ELLIPTIC MEASURES FOR DAHLBERG-KENIG-PIPHER OPERATORS: ASYMPTOTICALLY OPTIMAL ESTIMATES

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ABSTRACT. Questions concerning quantitative and asymptotic properties of the elliptic measure corresponding to a uniformly elliptic divergence form operator have been the focus of recent studies. In this setting we show that the elliptic measure of an operator with coefficients satisfying a vanishing Carleson condition in the upper half space is an asymptotically optimal A_∞ weight. In particular, for such operators the logarithm of the elliptic kernel is in the space of (locally) vanishing mean oscillation.

To achieve this, we prove local, quantitative estimates on a quantity (introduced by Fefferman, Kenig and Pipher) that controls the A_∞ constant. Our work uses recent results obtained by David, Li and Mayboroda. These quantitative estimates may offer a new framework to approach similar problems.

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1. Introduction

In this article we investigate the qualitative and quantitative properties of elliptic measures ω_L associated to divergence form uniformly elliptic operators $L = -\operatorname{div} A\nabla$ with certain variable coefficients in the half space $\mathbb{R}^{n+1}$. The coefficients satisfy the so-called weak Dahlberg-Kenig-Pipher (DKP) condition, which is a Carleson measure condition on the L^2-oscillation of the coefficients on Whitney regions (see Definition 2.8). A very closely related condition was introduced by Dahlberg. It was shown to be sufficient for L^p solvability of the Dirichlet problem for some $p > 1$ by Kenig and Pipher in [KP01] (see also [HL01]). Subsequently, it was shown by Dindos, Petermichl and Pipher [DPP07] that sufficient smallness in a similar[1] DKP-type condition allows one to solve the L^p-Dirichlet problem for $p > 1$ close to 1 (the 'smallness' depends on p).

We are particularly interested in the endpoint case when the Carleson norm defining the DKP condition has 'vanishing trace'. Under this assumption we study the BMO norm of the logarithm of the elliptic kernel, k_L at small scales (here $k_L = \frac{d\omega_L}{dx}$). For any non-negative locally integrable function w if $\log w$ has small norm BMO norm then, roughly speaking, w is 'almost' an 'optimal' A_∞ weight. The connection between the space of BMO (or VMO) and A_∞ (Muckenhoupt) weights is well documented [GCRdF85, Sar75, Kor98a, Kor98b] as is the connection between the A_∞ condition for the elliptic measure and the solvability of an L^p-Dirichlet problem [HL18].

The key new ingredient in this work is the quantitative control we obtain on a Carleson measure ν built from the elliptic measure, defined by

$$d\nu(x,r) = \frac{|\omega_L * (\nabla\varphi)_r(x)|^2}{|\omega_L * \varphi_r(x)|^2}\,\frac{dx\,dr}{r}, \tag{1.1}$$

where φ_r is a standard approximation of the identity. The control on ν is in the form of a point-wise density bound, which implies that ν is a Carleson measure, see (4.11) and Theorem 4.1. The kind of measure in (1.1) was introduced by Fefferman, Kenig and Pipher [FKP91] where they showed that a doubling weight w is in the Muckenhoupt A_∞ class if and only if ν_w is a Carleson measure, where ν_w is the measure formed by replacing ω_L by w in (1.1). Later, Korey [Kor98a] investigated the case where the ν_w was a 'vanishing' Carleson measure. Using this work of Korey we are able to show the following.

Theorem 1.2. *Let $L = -\operatorname{div} A\nabla$ be a divergence form uniformly elliptic operator on $\mathbb{R}^{n+1}_+$, whose coefficient matrix A satisfies the vanishing weak DKP condition (see Definition 2.8) . If k_L^∞ is the elliptic kernel associated to L (in $\mathbb{R}^{n+1}_+$) with pole at infinity then $\log k_L^\infty \in VMO(\mathbb{R}^n)$. Moreover, if k_L^X is the elliptic kernel associated to L with pole at $X \in \mathbb{R}^{n+1}_+$ then $\log k_L^X \in VMO_{loc}(\mathbb{R}^n)$. Here VMO is the space of vanishing mean oscillation and VMO_{loc} is a local version of VMO (see Definition 2.12).*

Remark 1.3. By a simple change of variable argument, we can show the same conclusion holds when we replace $\mathbb{R}^{n+1}_+$ by a C^1-square Dini domain. We say a

[1] See Section 6.

domain Ω is C^1-square Dini if locally Ω is the region above the graph of a C^1 function $\varphi : \mathbb{R}^n \to \mathbb{R}$, which satisfies that

$$|\nabla\varphi(x) - \nabla\varphi(y)| \le \theta(|x-y|), \qquad \text{for any } x, y \in \mathbb{R}^n, \tag{1.4}$$

and

$$\int_0^{*} \theta(r)^2 \frac{dr}{r} < +\infty. \tag{1.5}$$

We defer the proof of this remark to the Appendix.

Theorem 5.2, which constitutes a 'large constant' version of Theorem 1.2, is also new. In particular, in [DPP07] the authors use a slightly stronger assumption on the L^∞-oscillation of the coefficient matrix, and in [KP01] an even stronger condition on the gradient (see the DKP condition in Definition (2.8)) is imposed. The proof of the 'small constant' version of Theorem 1.2 requires revisiting the work of Fefferman, Kenig and Pipher [FKP91], see [BES21] and Section 6.

The advantage of using the measure ν as in (1.1) is that it allows us to use the 'Riesz formula' to shift the analysis from the elliptic measure (on the boundary) to the Green function (in the domain). For some time this approach seemed promising to the authors, but the necessary tools to complete the argument were lacking. The recent work of David, Li and Mayboroda [DLM] provides the missing tools. In [DLM], the authors show that the gradient of the Green function is almost purely in the transversal direction in terms of a Carleson measure (see Theorem 3.10). They also prove a 'Hardy inequality'-type lemma for weak DKP coefficients (see Lemma 2.10). The estimates in [DLM] in conjuction with the aforementioned point-wise density estimate (4.11) are used to prove Theorem 4.1. To the best of our knowledge this is the first time the measure defined in (1.1) has been used in this way. In [FKP91], these Carleson measures were used to produce counterexamples (see Section 4 therein), not to prove that the elliptic measure was an A_∞ weight.

To put our result in context it should be noted that the closely related work of [DPP07] places the elliptic kernel k_L in the (local) reverse Hölder class RH_p for all $p > 1$, when L satisfies a condition slightly stronger than in Theorem 1.2. As RH_p is a stronger condition for larger p and $k_L \in RH_p$ implies $k_L\, dx \in A_\infty$ one might be led to believe that our work could be deduced from this fact, under this slightly stronger hypothesis. This is not the case, as there are weights $w = f\, dx$ that are in every reverse Hölder class that fail to have the property that $\log f \in VMO(\mathbb{R}^n)$ (or $f \in VMO_{loc}(\mathbb{R}^n)$). (An equivalent way to phrase this is $A_{\infty,as} \subsetneq \bigcap_{p>1} RH_p$, see the characterizations of 'asymptotic A_∞' in Theorem 2.30 below.) As an example one can take $f(x) = 1_H(x) + (1+\epsilon)1_{H^c}(x)$ for any half space H and $\epsilon > 0$. On the other hand, the role of the RH_p condition is known, even in very rough settings, to be equivalent to the solvability of the $L^{p'}$ Dirichlet problem where $p' = p/(p-1)$ is the dual exponent (see [HL18, Proposition 4.5]). The condition that $\log k_L$ has small BMO norm implies that $k \in RH_p$ for large p and hence the L^p-Dirichlet problem is solvable for a wider range of p; however, by the example above, the converse is not true. For this reason, $\log k_L \in VMO(\mathbb{R}^n)$ has been considered an 'asymptotically optimal' condition for the elliptic kernel.

The condition $\log k_L \in VMO$ has appeared in many works and we give a few important examples here. For the Poisson kernel ($L = -\Delta$), the condition $\log k \in VMO$ was shown by Jerison and Kenig [JK82b] for C^1 domains, and Kenig and Toro [KT97] under 'vanishing flatness' condition for the domains. This is the natural endpoint to the work of Alt and Caffarelli, and Jerison [AC81, Jer90]. For more general elliptic operators, it was shown by Escauriaza [Esc96] in Lipschitz domains and Milakis, Pipher and Toro [MPT14] in chord arc domains that the property that $\log k \in VMO$ is stable under 'vanishing perturbations' of the coefficients, measured by Carleson measure as in [FKP91]. In [BTZ] we showed that $\log k_L \in VMO$ when A is Hölder continuous and the corresponding operator L is defined in a vanishing chord arc domain. We view these operators as a perturbations of constant coefficient ones. All of the results mentioned above where $\log k_L \in VMO$ are of a perturbative nature. This is part of what made Theorem 1.2 somewhat elusive, as the conditions on the matrix A make it difficult to view it as a 'suitable' perturbation of a good operator at all scales. On the other hand, we point out David, Li and Mayboroda [DLM] are able to obtain their estimates by using a perturbative regime at each scale, and we use their results in our work.

The paper is organized as follows. In Section 2 we lay out the setting, notation and the analysis tools used throughout the paper. In Section 3 we describe the classical PDE tools used throughout as well as the main result of [DLM]. In Section 4 we prove Theorem 4.1, the foundation of our work. In Section 5 we prove Theorems 5.1 and 5.13, which combine to give Theorem 1.2. In Section 6 we contrast our work with [KP01] and [DPP07].

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2. Preliminaries and Notation

Throughout $n \in \mathbb{N}$, $n \geq 2$ is a fixed constant. We work in $\mathbb{R}^{n+1}_+ := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : t > 0\}$. We use capital letters X, Y, Z to denote points in $\mathbb{R}^{n+1}$ and lowercase letters x, y, z to refer to points in $\mathbb{R}^n \times \{0\}$ (often identified with $\mathbb{R}^n$). For two positive numbers, a and b, we write $a \lesssim b$ whenever there exists a constant $C \geq 0$ such that $a/b < C$ such that C depends only on the allowed structural constants in the statement of a definition, lemma, theorem, etc.. Similarly we write $a \approx b$ if there exists $C \geq 1$ such that $C^{-1} \leq a/b \leq C$.

The operators and matrices we work with satisfy an ellipticity condition.

Definition 2.1 (Elliptic Matrices and Operators)**.** Fix $\Lambda \geq 1$. We say a matrix-valued function $A : \mathbb{R}^{n+1}_+ \to M_{n+1}(\mathbb{R})$ is Λ-elliptic if $\|A\|_{L^\infty(\mathbb{R}^{n+1})} \leq \Lambda$ and

$$\langle A(X)\xi, \xi\rangle \geq \Lambda^{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^{n+1}, X \in \mathbb{R}^{n+1}_+.$$

We say A is elliptic if it is Λ-elliptic for some $\Lambda \geq 1$. The smallest constant $\Lambda \geq 1$ such that A is Λ-elliptic is called the ellipticity constant of A. We say L is a divergence form elliptic operator (on $\mathbb{R}^{n+1}_+$) if $L = -\operatorname{div} A\nabla$ (viewed in the weak sense) for an elliptic matrix A. In particular, $\Omega \subseteq \mathbb{R}^{n+1}_+$ open we say $u \in W^{1,2}_{loc}(\Omega)$ is a weak solution to $Lu = 0$ in Ω if

$$\iint A\nabla u \cdot \nabla F \, dX = 0, \qquad \forall F \in C_c^\infty(\Omega).$$

Remark 2.2. This definition of ellipticity is the most common in the literature; however we could just as well replace A by $\widetilde{A} = A/\|A\|_{L^\infty(\mathbb{R}^{n+1})}$ and the theorems would only depend on the 'lower ellipticity' of $\widetilde{A}$. This way there is no 'artificial' dependence on ellipticity that is introduced when A is multiplied by a constant (a function u is a solution to $-\operatorname{div} A\nabla u = 0$ if and only if it is a solution to $-\operatorname{div} cA\nabla u = 0$).

This work concerns a family of canonical measures associated to divergence form elliptic operators. These measures are known (together) as elliptic measure.

Definition 2.3 (Elliptic measure and the Green function)**.** Let $L = -\operatorname{div} A\nabla$ be a divergence form elliptic operator on $\mathbb{R}^{n+1}_+$. There exists a family of Borel measures on $\mathbb{R}^n$, $\{\omega_L^X\}_{X\in\mathbb{R}^{n+1}_+}$, such that for $f \in C_c^\infty(\mathbb{R}^n)$ the function

$$u(X) = \int_{\mathbb{R}^n} f(y)\, d\omega_L^X(y)$$

is the unique weak solution to the Dirichlet problem

$$(D)_L \begin{cases} Lu = 0 \in \mathbb{R}^{n+1}, \\ u|_{\mathbb{R}^n} = f \end{cases}$$

satisfying $u \in C(\overline{\mathbb{R}^{n+1}_+} \cup \{\infty\})$. In particular $u(X) \to 0$ as $|X| \to \infty$ in $\mathbb{R}^{n+1}_+$. We call the measure ω_L^X the elliptic measure with pole at X.

By [HMT17, Lemma 2.25], there is a Green function associated to L in $\mathbb{R}^{n+1}_+$, $G_L(X,Y) : \mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+ \setminus \operatorname{diag}(\mathbb{R}^{n+1}_+) \to \mathbb{R}$, which satisfies the following. For fixed $X \in \mathbb{R}^{n+1}$ the Green function can be extended, as a function in Y, to a function that vanishes continuously on the boundary $\mathbb{R}^n$. The following 'Riesz formula' holds and connects the elliptic measure and the Green function: If $f \in C_c^\infty(\mathbb{R}^n)$ and $F \in C_c^\infty(\mathbb{R}^{n+1})$ are such that $F(y,0) = f(y)$ then

$$\int_{\mathbb{R}^n} f(y)\, d\omega_L^X(y) - F(X) = -\iint A^T(y,s)\nabla_{y,s} G_L(X,(y,s)) \cdot \nabla_{y,s} F(y,s)\, dy\, ds. \tag{2.4}$$

Here, and in the sequel, A^T is the transpose of A. In our applications of (2.4), $F(X)$ will be equal to zero. We emphasize that (2.4) also implies $u(\cdot) = G_L(X,\cdot)$ is a solution to $L^T u = 0$ in $\mathbb{R}^{n+1}_+ \setminus \{X\}$ and we have remarked that u vanishes continuously on $\mathbb{R}^n$.

In order to define the (weak) DKP condition we need some more notation.

- We write $|E|$ for the Lebesgue measure of a set E.
- We define the integral averages $\fint_{E'} f\, dx = \frac{1}{|E'|}\int_{E'} f\, dx$ and $⨌_E F\, dX = \frac{1}{|E|}\iint_E F\, dX$, for $E' \subset \mathbb{R}^n$ and $E \subset \mathbb{R}^{n+1}$ are sets of positive and finite measure (here $|\cdot|$ is the Lebesgue measure in the appropriate dimension).
- For $x \in \mathbb{R}^n$ and $r > 0$ we define $\Delta(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\}$, as usual we naturally identify $\Delta(x,r)$ as a subset of $\mathbb{R}^n \times \{0\}$. When we make this identification we call $\Delta(x,r) := \{(y,0) \in \mathbb{R}^n : |x-y| < r\}$ a surface ball.

- Given $x \in \mathbb{R}^n$ and $r > 0$ we define the Whitney region
$$W(x,r) := \Delta(x,r) \times (r/2, r].$$
- For $X \in \mathbb{R}^{n+1}$ and $r > 0$ we let $B(X,r)$ denote the usual $n+1$ Euclidean ball.
- For $\Delta = \Delta(x_0, r_0)$ or $B = B(x_1, r_1)$ we use the notation $r(\Delta) = r_0$ and $r(B) = r_1$ to denote the radius.
- For $x \in \mathbb{R}^n$ (identified with $\mathbb{R}^n \times \{0\}$ and $r > 0$ we define the Carleson region
$$T(x,r) := B(x,r) \cap \mathbb{R}^{n+1}_+.$$
- If $\Delta = \Delta(x,r)$ we set $T_\Delta = T(x,r)$.
- For $\Lambda \geq 1$, we let $\mathfrak{A}(\Lambda)$ denote the collection of all *constant* Λ-elliptic matrices.

Definition 2.5 (Oscillation Coefficients)**.** Let A be a Λ-elliptic matrix-valued function on $\mathbb{R}^{n+1}$. We define the following coefficients which measure the oscillation of A on various regions. For $x \in \mathbb{R}^n$ and $r > 0$ we define:
$$\alpha_2(x,r) = \inf_{A_0 \in \mathfrak{A}(\Lambda)} \left(\fint\!\!\fint_{(y,s) \in W(x,r)} |A(y,s) - A_0|^2 \right)^{1/2},$$
and
$$\gamma(x,r) = \inf_{A_0 \in \mathfrak{A}(\Lambda)} \left(\fint\!\!\fint_{(y,s) \in T(x,r)} |A(y,s) - A_0|^2 \right)^{1/2}.$$
If, in addition, A is locally Lipschitz, we define for $x \in \mathbb{R}^n$ and $r > 0$
$$\tilde{\alpha}(x,r) = r \sup_{(y,s) \in W(x,r)} |\nabla A(y,s)|.$$
It holds that $\alpha_2(x,r) \lesssim \tilde{\alpha}(x,r)$ and $\alpha_2(x,r) \leq 2\gamma(x,r)$.

We need one more definition before we introduce the class of coefficients we work with.

Definition 2.6 (Carleson Measures)**.** Let μ be a Borel measure on $\mathbb{R}^{n+1}_+$. We say μ Carleson measure if
$$\|\mu\|_C := \sup_\Delta |\Delta|^{-1} \mu(T_\Delta) < \infty,$$
where the supremum is over all n-dimensional balls Δ in $\mathbb{R}^n$. Roughly speaking, this means that μ acts like an n-dimensional measure at the boundary. We call $\|\mu\|_C$ the Carleson norm of μ. We also define a localized Carleson norm. For Δ_0 a surface ball and ν a Borel measure on T_{Δ_0}, we define
$$\|\nu\|_{C(\Delta_0)} := \sup_{\Delta \subset \Delta_0} |\Delta|^{-1} \mu(T_\Delta)$$
and if $\|\nu\|_{C(\Delta_0)} < \infty$ we say ν is a Carleson measure on T_{Δ_0}.

We say μ is a Carleson measure with vanishing trace (or simply vanishing Carleson measure) if μ is a Carleson measure and
$$\lim_{r_0 \to 0^+} \big(\sup_{x_0 \in \mathbb{R}^n} \|\mu\|_{C(\Delta(x_0, r_0))} \big) = 0. \tag{2.7}$$

Definition 2.8 (DKP, weak DKP and vanishing weak DKP conditions)**.** Let A be a Λ-elliptic matrix-valued function defined on $\mathbb{R}^{n+1}_+$.

- We say A satisfies the DKP condition if A is locally Lipschitz and μ defined by
$$d\mu(x,r) = \tilde{\alpha}(x,r)^2 \frac{dx\,dr}{r}$$
is a Carleson measure.
- We say A satisfies the weak DKP condition if μ defined by
$$d\mu(x,r) = \alpha_2(x,r)^2 \frac{dx\,dr}{r}$$
is a Carleson measure.
- We say A satisfies the vanishing weak DKP condition if μ defined by
$$d\mu(x,r) = \alpha_2(x,r)^2 \frac{dx\,dr}{r}$$
is a Carleson measure with vanishing trace.
- We say A satisfies the weak-DKP condition on T_{Δ_0} if μ defined by
$$d\mu(x,r) = \alpha_2(x,r)^2 \frac{dx\,dr}{r}$$
is a Carleson measure on T_{Δ_0}.

As observed above $\alpha_2(x,r) \lesssim \tilde{\alpha}(x,r)$ so that the DKP condition implies the weak DKP condition. For this reason, we will only work with the weak DKP condition in the sequel.

Remark 2.9. The quantities $\alpha_2(x,r)$, $\tilde{\alpha}(x,r)$ and $\gamma(x,r)$ do not see the difference between A and A^T. In particular, any Carleson condition involving $\alpha_2(x,r)$, $\tilde{\alpha}(x,r)$ and $\gamma(x,r)$ (like those in Definition 2.8) holds for A if and only if it holds for A^T. This will be particularly important as we intend to apply the estimates from [DLM] to Green functions, $G(X,\cdot)$ which are solutions to L^T away from X.

Though $\alpha_2(x,r) \leq 2\gamma(x,r)$, $\gamma(x,r)$ is in general not bounded pointwise by $\alpha_2(x,r)$. However, their Carleson measures are essentially equivalent:

Lemma 2.10 ([DLM, Remark 4.22])**.** *Suppose that A is a Λ-elliptic matrix-valued function defined on $\mathbb{R}^{n+1}_+$. Then*
$$\left\| \gamma(x,r)^2 \frac{dx\,dr}{r} \right\|_{C(\Delta_0)} \leq C \left\| \alpha_2(x,r)^2 \frac{dx\,dr}{r} \right\|_{C(3\Delta_0)},$$
and
$$\gamma(x,r)^2 \leq C \left\| \alpha_2(x,r)^2 \frac{dx\,dr}{r} \right\|_{C(3\Delta_0)} \quad \forall (x,r) \in T_{\Delta_0},$$
where C only depends on dimension.

The measures on $\mathbb{R}^n$ we work with often satisfy a doubling condition.

Definition 2.11 (Doubling measures)**.** Given a non-trivial Radon measure ω on $\mathbb{R}^n$, we say ω is doubling if there exists a constant C such that

$$\omega(\Delta(x, 2r)) \leq C\omega(\Delta(x, r)), \quad \forall x \in \mathbb{R}^n, r > 0.$$

The smallest constant in the inequality above is called the doubling constant for ω, denoted by C_{doub}.

Later, we will want to verify that elliptic kernels ($\frac{d\omega_L}{dx}$) exist and are in the function spaces VMO or VMO_{loc}. We define these spaces now.

Definition 2.12 (BMO, VMO and VMO_{loc})**.** Let $f \in L^1_{loc}(\mathbb{R}^n)$. We say $f \in BMO(\mathbb{R}^n)$ (or f has bounded mean oscillation) if

$$\|f\|_{BMO} := \sup_{r>0} \sup_{x\in\mathbb{R}^n} \fint_{\Delta(x,r)} \left| f(z) - \fint_{\Delta(x,r)} f(y)\, dy \right| dz < \infty.$$

We say $f \in VMO$ (or f has vanishing mean oscillation) if f is in BMO and

$$\lim_{r_0\to 0^+} \sup_{r\in(0,r_0)} \sup_{x\in\mathbb{R}^n} \fint_{\Delta(x,r)} \left| f(z) - \fint_{\Delta(x,r)} f(y)\, dy \right| dz = 0.$$

We say $f \in VMO_{loc}(\mathbb{R}^n)$ if for every compact set $K \subset \mathbb{R}^n$

$$\lim_{r_0\to 0^+} \sup_{r\in(0,r_0)} \sup_{x\in K} \fint_{\Delta(x,r)} \left| f(z) - \fint_{\Delta(x,r)} f(y)\, dy \right| dz = 0.$$

Notice the $VMO_{loc}(\mathbb{R}^n)$ condition does not require f to be in $BMO(\mathbb{R}^n)$.

We also want to investigate when k_L is in the A_∞ class (locally). Basic facts about A_∞ weights can be found in [GCRdF85].

Definition 2.13 (A_∞ weights and A_∞ measures)**.** We say a function w is a weight if $w \in L^1_{loc}(\mathbb{R}^n)$ and $w \geq 0$. A weight w is said to be in A_∞ if there exists C such that

$$\fint_{\Delta(x,r)} w(z)\, dz \leq C \exp\left\{ \fint_{\Delta(x,r)} \log w(z)\, dz \right\}, \quad \forall x \in \mathbb{R}^n, r > 0. \tag{2.14}$$

The infimum over constants C such that the inequality (2.14) holds is called the A_∞ constant, written $[w]_{A_\infty}$.

If $w \in A_\infty$ then there exists $p > 1$ and C' both depending on dimension and $[w]_{A_\infty}$ such that w satisfies the reverse Hölder inequality, with exponent p, that is,

$$\left(\fint_{\Delta(x,r)} w^p\, dz \right)^{1/p} \leq C' \fint_{\Delta(x,r)} w\, dz, \quad \forall x \in \mathbb{R}^n, r > 0. \tag{2.15}$$

Conversely, any weight satisfying (2.15) for some $p > 1$ is an A_∞ weight with $[w]_{A_\infty}$ depending on C' and p.

We say a Radon measure ω on $\mathbb{R}^n$ is in the A_∞ class, if ω is absolutely continuous[2] with respect to Lebesgue measure in $\mathbb{R}^n$ and its density $w := \frac{d\omega}{dx}$ is an A_∞ weight.[3]

[2]Here we take the definition that $|E| = 0$ implies $\omega(E) = 0$, so that absolute continuity is equivalent to the existence of a locally integrable density.

[3]In particular, an A_∞ measure must be doubling. In fact as is well known in the theory of weights, a measure in the $A_\infty(dx)$ class satisfies the property that $\frac{|E|}{|\Delta|} \leq C\left(\frac{\omega(E)}{\omega(\Delta)}\right)^\theta$ for any surface ball $\Delta \subset \mathbb{R}^n$

We use the following characterization of A_∞, which is a modest improvement of a particular case of [FKP91, Theorem 3.1]. Let $\varphi \in C_c^\infty(\Delta(0,1))$ be a radial function with $\varphi \equiv 1$ on $\Delta(0,1/2)$ and $0 \le \varphi \le 1$. Set $\psi := \nabla\varphi$ and use the notation $f_r(x) := r^{-n} f(x/r)$.

Theorem 2.16. *Let ω be a Radon measure on $\mathbb{R}^n$. Then ω is in the A_∞ class if and only if ω is doubling (see Definition 2.11) and the measure μ on $\mathbb{R}^{n+1}_+$ defined by*

$$d\mu(x,r) := \frac{|\omega * \psi_r(x)|^2}{|\omega * \varphi_r(x)|^2} \frac{dx\,dr}{r} \tag{2.17}$$

is a Carleson measure. Here we use the standard definition of convolution against a measure, that is,

$$\omega * f(x) := \int_{\mathbb{R}^n} f(x-y) d\omega(y).$$

Moreover, the relationship between $[\omega]_{A_\infty}$ and $\|\mu\|_C$ is quantitative in the sense that if $w = \frac{d\omega}{dx}$, then $[w]_{A_\infty} \le F_1(n, C_{doub}, \varphi, \|\mu\|_C)$ and $\|\mu\|_C \le F_2(n, C_{doub}, \varphi, [w]_{A_\infty})$.

Remark 2.18. The above theorem is false when ω is not doubling, as is pointed out in [FKP91] by the example of the weight $w_k(x) := \min\{1/|x|, k\}$ for large values of k.

The original statement in [FKP91, Theorem 3.1] is for weights, or more specifically, under the hypothesis that the Radon measure ω is equal to $w\,dx$ for a locally integrable function w. Our contribution here is the observation that the Carleson condition on μ, in fact, implies the absolute continuity of ω with respect to Lebesgue measure. We give the proof of this fact below. This combined with [FKP91, Theorem 3.1] finishes the proof of Theorem 2.16.

Lemma 2.19. *Let ω be a Radon measure on $\mathbb{R}^n$ which satisfies the doubling property. Suppose that the measure μ defined in (2.17) is a Carleson measure. Then ω is absolute continuous with respect with the Lebesgue measure.*

Proof. By [FKP91], in particular[4] [FKP91, Lemma 3.12] and its converse direction, we may replace φ with the Gaussian kernel $\phi(x) = c_n e^{-|x|^2}$ and replace ψ with $\nabla\phi$. Let $\{\epsilon_i\}$ be a sequence of positive numbers tending to zero, and we define a weight ω_i as follows

$$\omega_i(x) := \phi_{\sqrt{\epsilon_i}} * \omega(x) = \frac{1}{\epsilon_i^{n/2}} \int \phi\left(\frac{x-y}{\sqrt{\epsilon_i}}\right) d\omega(y). \tag{2.20}$$

Clearly $\omega_i \rightharpoonup \omega$ as Radon measures, and each ω_i is a doubling measure with constant only depending on the doubling constant of ω. We claim that $\omega_i \in A_\infty(dx)$ and the A_∞ constants of the ω_i's are uniform in i, or equivalently, there exists a constant C (independent of i) such that for any surface ball $\Delta \subset \mathbb{R}^n$,

$$\left|\log\left(\fint_\Delta \omega_i\,dx\right) - \fint_\Delta \log\omega_i\,dx\right| \le C. \tag{2.21}$$

---

and any set $E \subset \Delta$. Thus the doubling of ω simply follows from the doubling of the Lebesgue measure.

[4]Note that while [FKP91, Lemma 3.12] is stated for (convolutions with) weights, the proof goes through without modification for (convolutions with) measures.

Notice that this inequality gives (2.14) with a constant e^C.

We first consider small scales, i.e. when the radius of the surface ball satisfies $r_\Delta \leq \sqrt{\epsilon_i}$. Since ω is a doubling measure, by the definition (2.20) and the rapid decay of ϕ we have that

$$\omega_i(x) \approx \frac{1}{\epsilon_i^{n/2}} \omega\left(\Delta(x, \sqrt{\epsilon_i})\right).$$

Hence for any pair $x, y \in \mathbb{R}^n$ such that $|x-y| \leq \sqrt{\epsilon_i}$, we have that

$$\omega_i(x) \approx \frac{1}{\epsilon_i^{n/2}} \omega\left(\Delta(x, \sqrt{\epsilon_i})\right) \lesssim \frac{1}{\epsilon_i^{n/2}} \omega\left(\Delta(y, 2\sqrt{\epsilon_i})\right) \lesssim \frac{1}{\epsilon_i^{n/2}} \omega(\Delta(y, \sqrt{\epsilon_i})) \approx \omega_i(y).$$

Therefore (2.21) follows with a constant only depending on the doubling constant of ω.

Next we prove (2.21) for large scales, i.e. when $r_\Delta \geq \sqrt{\epsilon_i}$. Let H_t denote the heat semigroup and $K(t, x, y)$ the heat kernel, i.e. for every $f \in C_c^\infty(\mathbb{R}^n)$

$$H_t \circ f(x) = \int_{\mathbb{R}^n} K(t, x, y) f(y)\, dy = \frac{c_n}{t^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{t}} f(y)\, dy.$$

By definition

$$H_t \circ f(x) = \phi_{\sqrt{t}} * f(x). \tag{2.22}$$

and thus its spatial derivative satisfies

$$\sqrt{t}\, \nabla \left(H_t \circ f\right)(x) = \sqrt{t}\, \nabla \left(\phi_{\sqrt{t}} * f\right)(x) = (\nabla \phi)_{\sqrt{t}} * f(x). \tag{2.23}$$

Moreover, (2.22) and (2.23) also hold when we replace f by a doubling measure. Denote

$$u(x, t) := H_t \circ \omega(x), \quad u_i(x, t) := H_t \circ \omega_i(x).$$

By the definition of ω_i and the semigroup property of the heat kernel, we have that

$$u_i(x, t) = H_t \circ \left(\phi_{\sqrt{\epsilon_i}} * \omega\right)(x) = H_t \circ \left(H_{\epsilon_i} \circ \omega\right)(x) = H_{t+\epsilon_i} \circ \omega(x) = u(x, t+\epsilon_i). \tag{2.24}$$

Hence the spatial derivative satisfies

$$\nabla u_i(x, t) = \nabla u(x, t+\epsilon_i). \tag{2.25}$$

By (2.22), (2.23) and a change of variable ($t = \sqrt{r}$), it is easy to show that $d\mu$ defined in (2.17) (with the Gaussian in place of φ) is a Carleson measure if and only if

$$C(u) := \sup_{\substack{x_0 \in \mathbb{R}^n \\ s>0}} \frac{1}{s^n} \int_0^{s^2} \int_{B_s(x_0)} \frac{|\nabla u(x, t)|^2}{u(x, t)^2}\, dx\, dt < +\infty, \tag{2.26}$$

and moreover these two Carleson measure norms are equivalent. By (2.24) and (2.25), we have

$$\begin{aligned} \int_0^{s^2} \int_{\Delta(x_0, s)} \frac{|\nabla u_i(x, t)|^2}{u_i(x, t)^2}\, dx\, dt &= \int_0^{s^2} \int_{\Delta(x_0, s)} \frac{|\nabla u(x, t+\epsilon_i)|^2}{u(x, t+\epsilon_i)^2}\, dx\, dt \\ &= \int_{\epsilon_i}^{s^2+\epsilon_i} \int_{\Delta(x_0, s)} \frac{|\nabla u(x, \tau)|^2}{u(x, \tau)^2}\, dx\, d\tau \end{aligned}$$

$$\leq \int_0^{s^2+\epsilon_i} \int_{\Delta(x_0,\sqrt{s^2+\epsilon_i})} \frac{|\nabla u(x,\tau)|^2}{u(x,\tau)^2} dx\, d\tau$$
$$\leq (s^2+\epsilon_i)^{n/2} \cdot C(u).$$

Therefore as long as $s \geq \sqrt{\epsilon_i}$, we have that

$$\frac{1}{s^n} \int_0^{s^2} \int_{B_s(x_0)} \frac{|\nabla u_i(x,t)|^2}{u_i(x,t)^2} dx\, dt \leq C(u) \frac{(s^2+\epsilon_i)^{n/2}}{s^n} \leq 2^{n/2} C(u). \tag{2.27}$$

Applying the same argument as in the proof of [FKP91, Theorem 3.4] to the weight ω_i, we conclude that the Carleson-type estimate in (2.27) implies the A_∞-type estimate (2.21), also for large scales $r_\Delta \geq \sqrt{\epsilon_i}$. This finishes the proof of the claim.

Since $\omega_i \in A_\infty(dx)$ with a constant independent of i, the following holds: For any $\epsilon > 0$, there exists $\delta > 0$ (depending only on ϵ, not on i or R) such that for every $R > 0$,

$$\text{for any set } E \subset \Delta_{2R} \text{ satisfying } \frac{|E|}{|\Delta_{2R}|} < \delta, \text{ we have } \frac{\omega_i(E)}{\omega_i(\Delta_{2R})} < \epsilon, \tag{2.28}$$

where we use $|\cdot|$ to denote the Lebesgue measure in $\mathbb{R}^n$ and the notation $\Delta_R := \Delta(0,R)$.

Fix $R > 0$ and $\epsilon > 0$, and let E be an arbitrary set in Δ_R such that $|E| < \delta|\Delta_R|$, where δ is as above. By the outer approximation by open sets, there exists an open set $U \supset E$ such that $|U| < 2^n\delta|\Delta_R| = \delta|\Delta_{2R}|$. Without loss of generality we may assume that $U \subset \Delta_{2R}$. It follows from (2.28) that

$$\omega_i(U) < \epsilon\omega_i(\Delta_{2R}) \leq \epsilon\omega_i(\overline{\Delta_{2R}})$$

for every i. Since $\omega_i \rightharpoonup \omega$, U is open and $\overline{\Delta_{2R}}$ is compact, we have

$$\omega(E) \leq \omega(U) \leq \liminf_{i\to\infty} \omega_i(U) \leq \epsilon \limsup_{i\to\infty} \omega_i(\overline{\Delta_{2R}}) \leq \epsilon\omega(\overline{\Delta_{2R}}) \leq C\epsilon\omega(\Delta_R),$$

where we use the doubling property of ω in the last inequality. Thus, we have shown that for a fixed $R > 0$ and for every $\epsilon > 0$, there exists $\delta > 0$ so that the following holds:

$$\text{Every } E \subset \Delta_R \text{ with } |E| < \delta|\Delta_R| \text{ satisfies } \omega(E) \leq C\epsilon\omega(\Delta_R). \tag{2.29}$$

In particular, this indicates that ω is absolute continuous with respect to the Lebesgue measure within the ball Δ_R for every $R > 0$, and thus ω is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$. In fact, a similar argument shows directly that ω is in the class $A_\infty(dx)$ □

We also have the following characterizations for a weight to be *'asymptotic A_∞'*. This was observed by Sarason [Sar75] and thoroughly investigated by Korey [Kor98a, Kor98b].

Theorem 2.30 ([Kor98a, Theorem 1], [Kor98b, Theorem 10])**.** *Let w be a weight. The following are equivalent,*

(1) *$w \in A_\infty$ and there exists $p > 1$ such that*

$$\lim_{r_0 \to 0^+} \sup_{x \in \mathbb{R}^n} \sup_{r \in (0,r_0)} \frac{\left(\fint_{\Delta(x,r)} w^p \, dz \right)^{1/p}}{\fint_{\Delta(x,r)} w \, dz} = 1.$$

(2) *$w \in A_\infty$ and for <u>any</u> $p > 1$*

$$\lim_{r_0 \to 0^+} \sup_{x \in \mathbb{R}^n} \sup_{r \in (0,r_0)} \frac{\left(\fint_{\Delta(x,r)} w^p \, dz \right)^{1/p}}{\fint_{\Delta(x,r)} w \, dz} = 1.$$

(3) *$w \in A_\infty$ and*

$$\lim_{r_0 \to 0^+} \sup_{x \in \mathbb{R}^n} \sup_{r \in (0,r_0)} \frac{\fint_{\Delta(x,r)} w \, dz}{\exp \left\{ \fint_{\Delta(x,r)} \log w \, dz \right\}} = 1.$$

(4) *The measure $w\,dx$ is doubling ($\int_{\Delta(x,2r)} w\,dz \leq C_{doub} \int_{\Delta(x,r)} w\,dz$) and measure μ on $\mathbb{R}^{n+1}_+$ defined by*

$$d\mu(x,r) := \frac{|w * \psi_r(x)|^2}{|w * \phi_r(x)|^2} \frac{dx\,dr}{r}$$

as in Theorem 2.16 is a <u>vanishing</u> Carleson measure.

(5) $\log w \in VMO$.

By inspection, [Kor98b, Theorem 10] can be localized and we will use the following.

Theorem 2.31 ([Kor98b, Theorem 10]). *Let w be a weight. Then $\log w \in VMO_{loc}$ if and only if for every $R > 0$*

$$\lim_{r_0 \to 0^+} \sup_{x \in \Delta(0,R)} \sup_{r \in (0,r_0)} \frac{\left(\fint_{\Delta(x,r)} w^2 \, dz \right)^{1/2}}{\fint_{\Delta(x,r)} w \, dz} = 1.$$

3. Classical Estimates and the [DLM] energy estimates

We begin this section by recalling some classical estimates for positive solutions to divergence form elliptic equations in the upper half space that vanish at the boundary. After doing so, we will specialize to the case of operators whose coefficient matrix satisfies a (local) weak DKP condition and introduce the energy estimates proved in [DLM]. The following lemma is explicitly stated and proved in [DLM, Lemma 2.8].

Lemma 3.1. *Let $L = -\operatorname{div} A\nabla$ be a divergence form elliptic operator (on $\mathbb{R}^{n+1}_+$). Suppose that $x \in \mathbb{R}^n$, $r > 0$ and $u \in W^{1,2}(T(x,2r))$ is a non-negative weak solution to $Lu = 0$ in $T(x,2r)$ which vanishes continuously on $\Delta(x,2r)$. Then there exist implicit constants, depending only on n and the ellipticity constant of A such that*

$$\iint_{T(x,r)} |\nabla u(Y)|^2 \, dY \approx \frac{u(x,r)^2}{r^2}.$$

The following is a well-known estimate, often called the CFMS estimate.

Lemma 3.2 ([CFMS81])**.** *Let L be a divergence form elliptic operator (on $\mathbb{R}^{n+1}_+$). If $\omega_L^{X_0}$ is the elliptic measure for L with pole $X_0 \in \mathbb{R}^{n+1}_+$ and $G_L(X,Y)$ is the Green function for L then*

$$\frac{\omega_L^{X_0}(\Delta(x,r))}{|\Delta(x,r)|} \approx \frac{G_L(X_0,(x,r))}{r}$$

for $x \in \mathbb{R}^n$ and $r > 0$, provided $X_0 \notin T(x,2r)$. Here the implicit constants depend on n and the ellipticity constant of A.

Combining the previous two lemmas we obtain the following.

Lemma 3.3. *Let L be a divergence form elliptic operator (on $\mathbb{R}^{n+1}_+$). If $\omega_L^{X_0}$ is the elliptic measure for L with pole $X_0 \in \mathbb{R}^{n+1}_+$ and $G_L(X,Y)$ is the Green function[5] for L then*

$$\frac{\omega_L^{X_0}(\Delta(x,r))}{|\Delta(x,r)|} \approx \left(\fint\!\!\fint_{T(x,r)} |\nabla_Y G_L(X_0,Y)|^2\, dY \right)^{1/2}$$

for $x \in \mathbb{R}^n$ and $r > 0$, provided $X_0 \notin T(x,4r)$. Here the implicit constants depend on n and the ellipticity constant of A. More generally

$$\frac{\omega_L^{X_0}(\Delta(x,r))}{|\Delta(x,r)|} \approx_M \left(\fint\!\!\fint_{T(x,Mr)} |\nabla_Y G_L(X_0,Y)|^2\, dY \right)^{1/2}$$

for $x \in \mathbb{R}^n$ and $r > 0$, provided $X_0 \notin T(x,4Mr)$, the implicit constants depend on M, n and the ellipticity constant of A.

In the previous lemma, the second estimate follows from the first, Lemma 3.1 and the Harnack inequality (applied to $G(X_0,\cdot)$ a solution to $L^T u = 0$ away from X_0). We also have the following doubling property for harmonic measure, which can be deduced from Lemma 3.2 and the Harnack inequality.

Lemma 3.4. *Let L be a divergence form elliptic operator (on $\mathbb{R}^{n+1}_+$). If $\omega_L^{X_0}$ is the elliptic measure for L with pole $X_0 \in \mathbb{R}^{n+1}_+$ then*

$$\omega_L^{X_0}(\Delta(x,2r)) \lesssim \omega_L^{X_0}(\Delta(x,r))$$

provided that $X_0 \notin T(x,4r)$. Here the implicit constants depend on n and the ellipticity constant of A.

Notice the previous lemma does not give *global* doubling of the measure. The lemma only gives local doubling, up to the scale of the distance from the pole to the boundary. Later we would like to work with "the" Green function and elliptic measure with pole at infinity and we introduce them with the following lemma, which can be proved just as in [KT99, Corollary 3.2].

Lemma 3.5 (Green function and elliptic measure at infinity)**.** *Let L be a divergence form elliptic operator (on $\mathbb{R}^{n+1}_+$) with Green function $G_L(X,Y)$. Define the sequence of functions $u_k : \overline{\mathbb{R}^{n+1}_+} \to \mathbb{R}$ by*

$$u_k(Y) := \frac{G_L((0,2^k),Y)}{G_L((0,2^k),(0,1))},$$

[5] Recall $G_L(X_0,\cdot)$ satisfies (2.4) and $L^T G_L(X,\cdot) = \delta_X$.

where we have extended u_k to the boundary by zero ($u_k(y,0) = 0$, $\forall y \in \mathbb{R}^n$). There exists a subsequence u_{k_j} such that u_{k_j} converges uniformly on compact subsets of $\overline{\mathbb{R}^{n+1}_+}$ to a function U with the following properties.

- $U(y,0) = 0$ *for all* $y \in \mathbb{R}^n$.
- $U(0,1) = 1$.
- $U(Y) > 0$ *for all* $Y \in \mathbb{R}^{n+1}_+$.
- $U \in C(\overline{\mathbb{R}^{n+1}_+})$.
- U *solves* $L^T U = 0$ *in* $\mathbb{R}^{n+1}_+$.

Moreover, there exists a locally finite measure ω_L^∞ on $\mathbb{R}^n$ with

$$\frac{1}{G((0,2^{k_j}),(0,1))}\omega^{(0,2^{k_j})} \rightharpoonup \omega_L^\infty$$

such that the following Riesz formula holds

$$\int_{\mathbb{R}^n} f(y)\, d\omega_L^\infty(y) = -\iint A^T(y,s)\nabla_{y,s}U(y,s)\cdot\nabla_{y,s}F(y,s)\, dy\, ds, \tag{3.6}$$

whenever $f \in C_c^\infty(\mathbb{R}^n)$ and $F \in C_c^\infty(\mathbb{R}^{n+1})$ are such that $F(y,0) = f(y)$ We call ω_L^∞ the **elliptic measure with pole at infinity** *and U the* **Green function with pole at infinity**.

The estimates for G in Lemma 3.3 hold for U (globally) and the measure ω_L^∞ is *globally* doubling. We summarize these facts in the following Lemma.

Lemma 3.7. *Let L be a divergence form elliptic operator (on $\mathbb{R}^{n+1}_+$). Let U be the Green function with pole at infinity and ω_L^∞ is the elliptic measure with pole at infinity then the following hold:*

- *For $x \in \mathbb{R}^n$ and $r > 0$*
$$\frac{\omega_L^\infty(\Delta(x,r))}{|\Delta(x,r)|} \approx_M \left(\fint\!\!\fint_{T(x,Mr)} |\nabla U(Y)|^2\, dY \right)^{1/2},$$
where the implicit constants depend on M, n and the ellipticity constant of A.
- *For $x \in \mathbb{R}^n$ and $r > 0$ it holds $\omega_L^\infty(\Delta(x,2r)) \lesssim \omega_L^\infty(\Delta(x,r))$, where the implicit constant depends on n and the ellipticity of A.*

Next, we need to define some local energies as in [DLM] (we define some additional objects as well).

Definition 3.8 (Local energies)**.** Let $x \in \mathbb{R}^n$ and $r > 0$. Suppose $u \in W^{1,2}(T(x,r))$ and $\iint_{T(x,r)} |\nabla u|^2\, dz\, dt$ is non-zero. Define

$$E_u(x,r) := \fint\!\!\fint_{T(x,r)} |\nabla u|^2\, dz\, dt,$$

and for $i = 1, \dots, n$

$$E_{u,i}(x,r) := \fint\!\!\fint_{T(x,r)} |\partial_{x_i} u|^2\, dz\, dt.$$

Let

$$\lambda(x,r) := \fint\!\!\fint_{T(x,r)} \partial_t u \, dz\, dt.$$

We define, as in [DLM],

$$J_u(x,r) = \fint\!\!\fint_{T(x,r)} |(\nabla_{x,t} u) - \lambda(x,r) e_{n+1}|^2 \, dz, dt,$$

which essentially measures how far $\nabla_{x,t} u$ is from its vertical component, and define

$$\beta_u(x,r) := \frac{J_u(x,r)}{E_u(x,r)}.$$

A simple computation shows that

$$\beta_{u,i}(x,r) := \frac{E_{u,i}(x,r)}{E_u(x,r)} \le \beta_u(x,r). \tag{3.9}$$

We will need the following main theorem proven in [DLM].

Theorem 3.10 ([DLM, Theorem 1.15]). *Let $x_0 \in \mathbb{R}^n$ and $R > 0$ and A be a Λ-elliptic matrix satisfying the weak DKP condition on $T(x_0, R)$. Suppose u is a positive solution to $Lu := -\operatorname{div} A\nabla u = 0$ in $T(x_0, R)$ and u vanishes on $\Delta(x_0, R)$. Then for any $\tau \in (0, 1/20]$ it holds*

$$\left\| \beta_u(x,r) \frac{dx\,dr}{r} \right\|_{C(\Delta(x_0,\tau R))} \le C \left(\tau^\eta + \left\| \alpha_2(x,r)^2 \frac{dx\,dr}{r} \right\|_{C(\Delta(x_0,R))} \right),$$

where the constants C and η depend only on Λ and n.

As was observed above in (3.9), $\beta_{u,i}(x,r) \le \beta_u(x,r)$, giving the following corollary.

Corollary 3.11. *Let $x_0 \in \mathbb{R}^n$ and $R > 0$ and A be a Λ-elliptic matrix satisfying the weak DKP condition on $T(x_0, R)$. Suppose u is a positive solution to $Lu := -\operatorname{div} A\nabla u = 0$ in $T(x_0, R)$ and u vanishes on $\Delta(x_0, R)$. Then for any $\tau \in (0, 1/20]$ it holds*

$$\left\| \beta_{u,i}(x,r) \frac{dx\,dr}{r} \right\|_{C(\Delta(x_0,\tau R))} \le C \left(\tau^\eta + \left\| \alpha_2(x,r)^2 \frac{dx\,dr}{r} \right\|_{C(\Delta(x_0,R))} \right),$$

where the constants C and η depend only on Λ and n.

4. A local quantitative estimate on the FKP Carleson measure for ω_L

In this section we prove Theorem 4.1, from which we will derive all of our other results. Below $\varphi \in C_c(B(0,1))$ is a fixed radial function with $\varphi(x) = 1$ for $|x| \le 1/2$, $0 \le \varphi \le 1$. For a function f and $r > 0$ we use the notation $f_r(y) := r^{-n} f(y/r)$.

Theorem 4.1. *Suppose $x_0 \in \mathbb{R}^n$ and $R > 0$. Let A be a Λ-elliptic matrix satisfying the weak DKP condition on $T(x_0, 100R)$. Let $Z_0 = (z_0, t_0) \in \mathbb{R}^{n+1}_+$ be any point with $t_0 > 100R$ and $\omega = \omega_L^{Z_0}$ be the elliptic measure associated to $L = -\operatorname{div} A\nabla$ on $\mathbb{R}^{n+1}_+$ with pole at Z_0. Then the measure*

$$d\nu(x,r) = \frac{|\omega * (\nabla\varphi)_r(x)|^2}{|\omega * \varphi_r(x)|^2} \frac{dx\,dr}{r}$$

is a Carleson measure in $\Delta(x_0, R)$. *Moreover, there exists* $\tau_0 \in (0, 1/20]$ *depending on dimension and* $\eta = \eta(n, \Lambda)$ *such that for all* $\tau \in (0, \tau_0)$

$$\|\nu\|_{C(\Delta(x_0,\tau R))} \leq C \left(\tau^\eta + \left\| \alpha_2(x,r)^2 \frac{dx\, dr}{r} \right\|_{C(\Delta(x_0,100R))} \right), \tag{4.2}$$

where $C = C(n, \Lambda)$. *If* A *satisfies the (global) weak DKP condition, then the estimate* (4.2) *as well as*

$$\|\nu\|_C \leq C \left\| \alpha_2(x,r)^2 \frac{dx\, dr}{r} \right\|_C, \tag{4.3}$$

holds for

$$d\nu(x,r) = \frac{|\omega_L^\infty * (\nabla\varphi)_r(x)|^2}{|\omega_L^\infty * \varphi_r(x)|^2} \frac{dx\, dr}{r},$$

where ω_L^∞ *is the elliptic measure with pole at infinity*

Proof. We prove (4.2) in the finite pole case. To show (4.2) in the infinite pole case one can replace the use of both Lemma 3.3 and Lemma 3.4 with Lemma 3.7. Let $x \in \mathbb{R}^n$ and $r > 0$ be such that $\Delta(x,r) \subset \Delta(x_0, R)$. Set $G(Y) = G(Z_0, Y)$ the Green function for operator L with pole at Z_0 and $\omega := \omega_L^{Z_0}$. We note that $G(Y)$ solves $-\operatorname{div} A^T \nabla G = \delta_{Z_0}$, so that G is a solution to $L^T u = 0$ away from Z_0 vanishing on the boundary. In particular, since the matrix A^T has the same α_2 and γ numbers as A (see Remark 2.9), we may apply Corollary 3.11 and Lemma 2.10 to $u(Y) = G(Y)$ and we will do this later.

We will estimate the density

$$\frac{|\omega * (\nabla\varphi)_r(x)|^2}{|\omega * \varphi_r(x)|^2}.$$

We start with replacing the denominator by the energy $E_G(x,r)$. By the local doubling property of ω (Lemma 3.4) and the properties of φ we have

$$\omega * \varphi_r(x) \approx \frac{\omega(\Delta(x,r))}{|\Delta(x,r)|}.$$

Then using Lemma 3.3 (with L^T in place of L) we have

$$\omega * \varphi_r(x) \approx \left(\fint\!\!\fint_{T(x,2r)} |\nabla_Y G(Y)|^2\, dY \right)^{1/2} = E_G(x, 2r)^{1/2}.$$

Thus,

$$\begin{aligned} \frac{|\omega * (\nabla\varphi)_r(x)|^2}{|\omega * \varphi_r(x)|^2} &\approx |\omega * (\nabla\varphi)_r(x)|^2 E_G(x,2r)^{-1} \\ &\approx \sum_{i=1}^n |\omega * (\partial_{x_i}\varphi)_r(x)|^2 E_G(x,2r)^{-1}. \end{aligned} \tag{4.4}$$

The following claim will essentially prove the theorem.

Claim 4.5.

$$|\omega * (\partial_{x_i}\varphi)_r(x)|^2 \lesssim \gamma(x,2r)^2 E_G(x,2r) + \sum_{\ell=1}^n E_{G,\ell}(x,2r).$$

Proof of Claim 4.5. We need to make a few simple observations. We have $(\partial_{x_i}\varphi)_r(x) = r\partial_{x_i}\varphi_r(x)$. Let $g \in C_0^\infty([-1,1])$, $0 \le g \le 1$, $g = 1$ on $[-1/2, 1/2]$ then define

$$\Phi_{x,r}(y,s) := \varphi_r(x-y)g(s/r).$$

Then $\partial_{y_i}\Phi_{x,r}(y,s)$ is a smooth extension of $\partial_{y_i}\varphi_r(x-y)$, that is, $\partial_{y_i}\Phi_{x,r} \in C_0^\infty(B(x,2r))$ and $\partial_{y_i}\Phi_{x,r}(\cdot,0) = \partial_{y_i}\varphi_r(x-y)$. Moreover, for $j = 1,\dots n$, we have

$$|\partial_t\partial_{y_i}\Phi_{x,r}(y,s)|, |\partial_{y_j}\partial_{y_i}\Phi_{x,r}(y,s)| \lesssim \frac{1}{r^{n+2}}\chi_{B(x,2r)}(y,s), \tag{4.6}$$

where the implicit constants depend on dimension alone. The Riesz formula (2.4) gives

$$\begin{aligned}\omega * (\partial_{x_i}\varphi)_r(x) &= r\int_{\mathbb{R}^n} \partial_{x_i}\varphi_r(x-y)\,d\omega(y)\\ &= r\iint_{\mathbb{R}^{n+1}_+} A^T(Y)\nabla_{y,s}G(y,s)\cdot\nabla_{y,s}\partial_{y_i}\Phi_{x,r}(y,s)\,dy\,ds,\end{aligned}$$

where we used that $\partial_{y_i}\Phi_{x,r}(Z_0) = 0$ and we are using the convention that $G = 0$ in the lower half space $\mathbb{R}^{n+1}_-$. Now we let $A_0 = A_{x,r}$ be a constant matrix attaining the infimum in the definition of $\gamma(x,r)$. We write

$$\begin{aligned}&\iint_{\mathbb{R}^{n+1}_+} A^T(Y)\nabla_{y,s}G(y,s)\cdot\nabla_{y,s}\partial_{y_i}\Phi_{x,r}(y,s)\,dy\,ds\\ &= \iint_{\mathbb{R}^{n+1}_+} (A^T(Y)-A_0^T)\nabla_{y,s}G(y,s)\cdot\nabla_{y,s}\partial_{y_i}\Phi_{x,r}(y,s)\,dy\,ds\\ &\quad + \iint_{\mathbb{R}^{n+1}_+} A_0^T\nabla_{y,s}G(y,s)\cdot\nabla_{y,s}\partial_{y_i}\Phi_{x,r}(y,s)\,dy\,ds.\end{aligned}$$

In summary, we have shown

$$|\omega * (\partial_{x_i}\varphi)_r(x)| \le I + II, \tag{4.7}$$

where

$$I := r\left|\iint_{\mathbb{R}^{n+1}_+} (A^T(Y)-A_0^T)\nabla_{y,s}G(y,s)\cdot\nabla_{y,s}\partial_{y_i}\Phi_{x,r}(y,s)\,dy\,ds\right|$$

and

$$II := r\left|\iint_{\mathbb{R}^{n+1}_+} A_0^T\nabla_{y,s}G(y,s)\cdot\nabla_{y,s}\partial_{y_i}\Phi_{x,r}(y,s)\,dy\,ds\right|.$$

First we handle I. By the Cauchy-Schwarz inequality and (4.6)

$$\begin{aligned}I &\lesssim \left(\fint\!\!\fint_{T(x,2r)} |A-A_0|^2\,dY\right)^{1/2}\left(\fint\!\!\fint_{T(x,2r)} |\nabla_Y G(Y)|^2\,dY\right)^{1/2}\\ &\le \gamma(x,2r)E_G(x,2r)^{1/2}.\end{aligned} \tag{4.8}$$

To handle term II we write $\nabla_{y,s}G(y,s) = (\nabla_y G(y,s),0)^T + (0,\partial_s G(y,s))^T$ to see

$$\begin{aligned}II \le\ & r\left|\iint_{\mathbb{R}^{n+1}_+} A_0^T(\nabla_y G(y,s),0)^T\cdot\nabla_{y,s}\partial_{y_i}\Phi_{x,r}(y,s)\,dy\,ds\right|\\ &+ r\left|\iint_{\mathbb{R}^{n+1}_+} A_0^T(0,\partial_s G(y,s))^T\cdot\nabla_{y,s}\partial_{y_i}\Phi_{x,r}(y,s)\,dy\,ds\right|\end{aligned}$$

$$= II_1 + II_2.$$

To handle II_1 we use the boundedness of A and (4.6) to see

$$II_1 \lesssim \iint_{T(x,2r)} |\nabla_y G(y,s)|\, dy\, ds \lesssim \sum_{\ell=1}^{n} E_{G,\ell}(x,2r)^{1/2}.$$

To handle II_2 we integrate by parts in s noting that $G = 0$ on $\mathbb{R}^n \times \{0\}$ and $\Phi \in C_0^\infty(B(x,2r))$, but first we write out the matrix multiplication using the notation $(A_0)_{i,j} =: a^0_{i,j}$. We obtain

$$\begin{aligned} II_2 &= r\left|\sum_{j=1}^{n+1} \iint a^0_{n+1,j} \partial_s G(y,s)(\partial_{y_j}\partial_{y_i}\Phi_{x,r}(y,s))\, dy\, ds\right| \\ &= r\left|\sum_{j=1}^{n+1} \iint a^0_{n+1,j} G(y,s)(\partial_s\partial_{y_j}\partial_{y_i}\Phi_{x,r}(y,s))\, dy\, ds\right|, \end{aligned}$$

where we used the notation $\partial_{y_{n+1}} = \partial_s$ and integrated by parts in s. Now we integrate by parts in the y_i variable using that $\Phi \in C_0^\infty(B(x,2r))$ and that $i \in \{1,\ldots,n\}$ to obtain

$$II_2 = r\left|\sum_{j=1}^{n+1} \iint a^0_{j,n+1} \partial_{y_i} G(y,s)(\partial_s\partial_{y_j}\Phi_{x,r}(y,s))\, dy\, ds\right|. \tag{4.9}$$

Now using the boundedness of A, (4.9) and (4.6) we obtain

$$II_2 \lesssim E_{G,i}(x,2r)^{1/2}.$$

Putting together our estimates for II_1 and II_2 we have

$$II \lesssim \sum_{\ell=1}^{n} E_{G,\ell}(x,2r)^{1/2}.$$

Combining this bound with (4.8) and plugging into (4.7) gives

$$|\omega * (\partial_{x_i}\varphi)_r(x)| \lesssim \gamma(x,2r) E_G(x,2r)^{1/2} + \sum_{i=1}^{n} E_{G,i}(x,2r)^{1/2},$$

which proves the claim. □

Now using Claim 4.5 and (4.4), we have

$$\begin{aligned} \frac{|\omega * (\nabla\varphi)_r(x)|^2}{|\omega * \varphi_r(x)|^2} &\lesssim \gamma(x,2r)^2 + \sum_{i=1}^{n} \frac{E_{G,i}(x,2r)}{E_G(x,2r)} \\ &\lesssim \gamma(x,2r)^2 + \sum_{i=1}^{n} \beta_i(x,2r). \end{aligned} \tag{4.10}$$

Returning the the measure ν we have

$$d\nu(x,r) \lesssim \left(\gamma(x,2r)^2 + \sum_{i=1}^{n} \beta_i(x,2r)\right) \frac{dx\, dr}{r}. \tag{4.11}$$

The estimate (4.2) then follows by Corollary 3.11 and Lemma 2.10 (applied with $u = G$ and the operator $L^T = -\operatorname{div} A^T\nabla$, see Remark 2.9). To obtain (4.3), we note that (4.2) holds for $x_0 = 0$ and any $R > 0$ since the pole is infinite. Then for any fixed $\tau \in (0, \tau_n]$ it holds

$$\|\nu\|_C = \sup_{R>0} \|\nu\|_{C(\Delta(0,\tau R))} \leq C\left(\tau^\eta + \left\|\alpha_2(x,r)^2 \frac{dx\,dr}{r}\right\|_C\right).$$

As τ can be taken arbitrarily small, this shows (4.3).

□

5. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We will first 'prove' the infinite pole case, which is immediate from Theorem 4.1, Lemma 2.19 and the work of Korey [Kor98a]. To prove the finite pole case, we will need to introduce a change of pole argument and the 'kernel function'.

Theorem 5.1. *Let $L = -\operatorname{div} A\nabla$ be a divergence form elliptic operator on $\mathbb{R}^{n+1}_+$, whose coefficient matrix A satisfies the vanishing weak DKP condition. Let ω_L^∞ be the elliptic measure with pole at infinity. Then $\omega_L^\infty \ll \mathcal{L}^n$, $\omega_L^\infty \in A_\infty$ and $k_L^\infty(y) := \frac{d\omega_L^\infty}{dx}(y)$ has the property that $\log k_L^\infty \in VMO(\mathbb{R}^n)$.*

The function $k_L^\infty(y)$ is often referred to as the elliptic kernel with pole at infinity.

Proof of Theorem 5.1. By Theorem 4.1 and the fact that A satisfies the weak DKP condition it holds that the measure ν defined by

$$d\nu(x,r) = \frac{|\omega_L^\infty * (\nabla\varphi)_r(x)|^2}{|\omega_L^\infty * \varphi_r(x)|^2} \frac{dx\,dr}{r}$$

is a Carleson measure. Since ω_L^∞ is a doubling measure (see Lemma 3.7), Theorem 2.16 implies that ω_L^∞ is an $A_\infty(dx)$ measure, that is, $k_L^\infty = \frac{d\omega_L^\infty}{dx}$ exists and is an A_∞ weight. It follows that $\log k_L^\infty \in BMO(\mathbb{R}^n)$ [GCRdF85]. Moreover, since A satisfies the vanishing weak DKP condition Theorem 4.1 implies that ν is a Carleson measure with vanishing trace. Therefore, by [Kor98a, Theorem 1] $\log k_L^\infty \in VMO(\mathbb{R}^n)$ (see Theorem 2.30 (4) implies (5)). □

We observe that in the proof above, we did not use the *vanishing* assumption in the weak DKP condition to obtain that ω_L^∞ is an A_∞ weight and that $\log k_L^\infty \in BMO(\mathbb{R}^n)$. In particular, we have also proven the following theorem.

Theorem 5.2. *Let $L = -\operatorname{div} A\nabla$ be a divergence form elliptic operator on $\mathbb{R}^{n+1}_+$, whose coefficient matrix A satisfies the weak DKP condition. Let ω_L^∞ be the elliptic measure with pole at infinity. Then $\omega_L^\infty \ll \mathcal{L}^n$, $\omega_L^\infty \in A_\infty(dx)$, and $k_L^\infty(y) := \frac{d\omega_L^\infty}{dx}(y)$ has the property that $\log k \in BMO(\mathbb{R}^n)$. The implicit constants in the statements $\omega_L^\infty \in A_\infty$ and $\log k \in BMO(\mathbb{R}^n)$ are each bounded by a constant depending on n, ellipticity and $\left\|\alpha_2(x,r)^2 \frac{dx\,dr}{r}\right\|_C$.*

In order to prove the second half of Theorem 1.2 (i.e. for finite poles), we need to move the pole in Theorem 5.1 from infinity to a point $X \in \mathbb{R}^{n+1}_+$. To do so we need some standard estimates for the quotient of solutions to divergence form elliptic equations. Most of these estimates can be found in [JK82a, Ken94], where they are stated for harmonic functions or operators with symmetric coefficients. But upon inspection the proofs do not rely on the symmetry when we use the appropriate notion of Green's functions for L (see Definition 2.3 and also [HMT]).

Lemma 5.3 (Comparison principle)**.** *Let L be a divergence form elliptic operator with ellipticity Λ and $x \in \mathbb{R}^n$ and $r > 0$.. If $Lu = Lv = 0$, $u, v \geq 0$ in $T(x, 2r)$, u and v are non-trivial functions which vanish continuously on $\Delta(x, 2r)$ then*

$$\frac{u(X)}{v(X)} \approx \frac{u((x,r))}{v((x,r))}, \quad \forall X \in T(x,r),$$

where the implicit constants depend on n, Λ.

Lemma 5.4 (Quotients of non-negative solutions)**.** *Let L be a divergence form elliptic operator with ellipticity Λ and $x \in \mathbb{R}^n$ and $r > 0$. If $Lu = Lv = 0$, $u, v \geq 0$ in $T(x, 2r)$, and u and v vanish continuously on $\Delta(x, 2r)$, then u/v is Hölder continuous of order $\gamma = \gamma(n, \Lambda)$ in $\overline{B(x,r) \cap \mathbb{R}^{n+1}_+}$. In particular, $\lim_{Y \to y}(u/v)(Y)$ exists[6] and, moreover,*

$$\left| \frac{u(X)}{v(X)} - \frac{u((x,r))}{v((x,r))} \right| \leq C \left(\frac{|X - x|}{r} \right)^{\gamma} \frac{u((x,r))}{v((x,r))}, \quad \forall X, Y \in T(x,r), \tag{5.5}$$

where the constant $C > 0$ and $\mu \in (0, 1)$ depend on n, Λ.

Lemma 5.6 (Kernel function [Ken94])**.** *Let L be a divergence form elliptic operator with ellipticity Λ and $x \in \mathbb{R}^n$ and $r > 0$. For every $X_0 = (x_0, t_0)$ and $X_1 = (x_1, t_1)$ there exists a kernel function $H(X_0, X_1, z)$ (a function of z) defined by*

$$H(X_0, X_1, z) := \frac{d\omega_L^{X_0}}{d\omega_L^{X_1}}(z).$$

The kernel function is given by

$$H(X_0, X_1, z) = \lim_{Z \to z} \frac{G_L(X_0, Z)}{G_L(X_1, Z)},$$

where the limit is taken inside of $\mathbb{R}^{n+1}_+$. In particular, by Lemmas 5.3 and 5.4 for $t' := \min\{t_0, t_1\}/10$ and $|z - z'| < t'$ it holds

$$|H(X_0, X_1, z) - H(X_0, X_1, z')| \leq C \frac{G_L(X_0, (z, t'))}{G_L(X_1, (z, t'))} \left(\frac{|z - z'|}{t'} \right)^{\gamma}$$

and

$$C^{-1} \frac{G_L(X_0, (z, t'))}{G_L(X_1, (z, t'))} \leq H(X_0, X_1, z) \leq C \frac{G_L(X_0, (z, t'))}{G_L(X_1, (z, t'))}.$$

Here C and γ depend on n and Λ.

We need a kernel function that takes X_1 to infinity, in order to compare elliptic measure with fixed poles to that with pole at infinity. We produce this function with the following lemma.

[6] Here the limit is taken within $\mathbb{R}^{n+1}_+$.

Lemma 5.7 (Kernel function with infinite argument)**.** *Let L be a divergence form elliptic operator with ellipticity Λ. Given $X_0 = (x_0, t_0) \in \mathbb{R}^{n+1}_+$ the kernel function (as a function of z)*

$$H_\infty(X_0, z) = \frac{d\omega_L^{X_0}}{d\omega_L^\infty}(z)$$

exists as a locally Hölder continuous function of order $\gamma = \gamma(n, \Lambda)$. Moreover, for every $\kappa > 1$ there exists $C_\kappa = C_\kappa(\kappa, n, \Lambda)$ such that

$$|H_\infty(X_0, z) - H_\infty(X_0, z')| \leq C_\kappa \frac{G_L(X_0, (x_0, t_0/4))}{U(X_0)} \left(\frac{|z - z'|}{t_0}\right)^\gamma \tag{5.8}$$

for all $z, z' \in \Delta(x_0, 5\kappa t_0)$, $|z - z'| < t_0/4$ and

$$(C_\kappa)^{-1} \frac{G_L(X_0, (x_0, t_0/4))}{U(X_0)} \leq H_\infty(X_0, z) \leq C_\kappa \frac{G_L(X_0, (x_0, t_0/4))}{U(X_0)}. \tag{5.9}$$

for all $z \in \Delta(x_0, 5\kappa t_0)$. Here U is the Green function for L with pole at infinity, see Lemma 3.5.

Proof. We drop the subscript L from the Green function and the elliptic measure throughout the proof. Let $\kappa > 1$. We recall that $U(Y) = \lim_{j\to\infty} u_{k_j}(Y) := \lim_{j\to\infty} \frac{G((0,2^{k_j}),Y)}{G((0,2^{k_j}),(0,1))}$ and ω^∞ is the weak limit of $\frac{\omega^{(0,2^{k_j})}}{G((0,2^{k_j}),(0,1))}$. Set $X_j := (0, 2^{k_j})$, $u_j = u_{k_j}$ and $\omega_j = \frac{\omega^{(0,2^{k_j})}}{G((0,2^{k_j}),(0,1))} = \frac{\omega^{X_j}}{G((0,2^{k_j}),(0,1))}$. We will often use that the t-coordinate of X_j tends to infinity.

Now, by Lemma 5.6,

$$\frac{d\omega^{X_0}}{d\omega^{X_j}}(z) = H(X_0, X_j, z)$$

exists as a Hölder continuous function. Multiplying, by $G(X_j, (0, 1))$ $(= G((0, 2^{k_j}), (0, 1)))$ we have

$$\frac{d\omega^{X_0}}{d\omega_j}(z) = G(X_j, (0, 1))H(X_0, X_j, z) =: H_j(X_0, z)$$

is locally Hölder continuous. More specifically, provided $2^{k_j} > t_0$, Lemma 5.6 gives the estimates

$$|H_j(X_0, z) - H_j(X_0, z')| \leq M_j \left(\frac{|z - z'|}{t_0}\right)^\gamma$$

for $z, z' \in \Delta(x_0, 10\kappa t_0)$, $|z - z'| < t_0/4$ and

$$m_j \leq |H_j(X_0, z)| \leq M_j$$

for $z \in \Delta(x_0, 10\kappa r)$, where M_j and m_j depend on κ, j, n and Λ and are defined by

$$M_j := \sup\left\{\frac{G(X_j, (0, 1))}{G(X_j, (y, t_0/4))} G(X_0, (y, t_0/4)) : y \in \Delta(x_0, 10\kappa t_0)\right\}$$

and

$$m_j := \inf\left\{\frac{G(X_j, (0, 1))}{G(X_j, (y, t_0/4))} G(X_0, (y, t_0/4)) : y \in \Delta(x_0, 10\kappa t_0)\right\}.$$

By the Harnack inequality

$$M_j \approx m_j \approx \frac{G(X_j, (0,1))}{G(X_j, (x_0, t_0/4))} G(X_0, (x_0, t_0/4))$$

where the implicit constants depend only on κ, n, Λ, but not on j. On the other hand,

$$\lim_{j\to\infty} \frac{G(X_j, (0,1))}{G(X_j, (x_0, t_0/4))} G(X_0, (x_0, t_0/4)) = \frac{G_L(X_0, (x_0, t_0/4))}{U(x_0, t_0/4))} \approx \frac{G_L(X_0, (x_0, t_0/4))}{U(X_0)}.$$

Thus, there exists C_κ such that for all sufficiently large j

$$|H_j(X_0, z) - H_j(X_0, z')| \le C_\kappa \frac{G_L(X_0, (x_0, t_0/4))}{U(X_0)} \left(\frac{|z - z'|}{t_0}\right)^\gamma \tag{5.10}$$

for all $z, z' \in \overline{\Delta(x_0, 5\kappa t_0)}$, $|z - z'| < t_0/4$ and

$$(C_\kappa)^{-1} \frac{G_L(X_0, (x_0, t_0/4))}{U(X_0)} \le H_j(X_0, z) \le C_\kappa \frac{G_L(X_0, (x_0, t_0/4))}{U(X_0)}. \tag{5.11}$$

for all $z \in \overline{\Delta(x_0, 5\kappa t_0)}$. Since $\kappa > 1$ is arbitrary, we may use the Arzela-Ascoli theorem to produce a subsequence $H_{j_m}(X_0, z)$ converging to a function $H_\infty(X_0, z)$ locally uniformly and such that for fixed $\kappa > 1$ estimates (5.10) and (5.11) hold. Moreover, by definition of H_j

$$d\omega^{X_0} = H_j(X_0, z) d\omega_j \tag{5.12}$$

and since $\omega_j \rightharpoonup \omega_\infty$ it holds that

$$d\omega^{X_0} = H_\infty(X_0, z) d\omega_\infty.$$

Indeed, fix $f \in C_c(\mathbb{R}^n)$ and $R > 0$ so that $\operatorname{supp} f \subset \Delta(0, R)$. Note that $f H_\infty(X_0, \cdot) \in C_c(\mathbb{R}^n)$. Then by (5.12) and the fact that $H_{j_m}(X_0, z)$ converges locally uniformly to $H_\infty(X_0, z)$

$$\begin{aligned}
&\int_{\mathbb{R}^n} f d\omega^{X_0}\, dx = \lim_{m\to\infty} \int_{\mathbb{R}^n} f H_{j_m}(X_0, z) d\omega_{j_m} \\
&= \lim_{m\to\infty} \int_{\mathbb{R}^n} [f H_\infty(X_0, z)] d\omega_{j_m} + \lim_{m\to\infty} \int_{\mathbb{R}^n} [f(H_{j_m}(X_0, z) - H_\infty(X_0, z))] d\omega_{j_m} \\
&= \int_{\mathbb{R}^n} f H_\infty(X_0, z)\, d\omega^\infty + 0,
\end{aligned}$$

where, to show the second limit in the second line was zero, we used

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} [f(H_{j_m}(X_0, z) - H_\infty(X_0, z))] d\omega_{j_m} \right| \\
&\qquad \le \|f\|_\infty \sup_{z \in \Delta(0,R)} |H_{j_m}(X_0, z) - H_\infty(X_0, z))| \omega_{j_m}(\Delta(0, R))
\end{aligned}$$

and that $\omega_{j_m}(\Delta(0, R))$ is uniformly bounded (in m) for sufficiently large m. To see the later fact, we first write

$$\omega_{j_m}(\Delta(0, R))) = \frac{\omega^{X_{j_m}}(\Delta(0, R))}{G(X_{j_m}, (0, 1))}$$

then the CFMS estimates (Lemma 3.2) and the Harnack inequality give

$$\frac{\omega^{X_{j_m}}(\Delta(0,R))}{G(X_{j_m},(0,1))} \lesssim R^{n-1}\frac{G(X_{j_m},(0,R))}{G(X_{j_m},(0,1))} \lesssim C_R$$

for sufficiently large m. As all of the desired properties of $H_\infty(X_0,z)$ have been demonstrated, this proves the lemma. □

Theorem 5.13. *Let $L = -\operatorname{div} A\nabla$ be a divergence form elliptic operator on $\mathbb{R}^{n+1}_+$, whose coefficient matrix A satisfies the vanishing weak DKP condition. Let $\omega_L^{X_0}$ be the elliptic measure with pole at X_0. Then $\omega_L^{X_0} \ll \mathcal{L}^n$, and $k_L^{X_0}(y) := \frac{d\omega_L^{X_0}}{dx}(y)$ has the property that $\log k_L^{X_0} \in VMO_{loc}(\mathbb{R}^n)$.*

Proof. To ease notation we drop the subscript L in the proof. By Theorem 5.1 $\omega^\infty \ll \mathcal{L}^n$ and $k^\infty := \frac{d\omega^\infty}{dx}$ satisfies $\log k^\infty \in VMO$. Then by Theorem 2.30 it holds that

$$\lim_{r_0\to 0^+}\sup_{x\in\mathbb{R}^n}\sup_{r\in(0,r_0)}\frac{\left(⨍_{\Delta(x,r)}(k^\infty(z))^2\,dz\right)^{1/2}}{⨍_{\Delta(x,r)}k^\infty(z)\,dz} = 1. \tag{5.14}$$

It then follows from Lemma 5.7 that $k^{X_0} = \frac{d\omega^X}{dx}$ exists $\mathcal{L}^n$ a.e. with

$$k^{X_0}(z) = \frac{d\omega^{X_0}}{dx}(z) = \frac{d\omega^{X_0}}{d\omega^\infty}(z)\frac{d\omega^\infty}{dx}(z) = H_\infty(X_0,z)k^\infty(z).$$

By Theorem 2.31 it suffices to show that for fixed $R > 0$ and $\epsilon > 0$ there exists $r_0 > 0$ such that

$$\sup_{x\in\Delta(0,R)}\sup_{r\in(0,r_0)}\frac{\left(⨍_{\Delta(x,r)}(k^{X_0}(z))^2\,dz\right)^{1/2}}{⨍_{\Delta(x,r)}k^{X_0}(z)\,dz} \le (1+\epsilon)^2. \tag{5.15}$$

To this end, let $R,\epsilon > 0$ be fixed. By (5.14) there exists r_1 such that

$$\sup_{x\in\mathbb{R}^n}\sup_{r\in(0,r_1)}\frac{\left(⨍_{\Delta(x,r)}(k^\infty(z))^2\,dz\right)^{1/2}}{⨍_{\Delta(x,r)}k^\infty(z)\,dz} \le 1+\epsilon. \tag{5.16}$$

Write $X_0 = (x_0,t_0)$ and let κ be large enough (depending on R) so that $\Delta(0,10R) \subset \Delta(x_0,\kappa t_0)$. By Lemma 5.7, that is, estimates (5.8) and (5.9) there exists a constant C', depending on κ, n and ellipticity, such that for $z,z' \in \Delta(x_0,5\kappa t_0)$ with $|z-z'| < t_0/4$

(5.17)
$$\left|\frac{H_\infty(X_0,z)}{H_\infty(X_0,z')}-1\right| = \frac{1}{H_\infty(X_0,z')}\left|H_\infty(X_0,z)-H_\infty(X_0,z')\right| \lesssim C'\left(\frac{|z-z'|}{t_0}\right)^\gamma.$$

Let $r_2 \in (0,t_0/4)$ be such that $C'\left(\frac{2r_2}{t_0}\right)^\gamma < \epsilon$. Then for $r < \min\{r_1,r_2,9R\} =: r_0$ and $x \in \Delta(0,R)$ it holds

$$\frac{\left(⨍_{\Delta(x,r)}(k^{X_0}(z))^2\,dz\right)^{1/2}}{⨍_{\Delta(x,r)}k^{X_0}(z)\,dz} = \frac{\left(⨍_{\Delta(x,r)}(H_\infty(X_0,z)k^\infty(z))^2\,dz\right)^{1/2}}{⨍_{\Delta(x,r)}H_\infty(X_0,z)k^\infty(z)\,dz}$$

$$\leq \frac{\sup_{z\in\Delta(x,r)} H(z)}{\inf_{z\in\Delta(x,r)} H(z')} \frac{\left(\fint_{\Delta(x,r)} (k^\infty(z))^2\,dz\right)^{1/2}}{\fint_{\Delta(x,r)} k^\infty(z)\,dz}$$
$$\leq (1+\epsilon)^2,$$

where we used the choice of r_0, (5.17) and (5.16). This shows (5.15) and hence $\log k^{X_0} \in VMO_{loc}(\mathbb{R}^n)$. □

Combining Theorems 5.1 and 5.13 gives Theorem 1.2.

6. Globalizing local DKP conditions and the works of Kenig and Pipher and Dindos, Petermichl and Pipher

In this section, we reflect on the relationship between our results and the related works [KP01, DPP07]. The results in [KP01, DPP07] are for Lipschitz domains, which requires one to obtain localized estimates; however, our Theorems 5.1 and 5.13 are for operators that are defined globally in a half space. To bridge the gap we show how to extend coefficients satisfying a local weak DKP condition to globally defined coefficients. We then proceed to show a local version of Theorem 5.2, which was originally shown in [KP01], under the hypothesis of a gradient condition on the coefficients. This condition trivially controls the weak DKP coefficients, that is, the α_2-numbers (and also the α_∞-numbers, defined in (6.1) below).

The result of [DPP07] is a small constant version of [KP01]. To be precise, in [DPP07] the authors show that the L^p-Dirichlet problem is solvable for any fixed $p > 1$ *provided* the Carleson norm in a related weak DKP-type condition is sufficiently small. In [DPP07] the authors use a condition that is comparable to using α_∞ coefficients defined by

$$\alpha_\infty(x,r) := \inf_{A_0\in\mathfrak{A}(\Lambda)} \sup_{(y,s)\in W(x,r)} |A(y,s) - A_0|. \tag{6.1}$$

(Note that $\alpha_2(x,r) \leq \alpha_\infty(x,r)$ so that the coefficients used in the current work are controlled by those in [DPP07].) Equivalently, they show a local $L^{p'}$ reverse-Hölder inequality for the elliptic kernel, under this smallness assumption. Recent work by the first author with Egert and Saari [BES21] seems to indicate that our main quantitative estimate, Theorem 4.1, provides an alternative approach to their result (and control on the constant in the reverse Hölder inequality) in the upper half-space. We discuss this briefly in Section 6.2.

6.1. **Extending local DKP conditions and an alternative approach to Kenig and Pipher's theorem.**

In this subsection we show how to extend coefficients A satisfying the weak DKP condition on a Carleson region $T(x,r)$ to coefficients $\widetilde{A}$ so that $\widetilde{A}$ agrees with A on a smaller Carleson region $T(x,cr)$ and satisfies a global weak DKP condition. We also want to ensure that we do not increase the the constant in the weak DKP condition 'too much'. Then we will show how to use this to show an analogue of the main result in [KP01].

For the purposes of constructing these extensions we write $\alpha_2(x, r, A)$ and $\alpha_2(x, r, \widetilde{A})$ to denote the α_2 coefficients for A and $\widetilde{A}$ respectively. For instance,

$$\alpha_2(x, r, \widetilde{A}) = \inf_{A_0 \in \mathfrak{A}(\Lambda)} \left(\fint\!\!\fint_{(y,s)\in W(x,r)} |\widetilde{A}(y, s) - A_0|^2 \right)^{1/2}.$$

We also define for $x \in \mathbb{R}^n$ and $r > 0$ the cylindrical region

$$\Gamma(x, r) = \Delta(x, r) \times (0, r).$$

Lemma 6.2. *Let A be a matrix that satisfies the weak DKP condition on $T(x_0, R_0)$ for some $x_0 \in \mathbb{R}^n$ and $R_0 > 0$. There exists $\widetilde{A}$ such that $\widetilde{A} = A$ on $\Gamma(x_0, cR_0)$ and $\widetilde{A}$ satisfies the weak DKP condition on $\mathbb{R}^{n+1}_+$, where c is an absolute constant. Moreover, we have the estimates*

$$\left\| \alpha_2(x, r, \widetilde{A})^2 \frac{dx\,dr}{r} \right\|_C \lesssim \left\| \alpha_2(x, r, A)^2 \frac{dx\,dr}{r} \right\|_{C(\Delta(x_0,R_0))} + \min\left\{ 1, \left\| \alpha_2(x, r, A)^2 \frac{dx\,dr}{r} \right\|^{4/(n+3)}_{C(\Delta(x_0,R_0))} \right\} \tag{6.3}$$

and for $r_0 < R_0$

$$\sup_{x\in\mathbb{R}^n} \left\| \alpha_2(x, r, \widetilde{A})^2 \frac{dx\,dr}{r} \right\|_{C(\Delta(x,r_0))} \lesssim \left\| \alpha_2(x, r, A)^2 \frac{dx\,dr}{r} \right\|_{C(\Delta(x_0,R_0))} + (r_0/R_0)^2, \tag{6.4}$$

where the implicit constants depend on dimension and ellipticity.

Proof. By translation we may assume $x_0 = 0$ and we set

$$C_A := \left\| \alpha_2(x, r, A)^2 \frac{dx\,dr}{r} \right\|_{C(\Delta(x_0,R_0))}.$$

We choose c an absolute constant so that $\Gamma(0, 10^{10}cR_0) \subset T(0, R_0)$. Set $R := 2cR_0$ and let A_0 be the constant coefficient matrix so that

$$\gamma(0, 50R, A) = \left(\fint\!\!\fint_{T(0,50R)} |A(y, s) - A_0|^2 \, dy\, ds \right)^{1/2}.$$

Note that the point-wise inequality on $\gamma(x, r)$ in Lemma 2.10 gives

$$\gamma(0, 50R, A) \lesssim (C_A)^{1/2}. \tag{6.5}$$

Now we set

$$\widetilde{A}(y, s) = \mathbb{1}_{\Gamma(0,R)}(y, s) \left[(1 - f(|y|))A(y, s) + f(|y|)A_0 \right] + \mathbb{1}_{(\Gamma(0,R))^c}(y, s)A_0,$$

where $f : [0, \infty) \to [0, 1]$ is the piece-wise defined function

$$f(a) := \begin{cases} 0 & \text{if } a \in [0, R/2] \\ (2/R)(a - R/2) & \text{if } a \in (R/2, R] \\ 1 & \text{if } a > R. \end{cases} \tag{6.6}$$

Note that f is $(2/R)$-Lipschitz. By inspection we see that $\widetilde{A} = A$ in $\Gamma(0, cR_0) = \Gamma(0, R/2)$, so we only need to verify the estimates on the Carleson norms.

To do this, we let $(x,r) \in \mathbb{R}^{n+1}_+$ and estimate $\alpha_2(x,r,\widetilde{A})$. We break our analysis up into cases and combine them later. In the cases below $(y,s) \in W(x,r)$.
Case 0: $W(x,r)$ does not meet $\Gamma(0,R)$. In this case $A(y,s) = A_0$ a constant, Λ-elliptic matrix in $W(x,r)$. Thus, $\alpha_2(x,r,\widetilde{A}) = 0$.
Case 1: $W(x,r)$ is contained in $\Gamma(0,R)$. In this case $r \le R$ and we let $A_{x,r}$ be a constant, Λ-elliptic matrix such that

$$\alpha(x,r,A) = \left(\fint\!\!\fint_{W(x,r)} |A(y,s) - A_{x,r}|^2 \, dy \, ds \right)^{1/2}.$$

Now set

$$\widetilde{A}_{x.r} := [1 - f(|x|)]A_{x,r} - f(|x|)A_0,$$

a constant, Λ-elliptic matrix We make the estimate

$$\begin{aligned}
|\widetilde{A}(y,s) - \widetilde{A}_{x,r}| &\le |[1 - f(|y|)]A(y,s) - [1 - f(|x|)]A_{x,r}| \\
&\quad + |[f(|x|) - f(|y|)]A_0| \\
&\le |[1 - f(|y|)](A(y,s) - A_{x,r})| + |f(|x|) - f(|y|)|(|A_0| + |A_{x,r}|) \\
&\le |A(y,s) - A_{x,r}| + \frac{8r}{R}\Lambda,
\end{aligned}$$

where we used that $f(|x|) \in [0,1]$, f is $(2/R)$-Lipschitz, $|x - y| < 2r$ and A_0 and $A_{x,r}$ are Λ-elliptic. By using $\widetilde{A}_{x.r}$ in the definition of $\alpha_2(x,r,\widetilde{A})$ we obtain

$$\alpha_2(x,r,\widetilde{A}) \le \alpha_2(x,r,A) + \frac{8r}{R}\Lambda.$$

Case 2: $W(x,r)$ meets both $\Gamma(0,R)$ and $\Gamma(0,R)^c$. In this case we take $\widetilde{A}_{x,r} = A_0$ and we estimate

$$|\widetilde{A}(y,s) - \widetilde{A}_{x,r}| = \mathbb{1}_{\Gamma(0,R)}(y,s)|[1 - f(|y|)]A(y,s) - [1 - f(|y|)]A_0| \tag{6.7}$$

We break into further cases, setting $M = \max\{100, C_A^{-2/(n+3)}\}$.
Case 2a: $W(x,r)$ meets both $\Gamma(0,R)$ and $\Gamma(0,R)^c$ and $r > R/M$. If $M = 100$ we use that $|1 - f(|y|)| \le 1$ and that A and A_0 are Λ-elliptic to deduce from (6.7) that

$$|\widetilde{A}(y,s) - \widetilde{A}_{x,r}| \le 2\Lambda \le 200\Lambda(r/R)$$

Otherwise, $M = C_A^{-2/(n+3)}$ and we deduce from (6.5)

$$\alpha(x,r,\widetilde{A}) = |W(x,r)|^{-1/2} \left(\iint_{W(x,r)\cap\Gamma(0,R)} |A(y,s) - A_0|^2 \right)^{1/2} \lesssim (R/r)^{(n+1)/2}(C_A)^{1/2}.$$

Here we used that $W(x,r) \subset T(0,50R)$ since $W(x,r)$ meets both $\Gamma(0,R)$ implies $r/2 < R$.
Case 2b: $W(x,r)$ meets both $\Gamma(0,R)$ and $\Gamma(0,R)^c$ and $r \le R/M$. In this case, $W(x,r)$ must meet the 'side' of $\partial\Gamma(0,R)$ since $W(x,r) \cap \{(y,R) : y \in \mathbb{R}^n\} = \emptyset$. Then there exists $y_0 \in W(x,r)$ with $|y_0| = R$ and hence for $(y,s) \in W(x,r)$ we have

$$|1 - f(|y|)| \le |1 - f(|y_0|)| + |f(|y_0|) - f(|y|)| \le 0 + 6r/R,$$

where we used that f is $(2/R)$-Lipschitz. Thus, using (6.7) we find

$$|\widetilde{A}(y,s) - \widetilde{A}_{x,r}| \le \frac{12r}{R}\Lambda,$$

where we used A and A_0 are Λ-elliptic.
Combining the the cases we have (by choice of c)

$$\alpha_2(x,r,\widetilde{A}) \le C\mathbb{1}_{T(0,10^{-3}R_0)}(x,r)\,[\alpha(x,r,A)+h(r,R,C_A)]\,,$$

where C depends on dimension and ellipticity and the function $h(r,R,C_A)$ is given by

$$h(r,R,C_A) = \begin{cases} (R/r)^{(n+1)/2}(C_A)^{1/2} & \text{if } r > R/M \\ (r/R) & \text{if } r \le R/M \end{cases}$$

if $M = (C_A)^{-2/(n+3)}$ and $h(r,R,C_A) = 200\Lambda(r/R)$ if $M = 100$. (Recall $M = M(C_A) = \max\{100,(C_A)^{-2/(n+3)}\}$.) The Carleson measure bounds follow from this estimate and this proves the Lemma. □

Now that we have a matrix satisfying the (global) weak DKP condition, we can apply the results of Section 5 to $\widetilde{A}$; however, we want to say things about the elliptic measure/kernel for $L = -\operatorname{div} A\nabla$, not $\widetilde{L} = -\operatorname{div}\widetilde{A}\nabla$. The following lemma allows us to pass estimates on $k_{\widetilde{L}}$ to k_L, albeit in a rough manner. [7]

Lemma 6.8 ([BTZ, Lemma 2.22]). *Suppose that $L_i = -\operatorname{div} A_i\nabla$, $i = 1,2$ are two divergence form Λ-elliptic operators with $A_1 = A_2$ on $T(x,10r)$. Let $\omega_i^{X_0}$ be the L_i-elliptic measure with pole at $X_0 \in T(x,8r)\setminus T(x,4r)$. Then $\omega_1^{X_0}|_{\Delta(x,r)}$ and $\omega_2^{X_0}|_{\Delta(x,r)}$ are mutually absolutely continuous. In particular, if $\omega_1^{X_0}|_{\Delta(x,r)}$ and $\mathcal{L}^n|_{\Delta(x,r)}$ are mutually absolutely continuous then so are $\omega_2^{X_0}|_{\Delta(x,r)}$ and $\mathcal{L}^n|_{\Delta(x,r)}$. Moreover,*

$$\frac{d\omega_1^{X_0}}{d\omega_2^{X_0}}(y) \approx 1, \quad \omega_2^{X_0} - a.e.\ y \in \Delta(x,r/2),$$

where the implicit constants depend on n, Λ.

In what follows, for $\Delta = \Delta(x,r)$ we set $X_\Delta = (x,12r)$. We give an alternative proof to the following quantitative result from [KP01].

Theorem 6.9. *Suppose that A satisfies the weak DKP condition on $T(x_0,R_0)$ for some $x_0 \in \mathbb{R}^n$ and $R_0 > 0$, that is,*

$$\left\| \alpha_2(x,r,A)^2\,\frac{dx\,dr}{r} \right\|_{\mathcal{C}(\Delta(x_0,R_0))} =: C_A < \infty.$$

Then there is an absolute constant c' such that for every Δ such that $\Delta \subset \Delta(x_0,c'R_0)$ it holds

$$\left(\fint_\Delta (k_L^{X_\Delta}(z))^p\,dz \right)^{1/p} \le C \fint_\Delta k_L^{X_\Delta}(z)\,dz, \tag{6.10}$$

where the constants $C, p > 1$ depend on n, Λ and C_A.

[7] In some applications, for example as we will discuss in the Section 6.2, we need a sharper, more quantitative estimate of $d\omega_1^{X_0}/d\omega_2^{X_0}$. In that case we appeal instead to [BTZ, Lemma 5.1 and Remark 5.11].

The estimate (6.10) is often referred to as the ‘reverse Holder inequality for the elliptic kernel’ and is equivalent to a local A_∞-type condition for the elliptic measure. If (6.10) holds for all Δ, we have that the $L^{p'}$ Dirichlet problem is solvable on $\mathbb{R}^{n+1}_+$ (see e.g. [HL18, Proposition 4.5]). The flexibility of our ‘globalization’ lemma (Lemma 6.2) and a ‘standard’ pullback mechanism[8] allows one extend the above theorem to the setting of Lipschitz domains as was done by Kenig and Pipher [KP01].

Proof of Theorem 6.9. By translation we may assume $x_0 = 0$. Let A be as in the statement of the theorem. Let $\widetilde{A}$ be the operator produced by Lemma 6.2 and $\widetilde{L} = -\operatorname{div} \widetilde{A}\nabla$. Then $\widetilde{A}$ satisfies the *global* weak DKP condition with

$$\left\| \alpha_2(x,r,\widetilde{A})^2 \frac{dx\,dr}{r} \right\|_C \lesssim C_A + 1.$$

By Theorem 5.2, $\omega^\infty_{\widetilde{L}} \in A_\infty$, where $\omega^\infty_{\widetilde{L}}$ is the elliptic measure for $\mathcal{L}$ with pole at infinity. Moreover, the constants in the A_∞ condition are controlled by C_A, n and ellipticity. It follows from basic properties of A_∞ weights (see Definition 2.13) that the kernel $k^\infty_{\widetilde{L}} = \frac{d\omega^\infty_{\widetilde{L}}}{dx}$ satisfies a reverse Hölder condition, that is,

$$\left(\fint_{\Delta(x,r)} (k^\infty_{\widetilde{L}}(z))^p\, dz \right)^{1/p} \leq \widetilde{C} \fint_{\Delta(x,r)} k^\infty_{\widetilde{L}}(z)\, dz, \quad \forall x \in \mathbb{R}^n, r > 0,$$

where $\widetilde{C}, p > 1$ depend only on n, ellipticity and C_A.

Next, we move the pole to a finite one. Using Lemma 5.7 we obtain[9]

$$\left(\fint_{\Delta(x,r)} (k^{X_\Delta}_{\widetilde{L}}(z))^p\, dz \right)^{1/p} \leq C' \fint_{\Delta(x,r)} k^{X_\Delta}_{\widetilde{L}}(z)\, dz, \tag{6.11}$$

where $p > 1$ is as above and C' depends only on n, Λ and C_A. To conclude, we need to change the operator $\widetilde{L}$ to L. Recall that in Lemma 6.2 $\widetilde{A} = A$ on $\Gamma(0, cR_0)$. Now we choose c' so small to ensure that $\Delta(x,r) \subset \Delta(0, c'R_0)$ implies $T(x, 50r) \subset \Gamma(0, cR_0)$. This allows us to apply Lemma 6.8 to any such $\Delta(x,r)$, so that we obtain

$$k^{X_\Delta}_{\widetilde{L}}(z) \approx k^{X_\Delta}_{L}(z), \quad \forall \Delta \subset \Delta(x_0, c'R), z \in \Delta,$$

with constants depending on n and ellipticity. Using this estimate in conjunction with (6.11) yields (6.10) and proves the theorem. □

6.2. **Remarks concerning the work of Dindos, Petermichl and Pipher.** In a few places we have made some remarks regarding how our work can be used to complement the work of [DPP07]. This subsection clarifies these remarks. In [DPP07] the authors show that for any fixed $p > 1$ the conclusion of Theorem 6.9, that is, (6.10) holds *provided* C_A therein is sufficiently small. We do not attempt to reprove this here. One reason is that we use [FKP91], which gives *rough* bounds for the A_∞ constant of a doubling weight in terms of the Carleson norm of the measure

[8]This pullback is called sometimes referred to as the Dahlberg-Kenig-Stein pullback, see [KP01, Dah86].

[9]This time we do not need to be as precise as we were in Theorem 5.13. Instead, we only need the bound (5.9), and we do not use (5.8).

μ and its doubling constant as in Theorem 2.16. In particular, the argument in [FKP91] as is written does not provide the estimates we ask for below. It should be noted that the work of Korey does *not* explicitly treat the small constant case, but rather the vanishing constant case, where the measure μ in Theorem 2.16 is a Carleson measure with vanishing trace (recall the definition in (2.7)).

After this article was completed, the first author, Egert and Saari [BES21] showed the following. Suppose w is a weight that is doubling with constant C_{doub}. For $\epsilon > 0$, there exists $\delta = \delta(n, C_{doub}, \epsilon) > 0$ such that if the measure μ in Theorem 2.16 with $\omega = w\,dx$ satisfies $\|\mu\|_C \le \delta$ then $[w]_{A_\infty} \le 1 + \epsilon$.

This allows one to show Theorem 6.9 for specific $p > 1$, *provided* the Carleson norm in the weak DKP condition is sufficiently small. In particular, this might give an alternative proof of the results in [DPP07]. Some indication of how to proceed to treat more general geometric settings, as was done in [DPP07], is contained in [BES21, Section 5].

We did not investigate whether one can control 'local A_∞ constants' with a localized estimate on the Carleson norm of μ, that is, $\|\mu\|_{C(\Delta_0)} < \delta$ implies (2.14) with $C = (1 + \epsilon)$ on all balls $\Delta(x, r)$ sufficiently small and well contained in Δ_0. This would allow one to use Theorem 4.1 more directly to prove the Theorems that follow it. In particular, it would (conveniently) eliminate the need for using the pole change arguments to treat the finite pole case.

Appendix. An extension of Theorem 1.2

For a more general domain Ω (even if not graphical), we may still define an elliptic matrix in Ω satisfying the weak DKP condition (resp. with vanishing trace). In fact, let $A(\cdot)$ be an elliptic matrix in Ω. We define the oscillation of $A(\cdot)$ similar as in Definition 2.5: For any $X \in \Omega$, let

$$\alpha_2(X) = \inf_{A_0 \in \mathfrak{A}(\Lambda)} \left(\fint\!\!\fint_{Y \in B(X, \delta(X)/2)} |A(Y) - A_0|^2 \, dY \right)^{1/2}, \tag{A.12}$$

where $\delta(X) := \mathrm{dist}(X, \partial\Omega)$. We say $A(\cdot)$ satisfies the weak DKP condition (resp. with vanishing trace), if the measure

$$d\mu(X) = \frac{\alpha_2(X)^2}{\delta(X)} dX$$

is a Carleson measure (resp. with vanishing trace).

Even though Theorem 1.2 is proven for the upper half space, it is not hard to show that the analogue also holds for more general classes of domains.

Theorem A.13. *Let Ω be a C^1-square Dini domain in $\mathbb{R}^{n+1}$. Let $A(\cdot)$ be an elliptic matrix in Ω which satisfies the weak DKP condition with vanishing trace. Then for any $X_0 \in \Omega$, the elliptic measure $\omega_\Omega^{X_0}$ is absolute continuous with respect to the boundary surface measure $\sigma := \mathcal{H}^n|_{\partial\Omega}$, and moreover, the Poisson kernel $k_\Omega := \frac{d\omega_\Omega^{X_0}}{d\sigma}$ satisfies* $\log k_\Omega \in VMO_{loc}(\partial\Omega)$.

When we say Ω is a C^1-square Dini domain, it means there exist $R > 0$, finitely many boundary points $x_i \in \partial\Omega$ and cylindrical regions $C(x_i, R)$[10] centered at x_i such that $\Omega \subset \cup_i C(x_i, R/2)$ and for each i, $\Omega \cap C(x_i, R)$ is the region above the graph of a C^1-square Dini function φ_i. Assume without loss of generality that $X_0 \in C(x_i, R) \setminus C(x_i, R/2)$ for each i. It is not hard to see that (with the help of a cut-off function) we can extend φ_i to a globally-defined function $\varphi : \mathbb{R}^n \to \mathbb{R}$, such that the modulus of continuity of $\nabla\varphi$ is also bounded above by θ. Moreover, applying [BTZ, Lemma 5.1] (in particular, see [BTZ, Remark 5.11]) to C^1 domains, we can show that if two elliptic operators L_1 and L_2 agree in $\Omega \cap C(x_i, R)$, then the ratio of their elliptic measures $d\omega_{L_2}^{X_0}/d\omega_{L_1}^{X_0}$ has small oscillation in a surface ball, whose radius is much smaller compared to R. In particular, it implies by a similar proof as that of Lemma 5.13 that

$$\log k_{L_1} \in VMO(\partial\Omega \cap C(x_i, R/2)) \iff \log k_{L_2} \in VMO(\partial\Omega \cap C(x_i, R/2)). \tag{A.14}$$

Therefore the proof of Theorem A.13 is reduced to the setting where Ω is the region above the graph of a single function $\varphi : \mathbb{R}^n \to \mathbb{R}$.

More precisely, let $\Omega \subset \mathbb{R}^{n+1}$ be the domain above the graph of $\varphi : \mathbb{R}^n \to \mathbb{R}$, where the modulus of continuity for $\nabla\varphi$ satisfies the square Dini condition (see (1.5)). Assume without loss of generality that $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$. Let $A(x,t)$ be a uniformly elliptic coefficient matrix in the graphical domain Ω, which satisfies the weak DKP condition with vanishing trace, which means the following[11] For any $x_0 \in \mathbb{R}^n$ fixed, we define the Whitney region

$$W_\Omega(x_0, r) := \left\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : x \in \Delta_r(x_0),\ \varphi(x) + \frac{r}{2} < t \leq \varphi(x) + r\right\},$$

and denote the L^2-oscillation of the matrix A as

$$\alpha_A(x_0, r) := \inf_{A_0 \in \mathfrak{A}(\Lambda)} \left(\frac{1}{|W_\Omega(x_0,r)|} \iint_{W_\Omega(x_0,r)} |A(x,t) - A_0|^2 dxdt \right)^{1/2}, \tag{A.15}$$

where, as in Definition 2.5 before, the infimum ranges over all constant coefficient matrices. We say A satisfies the weak DKP condition with vanishing trace, if

$$d\mu_A(x,r) = \alpha_A(x,r)^2 \frac{dxdr}{r} \tag{A.16}$$

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[10] Modulo an orthogonal transformation $C(x_i, r)$ is defined as $\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |x - x_i| < R, -f(R)R < t < f(R)R\}$ where $f(R) = \max\{1, 2\theta(R)\}$. The choice of $f(R)$ guarantees that the graph of φ on the ball $\Delta(x_i, R))$ is completely contained in the cylinder.

[11] Notice that our definition of $\alpha_A(\cdot)$ in (A.15) for graphical domains is slightly different from the general definition $\alpha_2(\cdot)$ in (A.12). We will justify that for C^1-square Dini domains (or any graphical domain whose tangent has small oscillations), they are in fact equivalent in any fixed graphical chart $C(x_i, R/2)$ of Ω. On one hand, let $(x_0, r) \in C(x_i, R/2)$ be arbitrary. Consider $X = (y, \varphi(y) + s)$ for some $y \in \Delta(x_0, r)$ and $\frac{r}{2} \leq s \leq r$. Assuming R is sufficiently small, we have that $\frac{r}{4} \leq \delta(X) \leq r$. Moreover $W_\Omega(x_0, r) \subset \cup_{j,k} B(X_{j,k}, r/8) \subset \cup_{j,k} B(X_{j,k}, \delta(X_i)/2)$, where $X_{j,k} = (y_k, \varphi(y_k) + \frac{r}{2} + \frac{r}{8} j)$, $j = 0, 1, \cdots, 4$ and $y_k = x \pm \frac{r}{8} k$, $k = 0, 1, \cdots, 8$. Thus $\alpha_A(x_0, r)^2 \lesssim \sum_{j,k} \alpha_2(X_{j,k})^2$. On the other hand, let $X \in C(x_i, R/2)$ be arbitrary. Suppose that $X = (x, \varphi(x) + s)$ for some $s > 0$. Assuming R is sufficiently small we have that $s/2 \leq \delta(X) \leq s$. Hence $B(X, \delta(X)/2) \subset W_\Omega(x, \frac{5}{2}\delta(X)) \cup W_\Omega(x, \frac{5}{4}\delta(X)) \cup W_\Omega(x, \frac{5}{8}\delta(X))$, and therefore $\alpha_2(X)^2 \lesssim \sum_{j=1}^{3} \alpha_A(x, \frac{5}{2j}\delta(X))^2$.

is a Carleson measure in $\mathbb{R}^{n+1}_+$ with vanishing trace (see Definition 2.6). Let u be a solution to the elliptic equation

$$-\operatorname{div}(A(x,t)\nabla u) = 0 \text{ in } \Omega.$$

We consider the flattening map

$$\Phi : (y,s) \in \mathbb{R}^{n+1}_+ \mapsto (y, s+\varphi(y)) =: (x,t) \in \Omega,$$

and a function $\tilde{u} : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ defined by $\tilde{u}(y,s) := u\circ\Phi(y,s)$. A simple computation shows that $\tilde{u}$ is the solution to the elliptic operator $-\operatorname{div}(B(y,s)\nabla\tilde{u}) = 0$ in $\mathbb{R}^{n+1}_+$, where the coefficient matrix $B(y,s)$ is given by

$$\begin{aligned} B(y,s) &= \det D\Phi \cdot (D\Phi(y,s))^{-1} A(\Phi(y,s)) \left(D\Phi^T(y,s)\right)^{-1} \\ &= \begin{pmatrix} \mathrm{Id}_n & 0 \\ (-\nabla\varphi(y))^T & 1 \end{pmatrix} A(\Phi(y,s)) \begin{pmatrix} \mathrm{Id}_n & -\nabla\varphi(y) \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{A.17}$$

We may define the L^2-oscillation of the matrix B as in (A.15), except to replace the integration region by the corresponding Whitney region in $\mathbb{R}^{n+1}_+$

$$W(x_0,r) := \Delta_r(x_0) \times \left(\frac{r}{2}, r\right].$$

Let A_0 be a constant coefficient matrix which achieve the infimum for $\alpha_A(x_0,r)$. In particular, A_0 has the same constants of ellipticity as $A(\cdot)$. We define

$$B_0 := \begin{pmatrix} \mathrm{Id}_{d-1} & 0 \\ (-\nabla\varphi(x_0))^T & 1 \end{pmatrix} A_0 \begin{pmatrix} \mathrm{Id}_{d-1} & -\nabla\varphi(x_0) \\ 0 & 1 \end{pmatrix}.$$

For any $(y,s) \in W(x_0,r)$, by the formula (A.17) as well as (1.4) we get

$$|B(y,s) - B_0| \lesssim |A(\Phi(y,s)) - A_0| + |\nabla\varphi(y) - \nabla\varphi(x_0)||A_0| \lesssim |A(\Phi(y,s)) - A_0| + \theta(r),$$

where the constant depends on $\|A(x,t)\|_\infty$. Therefore

$$\begin{aligned} |\alpha_B(x_0,r)|^2 &\le \frac{1}{|W(x_0,r)|} \iint_{W(x_0,r)} |B(y,s) - B_0|^2 dyds \\ &\lesssim \frac{1}{|W(x_0,r)|} \iint_{W(x_0,r)} |A(\Phi(y,s)) - A_0|^2 dyds + \theta(r)^2 \\ &\lesssim \frac{1}{|W_\Omega(x_0,r)|} \iint_{W_\Omega(x_0,r)} |A(x,t) - A_0|^2 dxdt + \theta(r)^2 \\ &= |\alpha_A(x_0,r)|^2 + \theta(r)^2. \end{aligned} \tag{A.18}$$

In the penultimate inequality of (A.18), we use the fact that $\Phi(W(x_0,r)) = W_\Omega(x_0,r)$. Similarly to (A.16), we define

$$d\mu_B(x,r) := \alpha_B(x,r)^2 \frac{dxdr}{r}.$$

Then we may compute its Carleson norm on each surface ball $\Delta \subset \partial\mathbb{R}^{n+1}_+$ and

$$\|\mu_B\|_{C(\Delta)} \lesssim \|\mu_A\|_{C(\Delta)} + \int_0^{r_\Delta} \theta(r)^2 \frac{dr}{r}.$$

In particular, if θ satisfies the square Dini condition, then

$$\|\mu_B\|_{C(\Delta)} \to 0 \text{ as } r_\Delta \to 0.$$

However μ_B may not be a Carleson measure at large scales because of the extra $\theta(r)^2$ term. To remedy that, let $R > 0$ be fixed and we use a similar construction as in Lemma 6.2 to define a new coefficient matrix $\widetilde{B}(\cdot)$, so that $\widetilde{B} \equiv B$ in $\Gamma(0, R/2)$, and $\widetilde{B}(\cdot)$ is a constant coefficient matrix in $\mathbb{R}^{n+1}_+ \setminus \Gamma(0, R)$. To be more precise, let $\widetilde{B}_0$ be a constant matrix which achieves the minimum of $\gamma(0, 100cR)$ for the matrix $B(\cdot)$. We define

$$\widetilde{B}(y, s) = \mathbb{1}_{\Gamma(0,R)}(y, s)\left[(1 - f(|y|))B(y, s) + f(|y|)\widetilde{B}_0\right] + \mathbb{1}_{(\Gamma(0,R))^c}(y, s)\widetilde{B}_0,$$

where f is a piece-wise linear function defined as in (6.6). Then Lemma 6.2 implies $\widetilde{B}(\cdot)$ is indeed a Carleson measure with vanishing trace in $\mathbb{R}^{n+1}_+$.

Let $\omega_\Omega^{X_0}$ denote the elliptic measures corresponding to the matrix A in Ω. Let ω^{Y_0} and $\widetilde{\omega}^{Y_0}$ denote the elliptic measure corresponding to the matrix B and $\widetilde{B}$, respectively, in $\mathbb{R}^{n+1}_+$. Theorem 1.2 gives that $\widetilde{\omega}^{Y_0} \ll \mathcal{L}^n = dx$ and the Poisson kernel $\widetilde{k}(x) := \frac{d\widetilde{\omega}^{Y_0}}{dx}(x)$ satisfies $\log \widetilde{k} \in VMO_{loc}(\mathbb{R}^n)$. Similar to the discussions before (A.14), this implies that $\omega^{Y_0} \ll \mathcal{L}^n$ in $B(0, R/2)$, and moreover the Poisson kernel $k(x) = \frac{d\omega^{Y_0}}{dx}(x)$ satisfies $\log k \in VMO(\mathbb{R}^n \cap B(0, R/2))$.

A simple change of variable shows that

$$\omega_\Omega^{X_0}(B_r(x, \varphi(x))) = \omega^{\Phi^{-1}(X_0)}(\Phi^{-1}(B_r(x, \varphi(x)))). \tag{A.19}$$

Besides, for each $x \in \overline{B(0, R/2)}$ there exists a constant $M > 1$ which only depend on $\|\nabla\varphi\|_{L^\infty(\overline{B(0,R)})}$ such that

$$B_{r/M}(x, 0) \subset \Phi^{-1}(B_r(x, \varphi(x))) \subset B_{Mr}(x, 0).$$

Let $\Pi_n : \mathbb{R}^{n+1} \to \partial\mathbb{R}^{n+1}_+ \approx \mathbb{R}^n$ denote the projection onto $\mathbb{R}^n$. Using (A.19), the fact that $\partial\Omega$ is a graph, and the Lebesgue differentiation theorem, we have

$$\begin{aligned}
&\frac{\omega_\Omega^{X_0}(B_r(x, \varphi(x)))}{\mathcal{H}^n_{\partial\Omega}(B_r(x, \varphi(x)))} \\
&= \frac{\omega^{\Phi^{-1}(X_0)}(\Phi^{-1}(B_r(x, \varphi(x))))}{\int_{\Pi_n(B_r(x,\varphi(x)))} \sqrt{1 + |\nabla\varphi(z)|^2} dz} \\
&= \frac{⨍_{\Phi^{-1}(B_r(x,\varphi(x)))\cap\partial\mathbb{R}^{n+1}_+} k(z) dz}{⨍_{\Pi_n B_r(x,\varphi(x))} \sqrt{1 + |\nabla\varphi(z)|^2} dz} \cdot \frac{\mathcal{L}^n(\Phi^{-1}(B_r(x, \varphi(x))) \cap \partial\mathbb{R}^{n+1}_+)}{\mathcal{L}^n(\Pi_n B_r(x, \varphi(x)))} \\
&\to \frac{k(x)}{\sqrt{1 + |\nabla\varphi(x)|^2}} \qquad \text{as } r \to 0+.
\end{aligned}$$

Therefore $\omega_\Omega^{X_0} \ll \mathcal{H}^n_{\partial\Omega}$ and the corresponding Poisson kernel in Ω

$$k_\Omega(x, \varphi(x)) := \frac{d\omega_\Omega^{X_0}}{d\mathcal{H}^n_{\partial\Omega}}(x, \varphi(x))$$

satisfies

$$k_\Omega(x, \varphi(x)) = \frac{k(x)}{\sqrt{1 + |\nabla\varphi(x)|^2}}.$$

Since $\sqrt{1+|\nabla\varphi(x)|^2}$ is continuous and (locally) bounded above and below, it follows that $\log k_\Omega \in VMO_{loc}(\partial\Omega \cap B(0,R/3))$. Therefore we have proven Theorem A.13.

Declarations

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