claim.

For this example, the update matrix

$$M = \begin{bmatrix} I - \frac{1}{3}P_4 - \frac{1}{3}P_3 & \frac{1}{3}P_4 & \frac{1}{3}P_3\\ \frac{1}{2}P_1 & I - \frac{1}{2}P_1 & 0\\ 0 & \frac{1}{2}P_2 & I - \frac{1}{2}P_2 \end{bmatrix}$$

where $P_i = C_i'(C_iC_i')^{-1}C_i$. However, its eigenspace of eigenvalue one can be larger than span $\bar{I}$ even when $\bar{\mathbb{N}}$ is well-configured. To see this, set $C_1 = C_2$, $C_3 = C_4$ and $\mathcal{K}_2 \cap \mathcal{K}_3 = 0$, which implies that $\{\mathcal{K}_1 \cap \mathcal{K}_4, \mathcal{K}_2, \mathcal{K}_3\}$ is an independent family, and thus $\bar{\mathbb{N}}$ is well-configured. Picking any nonzero $y \in \mathcal{K}_1$, it is easy to verify that

$$x = \begin{bmatrix} 0\\ y\\ -y \end{bmatrix} \in \{x : Mx = x\}$$

which implies that span $\bar{I}$ is a proper subset of $\{x : Mx = x\}$. Thus $x(t + 1) = Mx(t)$ may converge to a non-consensus state, which has also been validated by simulations.

Applying the algorithm (15) to Example 1 in Section 3, the update matrix

$$M = \begin{bmatrix} I - \frac{1}{3}P_{21} - \frac{1}{3}P_{31} & \frac{1}{3}P_{21} & \frac{1}{3}P_{31}\\ \frac{1}{3}P_{12} & I - \frac{1}{3}P_{12} - \frac{1}{3}P_{32} & \frac{1}{3}P_{32}\\ 0 & 0 & I \end{bmatrix}$$

where $P_{ij} = C_{ij}'(C_{ij}C_{ij}')^{-1}C_{ij}$. It is not hard to show that for this matrix M its eigenspace of eigenvalue one is span $\bar{I}$ and all the remaining eigenvalues are strictly less than one in magnitude.

Note that in this example the neighbor graph is rooted with a single root vertex 3, which makes the analysis easier.

Therefore the following two questions remain open. First, what are the graphical conditions on $\mathbb{N}$ under which algorithm (15) will lead all the agents to reach a consensus for arbitrary initial states? Second, how one can construct a distributed algorithm for each agent which will drive the system from arbitrary start states to a consensus for any strongly connected graphs?

It is well known that a conventional consensus (with all $C_{ij} = I$) can be achieved if and only if the neighbor graph is rooted, i.e., contains a directed spanning tree. The results and examples in this paper show that it is not the case for consensus with limited information. The feasibility of consensus with limited information depends on not only graph connectivity but also weight matrices C_{ij}.

6. Conclusion

In this paper, we have studied the problem of achieving a consensus in the face of limited information transfer between agents, in which each agent receives a linear function of the state of each of its neighbors; in the case when the linear function is realized by a matrix whose kernel is nonzero, the neighbor's state cannot be determined by the information transferred. From this perspective, the problem studied here is related to so-called privacy preserving consensus problems [21], which typically rely on carefully designed additive noise. The limited information idea here can be used to protect the privacy of agents' states without adding noise. The problem is also related to the compressed communication techniques which have been recently used to address the communication bottleneck in distributed optimization and machine learning [22].

The feasibility of the problem of interest has been termed as well-configuration. Sufficient conditions for a multi-agent system to be well-configured have been provided for different types of directed graphs. For well-configured multi-agent systems, provably correct distributed algorithms have been developed for a number of special cases of the problem. It turns out that the state forms of these algorithms share similarity with so-called matrix-weighted consensus processes [23,24]. Our results imply that the existing sufficient conditions for matrix-weighted consensus, which usually require a tree whose matrix-valued weights are all positive definite, can be significantly relaxed.

In addition to the two open questions stated at the end of the preceding section, there are a number of directions of future work, including to establish necessary and sufficient conditions for a strongly connected system to be well-configured, to study well-configuration for general rooted graphs, and to derive convergence rates for the proposed algorithms.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

[1] M. Pease, R. Shostak, L. Lamport, Reaching agreement in the presence of faults, J. ACM 27 (2) (1980) 228–234.

[2] S. Mou, J. Liu, A.S. Morse, A distributed algorithm for solving a linear algebraic equation, IEEE Trans. Automat. Control 60 (11) (2015) 2863–2878.

[3] A. Nedić, A. Ozdaglar, Distributed subgradient methods for multi-agent optimization, IEEE Trans. Automat. Control 54 (1) (2009) 48–61.

[4] R. Olfati-Saber, Distributed Kalman filtering for sensor networks, in: Proceedings of the 46th IEEE Conference on Decision and Control, 2007, pp. 5492–5498.

[5] G. Antonelli, Interconnected dynamic systems: An overview on distributed control, IEEE Control Syst. Mag. 33 (1) (2013) 76–88.

[6] A. Kashyap, T. Başar, R. Srikant, Quantized consensus, Automatica 43 (7) (2007) 1192–1203.

[7] M. Cao, A.S. Morse, B.D.O. Anderson, Agreeing asynchronously, IEEE Trans. Automat. Control 53 (8) (2008) 1826–1838.

[8] L. Moreau, Stability of multi-agent systems with time-dependent communication links, IEEE Trans. Automat. Control 50 (2) (2005) 169–182.

[9] M.J. Fischer, N.A. Lynch, M.S. Paterson, Impossibility of distributed consensus with one faulty process, J. ACM 32 (2) (1985) 347–382.

[10] J.N. Tsisiklis, Problems in Decentralized Decision Making and Computation (Ph.D. thesis), Department of Electrical Engineering and Computer Science, M.I.T., 1984.

[11] M.E. Chamie, J. Liu, T. Başar, Design and analysis of distributed averaging with quantized communication, IEEE Trans. Automat. Control 61 (12) (2016) 3870–3884.

[12] L. Xiao, S. Boyd, Fast linear iterations for distributed averaging, Systems Control Lett. 53 (1) (2004) 65–78.

[13] J. Liu, S. Mou, A.S. Morse, B.D.O. Anderson, C. Yu, Deterministic gossiping, Proc. IEEE 99 (9) (2011) 1505–1524.

[14] C. Godsil, G. Royle, Algebraic Graph Theory, Springer, 2001.

[15] J. Bang-Jensen, G. Gutin, Digraphs: Theory, Algorithms and Applications, Springer Science & Business Media, 2008.

[16] H.E. Robbins, A theorem on graphs, with an application to a problem of traffic control, Amer. Math. Monthly 46 (5) (1939) 281–283.

[17] L. Xiao, S. Boyd, S. Lall, A scheme for robust distributed sensor fusion based on average consensus, in: Proceedings of the 4th International Conference on Information Processing in Sensor Networks, 2005, pp. 63–70.

[18] L. Elsner, I. Koltracht, M. Neumann, On the convergence of asynchronous paracontractions with applications to tomographic reconstruction from incomplete data, Linear Algebra Appl. 130 (1990) 65–82.

[19] M. Cao, A.S. Morse, B.D.O. Anderson, Reaching a consensus in a dynamically changing environment: A graphical approach, SIAM J. Control Optim. 47 (2) (2008) 575–600.

[20] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, 1985.

[21] Y. Mo, R.M. Murray, Privacy preserving average consensus, IEEE Trans. Automat. Control 62 (2) (2017) 753–765.

[22] A. Koloskova, S.U. Stich, M. Jaggi, Decentralized stochastic optimization and gossip algorithms with compressed communication, in: Proceedings of the 36th International Conference on Machine Learning, 2019, pp. 3478–3487.

[23] P. Barooah, J.P. Hespanha, Graph effective resistance and distributed control: Spectral properties and applications, in: Proceedings of the 45th IEEE Conference on Decision and Control, 2006, pp. 3479–3485.

[24] Q.V. Tran, M.H. Trinh, H.-S. Ahn, Discrete-time matrix-weighted consensus, IEEE Trans. Control Netw. Syst. 8 (4) (2021) 1568–1578.