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K-stability of cubic fourfolds

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Abstract. We prove that the K-moduli space of cubic fourfolds is identical to their GIT moduli space. More precisely, the K-(semi/poly)stability of cubic fourfolds coincide to the corresponding GIT stabilities, which was studied in detail by Laza. In particular, this implies that all smooth cubic fourfolds admit Kähler–Einstein metrics. Key ingredients are local volume estimates in dimension three due to Liu and Xu, and Ambro–Kawamata's non-vanishing theorem for Fano fourfolds.

1. Introduction

K-stability is an algebro-geometric stability condition introduced by Tian [70] and later reformulated algebraically by Donaldson [26] to detect the existence of Kähler–Einstein (KE) metrics on Fano varieties. The Yau–Tian–Donaldson (YTD) Conjecture predicts that the existence of a KE metric on a Fano variety X is equivalent to the K-polystability of X. The relatively easier direction of the YTD Conjecture that KE metrics implies K-polystability was confirmed in [8]. When X is smooth, the YTD Conjecture was proved in the celebrated works [20–22] and [72] using Cheeger–Colding–Tian theory. Later, a different approach to the YTD Conjecture, namely the variational approach, has been developed. Combining the analytic works [9, 45, 47] and the algebraic work [55], this approach gives a full proof of the YTD Conjecture for all (possibly singular) Fano varieties. However, it is often a challenging problem to check K-(semi/poly)stability of an explicit Fano variety.

In recent years, the algebraic study of K-stability has successfully led to a new theory, known as the *K-moduli theory*, that produces an algebraic construction of compact moduli spaces of Fano varieties. The Fano K-moduli theorem, proved in a combination of works [2, 11, 13, 16, 23, 37, 49, 55, 73–75], states that given dimension n and volume V, there exists an Artin stack $\mathcal{M}^{\mathrm{Kss}}_{n,V}$ of finite type parametrizing K-semistable Fano varieties, called the *K-moduli stack*, and $\mathcal{M}^{\mathrm{Kss}}_{n,V}$ admits a projective good moduli space $M^{\mathrm{Kps}}_{n,V}$ parametrizing K-polystable Fano varieties, called the *K-moduli space*. In the $\mathbb{Q}$-Gorenstein smoothable case, the Fano K-moduli theorem was proved earlier in [48, 74] (see also [61, 67]) based on analytic results from the works [20–22, 72].

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The K-moduli theory not only lays the foundation for these K-moduli spaces, but also provides strong tools to verify K-stability for explicit Fano varieties (e.g. Fano hypersurfaces or complete intersections). One notable strategy along this direction, namely the *moduli continuity method*, goes as follows. Using α-invariants and group actions, it is usually easy to find a K-stable Fano manifold X in the given family (e.g. Fermat hypersurfaces [4, 71, 78]). Then by openness of K-(semi)stability [12, 13, 55, 73], there exists an open neighborhood of $[X]$ in the parameter space which parametrizes K-stable Fano varieties $\mathcal{X}_t$. By the Fano K-moduli theorem, there exists a component M^{K} of the K-moduli space $M^{\mathrm{Kps}}_{n,V}$ that parametrizes those $\mathcal{X}_t$ and their K-polystable $\mathbb{Q}$-Gorenstein limits, where $n = \dim(X)$ and $V = (-K_X)^n$. Hence the K-moduli space M^{K} is birational to the GIT moduli space M^{GIT}. Using the local-to-global volume comparison in [53], if the global volume of X is relatively large, then we can often get a good control of the singularities appearing in the boundary of M^{K}. This enables us to give an explicit description of the birational map $M^{\mathrm{K}} \dashrightarrow M^{\mathrm{GIT}}$, and in some cases to even show that it is an isomorphism. This strategy first appeared implicitly in [68] where Tian showed that all smooth del Pezzo surfaces with reductive automorphism groups are K-polystable. Later, it was used to construct explicit K-moduli compactifications of del Pezzo surfaces of degree 4 in [58] and of degree ≤ 3 in [62] where the latter work was more focused on the stability study. In higher dimensions, it is shown that M^{K} is isomorphic to M^{GIT} for complete intersections of two quadric hypersurfaces in [66] (where smooth ones were shown to be K-stable earlier in [4]) and cubic threefolds in [54]. Some cases for log Fano pairs have been worked out as well, see e.g. [5–7, 32, 35]. Despite these results, much less is known in higher dimensions.

In this paper, we carry out this strategy for cubic fourfolds by showing that their K-moduli space is isomorphic to their GIT moduli space. In other words, the K-(semi/poly)stability of cubic fourfolds are the same as their GIT (semi/poly)stability.

Theorem 1.1. *Let $X \subset \mathbb{P}^5$ be a (possibly singular) cubic hypersurface. Then X is K-(semi/poly)stable if and only if $X \subset \mathbb{P}^5$ is GIT (semi/poly)stable. In particular, the K-moduli space M^{K} parametrizing K-polystable $\mathbb{Q}$-Fano varieties admitting $\mathbb{Q}$-Gorenstein smoothings to smooth cubic fourfolds is isomorphic to the GIT moduli space M^{GIT} of cubic fourfolds.*

The "only if" direction in Theorem 1.1 follows from the general fact in [63, 69] that a K-(semi/poly)stable Fano hypersurface is always GIT (semi/poly)stable. On the other hand, the "if" direction for general Fano hypersurfaces $X \subset \mathbb{P}^{n+1}$ is expected to be true only when $\deg X = 3$. In fact, if $\deg X \geq 4$, there are non-reduced, hence K-unstable, GIT polystable hypersurfaces X, e.g. a multiple of a smooth hyperquadric. In addition, the "if" direction can also fail for non-hypersurface Fano varieties, e.g. quartic double solids [7, Theorem 1.4].

The GIT of cubic fourfolds was studied in detail by Laza [42] (see also [76]). As a consequence, we have the following result which, together with [20–22, 72], implies that any smooth cubic fourfold admits a KE metric. We also obtain a result on singularities of GIT semistable cubic fourfolds without involving direct GIT calculation, which answers affirmatively a question of Spotti and Sun [66, Question 5.8] in dimension 4.

Corollary 1.2. *The following statements hold:*

(1) *All smooth cubic fourfolds are K-stable.*

(2) *All cubic fourfolds with simple singularities are K-stable.*

(3) *All GIT polystable cubic fourfolds are K-polystable. For a list of generic singularities of GIT polystable cubic fourfolds with non-simple singularities, see* [42, Theorem 1.2 and Table 3].

(4) *Any GIT semistable cubic fourfold has Gorenstein canonical singularities.*

In particular, each cubic fourfold in (1), (2) or (3) admits a (weak) KE metric.

We note that combined Corollary 1.2 with [18, 19] and [31, Corollary 1.4] on K-stability of smooth quintic fourfolds and the very recent result [1, Theorem 1.1] on K-stability of smooth quartic fourfolds, we answer affirmatively the folklore conjecture that all smooth Fano hypersurfaces have KE metrics in dimension 4.

In the process of proving Theorem 1.1, we confirm the ODP Gap conjecture of local volumes [66, Conjecture 5.5] (see also Conjecture 2.10) for all local complete intersection singularities. The proof uses the lower semicontinuity of local volumes [15]. We note that this conjecture was confirmed in dimension at most 3 [46, 54].

Theorem 1.3 (= Theorem 2.12). *Let $(x \in X)$ be an n-dimensional non-smooth local complete intersection klt singularity. Then*

$$\widehat{\mathrm{vol}}(x, X) \leq 2(n-1)^n,$$

and equality holds if and only if it is an ordinary double point.

Our proof of Theorem 1.1 starts from parallel arguments in [54] where the author and Xu showed the similar result to Theorem 1.1 for cubic threefolds. Suppose X is an n-dimensional K-semistable $\mathbb{Q}$-Fano variety admitting a $\mathbb{Q}$-Gorenstein smoothing to cubic hypersurfaces $\mathcal{X}_t$. Let L be the $\mathbb{Q}$-Cartier Weil divisor on X as the limit of hyperplane sections of $\mathcal{X}_t$. By the moduli continuity method in [54, 66], the analogous result to Theorem 1.1 for cubic n-folds would follow from showing X is a (possibly singular) cubic hypersurface. By [34] this reduces to showing L is Cartier. Using the local-to-global volume comparison from [53], the divisor L being Cartier would follow from the ODP Gap Conjecture of local volumes in dimension n (see Conjecture 2.10), especially applied to the index 1 cover of a singular point $x \in X$, where L is not Cartier. In [54], the author and Xu verified the ODP Gap conjecture in dimension 3, thus proving that K-(semi/poly)stability of cubic threefolds coincide with corresponding GIT stabilities. Such an approach depends heavily on the classification of terminal and canonical singularities in dimension 3, and is currently out of reach in dimension 4 or higher.

To overcome this difficulty, we study the explicit geometry of the linear systems $|L|$ and $|2L|$ on X under the assumption that L is not Cartier. Using the local-to-global volume estimates from [53, 54] and a Bertini-type result for local volumes (see Theorem 2.16), we show that $2L$ is Cartier, and L is Cartier away from finitely many points. Then using Ambro–Kawamata's non-vanishing theorem for fundamental divisors on Fano varieties with large Fano index [3, 38], we show that for general elements $D \in |2L|$ and $H \in |L|$, the complete intersection $(G = D \cap H, L|_G)$ is a polarized K3 surface with Du Val singularities of degree 6. Then classical results on linear systems of K3 surfaces [59] implies that $|2L|$ is base point free, and $(G, L|_G)$ is either a complete intersection, hyperelliptic, or unigonal. In the first case, we show that the index 1 cover of a non-Cartier point $x \in X$ of L must be a local complete intersection which satisfies the ODP Gap Conjecture (see Theorem 1.3). Thus similar arguments to [54] show that X is K-unstable. In the last two cases, by analyzing the rational map $\phi_{|L|} : X \dashrightarrow \mathbb{P}^5$ we show that $\alpha(X) < \frac{1}{5}$ which implies that X is K-unstable by [33, Theorem 3.5]. Given these contradictions, we conclude that L is Cartier on X, hence proving Theorem 1.1.

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2. K-stability and local volumes

Throughout this paper, we work over the field $\mathbb{C}$. We follow the standard convention from [39,41]. A pair (X, Δ) is a normal variety X together with an effective $\mathbb{Q}$-divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. A pair (X, Δ) is called a log Fano pair if X is projective, (X, Δ) is klt, and $-K_X - \Delta$ is ample. We call X a $\mathbb{Q}$-Fano variety if $(X, 0)$ is a log Fano pair. We call X a lc Fano variety if X is normal projective with log canonical singularities, and $-K_X$ is $\mathbb{Q}$-Cartier ample. A klt singularity $x \in X$ is a closed point x on a normal variety X with klt singularities.

2.1. Valuative criteria for K-stability.

Definition 2.1. Let X be a normal variety. A prime divisor E over X is a prime divisor E on a normal variety Y together with a proper birational morphism $\mu : Y \to X$. The center of E on X is $\mu(E)$. Moreover, if K_X is $\mathbb{Q}$-Cartier, then we define the log discrepancy of E as

$$A_X(E) := 1 + \mathrm{coeff}_E(K_Y - \mu^* K_X).$$

From the definition we know that $A_X(E) > 0$ (resp. ≥ 0) if X has klt (resp. log canonical) singularities.

Definition 2.2. Let X be an n-dimensional $\mathbb{Q}$-Fano variety. Let E be a prime divisor over E. The pseudo-effective threshold of E is defined as

$$T_X(E) := \sup\{t \in \mathbb{R} \mid \mu^*(-K_X) - tE \text{ is big}\}.$$

The S-invariant of E, first introduced in [14], is defined as

$$S_X(E) := \frac{1}{(-K_X)^n} \int_0^{T_X(E)} \mathrm{vol}_Y(\mu^*(-K_X) - tE)\, dt.$$

The β-invariant of E, first introduced in [30], is defined as

$$\beta_X(E) := A_X(E) - S_X(E).$$

The original definition of K-(poly/semi)stability introduced by [26, 70] is by checking the sign of generalized Futaki invariants of test configurations. In this paper, we will use the valuative criterion for K-(semi)stability invented by Fujita [30] and Li [43] with complementary result by Blum and Xu [16].

Let X be a $\mathbb{Q}$-Fano variety (resp. a lc Fano variety) with $-rK_X$ Cartier for $r \in \mathbb{Z}_{>0}$. Recall from [49,50] that a test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ of $(X, -rK_X)$ is special (resp. weakly special) if $(\mathcal{X}, \mathcal{X}_0)$ is plt (resp. log canonical) and $\mathcal{L} \sim_{\mathbb{Q}} -rK_{\mathcal{X}/\mathbb{A}^1}$, and we say that $\mathcal{X}_0$ is a special degeneration (resp. a weakly special degeneration) of X.

Theorem–Definition 2.3. *Let X be a $\mathbb{Q}$-Fano variety. Then:*

(1) [30,43] *X is K-semistable if and only if $\beta_X(E) \geq 0$ for any prime divisor E over X,*

(2) [16] *X is K-stable if and only if $\beta_X(E) > 0$ for any prime divisor E over X,*

(3) [49] *X is K-polystable if and only if any K-semistable special degeneration of X is isomorphic to itself,*

(4) *X is K-unstable if it is not K-semistable.*

Definition 2.4. Let X be a $\mathbb{Q}$-Fano variety. The α-invariant of X is defined as

$$\alpha(X) = \inf\{\mathrm{lct}(X; D) \mid 0 \leq D \sim_{\mathbb{Q}} -K_X\}.$$

By [14], we know that

$$\alpha(X) = \inf_E \frac{A_X(E)}{T_X(E)},$$

where the infimum is taken over all prime divisors E over X.

Theorem 2.5 ([33, Theorem 3.5]). *Let X be an n-dimensional K-semistable $\mathbb{Q}$-Fano variety. Then we have $\alpha(X) \geq \frac{1}{n+1}$.*

2.2. Local volumes. In this subsection, we recall the concept of normalized volume of valuations over a klt singularity first introduced by C. Li [44]. For simplicity, we restrict ourselves to divisorial valuations.

Definition 2.6 ([44]). Let $x \in X$ be an n-dimensional klt singularity. For a prime divisor E over X centered at x, we define (following [27]) the volume of E over $(x \in X)$ to be

$$\mathrm{vol}_{X,x}(E) := \lim_{m\to\infty} \frac{\ell(\mathcal{O}_{X,x}/\mathfrak{a}_m(E))}{m^n/n!}.$$

Here $\mathfrak{a}_m(E) := \{f \in \mathcal{O}_{X,x} \mid \mathrm{ord}_E(f) \geq m\}$ and ℓ denotes the length of an Artinian module. The normalized volume of E over $(x \in X)$ is defined as

$$\widehat{\mathrm{vol}}_{X,x}(E) := A_X(E)^n \cdot \mathrm{vol}_{X,x}(E).$$

The local volume of $x \in X$ is defined as

$$\widehat{\mathrm{vol}}(x, X) := \inf_E \widehat{\mathrm{vol}}_{X,x}(E),$$

where the infimum runs over all prime divisors E over X centered at x.

There are alternative characterizations of local volumes. We provide two of them using ideals and Kollár components which are useful.

Theorem 2.7 ([10, 53]). *Let $x \in X$ be an n-dimensional klt singularity. Denote*

$$(R, \mathfrak{m}) := (\mathcal{O}_{X,x}, \mathfrak{m}_{X,x}).$$

Then we have

$$\widehat{\mathrm{vol}}(x, X) = \inf_{\mathfrak{a}\colon \mathfrak{m}\text{-primary}} \mathrm{lct}(X; \mathfrak{a})^n \cdot \mathrm{e}(\mathfrak{a}) = \min_{\mathfrak{a}_\bullet\colon \mathfrak{m}\text{-primary}} \mathrm{lct}(X; \mathfrak{a}_\bullet)^n \cdot \mathrm{e}(\mathfrak{a}_\bullet).$$

Here $\mathfrak{a}$ (resp. $\mathfrak{a}_\bullet$) represents an ideal (resp. a multiplicative graded sequence of ideals) of R, and e denotes the Hilbert–Samuel multiplicity.

Definition 2.8. Let $x \in X$ be a klt singularity. We say that a proper birational morphism $\mu : Y \to X$ from a normal variety X provides a Kollár component S over $(x \in X)$ if μ is an isomorphism over $X \setminus \{x\}$, the preimage $\mu^{-1}(x) = S$ is a $\mathbb{Q}$-Cartier prime divisor on Y, the pair (Y, S) is plt, and $-S$ is μ-ample.

Theorem 2.9 ([52]). *For any klt singularity $x \in X$, we have*

$$\widehat{\mathrm{vol}}(x, X) = \inf_S \widehat{\mathrm{vol}}_{X,x}(S),$$

where the infimum runs over all Kollár components S over X centered at x.

The following conjecture was asked in [66]. It was confirmed in dimension 2 and 3 by [46, Proposition 4.10] and [54, Theorem 1.3], respectively.

Conjecture 2.10 (ODP Gap Conjecture). *Let $(x \in X)$ be an n-dimensional non-smooth klt singularity. Then*

$$\widehat{\mathrm{vol}}(x, X) \leq 2(n-1)^n,$$

and equality holds if and only if it is an ordinary double point.

Theorem 2.11. *The following statements hold:*

(1) [46, 54] *Conjecture* 2.10 *holds when $n \leq 3$.*

(2) [54, Theorem 1.6] *Let $x \in X$ be an n-dimensional klt singularity. Then we have*

$$\widehat{\mathrm{vol}}(x, X) \leq n^n,$$

and equality holds if and only if it is smooth.

The following result verifies Conjecture 2.10 for local complete intersection singularities.

Theorem 2.12. *Conjecture* 2.10 *holds for all local complete intersection singularities.*

Proof. Let $x \in X$ be an n-dimensional local complete intersection singularity. Since Conjecture 2.10 holds in dimension ≤ 3 by Theorem 2.11 (1), we may assume that $n \geq 4$. Then there exists a locally closed immersion $X \hookrightarrow Z$ into a smooth variety Z of dimension $n + r$ such that $X = V(f_1, f_2, \ldots, f_r)$ for some $f_i \in \mathcal{O}_{Z,x}$. We may assume that r achieves its minimum, i.e. $r := \mathrm{edim}(x, X) - n$, where $\mathrm{edim}(x, X) := \ell(\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2)$ denotes the embedding dimension. In particular, $\mathrm{ord}_x(f_i) \geq 2$ for any $1 \leq i \leq r$. Since $x \in X$ is non-smooth, we have $r \geq 1$.

If $r \geq 2$, we choose $g_1, g_2 \in \mathfrak{m}_{Z,x}^2$ and $g_3, \ldots, g_n \in \mathfrak{m}_{Z,x}$, so that $\bar{g}_1, \bar{g}_2 \in \mathfrak{m}_{Z,x}^2/\mathfrak{m}_{Z,x}^3$ and $\bar{g}_i \in \mathfrak{m}_{Z,x}/\mathfrak{m}_{Z,x}^2$ are general for $i \geq 3$. Let $X_t := V(f_1 + tg_1, f_2 + tg_2, \ldots, f_r + tg_r)$. Then it is clear that $(x \in X_t)_{t\in\mathbb{A}^1}$ is a $\mathbb{Q}$-Gorenstein flat family of klt singularities. Moreover, for a general $t \in \mathbb{A}^1$ we know that $x \in X_t$ is a local complete intersection of two quadrics with smooth projective tangent cone. Let S_t be the exceptional divisor of the ordinary blow up of $x \in X_t$ for general t. Then simple calculation shows that

$$A_{X_t}(S_t) = n - 2 \quad \text{and} \quad \mathrm{vol}_{X_t,x}(S_t) = \mathrm{e}(x, X_t) = 4.$$

Thus for general t we have

$$\widehat{\mathrm{vol}}(x, X_t) \leq A_{X_t}(S_t)^n \cdot \mathrm{vol}_{X_t,x}(S_t) = 4(n-2)^n < 2(n-1)^n.$$

Thus the lower semicontinuity of local volumes [15] implies that

$$\widehat{\mathrm{vol}}(x, X) \leq \widehat{\mathrm{vol}}(x, X_t) < 2(n-1)^n.$$

If $r = 1$, then $x \in X = V(f)$ is a hypersurface singularity for some $f \in \mathcal{O}_{Z,x}$. By [54, Lemma 3.1], we know that

$$\widehat{\mathrm{vol}}(x, X) \leq (n + 1 - \mathrm{ord}_x f)^n \cdot \mathrm{ord}_x f \leq 2(n-1)^n.$$

If the equality holds, then we must have $\mathrm{ord}_x f = 2$. Assume to the contrary that $x \in X$ is not an ordinary double point. By choosing a suitable algebraic coordinates $(z_0, \ldots, z_n)$ at $x \in Z$, we may assume that $f - (z_0^2 + \cdots + z_m^2) \in \mathfrak{m}_{Z,x}^3$, where $m < n$. Let

$$g = z_0^2 + \cdots + z_{n-1}^2 + z_n^3 \in \mathfrak{m}_{Z,x}^2,$$

and let $X_t := V(f + tg)$. Then by [15] we know that $\widehat{\mathrm{vol}}(x, X) \leq \widehat{\mathrm{vol}}(x, X_t)$ for general t. Since the degree 2 part of $f + tg$ has rank n for general t, we know that $x \in X_t$ is an n-dimensional A_2-singularity. Hence [44, Example 5.3] for $n \geq 4$ implies that

$$\widehat{\mathrm{vol}}(x, X) \leq \widehat{\mathrm{vol}}(x, X_t) \leq \frac{2n^n(n-2)^{n-1}}{(n-1)^{n-1}} < 2(n-1)^n.$$

Hence we get a contradiction. This finishes the proof. □

The following result from [75, Theorem 1.3] on finite degree formula of local volumes is very useful. Note that when X is a Gromov–Hausdorff limit of Kähler–Einstein Fano manifolds, such a result was proven earlier in [51, Theorem 1.7].

Theorem 2.13 ([75]). *Let $\tau : (\tilde{x} \in \widetilde{X}) \to (x \in X)$ be a finite quasi-étale Galois morphism between klt singularities. Then*

$$\widehat{\mathrm{vol}}(\tilde{x}, \widetilde{X}) = \deg(\tau) \cdot \widehat{\mathrm{vol}}(x, X).$$

The following theorem is one of the key ingredients in the moduli continuity method. It is a generalization of [29].

Theorem 2.14 ([53]). *Let X be an n-dimensional K-semistable $\mathbb{Q}$-Fano variety. Then for any closed point $x \in X$ we have*

$$(-K_X)^n \leq \left(1 + \frac{1}{n}\right)^n \widehat{\mathrm{vol}}(x, X).$$

The following lemma is well known to experts.

Lemma 2.15. *Let $(x \in X)$ and $(x' \in X')$ be two klt singularities that are analytically isomorphic, that is, $\widehat{\mathcal{O}_{X,x}} \cong \widehat{\mathcal{O}_{X',x'}}$. Then $\widehat{\mathrm{vol}}(x, X) = \widehat{\mathrm{vol}}(x', X')$.*

Proof. Denote $(R, \mathfrak{m}) := (\mathcal{O}_{X,x}, \mathfrak{m}_{X,x})$ and $(R', \mathfrak{m}') := (\mathcal{O}_{X',x'}, \mathfrak{m}_{X',x'})$. Denote by $\iota : \widehat{R} \to \widehat{R}'$ the ring isomorphism. Then any $\mathfrak{m}$-primary ideal $\mathfrak{a}$ of R corresponds to a unique $\mathfrak{m}'$-primary ideal $\mathfrak{a}'$ of R' via $\mathfrak{a}' = \iota(\widehat{\mathfrak{a}}) \cap R'$. Under this correspondence, it is easy to see $\mathrm{e}(\mathfrak{a}) = \mathrm{e}(\mathfrak{a}')$. By [24, Proposition 2.11] we know that $\mathrm{lct}(X; \mathfrak{a}) = \mathrm{lct}(X'; \mathfrak{a}')$. Hence the statement follows from Theorem 2.7. □

2.3. Bertini-type result for local volumes. In this subsection, we prove the following Bertini-type result for local volumes.

Theorem 2.16. *Let $x \in X$ be a non-isolated n-dimensional klt singularity. Then there exists a non-smooth $(n-1)$-dimensional klt singularity $y \in Y$ such that*

$$\frac{\widehat{\mathrm{vol}}(x, X)}{n^n} \leq \frac{\widehat{\mathrm{vol}}(y, Y)}{(n-1)^{n-1}}.$$

Moreover, Y can be chosen as a general hyperplane section of X.

Before presenting the proof of Theorem 2.16, we need the following result on the local volume of fibrations.

Proposition 2.17. *Let $\pi : \mathcal{X} \to B$ together with a section $\sigma : B \to \mathcal{X}$ be a $\mathbb{Q}$-Gorenstein flat family of klt singularities over a smooth curve B. Then for a general point $b \in B$ we have*

$$\frac{\widehat{\mathrm{vol}}(\sigma(b), \mathcal{X})}{n^n} = \frac{\widehat{\mathrm{vol}}(\sigma(b), \mathcal{X}_b)}{(n-1)^{n-1}}.$$

Proof. By the adjunction of local volumes [57, Theorem 1.7], we always have

$$\frac{\widehat{\mathrm{vol}}(\sigma(b), \mathcal{X})}{n^n} \geq \frac{\widehat{\mathrm{vol}}(\sigma(b), \mathcal{X}_b)}{(n-1)^{n-1}}$$

for any $b \in B$. Thus it suffices to show the reverse inequality for general $b \in B$.

Denote by $\bar{\eta}$ the geometric generic point of B. Then by [73] we know that there for a general point $b \in B$ we have $\widehat{\mathrm{vol}}(\sigma(b), \mathcal{X}_b) = \widehat{\mathrm{vol}}(\sigma(\bar{\eta}), \mathcal{X}_{\bar{\eta}})$. Let us fix an arbitrary $\epsilon > 0$. By Theorem 2.9, there exists a Kollár component $\mathcal{S}_{\bar{\eta}}$ over $\sigma(\bar{\eta}) \in \mathcal{X}_{\bar{\eta}}$ such that

$$\widehat{\mathrm{vol}}_{\mathcal{X}_{\bar{\eta}}, \sigma(\bar{\eta})}(\mathcal{S}_{\bar{\eta}}) \leq \widehat{\mathrm{vol}}(\sigma(\bar{\eta}), \mathcal{X}_{\bar{\eta}}) + \epsilon. \tag{2.1}$$

Denote by $\mu_{\bar{\eta}} : \mathcal{Y}_{\bar{\eta}} \to \mathcal{X}_{\bar{\eta}}$ the plt blow up extracting $\mathcal{S}_{\bar{\eta}}$. Hence there exists an étale morphism $\widetilde{B} \to B$ such that $\mu_{\bar{\eta}}$ extends over $\mathcal{X} \times_B \widetilde{B}$ which provides a flat family of Kollár components (see [54, Definition A.2]). Since the local volume is preserved under étale morphism by Lemma 2.15, we may replace B by $\widetilde{B}$. Thus there is a birational morphism $\mu : \mathcal{Y} \to \mathcal{X}$ which provides a flat family $\mathcal{S}$ of Kollár components over $\mathcal{X}$ centered at $\sigma(B)$. Denote by $\mathfrak{b}_m := \mu_* \mathcal{O}_{\mathcal{Y}}(-m\mathcal{S})$. Since $-\mathcal{S}$ is a μ-ample $\mathbb{Q}$-Cartier divisor, we know that $\mathfrak{b}_m$ is a flat family of ideals over B for $m \gg 1$ (i.e. $\mathcal{O}_{\mathcal{X}}/\mathfrak{b}_m$ is flat over B). After replacing B with a dense open set, we may assume that $\mathfrak{b}_m$ is a flat family of ideals over B for any $m \in \mathbb{Z}_{\geq 0}$. For $p \in \mathbb{R}_{\geq 0}$, we define $\mathfrak{b}_p := \mathfrak{b}_{\lceil p \rceil} = \mu_* \mathcal{O}_{\mathcal{Y}}(-\lceil p\mathcal{S} \rceil)$. Let $\mathcal{S}_b$ and $\mathfrak{b}_{p,b}$ be the restriction of $\mathcal{S}$ and $\mathfrak{b}_p$ on $\mathcal{X}_b$. By flatness of $\mathcal{S}$ and $\mathfrak{b}_\bullet$ over B, we have $\mathfrak{b}_{p,b} = \mathfrak{a}_p(\mathcal{S}_b)$ for any $b \in B$.

Next, we estimate the local volume $\widehat{\mathrm{vol}}(\sigma(b), \mathcal{X})$. Fix a general $b \in B$. Let $t \in \mathcal{O}_{b,B}$ be its uniformizer. For any $s \in \mathbb{R}_{>0}$, consider the ideal sequence $\mathcal{I}_{\bullet,s}$ as

$$\mathcal{I}_{m,s} := \mathfrak{b}_{ms} + \mathfrak{b}_{(m-1)s} t + \mathfrak{b}_{(m-2)s} t^2 + \cdots + \mathfrak{b}_s t^{m-1} + (t^m).$$

By the definition of $\mathfrak{b}_p$ we know that $\mathcal{I}_{\bullet,s}$ is a multiplicative ideal sequence of $\mathcal{O}_{\mathcal{X},\sigma(b)}$ cosupported at $\sigma(b)$. Then we know that

$$\ell(\mathcal{O}_{\mathcal{X},\sigma(b)}/\mathcal{I}_{m,s}) = \sum_{i=1}^{m} \ell(\mathcal{O}_{\mathcal{X}_b,\sigma(b)}/\mathfrak{b}_{is,b}).$$

Since $\ell(\mathcal{O}_{\mathcal{X}_b,\sigma(b)}/\mathfrak{b}_{p,b}) = \frac{1}{(n-1)!} \mathrm{vol}_{\mathcal{X}_b,\sigma(b)}(\mathcal{S}_b) p^{n-1} + O(p^{n-2})$, we know that

$$\ell(\mathcal{O}_{\mathcal{X},\sigma(b)}/\mathcal{I}_{m,s}) = \frac{1}{n!} \mathrm{vol}_{\mathcal{X}_b,\sigma(b)}(\mathcal{S}_b) s^{n-1} m^n + O(m^{n-1}).$$

This implies that

$$\mathrm{e}(\mathcal{I}_{\bullet,s}) = \mathrm{vol}_{\mathcal{X}_b,\sigma(b)}(\mathcal{S}_b) s^{n-1}. \tag{2.2}$$

Let v_s be the valuation of $\mathbb{C}(\mathcal{X})$ as the quasi-monomial combination of $\mathcal{Y}_b$ and $\mathcal{S}$ of weight s and 1, respectively. Then it is clear that

$$A_{\mathcal{X}}(v_s) = s + A_{\mathcal{X}}(\mathcal{S}) = s + A_{\mathcal{X}_b}(\mathcal{S}_b) \quad \text{and} \quad v_s(\mathcal{I}_{m,s}) \geq ms.$$

Hence we have

$$\mathrm{lct}(\mathcal{X}; \mathcal{I}_{\bullet,s}) \leq \frac{A_{\mathcal{X}}(v_s)}{v_s(\mathcal{I}_{\bullet,s})} \leq 1 + s^{-1} A_{\mathcal{X}_b}(\mathcal{S}_b). \tag{2.3}$$

Combining Theorem 2.7, (2.2) and (2.3), we obtain

$$\widehat{\mathrm{vol}}(\sigma(b), \mathcal{X}) \leq \mathrm{lct}(\mathcal{X}; \mathcal{I}_{\bullet,s})^n \cdot \mathrm{e}(\mathcal{I}_{\bullet,s}) \leq (1 + s^{-1} A_{\mathcal{X}_b}(\mathcal{S}_b))^n \cdot \mathrm{vol}_{\mathcal{X}_b,\sigma(b)}(\mathcal{S}_b) s^{n-1}.$$

Since s is arbitrary, we may choose $s = \frac{A_{\mathcal{X}_b}(\mathcal{S}_b)}{n-1}$ which minimizes the right-hand-side of the above inequality. Hence we have

$$\widehat{\mathrm{vol}}(\sigma(b), \mathcal{X}) \leq \frac{n^n}{(n-1)^{n-1}} \widehat{\mathrm{vol}}_{\mathcal{X}_b,\sigma(b)}(\mathcal{S}_b) \leq \frac{n^n}{(n-1)^{n-1}} (\widehat{\mathrm{vol}}(\sigma(\bar{\eta}), \mathcal{X}_{\bar{\eta}}) + \epsilon).$$

Here the second inequality follows from (2.1) and

$$\widehat{\mathrm{vol}}_{\mathcal{X}_b,\sigma(b)}(\mathcal{S}_b) = \widehat{\mathrm{vol}}_{\mathcal{X}_{\bar{\eta}},\sigma(\bar{\eta})}(\mathcal{S}_{\bar{\eta}})$$

by flatness of $\mathcal{S}$ and $\mathfrak{b}_\bullet$ over B. By [73] we know that

$$\widehat{\mathrm{vol}}(\sigma(\bar{\eta}), \mathcal{X}_{\bar{\eta}}) = \widehat{\mathrm{vol}}(\sigma(b), \mathcal{X}_b)$$

for general b. Hence by letting $\epsilon \to 0$, we prove the reverse inequality that

$$\frac{\widehat{\mathrm{vol}}(\sigma(b), \mathcal{X})}{n^n} \leq \frac{\widehat{\mathrm{vol}}(\sigma(b), \mathcal{X}_b)}{(n-1)^{n-1}}$$

for general $b \in B$. The proof is finished. □

Proof of Theorem 2.16. For simplicity, we assume that X is affine. Let $C \subset X_{\mathrm{sing}}$ be an integral curve through x. Let $\pi : X \to \mathbb{A}^1$ be a general linear projection. Then $\pi|_C : C \to \mathbb{A}^1$ is quasi-finite. Let $y \in C$ be a general point. Let Y be the fiber of π containing y. Then after taking base change of π to the normalization of C, Lemma 2.15 and Proposition 2.17 implies that

$$\frac{\widehat{\mathrm{vol}}(y, X)}{n^n} = \frac{\widehat{\mathrm{vol}}(y, Y)}{(n-1)^{n-1}}.$$

Since $y \in C$ is general, we have $\widehat{\mathrm{vol}}(x, X) \leq \widehat{\mathrm{vol}}(y, X)$ by [15]. This finishes the proof. □

Corollary 2.18. *Let $x \in X$ be a non-smooth klt singularity of dimension $n \geq 3$. Assume that* $\dim_x X_{\mathrm{sing}} \geq n-3$. *Then*

$$\frac{\widehat{\mathrm{vol}}(x, X)}{n^n} \leq \frac{16}{27}.$$

In particular, Conjecture 2.10 *holds for $x \in X$.*

Proof. We will focus on the first inequality $\frac{\widehat{\mathrm{vol}}(x,X)}{n^n} \leq \frac{16}{27}$, as the statement on Conjecture 2.10 is a consequence of this inequality by a simple computation $\frac{16}{27}n^n < 2(n-1)^n$ whenever $n \geq 4$, and the conjecture holds in dimension 3 by Theorem 2.11 (1).

We do induction on $n \geq 3$. When $n = 3$, the first inequality is precisely Theorem 2.11 (1). Assume that the first inequality is true in dimension $n-1$ with $n \geq 4$. Let $x \in X$ be a non-smooth klt singularity of dimension n with $\dim_x X_{\mathrm{sing}} \geq n-3$. Let V be an irreducible component of X_{sing} such that $x \in V$ and $\dim V \geq n-3$. Let $C \subset V$ be an integral curve through x. Then the proof of Theorem 2.16 implies that there exists a hyperplane section $Y \subset X$ and a closed point $y \in Y \cap C$ such that

$$\frac{\widehat{\mathrm{vol}}(x, X)}{n^n} \leq \frac{\widehat{\mathrm{vol}}(y, Y)}{(n-1)^{n-1}}.$$

Furthermore, since $Y_{\mathrm{sing}} \supseteq X_{\mathrm{sing}} \cap Y \supseteq V \cap Y$, we know that

$$\dim_y Y_{\mathrm{sing}} \geq \dim_y (V \cap Y) \geq \dim_y V - 1 \geq n-4.$$

By induction hypothesis, we have $\frac{\widehat{\mathrm{vol}}(y,Y)}{(n-1)^{n-1}} \leq \frac{16}{27}$. Hence we have

$$\frac{\widehat{\mathrm{vol}}(x, X)}{n^n} \leq \frac{\widehat{\mathrm{vol}}(y, Y)}{(n-1)^{n-1}} \leq \frac{16}{27}.$$

Thus the proof is finished. □

3. Local-to-global volume estimates

In order to prove our main result Theorem 1.1, we follow the strategy from [54, 66], that is, to show that any K-semistable $\mathbb{Q}$-Gorenstein limits of cubic hypersurfaces is again a cubic hypersurface. The following result on K-semistable degeneration of higher dimensional cubic hypersurfaces is an easy consequence of arguments therein.

Theorem 3.1. *Let $n \geq 4$ be an integer. Let $\mathcal{X} \to B$ be a K-semistable $\mathbb{Q}$-Fano family over a smooth pointed curve $0 \in B$ such that over $B^\circ := B \setminus \{0\}$ it is a smooth family of cubic n-folds. Then there exists a $\mathbb{Q}$-Cartier integral Weil divisor class L on $X := \mathcal{X}_0$ such that the following properties hold:*

(1) *$\mathcal{O}_X(mL)$ is Cohen–Macaulay for any $m \in \mathbb{Z}$.*

(2) *$-K_X \sim (n-1)L$ and $(L^n) = 3$.*

(3) *$h^i(X, \mathcal{O}_X(mL)) = h^i(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}(m))$ for any $m \in \mathbb{Z}$, $i \geq 0$, and $b \in B^\circ$. Moreover, one has $h^0(X, \mathcal{O}_X(L)) = n+2$, and $h^j(X, \mathcal{O}_X(mL)) = 0$ for any $m \in \mathbb{Z}$ and $1 \leq j \leq n-1$.*

(4) *Any $\mathbb{Q}$-Cartier Weil divisor D on X satisfies that $2D$ is Cartier. In particular, $2L$ is Cartier.*

Proof. Denote by $\mathcal{X}^\circ := \mathcal{X} \setminus \mathcal{X}_0$. By base change to a finite cover of B, we can find a hyperplane section $\mathcal{L}^\circ \sim_B \mathcal{O}_{\mathcal{X}^\circ}(1)$ and taking Zariski closure yields a Weil divisor $\mathcal{L}$ on $\mathcal{X}$. It is clear that

$$-K_{\mathcal{X}^\circ/B^\circ} \sim_{B^\circ} (n-1)\mathcal{L}^\circ.$$

Since $\mathcal{X}_0 \sim_B 0$ is integral, we know that

$$-K_{\mathcal{X}/B} \sim_B (n-1)\mathcal{L}$$

which implies that $\mathcal{L}$ is $\mathbb{Q}$-Cartier. By assumption $\mathcal{X}$ has klt singularities, so [41, Corollary 5.25] implies that $\mathcal{O}_{\mathcal{X}}(m\mathcal{L})$ is Cohen–Macaulay for any $m \in \mathbb{Z}$. Thus $L := \mathcal{L}|_{\mathcal{X}_0}$ is a $\mathbb{Q}$-Cartier Weil divisor satisfying that

$$\mathcal{O}_{\mathcal{X}_0}(mL) \cong \mathcal{O}_{\mathcal{X}}(mL) \otimes \mathcal{O}_{\mathcal{X}_0}$$

is Cohen–Macaulay for any $m \in \mathbb{Z}$, and $-K_X \sim (n-1)L$. The fact that $(L^n) = 3$ comes from $(\mathcal{O}_{\mathcal{X}_b}(1)^n) = 3$ and $\mathcal{L}$ is $\mathbb{Q}$-Cartier. Hence we have shown (1) and (2).

For part (3), notice that $\mathcal{O}_{\mathcal{X}}(m\mathcal{L})$ is flat over B for any $m \in \mathbb{Z}$ whose fiber over b and 0 are $\mathcal{O}_{\mathcal{X}_b}(m)$ and $\mathcal{O}_X(mL)$ respectively. If $m \geq 2-n$, then $mL - K_X \sim (m+n-1)L$ and $m\mathcal{L}_b - K_{\mathcal{X}_b} \sim \mathcal{O}_{\mathcal{X}_b}(m+n-1)$ are both ample. Hence Kawamata–Viehweg vanishing implies that

$$H^i(X, \mathcal{O}_X(mL)) = H^i(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}(m)) = 0 \quad \text{for any } i \geq 1 \text{ and } m \geq 2-n.$$

By the flatness of $\mathcal{O}_{\mathcal{X}}(m\mathcal{L})$, we know that

$$h^0(X, \mathcal{O}_X(mL)) = h^0(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}(m)) \quad \text{for any } m \geq 2-n.$$

On the other hand, by Serre duality for CM sheaves [41, Theorem 5.71], we know that

$$H^i(X, \mathcal{O}_X(mL)) = H^i(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}(m)) = 0 \quad \text{for any } i \leq n-1$$

and $h^n(X, \mathcal{O}_X(mL)) = h^n(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}(m))$ whenever $-mL$ and $-m\mathcal{L}_b$ are ample, i.e. $m \leq -1$. Thus part (3) is proven.

For part (4), we use the local volume estimates. Let $x \in X$ be a point where D is not Cartier. Denote by $\mathrm{ind}(x, D)$ the Cartier index of D at x. Then by Theorems 2.13, 2.11 (2), and 2.14, we know that

$$\frac{3n^n(n-1)^n}{(n+1)^n} = \frac{n^n}{(n+1)^n}(-K_X)^n \leq \widehat{\mathrm{vol}}(x, X) \leq \frac{n^n}{\mathrm{ind}(x, D)}.$$

In particular, we get $\mathrm{ind}(x, D) \leq \frac{(n+1)^n}{3(n-1)^n} < 3$. This implies that $\mathrm{ind}(x, D) = 2$. □

From now on, we restrict our focus to cubic fourfolds, i.e. $n = 4$. We will always denote by X a K-semistable $\mathbb{Q}$-Fano variety admitting a $\mathbb{Q}$-Gorenstein smoothing to cubic fourfolds. By [54, 66], to prove Theorem 1.1 the main challenge is to show that L is Cartier on X. This would follow from Conjecture 2.10 in dimension 4 as indicated by Proposition 3.3. However, currently we are unable to confirm Conjecture 2.10 for $n \geq 4$. In the below, we provide some partial results using the local volume estimates from Section 2. We will study the global geometry of (X, L) in Section 4.

Proposition 3.2. *With the above notation, we have that L is Cartier away from a finite subset $\Sigma \subset X$. Moreover, any $x \in \Sigma$ is an isolated singularity of X.*

Proof. Let Σ be the closed subset of X, where L is not Cartier. Clearly, $\Sigma \subset X_{\mathrm{sing}}$. First of all, assume to the contrary that Σ contains a curve C. Let $x \in C$ be a point. Let $\tau : (\tilde{x} \in \widetilde{X}) \to (x \in X)$ be the index 1 cover with respect to L. Let $\widetilde{C} := \pi^{-1}(C)$. Then by finite degree formula we know that

$$\frac{\widehat{\mathrm{vol}}(\tilde{x}', \widetilde{X})}{4^4} \geq \frac{2 \cdot 3^5}{5^4}$$

for any $\tilde{x}' \in \widetilde{C}$. If $\widetilde{X}$ is singular along $\widetilde{C}$, then by Corollary 2.18 we know that

$$\frac{\widehat{\mathrm{vol}}(\tilde{x}', \widetilde{X})}{4^4} \leq \frac{16}{27}.$$

This is a contradiction since $\frac{2\cdot 3^5}{5^4} > \frac{16}{27}$. Hence $\widetilde{X}$ is smooth at the generic point of $\widetilde{C}$. This implies that $(x' \in X)$ is a quotient singularity of order 2 for a general point $x' \in C$. Since C is contained in the ramification locus of τ, we know that $(x' \in X)$ has type $\frac{1}{2}(1,1,1,0)$ or $\frac{1}{2}(1,1,0,0)$.

Next, we will show that neither quotient type is possible. The argument is similar to [54, proof of Lemma 3.16]. If $(x' \in X)$ has type $\frac{1}{2}(1,1,1,0)$, we pick a general hyperplane section $\mathcal{H}$ through x' of $\mathcal{X}$ embedded in some projective space. Then clearly $\mathcal{H}_0 = \mathcal{H} \cap X$ has a quotient singularity of type $\frac{1}{2}(1,1,1)$, while $\mathcal{H}_b = \mathcal{H} \cap \mathcal{X}_b$ is smooth for $b \in B^\circ$. This contradicts the rigidity theorem of Schlessinger [65]. If $(x' \in X)$ has type $\frac{1}{2}(1,1,0,0)$, then we know that X has hypersurface singularities near x', and so does $\mathcal{X}$. We pick two general hyperplane sections $\mathcal{H}_1$ and $\mathcal{H}_2$ through x' of $\mathcal{X}$. Then clearly $(x' \in \mathcal{H}_1 \cap \mathcal{H}_2)$ is a normal isolated hypersurface singularity of dimension 3. Since $\mathcal{O}_{\mathcal{X}}(\mathcal{L})$ is Cohen–Macaulay, there is a well-defined $\mathbb{Q}$-Cartier Weil divisor class $\mathcal{L}|_{\mathcal{H}_1 \cap \mathcal{H}_2}$ such that

$$\mathcal{O}_{\mathcal{H}_1 \cap \mathcal{H}_2}(\mathcal{L}|_{\mathcal{H}_1 \cap \mathcal{H}_2}) \cong \mathcal{O}_{\mathcal{X}}(\mathcal{L}) \otimes \mathcal{O}_{\mathcal{H}_1 \cap \mathcal{H}_2}.$$

By the local Grothendieck–Lefschetz theorem [64], the local class group of $(x' \in \mathcal{H}_1 \cap \mathcal{H}_2)$ is torsion free which implies that $\mathcal{L}|_{\mathcal{H}_1 \cap \mathcal{H}_2}$ is Cartier at x'. Hence this implies that $\mathcal{L}$ is Cartier at x', which implies L is also Cartier at x'. This is a contradiction.

Finally, we show that Σ consists only of isolated singularities. Assume to the contrary that $x \in \Sigma$ is not isolated. Let $C' \subset X_{\mathrm{sing}}$ be a curve through x. Again, let $(\tilde{x} \in \widetilde{X})$ be the index one cover of $(x \in X)$ with respect to L. Since L is Cartier at the generic point of C', we know that $\widetilde{C}' := \pi^{-1}(C')$ is contained in $\widetilde{X}_{\mathrm{sing}}$. Hence by Theorems 2.13, 2.14, and Corollary 2.18 we know that

$$\frac{2 \cdot 3^5}{5^4} \le \frac{\widehat{\mathrm{vol}}(\tilde{x}, \widetilde{X})}{4^4} \le \frac{16}{27},$$

again a contradiction. □

Proposition 3.3. *With the above notation, if L is not Cartier at $x \in X$, then the index* 1 *cover $\tilde{x} \in \widetilde{X}$ violates Conjecture* 2.10.

Proof. By Theorems 2.13 and 2.14, we have

$$\frac{3^5 \cdot 4^4}{5^4} = \frac{4^4}{5^4}(-K_X)^4 \le \widehat{\mathrm{vol}}(x, X) = \frac{\widehat{\mathrm{vol}}(\tilde{x}, \widetilde{X})}{\mathrm{ind}(x, L)}.$$

Since $2L$ is Cartier at $x \in X$ by Theorem 3.1, we know that $\mathrm{ind}(x, L) = 2$. Hence

$$\widehat{\mathrm{vol}}(\tilde{x}, \widetilde{X}) \ge \frac{2 \cdot 3^5 \cdot 4^4}{5^4} > 2 \cdot 3^4.$$

If $\tilde{x} \in \widetilde{X}$ is smooth, then Proposition 3.2 implies that $\tau : \widetilde{X} \to X$ is ramified only at x which implies that $(x \in X)$ is an isolated quotient singularity of order 2 admitting a $\mathbb{Q}$-Gorenstein smoothing. This contradicts [65]. Hence $\tilde{x} \in \widetilde{X}$ violates Conjecture 2.10. □

4. Ambro–Kawamata non-vanishing approach

In this section, we use the following non-vanishing theorem of Ambro [3, Main Theorem] and Kawamata [38, Theorem 5.1] to study the geometry of K-semistable $\mathbb{Q}$-Gorenstein limits of cubic fourfolds.

Theorem 4.1 ([3,38]). *Let (Y, Δ) be a projective klt pair. Let M be a nef Cartier divisor over Y such that $M - K_Y - \Delta$ is nef and big. Assume that there exists a rational number $r > \dim(Y) - 3 \ge 0$ such that $-K_Y - \Delta \sim_{\mathbb{Q}} rM$. Then $H^0(Y, M) \neq 0$, and for a general member $D \in |M|$ the pair $(Y, \Delta + D)$ is plt.*

In the rest of this paper, we adapt the notation of Theorem 3.1 and assume $n = 4$. In particular, X is a K-semistable $\mathbb{Q}$-Gorenstein limit of cubic fourfolds, and L is an ample $\mathbb{Q}$-Cartier Weil divisor on X such that $-K_X \sim 3L$. Denote by Σ the non-Cartier locus of L on X which is a finite set by Proposition 3.2.

Next, we apply Ambro–Kawamata's non-vanishing theorem to our study on the geometry of linear systems $|2L|$ and $|L|$. Our goal is to show that L is Cartier on X.

Proposition 4.2. *Let D_1, D_2 be two general member of $|2L|$ on X. Then both D_i $(i = 1, 2)$ and their complete intersection $S := D_1 \cap D_2$ are Gorenstein canonical. Moreover,* $\mathrm{Bs}\,|2L|$ *is disjoint from* Σ.

Proof. We first show that both D_i and S are klt of Gorenstein index at most 2. By Theorem 3.1, we know that $2L$ is Cartier and ample. By applying Theorem 4.1 to $2L$ on X, we see that $-K_X \sim_{\mathbb{Q}} \frac{3}{2}(2L)$, so $r = \frac{3}{2} > 1 = \dim(X) - 3 \geq 0$. So we have that (X, D_i) is plt, hence D_i is klt. Next, we apply Theorem 4.1 to $2L|_{D_i}$ on D_i. By adjunction it is clear that $-K_{D_i} \sim L|_{D_i}$, so $r = \frac{1}{2} \geq 0$. Also, by Theorem 3.1 (3) we have an exact sequence

$$H^0(X, 2L) \to H^0(D_i, 2L|_{D_i}) \to H^1(X, \mathcal{O}_X) = 0.$$

Hence the general divisor $D_2 \in |2L|$ restricts to a general divisor $S \in |2L|_{D_1}|$. In particular, S is also klt. By adjunction, we know that $K_S \sim L|_S$, so both D_i and S have Gorenstein index at most 2.

Next we show that S is Gorenstein. Assume to the contrary that $x \in S$ has Gorenstein index 2. Then clearly $x \in \Sigma$. Let $\tau : (\tilde{x} \in \widetilde{X}) \to (x \in X)$ be the index 1 cover of L. Since $\widetilde{X}$ is Gorenstein, the preimage $\widetilde{S} := \tau^{-1}(S)$ is also Gorenstein as it is a complete intersection in $\widetilde{X}$. Thus $\tilde{x} \in \widetilde{S}$ is a Du Val singularity. This implies that $\mathrm{edim}(\tilde{x}, \widetilde{X}) \leq 5$, i.e. $\tilde{x} \in \widetilde{X}$ is a hypersurface singularity. But then the ODP conjecture holds for $\tilde{x} \in \widetilde{X}$ by Theorem 1.3, and we get a contradiction to Proposition 3.3. This shows that S is Gorenstein.

Since both D_i and S are Cohen–Macaulay, we know that

$$\mathcal{O}_X(L) \otimes \mathcal{O}_S \cong \mathcal{O}_S(L|_S) \cong \omega_S.$$

In particular, this implies that $S \cap \Sigma = \emptyset$. It is clear that $\mathrm{Bs}\,|2L| \subset S$, so $\mathrm{Bs}\,|2L| \cap \Sigma = \emptyset$. □

Proposition 4.3. *Let $D \in |2L|$ and $H \in |L|$ be general divisors on X. Let $G := D \cap H$ be their complete intersection. Then $(G, L|_G)$ is a polarized K3 surface with Du Val singularities of degree* 6. *Moreover, $|2L|$ is base point free, and the connected components of* $\mathrm{Bs}\,|L|$ *have dimension* 0 *or* 2.

Proof. By Proposition 4.2, we know that $L|_D \sim -K_D$ is Cartier. Since $\mathcal{O}_X(-L)$ and $\mathcal{O}_X(L)$ are both Cohen–Macaulay by Theorem 3.1 (1), we know that

$$\mathcal{O}_D(L|_D) \cong \mathcal{O}_X(L)/\mathcal{O}_X(-L).$$

Hence we have an exact sequence $H^0(X, L) \to H^0(D, L|_D) \to H^1(X, -L) = 0$ by Theorem 3.1 (3). Hence G is a general divisor in $|L|_D|$ which implies that (D, G) is plt by Theorem 4.1. Since $G \sim L|_D$ is Cartier and D is Gorenstein canonical, we know that G has Gorenstein canonical singularities as well. By adjunction, we have $K_G \sim 0$. We claim that $H^1(G, \mathcal{O}_G) = 0$. Since $L|_D$ is Cartier, there is an exact sequence

$$H^1(D, \mathcal{O}_D) \to H^1(G, \mathcal{O}_G) \to H^2(D, -L|_D).$$

Thus it suffices to show that both $H^1(D, \mathcal{O}_D)$ and $H^2(D, -L|_D)$ vanish. As D is Gorenstein canonical, we have

$$\mathcal{O}_D(mL|_D) \cong \mathcal{O}_X(mL) \otimes \mathcal{O}_D.$$

Hence we have the following exact sequences for $i = 1, 2$:

$$0 = H^i(X, mL) \to H^i(D, mL|_D) \to H^{i+1}(X, (m-2)L) = 0.$$

Here we use Theorem 3.1 (3). Thus both $H^1(D, \mathcal{O}_D)$ and $H^2(D, -L|_D)$ vanish, and the claim follows. Hence G is a K3 surface with Du Val singularities. The polarization $L|_G$ has degree 6 since $(L|_G^2) = (L^2 \cdot D \cdot H) = 2(L^4) = 6$.

Since $\mathrm{Bs}\,|2L| \subset \mathrm{Bs}\,|L|$, we know that $\mathrm{Bs}\,|2L| \subset G$. Moreover, we can show that

$$H^0(X, 2L) \to H^0(G, 2L|_G)$$

is surjective by tracing exact sequences as below and using Theorem 3.1 (3):

$$H^0(X, 2L) \to H^0(D, 2L|_D) \to H^1(X, \mathcal{O}_X) = 0$$

and

$$H^0(D, 2L|_D) \to H^0(G, 2L|_G) \to H^1(D, L|_D) = 0.$$

Hence we have $\mathrm{Bs}\,|2L| = \mathrm{Bs}\,|2L|_G|$. By classical result on linear system of K3 surfaces (see [36, Remark 3.4]), we know that $2L|_G$ is always base point free. Hence $|2L|$ is also base point free.

By similar argument on tracing exact sequences, we know that

$$H^0(X, 2L) \to H^0(H, 2L|_H)$$

is surjective. Since G is a general divisor of the ample base point free linear system $|2L|_H|$ on H, we know that H is integral which implies that $|L|$ has no fixed component. By tracing exact sequences as above, we know that $H^0(X, L) \to H^0(H, L|_H) \to H^0(G, L|_G)$ are both surjective. By Mayer's theorem [59] (see also [36, Corollary 3.15]), we know that $L|_G$ is either base point free or has a fixed component isomorphic to $\mathbb{P}^1$. Thus any connected component of $\mathrm{Bs}\,|L| = \mathrm{Bs}\,|L|_H|$ is either an isolated point or a surface. □

Proposition 4.4. *Let $H \in |L|$ be a general divisor. Then H is Gorenstein log canonical. Moreover, H admits a weakly special test configuration with central fiber isomorphic to the projective cone $C_p(G, 2L|_G)$.*

Proof. By the proof of Proposition 4.3, we know that H is integral. Since the ideal sheaf of H in X is $\mathcal{O}_X(-L)$, which is Cohen–Macaulay by Theorem 3.1 (1), we know that H is Cohen–Macaulay as well. Since a general member G of the base point free linear system $|2L|_H|$ is normal, we know that H is R_1 hence normal as well. By adjunction, we have that $K_H = (K_X + H)|_H \sim -2L|_H$ is Cartier. Hence H is Gorenstein normal.

Next we construct the weakly special test configuration. The idea is by degeneration to the normal cone of G. Let $R = \bigoplus_{m=0}^{\infty} R_m := H^0(H, 2mL|_H)$ be the section ring of $(H, 2L|_H)$. Consider the $\mathbb{N}$-filtration $\mathcal{F}$ on R (see e.g. [16, Section 2.3] for backgrounds) as

$$\mathcal{F}^p R_m := H^0(H, 2mL|_H - pG) \subset H^0(H, 2mL|_H) = R_m \quad \text{if } p \in \mathbb{Z}_{\geq 0}.$$

For $p \in \mathbb{Z}_{<0}$ we define $\mathcal{F}^p R_m = R_m$. Since $G \sim 2L|_H$, it is clear that $\mathcal{F}^\bullet R$ is a multiplicative, linearly bounded, finitely generated $\mathbb{N}$-filtration of R. Denote

$$\mathcal{H} := \mathrm{Proj}_{\mathbb{A}^1} \bigoplus_{m=0}^{+\infty} \bigoplus_{p=-\infty}^{+\infty} t^{-p} \mathcal{F}^p R_m,$$

where t is the parameter of $\mathbb{A}^1$, and the grading of $\mathcal{F}^p R_m$ is m. Then $(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(1)) \to \mathbb{A}^1$ is a test configuration of $(H, 2L|_H)$. The central fiber $\mathcal{H}_0$ is given by

$$\mathcal{H}_0 = \operatorname{Proj} \bigoplus_{m=0}^{+\infty} \bigoplus_{p=-\infty}^{+\infty} \mathcal{F}^p R_m / \mathcal{F}^{p+1} R_m.$$

It is clear that $\mathcal{F}^p R_m/\mathcal{F}^{p+1} R_m = 0$ for $p < 0$. Hence to show $\mathcal{H}_0 \cong C_p(G, 2L|_G)$ it suffices to show that $\mathcal{F}^p R_m/\mathcal{F}^{p+1} R_m \cong H^0(G, 2(m-p)L|_G)$ for $p \geq 0$. By tracing the exact sequence

$$\begin{aligned} 0 &\to H^0(H, 2mL|_H - (p+1)G) \to H^0(H, 2mL|_H - pG) \to H^0(G, 2(m-p)L|_G) \\ &\to H^1(H, 2mL|_H - (p+1)G) \cong H^1(H, 2(m-p-1)L|_H), \end{aligned}$$

it suffices to show that $H^1(H, 2qL|_H) = 0$ for any $q \in \mathbb{Z}$. This follows from the following exact sequence and Theorem 3.1 (3):

$$0 = H^1(X, 2qL) \to H^1(H, 2qL|_H) \to H^2(X, (2q-1)L) = 0.$$

Hence we have shown $\mathcal{H}_0 \cong C_p(G, 2L|_G)$. Since G is a K3 surface with canonical singularities, we know that $C_p(G, 2L|_G)$ is log canonical by [39, Lemma 3.1]. By inversion of adjunction, we know that $(\mathcal{H}, \mathcal{H}_0)$ is log canonical near $\mathcal{H}_0$, which implies that $(\mathcal{H}, \mathcal{H}_0)$ is log canonical as $\mathcal{H} \setminus \mathcal{H}_0 \cong H \times (\mathbb{A}^1 \setminus \{0\})$. Hence H is log canonical, and $\mathcal{H}$ is a weakly special test configuration of H. The proof is finished. □

Next, we divide the argument into cases based on the geometry of the polarized K3 surface $(G, L|_G)$, where G is a general complete intersection of $D \in |2L|$ and $H \in |L|$. By Mayer's theorem [59] (see also [36, Remark 3.8 and Corollary 3.15]), there are three cases based on the behavior of the linear system $|L|_G|$:

(1) (*unigonal*) $|L|_G|$ has a base curve $C_0 \cong \mathbb{P}^1$, $|L|_G - C_0|$ is base point free, and the map $\phi_{|L|_G - C_0|} : G \to \mathbb{P}^4$ is an elliptic fibration over a quartic rational normal curve.

(2) (*hyperelliptic*) $|L|_G|$ is base point free, and $\phi_{|L|_G|} : G \to \mathbb{P}^4$ is a double cover onto a non-degenerate rational surface of degree 3 in $\mathbb{P}^4$.

(3) (*complete intersection*) $|L|_G|$ is very ample, and the map $\phi_{|L|_G|} : G \hookrightarrow \mathbb{P}^4$ embeds G as a $(2,3)$-complete intersection in $\mathbb{P}^4$.

The next result shows that $(G, L|_G)$ cannot be unigonal.

Proposition 4.5. *The polarized K3 surface $(G, L|_G)$ is not unigonal. In particular,* $\operatorname{Bs}|L|$ *is a finite set.*

Proof. Assume to the contrary that $(G, L|_G)$ is unigonal of degree 6. Then we know that $L|_G \sim 4C_1 + C_0$, where $C_0 \cong \mathbb{P}^1$ is a (-2)-curve, and C_1 is a general fiber of the elliptic fibration $G \to \mathbb{P}^1$ induced by $|L|_G - C_0|$. In the proof of Proposition 4.3 we have shown that $H^0(X, L) \to H^0(H, L|_H) \to H^0(G, L|_G)$ are both surjective. Indeed, by the following exact sequence

$$0 = H^0(H, -L|_H) \to H^0(H, L|_H) \to H^0(G, L|_G) \to H^1(H, \mathcal{O}_H) = 0,$$

we know that $H^0(H, L|_H) \cong H^0(G, L|_G)$.

Next, we resolve the birational map $\phi_{|L|} : X \dashrightarrow \mathbb{P}^5$ as follows:

$$\begin{array}{ccccc} & & X' & & \\ & \overset{\pi}{\swarrow} & & \overset{\rho}{\searrow} & \\ X & & \overset{\phi_{|L|}}{\dashrightarrow} & & \mathbb{P}^5. \end{array}$$

Here X' is the normalization of the graph of $\phi_{|L|}$. From the above discussion, we know that for a general divisor $H \in |L|$, the image $\rho(\pi_*^{-1}H)$ is a general hyperplane section of $W := \rho(X')$. From the above surjectivity between H^0's, we know that the restrictions of $\phi_{|L|}$ to H and G are $\phi_{|L|_H|}$ and $\phi_{|L|_G|}$ respectively. We claim that $\rho(\pi_*^{-1}H)$ is a curve.

Assume to the contrary that $\dim(\rho(\pi_*^{-1}H)) \geq 2$. Denote by $|L|_H| = E + \Lambda_H$, where E is the base component and Λ_H is movable. Then it is clear that $\rho(\pi_*^{-1}H) = \phi_{\Lambda_H}(H)$. Since $\dim(\phi_{\Lambda_H}(H)) \geq 2$, Bertini's theorem implies that a general member $F \in \Lambda_H$ is an integral surface. Since G is ample on H, we know that $F \cap G$ is connected. Since $\mathrm{Bs}\,|L|_G| = C_0$ and G is a general member of the base point free linear system $|2L|_H|$, we know that $E|_G = C_0$. Hence $F|_G$ is a general member of $|L|_G - C_0|$ which is the sum of four distinct elliptic fibers. In particular, $F \cap G$ is disconnected. This is a contradiction. Thus the claim is proved.

Since $\rho(\pi_*^{-1}H)$ is a curve, it is the same as $\phi_{|L|_G|}(G)$ which is a rational normal curve of degree 4 in $\mathbb{P}^4$. Since $\rho(\pi_*^{-1}H)$ is a hyperplane section of W, we know that $W \subset \mathbb{P}^5$ is a non-degenerate surface of degree 4. By the classification of minimal degree varieties (see e.g. [28]), we know that W is isomorphic to either $\mathbb{P}(1,1,4)$ (cone over a quartic rational normal curve, $\phi_{|\mathcal{O}(4)|} : \mathbb{P}(1,1,4) \hookrightarrow \mathbb{P}^5$), $\mathbb{F}_{2,1}$ ($\phi_{|3f+e|} : \mathbb{F}_2 \hookrightarrow \mathbb{P}^5$, where f and e are a fiber and the negative section respectively), $\mathbb{F}_{0,2}$ ($\phi_{|\mathcal{O}(1,2)|} : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$), or the Veronese surface V_4 ($\phi_{|\mathcal{O}(2)|} : \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$). In each case, there exists a family $\{\mathcal{C}_t\}$ of non-reduced divisors in the linear system $|\mathcal{O}_W(1)|$ that covers W. Choose a general divisor $\mathcal{C}_t$, then we know $\pi_*\rho^*\mathcal{C}_t$ is a non-reduced divisor in $|L|$. Since $-K_X \sim 3L$, we know that $\alpha(X) \leq \frac{1}{6}$ which implies that X is K-unstable by Theorem 2.5. This is a contradiction. The conclusion on $\mathrm{Bs}\,|L|$ follows from the previous discussion since $|L|_G|$ is base point free in the non-unigonal cases. □

Next we treat the hyperelliptic case.

Proposition 4.6. *Assume $(G, L|_G)$ is hyperelliptic. Then X is K-unstable.*

Proof. We resolve the rational map $\phi_{|L|} = \rho \circ \pi^{-1}$ by $X \overset{\pi}{\leftarrow} X' \overset{\rho}{\rightarrow} W$, where X' is the normalization of the graph of $\phi_{|L|}$. Denote by $\pi^*|L| = \frac{1}{2}E + \Lambda$, where E is an effective Weil divisor on X' and Λ is base point free. Let $L' := \pi^*L - \frac{1}{2}E$ which is semiample on X'. Since $\rho(\pi_*^{-1}H)$ is a hyperplane section of W, and $\rho|_{G'}$ is a double cover for general G and $G' := \pi^{-1}(G)$, we know that $\dim(W) \geq 3$. We first show that $\dim(W) = 3$.

Assume to the contrary that $\dim(W) = 4$, i.e. ρ is generically finite. Since $\mathrm{Bs}\,|L|$ is a finite set, we know that $(\pi^*L \cdot E) = 0$ as a 2-cycle. Hence

$$\deg(W) \cdot \deg(\rho) = (L'^4) \leq (L'^3 \cdot \pi^*L) = ((\pi^*L - \tfrac{1}{2}E)^3 \cdot \pi^*L) = ((\pi^*L)^4) = 3.$$

Since W is non-degenerate, we have $\deg(W) \geq 2$ which implies that $\deg(\rho) = 1$, i.e. ρ is birational. Let $H' := \pi_*^{-1}H$ be a general divisor in Λ. Then we know that $\rho|_{H'}$ is also birational. However, $\rho|_{G'}$ has degree 2 for a general G' in the base point free linear system $(\pi^*|2L|)|_{H'}$. This is a contradiction. Thus we have $\dim(W) = 3$.

Next, we analyze the geometry of W. Notice that since $(G, L|_G) \cong (G', (\pi^*L)|_{G'})$ has degree 6, the image $\rho(G')$ is an integral surface of degree 3. Since $\dim(W) = 3$, we know that $\rho(H')$ is an integral surface for general $H' \in \Lambda$, hence $\rho(H') = \rho(G')$ has degree 3. This implies that $\deg(W) = 3$ as well. Now W is a non-degenerate threefold in $\mathbb{P}^5$ of degree 3. By [28], there are three possibilities of W: $\mathbb{P}(1,1,3,3)$ (a second iterated cone over a twisted cubic curve, $\phi_{|\mathcal{O}(3)|} : \mathbb{P}(1,1,3,3) \hookrightarrow \mathbb{P}^5$), the cone over $\mathbb{F}_{1,1}$ (the cone over the image of $\phi_{|2f+e|} : \mathbb{F}_1 \hookrightarrow \mathbb{P}^4$), or $\mathbb{P}^1 \times \mathbb{P}^2$ ($\phi_{|\mathcal{O}(1,1)|} : \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$). In the first two cases, W is covered by non-reduced hyperplane sections, which implies that $|L|$ contains a non-reduced element by similar arguments to the proof of Proposition 4.5. This shows $\alpha(X) \le \frac{1}{6}$ which implies that X is K-unstable by Theorem 2.5.

The only case left is when $(W, \mathcal{O}_W(1)) \cong (\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(1,1))$. We will show that this case cannot occur. Our argument is inspired by [25, Section 6].[1)] From the proof of Proposition 4.3, we see that $\phi_{|L|}$ restricted to D is the finite morphism $\phi_{|L|_D|} : D \to W$ as a double cover. Thus $\phi^*_{|L|_D|} : \mathrm{Pic}(W) \hookrightarrow \mathrm{Pic}(D)$ is an injection, which implies that

$$\mathrm{rk}\,\mathrm{Pic}(D) \ge \mathrm{rk}\,\mathrm{Pic}(W) = 2.$$

Since X admits a $\mathbb{Q}$-Gorenstein smoothing $f : \mathcal{X} \to B$ with $\mathcal{X}_0 \cong X$, where $\mathcal{X}_b$ is a smooth cubic fourfold for any $b \in B^\circ$, we know that $f_*\mathcal{O}_{\mathcal{X}}(2\mathcal{L})$ is flat by Theorem 3.1 (3). Hence after base change to a quasi-finite holomorphic map $\mathbb{D} \to B$ from the unit disc $\mathbb{D} \subset \mathbb{C}$, we can find a $\mathbb{Q}$-Gorenstein smoothing $\mathcal{D} \to \mathbb{D}$ of $D \cong \mathcal{D}_0$ such that $\mathcal{D}_t$ is a smooth $(2,3)$-complete intersection in $\mathbb{P}^5$ for any $t \in \mathbb{D}^\circ := \mathbb{D} \setminus \{0\}$. By Proposition 4.2 we know that $\mathcal{D}_0$ is a $\mathbb{Q}$-Fano variety with Gorenstein canonical singularity. Hence by Kawamata–Viehweg vanishing we have $H^i(\mathcal{D}_0, \mathcal{O}_{\mathcal{D}_0}) = 0$ for any $i > 0$. Similarly, since $\mathcal{D} \to \mathbb{D}$ is a $\mathbb{Q}$-Gorenstein flat family of $\mathbb{Q}$-Fano varieites, we have $H^i(\mathcal{D}, \mathcal{O}_{\mathcal{D}}) = 0$ for any $i > 0$. Hence from the exponential exact sequence, we obtain the following isomorphisms:

$$\mathrm{Pic}(\mathcal{D}_0) \xrightarrow{\cong} H^2(\mathcal{D}_0, \mathbb{Z}) \xleftarrow{\cong} H^2(\mathcal{D}, \mathbb{Z}) \xleftarrow{\cong} \mathrm{Pic}(\mathcal{D}).$$

Here the middle isomorphism follows from the topological fact that $\mathcal{D}_0 \hookrightarrow \mathcal{D}$ admits a deformation retraction. In particular, we know that

$$\mathrm{rk}\,\mathrm{Pic}(\mathcal{D}) = \mathrm{rk}\,\mathrm{Pic}(\mathcal{D}_0) \ge 2.$$

By Lefschetz hyperplane theorem, every fiber $\mathcal{D}_t$ of the smooth fibration $\mathcal{D}^\circ \to \mathbb{D}^\circ$ satisfies that

$$\mathrm{Pic}(\mathcal{D}_t) = \mathbb{Z} \cdot [-K_{\mathcal{D}_t}].$$

Hence [40, Conditions 12.2.1] hold for $\mathcal{D} \to \mathbb{D}$. By [40, Definitions 12.2.2, 12.2.4, and Propositions 12.2.3, 12.2.5], we know that there is a $\mathbb{Q}$-local system $\mathcal{GN}^1(\mathcal{D}/\mathbb{D})$ on $\mathbb{D}$ satisfying that

$$\mathcal{GN}^1(\mathcal{D}/\mathbb{D})(\mathbb{D}) \cong \mathrm{Pic}(\mathcal{D}) \otimes_{\mathbb{Z}} \mathbb{Q} \quad \text{and} \quad \mathcal{GN}^1(\mathcal{D}/\mathbb{D})|_t \cong \mathrm{Pic}(\mathcal{D}_t) \otimes_{\mathbb{Z}} \mathbb{Q}$$

for a very general $t \in \mathbb{D}^\circ$. Since $\mathbb{D}$ is contractible, we know that $\mathcal{GN}^1(\mathcal{D}/\mathbb{D})$ is a trivial $\mathbb{Q}$-local system, and

$$2 \le \mathrm{rk}\,\mathrm{Pic}(\mathcal{D}) = \mathrm{rk}\,\mathcal{GN}^1(\mathcal{D}/\mathbb{D}) = \mathrm{rk}\,\mathrm{Pic}(\mathcal{D}_t) = 1.$$

This is a contradiction. Thus the proof is finished. □

1) This argument is suggested by Chenyang Xu.

Finally, we treat the case of G being a complete intersection.

Proposition 4.7. *Suppose L is not Cartier at $x \in X$. Assume $(G, L|_G)$ is a $(2,3)$-complete intersection in $\mathbb{P}^4$. Then the index 1 cover $\tilde{x} \in \widetilde{X}$ of $x \in X$ with respect to L is a local complete intersection singularity.*

Proof. It is clear that $x \in \mathrm{Bs}\,|L| \subset H$. By Proposition 4.4, we know that there exists a weakly special test configuration $\mathcal{H}$ of H with central fiber $\mathcal{H}_0 \cong C_p(G, 2L|_G)$. Denote by $\mathcal{L}_{\mathcal{H}}$ the Zariski closure of $L|_H \times (\mathbb{A}^1 \setminus \{0\})$ in $\mathcal{H}$. Then it is clear that $\mathcal{L}_{\mathcal{H}}$ is a $\mathbb{G}_m$-invariant $\mathbb{Q}$-Cartier Weil divisor on $\mathcal{H}$. Let $\mathcal{L}_0$ be the restriction of $\mathcal{L}_{\mathcal{H}}$ on $\mathcal{H}_0$ which is also a $\mathbb{Q}$-Cartier Weil divisor. From the construction of $\mathcal{H}$, we know that the Zariski closure $\mathcal{G}$ of $G \times (\mathbb{A}^1 \setminus \{0\})$ in $\mathcal{H}$ is a trivial test configuration of G. Moreover, its central fiber $\mathcal{G}_0$ is precisely the section at infinity in the projective cone $C_p(G, 2L|_G)$. Thus we have $\mathcal{L}_0|_{\mathcal{G}_0} \cong L|_G$ under the natural identification of $\mathcal{G}_0 \cong G$. Since $\mathcal{L}_0$ is $\mathbb{G}_m$-invariant, we know that $\mathcal{L}_0$ is linearly equivalent to the cone over $L|_G$. In particular, we know that $\mathrm{ind}(o, \mathcal{L}_0) = \mathrm{ind}(o, \mathcal{L}_{\mathcal{H}}) = 2$, where $o \in \mathcal{H}_0$ is the cone vertex. Besides, since $L|_G$ is Cartier, we know that o is the only non-Cartier point of $\mathcal{L}_{\mathcal{H}}$ in $\mathcal{H}_0$.

From earlier discussions, we know that $L|_H$ is not Cartier at x. Thus $\mathcal{L}_{\mathcal{H}}$ is not Cartier at (x,t) for any $t \in \mathbb{A}^1 \setminus \{0\}$. This implies that the degeneration of (x,t) in $\mathcal{H}$ as $t \to 0$ is precisely o. Let $\tau_{\mathcal{H}} : (\tilde{o} \in \widetilde{\mathcal{H}}) \to (o \in \mathcal{H})$ be the index 1 cover of $\mathcal{L}_{\mathcal{H}}$. Then it is clear that $\tilde{o} \in \widetilde{\mathcal{H}}_0$ is isomorphic to the affine cone singularity $C_a(G, L|_G)$. Since G is a global complete intersection and $L|_G \cong \mathcal{O}_G(1)$, we know that $\tilde{o} \in \widetilde{\mathcal{H}}_0$ is a local complete intersection singularity. We denote by $\tau : (\tilde{x} \in \widetilde{X}) \to (x \in X)$ the index 1 cover of L. Denote by $\widetilde{H} := \tau^{-1}(H)$. Then it is clear that $\widetilde{\mathcal{H}}$ provides a $\mathbb{G}_m$-equivariant degeneration of $(\tilde{x} \in \widetilde{H})$ to $(\tilde{o} \in \widetilde{\mathcal{H}}_0)$ which is a local complete intersection singularity. By [17, Theorem 2.3.4], we know that $(\tilde{x} \in \widetilde{H})$ is a local complete intersection singularity. Since $\widetilde{H}$ is a Cartier divisor of $\widetilde{X}$, again using [17, Theorem 2.3.4] we conclude that $\tilde{x} \in \widetilde{X}$ is also a local complete intersection. The proof is finished. □

Remark 4.8 (Communicated with Ziquan Zhuang). There is an alternative way to prove Propositions 4.6 and 4.7 using higher codimensional α-invariants. Since X is K-semistable and $\mathrm{Bs}\,|L|$ is finite, by [77, Theorem 1.1] we know that

$$\alpha^{(4)}(X) \geq \frac{4}{5}.$$

Since $-K_X \sim 3L$, we know that

$$\mathrm{lct}(X; |L|) \geq \frac{12}{5}.$$

Hence $(X, H_1 + H_2 + \frac{2}{5}H_3)$ is log canonical for general members $H_i \in |L|$ $(1 \leq i \leq 3)$. Suppose $x \in \Sigma$ is a non-Cartier point of L with the index 1 cover $\tau : (\tilde{x} \in \widetilde{X}) \to (x \in X)$. Thus $(\widetilde{X}, \widetilde{H}_1 + \widetilde{H}_2 + \frac{2}{5}\widetilde{H}_3)$ is log canonical at $\tilde{x}$, where each $\widetilde{H}_i := \tau^* H_i$ is Cartier. Hence by adjunction we know that $(\widetilde{H}_1 \cap \widetilde{H}_2, \frac{2}{5}\widetilde{H}_1 \cap \widetilde{H}_2 \cap \widetilde{H}_3)$ is semi-log-canonical (slc). Hence $\tilde{x} \in \widetilde{H}_1 \cap \widetilde{H}_2$ is a Gorenstein slc surface singularity without an isolated lc center. By the classification of log canonical surface singularities (see e.g. [39, Section 3.3]), we know that $\tilde{x} \in \widetilde{H}_1 \cap \widetilde{H}_2$ is either Du Val or nodal. In particular, the point $\tilde{x} \in \widetilde{H}_1 \cap \widetilde{H}_2$ is a hypersurface singularity which implies that $\tilde{x} \in \widetilde{X}$ is also a hypersurface singularity. Then similar arguments to the proof of Theorem 4.9 implies that X is K-unstable, a contradiction.

To summarize, we have shown the following result which implies Theorem 1.1.

Theorem 4.9. *Let (X, L) be the K-semistable limit of cubic fourfolds as in Theorem* 3.1. *Then L is a very ample Cartier divisor, and $\phi_{|L|} : X \hookrightarrow \mathbb{P}^5$ embeds X as a (possibly singular) cubic fourfold.*

Proof. Assume to the contrary that L is not Cartier at $x \in X$. From the above discussions, we see that $(G, L|_G)$ is a polarized K3 surface with Du Val singularities of degree 6. If $(G, L|_G)$ is hyperelliptic or unigonal, then Propositions 4.5 and 4.6 imply that X is K-unstable, a contradiction. If $(G, L|_G)$ is a $(2, 3)$-complete intersection in $\mathbb{P}^4$, then Propositions 3.3 and 4.7 contradict each other since Conjecture 2.10 holds for local complete intersections by Theorem 1.3. Hence L must be Cartier on X. The rest of the statement directly follows from [34]. □

Proof of Theorem 1.1. The proof is almost the same as [54, proof of Theorem 1.1], with the following small modifications. By [60, Theorem 6.1], [71, pp. 85–87], [4], [56, Theorem 1.5], and [78, Corollary 1.4], there exists at least one smooth K-stable cubic fourfolds. We also replace [54, Lemma 3.17] by Theorem 4.9. Then the proof proceeds exactly the same as [54, proof of Theorem 1.1]. □

Proof of Corollary 1.2. For parts (1) and (2), by [42, Theorem 1.1] we know that cubic fourfolds with simple singularities are GIT stable. Hence the statements follow from Theorem 1.1. Part (3) follows directly from Theorem 1.1. For part (4), Theorem 1.1 implies that any GIT semistable cubic fourfold is K-semistable, hence it has klt singularities. A hypersurface with klt singularities must be Gorenstein canonical. The existence of (weak) KE metrics in (1)–(3) follows from the Yau–Tian–Donaldson Conjecture in the smooth case [20–22, 72] and the general case [9, 45, 47, 55]. Thus the proof is finished. □

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