

BOUNDARY UNIQUE CONTINUATION ON C^1 -DINI DOMAINS AND THE SIZE OF THE SINGULAR SET

CARLOS KENIG AND ZIHUI ZHAO

ABSTRACT. Let u be a harmonic function in a C^1 -Dini domain $D \subset \mathbb{R}^d$ such that u vanishes on a boundary surface ball $\partial D \cap B_{5R}(0)$. We consider an effective version of its singular set (up to boundary) $\mathcal{S}(u) := \{X \in \overline{D} : u(X) = |\nabla u(X)| = 0\}$ and give an estimate of its $(d-2)$ -dimensional Minkowski content, which only depends on the upper bound of some modified frequency function of u centered at 0. Such results are already known in the interior and at the boundary of convex domains, when the standard frequency function is monotone at every point. The novelty of our work on Dini domains is how to compensate for the lack of such monotone quantities at boundary as well as interior points.

CONTENTS

1. Introduction	2
2. Preliminaries	6
3. Monotonicity formula for elliptic operators	11
4. Reduction from a Dini domain to an almost convex domain	18
5. Blow up analysis and compactness for boundary points	20
5.1. Blow up analysis in Ω_{X_0} using the monotonicity formula	24
5.2. Compactness	27
6. Frequency function for interior points	30
6.1. Monotonicity formula of the frequency function	30
6.2. Doubling property and frequency function beyond the critical scale	32
7. Quantitative symmetry	36
8. Spatial variations of the frequency function	44

2010 *Mathematics Subject Classification.* 35J25, 42B37, 31B35.

The first author was supported in part by NSF grant DMS-1800082, and the second author was partially supported by NSF grant DMS-1902756.

This version of the article has been accepted for publication, after peer review but is not the Version of Record and does not reflect post-acceptance improvements, or any corrections. The Version of Record is available online at: <http://dx.doi.org/10.1007/s00205-022-01771-7>. Use of this Accepted Version is subject to the publisher's Accepted Manuscript terms of use <https://www.springernature.com/gp/open-research/policies/acceptedmanuscript-terms>.

9. L^2 -best approximation theorem and the frequency function	55
10. Covering of the quantitative stratum	67
11. Proof of Theorem 1.1	75
References	78

1. INTRODUCTION

In this paper, we study the size of the singular set when u is a harmonic function in the Dini domain $D \subset \mathbb{R}^d$, such that u vanishes on an open set of the boundary. (Dini domains are C^1 domains given locally by graphs of functions whose gradient has a Dini modulus of continuity, see Definition 2.1.) More precisely, suppose $u = 0$ on the surface ball $B_{5R}(0) \cap \partial D$ with $0 \in \partial D$, we show that inside a smaller ball $B_{\frac{3}{20}R}(0)$, the singular set $\mathcal{S}(u) := \{X \in \overline{D} : u(X) = |\nabla u(X)| = 0\}$ is $(d-2)$ -rectifiable, and moreover its $(d-2)$ -dimensional Minkowski content is bounded by a constant depending on the upper bound of $N_0(4R)$, the frequency function of u centered at 0.

Theorem 1.1. *Let $R, \Lambda > 0$ be fixed. There exists $r_c > 0$ such that for any $(u, D) \in \mathfrak{S}(R, \Lambda)$ (see Definition 2.4) and any $r_0 \in (0, r_c)$, the effective singular set $\tilde{\mathcal{C}}_{r_0}(u) \cap \mathcal{N}(u)$ (see the definitions in Section 2) has the $(d-2)$ -dimensional Minkowski bound*

$$\mathcal{M}^{d-2,*} \left(\tilde{\mathcal{C}}_{r_0}(u) \cap \mathcal{N}(u) \cap B_{\frac{R}{10}}(0) \right) \leq C(d, R, \Lambda).$$

In particular the singular set satisfies

$$\mathcal{M}^{d-2,*} \left(\mathcal{S}(u) \cap B_{\frac{R}{10}}(0) \right) \leq C(d, R, \Lambda),$$

and $\mathcal{S}(u) \cap B_{\frac{R}{10}}(0)$ is $(d-2)$ -rectifiable.

Let us first give some historical background for the result in Theorem 1.1. It seems that such results originate in problems in optimization and control theory, as is described in [Wec, SW], where the authors proved that when D is a smooth domain $\mathcal{S}(u) \cap \partial D$ has zero $(d-1)$ -dimensional Hausdorff measure. The authors of [SW] were inspired by the following problems:

Consider a harmonic function u in D , with $0 \leq u \leq 1$ in D . Fix two points X_0 and X_1 in D . We wish to minimize $u(X_1)$ among all such u with $u(X_0) = \frac{1}{2}$.

and

Conjecture ([SW]): The Bang-Bang Property. There exists a unique minimizer $\tilde{u}$ to the above problem, and moreover $\tilde{u}$ is “Bang-Bang”, namely $\tilde{u}|_{\partial D} = \chi_F$ for some $F \subset \partial D$.

It can be proven (see [SW, Ken]) that in order to establish the conjecture in a domain D , it suffices to show that, with u as in Theorem 1.1, the set $\mathcal{S}(u) \cap \partial D \cap B_{\frac{R}{10}}(0)$ has zero $(d-1)$ -dimensional Hausdorff measure, as was proved for smooth domains in [Wec, SW].

Another problem that arises in control theory, this time in the study of exact boundary controllability, is the following:

Suppose that u is the (Dirichlet) Laplacian eigenfunction in D with eigenvalue $\lambda > 0$ (that is to say $-\Delta v = \lambda v$ in D and $v = 0$ on ∂D). Assume that the normal derivative of v is zero on $E \subset \partial D$, with $\mathcal{H}^{d-1}(E) > 0$. Is v identically zero?

By considering the harmonic lifting $u(x, t) = e^{-\sqrt{\lambda}t}v(x)$ on $D \times \mathbb{R}$ of v , the above problem reduces again to the fact that for a non-zero u , the set $\mathcal{S}(u) \cap (\partial D \times \mathbb{R})$ has zero surface measure on $\partial D \times \mathbb{R}$. So this problem again has a positive answer in smooth domains by [Wec, SW]. The result was extended to $C^{1,1}$ domains by Lin in [Lin, Theorem 2.3], using a reflection argument and the interior unique continuation result for elliptic equations with Lipschitz coefficient matrices, proven in [AKS] and [GL] using different methods (see also [Kaz]). In fact, Lin was able to go further, using Federer's dimension reduction argument, and show that $\mathcal{S}(u) \cap \partial D$ has Hausdorff dimension $(d-2)$.

Another historical source for problems of this kind is a classical question of L. Bers. Consider a harmonic function u in the upper half-plane $\mathbb{R}_+^d$, which is in $C^1(\overline{\mathbb{R}_+^d})$. Assume that there exists $E \subset \mathbb{R}^{d-1} = \partial\mathbb{R}_+^d$ such that $u = 0$ and $\partial_d u = 0$ on E . Is it true that if u is not identically zero, the Lebesgue measure of E must be zero? It is not hard to see that for planar domains this is indeed the case because the logarithm of the modulus of the gradient of a harmonic function is sub-harmonic. However for $d > 2$, Wolff [Wol] showed this to be false, and Bourgain and Wolff [BW] showed this to be false even under the assumption that $u \in C^{1,\alpha}(\overline{\mathbb{R}_+^d})$ for some $\alpha > 0$. Given this failure, it then became of interest to study the “intermediate” case where, in the notation of Theorem 1.1, $u = 0$ on an *open* set $B_{5R}(0) \cap \partial D$ of the boundary and attempt to show that $\mathcal{S}(u) \cap \partial D \cap B_{\frac{R}{10}}(0)$ has zero $(d-1)$ -dimensional Hausdorff measure unless $u \equiv 0$. As mentioned earlier, this was proved in [Wec, SW] in smooth domains and in [Lin] for $C^{1,1}$ domains, where it was also proved that it was a $(d-2)$ -dimensional set.

The next progress in these problems (always assuming that u vanishes on a surface ball) was due to Adolfsson, Escauriaza and Kenig [AEK] (see also Kenig-Wang [KW] for an alternative proof) who proved that for convex domains $\mathcal{S}(u) \cap \partial D$ has zero $(d-1)$ -dimensional Hausdorff measure. This was then followed by works of Adolfsson-Escauriaza [AE] and Kukavica-Nyström [KN], who proved (using different methods) the result for Dini domains. In fact, Adolfsson and Escauriaza [AE] proved that for Dini domains, the set $\mathcal{S}(u) \cap \partial D$ has Hausdorff dimension $(d-2)$, using the Federer dimension reduction argument as in Lin [Lin]. Recently, after many years without progress, Tolsa [Tol] proved that for all C^1 domains (and even Lipschitz domains with small constant) and non-identically-zero harmonic functions u , the set $\mathcal{S}(u) \cap \partial D$ has zero $(d-1)$ -dimensional Hausdorff measure. The proof of this result was influenced by very important new methods developed by Logunov and Malinnikova in [Log1, Log2, LM].

In the interior, using the expansion of harmonic functions by homogeneous harmonic polynomials, Han [Han] proved the $(d - 2)$ -rectifiability of $S(u)$; moreover Naber and Valtorta [NV0] estimated the $(d - 2)$ -dimensional Minkowski content of an effective version of $S(u)$ and $C(u)$. These two results also hold for elliptic equations with Lipschitz coefficients, by a perturbation of the harmonic case. The same argument does not seem to apply easily to the boundary. Instead, inspired by the quantitative stratification method developed by Naber and Valtorta (see [NV1, NV3]) to study regularity results in geometric variational problems, McCurdy [McC] took up the study of $S(u)$ and proved the analogue of Theorem 1.1 for convex domains, thus establishing the $(d - 2)$ -rectifiability of $S(u)$ and a bound on its $(d - 2)$ -dimensional Minkowski content. Recall that it had been established before using Federer's dimension reduction argument that $S(u) \cap \partial D$ has Hausdorff dimension at most $(d - 2)$ (see [Lin, AE] for $C^{1,1}$ domains and Dini domains, respectively), but it was not even known that $\mathcal{H}^{d-2}(S(u) \cap \partial D) < +\infty$. So it was a big improvement to prove $(d - 2)$ -rectifiability and find an upper bound on the Minkowski content, not just for the singular set restricted to the boundary, but also as the singular set approaches the boundary from the interior. For the difference between the Minkowski content and Hausdorff measure, see Definition 2.7.

Our approach is also based on the techniques introduced by Naber and Valtorta in [NV1]. These techniques have been found useful in many problems in geometric measure theory and geometric analysis, as long as symmetry in the problem can be somewhat quantified by a monotone quantity ([CN, NV2, FS, EE] to name just a few examples). For the convenience of the reader, we highlight the main differences between our argument with that of Naber and Valtorta as well as other prior work using the method.

- Unlike the interior case (see [NV0]) or the boundary case for convex domains (see [McC]), the standard Almgren's frequency function is not monotone for boundary points in Dini domains. Instead fixing each boundary point $X \in \partial D$, we use the transformation map $\Psi_X : \Omega_X \rightarrow D$ introduced in [AE] so that the domain becomes *almost convex* at that point. This way we get a monotone quantity for X , which measures how close the harmonic function u is to be homogeneous near X .
- Not only do we study the singular set on the boundary, we also want to study how the singular set approaches the boundary from the interior. To bring interior points (which are very close to the boundary) in the picture, we also study Almgren's frequency function $N(X, r)$ for interior points, especially when the scale r is big so that $B_r(X)$ intersects the boundary. The standard frequency function $N(X, r)$ is monotone increasing up to some critical scale $r_{cs}(X)$, which depends on $\text{dist}(X, \partial D)$. Outside of this monotonic interval, we replace $N(X, r)$ by the frequency function (of the same scale) centered at a boundary point closest to X , and justify that we only need to pay a small price for this modification. This is mainly treated in Section 6.
- For different points $X, X' \in \overline{D}$, the degrees of homogeneity may be different. (Especially recall that the monotone quantities for boundary points are defined in different domains Ω_X and $\Omega_{X'}$.) This creates a difficulty when we need to quantitatively measure how far u is from being homogeneous (see Section 9) and when we study off-spinal points in the dimension reduction argument (see Proposition 7.3). So we need to study the variation of the monotone quantity (in small scales) as X changes. This

is dealt with mainly in Section 8. (We remark that various degrees of homogeneity also appear in the earlier work of [DMSV, FS].)

- As discussed above, for both interior and boundary points we have found some alternatives in place of Almgren's frequency function to measure how close u is from being homogeneous near that point. For a boundary point X we use a monotone quantity defined in the reduced almost-convex domain Ω_X ; for an interior point X we use the frequency function centered at a boundary point closest to X . We need to show the errors accrued by these alternatives do not accumulate when we count across possibly an infinite number of dyadic scales, see Theorem 9.7.

Lastly, we remark that Dini domains seem to be the optimal class of domains for which one can prove the $(d-2)$ -dimensional quantitative estimate of the singular set as in Theorem 1.1. In fact, in order to show $\mathcal{S}(u) \cap \partial D$ has zero boundary surface measure (see for example the work of [Tol]), one needs to at least know that ∇u exists $\mathcal{H}^{d-1}$ almost everywhere at the boundary. On the other hand, the $\mathcal{H}^{d-1}$ almost everywhere existence of ∇u at the boundary is not enough to measure the $(d-2)$ -dimensional size of the same set $\mathcal{S}(u) \cap \partial D$. In particular when $d = 2$, we need ∇u to be defined pointwise at the boundary in order to count the number of singular points (namely $\mathcal{H}^0(\mathcal{S}(u) \cap \partial D)$). Dini domains are the optimal class of domains to guarantee that harmonic functions vanishing on an open set of the boundary are continuously differentiable up to the boundary.

After introducing preliminary notation and definitions in Section 2, we will define the modified frequency function and prove its monotonicity for boundary points in Sections 3 and 4, and for interior points in Section 6. More precisely, for boundary points we first use the transformation map to change the Laplacian operator in a Dini domain D to a divergence-form elliptic operator in an *almost convex* domain Ω (see Section 4). (Here almost convex is with respect to the elliptic operator and a single boundary point, see (3.1).) Then we use *boundary convexity* to define the frequency function for solutions to a non-constant elliptic operator and prove its monotonicity as well as quantitative rigidity (see Section 3). For interior points we establish the monotonicity and quantitative rigidity of the standard Almgren's frequency function up to some scale. We also state a closeness result between the frequency function centered at $X \in D$ and the modified frequency function centered at $\tilde{X}$, which is a boundary point closest to X . In Section 5, we first analyze the frequency function and use quantitative rigidity to characterize the tangent functions as homogeneous harmonic polynomials in the upper half-space. We also use the uniform bound of the frequency function to prove compactness for rescalings of the harmonic function. Section 7 proves quantitative versions of the following two principles for harmonic functions: if the frequency function centered at a point X is constant, then the harmonic function is homogeneous with respect to X ; if the frequency functions centered at two distinct points X, X' are both constant, then the harmonic function is invariant along the line connecting X and X' , and it is homogeneous with respect to any point on the line (with the same degree of homogeneity). These two principles are used extensively throughout the paper. In Section 8 we estimate the spatial variation of the (modified) frequency function for both boundary and interior points. In the beginning of Section 9, we state two Reifenberg theorems, which are generalizations of the classical Reifenberg theorem and which connect the estimate of the L^2 β -number with packing estimates and rectifiability. The reason why Reifenberg-type theorem is relevant is that

we can relate the $(d - 2)$ -dimensional β -number to the change of the frequency functions centered at effective singular points, as proven in Theorem 9.7. Finally in Section 10, we cover the effective singular set $\mathcal{N}(u) \cap \widetilde{C}_r(u)$ inductively, with the stopping criterion being that every point in the ball has a definite frequency drop. By Proposition 7.3, in a fixed ball all such points with small frequency drop are either clustered to a small set (*small* means of dimension $\leq (d - 3)$), or they must lie in a tubular neighborhood of a $(d - 2)$ -dimensional affine plane. In the second case, we control their volume using the Reifenberg-type theorem.

2. PRELIMINARIES

Definition 2.1. Let $\theta : [0, +\infty) \rightarrow (0, +\infty)$ be a nondecreasing function verifying

$$\int_0^* \frac{\theta(r)}{r} < \infty. \quad (2.2)$$

A connected domain D in $\mathbb{R}^d$ is a *Dini domain* with parameter θ if for each point X_0 on the boundary of D there is a coordinate system $X = (x, x_d)$, $x \in \mathbb{R}^{d-1}$, $x_d \in \mathbb{R}$ such that with respect to this coordinate system $X_0 = (0, 0)$, and there are a ball B centered at X_0 and a Lipschitz function $\varphi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ verifying the following

- (1) $\|\nabla\varphi\|_{L^\infty(\mathbb{R}^{d-1})} \leq C_0$ for some $C_0 > 0$;
- (2) $|\nabla\varphi(x) - \nabla\varphi(y)| \leq \theta(|x - y|)$ for all $x, y \in \mathbb{R}^{d-1}$;
- (3) $D \cap B = \{(x, x_d) \in B : x_d > \varphi(x)\}$.

Remark 2.3. By shrinking the ball B if necessary, we may modify the coordinate system so that $\nabla\varphi(0) = 0$.

Definition 2.4. Let $R, \Lambda > 0$ be finite. For any domain D and any function u , we say $(u, D) \in \mathfrak{H}(R, \Lambda)$ if

- D is a Dini domain in $\mathbb{R}^d$ with parameter θ , and it satisfies $\partial D \ni 0$, D is graphical inside the ball $B_{5R}(0)$ and

$$\theta(8R) < \frac{1}{72}, \quad \int_0^{16R} \frac{\theta(s)}{s} ds \leq 1; \quad (2.5)$$

- u is a non-trivial harmonic function in $D \cap B_{5R}(0)$,
- $u = 0$ on $\partial D \cap B_{5R}(0)$,
- the frequency function for $v := u \circ \Psi_0$ satisfies $N_0(4R) \leq \Lambda < +\infty$ (see Section 4 for the definitions of Ψ_0 and the frequency function).

Remark 2.6. In the above definition we indicate how (u, D) depends on R , but we do not explicitly indicate its dependence on the Dini parameter θ . This is an abuse of notation. In fact, throughout the paper we allow θ to vary as long as it is bounded from above by a given Dini function θ_0 . We use the quantifier R , which is determined by the Dini parameter θ , in the notation above because R will appear explicitly in the statements of lemmas and theorems.

For any compact set K and $\tau > 0$, we use

$$B_\tau(K) := \{X : \text{dist}(X, K) < \tau\}$$

to denote the τ -neighborhood of K .

Definition 2.7 (Minkowski content). Let A be a bounded subset of $\mathbb{R}^d$. The s -dimensional upper and lower Minkowski contents of A are defined as

$$\begin{aligned}\mathcal{M}^{s,*}(A) &= \limsup_{r \rightarrow 0+} (2r)^{s-d} |B_r(A)|, \\ \mathcal{M}_*^s(A) &= \liminf_{r \rightarrow 0+} (2r)^{s-d} |B_r(A)|.\end{aligned}$$

Moreover the upper and lower Minkowski dimensions of A are defined as

$$\begin{aligned}\overline{\dim}_{\mathcal{M}} A &= \inf\{s > 0 : \mathcal{M}^{s,*}(A) = 0\} = \sup\{s > 0 : \mathcal{M}^{s,*}(A) > 0\}, \\ \underline{\dim}_{\mathcal{M}} A &= \inf\{s > 0 : \mathcal{M}_*^s(A) = 0\} = \sup\{s > 0 : \mathcal{M}_*^s(A) > 0\}.\end{aligned}$$

Remark 2.8. It is not hard to see that $\mathcal{H}^s(A) \lesssim \mathcal{M}_*^s(A) \leq \mathcal{M}^{s,*}(A)$, where strict inequalities are possible. In other words, bounding the Minkowski content is stronger than bounding the Hausdorff measure of the same dimension. In fact let $\mathbb{Q}^d \subset \mathbb{R}^d$ be the d -dimensional rational lattice, its Hausdorff dimension is 0 and yet its Minkowski dimension is d . See [Mat, Chapter 5] for more about Hausdorff measure, Minkowski content and packing measure.

For a harmonic function u in $D \cap B_{5R}(0)$ with vanishing boundary data and D being a Dini domain, we have that u is continuously differentiable up to boundary (see [DEK]). Let

$$\mathcal{N}(u) := \{X \in \overline{D} \cap B_{5R}(0) : u(X) = 0\} \quad (2.9)$$

be the nodal set of u . In particular by the above definition $\partial D \cap B_{5R}(0) \subset \mathcal{N}(u)$. We also define the critical set of u by

$$\mathcal{C}(u) := \{X \in \overline{D} \cap B_{5R}(0) : \nabla u(X) = 0\},$$

and define the singular set of u by

$$\mathcal{S}(u) := \mathcal{N}(u) \cap \mathcal{C}(u) = \{X \in \overline{D} \cap B_{5R}(0) : u(X) = |\nabla u(X)| = 0\}. \quad (2.10)$$

Definition 2.11. Let u be an arbitrary non-trivial L^2 function. We use the following notation to denote its rescalings centered at $X \in \overline{D}$ with scale $r > 0$:

$$T_{X,r}u(Y) := \frac{u(X + rY) - u(X)}{\left(\frac{1}{r^d} \iint_{B_r(X) \cap D} |u - u(X)|^2 dZ\right)^{\frac{1}{2}}}, \quad T_ru(Y) := \frac{u(rY) - u(0)}{\left(\frac{1}{r^d} \iint_{B_r(0) \cap D} |u - u(0)|^2 dZ\right)^{\frac{1}{2}}}.$$

Remark 2.12. Clearly the above rescaling satisfies

$$\iint_{B_1(0) \cap \frac{D-X}{r}} |T_{X,r}u|^2 dY = 1.$$

Remark 2.13. We also remark that for any $X \in \mathcal{N}(u)$, the above rescaling satisfies

$$T_{X,r}u(Y) = \frac{u(X + rY)}{\left(\frac{1}{r^d} \iint_{B_r(X) \cap D} u^2 dZ\right)^{\frac{1}{2}}} = \frac{u(X + rY)}{\left(\frac{1}{r^d} \iint_{B_r(X)} u^2 dZ\right)^{\frac{1}{2}}}.$$

Since u vanishes on the boundary $\partial D \cap B_{5R}(0)$, it is convenient to simply extend u by zero outside of $D \cap B_{5R}(0)$. This is what we did in the second equality.

In this paper we work with a quantitative version of the critical set, defined as follows. Let $\beta, \alpha_0 > 0$ be two small constants to be determined later^{*} (whose values depend on d, Λ). For any $r > 0$ we set

$$\begin{aligned} C_r(u) &:= \left\{ X \in \overline{D} : \inf_{B_\beta(0)} |\nabla T_{X,r} u| \leq \alpha_0 \right\} \\ &= \left\{ X \in \overline{D} : \inf_{B_{\beta r}(X)} r^2 |\nabla u|^2 \leq \alpha_0^2 \cdot \frac{1}{r^d} \iint_{B_r(X)} |u - u(X)|^2 dZ \right\}. \end{aligned} \quad (2.14)$$

Clearly $C(u) \subset C_r(u)$ for every $r > 0$, and $X \in \overline{D} \setminus C_r(u)$ means that $|\nabla u|$ has a uniform lower bound centered at X at scale r . Heuristically, near X one of the following two scenarios happen. Either u blows up to a homogeneous harmonic polynomial of degree $N \geq 2$, in which case $X \in C(u)$ and $\nabla T_{X,r} u(0) = 0$ for any $r > 0$. Or u blows up to a homogeneous harmonic polynomial of degree $N = 1$, which is just a linear function. Since the gradient of a linear function with unit L^2 -norm has constant modulus, by choosing α_0 sufficiently small $X \in C_r(u) \setminus C(u)$ just means that even though u blows up to a linear function, but at the scale r , u is a fixed distance away from linear functions.[†] Moreover, we define

$$\tilde{C}_r(u) := \left\{ X \in \overline{D} : \inf_{B_\beta(0)} |\nabla T_{X,s} u| \leq \alpha_0 \text{ for any } r \leq s \leq r_c \right\} = \bigcap_{r \leq s \leq r_c} C_s(u) \quad (2.15)$$

to make sure $\tilde{C}_r(u)$ is monotone with respect to r , i.e.

$$C(u) \subset \tilde{C}_{r_1}(u) \subset \tilde{C}_{r_2}(u), \quad \text{for any } 0 < r_1 \leq r_2. \quad (2.16)$$

The value of r_c is determined (depending on d, R, Λ) as in the end of Section 10.

Lemma 2.17. *Let h be a harmonic function in $\mathbb{R}^d$. Suppose h is N_1 -homogeneous with respect to the origin, and it is also N_2 -homogeneous with respect to $X \in \mathbb{R}^d \setminus \{0\}$ [‡]. Then $N_1 = N_2 \in \mathbb{N}$, and*

- either h is linear, i.e. $N_1 = N_2 = 1$;
- or h is invariant in the X -direction, i.e.

$$h(tX + Y) = h(Y), \quad \text{for any } Y \in \mathbb{R}^d \text{ and } t \in \mathbb{R}.$$

^{*}see Proposition 7.3 as well as Lemma 9.50. Heuristically as $r \rightarrow 0+$, $T_{X,r} u$ converges to a homogeneous harmonic polynomial P_N of degree $N \in \mathbb{N}$, and it satisfies $\iint_{B_1(0)} |P_N|^2 dZ = 1$. In particular it implies that $\iint_{B_1(0)} |\nabla P_N|^2 dZ = 2N + d$. In the special case when $N = 1$ and P_N is linear, $|\nabla P_N|$ equals a dimensional constant, denoted by α_d . Here α_0 is chosen to be strictly smaller than α_d . On the other hand when the degree $N > 1$, by homogeneity $|\nabla P_N(r\omega)| = O(r^{N-1})$ grows polynomially in the radial direction. Since $\iint_{B_1(0)} |\nabla P_N|^2 dZ = 2N + d$ and it has a uniform upper bound when $N \leq C(\Lambda)$, the polynomial growth of $|\nabla P_N|$ implies that we can choose $\beta < 1$ so that for any degree $N \leq C(\Lambda)$, $\sup_{B_\beta(0)} |\nabla P_N|$ is also strictly smaller than α_d . (It's certainly not the case that $|\nabla P_N| < \alpha_d$ on all of $B_1(0)$.)

[†]This can be made rigorous in the interior case. By [Han, Theorem 3.1] the harmonic function has an expansion of the form $u(X + Y) - u(X) = P_N(Y) + \Psi(Y)$, where P_N is a homogeneous harmonic polynomial of degree $N \in \mathbb{N}$ and $\Psi(Y) = O(|Y|^{N+\epsilon})$; moreover $\|\nabla T_{X,r} u - \nabla P_N\|_{L^p(B_1(0))} = O(r^{\epsilon + \frac{d}{p} - 1}) \rightarrow 0$ for any $p \in (1, d]$.

[‡]For any $X \in \mathbb{R}^d$, we say h is N -homogeneous with respect to X if $h - h(X)$ is N -homogeneous, namely $h(X + \lambda Z) - h(X) = \lambda^N (h(X + Z) - h(X))$ for any $Z \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}_+$.

Proof. Let $t \in [0, 1]$. By the assumption, we have

$$h((1-t)X) - h(0) = (1-t)^{N_1} [h(X) - h(0)],$$

and

$$h((1-t)X) - h(X) = t^{N_2} [h(0) - h(X)].$$

Combining these two equalities we get

$$h(0) + (1-t)^{N_1} [h(X) - h(0)] = h(X) + t^{N_2} [h(0) - h(X)],$$

or equivalently

$$[(1-t)^{N_1} + t^{N_2}] \cdot [h(X) - h(0)] = h(X) - h(0).$$

Since h is a harmonic function, the degrees of homogeneity N_1, N_2 must be positive integers. Therefore either $N_1 = N_2 = 1$, or $h(X) - h(0) = 0$.

Assume we are in the second case where $h(X) = h(0)$. In particular we have $h(tX) = h(0)$ for every $t \in \mathbb{R}$. By the assumption h is N_1 -homogeneous with respect to the origin, so

$$\langle \nabla h(Z), Z \rangle = N_1 [h(Z) - h(0)]. \quad (2.18)$$

Similarly we also have

$$\langle \nabla h(Z), Z - X \rangle = N_2 [h(Z) - h(X)]. \quad (2.19)$$

Subtracting (2.19) from (2.18), we get

$$\langle \nabla h(Z), X \rangle = (N_1 - N_2) [h(Z) - h(0)]. \quad (2.20)$$

Let $r > 0$ be fixed and we define

$$g(t) = \log N(tX, r) = \log \frac{rD(tX, r)}{H(tX, r)},$$

where

$$D(tX, r) = \iint_{B_r(tX)} |\nabla h|^2 dZ, \quad H(tX, r) = \int_{\partial B_r(tX)} |h - h(tX)|^2 d\mathcal{H}^{d-1}.$$

We want to show that $g'(t) = 0$. To that end, we compute

$$\begin{aligned} \frac{d}{dt} D(tX, r) &= \frac{d}{dt} \iint_{B_r(0)} |\nabla h(tX + Y)|^2 dY = 2 \iint_{B_r(tX)} \langle \nabla h(Z), \nabla^2 h(Z)(X) \rangle dZ \\ &= 2 \iint_{B_r(tX)} \langle \nabla h, \nabla \langle \nabla h, X \rangle \rangle dZ \\ &= 2 \iint_{B_r(tX)} \operatorname{div} (\langle \nabla h, X \rangle \nabla h) dZ \\ &= 2 \int_{\partial B_r(tX)} \langle \nabla h, X \rangle \frac{\partial h}{\partial n} d\mathcal{H}^{d-1}. \end{aligned}$$

And

$$\begin{aligned} \frac{d}{dt} H(tX, r) &= \frac{d}{dt} \int_{\partial B_r(0)} |h(tX + Y) - h(tX)|^2 d\mathcal{H}^{d-1}(Y) \\ &= 2 \int_{\partial B_r(tX)} \langle \nabla h, X \rangle (h - h(tX)) d\mathcal{H}^{d-1} \end{aligned}$$

$$\begin{aligned}
& -2\langle \nabla h(tX), X \rangle \cdot \int_{\partial B_r(tX)} (h - h(tX)) d\mathcal{H}^{d-1} \\
& = 2 \int_{\partial B_r(tX)} \langle \nabla h, X \rangle (h - h(tX)) d\mathcal{H}^{d-1},
\end{aligned}$$

where we use the mean value property in the last equality. Combined with (2.20), $h(tX) = h(0)$ and the equality

$$\iint_{B_r(tX)} |\nabla h|^2 dZ = \int_{\partial B_r(tX)} (h - h(tX)) \frac{\partial h}{\partial n} d\mathcal{H}^{d-1},$$

we obtain

$$\begin{aligned}
g'(t) &= \frac{\frac{d}{dt}D(tX, r)}{D(tX, r)} - \frac{\frac{d}{dt}H(tX, r)}{H(tX, r)} \\
&= \frac{2 \int_{\partial B_r(tX)} \langle \nabla h, X \rangle \frac{\partial h}{\partial n} d\mathcal{H}^{d-1}}{\int_{\partial B_r(tX)} (h - h(0)) \frac{\partial h}{\partial n} d\mathcal{H}^{d-1}} - \frac{2 \int_{\partial B_r(tX)} \langle \nabla h, X \rangle (h - h(0)) d\mathcal{H}^{d-1}}{\int_{\partial B_r(tX)} |h - h(0)|^2 d\mathcal{H}^{d-1}} \quad (2.21)
\end{aligned}$$

$$\begin{aligned}
&= 2(N_1 - N_2) - 2(N_1 - N_2) \\
&= 0. \quad (2.22)
\end{aligned}$$

Therefore the map $t \mapsto N(tX, r)$ is a constant. In particular

$$N_2 = N(X, r) = N(tX, r) = N(0, r) = N_1,$$

and h is N_1 -homogeneous with respect to the vertex tX , for any $t \in \mathbb{R}$. Moreover, by (2.20) we have that for every $Y \in \mathbb{R}^d$ and $t \in \mathbb{R}$,

$$h(tX + Y) - h(Y) = \int_0^t \frac{d}{ds} h(sX + Y) ds = \int_0^t \langle \nabla h(sX + Y), X \rangle ds = 0$$

□

The above theorem also has an analogue in the upper half-space:

Lemma 2.23. *Let h be a harmonic function in $\mathbb{R}_+^d$ and $h = 0$ on $\partial\mathbb{R}_+^d$. Suppose h is N_1 -homogeneous with respect to the origin, and it is also N_2 -homogeneous with respect to the vertex $X \in \partial\mathbb{R}_+^d \setminus \{0\}$. Then h is invariant in the X -direction, i.e.*

$$h(tX + Y) = h(Y) \quad \text{for any } Y \in \mathbb{R}^d \text{ and } t \in \mathbb{R}.$$

Moreover $N_1 = N_2 \in \mathbb{N}$.

Proof. Using $h(0) = h(X) = 0$, the proof of $N_1 = N_2$ is similar to that of Lemma 2.17. The only difference is that when we integrate by parts, we have an extra term on the boundary $B_r(tX) \cap \partial\mathbb{R}_+^d$. However, since h vanishes on the boundary and $X \in \partial\mathbb{R}_+^d$, we have that $\langle \nabla h, X \rangle = 0$. To prove the invariance in the X -direction, it suffices to study the first alternative when h is linear. Suppose h is linear and the invariant hyperplane is denoted by V . Either $V = \partial\mathbb{R}_+^d$, then h is invariant in the X -direction since $X \in V$; or V intersects $\partial\mathbb{R}_+^d$ transversally, in which case $h \equiv 0$ trivially and there is nothing to prove.

□

3. MONOTONICITY FORMULA FOR ELLIPTIC OPERATORS

Let Ω be a Lipschitz domain with $0 \in \partial\Omega$, and let A be a symmetric elliptic matrix defined on $\mathbb{R}^d$ satisfying the following assumptions (c.f. [AE, Theorem 1.1]):

- $A(0) = \text{Id}$ and

$$\langle A(X)X, n(X) \rangle \geq 0, \quad \text{for all } X \in \Delta_1 = B_1 \cap \partial\Omega, \quad (3.1)$$

where $n(X)$ denotes the unit outer normal to Ω at X ;

- There is a non-decreasing function $\theta : [0, +\infty) \rightarrow [0, +\infty)$ such that

$$|A(X) - A(0)| \leq \theta(|X|), \quad |\nabla A(X)| \leq \theta(|X|)/|X| \quad (3.2)$$

and

$$\int_0^* \frac{\theta(r)}{r} dr < +\infty. \quad (3.3)$$

Let v be a non-trivial solution to $Lv = \text{div}(A(X)\nabla v)$ satisfying $v = 0$ on the surface ball $\Delta_1 := B_1(0) \cap \partial\Omega$. The assumption (3.1) means that the domain Ω is L -convex with respect to the boundary point $0 \in \partial\Omega$. Inspired by Almgren's frequency function for harmonic functions, we want to define the frequency function for v centered at 0, and prove that it is almost monotone.

We first define a metric tensor $\bar{g} = \bar{g}_{ij}(X) dX_i \otimes dX_j$ by setting

$$(\bar{g}_{ij}(X)) := (\det A(X))^{\frac{1}{d-2}} A(X)^{-1}, \quad \text{where the indices } i, j \text{ are in } \{1, \dots, d\};$$

and its inverse metric is denoted by $(\bar{g}^{ij}) = (\bar{g}_{ij})^{-1}$. We are interested in a neighborhood of the origin, so we define

$$r(X)^2 := \bar{g}_{ij}(0)(X - 0)_i(X - 0)_j \quad \text{and} \quad \eta(X) := \bar{g}^{kl}(X) \frac{\partial r}{\partial x_k} \frac{\partial r}{\partial x_l}.$$

Now we define a new metric tensor $g = g_{ij}(X) dX_i \otimes dX_j$ by setting

$$g_{ij}(X) := \eta(X) \bar{g}_{ij}(X) = (\det A(X))^{\frac{1}{d-2}} \eta(X) (A(X)^{-1})_{ij}.$$

The advantage of modifying the metric $\bar{g}$ by $\eta(X)$ is that under the new metric, the geodesic distance from 0 to any X is the same as $r(X)$; moreover, under polar coordinates (r, ω) the metric g is simply

$$g(r, \omega) = dr \otimes dr + r^2 b_{kl}(r, \omega) d\omega^i \otimes d\omega^j, \quad \text{where } i, j \in \{1, \dots, d-1\}. \quad (3.4)$$

We refer interested readers to [AKS, Section 3] or [GL, Section 2]. We denote the inverse metric of g as $(g^{ij}(X)) = (g_{ij}(X))^{-1}$, and write $|g(X)| = |\det g_{ij}(X)|$. Let $\tilde{\eta}(X) = (\det A(X))^{\frac{1}{d-2}} \eta(X)$, and simple computations show

$$\tilde{\eta}(X) = \frac{\langle A(X)X, X \rangle}{|X|^2}, \quad g(X) = \tilde{\eta}(X) A(X)^{-1}.$$

By ellipticity and the assumption (3.2), we know that for all X in a small neighborhood of the origin we have

$$C_1 \leq \eta(X), \quad |b(r, \omega)| \leq C_2 \quad (3.5)$$

$$|\nabla\eta(X)|, |\nabla g_{ij}(X)|, |\nabla b_{kl}(X)| \lesssim |\nabla A(X)| \leq \theta(|X|)/|X| \quad (3.6)$$

where the constants depend on n and the ellipticity of the matrix A .

Under the metric g , we can write the divergence-form elliptic operator $Lv = \operatorname{div}(A(X)\nabla v)$ as

$$L_g v = \operatorname{div}_g(\mu(X)\nabla_g v), \quad \text{where } \mu(X) = \eta(X)^{-\frac{d-2}{2}}. \quad (3.7)$$

Here we denote by $\nabla_g v$ and $\operatorname{div}_g \vec{Y}$ the intrinsic gradient of a function v and the intrinsic divergence of a vector field $\vec{Y}$ in the metric g , i.e.

$$\nabla_g v = g^{ij} \frac{\partial v}{\partial x_i} \frac{\partial}{\partial x_j}, \quad \operatorname{div}_g \vec{Y} = \frac{1}{\sqrt{|g|}} \operatorname{div} \left(\sqrt{|g|} \vec{Y} \right)$$

For a solution v to (3.7), we define

$$D_g(r) = \iint_{B_r \cap \Omega} \mu |\nabla_g v|_g^2 dV_g, \quad H_g(r) = \int_{\partial B_r \cap \Omega} \mu v^2 dV_{\partial B_r}. \quad (3.8)$$

Here B_r denotes the geodesic ball centered at 0 in the metric g . But by the above discussion it is the same as the Euclidean ball centered at 0 of the same radius. Now we define the frequency function of u as

$$N_g(r) = \frac{r D_g(r)}{H_g(r)}. \quad (3.9)$$

For simplicity we often omit the dependence on the metric g and just write $D(r)$, $H(r)$ and $N(r)$. We will show the following:

Proposition 3.10. *There exists some $r_0 < 1$, such that the frequency function satisfies*

$$N'(r) = O\left(\frac{\theta(r)}{r}\right) N(r) + R_h(r) + R_b(r), \quad (3.11)$$

for all $r < r_0$. Here

$$R_h(r) = \frac{2r}{H^2(r)} \left[\left(\int_{\partial B_r \cap \Omega} \mu v^2 dV_{\partial B_r} \right) \cdot \left(\int_{\partial B_r \cap \Omega} \mu (v_\rho)^2 dV_{\partial B_r} \right) - \left(\int_{\partial B_r \cap \Omega} \mu v v_\rho dV_{\partial B_r} \right)^2 \right] \geq 0,$$

$$R_b(r) = \frac{1}{H(r)} \int_{\Delta_r} \frac{(\partial_n v)^2}{\tilde{\eta}(X)} \langle A(X)X, n_\Omega(X) \rangle \langle A(X)n_\Omega(X), n_\Omega(X) \rangle d\mathcal{H}^{d-1} \geq 0,$$

where $v_\rho = \langle \nabla_g v, X/|X| \rangle$ denotes the radial differentiation.* (We use $\langle \cdot, \cdot \rangle$ to denote Euclidean inner product, and use $\langle \cdot, \cdot \rangle_g$ to denote inner product with the metric g .)

In particular, there exists a constant $C > 0$ depending on n and the ellipticity of the matrix A , such that the modified frequency function

$$\tilde{N}(r) := N(r) \exp \left(C \int_0^r \frac{\theta(s)}{s} ds \right) \text{ is monotone increasing with respect to } r. \quad (3.12)$$

*Let $\vec{Y}$ be an arbitrary vector, then its projection onto the radial direction can be written as $\left\langle \vec{Y}, \frac{\frac{1}{2} \nabla_g |X|^2}{|X|} \right\rangle_g = \left\langle \vec{Y}, \frac{X}{|X|} \right\rangle_g$.

Remark 3.13. • Written in Euclidean metric, we have

$$D(r) = \iint_{B_r \cap \Omega} \langle A(X) \nabla v, \nabla v \rangle dX, \quad H(r) = \int_{\partial B_r \cap \Omega} \tilde{\eta}(X) v^2(X) d\mathcal{H}^{d-1}(X), \quad (3.14)$$

with $\tilde{\eta}(X) = \langle A(X)X, X \rangle / |X|^2$.

• For convenience, we can also write $R_h(r)$ as

$$R_h(r) = \frac{2r}{H(r)} \int_{\partial B_r \cap \Omega} \mu \left| v_\rho - N(r) \frac{v}{r} \right|^2 dV_{\partial B_r}. \quad (3.15)$$

Proof. We first compute $H'(r)$. Let $|b(r, \omega)| = |\det(b_{kl}(r, \omega))|$, then

$$dV_g(X) = \sqrt{|g(X)|} dX, \quad dV_{\partial B_r}(\omega) = r^{d-1} \sqrt{|b(r, \omega)|} d\mathcal{H}^{d-1}(\omega).$$

By extending u by zero in Ω^c , we can rewrite $H(r)$ as

$$H(r) = \int_{\partial B_r} \mu v^2 dV_{\partial B_r} = r^{d-1} \int_{\partial B_1} \mu(r, \omega) \sqrt{|b(r, \omega)|} v^2(r, \omega) d\mathcal{H}^{d-1}(\omega).$$

Taking the derivative in r , we get

$$H'(r) = \frac{d-1}{r} H(r) + 2 \int_{\partial B_r} \mu v v_\rho dV_{\partial B_r} + \int_{\partial B_r} \frac{1}{\sqrt{|b|}} \partial_\rho \left(\mu \sqrt{|b|} \right) v^2 dV_{\partial B_r}.$$

By (3.5) and (3.6), we can bound the last term and get

$$H'(r) = \left(\frac{d-1}{r} + O\left(\frac{\theta(r)}{r}\right) \right) H(r) + 2 \int_{\partial B_r} \mu v v_\rho dV_{\partial B_r}. \quad (3.16)$$

Moreover, since $\operatorname{div}_g(\mu \nabla_g v) = 0$ we have

$$\iint_{B_r \cap \Omega} \operatorname{div}_g(\mu \nabla_g v^2) dV_g = 2 \iint_{B_r \cap \Omega} \mu |\nabla_g v|_g^2 dV_g.$$

On the other hand by the divergence theorem and the assumption $v \equiv 0$ on Δ_r , we also have

$$\iint_{B_r \cap \Omega} \operatorname{div}_g(\mu \nabla_g v^2) dV_g = 2 \int_{\partial B_r \cap \Omega} \mu v v_\rho dV_{\partial B_r}.$$

Therefore we have

$$D(r) = \iint_{B_r \cap \Omega} \mu |\nabla_g v|_g^2 dV_g = \int_{\partial B_r \cap \Omega} \mu v v_\rho dV_{\partial B_r}. \quad (3.17)$$

Now we compute $D'(r)$. Since $D(r)$ is a solid integral in B_r , we easily get the formula

$$D'(r) = \int_{\partial B_r \cap \Omega} \mu |\nabla_g v|_g^2 dV_{\partial B_r}.$$

We need the following **Rellich identity**: if the function v satisfies $\operatorname{div}_g(\mu \nabla_g v) = 0$, then

$$\begin{aligned} \operatorname{div}_g \left(\mu |\nabla_g v|_g^2 \cdot \frac{1}{2} \nabla_g |X|^2 \right) &= 2 \operatorname{div}_g \left(\left\langle \frac{1}{2} \nabla_g |X|^2, \nabla_g v \right\rangle_g \cdot \mu \nabla_g v \right) + \operatorname{div}_g \left(\frac{1}{2} \nabla_g |X|^2 \right) \cdot \mu |\nabla_g v|_g^2 \\ &\quad - 2\mu |\nabla_g v|_g^2 + O(\theta(|X|)) \cdot \mu |\nabla_g v|_g^2. \end{aligned} \quad (3.18)$$

For comparison we remind the readers of the standard Rellich identity

$$\operatorname{div}(|\nabla v|^2 X) = 2 \operatorname{div}(\langle X, \nabla v \rangle \nabla v) + \operatorname{div}(X) |\nabla v|^2 - 2 |\nabla v|^2.$$

(Notice that in Euclidean metric, the position vector X is pointing to the radial direction; but under the metric g , the radial direction at the point X is computed by $\frac{1}{2} \nabla_g |X|^2$.) We claim the identity (3.18) holds, and show why it helps in the computation of $D'(r)$. We integrate the above identity in the domain $B_r \cap \Omega$, and see what each term becomes.

By the divergence theorem,

$$I := \iint_{B_r \cap \Omega} \operatorname{div}_g \left(\mu |\nabla_g v|_g^2 \cdot \frac{1}{2} \nabla_g |X|^2 \right) dV_g = \int_{\partial(B_r \cap \Omega)} \sqrt{|g|} \mu |\nabla_g v|_g^2 \left\langle \frac{1}{2} \nabla_g |X|^2, n \right\rangle d\mathcal{H}^{d-1},$$

$$\begin{aligned} II &:= \iint_{B_r \cap \Omega} \operatorname{div}_g \left(\left\langle \frac{1}{2} \nabla_g |X|^2, \nabla_g v \right\rangle_g \cdot \mu \nabla_g v \right) dV_g \\ &= \int_{\partial(B_r \cap \Omega)} \sqrt{|g|} \left\langle \frac{1}{2} \nabla_g |X|^2, \nabla_g v \right\rangle_g \langle \mu \nabla_g v, n \rangle d\mathcal{H}^{d-1}. \end{aligned}$$

On $\partial B_r \cap \Omega$, the unit outer normal is $n(X) = X/|X|$ and

$$\begin{aligned} \left\langle \frac{1}{2} \nabla_g |X|^2, \frac{X}{|X|} \right\rangle &= \frac{1}{\tilde{\eta}(X)} \frac{\langle A(X)X, X \rangle}{|X|} = |X| = r, \\ \left\langle \frac{1}{2} \nabla_g |X|^2, \nabla_g v \right\rangle_g &= \frac{1}{\tilde{\eta}(X)} \langle A(X) \nabla v, X \rangle = |X| v_\rho = r v_\rho, \quad \left\langle \nabla_g v, \frac{X}{|X|} \right\rangle = v_\rho. \end{aligned}$$

On $\Delta_r := B_r \cap \partial\Omega$, the unit outer normal vector is that of the domain Ω , denoted by $n_\Omega(X)$.

And

$$\begin{aligned} \left\langle \frac{1}{2} \nabla_g |X|^2, n \right\rangle &= \frac{\langle A(X)X, n_\Omega(X) \rangle}{\tilde{\eta}(X)} \geq 0, \\ \sqrt{|g|} \mu |\nabla_g v|_g^2 &= \langle A(X) \nabla v, \nabla v \rangle = (\partial_n v)^2 \langle A(X) n_\Omega(X), n_\Omega(X) \rangle \geq 0, \\ \langle \nabla_g v, n_\Omega \rangle &= \frac{1}{\tilde{\eta}(X)} \langle A(X) \nabla v, n_\Omega(X) \rangle = \frac{\partial_n v}{\tilde{\eta}(X)} \langle A(X) n_\Omega(X), n_\Omega(X) \rangle, \\ \left\langle \frac{1}{2} \nabla_g |X|^2, \nabla_g v \right\rangle_g &= \frac{1}{\tilde{\eta}(X)} \langle A(X) \nabla v, X \rangle = \frac{\partial_n v}{\tilde{\eta}(X)} \langle A(X)X, n_\Omega(X) \rangle. \end{aligned}$$

Here we have used $v \equiv 0$ on Δ_r , and thus ∇v only has normal component and $\nabla v = \partial_n v \cdot n_\Omega$.

Combined we conclude

$$I = r \int_{\partial B_r \cap \Omega} \mu |\nabla_g v|_g^2 dV_{\partial B_r} + \int_{\Delta_r} \frac{(\partial_n v)^2}{\tilde{\eta}(X)} \langle A(X)X, n_\Omega(X) \rangle \langle A(X) n_\Omega(X), n_\Omega(X) \rangle d\mathcal{H}^{d-1}, \quad (3.19)$$

$$II = r \int_{\partial B_r \cap \Omega} \mu (v_\rho)^2 dV_{\partial B_r} + \int_{\Delta_r} \frac{(\partial_n v)^2}{\tilde{\eta}(X)} \langle A(X)X, n_\Omega(X) \rangle \langle A(X) n_\Omega(X), n_\Omega(X) \rangle d\mathcal{H}^{d-1}. \quad (3.20)$$

On the other hand,

$$\operatorname{div}_g \left(\frac{1}{2} \nabla_g |X|^2 \right) = \frac{1}{\sqrt{|g|}} \operatorname{div} \left(\frac{\sqrt{|g|}}{\tilde{\eta}(X)} A(X)X \right)$$

$$\begin{aligned}
&= \frac{1}{\tilde{\eta}(X)} \operatorname{div} (A(X)X) + \frac{1}{\sqrt{|g|}} \left\langle \nabla \left(\frac{\sqrt{|g|}}{\tilde{\eta}(X)} \right), A(X)X \right\rangle \\
&= \frac{\operatorname{tr} A(X)}{\tilde{\eta}(X)} + O(\theta(|X|)),
\end{aligned} \tag{3.21}$$

where in the last equality we have used the bounds (3.5) and (3.6). It follows from $|A(X) - \operatorname{Id}| \leq \theta(|X|)$ that

$$\operatorname{tr} A(X) = d + O(\theta(|X|)),$$

and

$$\tilde{\eta}(X) = \frac{\langle A(X)X, X \rangle}{|X|^2} = 1 + \frac{\langle (A(X) - \operatorname{Id})X, X \rangle}{|X|^2} = 1 + O(\theta(|X|)).$$

Plugging into (3.21) we get

$$\operatorname{div}_g \left(\frac{1}{2} \nabla_g |X|^2 \right) = d + O(\theta(|X|)). \tag{3.22}$$

Combining (3.19), (3.20), (3.22) and the Rellich identity (3.18), we obtain

$$\begin{aligned}
r \int_{\partial B_r \cap \Omega} \mu |\nabla_g v|_g^2 dV_{\partial B_r} &= 2r \int_{\partial B_r \cap \Omega} \mu (v_\rho)^2 dV_{\partial B_r} + (d - 2 + O(\theta(r))) \iint_{B_r \cap \Omega} \mu |\nabla_g v|_g^2 dV_g \\
&\quad + \int_{\Delta_r} \frac{(\partial_n v)^2}{\tilde{\eta}(X)} \langle A(X)X, n_\Omega(X) \rangle \langle A(X)n_\Omega(X), n_\Omega(X) \rangle d\mathcal{H}^{d-1}.
\end{aligned}$$

In other words, we get the desired quantity $D'(r)$:

$$\begin{aligned}
D'(r) &= \frac{d - 2 + O(\theta(r))}{r} D(r) + 2 \int_{\partial B_r \cap \Omega} \mu (v_\rho)^2 dV_{\partial B_r} \\
&\quad + \frac{1}{r} \int_{\Delta_r} \frac{(\partial_n v)^2}{\tilde{\eta}(X)} \langle A(X)X, n_\Omega(X) \rangle \langle A(X)n_\Omega(X), n_\Omega(X) \rangle d\mathcal{H}^{d-1}.
\end{aligned} \tag{3.23}$$

Combining (3.16), (3.23) and (3.17), we get

$$\begin{aligned}
\frac{N'(r)}{N(r)} &= \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} \\
&= O\left(\frac{\theta(r)}{r}\right) + 2 \frac{\left(\int_{\partial B_r \cap \Omega} \mu v^2 dV_{\partial B_r} \right) \cdot \left(\int_{\partial B_r \cap \Omega} \mu (v_\rho)^2 dV_{\partial B_r} \right) - \left(\int_{\partial B_r \cap \Omega} \mu v v_\rho dV_{\partial B_r} \right)^2}{D(r)H(r)} \\
&\quad + \frac{1}{rD(r)} \int_{\Delta_r} \frac{(\partial_n v)^2}{\tilde{\eta}(X)} \langle A(X)X, n_\Omega(X) \rangle \langle A(X)n_\Omega(X), n_\Omega(X) \rangle d\mathcal{H}^{d-1}.
\end{aligned}$$

Notice that the second term is non-negative by the Cauchy-Scharz inequality, and it equals zero if and only if

$$v_\rho = \lambda_r v \text{ on } \partial B_r \cap \Omega.$$

The last terms is also non-negative by the assumption on the matrix $A(X)$. And since $\partial_n v \neq 0$, it equals zero if and only if $\langle A(X)X, n_\Omega(X) \rangle \equiv 0$ on Δ_r . To sum up we get

$$\frac{N'(r)}{N(r)} \geq -C \frac{\theta(r)}{r},$$

and hence

$$\tilde{N}(r) = N(r) \exp \left(C \int_0^r \frac{\theta(s)}{s} ds \right) \text{ is monotone increasing with respect to } r.$$

This finishes the proof of the proposition.

Now we return to *the proof of the Rellich identity (3.18)*. We will use two abstract formulae:

$$\operatorname{div}_g(a\vec{Y}) = a \operatorname{div}_g(\vec{Y}) + \langle \nabla_g a, \vec{Y} \rangle_g = a \operatorname{div}_g(\vec{Y}) + \langle \nabla a, \vec{Y} \rangle, \quad (3.24)$$

$$\langle \nabla_g \langle \vec{A}, \vec{B} \rangle_g, \vec{C} \rangle_g = \langle \nabla \langle \vec{A}, \vec{B} \rangle_g, \vec{C} \rangle = \frac{\partial A_j}{\partial x_l} g_{jk} B_k C_l + A_j g_{jk} \frac{\partial B_k}{\partial x_l} C_l + A_j \frac{\partial g_{jk}}{\partial x_l} B_k C_l. \quad (3.25)$$

By (3.24) the left hand side of (3.18) becomes

$$\operatorname{div}_g \left(\mu |\nabla_g v|_g^2 \cdot \frac{1}{2} \nabla_g |X|^2 \right) = \operatorname{div}_g \left(\frac{1}{2} \nabla_g |X|^2 \right) \cdot \mu |\nabla_g v|_g^2 + \left\langle \nabla_g \langle \mu \nabla_g v, \nabla_g v \rangle_g, \frac{1}{2} \nabla_g |X|^2 \right\rangle_g.$$

The first term on the right hand side also appears in the right hand side of (3.18), so it suffices to compute the second term. By (3.25) we compute

$$\begin{aligned} \left\langle \nabla_g \langle \mu \nabla_g v, \nabla_g v \rangle_g, \frac{1}{2} \nabla_g |X|^2 \right\rangle_g &= \frac{\partial (\mu \nabla_g v)_j}{\partial x_l} g_{jk} (\nabla_g v)_k \left(\frac{1}{2} \nabla_g |X|^2 \right)_l \\ &+ \mu (\nabla_g v)_j g_{jk} \frac{\partial (\nabla_g v)_k}{\partial x_l} \left(\frac{1}{2} \nabla_g |X|^2 \right)_l + \mu (\nabla_g v)_j \frac{\partial g_{jk}}{\partial x_l} (\nabla_g v)_k \left(\frac{1}{2} \nabla_g |X|^2 \right)_l \\ &= 2\mu \frac{\partial (\nabla_g v)_j}{\partial x_l} g_{jk} (\nabla_g v)_k \left(\frac{1}{2} \nabla_g |X|^2 \right)_l + \frac{\partial \mu}{\partial x_l} |\nabla_g v|_g^2 \left(\frac{1}{2} \nabla_g |X|^2 \right)_l + \mu (\nabla_g v)_j \frac{\partial g_{jk}}{\partial x_l} (\nabla_g v)_k \left(\frac{1}{2} \nabla_g |X|^2 \right)_l \\ &=: E_0 + E_1 + E_2. \end{aligned}$$

In the second equality we use the symmetry of the matrix g , which follows from the symmetry of the elliptic matrix A . On the other hand, by (3.24) and $\operatorname{div}_g(\mu \nabla_g v) = 0$ we have

$$\begin{aligned} \operatorname{div}_g \left(\left\langle \frac{1}{2} \nabla_g |X|^2, \nabla_g v \right\rangle_g \cdot \mu \nabla_g v \right) &= \left\langle \nabla_g \left\langle \frac{1}{2} \nabla_g |X|^2, \nabla_g v \right\rangle_g, \mu \nabla_g v \right\rangle_g \\ &= \frac{\partial \left(\frac{1}{2} \nabla_g |X|^2 \right)_j}{\partial x_l} g_{jk} (\nabla_g v)_k \mu (\nabla_g v)_l + \left(\frac{1}{2} \nabla_g |X|^2 \right)_j g_{jk} \frac{\partial (\nabla_g v)_k}{\partial x_l} \mu (\nabla_g v)_l \\ &\quad + \left(\frac{1}{2} \nabla_g |X|^2 \right)_j \frac{\partial g_{jk}}{\partial x_l} (\nabla_g v)_k \mu (\nabla_g v)_l \\ &=: E_3 + E'_0 + E_4. \end{aligned}$$

By the symmetry of g it is not hard to see $E_0 = 2 \cdot E'_0$. Therefore

$$\begin{aligned} \operatorname{div}_g \left(\mu |\nabla_g v|_g^2 \cdot \frac{1}{2} \nabla_g |X|^2 \right) - 2 \operatorname{div}_g \left(\left\langle \frac{1}{2} \nabla_g |X|^2, \nabla_g v \right\rangle_g \cdot \mu \nabla_g v \right) &= \operatorname{div}_g \left(\frac{1}{2} \nabla_g |X|^2 \right) \cdot \mu |\nabla_g v|_g^2 \\ &= E_1 + E_2 - 2 \cdot E_3 - 2 \cdot E_4. \end{aligned}$$

By (3.5) and (3.6) we can easily get

$$E_1, E_2, E_4 = O(\theta(|X|)) \cdot \mu |\nabla_g v|_g^2.$$

On the other hand, the product rule, (3.5), (3.6) and the symmetry of g yield

$$\begin{aligned} E_3 &= O(\theta(|X|)) \cdot \mu |\nabla_g v|_g^2 + \mu (\nabla_g v)_k \delta_{kl} (\nabla_g v)_l \\ &= O(\theta(|X|)) \cdot \mu |\nabla_g v|_g^2 + \mu \frac{\langle A \nabla v, A \nabla v \rangle}{\tilde{\eta}^2} \\ &= \mu |\nabla_g v|_g^2 + O(\theta(|X|)) \cdot \mu |\nabla_g v|_g^2, \end{aligned}$$

where we use the assumption (3.2) in the last equality. This finishes the proof of the Rellich identity. $\square$

To get an idea of the quantitative rigidity, we look at the sharp case.

Corollary 3.26 (Rigidity of the monotonicity formula). *Suppose $A(X) \equiv \text{Id}$ and $\theta(r) = 0$. Then $N(r)$ is a constant (denoted by N) if and only if*

- (1) $v(r, \omega) = r^N v(\omega)$ is a homogeneous harmonic function of degree N in Ω ;
- (2) Ω is a cone with vertex at the origin.

Remark 3.27. So far we do not know if N is a positive integer or not, so it is perhaps a misnomer to call it N . But in Subsection 5.1 we will show that if the domain Ω is C^1 , it blows up to an upper half-space and thus the homogeneity N of a harmonic function must be a positive integer.

Proof. By (3.11) and since $\theta(r) = 0$, $N(r)$ is a constant if and only if both $R_h(r)$ and $R_b(r)$ vanish for all r . By (3.15) $R_h(r)$ equals zero if and only if

$$v_\rho - N(r) \frac{v}{r} \equiv 0 \text{ on } \partial B_r \cap \Omega.$$

When $A(X) = \text{Id}$, we have $v_\rho = \langle \nabla_g v, \frac{X}{|X|} \rangle = \langle \nabla v, \frac{X}{|X|} \rangle$. Hence it follows that $v(r, \omega) = r^N v(\omega)$ is a homogeneous harmonic function of degree N . Recall that we assume v is a non-trivial function, hence $R_b(r)$ equals zero if and only if

$$\langle A(X)X, n_\Omega(X) \rangle \equiv 0,$$

that is to say, Ω is a cone (not necessarily convex) with vertex at the origin. $\square$

Corollary 3.28 (The doubling property of $H(r)$). *The quantity*

$$\tilde{H}(r) := \frac{H(r)}{r^{d-1}}$$

satisfies

$$\left(\frac{r_2}{r_1} \right)^{2\tilde{N}(r_1) \exp\left(-C \int_0^{r_2} \frac{\theta(s)}{s} ds\right)} \exp\left(-C \int_{r_1}^{r_2} \frac{\theta(s)}{s} ds\right) \leq \frac{\tilde{H}(r_2)}{\tilde{H}(r_1)} \quad (3.29)$$

$$\frac{\tilde{H}(r_2)}{\tilde{H}(r_1)} \leq \left(\frac{r_2}{r_1} \right)^{2\tilde{N}(r_2) \exp\left(-C \int_0^{r_1} \frac{\theta(s)}{s} ds\right)} \exp\left(C \int_{r_1}^{r_2} \frac{\theta(s)}{s} ds\right) \quad (3.30)$$

for any two radii r_1, r_2 sufficiently small satisfying $0 < r_1 < r_2$.

Proof. Combining (3.16) and (3.17) we get

$$\frac{H'(r)}{H(r)} = \frac{d-1}{r} + O\left(\frac{\theta(r)}{r}\right) + \frac{2D(r)}{H(r)} = \frac{d-1}{r} + O\left(\frac{\theta(r)}{r}\right) + \frac{2N(r)}{r}. \quad (3.31)$$

Therefore there exists a constant $C > 0$ such that

$$\frac{d}{dr} \log \left(\frac{H(r)}{r^{d-1}} \exp \left(C \int_0^r \frac{\theta(s)}{s} ds \right) \right) \geq \frac{2N(r)}{r} = \frac{2\tilde{N}(r)}{r} \exp \left(-C \int_0^r \frac{\theta(s)}{s} ds \right).$$

Integrating the above differential inequality on the interval $[r_1, r_2]$ and using the monotonicity of $\tilde{N}(r)$, we obtain

$$\begin{aligned} \log \tilde{H}(r_2) - \log \tilde{H}(r_1) + C \int_{r_1}^{r_2} \frac{\theta(s)}{s} ds &\geq \int_{r_1}^{r_2} \frac{2\tilde{N}(r)}{r} \exp \left(-C \int_0^r \frac{\theta(s)}{s} ds \right) dr \\ &\geq 2\tilde{N}(r_1) \exp \left(-C \int_0^{r_2} \frac{\theta(s)}{s} ds \right) (\log r_2 - \log r_1). \end{aligned}$$

The desired inequality (3.29) follows from taking the exponential of the above. The proof of (3.30) is similar, except that we use the upper bound $\tilde{N}(r) \leq \tilde{N}(r_2)$. $\square$

4. REDUCTION FROM A DINI DOMAIN TO AN ALMOST CONVEX DOMAIN

In this section, we recall the reduction in [AE] of the Laplacian operator in a Dini domain D to a divergence-form elliptic operator in a Lipschitz domain Ω , so that the elliptic operator satisfies the assumptions in the beginning of Section 3. We focus on how the reduction map changes relevant quantities of the solution, and in particular how the frequency function discussed in Section 3 controls the *symmetry* of the original harmonic function, modulo a Dini-type error term.

Let D be a Dini domain (see Definition 2.1) with parameter θ . We set

$$\tilde{\theta}(r) = \frac{1}{\log^2 2} \int_r^{2r} \frac{1}{t} \int_t^{2t} \frac{\theta(s)}{s} ds dt \quad (4.1)$$

to be a smoothed-out alternative function of $\theta(\cdot)$, and

$$\alpha(r) = 3 \frac{d}{dr} (r\tilde{\theta}(r)).$$

Simple computations show that

$$\theta(r) \leq \tilde{\theta}(r) \leq \theta(4r), \quad 3\theta(r) \leq \alpha(r) \leq 13\theta(4r).$$

Thus we can use $\tilde{\theta}(\cdot)$ and $\theta(\cdot)$ interchangeably, modulo multiplying the variable by a constant. Let u be a non-trivial solution to $\Delta u = 0$ in the domain D . Assume that $0 \in \partial D$, D is graphical inside the ball $B_{5R}(0)$ and $\theta(8R) < 1/13$ for some $R > 0$.^{*} Suppose u satisfies $u = 0$ on $B_{5R}(0) \cap \partial D$. We consider the following map

$$\Psi : X = (x, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \mapsto (x, x_d + 3|X|\tilde{\theta}(|X|)) \in \mathbb{R}^d. \quad (4.2)$$

^{*}Throughout the paper we will often require *the scale to be sufficiently small*. Unless otherwise specified it always means that the radius is less or equal to R satisfying the assumption here.

Let $X_0 = (x_0, \varphi(x_0)) \in B_R(0) \cap \partial D$ be arbitrary, and we define $\Psi_{X_0}(X) := X_0 + \Psi(X)$. Then

$$D\Psi_{X_0}(X) = D\Psi(X) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ \frac{\alpha(|X|)}{|X|}x_1 & \cdots & \frac{\alpha(|X|)}{|X|}x_{n-1} & 1 + \frac{\alpha(|X|)}{|X|}x_d \end{pmatrix} = \text{Id} + O(\alpha(|X|)). \quad (4.3)$$

In particular

$$1 - \alpha(|X|) \leq \det D\Psi(X) = 1 + \frac{\alpha(|X|)}{|X|}x_d \leq 1 + \alpha(|X|), \quad (4.4)$$

and hence Ψ is invertible in $B_{2R}(0)$. Set $v(X) = u(\Psi_{X_0}(X))$ and

$$\Omega_{X_0} = \{X = (x, x_d) \in \mathbb{R}^d : x_d > \varphi(x + x_0) - \varphi(x_0) - 3|X|\tilde{\theta}(|X|)\}.$$

We have that u is a solution to $\Delta u = 0$ in $D \cap B_{2R}(X_0)$ if and only if v is a solution to $\text{div}(A(X)\nabla v) = 0$ in $\Omega_{X_0} \cap B_{2R}(0)$, where

$$A(X) = \det D\Psi(X) \cdot (D\Psi(X))^{-1} ((D\Psi(X))^{-1})^t \quad (4.5)$$

is a symmetric, elliptic matrix whose ellipticity constants are $1 \pm O(\alpha(|X|))$. Moreover, following the same computation as in [AE] we know that $A(X)$ satisfies the desired assumption (3.2) (with a constant multiple of $\theta(4r)$ in place of $\theta(r)$) and that

$$\langle A(X)X, n_{\Omega_{X_0}}(X) \rangle \gtrsim |X|\theta(|X|) \geq 0.$$

To summarize Section 3 and the discussion above, we have proven the following. For any $X_0 \in B_R(0) \cap \partial D$, after the transformation map Ψ_{X_0} the function $v_{X_0} := u \circ \Psi_{X_0}$ satisfies the divergence-form elliptic operator with coefficient matrix $A(X)$, and $A(X)$ satisfies the assumptions in the beginning of Section 3. Therefore the frequency function for v_{X_0} centered at $0 = \Psi_{X_0}^{-1}(X_0)$ is almost monotone. We set the notation

$$N_{X_0} := \lim_{r \rightarrow 0+} \tilde{N}(v_{X_0}, r) \quad \text{and} \quad N_{X_0}(r) := \tilde{N}(v_{X_0}, r) \quad (4.6)$$

to emphasize the dependence of the modified frequency function $\tilde{N}$ on the corresponding base point $X_0 \in B_R(0) \cap \partial D$.

Recall that we studied the divergence-form elliptic operator in Section 3 using an intrinsic metric g . We will show that

$$|\langle \nabla_g v(Y), Y - 0 \rangle - \langle \nabla u(Z), Z - X_0 \rangle| \lesssim |Y|\theta(4|Y|)|\nabla v(Y)|. \quad (4.7)$$

where $Z = \Psi_{X_0}(Y) \in D$ is the image of $Y \in \Omega_{X_0}$. This estimate will be used in Section 9. By definitions

$$\tilde{\eta}(Y) = \frac{\langle A(Y)Y, Y \rangle}{|Y|^2} = \det D\Psi(Y) \frac{|((D\Psi(Y))^{-1})^t Y|^2}{|Y|^2} = \det D\Psi(Y) \cdot (1 + O(\alpha(|Y|))), \quad (4.8)$$

$$|Y - \Psi(Y)| \leq 3|Y|\tilde{\theta}(|Y|), \quad (4.9)$$

$$\nabla v(Y) = (D\Psi(Y))^t \nabla u(Z) = (\text{Id} + O(\alpha(|Y|))) \nabla u(Z)$$

Finally,

$$\begin{aligned}
\nabla_g v(Y) &= \frac{1}{\tilde{\eta}(Y)} A(Y) \nabla v(Y) \\
&= (1 + O(\alpha(|Y|))) \cdot (D\Psi(Y))^{-1} \left((D\Psi(Y))^{-1} \right)^t \cdot (D\Psi(Y))^t \nabla u(Z) \\
&= (1 + O(\alpha(|Y|))) \cdot (D\Psi(Y))^{-1} \nabla u(Z) \\
&= (1 + O(\alpha(|Y|))) \cdot (\text{Id} + O(\alpha(|Y|))) \nabla u(Z)
\end{aligned}$$

and thus

$$\begin{aligned}
\langle \nabla_g v(Y), Y \rangle &= \langle (1 + O(\alpha(|Y|))) \cdot (\text{Id} + O(\alpha(|Y|))) \nabla u(Z), \Psi(Y) + O(|Y| \tilde{\theta}(|Y|)) \rangle \\
&= \langle \nabla u(Z), Z - X_0 \rangle + O(|Y| \theta(4|Y|)) \cdot |\nabla u(Z)| \\
&= \langle \nabla u(Z), Z - X_0 \rangle + O(|Y| \theta(4|Y|)) \cdot |\nabla v(Y)|.
\end{aligned}$$

5. BLOW UP ANALYSIS AND COMPACTNESS FOR BOUNDARY POINTS

The main goal of this section is to show that the frequency function centered at $X_0 \in \partial D$ carries a lot of information about the local behavior of the harmonic function u near X . We also remark that the same analysis can also be carried out for interior points, after we establish the monotonicity formula of the frequency function in Proposition 6.3.

We first prove the following lemma, which essentially says the frequency function of scale r takes comparable values in an r -neighborhood. In particular, we get a upper bound on the frequency function.

Lemma 5.1. *There exist constants $C_2, C_3 > 0$ such that the following holds. Let $X_0, X'_0 \in B_R(0) \cap \partial D$ be arbitrary. Suppose they satisfy $|X_0 - X'_0| \leq \frac{r}{20}$ for some r sufficiently small (or to be precise, r small enough so that $r \leq 3R$ and $\theta(4r) < 1/30$). Then*

$$N_{X_0} \left(\frac{r}{4} \right) \leq C_2 + C_3 N_{X'_0} \left(\frac{4r}{3} \right),$$

as long as the right hand side is finite.

In particular, suppose the frequency function at the origin satisfies $N_0(4R) \leq \Lambda < +\infty$, then

$$N_{X_0} \left(\frac{3}{4} R \right) \leq C_2 + C_3 \Lambda, \quad \text{for any } X_0 \in B_{\frac{3}{20}R}(0) \cap \partial D. \quad (5.2)$$

Proof. We claim that

$$\frac{1}{2} \iint_{B_{\frac{s}{2}}(X_0)} u^2 dX \leq \iint_{B_s \cap \Omega_{X_0}} \mu v_{X_0}^2 dV_g \leq 2 \iint_{B_{2s}(X_0) \cap D} u^2 dX, \quad (5.3)$$

for all $X_0 \in B_R(0) \cap \partial D$ and all $s > 0$ small enough. By (4.8), a change of variables and (4.4),

$$\iint_{B_s \cap \Omega_{X_0}} \mu v_{X_0}^2 dV_g = \iint_{B_s \cap \Omega_{X_0}} \tilde{\eta} v_{X_0}^2 dY = (1 + O(\theta(4s))) \iint_{B_s \cap \Omega_{X_0}} u^2(\Psi_{X_0}(Y)) dY$$

$$= (1 + O(\theta(4s))) \iint_{\Psi_{X_0}(B_s) \cap D} u^2(X) (1 + O(\theta(4s)))^{-1} dX. \quad (5.4)$$

Notice that for any $\rho > 0$ sufficiently small, the transformation Ψ maps the sphere ∂B_ρ bijectively to the sphere $\partial B_\rho + 3\rho\tilde{\theta}(\rho)e_d$, which is the sphere ∂B_ρ shifted upwards by $3\rho\tilde{\theta}(\rho)$ in the positive e_d -direction. Hence

$$\Psi_{X_0}(B_s) = X_0 + \Psi(B_s) \subset B_{s+3s\tilde{\theta}(s)}(X_0) \subset B_{\frac{11}{10}s}(X_0)$$

as long as s is sufficiently small (so that $\theta(4s) < 1/30$). On the other hand, recall that Ψ is invertible in $B_{2R}(0)$. That is to say for any $X' \in B_{\frac{9}{10}s}$ we can find some X in the neighborhood of the origin such that $\Psi(X) = X'$. Moreover, such X satisfies

$$|X| \leq \|(D\Psi)^{-1}\|_\infty |X'| \leq \left(1 + \alpha \left(\frac{9}{10}s\right)\right) \frac{9}{10}s < s.$$

That is to say $B_{\frac{9}{10}s} \subset \Psi(B_s)$, and hence

$$B_{\frac{9}{10}s}(X_0) \subset X_0 + \Psi(B_s) = \Psi_{X_0}(B_s).$$

Plugging back into (5.4), we conclude that

$$\frac{1}{2} \iint_{A_{\frac{11}{20}r, \frac{9}{10}r}(X_0) \cap D} u^2 dX \leq \iint_{A_{\frac{r}{2}, r} \cap \Omega_{X_0}} \mu v_{X_0}^2 dV_g \leq 2 \iint_{A_{\frac{9}{20}r, \frac{11}{10}r}(X_0) \cap D} u^2 dX, \quad (5.5)$$

where we use $A_{r_1, r_2}(X_0)$ with $0 < r_1 < r_2$ to denote the annulus $B_{r_2}(X_0) \setminus B_{r_1}(X_0)$.

Let $\rho > 0$ be sufficiently small, and let $a > 1$ be a constant to be chosen later. By Corollary 3.28, we have that

$$\begin{aligned} \frac{H_{X_0}(a\rho)}{H_{X_0}(\rho)} &= a^{n-1} \frac{\tilde{H}_{X_0}(a\rho)}{\tilde{H}_{X_0}(\rho)} \leq a^{n-1} a^{2N_{X_0}(a\rho) \exp(-C \int_0^\rho \frac{\theta(4s)}{s} ds)} \exp\left(C \int_\rho^{a\rho} \frac{\theta(4s)}{s} ds\right) \\ &\leq C_1 a^{n-1} a^{2N_{X_0}(a\rho)}, \end{aligned} \quad (5.6)$$

where we write $C_1 = \exp\left(C \int_0^{4R} \frac{\theta(4s)}{s} ds\right) < +\infty$. Similarly, we also have the lower bound

$$\frac{H_{X_0}(a\rho)}{H_{X_0}(\rho)} \geq a^{n-1} a^{2N_{X_0}(\rho) \exp(-C \int_0^{a\rho} \frac{\theta(4s)}{s} ds)} \exp\left(-C \int_\rho^{2\rho} \frac{\theta(4s)}{s} ds\right) \geq C_1^{-1} a^{n-1} a^{2N_{X_0}(\rho)/C_1}. \quad (5.7)$$

Using (5.5) twice, we get

$$\frac{\iint_{A_{\frac{r}{2}, r} \cap \Omega_{X_0}} \mu v_{X_0}^2 dV_g}{\iint_{A_{\frac{r}{4}, \frac{r}{2}} \cap \Omega_{X_0}} \mu v_{X_0}^2 dV_g} \leq 4 \frac{\iint_{A_{\frac{9}{20}r, \frac{11}{10}r}(X_0) \cap D} u^2 dX}{\iint_{A_{\frac{11}{40}r, \frac{9}{20}r}(X_0) \cap D} u^2 dX} \leq 4 \frac{\iint_{A_{\frac{2}{5}r, \frac{23}{20}r}(X'_0) \cap D} u^2 dX}{\iint_{A_{\frac{13}{40}r, \frac{2}{5}r}(X'_0) \cap D} u^2 dX}$$

$$\leq 16 \frac{\iint_{A_{\frac{9}{25}r, \frac{253}{200}r} \cap \Omega_{X'_0}} \mu v_{X'_0}^2 dV_g}{\iint_{A_{\frac{143}{400}r, \frac{9}{25}r} \cap \Omega_{X'_0}} \mu v_{X'_0}^2 dV_g}. \quad (5.8)$$

Using the lower bound in (5.7) with $a = 2$ and the monotonicity of $N_{X_0}(\cdot)$, we have

$$\frac{\iint_{A_{\frac{r}{2}, r} \cap \Omega_{X_0}} \mu v_{X_0}^2 dV_g}{\iint_{A_{\frac{r}{4}, \frac{r}{2}} \cap \Omega_{X_0}} \mu v_{X_0}^2 dV_g} = \frac{\int_{\frac{r}{2}}^r H_{X_0}(\rho) d\rho}{\int_{\frac{r}{4}}^{\frac{r}{2}} H_{X_0}(\rho) d\rho} = \frac{2 \int_{\frac{r}{4}}^{\frac{r}{2}} H_{X_0}(2\rho) d\rho}{\int_{\frac{r}{4}}^{\frac{r}{2}} H_{X_0}(\rho) d\rho} \geq C_1^{-1} 2^n 2^{2N_{X_0}(\frac{r}{4})/C_1}. \quad (5.9)$$

Similarly, we want to use the upper bound in (5.6) to bound the right hand side of (5.8). However, since the integral region of the numerator is not exactly a rescaling of the integral region of the denominator, we need to first break down the integral on top. To do that we denote $\ell := \frac{9}{25}r - \frac{143}{400}r = \frac{1}{400}r$,

$$r_1 = \frac{9}{25}r \quad \text{and} \quad a_1 = a := \frac{r_1}{\frac{143}{400}r} = \frac{144}{143},$$

and inductively

$$r_j = r_1 + a\ell \cdot (j-1) \quad \text{and} \quad a_j = \frac{r_j}{\frac{143}{400}r} = \frac{144}{143} + \frac{144}{143^2}(j-1)$$

for $j = 1, \dots, 361$ (since r_{361} is the last one to be smaller than $\frac{253}{200}r$). For each j we can use the upper bound in (5.6) with coefficient a_j to bound

$$\frac{\iint_{A_{r_j, r_{j+1}} \cap \Omega_{X'_0}} \mu v_{X'_0}^2 dV_g}{\iint_{A_{\frac{143}{400}r, \frac{9}{25}r} \cap \Omega_{X'_0}} \mu v_{X'_0}^2 dV_g} = \frac{\int_{r_j}^{r_{j+1}} H_{X'_0}(\rho) d\rho}{\int_{\frac{143}{400}r}^{\frac{9}{25}r} H_{X'_0}(\rho) d\rho} \leq C_1 a_j^n a_j^{2N_{X'_0}(r_{j+1})} \leq C_1 4^n 4^{2N_{X'_0}(\frac{4}{3}r)}.$$

Finally

$$\frac{\iint_{A_{\frac{9}{25}r, \frac{253}{200}r} \cap \Omega_{X'_0}} \mu v_{X'_0}^2 dV_g}{\iint_{A_{\frac{143}{400}r, \frac{9}{25}r} \cap \Omega_{X'_0}} \mu v_{X'_0}^2 dV_g} \leq \sum_{j=1}^{361} \frac{\iint_{A_{r_j, r_{j+1}} \cap \Omega_{X'_0}} \mu v_{X'_0}^2 dV_g}{\iint_{A_{\frac{143}{400}r, \frac{9}{25}r} \cap \Omega_{X'_0}} \mu v_{X'_0}^2 dV_g} \leq 361 C_1 4^n 4^{2N_{X'_0}(\frac{4}{3}r)}. \quad (5.10)$$

Combining (5.8), (5.9) and (5.10), we conclude

$$N_{X_0}\left(\frac{r}{4}\right) \leq C_2 + C_3 N_{X'_0}\left(\frac{4r}{3}\right),$$

where the constants C_2, C_3 depends on n and the value of C_1 . $\square$

Following the proof of Lemma 5.1 and the bound (5.2) of the frequency function, we get the following lemma which will become useful later.

Lemma 5.11. *Assume $N_0(4R) \leq \Lambda < +\infty$. For any $X_0 \in \partial D \cap B_R(0)$, we have*

$$\frac{\iint_{B_{\tau\rho}(\partial B_\rho) \cap \Omega_{X_0}} \mu v_{X_0}^2 dV_g}{\iint_{B_\rho \cap \Omega_{X_0}} \mu v_{X_0}^2 dV_g} \leq C_1 \left[1 - \left(\frac{1-\tau}{1+\tau} \right)^{d+2N_{X_0}((1+\tau)\rho)} \right]. \quad (5.12)$$

In particular if $|X_0| \leq \frac{3}{20}R$ and $\rho < \frac{3}{4}R$,

$$\text{as } \tau \rightarrow 0, \text{ the ratio } \frac{\iint_{B_{\tau\rho}(\partial B_\rho) \cap \Omega_{X_0}} \mu v_{X_0}^2 dV_g}{\iint_{B_\rho \cap \Omega_{X_0}} \mu v_{X_0}^2 dV_g} \text{ converges to 0 uniformly for } X_0 \text{ and } \rho.$$

Proof. Using polar coordinates, we have

$$\frac{\iint_{B_{\tau\rho}(\partial B_\rho) \cap \Omega_{X_0}} \mu v_{X_0}^2 dV_g}{\iint_{B_\rho \cap \Omega_{X_0}} \mu v_{X_0}^2 dV_g} \leq \frac{\int_{(1-\tau)\rho}^{(1+\tau)\rho} H_{X_0}(s) ds}{\int_0^{(1-\tau)\rho} H_{X_0}(s) ds}. \quad (5.13)$$

Let $a = \frac{(1+\tau)\rho}{(1-\tau)\rho} = \frac{1+\tau}{1-\tau} > 1$, then

$$\begin{aligned} \frac{\int_0^{(1-\tau)\rho} H_{X_0}(s) ds}{\int_{(1-\tau)\rho}^{(1+\tau)\rho} H_{X_0}(s) ds} &= \sum_{j=0}^{\infty} \frac{\int_{a^{-(j+1)}(1-\tau)\rho}^{a^{-j}(1-\tau)\rho} H_{X_0}(s) ds}{\int_{(1-\tau)\rho}^{(1+\tau)\rho} H_{X_0}(s) ds} \geq \sum_{j=0}^{\infty} \frac{1}{C_1 a^j [d+2N_{X_0}((1+\tau)\rho)]} \\ &= \frac{1}{C_1} \frac{1}{1 - \left(\frac{1-\tau}{1+\tau}\right)^{d+2N_{X_0}((1+\tau)\rho)}}. \end{aligned} \quad (5.14)$$

Combining (5.13) and (5.14) we get the desired inequality (5.12).

By the assumption and (5.2), we have that the exponent

$$d + 2N_{X_0}((1+\tau)\rho) \leq C(d, \Lambda) < +\infty$$

for τ sufficiently small. Therefore the ratio tends to zero as $\tau \rightarrow 0$. $\square$

In the rest of this section, we first study what the monotonicity formula in Proposition 3.10 tells us about the tangent function of v . Then we switch gears to the harmonic function u in the Dini domain and study its tangent function, by looking at the transformation maps between u and v .

5.1. Blow up analysis in Ω_{X_0} using the monotonicity formula. Let $X_0 \in B_R(0) \cap \partial D$ be fixed. Since v_{X_0} has boundary value zero, we may extend it by zero across Ω_{X_0} . For any $r > 0$ we define

$$v_r(Y) = T_r v_{X_0} = \frac{v_{X_0}(rY)}{\left(\frac{1}{r^d} \iint_{B_r} v_{X_0}^2 dX\right)^{\frac{1}{2}}}, \quad \text{for } Y \in \frac{\Omega_{X_0}}{r} \cap B_5. \quad (5.15)$$

The normalizing factor in the denominator guarantees that

$$1 \leq \iint_{\frac{\Omega_{X_0}}{r} \cap B_5} |v_r|^2 dY = \frac{\iint_{B_{5r}} v_{X_0}^2 dX}{\iint_{B_r} v_{X_0}^2 dX} \leq C_1 5^{d+2N_{X_0}(5r)} \leq C(d, \Lambda) < +\infty, \quad (5.16)$$

by (3.30) and (5.2), as long as $|X_0| \leq \frac{3}{20}R$ and $r \leq \frac{3}{20}R$. In particular $\|v_r\|_{L^2}$ is uniformly bounded. Heuristically if the function v can be written as the sum of homogeneous functions, then as $r \rightarrow 0+$ we have v_r approaches the leading order homogeneous function (modulo a normalization factor). We first look at how the above rescaling by r affects the frequency function. Notice that not only is v_r defined in the domain $\frac{\Omega}{r}$, which is different from the domain Ω of v , it also satisfies a different divergence-form elliptic operator:

$$\operatorname{div}(A_r(Y)\nabla v_r) = 0 \text{ in } \frac{\Omega}{r}, \quad \text{where the matrix } A_r(Y) := A(rY).$$

So when we write the frequency function for v_r at the scale $s > 0$, to be more rigorous we should write $N(\Omega/r, A_r, v_r, s)$. But for the sake of simplicity we will just write $N(v_r, s)$ and keep in mind that the definition depends on the domain and elliptic operator (and thus the corresponding intrinsic metric) for the function v_r . By (3.14) and a change of variables, we get

$$N(v_r, s) = \frac{sD(v_r, s)}{H(v_r, s)} = \frac{\frac{s}{r^{d-2}}D(v_{X_0}, rs)}{\frac{1}{r^{d-1}}H(v_{X_0}, rs)} = \frac{rsD(v, rs)}{H(v, rs)} = N(v_{X_0}, rs). \quad (5.17)$$

Fix the scale $s > 0$, we get

$$N(v_{X_0}, rs) = \tilde{N}(v_{X_0}, rs) \exp\left(-C \int_0^{rs} \frac{\theta(4\tau)}{\tau} d\tau\right) \rightarrow \lim_{\rho \rightarrow 0+} \tilde{N}(v_{X_0}, \rho) =: N_{X_0}, \quad \text{as } r \rightarrow 0.$$

Hence

$$N(v_r, s) \rightarrow N_{X_0} \quad \text{as } r \rightarrow 0. \quad (5.18)$$

On the other hand, we have the following lemma:

Lemma 5.19. *Assume that $N_0(4R) \leq \Lambda + \infty$. Then there exists a universal constant $C' > 0$ (depending on the values of d, C_1 and Λ), such that for any $X_0 \in \partial D$ satisfying $|X_0| < \frac{3}{20}R$ and any $r < \frac{1}{36}R$, we have*

$$|\nabla v_r(Y)| \leq C', \quad \text{for all } Y \in \frac{\Omega_{X_0}}{r} \cap B_5.$$

Proof. By (4.3) and the chain rule,

$$|\nabla v_r(Y)| = \frac{|\nabla(u(X_0 + \Psi(rY)))|}{\left(\frac{1}{r^d} \iint_{B_r \cap \Omega_{X_0}} v_{X_0}^2 dX\right)^{\frac{1}{2}}} \lesssim \frac{r|\nabla u(Z)|}{\left(\frac{1}{r^d} \iint_{B_r \cap \Omega_{X_0}} v_{X_0}^2 dX\right)^{\frac{1}{2}}}, \quad (5.20)$$

where $Z = X_0 + \Psi(rY) \in B_{\frac{16}{3}r}(X_0)$. (The elliptic variant of [CK, Theorem 1.4.10] shows that $\nabla u \in C(\overline{D} \cap B_{5R}(0))$, with a modulus of continuity depending on the Dini parameter. In fact, the parabolic theorems in [CK] imply the elliptic ones by extending the elliptic solution independent of the t -variable.) We then use the gradient estimate in [CK, Theorem 1.4.3]. (In [CK] the authors work with a Dini-type elliptic operator in the upper half-space, and we work with the Laplacian in a Dini domain. These two settings are related by a simple change of variables $(x, x_d) \in D \mapsto (x, x_d - \varphi(x)) \in \mathbb{R}_+^d$, and the Laplacian in D becomes a divergence-form elliptic operator in $\mathbb{R}_+^d$ with coefficient matrix $\begin{pmatrix} \text{Id} & -\nabla\varphi \\ (-\nabla\varphi)^t & 1 + |\nabla\varphi|^2 \end{pmatrix}$.) After a rescaling, [CK, Theorem 1.4.3] implies

$$\sup_{B(X_0, \frac{16}{3}r) \cap D} |\nabla u| \lesssim \frac{1}{r} \left(\iint_{B(X_0, \frac{17}{3}r) \cap D} u^2 dX \right)^{\frac{1}{2}} \lesssim \frac{1}{r} \left(\iint_{B_{6r} \cap \Omega_{X_0}} \mu v_{X_0}^2 dV_g \right)^{\frac{1}{2}} \quad (5.21)$$

By the doubling property (3.30) and (5.2),

$$\frac{\iint_{B_{6r} \cap \Omega_{X_0}} \mu v_{X_0}^2 dV_g}{\iint_{B_r \cap \Omega_{X_0}} \mu v_{X_0}^2 dV_g} \leq C_1 6^{d+2N_{X_0}(6r)} \leq C(C_1, \Lambda) < +\infty \quad (5.22)$$

if $|X_0| < \frac{3}{20}R$ and $r < \frac{R}{8}$. Finally combining (5.20), (5.21), (4.8) and (5.22) we conclude that

$$|\nabla v_r(Y)| \leq C \left(\frac{\iint_{B_{6r} \cap \Omega_{X_0}} \mu v_{X_0}^2 dV_g}{\iint_{B_r \cap \Omega_{X_0}} v_{X_0}^2 dX} \right)^{\frac{1}{2}} \leq C' < +\infty, \quad \text{for all } |Y| \leq 5.$$

□

Let r_i be a sequence such that $r_i \rightarrow 0+$. Since

$$\Omega_{X_0} = \{(x, x_d) : x_d > \varphi(x + x_0) - \varphi(x_0) - 3|X|\tilde{\theta}(|X|)\},$$

with $X_0 = (x_0, \varphi(x_0))$, it is not hard to see that inside B_5 ,

$$\frac{\Omega_{X_0}}{r_i} \text{ converges graphically to the half-space } \Omega_\infty := \{(y, y_d) : y_d > \nabla\varphi(x_0) \cdot y\}.$$

Throughout the paper, we say a sequence of domains $D_j := \{(x, x_d) : x_d > \psi_j(x)\}$ converges graphically to a domain D_∞ , if ψ_j converges locally uniformly to a function ψ_∞ and D_∞ is the domain above the graph of ψ_∞ , i.e. $D_\infty = \{(x, x_d) : x_d > \psi_\infty(x)\}$.

Since we also have $v_{r_i}(0) = 0$, it follows from the Arzela-Ascoli Theorem that modulo passing to a subsequence

v_{r_i} converges uniformly to a function v_∞ in B_5 ,

and $v_\infty = 0$ on $B_5 \setminus \Omega_\infty$. Note that $v_\infty \not\equiv 0$ because of (5.16). Moreover we also have $\nabla v_{r_i} \rightharpoonup \nabla v_\infty$ in L^2 and thus v_∞ is a harmonic function in Ω_∞ . In fact, we also have that ∇v_{r_i} converges to ∇v_∞ strongly in L^2 . Firstly let K be a compact set in $\Omega_\infty \cap B_5$. Then K is also compactly contained in $\frac{\Omega_{X_0}}{r_i}$ for all i sufficiently large. In K , the function $v_{r_i} - v_\infty$ satisfies the differential equation

$$\operatorname{div} (A_{r_i}(Y) \nabla (v_{r_i} - v_\infty)) = \operatorname{div} ((\operatorname{Id} - A_{r_i}(Y)) \nabla v_\infty),$$

where $A_{r_i}(Y) = A(r_i Y)$. We apply the interior gradient estimate in [CK, Corollary 1.2.22], but this time with non-trivial right hand side (see the general form of the differential equation in [CK, (1.2.1)]). And we get

$\nabla v_{r_i} - \nabla v_\infty$ converges to 0 uniformly in K .

On the other hand, let $\tau > 0$ and we set $B_\tau(\partial\Omega_\infty) := \{X : \operatorname{dist}(X, \partial\Omega_\infty) < \tau\}$. Since ∇v_{r_i} is uniformly bounded, we may choose τ small enough so that

$$\iint_{B_\tau(\partial\Omega_\infty) \cap B_5} |\nabla v_{r_i}|^2 dX < \epsilon.$$

Then for an arbitrary $s \in (0, 5)$, we have

$$\limsup_{r_i \rightarrow 0+} \iint_{B_s \cap \frac{\Omega_{X_0}}{r_i}} |\nabla v_{r_i}|^2 dX \leq \lim_{r_i \rightarrow 0+} \iint_{B_s \setminus B_\tau(\Omega_\infty) \cap \frac{\Omega_{X_0}}{r_i}} |\nabla v_{r_i}|^2 dX + \epsilon \leq \iint_{B_s \cap \Omega_\infty} |\nabla v_\infty|^2 dX + \epsilon.$$

Combined with Fatou's Lemma we conclude that

$$\iint_{B_s \cap \frac{\Omega_{X_0}}{r_i}} |\nabla v_{r_i}|^2 dX \rightarrow \iint_{B_s \cap \Omega_\infty} |\nabla v_\infty|^2 dX.$$

In particular, it implies that

$$N(v_{r_i}, s) \rightarrow N(v_\infty, s) \text{ as } r_i \rightarrow 0.$$

Combined with (5.18), we obtain

$$N(v_\infty, s) \equiv N_{X_0} \quad \text{for all } 0 < s < 5.$$

By the rigidity of the monotonicity formula in Corollary 3.26, this implies that the harmonic function $v_\infty(r, \omega) = r^{N_{X_0}} v(\omega)$ is homogeneous of degree N_{X_0} . Hence $\lambda := N_{X_0}(N_{X_0} + d - 2)$ is an eigenvalue for the Laplace-Beltrami operator of the half-sphere $\Omega_\infty \cap \mathbb{S}^{n-1}$. We notice that any odd extension of a Dirichlet eigenfunction on the half-sphere is an eigenfunction on the entire sphere, which vanishes on the equator. Additionally, recall that eigenfunctions on the entire sphere are restrictions of harmonic polynomials. Therefore we have proven:

Lemma 5.23. $N_{X_0} := \lim_{r \rightarrow 0+} \tilde{N}(v_{X_0}, r)$ must take positive integer values.

5.2. Compactness. In this subsection, we will make sense of the following observation: if we blow up at a point in $p \in B_R(0) \cap \partial D$, the configuration at the limit is the same as the blow-up limit for the corresponding reduced domain Ω_p . In fact the observation also holds true for pseudo blow-ups, i.e. we zoom in not at a fixed point, but at a sequence of points contained in a compact set. To be more precise,

Proposition 5.24. *Let $R, \Lambda > 0$ be fixed, and $(u_i, D_i) \in \mathfrak{H}(R, \Lambda)$. Let p_i be a sequence of points in $B_{\frac{3}{20}R}(0) \cap \partial D_i$ and r_i be a sequence that converges to zero. Modulo passing to a subsequence,*

- (1) *the sequence of domains $\frac{D_i - p_i}{r_i}$ converges graphically to D_∞ , where $D_\infty = \{(y, y_d) : y_d > \nabla \varphi(x_\infty) \cdot y\}$ is an upper half-space and $p_\infty = (x_\infty, \varphi(x_\infty))$ is a limit point of the sequence $\{p_i\}$;*
- (2) *inside $B_5(0)$ the sequence $T_{p_i, r_i} u_i$ converges uniformly and in $W^{1,2}$ to a function u_∞ , which satisfies $\Delta u_\infty = 0$ in $D_\infty \cap B_5(0)$ and $u_\infty = 0$ in $B_5(0) \setminus D_\infty$.*

Moreover, consider the functions $v_{p_i} = u_i \circ \Psi_{p_i}$ in domains Ω_{p_i} respectively. We have that $\frac{\Omega_{p_i}}{r_i}$ also converges graphically to D_∞ , and inside B_5 the sequence $T_{r_i} v_{p_i}$ also converges uniformly and in $W^{1,2}$ to u_∞ .

Remark 5.25. In fact, in the proof we will see that to get compactness, it suffices to assume r_i 's are bounded (in that case the limiting domain D_∞ is no longer the upper half-space). We need $r_i \rightarrow 0$ to get that $T_{p_i, r_i} u_i$ and $T_{r_i} v_{p_i}$ converge to the same limiting function, and in particular to get that $T_{r_i} v_{p_i}$ converges to a harmonic function. (Note that $T_{r_i} v_{p_i}$ solves an elliptic equation with coefficient matrix $A_{r_i}(Y)$, which does not converge to the identity matrix if $r_i \not\rightarrow 0$.)

Proof. By the assumption $D_i \cap B_{5R}(0)$ is above the graph of a C^1 -Dini function φ_i , which satisfies $\varphi_i(0) = 0$, $\nabla \varphi_i(0) = 0$ (since ∂D_i is tangent to $\mathbb{R}^{d-1} \times \{0\}$ at the origin) and

$$|\nabla \varphi_i(x) - \nabla \varphi_i(y)| \leq \theta(|x - y|). \quad (5.26)$$

By Arzela-Ascoli Theorem φ_i converges in C^1 to a function $\varphi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$, which also satisfies (5.26). We write $p_i = (x_i, \varphi_i(x_i))$. Since each x_i is contained in $B_{\frac{3}{20}R}^{d-1}(0)$, modulo taking a subsequence x_i converges to some $x_\infty \in \mathbb{R}^{d-1}$. Hence $p_i = (x_i, \varphi_i(x_i))$ converges to some $p_\infty = (x_\infty, \varphi(x_\infty))$ satisfying $|p_\infty| \leq \frac{3}{20}R$. We let D denote the domain above the graph of φ . Clearly it is also a Dini domain with parameter θ , $\partial D \ni 0, p_\infty$ and ∂D is tangent to $\mathbb{R}^{d-1} \times \{0\}$ at the origin.

It follows that $\frac{D_i - p_i}{r_i}$ is locally above the graph of a function $\tilde{\varphi}_i : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$, where

$$\tilde{\varphi}_i(y) = \frac{1}{r_i} [\varphi(x_i + r_i y) - \varphi(x_i)], \quad \text{for } |y| \leq 5.$$

By the properties of φ in (5.26), we have

$$\begin{aligned} |\tilde{\varphi}_i(y) - \langle \nabla \varphi_i(x_\infty), y \rangle| &\leq |\tilde{\varphi}_i(y) - \langle \nabla \varphi_i(x_i), y \rangle| + |\langle \nabla \varphi_i(x_i), y \rangle - \langle \nabla \varphi_i(x_\infty), y \rangle| \\ &\leq \theta(r_i |y|) |y| + \theta(|x_i - x_\infty|) |y| \\ &\leq 5(\theta(5r_i) + \theta(|x_i - x_\infty|)). \end{aligned}$$

The last term tends to zero uniformly in $|y| \leq 5$. Therefore $\tilde{\varphi}_i(y)$ converges uniformly to a linear function $\varphi_\infty(y) = \langle \nabla \varphi(x_\infty), y \rangle$. Moreover, inside $B_5(0)$ the domain $\frac{D_i - p_i}{r_i}$ converges graphically to the domain

$$D_\infty := \{(y, y_n) : y_n > \varphi_\infty(y) = \langle \nabla \varphi(x_\infty), y \rangle\},$$

which is the region above a hyperplane $\{(y, y_n) : y_n = \langle \nabla \varphi(x_\infty), y \rangle\}$. This finishes the proof of (1).

For simplicity of notation we write $\tilde{u}_i = T_{p_i, r_i} u_i$ and $\tilde{v}_i = T_{r_i} v_{p_i}$. Notice that the uniform gradient bound proved in Lemma 5.19 is independent of the base point X_0 , or as in the current notation, the base points p_i 's. So we can use similar argument as in the previous subsection to show that $\tilde{u}_i$ converges uniformly and in $W^{1,2}$ to a function u_∞ in $B_5(0)$, such that $u_\infty = 0$ in $B_5(0) \setminus \Omega_\infty$ and $\Delta u_\infty = 0$ in $\Omega_\infty \cap B_5(0)$.

On the other hand, as in Section 4 setting

$$v_{p_i}(X) = u_i(\Psi_{p_i}(X)) = u_i(p_i + \Psi(X)), \quad \text{for } X \in \Omega_{p_i},$$

we know it satisfies $\text{div}(A(X)\nabla v_i) = 0$ where the matrix $A(X)$ is defined as in (4.5). Notice that the matrix $A(X)$ is independent of the base point p_i . Recall that for each $p_i = (x_i, \varphi(x_i))$, the domain Ω_{p_i} is above the graph of an implicit function $\phi_i : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ defined as

$$\phi_i(x) := \varphi_i(x_i + x) - \varphi_i(x_i) - 3|X|\tilde{\theta}(|X|), \quad \text{where } X = (x, \phi_i(x)).$$

Therefore $\frac{\Omega_{p_i}}{r_i}$ is locally above the graph of the implicit function

$$\tilde{\phi}_i(y) := \frac{1}{r_i} [\varphi_i(x_i + r_i y) - \varphi_i(x_i) - 3r_i|Y|\tilde{\theta}(r_i|Y|)], \quad \text{where } Y = (y, \tilde{\phi}_i(y)).$$

On the compact set $|y| \leq 5$, the function $\tilde{\phi}_i$ converges uniformly to a linear function $\phi_\infty(y) = \langle \nabla \varphi(x_\infty), y \rangle$, which is the same as $\varphi_\infty(y)$. Hence inside B_5 ,

$$\frac{\Omega_{p_i}}{r_i} \text{ also converges graphically to the domain } D_\infty.$$

Besides, for any test function ψ compactly supported in $D_\infty \cap B_5$, using a change of variables several times we know

$$\begin{aligned} & \iint_{\frac{\Omega_{p_i}}{r_i}} \langle A(r_i Y) \nabla \tilde{v}_i(Y), \nabla \psi(Y) \rangle dY \\ &= \frac{1}{\left(\frac{1}{r_i^d} \iint_{B_{r_i} \cap \Omega_{p_i}} v_{p_i}^2 dX \right)^{\frac{1}{2}}} \frac{1}{r_i^{d-2}} \iint_{\Omega_{p_i}} \langle A(X) \nabla v_{p_i}(X), \nabla \psi_i(X) \rangle dX \quad \psi_i(X) := \psi\left(\frac{X}{r_i}\right) \\ &= \frac{\left(\iint_{B_{r_i}(p_i) \cap D_i} u_i^2 dX \right)^{\frac{1}{2}}}{\left(\iint_{B_{r_i} \cap \Omega_{p_i}} v_{p_i}^2 dX \right)^{\frac{1}{2}}} \iint_{\frac{D_i - p_i}{r_i}} \langle \nabla \tilde{u}_i(Z), \nabla \tilde{\psi}_i(Z) \rangle dZ, \quad \tilde{\psi}_i(Z) = \psi\left(\frac{\Psi^{-1}(r_i Z)}{r_i}\right). \quad (5.27) \end{aligned}$$

By the definition of Ψ , we know that

$$B\left(0, \frac{s}{1+3\tilde{\theta}(s)}\right) \subset \Psi^{-1}(B_s(0)) \subset B\left(0, \frac{s}{1-3\tilde{\theta}(s)}\right).$$

Hence

$$\frac{\iint_{(\Psi^{-1}(B_{r_i}) \Delta B_{r_i}) \cap \Omega_{p_i}} \mu v_{p_i}^2 dV_g}{\iint_{B_{r_i} \cap \Omega_{p_i}} \mu v_{p_i}^2 dV_g} \leq \frac{\iint_{B_{\tau_i r_i}(\partial B_{r_i}) \cap \Omega_{p_i}} \mu v_{p_i}^2 dV_g}{\iint_{B_{r_i} \cap \Omega_{p_i}} \mu v_{p_i}^2 dV_g}, \quad \text{where } \tau_i = \frac{3\tilde{\theta}(r_i)}{1-3\tilde{\theta}(r_i)}.$$

By Lemma 5.19, the right hand side tends to zero as $r_i \rightarrow 0$ (and thus $\tau_i \rightarrow 0$). Using this and the definitions of Ω_{p_i} and v_{p_i} , we conclude the factor

$$\frac{\left(\iint_{B_{r_i}(p_i) \cap D_i} u_i^2 dX\right)^{\frac{1}{2}}}{\left(\iint_{B_{r_i} \cap \Omega_{p_i}} v_{p_i}^2 dX\right)^{\frac{1}{2}}} \rightarrow 1 \text{ as } r_i \rightarrow 0. \quad (5.28)$$

Let $Y = \frac{\Psi^{-1}(r_i Z)}{r_i}$, then by the definition (4.2) of the map Ψ we have

$$Z = (y, y_d - 3|Y|\tilde{\theta}(r_i|Y|)) = Y - 3|Y|\tilde{\theta}(r_i|Y|)e_d.$$

Since ψ is a smooth function whose support is contained in on B_5 , it follows that

$$\begin{aligned} |\nabla \tilde{\psi}_i(Z) - \nabla \psi(Y)| &\leq \|\partial_d \tilde{\psi}_i\|_{\infty} (3\tilde{\theta}(r_i|Y|) + 3r_i|Y|\tilde{\theta}'(r_i|Y|)), \\ |\nabla \psi(Z) - \nabla \psi(Y)| &\leq \|\nabla^2 \psi\|_{\infty} |Y - Z| \leq 3|Y| \|\nabla^2 \psi\|_{\infty} \tilde{\theta}(r_i|Y|). \end{aligned}$$

Hence

$$|\nabla \psi(Z) - \nabla \tilde{\psi}_i(Z)| \rightarrow 0 \text{ uniformly for } |Z| \leq 5.$$

Therefore

$$\begin{aligned} &\left| \iint_{\frac{D_i - p_i}{r_i}} \langle \nabla \tilde{u}_i(Z), \nabla \tilde{\psi}_i(Z) \rangle dZ - \iint_{D_{\infty}} \langle \nabla u_{\infty}(Z), \nabla \psi(Z) \rangle dZ \right| \\ &\leq \left| \iint_{\frac{D_i - p_i}{r_i}} \langle \nabla \tilde{u}_i(Z), \nabla \tilde{\psi}_i(Z) \rangle dZ - \iint_{\frac{D_i - p_i}{r_i}} \langle \nabla \tilde{u}_i(Z), \nabla \psi(Z) \rangle dZ \right| \\ &\quad + \left| \iint_{\frac{D_i - p_i}{r_i}} \langle \nabla \tilde{u}_i(Z), \nabla \psi(Z) \rangle dZ - \iint_{D_{\infty}} \langle \nabla u_{\infty}(Z), \nabla \psi(Z) \rangle dZ \right| \end{aligned}$$

also converges to zero. Combined with (5.27) and (5.28) we get

$$\iint_{\frac{\Omega_{p_i}}{r_i}} \langle A(r_i Y) \nabla \tilde{v}_i, \nabla \psi(Y) \rangle dY \rightarrow \iint_{D_{\infty}} \langle \nabla u_{\infty}(Z), \nabla \psi(Z) \rangle dZ.$$

Since $A(r_i Y)$ converges to the identity map uniformly in $|Y| \leq 5$, it follows that $\tilde{v}_i$ also converges uniformly and in $W^{1,2}$ to the function u_{∞} . In particular $\tilde{u}_i$ and $\tilde{v}_i$ converge to the same harmonic function u_{∞} in $D_{\infty} \cap B_5$.

□

6. FREQUENCY FUNCTION FOR INTERIOR POINTS

6.1. Monotonicity formula of the frequency function. For any interior point p in a C^1 domain D , we will show that the frequency function $N(p, r)$ is monotone increasing for radius r in a finite interval, the length of which depends on $\text{dist}(p, \partial D)$. More precisely, for any $r > 0$ let

$$D(p, r) = \iint_{B_r(p) \cap D} |\nabla u|^2 dX \quad \text{and} \quad H(p, r) = \int_{\partial B_r(p) \cap D} u^2 d\mathcal{H}^{d-1}; \quad (6.1)$$

and let the frequency function centered at p be defined as

$$N(p, r) = \frac{rD(p, r)}{H(p, r)}. \quad (6.2)$$

For simplicity of notation, when there is no confusion we often drop the dependence on p and simply write $D(r)$, $H(r)$ and $N(r)$.

Proposition 6.3. *Let D be a C^1 domain, in the sense that*

$$D \cap B_{5R}(0) = \{(x, x_d) \in \mathbb{R}^d \times \mathbb{R} : x_d > \varphi(x)\} \cap B_{5R}(0)$$

where φ is a C^1 function and $0 \in \partial D$. Let u be a harmonic function in D such that $u = 0$ on $\partial D \cap B_{5R}(0)$. Then for any $p \in D \cap B_{2R}(0)$, the frequency function $N(p, r)$ satisfies

$$N'(r) = R_h(r) + R_b(r),$$

where

$$R_h(r) := \frac{2r}{H(r)} \int_{\partial B_r(p) \cap D} \left| \partial_\rho u - \frac{N(r)}{r} u \right|^2 d\mathcal{H}^{d-1},$$

$$R_b(r) := \frac{1}{H(r)} \int_{B_r(p) \cap \partial D} (\partial_n u)^2 \langle X - p, n_D(X) \rangle d\mathcal{H}^{d-1}.$$

Here $\partial_\rho u$ denotes the radial derivative of u with center p , and $\partial_n u$ denotes the normal derivative of u pointing away from D .

In particular $N(r)$ is monotone increasing with respect to r , as long as

$$r\theta(r) \leq \text{dist}(p, \partial D), \quad (6.4)$$

where $\theta(\cdot)$ is the modulus of continuity for the function $\nabla \varphi$.

Proof. The proof is similar to the proof of Proposition 3.10 for the case of the Laplacian operator, so we will just sketch the proof emphasizing the differences. Using the assumptions that $\Delta u = 0$ and u vanishes on the boundary, we get

$$H'(r) = \frac{d-1}{r} H(r) + 2 \int_{\partial B_r(p) \cap D} u \partial_\rho u d\mathcal{H}^{d-1} = \frac{d-1}{r} H(r) + 2D(r), \quad (6.5)$$

and

$$D'(r) = \frac{d-2}{r} D(r) + 2 \int_{\partial B_r(p) \cap D} (\partial_\rho u)^2 d\mathcal{H}^{d-1} + \frac{1}{r} \int_{B_r(p) \cap \partial D} (\partial_n u)^2 \langle X - p, n_D(X) \rangle d\mathcal{H}^{d-1}.$$

Therefore

$$\begin{aligned} \frac{N'(r)}{N(r)} &= \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} \\ &= \frac{2}{D(r)} \int_{\partial B_r(p) \cap D} \left| \partial_\rho u - \frac{N(r)}{r} u \right|^2 d\mathcal{H}^{d-1} + \frac{1}{rD(r)} \int_{B_r(p) \cap \partial D} (\partial_n u)^2 \langle X - p, n_D(X) \rangle d\mathcal{H}^{d-1}. \end{aligned}$$

Or equivalently,

$$N'(r) = \frac{2r}{H(r)} \int_{\partial B_r(p) \cap D} \left| \partial_\rho u - \frac{N(r)}{r} u \right|^2 d\mathcal{H}^{d-1} + \frac{1}{H(r)} \int_{B_r(p) \cap \partial D} (\partial_n u)^2 \langle X - p, n_D(X) \rangle d\mathcal{H}^{d-1}. \quad (6.6)$$

We claim that if the condition (6.4) holds, then

$$\langle X - p, n_D(X) \rangle \geq 0, \quad \text{for every } X \in B_r(p) \cap \partial D.$$

It then follows that $R_b(r) \geq 0$. In fact, we denote $p = (x_0, z_0)$, where $x_0 \in \mathbb{R}^{d-1}$ and $z_0 \in \mathbb{R}$ satisfies $z_0 - \varphi(x_0) \geq \text{dist}(p, \partial D)$. Let $X = (x, \varphi(x))$ be an arbitrary boundary point in $B_r(p)$. then

$$\begin{aligned} \langle X - p, n_D(X) \rangle &= \left\langle (x - x_0, \varphi(x) - z_0), \frac{(\nabla \varphi(x), -1)}{\sqrt{1 + |\nabla \varphi(x)|^2}} \right\rangle \\ &= \frac{1}{\sqrt{1 + |\nabla \varphi(x)|^2}} [\langle \nabla \varphi(x), x - x_0 \rangle + z_0 - \varphi(x)] \\ &= \frac{1}{\sqrt{1 + |\nabla \varphi(x)|^2}} [(z_0 - \varphi(x_0)) + (\varphi(x_0) - \varphi(x) - \langle \nabla \varphi(x), x_0 - x \rangle)] \\ &\geq \frac{1}{\sqrt{1 + |\nabla \varphi(x)|^2}} [\text{dist}(p, \partial D) - \theta(r)r] \\ &\geq 0. \end{aligned}$$

Since we also have $R_h(r) \geq 0$, by (6.6) we know $N(r)$ is monotone increasing with respect to r . $\square$

As in the purely interior case, the limit of the frequency function $N(p, 0) := \lim_{r \rightarrow 0+} N(p, r)$ measures the vanishing order of u at p . By the blow up analysis similar to Subsection 5.1, when $p \in \mathcal{N}(u)$ we know that $N(p, 0) \in \mathbb{N}$. *

Since $\tilde{\theta}(\cdot) > 0$ is a nondecreasing and continuous function, the map $r \in [0, R] \mapsto r\tilde{\theta}(r) \in [0, R\tilde{\theta}(R)]$ is bijective. In particular it gives a decomposition of the interval $[0, R\tilde{\theta}(R)]$, and for any real number $x \in [0, R\tilde{\theta}(R)]$ we can find $r \in [0, R]$ such that $x = r\tilde{\theta}(r)$. Thus for any $p \in D \cap B_{2R}(0)$ sufficiently close to the boundary, we can always find a unique $r > 0$ such that

$$\text{dist}(p, \partial D) = r\tilde{\theta}(r). \quad (6.7)$$

*If $u(p) \neq 0$, then the leading order of u near p is simply the non-trivial constant $u(p)$. In this case, we need to define the frequency function centered at p using the harmonic function $u - u(p)$, so as to capture the leading order term of $u - u(p)$.

Thus we will refer to such r as the *critical scale* for p and denote it by $r_{cs}(p)$, and say that $[0, r_{cs}(p)]$ is the *monotonic interval* for p . To unify the notation we use the convention that $r_{cs}(p) = 0$ if $p \in \partial D$.

6.2. Doubling property and frequency function beyond the critical scale. For interior points, similar to Lemma 5.1 we also get a uniform bound (depending on d, Λ) on the frequency function.

Lemma 6.8. *Assume that $N_0(4R) \leq \Lambda < +\infty$. Suppose $X \in D \cap B_{\frac{R}{10}}(0)$ is such that $\text{dist}(X, \partial D) \leq \frac{3}{5}R\tilde{\theta}(\frac{3}{5}R)$. Then for any $r \leq \frac{3}{5}R$, we have*

$$N(X, r) \leq C(\Lambda)N_{\tilde{X}}\left(\frac{3}{4}R\right) \leq C'(\Lambda), \quad (6.9)$$

where $\tilde{X} \in \partial D$ satisfies $|X - \tilde{X}| = \text{dist}(X, \partial D)$.

Remark 6.10. Notice that the above upper bound holds even outside of the monotonic interval $[0, r_{cs}(X)]$, as long as we have $r \leq \frac{3}{5}R$.

Proof. Since the frequency function $N(X, \cdot)$ is monotone increasing on the interval $[0, r_{cs}(X)]$, it suffices to prove (6.9) for $r_{cs}(X) \leq r \leq \frac{3}{5}R$. By (2.5), we have

$$|X - \tilde{X}| = \text{dist}(X, \partial D) < \frac{r_{cs}(X)}{4} \leq \frac{r}{4}.$$

By the sub-harmonicity of u^2 and the doubling property of $H(\tilde{X}, \cdot)$, we have

$$H(X, r) \gtrsim \frac{1}{r} \iint_{A_{\frac{r}{4}, r}(X)} u^2 dZ \geq \frac{1}{r} \iint_{A_{\frac{r}{2}, \frac{3}{4}r}(\tilde{X})} u^2 dZ \gtrsim \int_{\partial B_{\frac{r}{2}}(\tilde{X})} u^2 d\mathcal{H}^{d-1} \gtrsim_{\Lambda} H\left(\tilde{X}, \frac{5}{4}r\right).$$

Therefore

$$N(X, r) = \frac{r \iint_{B_r(X)} |\nabla u|^2 dZ}{H(X, r)} \lesssim \frac{r \iint_{B_{\frac{5}{4}r}(\tilde{X})} |\nabla u|^2 dZ}{H(\tilde{X}, \frac{5}{4}r)} \leq N_{\tilde{X}}\left(\frac{5}{4}r\right) \lesssim N_{\tilde{X}}\left(\frac{3}{4}R\right).$$

□

The above frequency bound yields the doubling property for the L^2 norm, similar to the boundary case in Corollary 3.28.

Corollary 6.11. *Assume that $N_0(4R) \leq \Lambda < +\infty$. Then for any $X \in D \cap B_{\frac{R}{10}}(0)$ such that $\text{dist}(X, \partial D) \leq \frac{3}{5}R\tilde{\theta}(\frac{3}{5}R)$, and any $\rho > 0$, $a > 1$ such that $a\rho \leq \frac{3}{5}R$, we have*

$$\frac{\iint_{B_{a\rho}(X)} u^2 dZ}{\iint_{B_{\rho}(X)} u^2 dZ} \leq a^{d+C(\Lambda)}. \quad (6.12)$$

Proof. Recall we have computed the derivative of $H(X, s)$ in (6.5), which can be reformulated into

$$\left(\log \frac{H(X, s)}{s^{d-1}} \right)' = \frac{2N(X, s)}{s}.$$

Integrating the above equality and by the upper bound of the frequency function, we get

$$\frac{H(X, as)}{H(X, s)} = a^{d-1} \exp \left[\int_s^{as} \frac{2N(X, \tau)}{\tau} d\tau \right] \leq a^{d-1+C(\Lambda)}. \quad (6.13)$$

Therefore

$$\frac{\iint_{B_{ap}(X)} u^2 dZ}{\iint_{B_p(X)} u^2 dZ} = \frac{a \int_0^p H(X, as) ds}{\int_0^p H(X, s) ds} \leq a^{d+C(\Lambda)}.$$

□

Lemma 6.14. *Assume that $N_0(4R) \leq \Lambda + \infty$. For any $X \in D \cap B_{\frac{R}{10}}(0) \cap \mathcal{N}(u)$ such that $\text{dist}(X, \partial D) \leq \frac{3}{5}R\tilde{\theta}(\frac{3}{5}R)$, and any $r \leq \frac{3}{35}R$, we have*

$$|\nabla T_{X,r}u(Y)| \leq C, \quad \text{for all } Y \in \frac{D-X}{r} \cap B_5.$$

Proof. Recalling the definition of $T_{X,r}u$ in Definition 2.11, we have

$$|\nabla T_{X,r}u(Y)|^2 = \frac{(r|\nabla u(X + rY)|)^2}{\frac{1}{r^d} \iint_{B_r(X)} u^2 dZ}.$$

We first consider the purely interior case when $6r < \text{dist}(X, \partial D)$. In this case, by the interior gradient estimate, $u(X) = 0$ and the doubling property in Corollary 6.11, we have

$$|\nabla T_{X,r}u(Y)| \leq \frac{\sup_{B_{5r}(X)} (r|\nabla u|)^2}{\frac{1}{r^d} \iint_{B_r(X)} u^2 dZ} \lesssim \frac{\iint_{B_{6r}(X)} u^2 dZ}{\iint_{B_r(X)} u^2 dZ} \leq 6^{d+C(\Lambda)}$$

is uniformly bounded. Now assume $\text{dist}(X, \partial D) \leq 6r$. Let $\tilde{X} \in \partial D$ such that $|X - \tilde{X}| = \text{dist}(X, \partial D)$. We bound the numerator by the boundary gradient estimate of u :

$$(r|\nabla u(X + rY)|)^2 \leq \sup_{B_{6r}(\tilde{X}) \cap D} (r|\nabla u|)^2 \lesssim \frac{1}{r^d} \iint_{B_{7r}(\tilde{X})} u^2 dZ. \quad (6.15)$$

To bound the denominator, we use the doubling property (6.13) and $|X - \tilde{X}| \leq 6r$, and get

$$\frac{1}{r^d} \iint_{B_r(X)} u^2 dZ \gtrsim_{\Lambda} \frac{1}{r^d} \iint_{B_{7r}(X)} u^2 dZ \gtrsim \frac{1}{r^d} \iint_{B_r(\tilde{X})} u^2 dZ. \quad (6.16)$$

Therefore combining (6.15) and (6.16), it follows from Corollary 3.28 and the bound on the frequency function that

$$|\nabla T_{X,r} u|^2 \lesssim \frac{\iint_{B_{7r}(\tilde{X})} u^2 dZ}{\iint_{B_r(\tilde{X})} u^2 dZ} \leq 7^{d+2N_{\tilde{X}}(7r)} \leq C(\Lambda).$$

□

For an interior point $X \in D$, since the frequency function $N(X, \cdot)$ is only monotone increasing in the interval $[0, r_{cs}(X)]$, for large radius we will replace $N(X, r)$ by the corresponding frequency function centered at a boundary point $\tilde{X} \in \partial D$, which satisfies $|X - \tilde{X}| = \text{dist}(X, \partial D)$. The following lemma justifies this choice.

Lemma 6.17. *Let $R, \Lambda > 0$, $\rho \in (0, 1/6]$ and $\delta_{in} > 0$ be fixed. There exists $r_{in} = r_{in}(\delta_{in}, \rho) > 0$ such that the following holds for any $(u, D) \in \mathfrak{H}(R, \Lambda)$. Suppose $p \in D \cap B_{\frac{R}{10}}(0) \cap N(u)$ with $\rho r_{cs}(p) \leq r_{in}$. Let $q \in \partial D$ satisfy $|q - p| \leq 2 \text{dist}(p, \partial D)$. Then for any radius $\rho r_{cs}(p) \leq r \leq r_{in}$ we have*

$$|N(p, r) - N_q(r)| \leq \delta_{in}.$$

Proof. We argue by contradiction. To that end we assume there exist sequences $(u_i, D_i) \in \mathfrak{H}(R, \Lambda)$, $p_i \in D_i \cap B_{\frac{R}{10}}(0) \cap N(u_i)$ with $\rho r_{cs}(p_i) \leq r_i \rightarrow 0$ and

$$d_{p_i} := \text{dist}(p_i, \partial D_i) = r_{cs}(p_i) \tilde{\theta}(r_{cs}(p_i)) \leq \frac{r_i}{\rho} \tilde{\theta}\left(\frac{r_i}{\rho}\right), \quad (6.18)$$

and $q_i \in \partial D_i$ with $|p_i - q_i| \leq 2d_{p_i}$, such that

$$|N(p_i, r_i) - N_{q_i}(r_i)| > \delta_{in} > 0. \quad (6.19)$$

By assumption

$$D_i \cap B_{5R}(0) \subset \{(x, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : x_d > \varphi_i(x)\}$$

for some C^1 function φ_i with Dini parameter θ . Without loss of generality we assume $\varphi_i(0) = 0$ and $\nabla \varphi_i(0) = 0$. Hence

$$|\nabla \varphi_i(x)| = |\nabla \varphi_i(x) - \nabla \varphi_i(0)| \leq \theta(|x|) \leq \theta(5R)$$

is uniformly bounded, and by Arzela-Ascoli φ_i converges uniformly to a function φ which satisfies the same properties as φ_i .

We denote $p_i = (x_i, z_i) \in \mathbb{R}^{d-1} \times \mathbb{R}$. By assumption we have $|p_i| < \frac{R}{10}$. Simple geometry shows that

$$d_{p_i} \leq z_i - \varphi_i(x_i) \leq d_{p_i} \sqrt{1 + |\theta(R)|^2} + \theta(R)d_{p_i} \leq 2d_{p_i}. \quad (6.20)$$

Simple computations show that $\frac{D_i - p_i}{r_i} \cap B_5(0)$ corresponds to the region above the graph of the function $\psi_i : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$, defined as

$$\psi_i(y) := \frac{1}{r_i} [\varphi_i(x_i + r_i y) - \varphi_i(x_i) - (z_i - \varphi_i(x_i))].$$

By (6.20) and the assumption (6.18), we have that $\psi_i(y)$, modulo passing to a subsequence, converges uniformly to a linear function $\varphi_\infty(y) := \langle \nabla \varphi(x_\infty), y \rangle$, where x_∞ is a cluster point for $\{x_i\} \subset B_R(0)$. In other words, inside $B_5(0)$ the sequence of domains

$$\frac{D_i - p_i}{r_i} \text{ converges to the upper half space } D_\infty := \{(y, y_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : y_d > \varphi_\infty(y)\}.$$

By Lemma 6.14, $T_{p_i, r_i} u_i(0) = 0$ and compactness, we get that the sequence

$$T_{p_i, r_i} u_i(Y) := \frac{u_i(p_i + r_i Y)}{\left(\frac{1}{r_i^d} \iint_{B_{r_i}(p_i)} u_i^2 dZ \right)^{\frac{1}{2}}}$$

converges uniformly and in $W^{1,2}$ to a harmonic function u_∞ in $D_\infty \cap B_5(0)$. On the other hand, by Proposition 5.24 the sequence

$$T_{q_i, r_i} u_i(Y) := \frac{u_i(q_i + r_i Y)}{\left(\frac{1}{r_i^d} \iint_{B_{r_i}(q_i)} u_i^2 dZ \right)^{\frac{1}{2}}}$$

(and the sequence $T_{r_i} v_{q_i}$) also converges uniformly and in $W^{1,2}$ to a harmonic function $\tilde{u}_\infty$ in the same upper half space D_∞ . Moreover we claim there exists some $a_i \approx 1$ such that

$$|a_i T_{p_i, r_i} u_i(Y) - T_{q_i, r_i} u_i(Y)| \rightarrow 0 \text{ uniformly for } Y \in B_5(0). \quad (6.21)$$

Assuming the claim is true, then $\tilde{u}_\infty = u_\infty$. (A priori $\tilde{u}_\infty$ is a constant multiple of u_∞ , and the constant must be 1 since they both have unit L^2 norm on $B_1(0)$.) Hence

$$\begin{aligned} \left| N(u_i, p_i, r_i) - \tilde{N}(v_{q_i}, r_i) \right| &= \left| N(T_{p_i, r_i} u_i, 0, 1) - N(T_{r_i} v_{q_i}, 0, 1) \exp \left(C \int_0^{r_i} \frac{\theta(4s)}{s} ds \right) \right| \\ &\rightarrow |N(u_\infty, 0, 1) - N(\tilde{u}_\infty, 0, 1)| = 0, \end{aligned}$$

which contradicts the assumption (6.19).

Proof of the claim (6.21). By the assumption (6.18) and $|q_i - p_i| \leq 2d_{p_i} \ll r_i$, we know

$$\frac{1}{r_i^d} \iint_{B_{r_i}(p_i)} u_i^2 dZ \approx \frac{1}{r_i^d} \iint_{B_{r_i}(q_i)} u_i^2 dZ.$$

We denote

$$a_i = \left(\frac{\frac{1}{r_i^d} \iint_{B_{r_i}(p_i)} u_i^2 dZ}{\frac{1}{r_i^d} \iint_{B_{r_i}(q_i)} u_i^2 dZ} \right)^{\frac{1}{2}} \approx 1.$$

By the boundary gradient estimate in [CK, Theorem 1.4.3], the doubling property (3.30) at q_i , and the assumption (6.18), we get

$$\frac{|u_i(p_i + r_i Y) - u_i(q_i + r_i Y)|}{\left(\frac{1}{r_i^d} \iint_{B_{r_i}(q_i)} u_i^2 dZ \right)^{\frac{1}{2}}} \leq \frac{|p_i - q_i|}{\left(\frac{1}{r_i^d} \iint_{B_{r_i}(q_i)} u_i^2 dZ \right)^{\frac{1}{2}}} \sup_{B_{\frac{11}{2}r_i}(q_i) \cap D_i} |\nabla u_i|$$

$$\begin{aligned}
&\leq \frac{|p_i - q_i|}{\left(\frac{1}{r_i^d} \iint_{B_{r_i}(q_i)} u_i^2 dZ\right)^{\frac{1}{2}}} \cdot \frac{1}{r_i} \left(\frac{1}{r_i^d} \iint_{B_{6r_i}(q_i)} u_i^2 dZ\right)^{\frac{1}{2}} \\
&\lesssim \frac{|p_i - q_i|}{r_i} \left(\frac{\frac{1}{r_i^d} \iint_{B_{6r_i}(q_i)} u_i^2 dZ}{\frac{1}{r_i^d} \iint_{B_{r_i}(q_i)} u_i^2 dZ}\right)^{\frac{1}{2}} \\
&\lesssim \frac{d_{p_i}}{r_i} \cdot 6^{N(q_i, 6r_i)} \\
&\lesssim \tilde{\theta} \left(\frac{r_i}{\rho}\right) \cdot 6^{C(\Lambda)} \rightarrow 0.
\end{aligned} \tag{6.22}$$

This finishes the proof of (6.21). $\square$

The above lemma means that for any interior point $p \in D$ close enough to the boundary, whenever r is out of the monotonic interval for p , we can always replace the frequency function $N(p, r)$ by that of a nearby boundary point q , with a small price to pay. Therefore we define

$$N_p(r) := \begin{cases} N(p, r), & r \leq r_{cs}(p) \\ N_q(r) = \tilde{N}(v_q, r), & r > r_{cs}(p) \end{cases}$$

where $q \in \partial D$ is such that $|q - p| = \text{dist}(p, \partial D)$. Hence $N_p(r)$ is monotone increasing in the disjoint intervals $[0, r_{cs}(p)]$ and $(r_{cs}(p), +\infty)$, with a possible jump of δ_{in} since Lemma 6.17 and the monotonicity of $N_q(\cdot)$ imply that

$$N_p(r_{cs}(p)) = N(p, r_{cs}(p)) \leq N_q(r_{cs}(p)) + \delta_{in} \leq N_q(r_{cs}(p) +) + \delta_{in} = N_p(r_{cs}(p) +) + \delta_{in}.$$

To summarize Sections 4 and 6, for any $(u, D) \in \mathfrak{H}(R, \Lambda)$ we have defined the frequency function for any point $X \in \overline{D}$ as follows:

$$N_X(r) = N_X^u(r) = \begin{cases} \tilde{N}(v_X, r), & \text{if } X \in \partial D \\ N(X, r), & \text{if } X \in D \text{ and } r \leq r_{cs}(X) \\ N_{\tilde{X}}(r) = \tilde{N}(v_{\tilde{X}}, r), & \text{if } X \in D \text{ and } r > r_{cs}(X) \end{cases} \tag{6.23}$$

where $\tilde{X} \in \partial D$ is such that $|X - \tilde{X}| = \text{dist}(X, \partial D)$.

7. QUANTITATIVE SYMMETRY

In this section, we use compactness to prove quantitative cone-splitting and dimension reduction.

Heuristically, the intuition for dimension reduction is the following simple observation. Suppose C is a cone in $\mathbb{R}^d$ which is translation-invariant along a k -dimensional linear subspace V . In other words we may identify V with $\{0\} \times \mathbb{R}^k$ and identify C with a cylindrical cone $C_0 \times \mathbb{R}^k$, where C_0 is a cone in $\mathbb{R}^{d-k}$. Then any point X away from the spine $\{0\} \times \mathbb{R}^k$ is symmetric in $(k+1)$ directions (i.e. along the direction of the spine V and the radial direction

in C_0). To prove quantitative cone-splitting and dimension reduction in our setting requires two non-trivial adaptations of the above observation. Firstly at each boundary point X_0 the frequency function is only monotone for $v_{X_0} = u \circ \Psi_{X_0}$, and we need to combine the information of v_{X_j} 's with different base points X_j 's to produce an invariant subspace. Secondly, along the radial direction the limit function is not constant, but grows polynomially. So we need a different approach to distinguish points on the spine and away from the spine.

Definition 7.1. Let $Y_0, \dots, Y_k$ be arbitrary points in $B_r(p) \subset \mathbb{R}^k$. If for all $i = 1, \dots, k$, they satisfy

$$Y_i \notin B_{\tau r} \left(Y_0 + \text{span}\{Y_1 - Y_0, \dots, Y_{i-1} - Y_0\} \right),$$

we say that these points τr -effectively span the k -dimensional affine subspace $V := Y_0 + \text{span}\{Y_1 - Y_0, \dots, Y_k - Y_0\}$ in $B_r(p)$.

Given a set $F \subset B_r(p)$, we say that F τr -effectively spans a k -dimensional affine subspace if there exists a $(k+1)$ -tuple $\{Y_0, \dots, Y_k\} \subset F$ which τr -effectively spans a k -dimensional affine subspace.

Remark 7.2. Unlike the notion of *linear independence*, the above effective notion is preserved under limits.

Proposition 7.3. Let $R, \Lambda > 0$ and $\delta_0, \rho, \tau \in (0, 1)$ be fixed. There exist $\delta > 0$, $\beta > 0$ and $r_m \leq r_{in}(\delta)$ (where $r_{in}(\delta)$ denotes the radius in Lemma 6.17 if we take $\delta_{in} = \delta$ and $\rho = 1/6$) such that the following holds for any $(u, D) \in \mathfrak{H}(R, \Lambda)$ and $r \leq r_m$. Suppose

$$\sup_{X \in B_{2r}(0) \cap \mathcal{N}(u)} N_X(r) \leq \tilde{\Lambda} \leq C(\Lambda). \quad (7.4)$$

If the set

$$F = \{X \in B_{2r}(0) \cap \mathcal{N}(u) : N_X(\rho r) \geq \tilde{\Lambda} - \delta\}$$

$2\tau r$ -effectively spans a $(d-2)$ -dimensional affine subspace V , then

$$\text{for every } X \in B_{2\beta r}(V) \cap B_{2r}(0) \cap \mathcal{N}(u), \quad N_X(\rho r) \geq \tilde{\Lambda} - \delta_0; \quad (7.5)$$

and

$$C_r(u) \cap B_{2r}(0) \cap \mathcal{N}(u) \subset B_{2\beta r}(V). \quad (7.6)$$

Proof. Proof of (7.5). We remark that $B_{2\beta r}(V) \cap B_{2r}(0) \cap \mathcal{N}(u)$ can not be empty, since it contains at least $(d-1)$ points generating the subspace V . We also remark that

$$F \subset \{X \in B_{2r}(0) \cap \mathcal{N}(u) : |N_X(r) - N_X(\rho r)| \leq \delta\}. \quad (7.7)$$

Indeed it is clearly true if $X \in \partial D$, since

$$0 \leq N_X(r) - N_X(\rho r) \leq \tilde{\Lambda} - (\tilde{\Lambda} - \delta) = \delta.$$

When $X \in D$, (7.7) is true if $N_X(r) \geq N_X(\rho r)$. If not (which can only happen when $\rho r \leq r_{cs}(X) < r$), by the definition (6.23), $r_{cs}(X) < r \leq r_{in}(\delta)$ and Lemma 6.17 we have

$$N_X(r) - N_X(\rho r) = N_{\tilde{X}}(r) - N(X, \rho r) \geq N_{\tilde{X}}(r) - (N_{\tilde{X}}(\rho r) + \delta) \geq -\delta,$$

where $\tilde{X} \in \partial D$ is such that $|X - \tilde{X}| = \text{dist}(X, \partial D)$. By the assumption (7.4) and the definition of F ,

$$N_X(r) - N_X(\rho r) \leq \tilde{\Lambda} - (\tilde{\Lambda} - \delta) = \delta.$$

Hence (7.7) is true.

We argue by contradiction. That is, we assume there are $(u_i, D_i) \in \mathfrak{S}(R, \Lambda)$ and $\delta_i, r_i, \beta_i \rightarrow 0$ satisfying $r_i \leq r_{in}(\delta_i)$, such that

$$\sup_{X \in B_{2r_i}(0) \cap \mathcal{N}(u_i)} N_X^{u_i}(r_i) \leq \tilde{\Lambda}_i \leq C(\Lambda),$$

the set

$$F_i = \{X \in B_{2r_i}(0) \cap \mathcal{N}(u_i) : N_X^{u_i}(\rho r_i) \geq \tilde{\Lambda}_i - \delta_i\}$$

$2\tau r_i$ -effectively spans a $(d-2)$ -dimensional affine subspace, denoted by

$$V_i = X_i^0 + \text{span}\{X_i^1 - X_i^0, \dots, X_i^{d-2} - X_i^0\}, \quad \text{with } X_i^j \in F_i \text{ for all } j \in \{0, \dots, d-2\};$$

and yet there exists $X_i \in B_{2\beta_i r_i}(V_i) \cap B_{2r_i}(0) \cap \mathcal{N}(u_i)$ such that $N_{X_i}^{u_i}(\rho r_i) < \tilde{\Lambda}_i - \delta_0$.

Step 1. Since $0 < \tilde{\Lambda}_i \leq C(\Lambda)$, modulo passing to a subsequence $\tilde{\Lambda}_i$ converges to some $\tilde{\Lambda} \in [0, C(\Lambda)]$. By Proposition 5.24 for each $j \in \{0, \dots, d-2\}$ fixed, the sequence $T_{X_i^j, r_i} u_i$ converges uniformly and in $W^{1,2}$ to some harmonic function u_∞^j in $D_\infty^j \cap B_5(0)$, which vanishes on the boundary. With j fixed, we consider two cases. Case (i): there are infinitely many boundary points in the sequence $\{X_i^j\}_i$. Then we throw away all interior points in the sequence. By Proposition 5.24 $T_{r_i} v_{X_i^j}$ also converges to the same function u_∞^j . Since $X_i^j \in B_{2r_i}(0)$, we have

$$X_i^j \rightarrow 0 \text{ as } i \rightarrow +\infty, \quad \text{for each } j \text{ fixed.}$$

Hence the domain D_∞^j is in fact the upper half-space $\mathbb{R}_+^d$. Let $\rho \in (0, 1)$ be arbitrary. Then for any i sufficiently large so that $\rho_i \leq \rho$, by the scale-invariance of the frequency function we have

$$\begin{aligned} & N(T_{r_i} v_{X_i^j}, 1) - N(T_{r_i} v_{X_i^j}, \rho) \\ &= N(v_{X_i^j}, r_i) - N(v_{X_i^j}, \rho r_i) \\ &= \tilde{N}(v_{X_i^j}, r_i) \exp\left(-C \int_0^{r_i} \frac{\theta(4\tau)}{\tau} d\tau\right) - \tilde{N}(v_{X_i^j}, \rho r_i) \exp\left(-C \int_0^{\rho r_i} \frac{\theta(4\tau)}{\tau} d\tau\right) \\ &= \tilde{N}(v_{X_i^j}, r_i) \left[\exp\left(-C \int_0^{r_i} \frac{\theta(4\tau)}{\tau} d\tau\right) - \exp\left(-C \int_0^{\rho r_i} \frac{\theta(4\tau)}{\tau} d\tau\right) \right] \\ &\quad + \left[\tilde{N}(v_{X_i^j}, r_i) - \tilde{N}(v_{X_i^j}, \rho r_i) \right] \exp\left(-C \int_0^{\rho r_i} \frac{\theta(4\tau)}{\tau} d\tau\right) \\ &\rightarrow 0, \quad \text{as } r_i \rightarrow 0. \end{aligned}$$

In the last line we used the assumption $X_i^j \in F_i$, the fact that $N_{X_i^j}(r_i)$'s are uniformly bounded by $C(\Lambda)$ and $\int_0^{r_i} \frac{\theta(4\tau)}{\tau} d\tau \rightarrow 0$ (see Remark 2.6). Hence

$$N(u_\infty^j, 1) - N(u_\infty^j, \rho) = \lim_{r_i \rightarrow 0} N(T_{r_i} v_{X_i^j}, 1) - N(T_{r_i} v_{X_i^j}, \rho) = 0.$$

Since u_∞^j is a harmonic function in the upper half-space, by the rigidity of the monotonicity formula in Corollary 3.26 u_∞^j is a homogeneous harmonic polynomial in $B_1(0)$. Thus by

the unique continuation theorem u_∞^j is a homogeneous harmonic polynomial in the upper half-space.

Otherwise, we fall into Case (ii): there are infinitely many interior points in the sequence $\{X_i^j\}_i$. Then we throw away the finitely many boundary points (if they exist) in the sequence. Since $X_i^j \in B_{2r_i}(0)$, we know that

$$\text{dist}(X_i^j, \partial D_i) \leq |X_i^j - 0| < 2r_i.$$

In particular, modulo passing to a subsequence, the sequence of real numbers $\text{dist}(X_i^j, \partial D_i)/r_i$ converges to some $d_j \in [0, 2]$. A simple computation then shows that $\frac{D_i - X_i^j}{r_i}$ converges graphically to the domain

$$D_\infty^j = \{(y, y_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : y_d > -d_j\},$$

which is the upper half-space (when $d_j = 0$), or the upper half-space shifted down by d_j in the direction of $-e_d$.

We claim that

$$\left| N(T_{X_i^j, r_i} u_i, 0, 1) - N(T_{X_i^j, r_i} u_i, 0, \rho) \right| = \left| N(u_i, X_i^j, r_i) - N(u_i, X_i^j, \rho r_i) \right| \rightarrow 0. \quad (7.8)$$

This does not follow obviously from the assumption $X_i^j \in F_i$ and the observation (7.7): Recalling the definition (6.23), the frequency function $N_{X_i^j}^{u_i}(\cdot)$ that appears in F_i is not always the same as $N(u_i, X_i^j, \cdot)$ straightforwardly; it is defined as follows

$$N_{X_i^j}^{u_i}(r) = \begin{cases} N(u_i, X_i^j, r), & r \leq r_{cs}(X_i^j) \\ N_{\tilde{X}_i^j}^{u_i}(r) = \tilde{N} \left(v_{\tilde{X}_i^j} = \Psi_{\tilde{X}_i^j} \circ u_i, r \right) & r > r_{cs}(X_i^j) \end{cases} \quad (7.9)$$

where $r_{cs}(X_i^j)$ denotes the critical scale for u_i at the interior point X_i^j , and $\tilde{X}_i^j \in \partial D$ is such that $|X_i^j - \tilde{X}_i^j| = \text{dist}(X_i^j, \partial D_i)$.

In fact, if $r_i \leq r_{cs}(X_i^j)$, then by the definition (7.9), $X_i^j \in F_i$ and the observation (7.7), we have

$$\left| N(u_i, X_i^j, r_i) - N(u_i, X_i^j, \rho r_i) \right| \leq \left| N_{X_i^j}^{u_i}(r_i) - N_{X_i^j}^{u_i}(\rho r_i) \right| \leq \delta_i;$$

if $\rho r_i \leq r_{cs}(X_i^j) < r_i$, then by the definition (7.9), $r_{cs}(X_i^j) < r_i \leq r_{in}(\delta_i)$, Lemma 6.17, $X_i^j \in F_i$ and the observation (7.7), we have

$$\begin{aligned} \left| N(u_i, X_i^j, r_i) - N(u_i, X_i^j, \rho r_i) \right| &= \left| N(u_i, X_i^j, r_i) - N_{X_i^j}^{u_i}(\rho r_i) \right| \\ &\leq \left| N(u_i, X_i^j, r_i) - N_{\tilde{X}_i^j}^{u_i}(r_i) \right| + \left| N_{\tilde{X}_i^j}^{u_i}(r_i) - N_{X_i^j}^{u_i}(\rho r_i) \right| \\ &\leq 2\delta_i; \end{aligned}$$

if $\rho r_i > r_{cs}(X_i^j)$, then by $r_{cs}(X_i^j) < \rho r_i < r_i \leq r_{in}(\delta_i)$, Lemma 6.17, the definition (7.9), $X_i^j \in F_i$ and the observation (7.7) we have

$$\begin{aligned} & \left| N(u_i, X_i^j, r_i) - N(u_i, X_i^j, \rho r_i) \right| \\ & \leq \left| N(u_i, X_i^j, r_i) - N_{\tilde{X}_i^j}^{u_i}(r_i) \right| + \left| N_{\tilde{X}_i^j}^{u_i}(r_i) - N_{\tilde{X}_i^j}^{u_i}(\rho r_i) \right| + \left| N_{\tilde{X}_i^j}^{u_i}(\rho r_i) - N(u_i, X_i^j, \rho r_i) \right| \\ & \leq 2\delta_i + \left| N_{X_i^j}^{u_i}(r_i) - N_{X_i^j}^{u_i}(\rho r_i) \right| \\ & \leq 3\delta_i. \end{aligned}$$

So for each i we always have

$$\left| N(u_i, X_i^j, r_i) - N(u_i, X_i^j, \rho r_i) \right| \leq 3\delta_i \rightarrow 0, \quad (7.10)$$

which finishes the proof of the claim (7.8). Therefore

$$\left| N(u_\infty^j, 0, 1) - N(u_\infty^j, 0, \rho) \right| = \lim_{i \rightarrow \infty} \left| N(T_{X_i^j, r_i} u_i, 0, 1) - N(T_{X_i^j, r_i} u_i, 0, \rho) \right| = 0. \quad (7.11)$$

If $d_j = 0$ and thus $0 \in \partial D_\infty^j$, we appeal to the rigidity of the monotonicity formula for boundary points (Corollary 3.26) to conclude that u_∞^j is homogeneous with respect to the origin. Alternatively if $d_j > 0$, then $0 \in D_\infty^j$ and moreover $\text{dist}(0, \partial D_\infty^j) = d_j$. Since u_∞^j vanishes on the boundary of D_∞^j , by the rigidity of the monotonicity formula for interior points (see Proposition 6.3) we conclude that u_∞^j is homogeneous with respect to 0. Since $u_\infty^j(0) = 0$, by homogeneity and the unique continuation property $u_\infty^j \equiv 0$, which is impossible since $\iint_{B_1(0)} |u_\infty^j|^2 dZ = 1$. Therefore we can only have $d_j = 0$ and $D_\infty^j = \mathbb{R}_+^d$.

Moreover, in both cases, since $\tilde{\Lambda}_i \rightarrow \tilde{\Lambda}$ and $\delta_i \rightarrow 0$ we can show that

$$N(u_\infty^j, 0, 1) = \lim_{i \rightarrow \infty} N(u_i, X_i, r_i) \leq \tilde{\Lambda},$$

and

$$N(u_\infty^j, 0, \rho) = \lim_{i \rightarrow \infty} N(u_i, X_i, \rho r_i) \geq \tilde{\Lambda}.$$

Therefore the degree of homogeneity of u_∞^j with respect to the origin satisfies

$$N(u_\infty^j, 0, 0) = N(u_\infty^j, 0, 1) = N(u_\infty^j, 0, \rho) = \tilde{\Lambda}. \quad (7.12)$$

Step 2. Let

$$Y_i^j = \frac{X_i^j - X_i^0}{r_i} \in \frac{\overline{D}_i - X_i^0}{r_i}.$$

By the assumption of effective spanning, we have

$$2\tau \leq |Y_i^j| \leq 4 \quad \text{and} \quad \left| Y_i^j - Y_i^{j'} \right| = \frac{|X_i^j - X_i^{j'}|}{r_i} \geq 2\tau \text{ if } j \neq j'.$$

Modulo passing to a subsequence we have that $\{Y_i^0, Y_i^1, \dots, Y_i^{d-2}\}$ converges to $\{0, Y^1, \dots, Y^{d-2}\}$, which 2τ -effectively spans a $(d-2)$ -dimensional linear subspace V_∞ . Moreover, since

$$\text{dist}\left(Y_i^j, \frac{\partial D_i - X_i^0}{r_i}\right) = \frac{\text{dist}(X_i^j, \partial D_i)}{r_i} \rightarrow d_j,$$

it follows that

$$\text{dist}(Y^j, \partial D_\infty^0) = \lim_{i \rightarrow \infty} \text{dist}\left(Y_i^j, \frac{\partial D_i - X_i^0}{r_i}\right) = d_j = 0, \quad (7.13)$$

i.e. $Y^j \in \partial D_\infty^0$.

We claim that for any $j \in \{1, \dots, d-2\}$, there is a constants $a_j \approx 1$ (depending on the values of d, R and Λ) such that

$$u_\infty^0(Y^j + Y) = a_j u_\infty^j(Y), \quad \text{for every } Y \in D_\infty^j. \quad (7.14)$$

For simplicity of notation we just write down the proof of the claim when $j = 1$. Recall that

$$T_{X_i^1, r_i} u_i(Y) = \frac{u_i(X_i^1 + r_i Y)}{\left(\frac{1}{r_i^d} \iint_{B_{r_i}(X_i^1)} u_i^2 dX\right)^{\frac{1}{2}}},$$

and

$$T_{X_i^0, r_i} u_i(Y_i^1 + Y) = \frac{u_i(X_i^0 + r_i Y_i^1 + r_i Y)}{\left(\frac{1}{r_i^d} \iint_{B_{r_i}(X_i^0)} u_i^2 dX\right)^{\frac{1}{2}}} = \frac{u_i(X_i^1 + r_i Y)}{\left(\frac{1}{r_i^d} \iint_{B_{r_i}(X_i^0)} u_i^2 dX\right)^{\frac{1}{2}}}.$$

Since $|X_i^1 - X_i^0| < 4r_i$, using the assumption $X_i^0, X_i^1 \in \mathcal{N}(u_i)$ and the doubling property in (3.30) (for boundary points) and (6.12) (for interior points), we get

$$\iint_{B_{r_i}(X_i^1)} u_i^2 dX \approx \iint_{B_{r_i}(X_i^0)} u_i^2 dX, \quad (7.15)$$

where the constant depends on d, Λ . Hence modulo passing to a subsequence their ratio converges to a positive number. This implies the claim (7.14). Recall that each u_∞^j is a homogeneous polynomial with respect to the origin. The claim then implies that u_∞^0 is a homogeneous polynomial with respect to $Y^1, \dots, Y^{d-2}$. Therefore either u_∞^0 is linear, or u_∞^0 is invariant along the $(d-2)$ -dimensional linear subspace $V_\infty = 0 + \text{span}\{Y^1, \dots, Y^{d-2}\} \subset \partial D_\infty^0$.

Step 3. On the other hand, let

$$Y_i = \frac{X_i - X_i^0}{r_i} \in \frac{\overline{D_i} - X_i^0}{r_i}.$$

Then modulo passing to a subsequence Y_i converges to some point $Y_\infty \in \overline{\mathbb{R}_+^d} \cap \overline{B_4(0)}$. By compactness we have $T_{X_i, r_i} u_i$ converges to some harmonic function (not necessarily homogeneous) u_∞ in $\mathbb{R}_+^d - Y_\infty \cap B_5(0)$. Moreover, let

$$\widetilde{V}_i := 0 + \text{span}\{Y_i^1, \dots, Y_i^{d-2}\}$$

be a $(d-2)$ -dimensional linear subspace. Since $X_i \in B_{2\beta_i r_i}(V_i)$, we have

$$\text{dist}(Y_i, \tilde{V}_i) = \frac{\text{dist}(X_i, V_i)}{r_i} < 2\beta_i \rightarrow 0.$$

Since $\tilde{V}_i \rightarrow V_\infty$, this implies that $Y_\infty \in V_\infty \subset \partial\mathbb{R}_+^d$. Hence the domain $\mathbb{R}_+^d - Y_\infty$ is exactly the upper half-space $\mathbb{R}_+^d$. This in particular implies that $\text{dist}(X_i, \partial D_i)/r_i \rightarrow 0$, and similar to the proof of (7.14), we can show there is a constant $a \approx 1$ such that

$$u_\infty^0(Y_\infty + Y) = a u_\infty(Y). \quad (7.16)$$

Since u_∞^0 is invariant along the plane V_∞ and $Y_\infty \in V_\infty$, it follows from (7.16) that u_∞ is homogeneous with respect to the origin, and moreover

$$N(u_\infty, 0, \rho) = N(u_\infty, 0, 0) = N(u_\infty^0, Y_\infty, 0) = N(u_\infty^0, 0, 0) = \tilde{\Lambda}, \quad (7.17)$$

where we use (7.12) in the last equality. However, by the assumption on X_i and Lemma 6.17

$$N(u_i, X_i, \rho r_i) \leq N_{X_i}^{u_i}(\rho r_i) + \delta_i < (\tilde{\Lambda}_i - \delta_0) + \delta_i,$$

and thus

$$N(u_\infty, 0, \rho) = \lim_{i \rightarrow \infty} N(T_{X_i, r_i} u_i, 0, \rho) = \lim_{i \rightarrow \infty} N(u_i, X_i, \rho r_i) \leq \tilde{\Lambda} - \delta_0.$$

This is in contradiction with (7.17). Therefore we have proven (7.5).

Proof of (7.6). Notice that if the set on the left hand side is empty, there is nothing to prove. So we assume that is not the case. We argue by contradiction as before, but this time with β fixed to be the value we just found. That is, we assume there are $(u_i, D_i) \in \mathfrak{S}(R, \Lambda)$ and $\delta_i, r_i \rightarrow 0$ and $r_i \leq r_{in}(\delta_i)$ such that

$$\sup_{X \in B_{2r_i}(0) \cap \mathcal{N}(u_i)} N_X^{u_i}(r_i) \leq \tilde{\Lambda}_i \leq C(\Lambda),$$

the set

$$F_i = \{X \in B_{2r_i}(0) \cap \mathcal{N}(u_i) : N_X^{u_i}(\rho r_i) \geq \tilde{\Lambda}_i - \delta_i\}$$

$2\tau r_i$ -effectively spans a $(d-2)$ -dimensional linear subspace V_i , and yet there exists some $X'_i \in C_{r_i}(u_i) \cap B_{2r_i}(0) \cap \mathcal{N}(u_i)$ which satisfies $\text{dist}(X'_i, V_i) \geq 2\beta r_i$.

The first part of the proof is the same as in Step 1 and Step 2 above, and we obtain that $T_{X'_i, r_i} u_i$ converges to u_∞^0 , which is a harmonic homogeneous polynomial and is invariant along a $(d-2)$ -dimensional linear subspace V_∞ .

Similarly to Step 3 above, we let

$$Y'_i = \frac{X'_i - X_i^0}{r_i} \in \frac{\overline{D_i} - X_i^0}{r_i}.$$

Recall we have shown in Step 1 in the proof of (7.5) that the sequence of domains $\frac{\overline{D_i} - X_i^0}{r_i} \rightarrow \overline{\mathbb{R}_+^d}$. Then modulo passing to a subsequence Y'_i converges to some point $Y' \in \overline{\mathbb{R}_+^d} \cap \overline{B_4(0)}$;

and $T_{X'_i, r_i} u_i$ converges to some harmonic function (not necessarily homogeneous) u'_∞ in $(\mathbb{R}_+^d - Y') \cap B_5(0)$. Additionally, since $\text{dist}(X'_i, V_i) \geq 2\beta r_i$, we have

$$\text{dist}(Y'_i, \tilde{V}_i) = \frac{\text{dist}(X'_i, V_i)}{r_i} \geq 2\beta,$$

and thus

$$\text{dist}(Y', V_\infty) \geq 2\beta > 0. \quad (7.18)$$

Moreover, similarly to the proof before we can show there is a constant $a' \approx 1$ such that $u_\infty^0(Y' + Y) = a' u'_\infty(Y)$, and thus

$$\nabla u_\infty^0(Y' + Y) = a' \nabla u'_\infty(Y). \quad (7.19)$$

Since u_∞^0 is invariant along the $(d-2)$ -dimensional linear subspace $V_\infty \subset \partial\mathbb{R}_+^d$, by identifying the coordinate system in $V_\infty^\perp \times V_\infty$ with $\mathbb{R}^2 \times \mathbb{R}^{d-2}$ we have

- either u_∞^0 is a harmonic function in one variable, and then necessarily u_∞^0 is linear in that variable and (modulo a change of coordinates in $V_\infty^\perp$)

$$u_\infty^0(x_1, x_2, y) = c(x_1)_+, \quad \text{where } x_1 \in V_\infty^\perp \cap \mathbb{R}_+^d, x_2 \in V_\infty^\perp \cap \partial\mathbb{R}_+^d \text{ and } y \in V_\infty.$$

- or u_∞^0 is a harmonic function in two variables, and then necessarily we have (modulo a change of coordinates in $V_\infty^\perp$) in polar coordinates

$$u_\infty^0(r, \omega, y) = \tilde{c} r^N \sin(N\omega), \quad \text{where } r \geq 0, \omega \in [0, \pi], y \in V_\infty. \quad (7.20)$$

Here the degree of homogeneity is determined as $N = N(u_\infty^0, 1) = N(u_\infty^0, 0)$ and $N \geq 2$.

Since u_∞^0 satisfies $\iint_{B_1(0)} |u_\infty^0|^2 dY = 1$, the constant $c = \alpha_d^1$ only depends on the dimension d , where α_d^1 satisfies $\omega_{d-1} |\alpha_d^1|^2 \int_0^1 t^2 (1-t^2)^{\frac{d-1}{2}} dt = 1$ with ω_{d-1} being the volume of $(d-1)$ -dimensional unit ball. And the constant $\tilde{c}$ satisfies

$$\frac{\pi}{2} |\tilde{c}|^2 \omega_{d-2} \int_0^1 t^{2N} (1-t^2)^{\frac{d-2}{2}} dt = 1,$$

and thus it depends on the dimension as well as the degree N . However $\tilde{c}$ has a uniform lower bound independent of N , which we denote by $|\tilde{c}| \geq \alpha_d^2$.

The assumption $X'_i \in C_{r_i}(u_i)$ implies that

$$\inf_{B_\beta(0)} |\nabla T_{X'_i, r_i} u_i| \leq \alpha_0.$$

Recall that $\nabla T_{X'_i, r_i} u_i \rightarrow \nabla u'_\infty$ in L^2 , then modulo passing to a subsequence we have almost everywhere convergence. This, combined with the continuity of $\nabla T_{X'_i, r_i} u_i$ and $\nabla u'_\infty$, implies that

$$\inf_{B_\beta(0)} |\nabla u'_\infty| \leq \alpha_0.$$

By (7.19), it follows that

$$\inf_{B_\beta(Y')} |\nabla u_\infty^0| = |a'| \inf_{B_\beta(0)} |\nabla u'_\infty| \leq \tilde{\alpha}_0, \quad (7.21)$$

where $\tilde{\alpha}_0$ is α_0 multiplied by the upper bound of $|a'|$, which is a dimensional constant. In the first case we have $|\nabla u_\infty^0| \equiv \alpha_d^1$. So by choosing $\tilde{\alpha}_0 < \alpha_d^1$, we can guarantee the first case does not happen. In the second case, we have that for any $(r, \omega) \in \mathbb{R}_+ \times [0, \pi]$

$$|\nabla u_\infty^0|^2 = \tilde{c}^2 [|Nr^{N-1} \sin(N\omega)|^2 + |Nr^{N-1} \cos(N\omega)|^2] = \tilde{c}^2 N^2 r^{2(N-1)}.$$

We identify $Y' \in \overline{\mathbb{R}_+^d}$ with (r_0, ω_0, y_0) . (The angle $\omega_0 = 0$ or π if $Y' \in \partial\mathbb{R}_+^d$, and $\omega_0 \in (0, \pi)$ if $Y' \in \mathbb{R}_+^d$.) Then by (7.18), we have

$$r_0 = \text{dist}(Y', V_\infty) \geq 2\beta > 0.$$

Therefore

$$\inf_{B_\beta(Y')} |\nabla u_\infty^0| = |\tilde{c}| N(r_0 - \beta)^{N-1} \geq 2\alpha_d^2 \cdot \beta^{C(\Lambda)-1}.$$

By choosing $\tilde{\alpha}_0 < 2\alpha_d^2 \cdot \beta^{C(\Lambda)-1}$, we get a contradiction with the bound in (7.21). $\square$

8. SPATIAL VARIATIONS OF THE FREQUENCY FUNCTION

The following lemma gives an estimate of the spatial variation of the frequency function (at the same scale) for harmonic functions on the upper half space. It can be viewed as a quantitative version of Lemma 2.23.

Lemma 8.1. *Let u be a harmonic function in $\mathbb{R}_+^d$ such that u vanishes on $\partial\mathbb{R}_+^d$. Let $N(X, r)$ denote the frequency function of u centered at $X \in \partial\mathbb{R}_+^d$ and at scale $r > 0$. Suppose $N(0, 3) \leq \Lambda$. Then there exists a constant C (depending on d and Λ), such that for any $0 < r < 1$ and any $X_1, X_2 \in B_1(0) \cap \partial\mathbb{R}_+^d$ with $|X_1 - X_2| \leq r/2$, we have*

$$|N(X_1, r) - N(X_2, r)| \leq C \left(W^{\frac{1}{2}}(X_1, r) + W^{\frac{1}{2}}(X_2, r) \right), \quad (8.2)$$

where $W(X_j, r) = N(X_j, 3r/2) - N(X_j, r/2)$ for $j = 1, 2$.

Proof. Let $Y = X_2 - X_1$. Any point on the line segment $[X_1, X_2]$ can be written as $X = X_1 + tY \in \partial\mathbb{R}_+^d$, where $t \in [0, 1]$. Let

$$\rho_X(Z) = \frac{Z - X}{|Z - X|}$$

denote the radial direction with center X . Similarly to (2.21), we have

$$\begin{aligned} & \frac{d}{dt} N(X_1 + tY, r) \\ &= \frac{d}{dt} \log N(X_1 + tY, r) \cdot N(X_1 + tY, r) \\ &= \frac{2}{H(X, r)} \left[r \int_{\partial B_r(X)} \langle \nabla u(Z), Y \rangle \partial_{\rho_X} u(Z) d\mathcal{H}^{d-1} - N(X, r) \cdot \int_{\partial B_r(X)} \langle \nabla u(Z), Y \rangle u(Z) d\mathcal{H}^{d-1} \right]. \end{aligned} \quad (8.3)$$

Recall that $u \in C^1(\overline{\mathbb{R}_+^d})$ and we extend u by zero in its complement, so for simplicity we drop $\mathbb{R}_+^d$ in the integration region. Let

$$\mathcal{E}_j(Z) = \langle \nabla u(Z), Z - X_j \rangle - N(X_j, |Z - X_j|) \cdot u(Z), \quad \text{for } j = 1, 2.$$

Then

$$\langle \nabla u(Z), Y \rangle = \mathcal{E}_1(Z) - \mathcal{E}_2(Z) + N(X_1, |Z - X_1|) \cdot u(Z) - N(X_2, |Z - X_2|) \cdot u(Z).$$

Inserting this into the equality (8.3) we get

$$\begin{aligned} & \frac{d}{dt} N(X_1 + tY, r) \\ &= \frac{2}{H(X, r)} \int_{\partial B_r(X)} (\mathcal{E}_1(Z) - \mathcal{E}_2(Z)) [r \partial_{\rho_X} u(Z) - N(X, r) \cdot u(Z)] d\mathcal{H}^{d-1} \\ &+ \frac{2}{H(X, r)} \int_{\partial B_r(X)} (N(X_1, |Z - X_1|) - N(X_2, |Z - X_2|)) [r \partial_{\rho_X} u(Z) - N(X, r) \cdot u(Z)] u(Z) d\mathcal{H}^{d-1} \\ &=: I_1(X) + I_2(X) \end{aligned} \quad (8.4)$$

We write

$$\begin{aligned} & N(X_1, |Z - X_1|) - N(X_2, |Z - X_2|) \\ &= (N(X_1, |Z - X_1|) - N(X_1, r)) + (N(X_1, r) - N(X_2, r)) + (N(X_2, r) - N(X_2, |Z - X_2|)) \\ &=: \mathcal{S}_1(Z) + \mathcal{S} + \mathcal{S}_2(Z), \end{aligned}$$

where the middle term $\mathcal{S} = N(X_1, r) - N(X_2, r)$ is independent of Z . For any X on the line segment $[X_1, X_2]$ and any $Z \in \partial B_r(X)$, we have

$$\frac{r}{2} \leq |Z - X_j| \leq \frac{3}{2}r, \quad \text{for } j = 1, 2.$$

Hence

$$|\mathcal{S}_j(Z)| = |N(X_j, |Z - X_j|) - N(X_j, r)| \leq W(X_j, r).$$

On the other hand, plugging $\mathcal{S}$ into the second term of (8.4) we get

$$\begin{aligned} & \mathcal{S} \left[\int_{\partial B_r(X)} r u \partial_{\rho_X} u d\mathcal{H}^{d-1}(Z) - N(X, r) \int_{\partial B_r(X)} u^2 d\mathcal{H}^{d-1}(Z) \right] \\ &= \mathcal{S} [rD(X, r) - N(X, r)H(X, r)] = 0. \end{aligned}$$

Notice that the above argument is exactly the same as the proof (2.22), with $\mathcal{S} = N_1 - N_2$.

We can estimate the remaining terms of $I_2(X)$ as follows:

$$\begin{aligned} |I_2(X)| &\leq \frac{2}{H(X, r)} (W(X_1, r) + W(X_2, r)) \left[(r + N(X, r)) \int_{\partial B_r(X)} u^2 d\mathcal{H}^{d-1} + r \int_{\partial B_r(X)} |\nabla u|^2 d\mathcal{H}^{d-1} \right] \\ &\leq 2 (W(X_1, r) + W(X_2, r)) \left[C(\Lambda) + \frac{r \int_{\partial B_r(X)} |\nabla u|^2 d\mathcal{H}^{d-1}}{H(X, r)} \right]. \end{aligned} \quad (8.5)$$

We estimate $I_1(X)$ by Cauchy-Schwartz inequality, and obtain

$$|I_1(X)| \lesssim \left[r \left(\frac{\int_{\partial B_r(X)} |\nabla u|^2 d\mathcal{H}^{d-1}}{H(X, r)} \right)^{\frac{1}{2}} + N(X, r) \right] \cdot \left(\frac{1}{H(X, r)} \int_{\partial B_r(X)} |\mathcal{E}_1(Z)|^2 + |\mathcal{E}_2(Z)|^2 d\mathcal{H}^{d-1} \right)^{\frac{1}{2}}. \quad (8.6)$$

To further estimate $I_1(X)$ and $I_2(X)$, we make the following observation. Since u is a harmonic function which vanishes on the boundary, we have u^2 is sub-harmonic in $\mathbb{R}^d$. Hence

$$\oint_{\partial B_\rho(X)} u^2 d\mathcal{H}^{d-1} \text{ is increasing with respect to } \rho.$$

It follows that

$$\begin{aligned} H(X_j, r) &= \int_{\partial B_r(X_j)} u^2 d\mathcal{H}^{d-1} \leq \frac{2}{r} \iint_{A_{r, \frac{3}{2}r}(X_j)} u^2 dZ \\ &\leq \frac{2}{r} \iint_{A_{\frac{r}{2}, 2r}(X)} u^2 dZ \\ &\leq 3 \int_{\partial B_{2r}(X)} u^2 d\mathcal{H}^{d-1} = 3H(X, 2r). \end{aligned}$$

On the other hand by the monotonicity formula (or (3.30) for the Laplacian) we have

$$\frac{H(X, 2r)}{H(X, \frac{3}{2}r)} \leq \left(\frac{4}{3}\right)^{d-1+N(X, 2r)} \leq C(d, \Lambda).$$

In the second inequality we use Lemma 5.1 (for the Laplacian operator in the upper half-space) to bound $N(X, 2r)$ from above by $N(0, 3)$. Therefore

$$H(X, r) \geq c(d, \Lambda) H\left(X_j, \frac{3}{2}r\right) \quad \text{for } j = 1, 2. \quad (8.7)$$

Now we allow $X = X_t$ to move in the line segment $[X_1, X_2]$, as t varies in $[0, 1]$. Since $|X_1 - X_2| \leq r/2$, the integration region satisfies

$$\bigcup_{t \in [0, 1]} \partial B_r(X_t) \subset A_{\frac{\sqrt{3}}{2}r, \frac{3}{2}r}(X_j).$$

By (8.5) and (8.7), we have

$$\begin{aligned} \int_0^1 |I_2(X_t)| dt &\leq 2(W(X_1, r) + W(X_2, r)) \left[C_1 + \frac{C_2 r}{H(X_1, r)} \int_0^1 \int_{\partial B_r(X_t)} |\nabla u|^2 d\mathcal{H}^{d-1} dt \right] \\ &\leq 2(W(X_1, r) + W(X_2, r)) \left[C_1 + C_2 \frac{rD(X_1, \frac{3}{2}r)}{H(X_1, \frac{3}{2}r)} \right] \\ &\leq C(\Lambda) (W(X_1, r) + W(X_2, r)). \end{aligned} \quad (8.8)$$

Next we use the Cauchy-Schwarz inequality to estimate

$$\begin{aligned} \int_0^1 |I_1(X_t)| dt &\lesssim \left\{ \int_0^1 \left[\frac{r^2}{H(X_t, r)} \int_{\partial B_r(X_t)} |\nabla u|^2 d\mathcal{H}^{d-1} + N^2(X_t, r) \right] dt \right\}^{\frac{1}{2}} \\ &\quad \times \left\{ \int_0^1 \frac{1}{H(X_t, r)} \int_{\partial B_r(X_t)} |\mathcal{E}_1(Z)|^2 + |\mathcal{E}_2(Z)|^2 d\mathcal{H}^{d-1} dt \right\}^{\frac{1}{2}} \end{aligned}$$

$$\lesssim \left[\frac{r^2}{H(X_1, \frac{3}{2}r)} \cdot D(X_1, \frac{3}{2}r) + C(\Lambda) \right]^{\frac{1}{2}} \\ \times \left[\sum_{j=1}^2 \frac{1}{H(X_j, \frac{3}{2}r)} \iint_{A_{\frac{\sqrt{3}}{2}r, \frac{3}{2}r}(X_j)} |\mathcal{E}_j(Z)|^2 dZ \right]^{\frac{1}{2}}.$$

Recall Proposition 3.10 (for the Laplace operator), in particular (3.11) and (3.15). We have

$$\begin{aligned} & \frac{1}{H(X_j, \frac{3}{2}r)} \iint_{A_{\frac{\sqrt{3}}{2}r, \frac{3}{2}r}(X_j)} |\mathcal{E}_j(Z)|^2 dZ \\ &= \frac{1}{H(X_j, \frac{3}{2}r)} \iint_{A_{\frac{\sqrt{3}}{2}r, \frac{3}{2}r}(X_j)} \left| \langle \nabla u(Z), Z - X_j \rangle - N(X_j, |Z - X_j|) \cdot u(Z) \right|^2 dZ \\ &\lesssim r \int_{\frac{\sqrt{3}}{2}r}^{\frac{3}{2}r} \frac{\rho}{H(X_j, \rho)} \int_{\partial B_\rho(X_j)} \left| \partial_{\rho X_j} u(Z) - \frac{N(X_j, \rho)}{\rho} u(Z) \right|^2 d\mathcal{H}^{d-1} d\rho \\ &\lesssim r \int_{\frac{\sqrt{3}}{2}r}^{\frac{3}{2}r} R_h(X_j, \rho) d\rho \\ &\lesssim W(X_j, r). \end{aligned}$$

Therefore

$$\int_0^1 |I_1(X_t)| dt \leq C(\Lambda) \left(W^{\frac{1}{2}}(X_1, r) + W^{\frac{1}{2}}(X_2, r) \right). \quad (8.9)$$

Combining (8.4), (8.8) and (8.9), we get

$$|N(X_2, r) - N(X_1, r)| \leq \int_0^1 |I_1(X_t)| + |I_2(X_t)| dt \leq C_{sv} \left(W^{\frac{1}{2}}(X_1, r) + W^{\frac{1}{2}}(X_2, r) \right),$$

where the constant C_{sv} depends on d and Λ . $\square$

Next we estimate the spatial variation of the frequency function for harmonic functions in Dini domains. Firstly, recall that for boundary points we define the frequency function $N_X(r)$ using the transformation Ψ_X in Section 4, i.e.

$$N_X(r) := \tilde{N}(v_X, r) = N(v_X, r) \exp \left(C \int_0^r \frac{\theta(4s)}{s} ds \right);$$

and the modified frequency function $\tilde{N}(v_X, r)$ is monotone increasing with respect to r . We define the frequency drop accordingly as

$$W_X(r) := N_X(3r/2) - N_X(r/2).$$

Proposition 8.10. *Let $R, \Lambda > 0$ be fixed. There exists a constant $C_{sv} > 0$ such that the following holds. Suppose $(u, D) \in \mathfrak{H}(R, \Lambda)$ and X_1, X_2 are two points in $B_R(0) \cap \partial D$ with $|X_1 - X_2| \leq r/3$ and $0 < r \leq 2R$. Then we have*

$$|N(v_{X_1}, r) - N(v_{X_2}, r)| \leq C_{sv} \left(W_{X_1}^{\frac{1}{2}}(r) + W_{X_2}^{\frac{1}{2}}(r) + \theta(4r) \right). \quad (8.11)$$

Remark 8.12. In fact, for the purpose of this paper, it suffices to use the rough estimate

$$|N(v_{X_1}, r) - N(v_{X_2}, r)| \leq C \left(r^{\frac{1}{2}} + \theta(4r) \right). \quad (8.13)$$

The proof of (8.13) is similar to that of Lemma 8.21, where we simply bound each term in (8.17) by taking absolute values inside. Here we include the proof of the sharper estimate (8.11).

Proof. Notice that unlike the case of the upper half space in Lemma 8.1, here ∂D is not flat. So we need to be more careful in defining the line in ∂D which connects X_1 to X_2 . Assume $X_1 = (x_1, \varphi(x_1))$ and $X_2 = (x_2, \varphi(x_2))$. Let $y = x_2 - x_1 \in \mathbb{R}^{d-1}$. We define for any $t \in [0, 1]$

$$X_t = (x_t, \varphi(x_t)) := (x_1 + ty, \varphi(x_1 + ty)) \in \partial D.$$

We also remark that

$$\frac{d}{dt} X_t = (y, \langle \nabla \varphi(x_t), y \rangle) \text{ points in the tangential direction of } \partial D \text{ at } X_t. \quad (8.14)$$

Recall that for any $X \in \partial D$, we define

$$D(v_X, r) = \iint_{B_r \cap \Omega_X} \mu |\nabla_g v_X|_g^2 dV_g = \iint_{\Psi_X(B_r) \cap D} |\nabla u|^2 dZ =: \widehat{D}(X, r);$$

and

$$H(v_X, r) = \int_{\partial B_r \cap \Omega_X} \mu v_X^2 d\mathcal{H}^{d-1} = (1 + O(\theta(4r))) \int_{\Psi_X(\partial B_r) \cap D} u^2 d\mathcal{H}^{d-1} = (1 + O(\theta(4r))) \widehat{H}(X, r),$$

where we introduce the definition

$$\widehat{H}(X, r) := \int_{\Psi_X(\partial B_r) \cap D} u^2 d\mathcal{H}^{d-1} = \int_{\Psi_X(\partial B_r)} u^2 d\mathcal{H}^{d-1}.$$

Let

$$\widehat{N}(X, r) := \frac{r \widehat{D}(X, r)}{\widehat{H}(X, r)},$$

then the frequency function of v_X satisfies

$$N(v_X, r) = \frac{r D(v_X, r)}{H(v_X, r)} = (1 + O(\theta(4r))) \frac{r \widehat{D}(X, r)}{\widehat{H}(X, r)} = (1 + O(\theta(4r))) \widehat{N}(X, r), \quad (8.15)$$

We claim that

$$\partial \Psi_X(B_r) = \partial B_r(X + 3r\tilde{\theta}(r)e_d) = \Psi_X(\partial B_r). \quad (8.16)$$

In fact by the definition of the transformation map Ψ (see (4.2)), it is clear that

$$\Psi(\partial B_r) = \partial B_r + 3r\tilde{\theta}(r)e_d = \partial B_r(3r\tilde{\theta}(r)e_d).$$

To understand what the set $\partial \Psi(B_r)$ is, we first study the set $\Psi(B_r)$. Clearly

$$\Psi(B_r) = \bigcup_{\rho \in [0, r)} \Psi(\partial B_\rho) = \bigcup_{\rho \in [0, r)} \partial B_\rho(3\rho\tilde{\theta}(\rho)e_d).$$

Consider the function

$$f : \rho \in [0, r) \mapsto -\rho + 3\rho\tilde{\theta}(\rho),$$

which corresponds to the height of the lower-most point of the (shifted) ball $\partial B_\rho + 3\rho\tilde{\theta}(\rho)e_d$. A simple computation shows that f is a continuous function, and

$$\begin{aligned} f'(\rho) &= -1 + 3\tilde{\theta}(\rho) + 3\rho\tilde{\theta}'(\rho) = -1 + 3\tilde{\theta}(\rho) + \frac{3}{\log^2 2} \int_\rho^{2\rho} \frac{\theta(2s) - \theta(s)}{s} ds \\ &\leq -1 + 3\theta(4\rho) + \frac{3}{\log 2} \theta(4\rho) \\ &\leq -1 + 13\theta(4r). \end{aligned}$$

By choosing r sufficiently small so that $\theta(4r) < 1/26$, we can guarantee that f is a monotone decreasing function. In particular, this implies that the balls $\Psi(\partial B_\rho) = \partial B_\rho + 3\rho\tilde{\theta}(\rho)e_d$ with $\rho \in [0, r)$ are nested, i.e.

$$B_\rho(3\rho\tilde{\theta}(\rho)e_d) \subset B_{\rho'}(3\rho'\tilde{\theta}(\rho')e_d), \quad \text{if } \rho \leq \rho'.$$

In fact, let $X \in B_\rho(3\rho\tilde{\theta}(\rho)e_d)$ be arbitrary. Then

$$\begin{aligned} |X - 3\rho'\tilde{\theta}(\rho')e_d| &\leq |X - 3\rho\tilde{\theta}(\rho)e_d| + (3\rho'\tilde{\theta}(\rho') - 3\rho\tilde{\theta}(\rho)) \\ &< \rho + f(\rho') + \rho' - (f(\rho) + \rho) \\ &= \rho' + (f(\rho') - f(\rho)) \\ &\leq \rho'. \end{aligned}$$

Hence $X \in B_{\rho'}(3\rho'\tilde{\theta}(\rho')e_d)$. Moreover by the intermediate value theorem $f(\rho)$ assumes all values between $\lim_{\rho \rightarrow r-} f(\rho) = -r + 3r\tilde{\theta}(r)$ and $\lim_{\rho \rightarrow 0+} f(\rho) = 0$. This finishes the proof of the claim (8.16).

As in (8.3) in the proof of Lemma 8.1, we have

$$\begin{aligned} &\frac{d}{dt} \widehat{N}(X_t, r) \\ &= \frac{d}{dt} \log \widehat{N}(X_t, r) \cdot \widehat{N}(X_t, r) \\ &= \frac{1}{\widehat{H}(X_t, r)} \left[r \frac{d}{dt} \widehat{D}(X_t, r) - \widehat{N}(X_t, r) \cdot \frac{d}{dt} \widehat{H}(X_t, r) \right] \\ &= \frac{2}{\widehat{H}(X_t, r)} \left[r \int_{X_t + \partial \Psi(B_r)} \left\langle \nabla u(Z), \frac{d}{dt} X_t \right\rangle \partial_n u(Z) d\mathcal{H}^{d-1} \right. \\ &\quad \left. + r \int_{X_t + \Psi(B_r) \cap \partial D} \left\langle \nabla u(Z), \frac{d}{dt} X_t \right\rangle \partial_n u(Z) d\mathcal{H}^{d-1} \right. \\ &\quad \left. - \widehat{N}(X_t, r) \cdot \int_{X_t + \Psi(\partial B_r)} \left\langle \nabla u(Z), \frac{d}{dt} X_t \right\rangle u(Z) d\mathcal{H}^{d-1} \right] \\ &= \frac{2}{\widehat{H}(X_t, r)} \left[r \int_{X_t + \Psi(\partial B_r)} \left\langle \nabla u(Z), \frac{d}{dt} X_t \right\rangle \partial_n u(Z) d\mathcal{H}^{d-1} \right. \\ &\quad \left. + r \int_{X_t + \Psi(B_r) \cap \partial D} \left\langle \nabla u(Z), \frac{d}{dt} X_t \right\rangle \partial_n u(Z) d\mathcal{H}^{d-1} \right] \end{aligned}$$

$$\begin{aligned}
& - \widehat{N}(X_t, r) \cdot \int_{X_t + \Psi(\partial B_r)} \left\langle \nabla u(Z), \frac{d}{dt} X_t \right\rangle u(Z) d\mathcal{H}^{d-1} \Big] \\
& =: I_1(X_t) + I_2(X_t) + I_3(X_t).
\end{aligned} \tag{8.17}$$

In the second to last equality we simply use (8.16).

Since u vanishes on the boundary and at any $Z = (x, \varphi(x)) \in \partial D$, the vector $(y, \langle \nabla \varphi(x), y \rangle)$ points in the tangential direction of ∂D , we have

$$\langle \nabla u(Z), (y, \langle \nabla \varphi(x), y \rangle) \rangle = 0.$$

Hence by (8.14), we know

$$\left\langle \nabla u(Z), \frac{d}{dt} X_t \right\rangle = \left\langle \nabla u(Z), \frac{d}{dt} X_t - (y, \langle \nabla \varphi(x), y \rangle) \right\rangle = \langle \nabla u(Z), (0, \langle \nabla \varphi(x_t) - \nabla \varphi(x), y \rangle) \rangle.$$

Thus

$$\left| \left\langle \nabla u(Z), \frac{d}{dt} X_t \right\rangle \right| \leq |\partial_d u(Z)| \cdot \theta(|x_t - x|)|y| \leq |\partial_d u(Z)| \cdot \theta(r)|X_2 - X_1|.$$

Therefore we can estimate $I_2(X_t)$ as follows

$$\begin{aligned}
|I_2(X_t)| & \leq \frac{2r}{\widehat{H}(X_t, r)} \int_{\Psi_{X_t}(B_r) \cap \partial D} |\partial_d u|^2 \theta(r) |X_2 - X_1| d\mathcal{H}^{d-1} \\
& \lesssim \frac{r^2 \theta(r)}{\widehat{H}(X_t, r)} \int_{\Psi_{X_t}(B_r) \cap \partial D} |\nabla u|^2 d\mathcal{H}^{d-1} \\
& \lesssim \frac{\theta(r)}{\widehat{H}(X_t, r)} \mathcal{H}^{d-1}(\Psi_{X_t}(B_r) \cap \partial D) \cdot \iint_{\Psi_{X_t}(B_{3r/2})} u^2 dZ \\
& \lesssim \theta(r).
\end{aligned} \tag{8.18}$$

Since u^2 is subharmonic (after extending by zero) and

$$|X_t - X_1| \leq (1 + \theta(R))|X_2 - X_1| < \frac{5}{12}r,$$

we have

$$\begin{aligned}
\widehat{H}(X_t, r) & = \int_{\partial B_r(X_t + 3r\bar{\theta}(r)e_d)} u^2 d\mathcal{H}^{d-1} \geq \frac{12}{11r} \iint_{A_{\frac{r}{12}, r}(X_t + 3r\bar{\theta}(r)e_d)} u^2 dZ \\
& \geq \frac{12}{11r} \iint_{A_{\frac{5}{48}r, \frac{47}{48}r}(X_t)} u^2 dZ \\
& \geq \frac{12}{11r} \iint_{A_{\frac{25}{48}r, \frac{27}{48}r}(X_1)} u^2 dZ \\
& \geq \frac{1}{22} \int_{\partial B_{\frac{r}{2}}(X_1)} u^2 d\mathcal{H}^{d-1}.
\end{aligned} \tag{8.19}$$

An elementary computation shows that

$$\begin{aligned}
X_2 - X_1 - (y, \langle \nabla \varphi(x_t), y \rangle) & = (x_2 - x_1, \varphi(x_2) - \varphi(x_1)) - (x_2 - x_1, \langle \nabla \varphi(x_t), x_2 - x_1 \rangle) \\
& = (x_2 - x_1, \langle \nabla \varphi(x) - \nabla \varphi(x_t), x_2 - x_1 \rangle),
\end{aligned}$$

where x is some point on the line segment $[x_1, x_2] \subset \mathbb{R}^{d-1}$. Therefore by the assumption of φ in Definition 2.1, we have

$$|X_2 - X_1 - (y, \langle \nabla \varphi(x_t), y \rangle)| \leq \theta(|x_2 - x_1|) \cdot |x_2 - x_1|^2.$$

Hence

$$\begin{aligned} \langle \nabla u(Z), (y, \langle \nabla \varphi(x_t), y \rangle) \rangle &= \langle \nabla u(Z), X_2 - X_1 \rangle - \langle \nabla u(Z), X_2 - X_1 - (y, \langle \nabla \varphi(x_t), y \rangle) \rangle \\ &= \langle \nabla u(Z), X_2 - X_1 \rangle + O(r^2 \theta(r)) \cdot |\nabla u(Z)| \end{aligned}$$

Inserting $\langle \nabla u(Z), X_2 - X_1 \rangle$ into the equality (8.17), we can use the same argument as in the proof of Lemma 8.1 (as well as (8.19)) for the estimate, so we will not repeat the argument here.

The remaining term satisfies

$$\begin{aligned} \mathcal{E}(X_t) &\leq r^2 \theta(r) \left[\frac{2r}{\widehat{H}(X_t, r)} \cdot \int_{X_t + \Psi(\partial B_r)} |\nabla u|^2 d\mathcal{H}^{d-1} \right. \\ &\quad \left. + \frac{C(\Lambda)}{\widehat{H}(X_t, r)} \left(\int_{X_t + \Psi(\partial B_r)} |\nabla u|^2 d\mathcal{H}^{d-1} \right)^{\frac{1}{2}} \left(\int_{X_t + \Psi(\partial B_r)} u^2 d\mathcal{H}^{d-1} \right)^{\frac{1}{2}} \right] \\ &\leq r^{\frac{3}{2}} \theta(r) \left[\frac{2r \int_{X_t + \Psi(\partial B_r)} |\nabla u|^2 d\mathcal{H}^{d-1}}{\int_{\partial B_{\frac{r}{2}}(X_1)} u^2 d\mathcal{H}^{d-1}} + C(\Lambda) \left(\frac{r \int_{X_t + \Psi(\partial B_r)} |\nabla u|^2 d\mathcal{H}^{d-1}}{\int_{\partial B_{\frac{r}{2}}(X_1)} u^2 d\mathcal{H}^{d-1}} \right)^{\frac{1}{2}} \right]. \end{aligned}$$

Hence by integrating on the interval $t \in [0, 1]$, we get

$$\begin{aligned} \int_0^1 \mathcal{E}(X_t) dt &\leq r^{\frac{3}{2}} \theta(r) \left[\frac{2r \iint_{A_{\frac{r}{2}, \frac{3}{2}r}(X_1)} |\nabla u|^2 d\mathcal{H}^{d-1}}{\int_{\partial B_{\frac{r}{2}}(X_1)} u^2 d\mathcal{H}^{d-1}} + C(\Lambda) \left(\frac{r \iint_{A_{\frac{r}{2}, \frac{3}{2}r}(X_1)} |\nabla u|^2 d\mathcal{H}^{d-1}}{\int_{\partial B_{\frac{r}{2}}(X_1)} u^2 d\mathcal{H}^{d-1}} \right)^{\frac{1}{2}} \right] \\ &\leq C'(\Lambda) r^{\frac{3}{2}} \theta(r). \end{aligned} \tag{8.20}$$

Finally, combining (8.17), (8.30) and (8.20) we conclude that

$$\begin{aligned} |\widehat{N}(X_2, r) - \widehat{N}(X_1, r)| &\leq C \left(W_{X_1}^{\frac{1}{2}}(r) + W_{X_2}^{\frac{1}{2}}(r) + \theta(r) \right) + \int_0^1 \mathcal{E}(X_t) dt \\ &\leq C' \left(W_{X_1}^{\frac{1}{2}}(r) + W_{X_2}^{\frac{1}{2}}(r) + \theta(r) \right). \end{aligned}$$

Therefore by (8.15), we get

$$\begin{aligned} |N(v_{X_2}, r) - N(v_{X_1}, r)| &\leq |\widehat{N}(X_2, r) - \widehat{N}(X_1, r)| + C\theta(4r) (N(v_{X_1}, r) + N(v_{X_2}, r)) \\ &\leq C''(\Lambda) \left(W_{X_1}^{\frac{1}{2}}(r) + W_{X_2}^{\frac{1}{2}}(r) + \theta(4r) \right). \end{aligned}$$

□

Next we estimate the spatial variation for interior points. Compared to the spatial variation for boundary points in Proposition 8.10, here we use a similar idea but it suffices to get a rough estimate.

Lemma 8.21. *Let $R, \Lambda > 0$ be fixed. Suppose $(u, D) \in \mathfrak{H}(R, \Lambda)$ and $p \in D \cap B_{\frac{R}{10}}(0) \cap \mathcal{N}(u)$ satisfies $\text{dist}(p, \partial D) \leq \frac{3}{5}R\tilde{\theta}(\frac{3}{5}R)$. Then for any $r \leq \frac{12}{25}R$ we have*

$$|N(v_q, r) - N(p, r)| \leq C \left(r^{\frac{1}{2}} + \theta(4r) \right), \quad \text{for every } q \in B_{\frac{r}{5}}(p) \cap \partial D \cap \mathcal{N}(u); \quad (8.22)$$

and

$$|N(q, r) - N(p, r)| \leq Cr^{\frac{1}{2}}, \quad \text{for every } q \in B_{\frac{r}{5}}(p) \cap D \cap \mathcal{N}(u). \quad (8.23)$$

Here the constant $C > 0$ depends on d and Λ .

Proof. The idea of the proof is similar to that of Proposition 8.10, with $p \in D$ playing the role of X_1 and

$$\tilde{q} := \begin{cases} q, & \text{if } q \in D \\ q + 3r\tilde{\theta}(r)e_d, & \text{if } q \in \partial D \end{cases}$$

playing the role of X_2 . This way it suffices to estimate $|N(\tilde{q}, r) - N(p, r)|$ for both the interior and boundary cases.

For $t \in [0, 1]$ we first define $X_t^0 := p + t(\tilde{q} - p)$. However, the line segment $[\tilde{q}, p]$ may not be completely contained in D . Let $A = \{t \in [0, 1] : X_t \notin D\}$. Since ∂D is a C^1 graph, A is a finite union of closed intervals and can be written as

$$A = \bigcup_{k=1}^m [t_{2k-1}, t_{2k}].$$

To unify the notation at the endpoints, we also set $t_0 = 0$ and $t_{2m+1} = 1$. We revise the definition of X_t^0 for all $t \in A$ by replacing X_t^0 by its projection onto ∂D , i.e.

$$X_t^0 := (x_1 + t(x_2 - x_1), \varphi(x_1 + t(x_2 - x_1))) \in \partial D,$$

where we denote $p = (x_1, z_1) \in D$ and $\tilde{q} = (x_2, z_2) \in D$. Notice that for every $t, t' \in [0, 1]$, we have

$$\begin{aligned} |X_t^0 - X_{t'}^0| &\leq \left| ((t - t')(x_2 - x_1), \varphi(x_1 + t(x_2 - x_1)) - \varphi(x_1 + t'(x_2 - x_1))) \right| \\ &\leq |t - t'| \sqrt{|x_2 - x_1|^2 + (\theta(R) \cdot |x_2 - x_1|)^2} \\ &\leq \frac{10}{9} |t - t'| |x_2 - x_1| \\ &\leq \frac{10}{9} |t - t'| |q - p|. \end{aligned}$$

Finally we let

$$X_t := \begin{cases} X_t^0, & t = 0 \text{ or } 1 \\ X_t^0 + 3r\tilde{\theta}(r)e_d, & t \in A \\ X_{2k-2} + \frac{t-t_{2k-2}}{t_{2k-1}-t_{2k-2}} (X_{2k-1} - X_{2k-2}) & t \in [t_{2k-2}, t_{2k-1}]. \end{cases}$$

After such modification we guarantee that $X_t \in D$. It is not hard to see that the total length ℓ of the curve $\{X_t\}_{t \in [0,1]}$ satisfies

$$\ell \leq \frac{10}{9}|q - p| + 2 \cdot 3r\tilde{\theta}(r) < \frac{r}{4}.$$

We then reparametrize it (still denoted by $\{X_t\}_{t \in [0,1]}$) so that it has unit speed, and thus

$$\left| \frac{d}{dt} X_t \right| \leq \ell < \frac{r}{4}. \quad (8.24)$$

Moreover, for every $t, t' \in [0, 1]$ we have

$$|X_t - X_{t'}| \leq \ell < \frac{r}{4}. \quad (8.25)$$

As in (8.17), we have

$$\begin{aligned} & \frac{d}{dt} N(X_t, r) \\ &= \frac{1}{H(X_t, r)} \left[r \frac{d}{dt} D(X_t, r) - N(X_t, r) \cdot \frac{d}{dt} H(X_t, r) \right] \\ &= \frac{2}{H(X_t, r)} \left[r \int_{\partial B_r(X_t)} \left\langle \nabla u, \frac{d}{dt} X_t \right\rangle \partial_n u \, d\mathcal{H}^{d-1} + r \int_{B_r(X_t) \cap \partial D} \left\langle \nabla u, \frac{d}{dt} X_t \right\rangle \partial_n u \, d\mathcal{H}^{d-1} \right. \\ & \quad \left. - N(X_t, r) \cdot \int_{\partial B_r(X_t)} \left\langle \nabla u, \frac{d}{dt} X_t \right\rangle u \, d\mathcal{H}^{d-1} \right] \\ &=: I_1(X_t) + I_2(X_t) + I_3(X_t). \end{aligned} \quad (8.26)$$

(There is a slight abuse of notation in the last equality, where $\partial_n u$ denotes the derivative pointing away from the sphere $\partial B_r(X_t)$ in the first integral, and $\partial_n u$ denotes the normal derivative pointing away from ∂D in the second integral.) We remark that the main difference with the boundary case (8.17) is that (8.17) does not have the second term $I_2(X_t)$ since there $\frac{d}{dt} X_t$ is in the tangential direction on ∂D and thus $\langle \nabla u, \frac{d}{dt} X_t \rangle = 0$. Here, on the other hand, the vector $\frac{d}{dt} X_t$ does not always point in the tangential direction of ∂D . We first estimate this term using an idea similar to the Rellich identity.

It is not hard to see that

$$\operatorname{div}(|\nabla u|^2 e_d) = 2 \langle \nabla(\partial_d u), \nabla u \rangle = 2 (\operatorname{div}(\partial_d u \cdot \nabla u) - \partial_d u \cdot \Delta u) = 2 \operatorname{div}(\partial_d u \cdot \nabla u). \quad (8.27)$$

Integrating both sides of (8.27) on the region $B_r(X_t) \cap D$ and using the divergence theorem, we get

$$\begin{aligned} & \int_{\partial B_r(X_t)} |\nabla u|^2 \left\langle e_d, \frac{Z - X_t}{|Z - X_t|} \right\rangle \, d\mathcal{H}^{d-1} + \int_{B_r(X_t) \cap \partial D} |\nabla u|^2 \langle e_d, n_D(Z) \rangle \, d\mathcal{H}^{d-1} \\ &= 2 \int_{\partial B_r(X_t)} \partial_d u \left\langle \nabla u, \frac{Z - X_t}{|Z - X_t|} \right\rangle \, d\mathcal{H}^{d-1} + 2 \int_{B_r(X_t) \cap \partial D} \partial_d u \cdot \partial_n u \, d\mathcal{H}^{d-1}. \end{aligned} \quad (8.28)$$

On every boundary point $Z = (x, \varphi(x)) \in \partial D$, we have $\nabla u(Z) = (\partial_n u) n_D(Z)$, where

$$n_D(Z) = \frac{(\nabla \varphi(x), -1)}{\sqrt{1 + |\nabla \varphi(x)|^2}} \text{ is the unit outer normal vector.}$$

Hence

$$\partial_d u \cdot \partial_n u = \langle \nabla u, e_d \rangle \partial_n u = |\partial_n u|^2 \langle n_D(Z), e_d \rangle = \frac{-(\partial_n u)^2}{\sqrt{1 + |\nabla \varphi(x)|^2}},$$

and (8.28) is simplified to

$$\begin{aligned} & \int_{B_r(X_t) \cap \partial D} \frac{(\partial_n u)^2}{\sqrt{1 + |\nabla \varphi(x)|^2}} \\ &= 2 \int_{\partial B_r(X_t)} \partial_d u \left\langle \nabla u, \frac{Z - X_t}{|Z - X_t|} \right\rangle d\mathcal{H}^{d-1} - \int_{\partial B_r(X_t)} |\nabla u|^2 \left\langle e_d, \frac{Z - X_t}{|Z - X_t|} \right\rangle d\mathcal{H}^{d-1} \\ &\leq 3 \int_{\partial B_r(X_t)} |\nabla u|^2 d\mathcal{H}^{d-1}. \end{aligned} \quad (8.29)$$

On the other hand by (8.24), for every $Z \in \partial D$ we have

$$\begin{aligned} \left| \left\langle \nabla u, \frac{d}{dt} X_t \right\rangle \right| &= \left| \partial_n u \left\langle n_D(Z), \frac{d}{dt} X_t \right\rangle \right| = \frac{|\partial_n u|}{\sqrt{1 + |\nabla \varphi(x)|^2}} \left| \left\langle (\nabla \varphi(x), -1), \frac{d}{dt} X_t \right\rangle \right| \\ &\leq \frac{r}{4} \frac{|\partial_n u|}{\sqrt{1 + |\nabla \varphi(x)|^2}}. \end{aligned}$$

Therefore

$$\begin{aligned} |I_2(X_t)| &= \frac{2r}{H(X_t, r)} \int_{B_r(X_t) \cap \partial D} \left| \left\langle \nabla u, \frac{d}{dt} X_t \right\rangle \partial_n u \right| d\mathcal{H}^{d-1} \\ &\leq \frac{r^2}{2H(X_t, r)} \int_{B_r(X_t) \cap \partial D} \frac{(\partial_n u)^2}{\sqrt{1 + |\nabla \varphi(x)|^2}} d\mathcal{H}^{d-1} \\ &\leq \frac{2r^2}{H(X_t, r)} \int_{\partial B_r(X_t)} |\nabla u|^2 d\mathcal{H}^{d-1}. \end{aligned} \quad (8.30)$$

Notice that

$$|X_t - p| = |X_t - X_1| \leq \ell < \frac{r}{4}.$$

By the sub-harmonicity of u^2 , we have

$$H(X_t, r) \gtrsim \frac{1}{r} \iint_{A_{\frac{r}{4}, r}(X_t)} u^2 dZ \geq \frac{1}{r} \iint_{A_{\frac{r}{2}, \frac{3}{4}r}(p)} u^2 dZ \gtrsim H\left(p, \frac{r}{2}\right).$$

Therefore it follows from (8.24), (8.30) and Hölder's inequality that

$$|I_1(X_t)|, |I_2(X_t)| \lesssim \frac{r^2}{H(p, \frac{r}{2})} \int_{\partial B_r(X_t)} |\nabla u|^2 d\mathcal{H}^{d-1},$$

and

$$|I_3(X_t)| \leq \frac{C(\Lambda)r}{H(X_t, r)} \left(\int_{\partial B_r(X_t)} |\nabla u|^2 d\mathcal{H}^{d-1} \right)^{\frac{1}{2}} \left(\int_{\partial B_r(X_t)} u^2 d\mathcal{H}^{d-1} \right)^{\frac{1}{2}}$$

$$\leq C(\Lambda) r^{\frac{1}{2}} \left(\frac{r \int_{\partial B_r(X_t)} |\nabla u|^2 d\mathcal{H}^{d-1}}{H(p, \frac{r}{2})} \right)^{\frac{1}{2}}.$$

Integrating on the interval $t \in [0, 1]$ and using the fact that

$$\bigcup_{t \in [0, 1]} \partial B_r(X_t) \subset B_{\frac{5}{4}r}(p),$$

we get

$$\begin{aligned} & |N(\tilde{q}, r) - N(p, r)| \\ & \leq \int_0^1 |I_1(X_t)| + |I_2(X_t)| + |I_3(X_t)| dt \\ & \lesssim \frac{r^2}{H(p, \frac{r}{2})} \iint_{B_{\frac{5}{4}r}(p)} |\nabla u|^2 dZ + C(\Lambda) r^{\frac{1}{2}} \left(\frac{r \iint_{B_{\frac{5}{4}r}(p)} |\nabla u|^2 d\mathcal{H}^{d-1}}{H(p, \frac{r}{2})} \right)^{\frac{1}{2}} \\ & \lesssim C'(\Lambda) r^{\frac{1}{2}}, \end{aligned}$$

where we use the doubling property and the upper bound of $N(p, \frac{5}{4}r)$ in the last inequality.

In the case of $q \in \partial D$, it remains to estimate $|N(\tilde{q}, r) - N(v_q, r)|$. By (8.15) and the properties of the transformation map Ψ_X , we know

$$N(v_q, r) = (1 + O(\theta(4r))) \hat{N}(q, r) = (1 + O(\theta(4r))) N(\tilde{q}, r).$$

Therefore

$$|N(\tilde{q}, r) - N(v_q, r)| \lesssim_{\Lambda} \theta(4r),$$

and it follows that

$$|N(v_q, r) - N(p, r)| \lesssim_{\Lambda} r^{\frac{1}{2}} + \theta(4r).$$

□

9. L^2 -BEST APPROXIMATION THEOREM AND THE FREQUENCY FUNCTION

Let μ be a Radon measure. We define the k -dimensional L^2 beta-number of μ in $B_r(p)$ as

$$\beta_{\mu}^k(p, r) = \inf_{L^k} \left(\frac{1}{r^{2+k}} \int_{B_r(p)} \text{dist}^2(Y, L^k) d\mu(Y) \right)^{\frac{1}{2}}, \quad (9.1)$$

where the infimum is taken over all k -dimensional affine planes. We remark that the scaling factor r^{2+k} is to make sure the above definition is scale-invariant, when the measure μ is k -dimensional. In Section 10 we will use the following Reifenberg theorem with dimension $k = d - 2$ to give uniform volume bounds on $C_r(u) \cap \mathcal{N}(u)$.

Theorem 9.2 (Discrete Reifenberg Theorem [NV1]). *There exist constants $\eta_{dr} > 0$ and $C_{dr} > 0$ depending only on the dimension such that the following holds. Let $\{B_{r_X/5}(X)\}_{X \in \mathcal{C}} \subset B_3(0) \subset \mathbb{R}^d$ be a collection of pairwise disjoint balls with their centers $X \in B_1(0)$, and let $\mu = \sum_{X \in \mathcal{C}} \omega_k r_X^k \delta_X$ be the associated packing measure. Assume that for each $B_s(p_0) \subset B_2(0)$*

$$\iint_{B_s(p_0)} \int_0^s |\beta_\mu^k(p, r)|^2 \frac{dr}{r} d\mu(p) \leq \eta_{dr} s^k.$$

Then we have the uniform estimate

$$\sum_{X \in \mathcal{C}} r_X^k \leq C_{dr}. \quad (9.3)$$

Remark 9.4. For the discrete measure above associated to the covering, we notice that for any $X \in \mathcal{C} \subset \text{spt } \mu$, if $r \leq r_X/5$ then $B_r(X) \cap \text{spt } \mu = \{X\}$, and thus by definition $\beta_\mu^k(X, r) = 0$.

We will also need a variant of it, the Rectifiable Reifenberg Theorem.

Theorem 9.5 (Rectifiable Reifenberg Theorem [NV1]). *There exist constants $\eta_{rr} > 0$ and $C_{rr} > 0$ depending only on the dimension such that the following holds. Let $E \subset B_2(0)$ be a $\mathcal{H}^k$ -measurable set, and assume that for each $B_s(p_0) \subset B_2(0)$*

$$\iint_{B_s(p_0)} \int_0^s |\beta_{\mathcal{H}^k \llcorner E}^k(p, r)|^2 \frac{dr}{r} d\mathcal{H}^k \llcorner E \leq \eta_{rr} s^k.$$

Then E is k -rectifiable, and for each $p_0 \in E \cap B_1(0)$ and $0 < s < 1$ we have $\mathcal{H}^k(B_s(p_0)) \leq C_{rr} s^k$.

Remark 9.6. There have been many generalizations of the classical Reifenberg theorem, see for instance [Tol, DT]. Reifenberg theorems of the form in Theorems 9.2 and 9.5 first appear in [NV1, Theorems 3.4 and 3.3] (see also [NV3, Theorems 40 and 42]). We refer interested readers to [ENV] for a review and generalization of Reifenberg-type theorems.

The main goal of this section is to control the $(d-2)$ -dimensional beta number $\beta_\mu^{d-2}(p, r)$ by the drop of the frequency function inside the ball $B_r(p)$.

Theorem 9.7. *Let $R, \Lambda > 0$ and $\delta_{in} \in (0, 1)$ be fixed. Let $r_b := \min\{r_{in}, r_s, \frac{R}{10}\} > 0$ where r_{in} is given in Lemma 6.17 with parameters δ_{in} and $\rho = \frac{1}{6}$, and r_s is given in Lemma 9.50. Then for any $(u, D) \in \mathfrak{H}(R, \Lambda)$, any $p \in C_r(u) \cap N(u) \cap B_{\frac{R}{10}}(0)$ with $\text{dist}(p, \partial D) \leq 6r_{in}\tilde{\theta}(6r_{in})$, and any $0 < r \leq r_b$, the $(d-2)$ -dimensional L^2 β -number in $B_r(p)$ satisfies*

$$\begin{aligned} |\beta_\mu^{d-2}(p, r)|^2 &\leq C_b \left[\frac{1}{r^{d-2}} \int_{B_r(p)} \left(\tilde{W}_X(r) + \delta_{in} \cdot \chi_{r_{cs}(X)/6 < r \leq r_{cs}(X)} \right) d\mu(X) \right. \\ &\quad \left. + \frac{\mu(B_r(p))}{r^{d-2}} (r + \theta(24r)) \right], \end{aligned} \quad (9.8)$$

where μ is an arbitrary Radon measure such that

$$\text{spt } \mu \subset N(u) \cap B_{6r_{in}\tilde{\theta}(6r_{in})}(\partial D),$$

and $\tilde{W}_X(r) := N_X(6r) - N_X(r)$. The constant C_b depends on d , the uniform frequency bound $C(\Lambda)$ and C_s in Lemma 9.50. (In particular it is independent of the choice of δ_{in} .)

- Remark 9.9.* • Heuristically, the above estimate (9.8) says the following. Suppose the right hand side is small, that is to say, (on an average sense) points on the supports of the measure μ have small frequency drop at scale r (i.e. if $\tilde{W}_X(r) \ll 1$ when $X \in \text{spt} \mu$). Then $\text{spt} \mu$ must be distributed in a small neighborhood of a $(d-2)$ -dimensional affine subspace. (In particular $\text{spt} \mu$ can be distributed near a k -dimensional affine subspace with $k < d-2$.) Intuitively, the reason is that if we look at the tangent function centered at any singular point $p \in C_r(u) \cap \mathcal{N}(u)$, it can be almost invariant along an at most $(d-2)$ -dimensional affine subspace.
- The second term in the right of the inequality in (9.8) comes from Lemma 6.17. Namely for an interior point $X \in D$, its frequency function $N_X(\cdot)$ is monotone in the intervals $[0, r_{cs}(X)]$ and $(r_{cs}(X), +\infty)$, and changes by at most δ_{in} at the critical scale $r_{cs}(X)$.

Step 1. Let C_μ denote the center of mass of the measure μ on $B_r(p)$, i.e.

$$C_\mu := \int_{B_r(p)} X d\mu(X).$$

To compute the beta-number $\beta_\mu^k(p, r)$ we consider the bilinear quadratic form $Q(v, w)$ associated to $\mu \llcorner_{B_r(p)}$, which is defined as

$$Q(v, w) := \frac{1}{r^{2+k}} \int_{B_r(p)} \langle v, Y - C_\mu \rangle \langle w, Y - C_\mu \rangle d\mu(Y).$$

Let $\lambda_1 \geq \dots \geq \lambda_d \geq 0$ be the eigenvalues of Q in decreasing order, and let $w_1, \dots, w_d$ be the corresponding eigenvectors with unit norm. Then

$$\lambda_j = \sup \left\{ \frac{1}{r^{2+k}} \int_{B_r(p)} |\langle w, Y - C_\mu \rangle|^2 d\mu(Y) \text{ s.t. } |w|^2 = 1 \text{ and } \langle w, w_i \rangle = 0 \text{ for all } i < j \right\}. \quad (9.10)$$

Then it is not hard to see that the affine k -plane $L^k := C_\mu + \text{span}\{w_1, \dots, w_k\}$ achieves the infimum in the definition (9.1) of the beta number, and moreover

$$|\beta_\mu^k(p, r)|^2 = \frac{1}{r^{2+k}} \int_{B_r(p)} \text{dist}^2(Y, L^k) d\mu(Y) = \lambda_{k+1} + \dots + \lambda_d. \quad (9.11)$$

Step 2. Let $(u, D) \in \mathfrak{H}(R, \Lambda)$ and $p \in \text{spt} \mu$. Without loss of generality we assume u is normalized (with respect to the ball $B_r(p)$) so that

$$\frac{1}{r^d} \iint_{B_r(p)} u^2 dZ = 1. \quad (9.12)$$

Since a constant multiple of a function does not change its frequency function, this normalization does not affect (9.8).

- Suppose $X \in B_r(p) \cap D$ and $\rho \in [r, 6r]$. We consider two cases: either $10r \leq r_{cs}(p)$ so that we can use the monotonicity formula for p , or $10r > r_{cs}(p)$. In the first case,

we use (9.12) and the monotonicity formula for p , and get

$$H(X, \rho) \leq \frac{2}{\rho} \iint_{A_{\rho, \frac{3}{2}\rho}(X) \cap D} u^2 dZ \lesssim \frac{1}{r} \iint_{B_{10r}(p) \cap D} u^2 dZ \lesssim r^{d-1}. \quad (9.13)$$

In the second case, let $q \in \partial D$ so that $|q - p| = \text{dist}(p, \partial D)$. By the definition of the critical scale we know $\text{dist}(p, \partial D) \ll r_{cs}(p) < 10r$. In particular by choosing the scale sufficiently small we can guarantee that $|q - p| = \text{dist}(p, \partial D) < r/2$. Then by the monotonicity formula centered at q , we have

$$\begin{aligned} H(X, \rho) &\leq \frac{2}{\rho} \iint_{A_{\rho, \frac{3}{2}\rho}(X) \cap D} u^2 dZ \lesssim \frac{1}{r} \iint_{B_{10r}(q) \cap D} u^2 dZ \\ &\lesssim \frac{1}{r} \iint_{B_{r/2}(q) \cap D} u^2 dZ \\ &\leq \frac{1}{r} \iint_{B_r(p) \cap D} u^2 dZ = r^{d-1}. \end{aligned} \quad (9.14)$$

- Suppose $X \in B_{2r}(p) \cap \partial D$ and $\rho \in [r, 6r]$. Since $B_r(X) \subset B_{3r}(p)$, similar as above we consider the cases $3r \leq r_{cs}(p)$ and $3r > r_{cs}(p)$. Then using (9.12) and the monotonicity formulae centered at X and p in the first case, and the monotonicity formulae centered at X and $q \in \partial D$ in the second case, we deduce that

$$\begin{aligned} H(v_X, \rho) &\approx \frac{1}{\rho} \iint_{A_{\frac{\rho}{2}, \frac{3}{2}\rho}(0) \cap \Omega_X} \mu v_X^2 dV_g \quad \text{by (3.30)} \\ &\lesssim \frac{1}{r} \iint_{B_r(X) \cap D} u^2 dZ \\ &\lesssim \frac{1}{r} \iint_{B_r(p) \cap D} u^2 dZ \\ &= r^{d-1}. \end{aligned} \quad (9.15)$$

We omit some details here since the arguments in the two cases are very similar to the above.

The constants in the above inequalities depend on Λ (in fact, the constant grows exponentially as Λ grows).

For any $Z \in D$ and $j = 1, \dots, d$ we have

$$\lambda_j \langle w_j, \nabla u(Z) \rangle = Q(w_j, \nabla u(Z)) = \frac{1}{r^{2+k}} \int_{B_r(p)} \langle w_j, X - C_\mu \rangle \langle \nabla u(Z), X - C_\mu \rangle d\mu(X). \quad (9.16)$$

Since

$$\int_{B_r(p)} (X - C_\mu) d\mu(X) = 0,$$

we may add the term

$$\frac{1}{r^{2+k}} \left(\int_{B_r(p)} \langle w_j, X - C_\mu \rangle d\mu(X) \right) \cdot (\langle \nabla u(Z), C_\mu - Z \rangle + cu(Z))$$

to the right hand side of (9.16) without changing the equality:

$$\lambda_j \langle w_j, \nabla u(Z) \rangle = \frac{1}{r^{2+k}} \int_{B_r(p)} \langle w_j, X - C_\mu \rangle [\langle \nabla u(Z), X - Z \rangle + cu(Z)] d\mu(X).$$

Here c is a constant whose value is to be determined later. Thus by Hölder's inequality and (9.10),

$$\begin{aligned} & |\lambda_j \langle w_j, \nabla u(Z) \rangle|^2 \\ & \leq \left(\frac{1}{r^{2+k}} \int_{B_r(p)} |\langle w_j, X - C_\mu \rangle|^2 d\mu(X) \right) \cdot \left(\frac{1}{r^{2+k}} \int_{B_r(p)} |\langle \nabla u(Z), Z - X \rangle - cu(Z)|^2 d\mu(X) \right) \\ & = \lambda_j \left(\frac{1}{r^{2+k}} \int_{B_r(p)} |\langle \nabla u(Z), Z - X \rangle - cu(Z)|^2 d\mu(X) \right). \end{aligned}$$

Assume $\lambda_j > 0$, otherwise there is nothing to prove. It then simplifies to

$$\lambda_j |\langle \nabla u(Z), w_j \rangle|^2 \leq \frac{1}{r^{2+k}} \int_{B_r(p)} |\langle \nabla u(Z), Z - X \rangle - cu(Z)|^2 d\mu(X).$$

Next we integrate this inequality on the annulus region $A_{3r,4r}(p) \cap D = \{Z \in D : 3r < |Z - p| < 4r\}$. We get

$$\begin{aligned} & \frac{\lambda_j}{r^{d-2}} \iint_{A_{3r,4r}(p) \cap D} |\langle \nabla u(Z), w_j \rangle|^2 dZ \\ & \leq \frac{1}{r^{d-2}} \iint_{A_{3r,4r}(p) \cap D} \frac{1}{r^{2+k}} \int_{B_r(p)} |\langle \nabla u(Z), Z - X \rangle - cu(Z)|^2 d\mu(X) dZ \\ & \leq \frac{1}{r^k} \int_{B_r(p)} \frac{1}{r^d} \iint_{A_{2r,5r}(X) \cap D} |\langle \nabla u(Z), Z - X \rangle - cu(Z)|^2 dZ d\mu(X). \end{aligned} \quad (9.17)$$

Now we fix the constant c as

$$c = \begin{cases} N(v_p, 6r), & \text{if } p \in \partial D \\ N(p, 6r), & \text{if } p \in D \text{ and } 6r \leq r_{cs}(p) \\ N(v_q, 6r), & \text{if } p \in D \text{ and } 6r > r_{cs}(p) \end{cases} \quad (9.18)$$

where in the last line $q \in \partial D$ is such that $|p - q| = \text{dist}(p, \partial D)$. (Notice that c can not depend on $X \in \text{spt } \mu \cap B_r(p)$, but it can depend on the fixed point p .)

We start with the more complicated case (i): when $X \in \partial D$. Recall (4.7), we have

$$|\langle \nabla u(Z), Z - X \rangle - cu(Z)| \leq |\langle \nabla_g v_X(Y), Y \rangle - cv_X(Y)| + C|Y|\theta(4|Y|) \cdot |\nabla v_X(Y)|$$

where $Y = \Psi_X^{-1}(Z) \in \Omega_X$ is the preimage of $Z \in D$ and $v_X(Y) = u(\Psi_X(Y)) = u(Z)$. (The constant C is independent of the choice of $X \in \partial D$.) Hence by a change of variable and (4.4), (4.9), the integrand becomes

$$\begin{aligned} & \frac{1}{r^d} \iint_{A_{2r,5r}(X) \cap D} |\langle \nabla u(Z), Z - X \rangle - cu(Z)|^2 dZ \\ & \leq (1 + O(\theta(24r))) \frac{1}{r^d} \iint_{A_{r,6r}(0) \cap \Omega_X} |\langle \nabla_g v_X(Y), Y \rangle - cv_X(Y)|^2 dY \end{aligned}$$

$$+ C \frac{\theta(24r)}{r^{d-2}} \iint_{A_{r,6r}(0) \cap \Omega_X} |\nabla v_X(Y)|^2 dY. \quad (9.19)$$

We denote the second term as $\mathcal{E}$, and we have

$$\mathcal{E} \lesssim \frac{\theta(24r)}{r^{d-2}} D(v_X, 6r) \lesssim \frac{\theta(24r)}{r^{d-1}} N_X(6r) \cdot H(v_X, 6r) \lesssim \theta(24r), \quad (9.20)$$

with constants depending on d, r_* and Λ . We have used (9.15) in the last inequality.

We rewrite the leading order term in the right hand side of (9.19) as follows

$$\begin{aligned} & \frac{1}{r^d} \iint_{A_{r,6r}(0) \cap \Omega_X} |\langle \nabla_g v_X(Y), Y \rangle - c v_X(Y)|^2 dY \\ &= \frac{1}{r^d} \int_r^{6r} \rho^2 \int_{\partial B_\rho \cap \Omega_X} \left| \partial_\rho v_X - c \frac{v_X}{\rho} \right|^2 d\mathcal{H}^{d-1}(Y) d\rho \\ &\lesssim (1 + O(\theta(24r))) \int_r^{6r} \frac{1}{\rho^{d-2}} \int_{\partial B_\rho \cap \Omega_X} \mu \left| \partial_\rho v_X - c \frac{v_X}{\rho} \right|^2 dV_{\partial B_\rho} d\rho \end{aligned} \quad (9.21)$$

Recall we have shown in (3.15) that

$$R_h(v_X, \rho) = \frac{2\rho}{H(v_X, \rho)} \int_{\partial B_\rho \cap \Omega_X} \mu \left| \partial_\rho v_X - N(v_X, \rho) \frac{v_X}{\rho} \right|^2 dV_{\partial B_\rho} \leq \frac{d}{dr} \Big|_{r=\rho} N_X(r). \quad (9.22)$$

If $p \in \partial D$, we fix the constant c to be $N(v_p, 6r)$. We need to estimate the difference

$$|N(v_X, \rho) - c| \leq |N(v_X, \rho) - N(v_X, 6r)| + |N(v_X, 6r) - N(v_p, 6r)|. \quad (9.23)$$

By (8.13) in Proposition 8.10, we have

$$|N(v_p, 6r) - N(v_X, 6r)| \leq C \left(r^{\frac{1}{2}} + \theta(24r) \right), \quad (9.24)$$

where the constant C depends on Λ . On the other hand, by the monotonicity of the modified frequency function, for any $\rho \in [r, 6r]$ we have

$$\begin{aligned} & |N(v_X, \rho) - N(v_X, 6r)| \\ &= \left| N_X(\rho) \exp \left(-C \int_0^\rho \frac{\theta(4\tau)}{\tau} d\tau \right) - N_X(6r) \exp \left(-C \int_0^{6r} \frac{\theta(4\tau)}{\tau} d\tau \right) \right| \\ &\leq |N_X(\rho) - N_X(6r)| + N_X(6r) \left| \exp \left(-C \int_0^\rho \frac{\theta(4\tau)}{\tau} d\tau \right) - \exp \left(-C \int_0^{6r} \frac{\theta(4\tau)}{\tau} d\tau \right) \right| \\ &\leq N_X(6r) - N_X(r) + C(\Lambda) \int_r^{6r} \frac{\theta(4\tau)}{\tau} d\tau \\ &\leq N_X(6r) - N_X(r) + C'(\Lambda) \theta(24r). \end{aligned} \quad (9.25)$$

Combining (9.23), (9.24), (9.25) and using the monotonicity of the modified frequency function, we get

$$|N(v_X, \rho) - c| \leq \widetilde{W}_X(r) + C \left(r^{\frac{1}{2}} + \theta(24r) \right), \quad (9.26)$$

where we use the notation $\widetilde{W}_X(r) := N_X(6r) - N_X(r)$ to denote the frequency drop.

If $p \in D$ and $r \leq r_{cs}(p)/6$, recall that we have fixed $c = N(p, 6r)$ in (9.18). By Lemma 8.21 we have

$$|N(v_X, 6r) - N(p, 6r)| \leq C(r^{\frac{1}{2}} + \theta(24r)).$$

Therefore

$$\begin{aligned} |N(v_X, \rho) - c| &\leq |N(v_X, \rho) - N(v_X, 6r)| + |N(v_X, 6r) - N(p, 6r)| \\ &\leq N(v_X, 6r) - N(v_X, r) + C(r^{\frac{1}{2}} + \theta(24r)) \\ &\leq \tilde{W}_X(r) + C(r^{\frac{1}{2}} + \theta(24r)). \end{aligned} \quad (9.27)$$

If $p \in D$ and $r > r_{cs}(p)/6$, recall by (9.18) and (6.23) we have fixed $c = N(v_q, 6r)$, where $q \in \partial D$ is such that $|p - q| = \text{dist}(p, \partial D)$. Since every $X \in B_r(p)$ satisfies

$$|X - q| \leq |X - p| + |p - q| < 2r,$$

by (8.13) in Proposition 8.10 we have

$$|N(v_X, 6r) - N(v_q, 6r)| \leq C \left(r^{\frac{1}{2}} + \theta(24r) \right).$$

Hence

$$|N(v_X, \rho) - c| \leq |N(v_X, \rho) - N(v_X, 6r)| + |N(v_X, 6r) - N(v_q, 6r)| \leq \tilde{W}_X(r) + C(r^{\frac{1}{2}} + \theta(24r)).$$

Combining any one of (9.26), (9.27) with (9.21) and (9.31), we conclude

$$\begin{aligned} &\frac{1}{r^d} \iint_{A_{r,6r}(0) \cap \Omega_X} |\langle \nabla_g v_X(Y), Y \rangle - c v_X(Y)|^2 dY \\ &\lesssim \int_r^{6r} \frac{1}{\rho^{d-2}} \int_{\partial B_\rho \cap \Omega_X} \mu \left| \partial_\rho v_X - N(v_X, \rho) \frac{v_X}{\rho} \right|^2 dV_{\partial B_\rho} d\rho + \int_r^{6r} |N(v_X, \rho) - c|^2 \frac{H(v_X, \rho)}{\rho^d} d\rho \\ &\lesssim \int_r^{6r} R_h(v_X, \rho) \frac{H(v_X, \rho)}{\rho^{d-1}} d\rho + \left(\tilde{W}_X(r) + r + \theta(24r) \right) \int_r^{6r} \frac{d\rho}{\rho} \\ &\lesssim \tilde{W}_X(r) + r + \theta(24r), \end{aligned} \quad (9.28)$$

where we have used (9.15) in the last two inequalities.

Finally, combining (9.17), (9.19), (9.20) and (9.28), we get

$$\begin{aligned} &\frac{\lambda_j}{r^{d-2}} \iint_{A_{3r,4r}(p) \cap D} |\langle \nabla u(Z), w_j \rangle|^2 dZ \\ &\lesssim \frac{1}{r^k} \int_{B_r(p)} \left[\frac{1}{r^d} \iint_{A_{r,6r}(0) \cap \Omega_X} |\langle \nabla_g v_X(Y), Y \rangle - N_p(2r) \cdot v_X(Y)|^2 dY + \theta(24r) \right] d\mu(X) \\ &\lesssim \frac{1}{r^k} \int_{B_r(p)} \tilde{W}_X(r) d\mu(X) + \frac{\mu(B_r(p))}{r^k} (r + \theta(24r)). \end{aligned} \quad (9.29)$$

Case (ii): when $X \in D$ and $r \leq r_{cs}(X)/6$. We bound the integrand in (9.17) by

$$\frac{1}{r^d} \iint_{A_{2r,5r}(X) \cap D} |\langle \nabla u(Z), Z - X \rangle - cu(Z)|^2 dZ \lesssim \int_{2r}^{5r} \frac{1}{\rho^{d-2}} \int_{\partial B_\rho(X) \cap D} \left| \partial_{\rho_X} u - c \frac{u}{\rho} \right|^2 d\mathcal{H}^{d-1} d\rho \quad (9.30)$$

Recall we have shown in Proposition 6.3 that for any $\rho \leq r_{cs}(X)$, we have

$$R_h(X, \rho) = \frac{2\rho}{H(X, \rho)} \int_{\partial B_\rho(X) \cap D} \left| \partial_{\rho_X} u - N(X, \rho) \frac{u}{\rho} \right|^2 d\mathcal{H}^{d-1} \leq \frac{d}{dr} \Big|_{r=\rho} N(X, r). \quad (9.31)$$

Hence to estimate the right hand side of (9.30) it suffices to bound the difference $|N(X, \rho) - c|$.

If $p \in \partial D$, the constant $c = N(v_p, 6r)$ by (9.18). Hence by Lemma 8.21 and the assumption $6r \leq r_{cs}(X)$, we have

$$\begin{aligned} |N(X, \rho) - c| &\leq |N(X, \rho) - N(X, 6r)| + |N(X, 6r) - N(v_p, 6r)| \\ &\leq N(X, 6r) - N(X, r) + C(r^{\frac{1}{2}} + \theta(24r)) \\ &= \tilde{W}_X(r) + C(r^{\frac{1}{2}} + \theta(24r)). \end{aligned} \quad (9.32)$$

If $p \in D$ and $r \leq r_{cs}(p)/6$, the constant $c = N(p, 6r)$ by (9.18). By Lemma 8.21 we have

$$\begin{aligned} |N(X, \rho) - c| &\leq |N(X, \rho) - N(X, 6r)| + |N(X, 6r) - N(p, 6r)| \\ &\leq N(X, 6r) - N(X, r) + C(r^{\frac{1}{2}} + \theta(24r)) \\ &= \tilde{W}_X(r) + C(r^{\frac{1}{2}} + \theta(24r)). \end{aligned} \quad (9.33)$$

On the other hand if $p \in D$ and $r > r_{cs}(p)/6$, by (9.18) and (6.23) the constant is fixed at $c = N(v_q, 6r)$, where $q \in \partial D$ is such that $|p - q| = \text{dist}(p, \partial D)$. By Lemma 8.21, we have

$$\begin{aligned} |N(X, \rho) - c| &\leq |N(X, \rho) - N(X, 6r)| + |N(X, 6r) - N(v_q, 6r)| \\ &\leq \tilde{W}_X(r) + C(r^{\frac{1}{2}} + \theta(24r)). \end{aligned} \quad (9.34)$$

Plugging back any one of (9.33), (9.32), (9.34) into (9.30), we obtain

$$\begin{aligned} &\frac{1}{r^d} \iint_{A_{2r, 5r}(X) \cap D} |\langle \nabla u(Z), Z - X \rangle - cu(Z)|^2 dZ \\ &\lesssim \int_{2r}^{5r} \frac{1}{\rho^{d-2}} \int_{\partial B_\rho(X) \cap D} \left| \partial_{\rho_X} u - N(X, \rho) \frac{u}{\rho} \right|^2 d\mathcal{H}^{d-1} d\rho \\ &\quad + \int_{2r}^{5r} \frac{|N(X, \rho) - c|^2}{\rho^d} \int_{\partial B_\rho(X) \cap D} u^2 d\mathcal{H}^{d-1} d\rho \\ &\lesssim \int_{2r}^{5r} R_h(X, \rho) \frac{H(X, \rho)}{\rho^{d-1}} d\rho + \int_{2r}^{5r} |N(X, \rho) - c|^2 \frac{H(X, \rho)}{\rho^d} d\rho \\ &\lesssim \tilde{W}_X(r) + r + \theta(24r), \end{aligned} \quad (9.35)$$

where we have used (9.13) or (9.14) in the last inequality.

Case (iii): when $X \in D$ and $r > r_{cs}(X)/6$. Let $\tilde{X} \in \partial D$ be such that $|X - \tilde{X}| = \text{dist}(X, \partial D)$. By the definition of the critical value, we know that

$$\text{dist}(X, \partial D) = r_{cs}(X) \tilde{\theta}(r_{cs}(X)) \ll r_{cs}(X) < 6r. \quad (9.36)$$

In particular since $\tilde{\theta}(6r) \leq \tilde{\theta}(\frac{3}{5}R)$ is chosen sufficiently small, we can guarantee

$$|X - \tilde{X}| = \text{dist}(X, \partial D) < \frac{r}{5},$$

and thus

$$|p - \tilde{X}| \leq |p - X| + |X - \tilde{X}| < \frac{6}{5}r. \quad (9.37)$$

This time we bound the integrand in (9.17) by

$$\begin{aligned} & \frac{1}{r^d} \iint_{A_{2r,5r}(X) \cap D} |\langle \nabla u(Z), Z - X \rangle - cu(Z)|^2 dZ \\ & \leq \frac{1}{r^d} \iint_{A_{r,6r}(\tilde{X}) \cap D} |\langle \nabla u(Z), Z - X \rangle - cu(Z)|^2 dZ \\ & \leq \frac{1}{r^d} \iint_{A_{r,6r}(\tilde{X}) \cap D} |\langle \nabla u(Z), Z - \tilde{X} \rangle - cu(Z)|^2 dZ + \frac{|X - \tilde{X}|^2}{r^d} \iint_{A_{r,6r}(\tilde{X})} |\nabla u|^2 dZ \\ & =: I_1 + I_2. \end{aligned} \quad (9.38)$$

We can bound the error term I_2 very easily as follows

$$\begin{aligned} I_2 & \leq \frac{|X - \tilde{X}|^2}{r^d} \iint_{B_{6r}(\tilde{X})} |\nabla u|^2 dZ \leq C(\Lambda) \frac{|X - \tilde{X}|^2}{r^{d+1}} \int_{\partial B_{6r}(\tilde{X})} u^2 d\mathcal{H}^{d-1} \\ & \lesssim C'(\Lambda) \frac{|X - \tilde{X}|^2}{r^2} \\ & \lesssim \theta(24r), \end{aligned} \quad (9.39)$$

where we have used (9.15) and $\tilde{X} \in \partial D \cap B_{2r}(p)$ in the second to last inequality, and we used (9.36), the assumption $r_{cs}(X) < 6r$ and $\tilde{\theta}(s) \leq \theta(4s) < 1$ in the last inequality. To bound the main term I_1 , we need to estimate $|N(v_{\tilde{X}}, \rho) - c|$ for $\rho \in [r, 6r]$. Again we divide into three cases.

If $p \in \partial D$, then the constant $c = N(v_p, 6r)$ by (9.18). By (8.13) in Proposition 8.10, we have

$$\begin{aligned} |N(v_{\tilde{X}}, \rho) - c| & \leq |N(v_{\tilde{X}}, \rho) - N(v_{\tilde{X}}, 6r)| + |N(v_{\tilde{X}}, 6r) - N(v_p, 6r)| \\ & \leq \tilde{W}_{\tilde{X}}(r) + C(r^{\frac{1}{2}} + \theta(24r)). \end{aligned} \quad (9.40)$$

If $p \in D$ and $r \leq r_{cs}(p)/6$, the constant $c = N(p, 6r)$ by (9.18). By Lemma 8.21, we have

$$\begin{aligned} |N(v_{\tilde{X}}, \rho) - c| & \leq |N(v_{\tilde{X}}, \rho) - N(v_{\tilde{X}}, 6r)| + |N(v_{\tilde{X}}, 6r) - N(p, 6r)| \\ & \leq \tilde{W}_{\tilde{X}}(r) + C(r^{\frac{1}{2}} + \theta(24r)). \end{aligned} \quad (9.41)$$

If $p \in D$ and $r > r_{cs}(p)/6$, we fix the constant c to be $N(v_q, 6r)$, where $q \in \partial D$ is such that $|p - q| = \text{dist}(p, \partial D)$. Hence

$$|q - p| = \text{dist}(p, \partial D) \ll r_{cs}(p) < 6r.$$

In particular by choosing sufficiently small radius we can guarantee that

$$|q - \tilde{X}| \leq |q - p| + |p - \tilde{X}| < \frac{4}{5}r + \frac{6}{5}r = 2r.$$

Hence by (8.13) in Proposition 8.10, we have

$$|N(v_{\tilde{X}}, \rho) - c| \leq |N(v_{\tilde{X}}, \rho) - N(v_{\tilde{X}}, 6r)| + |N(v_{\tilde{X}}, 6r) - N(v_q, 6r)|$$

$$\leq \tilde{W}_{\tilde{X}}(r) + C(r^{\frac{1}{2}} + \theta(24r)). \quad (9.42)$$

Lastly, we claim that

$$\tilde{W}_{\tilde{X}}(r) \leq \tilde{W}_X(r) + \delta_{in}. \quad (9.43)$$

In fact, if $r > r_{cs}(X)$, by the definition (6.23) we have

$$\tilde{W}_{\tilde{X}}(r) = \tilde{N}(v_{\tilde{X}}, 6r) - \tilde{N}(v_{\tilde{X}}, r) = N_X(6r) - N_X(r) = W_X(r); \quad (9.44)$$

if $r_{cs}(X)/6 < r \leq r_{cs}(X)$, by the definition (6.23) and Lemma 6.17

$$\tilde{W}_{\tilde{X}}(r) = N_{\tilde{X}}(6r) - N_{\tilde{X}}(r) \leq N_X(6r) - (N(X, r) - \delta_{in}) = \tilde{W}_X(r) + \delta_{in}. \quad (9.45)$$

Similar to (9.19) in Case (i), we have

$$\begin{aligned} I_1 &= \frac{1}{r^d} \iint_{A_{r,6r}(\tilde{X}) \cap D} |\langle \nabla u(Z), Z - \tilde{X} \rangle - cu(Z)|^2 dZ \\ &\leq (1 + O(\theta(24r))) \frac{1}{r^d} \iint_{A_{r,6r}(0) \cap \Omega_{\tilde{X}}} |\langle \nabla_g v_{\tilde{X}}(Y), Y \rangle - cv_{\tilde{X}}(Y)|^2 dY \\ &\quad + C \frac{\theta(24r)}{r^{d-2}} \iint_{A_{r,6r}(0) \cap \Omega_{\tilde{X}}} |\nabla v_{\tilde{X}}(Y)|^2 dY \\ &\lesssim \frac{1}{r^d} \iint_{A_{r,6r}(0) \cap \Omega_{\tilde{X}}} |\langle \nabla_g v_{\tilde{X}}(Y), Y \rangle - cv_{\tilde{X}}(Y)|^2 dY + \theta(24r). \end{aligned} \quad (9.46)$$

Moreover

$$\begin{aligned} &\frac{1}{r^d} \iint_{A_{r,6r}(0) \cap \Omega_{\tilde{X}}} |\langle \nabla_g v_{\tilde{X}}(Y), Y \rangle - cv_{\tilde{X}}(Y)|^2 dY \\ &\lesssim \int_r^{6r} \frac{1}{\rho^{d-2}} \int_{\partial B_\rho \cap \Omega_{\tilde{X}}} \mu \left| \partial_\rho v_{\tilde{X}} - N(v_{\tilde{X}}, \rho) \frac{v_{\tilde{X}}}{\rho} \right|^2 dV_{\partial B_\rho} d\rho \\ &\quad + \int_r^{6r} \frac{|N(v_{\tilde{X}}, \rho) - c|^2}{\rho^d} \int_{\partial B_\rho \cap \Omega_{\tilde{X}}} \mu v_{\tilde{X}}^2 dV_{\partial B_\rho} d\rho \\ &\lesssim \int_r^{6r} R_h(v_{\tilde{X}}, \rho) \frac{H(v_{\tilde{X}}, \rho)}{\rho^{d-1}} d\rho + \int_r^{6r} |N(v_{\tilde{X}}, \rho) - c|^2 \frac{H(v_{\tilde{X}}, \rho)}{\rho^d} d\rho \\ &\lesssim \tilde{W}_{\tilde{X}}(r) + r + \theta(24r) \\ &\lesssim \tilde{W}_X(r) + r + \theta(24r) + \delta_{in} \cdot \chi_{r_{cs}(X)/6 < r \leq r_{cs}(X)}. \end{aligned} \quad (9.47)$$

where we used (9.15) in the penultimate inequality and (9.43) in the last inequality. Here we include the characteristic function to make sure when we sum them up in dyadic scales of r , the small constant δ_{in} appears at most finitely many times. Finally, combining (9.38), (9.39), (9.46) and (9.47), we obtain

$$\begin{aligned} &\frac{1}{r^d} \iint_{A_{2r,5r}(X) \cap D} |\langle \nabla u(Z), Z - X \rangle - cu(Z)|^2 dZ \\ &\lesssim \tilde{W}_X(r) + r + \theta(24r) + \delta_{in} \cdot \chi_{r_{cs}(X)/6 < r \leq r_{cs}(X)}. \end{aligned} \quad (9.48)$$

Plugging (9.35) (in case (ii)) or (9.48) (in case (iii)) back into (9.17), we get

$$\begin{aligned}
& \frac{\lambda_j}{r^{d-2}} \iint_{A_{3r,4r}(p) \cap D} |\langle \nabla u(Z), w_j \rangle|^2 dZ \\
& \leq \frac{1}{r^k} \int_{B_r(p)} \frac{1}{r^d} \iint_{A_{2r,5r}(X) \cap D} |\langle \nabla u(Z), Z - X \rangle - cu(Z)|^2 dZ d\mu(X) \\
& \lesssim \frac{1}{r^k} \int_{B_r(p)} \left(\tilde{W}_X(r) + \delta_{in} \cdot \chi_{r_{cs}(X)/6 < r \leq r_{cs}(X)} \right) d\mu(X) + \frac{\mu(B_r(p))}{r^k} (r + \theta(24r)). \quad (9.49)
\end{aligned}$$

Step 3. The inequalities (9.29) and (9.49) are helpful in our estimate of the beta number, because of the expression (9.11) and the following lemma.

Lemma 9.50. *Let $R, \Lambda > 0$ be fixed. There exist $C_s > 0$ and $r_s > 0$ such that the following holds for all $0 < r \leq r_s$. For any $(u, D) \in \mathfrak{H}(R, \Lambda)$ and any $p \in C_r(u) \cap \mathcal{N}(u) \cap B_{\frac{R}{10}}(0)$ satisfying $\text{dist}(p, \partial D) \leq \frac{3}{5}R\tilde{\theta}(\frac{3}{5}R)$, assume u is normalized so that*

$$\frac{1}{r^d} \iint_{B_r(p)} u^2 dZ = 1. \quad (9.51)$$

Then we have

$$\frac{1}{r^{d-2}} \iint_{A_{3r,4r}(p) \cap D} \sum_{j=1}^{d-1} |\langle \nabla u(Z), w_j \rangle|^2 dZ \geq C_s$$

for any $(d-1)$ -orthonormal vectors $\{w_1, \dots, w_{d-1}\}$.

Proof. Assume the statement is false. Then there exist sequences $\delta_i, r_i \rightarrow 0$, $(u_i, D_i) \in \mathfrak{H}(R, \Lambda)$, $p_i \in C_{r_i}(u_i) \cap \mathcal{N}(u_i) \cap \overline{D_i}$ and $(d-1)$ -orthonormal vectors $\{\omega_i^1, \dots, \omega_i^{d-1}\}$, such that

$$\frac{1}{r_i^{d-2}} \iint_{A_{3r_i,4r_i}(p_i) \cap D_i} \sum_{j=1}^{d-1} |\langle \nabla u_i(Z), \omega_i^j \rangle|^2 dZ < \delta_i. \quad (9.52)$$

We consider the rescalings $T_{p_i, r_i} u_i$ of u_i 's, see Definition 2.11. Then it follows from (9.52) and the normalization (9.51) that

$$\iint_{A_{3,4}(0) \cap \frac{D_i - p_i}{r_i}} \sum_{j=1}^{d-1} |\langle \nabla T_{p_i, r_i} u_i(Z), w_i^j \rangle|^2 dZ < \delta_i. \quad (9.53)$$

We claim that modulo passing to a subsequence

$T_{p_i, r_i} u_i$ converges uniformly and in $W^{1,2}$ to a harmonic function u_∞ in $D_\infty \cap B_5(0)$.

In fact, if infinitely many p_i 's are boundary points (i.e. $p_i \in \partial D_i$) we appeal to Proposition 5.24, and D_∞ is an (possibly tilted) upper half-space. If infinitely many p_i 's are interior points, then compactness follows from the estimate in Lemma 6.14 and the assumption

$$T_{p_i, r_i} u_i(0) = u(p_i) = 0.$$

Moreover, the limit domain D_∞ is an upper half-space with $0 \in \partial D_\infty$, an upper half-space with $0 \in D_\infty$, or the entire space $\mathbb{R}^d$, depending on

$\frac{\text{dist}(p_i, \partial D_i)}{r_i}$ converges to 0, a positive number, or ∞ , respectively.

On the other hand, since for each i the vectors $\{w_i^1, \dots, w_i^{d-1}\}$ form a $(d-1)$ -orthonormal basis, modulo passing to a subsequence, they converge to a $(d-1)$ -tuple of orthonormal vectors, denoted by $\{w_1, \dots, w_{d-1}\}$.

Therefore passing the inequality (9.53) to the limit, we get

$$\iint_{A_{3,4}(0) \cap D_\infty} \sum_{j=1}^{d-1} |\langle \nabla u_\infty(Z), w_j \rangle|^2 dZ = 0. \quad (9.54)$$

This implies that $\langle \nabla u(Z), w_j \rangle \equiv 0$ in the annulus $A_{3,4}(0)$, for every $j = 1, \dots, d-1$. Since $\langle \nabla u(Z), w_j \rangle$ is also a harmonic function in $D_\infty \cap A_5(0)$, by unique continuation theorem it vanishes everywhere. In other words u_∞ is invariant in the w_j -direction, for every $j = 1, \dots, d-1$. Since u_∞ is a harmonic function in D_∞ with $\iint_{B_1(0)} u_\infty^2 dZ = 1$, it follows that u_∞ is a (non-trivial) linear function. In particular u_∞ is homogeneous with respect to the origin of degree 1. The second case where D_∞ is an upper half-space with $0 \in D_\infty$ is impossible. Indeed u_∞ vanishes on $B_5(0) \cap \partial D_\infty$ and $u_\infty(0) = 0$ at an interior point 0 imply that u_∞ must be a trivial function. In the first case where D_∞ is an upper half-space with $0 \in \partial D_\infty$, by the same computation as in the proof of (7.6) we have $|\nabla u_\infty| \equiv \alpha_d^1$. In the third case where $D_\infty = \mathbb{R}^d$, by $u_\infty(0) = 0$ and symmetry we similarly get $|\nabla u_\infty| = \frac{\alpha_d^1}{\sqrt{2}}$.

On the other hand, recall that $p_i \in C_{r_i}(u_i)$ means that

$$\inf_{B_\beta(0)} |\nabla T_{p_i, r_i} u_i| \leq \alpha_0.$$

By choosing α_0 sufficiently small so that $\alpha_0 < \frac{\alpha_d^1}{\sqrt{2}} < \alpha_d^1$, we get a contradiction. □

Finally, combining (9.17), Lemma 9.50 and (depending on whether $X \in \partial D$ or $X \in D$) (9.29) or (9.49), we obtain

$$\begin{aligned} |\beta_\mu^{d-2}(p, r)|^2 &= \lambda_{d-1} + \lambda_d \\ &\leq \frac{2}{d-1} \frac{1}{C_s} \sum_{j=1}^{d-1} \frac{\lambda_j}{r^{d-2}} \iint_{A_{3r,4r}(p) \cap D} |\langle \nabla u(Z), w_j \rangle|^2 dZ \\ &\lesssim \frac{1}{r^{d-2}} \int_{B_r(p)} \left(\tilde{W}_X(r) + \delta_{in} \cdot \chi_{r_{cs}(X)/6 < r \leq r_{cs}(X)} \right) d\mu(X) + \frac{\mu(B_r(p))}{r^{d-2}} (r + \theta(24r)) \end{aligned}$$

which finishes the proof of Theorem 9.7.

10. COVERING OF THE QUANTITATIVE STRATUM

Lemma 10.1. *Let $R, \Lambda > 0$ and $\epsilon, \rho \in (0, 1/2)$ be fixed. There exist $\delta, r_c > 0$ such that the following holds. For any $(u, D) \in \mathfrak{H}(R, \Lambda)$, any radii r_0, r_* satisfying $0 < r_0 < r_* \leq r_c$, and any subset $\mathcal{S} \subset \tilde{\mathcal{C}}_{r_0}(u) \cap \mathcal{N}(u) \cap B_{r_c \tilde{\theta}(r_c)}(\partial D)$, let $\Lambda_* := \sup_{X \in \mathcal{S} \cap B_{2r_*}(0)} N_X(r_*)$.^{*} There exists a finite covering of $\mathcal{S} \cap B_{r_*}(0)$ such that*

$$\mathcal{S} \cap B_{r_*}(0) \subset \bigcup_{X \in \mathcal{C}} B_{r_X}(X), \quad \text{with } r_X \geq r_0;$$

$$B_{r_X/5}(X) \cap B_{r_{X'}/5}(X') = \emptyset, \quad \text{for every distinct pair } X, X' \in \mathcal{C}; \quad (10.2)$$

and

$$\sum_{X \in \mathcal{C}} r_X^{d-2} \leq C_1 C_V \cdot r_*^{d-2}. \quad (10.3)$$

Here $C_V > 0$ is a dimensional constant, and $C_1 > 0$ depends on the dimension as well as ρ . Moreover, for each center point $X \in \mathcal{C}$, one of the following alternatives holds:

- (i) terminal ball: $r_X = r_0$;
- (ii) small dimension: $r_X > r_0$ and moreover, the set of points

$$F_X = \{Y \in \mathcal{S} \cap B_{2r_X}(X) : N_Y(\rho r_X/10) \geq \Lambda_* - \delta\}$$

is contained in $B_{\rho r_X/5}(V_X)$, where V_X is some $(d-3)$ -dimensional affine subspace;

- (iii) definite frequency drop:

$$N_Y(r_X/10) < \Lambda_* - \delta, \quad \text{for any } Y \in \mathcal{S} \cap B_{2r_X}(X). \quad (10.4)$$

Remark 10.5. For convenience and without loss of generality, we assume that both r_0, r_* are some integer power of ρ , i.e.

$$r_0 = \rho^{j_0}, \quad r_* = \rho^{j_*}, \quad \text{where } j_0 > j_* \text{ are positive integers.}$$

Proof. We choose the parameters in the following order:

- Let $\delta_0 > 0$ be fixed whose value is to be determined later in (10.22).
- Let δ, r_m, β be the values determined in Proposition 7.3 with given values of R, Λ and $\delta_0, \rho/10, \rho/10$ (that is, we take $\rho/10$ in place of ρ and τ in the statement).
- Let $\delta_{in} \in (0, \delta_0/2]$ be fixed whose value is to be determined later in (10.23), and r_{in} as given in Lemma 6.17 with parameters $\rho/10$ and δ_{in} .
- Let $r_b = r_b(\epsilon, \delta_{in})$ be the radius determined in Theorem 9.7.
- Finally, let

$$\tilde{r}_c := \min \left\{ r_m, r_{in}, r_b, \frac{1}{80} R \right\}, \quad (10.6)$$

and we will determine the value of $r_c \leq \tilde{r}_c$ later in (10.21).

^{*}By Lemmas 5.1 and 6.8 we know the bound $\Lambda_* \leq C(\Lambda)$.

By Lemmas 5.1 and 6.8, the assumption $2\tilde{r}_c \leq \frac{R}{40}$ guarantees the uniform upper bound

$$N_{X_0}(\tilde{r}_c) \leq C(\Lambda, R), \quad \text{for any } X_0 \in B_{2\tilde{r}_c}(0) \cap \partial D.$$

Construction of the covering. To start let

$$F_0 := \{X \in \mathcal{S} \cap B_{2r_*}(0) : N_X(\rho r_*/10) \geq \Lambda_* - \delta\}.$$

The case when $F_0 = \emptyset$ is trivial: We just cover $\mathcal{S} \cap B_{r_*}(0)$ by a finite number of balls centered at $X \in \mathcal{S} \cap B_{r_*}(0)$ and with radius $r_X = \rho r_*$, and label the collection of centers as $\mathcal{C}_\emptyset$. Clearly

$$\sum_{X \in \mathcal{C}_\emptyset} r_X^{d-2} = (\rho r_*)^{d-2} \cdot \#\mathcal{C}_\emptyset \leq 10^d \rho^{-2} r_*^{d-2} =: C_1 r_*^{d-2}, \quad (10.7)$$

and for any $Y \in \mathcal{S} \cap B_{2\rho r_*}(X) \subset \mathcal{S} \cap B_{2r_*}(0)$,

$$N_Y(r_X/10) = N_Y(\rho r_*/10) < \Lambda_* - \delta.$$

Now we assume $F_0 \neq \emptyset$. If F_0 is contained in $B_{\rho r_*/5}(V_0) \cap B_{2r_*}(0)$, where V_0 is some $(d-3)$ -dimensional affine subspace, then the trivial cover $B_{r_*}(0)$ satisfies alternative (ii). We say the center $0 \in \mathcal{C}_{sd}$, where the subindex is to indicate the *small dimension* alternative. If not, then $F_0 \rho r_*/5$ -effectively spans a (at least) $(d-2)$ -dimensional affine subspace (denoted by L_0) in $B_{2r_*}(0)$. Then by Proposition 7.3 as well as the inclusion $\tilde{C}_{r_0}(u) \subset C_r(u)$ for any $r_0 \leq r \leq r_c$, we get

$$\mathcal{S} \cap B_{2r_*}(0) \subset B_{2\beta r_*}(L_0), \quad (10.8)$$

and

$$N_X(\rho r_*/10) \geq \Lambda_* - \delta_0 \quad \text{for every } X \in B_{2\beta r_*}(L_0) \cap B_{2r_*}(0) \cap \mathcal{N}(u). \quad (10.9)$$

We cover $\mathcal{S} \cap B_{r_*}(0)$ by a collection of balls centered at $X \in \mathcal{S} \cap B_{r_*}(0)$ with radius $r_X^1 = \rho r_*$ and label the collection of centers $\mathcal{C}_g^{(1)}$:

$$\mathcal{S} \cap B_{r_*}(0) \subset \bigcup_{X \in \mathcal{C}_g^{(1)}} B_{\rho r_*}(X), \quad \text{with } X \in \mathcal{S} \cap B_{r_*}(0); \quad (10.10)$$

and they satisfy

$$B_{\rho r_*/5}(X) \cap B_{\rho r_*/5}(X') = \emptyset, \quad \text{for any distinct } X, X' \in \mathcal{C}_g^{(1)}.$$

By (10.8) and (10.9) each $X \in \mathcal{C}_g^{(1)}$ has small frequency drop, in the sense that

$$N_X(\rho r_*/10) \geq \Lambda_* - \delta_0.$$

Because of this we call these balls the *good balls* of level 1.

Now we state the construction inductively. Let i be an integer with $0 \leq i < j_0 - j_*$ (see Remark 10.5). Let $X \in \mathcal{C}_g^{(i)}$ be fixed, and we consider the *good ball* $B_{r_X^i}(X)$ with $r_X^i = \rho^i r_* > r_0$ and the set

$$F_X^{(i)} := \{Y \in \mathcal{S} \cap B_{2r_X^i}(X) : N_Y(\rho r_X^i/10) \geq \Lambda_* - \delta\}.$$

(We use the convention that $\mathcal{C}_g^{(0)} = \{0\}$ and thus $B_{r_*}(0)$ is the ball in the base case.) Our goal is to cover $\mathcal{S} \cap B_{r_X^i}(X)$.

- $\mathcal{C}_\emptyset$: If the ball $B_{r_X^i}(X)$ satisfies that $F_X^{(i)} = \emptyset$, then we cover $\mathcal{S} \cap B_{r_X^i}(X)$ trivially by finitely many balls $\{B_{r_Y}(Y)\}$ centered at $Y \in \mathcal{S} \cap B_{r_X^i}(X)$ with radius $r_Y = \rho r_X^i = \rho^{i+1} r_*$, and label the collection of centers $\mathcal{C}_\emptyset^{(i+1)}(X)$. These balls satisfy the *definite frequency drop* alternative (iii). Moreover, similarly to (10.7) we have

$$\sum_{Y \in \mathcal{C}_\emptyset^{(i+1)}(X)} (r_Y)^{d-2} \leq C_1 (r_X^i)^{d-2}. \quad (10.11)$$

- $\mathcal{C}_{sd}$: If the ball $B_{r_X^i}(X)$ is such that $F_X^{(i)}$ satisfies the *small dimension* alternative (i.e. $F_X^{(i)}$ is contained in a $\rho r_X^i/5$ -tubular neighborhood of a $(d-3)$ -dimensional affine subspace), then we simply cover $\mathcal{S} \cap B_{r_X^i}(X)$ by $B_{r_X^i}(X)$. In this case we say

$$X \in \mathcal{C}_{sd}^{(i+1)} \quad \text{and} \quad r_X = r_X^i = \rho^i r_*.$$

- $\mathcal{C}_g$: Alternatively, the non-empty set $F_X^{(i)}$ $\rho r_X^i/5$ -effectively span a $(d-2)$ -dimensional affine subspace, denoted by L_X . Then as in (10.10) we cover $\mathcal{S} \cap B_{r_X^i}(X)$ by a collection of balls centered at $Y \in \mathcal{C}_g^{(i+1)}(X)$ with radius $r_Y^{i+1} = \rho r_X^i = \rho^{i+1} r_*$:

$$\mathcal{S} \cap B_{r_X^i}(X) \subset \bigcup_{Y \in \mathcal{C}_g^{(i+1)}(X)} B_{\rho r_X^i}(Y), \quad \text{with } Y \in \mathcal{S} \cap B_{r_X^i}(X); \quad (10.12)$$

and they satisfy

$$B_{r_Y^{i+1}}(Y) \cap B_{r_{Y'}^{i+1}}(Y') = \emptyset, \quad \text{for any distinct } Y, Y' \in \mathcal{C}_g^{(i+1)}(X).$$

Moreover, each $Y \in \mathcal{C}_g^{(i+1)}(X)$ has small frequency drop, in the sense that

$$N_X(\rho r_X^i/10) \geq \Lambda_* - \delta_0.$$

If $r_Y^{i+1} = r_0$, we call these balls the *terminal balls* and denote $Y \in \mathcal{C}_0$ and $r_Y = r_Y^{i+1} = r_0$. Otherwise, for any $Y \in \mathcal{C}_g^{(i+1)}(X)$ we call $B_{r_Y^{i+1}}(Y)$ a *good ball* of level $(i+1)$. We repeat this argument and keep subdividing the *good balls*.

By the above construction, after at most $(j_0 - j_*)$ levels all the remaining balls $B_{r_X}(X)$ fall into one of the three categories:

- $X \in \mathcal{C}_\emptyset^{(i)}$ for some $i \in \{1, \dots, j_0 - j_*\}$, $r_X = \rho^i r_*$ and $B_{r_X}(X)$ satisfies (iii).
- $X \in \mathcal{C}_{sd}^{(i)}$ for some $i \in \{1, \dots, j_0 - j_*\}$, $r_X = \rho^{i-1} r_*$ and $B_{r_X}(X)$ satisfies (ii).
- $X \in \mathcal{C}_0$ with $r_X = r_0$, and thus $B_{r_X}(X)$ satisfies (i).

At each level, by considering all the coverings as a whole, we can guarantee they are almost disjoint (as in (10.2)); by taking $\rho < 2/3$ and removing redundant new balls if necessary, we can guarantee they are also almost disjoint from all the previous balls with centers in $\mathcal{C}_\emptyset$ or $\mathcal{C}_{sd}$. It remains to prove the packing estimate (10.3). We denote

$$\mathcal{C}_\emptyset = \bigcup_{i=1}^{j_0-j_*} \mathcal{C}_\emptyset^{(i)}, \quad \mathcal{C}_{sd} = \bigcup_{i=1}^{j_0-j_*} \mathcal{C}_{sd}^{(i)}, \quad \mathcal{C} = \mathcal{C}_\emptyset \bigcup \mathcal{C}_{sd} \bigcup \mathcal{C}_0,$$

and we denote the intermediate balls

$$\mathcal{C}_g = \bigcup_{i=1}^{j_0-j_*-1} \mathcal{C}_g^{(i)}.$$

Moreover, for every point $Y \in \mathcal{C}_\emptyset$ or $Y \in \mathcal{C}_{sd}$, there must exist a precursor $X \in \mathcal{C}_g^{(i-1)}$ with $i-1 = 0, \dots, j_0-j_*$, such that $Y \in \mathcal{C}_\emptyset^{(i)}(X)$ or $Y \in \mathcal{C}_{sd}^{(i)}(X)$, respectively. In that case we denote $\tilde{r}_X = r_X^i > r_0$. We let $\mathcal{C}_{pre}$ be the collection of all such X 's and let $\mathcal{C}' := \mathcal{C}_{pre} \cup \mathcal{C}_0$. We denote $\tilde{r}_X = r_0$ if $X \in \mathcal{C}_0$. In particular we have the sets $B_{\tilde{r}_X/5}(X)$ with $X \in \mathcal{C}'$ are pairwise disjoint. Moreover, every $X \in \mathcal{C}'$ satisfies

$$N_X(\tilde{r}_X/10) \geq \Lambda_* - \delta_0. \quad (10.13)$$

By the above construction, we have

$$\begin{aligned} \sum_{X \in \mathcal{C}} (r_X)^{d-2} &\leq \sum_{X \in \mathcal{C}_\emptyset} (r_X)^{d-2} + \sum_{X \in \mathcal{C}_{sd}} (r_X)^{d-2} + \sum_{X \in \mathcal{C}_0} (r_X)^{d-2} \\ &\leq \max\{C_1, 1\} \sum_{X \in \mathcal{C}_{pre}} (\tilde{r}_X)^{d-2} + \sum_{X \in \mathcal{C}_0} (\tilde{r}_X)^{d-2} \\ &\leq C_1 \sum_{X \in \mathcal{C}'} (\tilde{r}_X)^{d-2}, \end{aligned}$$

since the constant C_1 defined in (10.7) is greater than 1. Therefore it suffices to show

$$\sum_{X \in \mathcal{C}'} (\tilde{r}_X)^{d-2} \leq C_V \cdot (r_*)^{d-2} \quad (10.14)$$

for some universal constant C_V . This way, we reduce the packing estimates to center points with small frequency drop, which are good points for our purpose.

Packing estimate. We first make the following remark: for any $X \in \mathcal{C}_g^{(i)}$, we have the rough estimate

$$\sum_{Y \in \mathcal{C}_g^{(i+1)}} (r_Y^{i+1})^{d-2} = (\rho r_X^i)^{d-2} \cdot \#\mathcal{C}_g^{(i+1)} \leq (\rho r_X^i)^{d-2} \cdot \left(\frac{10}{\rho}\right)^{d-2} = C_0 (r_X^i)^{d-2}. \quad (10.15)$$

Using this estimate we get

$$\sum_{X \in \mathcal{C}} (r_X)^{d-2} = \sum_{X \in \mathcal{C}_\emptyset} (r_X)^{d-2} + \sum_{X \in \mathcal{C}_{sd}} (r_X)^{d-2} + \sum_{X \in \mathcal{C}_0} (r_X)^{d-2} \lesssim C_0^{j_0-j_*} (r_*)^{d-2},$$

but the constant $C_0^{j_0-j_*}$ grows polynomially with the ratio r_*/r_0 .

Instead, we prove (10.14) inductively. Let $\mathcal{C}'_t = \{X \in \mathcal{C}' : \tilde{r}_X \leq t\}$, where the relevant cases for us are $t = r_0, r_0/\rho, r_0/\rho^2, \dots, r_*/6^*$. Here we denote $\tilde{r}_X = r_0$ if $X \in \mathcal{C}_0$. We also let

$$\mu = \omega_{d-2} \sum_{X \in \mathcal{C}'} (\tilde{r}_X)^{d-2} \delta_X \quad \text{and} \quad \mu_t = \omega_{d-2} \sum_{X \in \mathcal{C}'_t} (\tilde{r}_X)^{d-2} \delta_X \leq \mu_{r_*} = \mu.$$

If $r_/6 \neq r_0/\rho^i$ for any integer $i \in \mathbb{N}$, we just replace it by r_0/ρ^{i_*} where i_* is the largest integer such $r_0/\rho^{i_*} \leq r_*/6$.

We will prove the following estimate inductively on the level of $t = r_0, r_0/\rho, \dots, r_*/6$: there exists a universal constant

$$C'_V = \max\{11^d \omega_{d-2}, 2^{d-2} C_{dr} \omega_{d-2}\} > 0$$

such that

$$\mu_t(B_t(X)) \leq C'_V \cdot t^{d-2}, \quad \text{for every } X \in B_{r_*}(0).$$

Here C_{dr} is the dimensional constant in the Discrete Reifenberg Theorem 9.2. It is easy to see the above statement follows from

$$\mu_t(B_{2t}(X)) \leq C'_V \cdot t^{d-2}, \quad \text{for every } X \in B_{r_*}(0) \cap \text{spt } \mu_t. \quad (10.16)$$

We start from the base case $t = r_0$. In this case $\mathcal{C}'_{r_0}$ consists entirely of points in $\mathcal{C}_0$, and they satisfy

$$B_{r_0/5}(Y) \cap B_{r_0/5}(Y') = \emptyset \quad \text{for every distinct } Y, Y' \in \mathcal{C}_0.$$

Hence

$$\begin{aligned} \mu_{r_0}(B_{2r_0}(X)) &\leq \omega_{d-2}(r_0)^{d-2} \cdot \# \text{ of } \mathcal{C}_0 \text{ inside } B_{2r_0}(X) \\ &\leq 11^d \omega_{d-2}(r_0)^{d-2}. \end{aligned}$$

Now assuming we have proven the claim (10.16) for $t = r_0, \dots, r_0/\rho^i$, and we shall prove it for $t = r_0/\rho^{i+1}$. Let $\tilde{r} = r_0/\rho^{i+1}$. We can split the measure as follows

$$\mu_{\tilde{r}} = \mu_{\rho\tilde{r}} + \tilde{\mu}_{\tilde{r}} := \omega_{d-2} \sum_{X \in \mathcal{C}'_{\rho\tilde{r}}} (\tilde{r}_X)^{d-2} \delta_X + \omega_{d-2} \sum_{X \in \mathcal{C}': \rho\tilde{r} < \tilde{r}_X \leq \tilde{r}} (\tilde{r}_X)^{d-2} \delta_X.$$

We first have a rough estimate

$$\begin{aligned} \mu_{\tilde{r}}(B_{2\tilde{r}}(X)) &= \mu_{\rho\tilde{r}}(B_{2\tilde{r}}(X)) + \tilde{\mu}_{\tilde{r}}(B_{2\tilde{r}}(X)) \\ &\lesssim \rho^{-d} \cdot C'_V \cdot (\rho\tilde{r})^{d-2} + \rho^{-d} \cdot (\tilde{r})^{d-2} \\ &= C_2 C'_V \cdot (\tilde{r})^{d-2}, \end{aligned} \quad (10.17)$$

where the constant C_2 depends on the dimension and the value of ρ . The same rough estimate in fact holds for all $\rho\tilde{r} < t \leq \tilde{r}/\rho$.

Next we use the Discrete Reifenberg Theorem 9.2 to give the desired bound in (10.16). To simplify the notation we let

$$\bar{\mu} := \mu_{\tilde{r}} \llcorner B_{2\tilde{r}}(X), \quad \text{where } X \in \text{spt } \mu_{\tilde{r}} \text{ is fixed.}$$

Now let $q \in \text{spt } \bar{\mu}$ and $0 < s \leq 2\tilde{r}$ be arbitrary, and let $p \in B_s(q) \cap \text{spt } \bar{\mu}$ and $0 < r \leq s$. Since

$$\text{spt } \bar{\mu} \subset \mathcal{C}' \subset \mathcal{S} \subset \mathcal{N}(u),^*$$

by Theorem 9.7 we can bound the β -number as follows

$$\begin{aligned} |\beta_{\bar{\mu}}^{d-2}(p, r)|^2 &\leq C_b \left[\frac{1}{r^{d-2}} \int_{B_r(p)} \left(\tilde{W}_Y(r) + \delta_{in} \cdot \chi_{r_{cs}(Y)/6 < r \leq r_{cs}(Y)} \right) d\bar{\mu}(Y) \right. \\ &\quad \left. + \frac{\bar{\mu}(B_r(p))}{r^{d-2}} (r + \theta(24r)) \right] \end{aligned}$$

*The only exception is that it is possible in our covering argument (at the top level) that $0 \in \text{spt } \bar{\mu}$ even though $0 \notin \mathcal{S}$. But we ignore this case since this can only happen at a single point.

$$\leq C_b \left[\frac{1}{r^{d-2}} \int_{B_r(p)} \left(\tilde{W}_Y(r) + \delta_{in} \cdot \chi_{r_{cs}(Y)/6 < r \leq r_{cs}(Y)} \right) d\bar{\mu}(Y) \right. \\ \left. + C_2 C'_V \cdot (r + \theta(24r)) \right]. \quad (10.18)$$

In the second inequality, we use the rough estimate (10.17) when $\rho\bar{t} < r \leq \bar{2}t$; and when $r \leq \rho\bar{t}$ we use the inductive hypothesis for μ_r , the observations that $\bar{\mu} = \mu_r + \mu \llbracket X \in \mathcal{C}' : r < \tilde{r}_X \leq \bar{t} \rrbracket$ and $r > \tilde{r}_p/5$ (see Remark 9.4).

Next we want to estimate the following integral

$$\int_{B_s(q)} \int_0^s |\beta_{\bar{\mu}}^{d-2}(p, r)|^2 \frac{dr}{r} d\bar{\mu}(p) = \int_{B_s(q)} \int_{\tilde{r}_p/5}^s |\beta_{\bar{\mu}}^{d-2}(p, r)|^2 \frac{dr}{r} d\bar{\mu}(p).$$

Firstly, to estimate the integral of the second term in (10.18) we claim that

$$\int_{B_s(q)} \int_{\tilde{r}_p/5}^s \left(\frac{1}{r^{d-2}} \int_{B_r(p)} \chi_{r_{cs}(Y)/6 < r \leq r_{cs}(Y)} d\bar{\mu}(Y) \right) \frac{dr}{r} d\bar{\mu}(p) \lesssim \bar{\mu}(B_{2s}(q)). \quad (10.19)$$

In fact, since the variables of integration satisfy $Y \in B_r(p), p \in B_s(q)$ and $r \leq s$, it follows that $p \in B_r(Y)$ and $Y \in B_{r+s}(q) \subset B_{2s}(q)$. Hence we may change the order of integrations using Fubini theorem and get

$$\begin{aligned} & \int_{B_s(q)} \int_{\tilde{r}_p/5}^s \left(\frac{1}{r^{d-2}} \int_{B_r(p)} \chi_{r_{cs}(Y)/6 < r \leq r_{cs}(Y)} d\bar{\mu}(Y) \right) \frac{dr}{r} d\bar{\mu}(p) \\ &= \int_{B_{2s}(q)} \int_{\tilde{r}_p/5}^s \left(\frac{1}{r^{d-2}} \chi_{r_{cs}(Y)/6 < r \leq r_{cs}(Y)} \int_{B_r(Y)} d\bar{\mu}(p) \right) \frac{dr}{r} d\bar{\mu}(Y) \\ &= \int_{B_{2s}(q)} \int_{\tilde{r}_p/5}^s \left(\frac{\bar{\mu}(B_r(Y))}{r^{d-2}} \chi_{r_{cs}(Y)/6 < r \leq r_{cs}(Y)} \right) \frac{dr}{r} d\bar{\mu}(Y) \\ &\lesssim \int_{B_{2s}(q)} \int_{\tilde{r}_p/5}^s \chi_{r_{cs}(Y)/6 < r \leq r_{cs}(Y)} \frac{dr}{r} d\bar{\mu}(Y) \\ &\lesssim \bar{\mu}(B_{2s}(q)), \end{aligned}$$

where we use the rough estimate (10.17) and the inductive hypothesis, similar as in (10.18), to get the penultimate inequality. This finishes the proof of the claim (10.19). Therefore

$$\begin{aligned} & \int_{B_s(q)} \int_0^s |\beta_{\bar{\mu}}^{d-2}(p, r)|^2 \frac{dr}{r} d\bar{\mu}(p) \\ &= \int_{B_s(q)} \int_{\tilde{r}_p/5}^s |\beta_{\bar{\mu}}^{d-2}(p, r)|^2 \frac{dr}{r} d\bar{\mu}(p) \\ &\lesssim \int_{B_s(q)} \int_{\tilde{r}_p/5}^s \frac{1}{r^{d-2}} \int_{B_r(p)} \left(\tilde{W}_Y(r) + \delta_{in} \cdot \chi_{r_{cs}(Y)/6 < r \leq r_{cs}(Y)} \right) d\bar{\mu}(Y) \frac{dr}{r} d\bar{\mu}(p) \\ &\quad + s \bar{\mu}(B_s(q)) + \int_{B_s(q)} \int_0^s \theta(24r) \frac{dr}{r} d\bar{\mu}(p) \\ &\lesssim \int_{B_s(q)} \int_{\tilde{r}_p/5}^s \frac{1}{r^{d-2}} \int_{B_r(p)} \tilde{W}_Y(r) d\bar{\mu}(Y) \frac{dr}{r} d\bar{\mu}(p) \end{aligned}$$

$$\begin{aligned}
& + \int_{B_s(q)} \int_0^s \theta(24r) \frac{dr}{r} d\bar{\mu}(p) + s \bar{\mu}(B_s(q)) + \delta_{in} \bar{\mu}(B_{2s}(q)) \\
& =: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

By Fubini theorem, we get

$$I_1 \lesssim \int_{B_{2s}(q)} \int_0^s \left(\frac{1}{r^{d-2}} \int_{B_r(Y)} \chi_{\tilde{r}_p/5 \leq r} d\bar{\mu}(p) \right) \tilde{W}_Y(r) \frac{dr}{r} d\bar{\mu}(Y).$$

Let $Y \in B_{2s}(q) \cap \text{spt } \bar{\mu}$ be fixed. If the radius $r < \tilde{r}_Y/5$, then by almost-disjointness (as in (10.2)) we have

$$B_r(Y) \cap \text{spt } \bar{\mu} = \{Y\},$$

and thus

$$r \geq \tilde{r}_p/5 = \tilde{r}_Y/5.$$

This contradicts the assumption $r < \tilde{r}_Y/5$. Therefore we always have $r \geq \tilde{r}_Y/5$. It follows that

$$\begin{aligned}
I_1 & \lesssim \int_{B_{2s}(q)} \int_{\tilde{r}_Y/5}^s \left(\frac{1}{r^{d-2}} \int_{B_r(Y)} d\bar{\mu}(p) \right) \tilde{W}_Y(r) \frac{dr}{r} d\bar{\mu}(Y) \\
& \lesssim \int_{B_{2s}(q)} \int_{\tilde{r}_Y/5}^s \left(\frac{\bar{\mu}(B_r(Y))}{r^{d-2}} \right) \tilde{W}_Y(r) \frac{dr}{r} d\bar{\mu}(Y) \\
& \lesssim \int_{B_{2s}(q)} \int_{\tilde{r}_Y/5}^s \tilde{W}_Y(r) \frac{dr}{r} d\bar{\mu}(Y) \quad \text{by the rough estimate (10.17)} \\
& \lesssim \int_{B_{2s}(q)} N_Y(6s) - N_Y(\tilde{r}_Y/10) d\bar{\mu}(Y) \quad \text{by telescoping on } \tilde{W}_Y(r) \\
& \lesssim \delta_0(2s)^{d-2} \quad \text{by } 6s \leq r_*, (10.13), (10.17) \text{ and a simple covering} \\
& =: C_4 \delta_0 s^{d-2}.
\end{aligned}$$

Besides we have

$$I_2 \lesssim \left(\int_0^{24s} \frac{\theta(r)}{r} dr \right) \bar{\mu}(B_s(q)) =: C_5 \left(\int_0^{24s} \frac{\theta(r)}{r} dr \right) s^{d-2}.$$

Clearly by the rough estimate (10.17) and $s \leq r_*/6$, we have

$$I_3 \lesssim r_* s^{d-2} =: C_6 r_* s^{d-2}, \quad I_4 \lesssim \delta_{in}(2s)^{d-2} =: C_7 \delta_{in} s^{d-2}.$$

Let η_{dr} be the constant in Theorem 9.2. Let

$$\tau_c := \sup \left\{ \tau > 0 : \int_0^{24\tau} \frac{\theta(r)}{r} dr \leq \eta_{dr}/5C_5 \right\}, \quad (10.20)$$

and we choose

$$r_c = \min \left\{ \tilde{r}_c, 4\tau_c, \frac{\eta_{dr}}{5C_6} \right\}. \quad (10.21)$$

We also choose δ_0 (depending on $d, \Lambda, \epsilon, \rho$) so that

$$\delta_0 = \min \left\{ \frac{\eta_{dr}}{5C_3}, \frac{\eta_{dr}}{5C_4} \right\} \quad (10.22)$$

and choose

$$\delta_{in} = \min \left\{ \frac{\delta_0}{2}, \frac{\eta_{dr}}{5C_7} \right\}. \quad (10.23)$$

This way, we guarantee

$$I_j \leq \frac{\eta_{dr}}{5} s^{d-2}, \quad \text{for every } j = 1, \dots, 5.$$

Therefore

$$\int_{B_s(q)} \int_0^s |\beta_{\bar{\mu}}^{d-2}(p, r)|^2 \frac{dr}{r} d\bar{\mu}(p) \leq \eta_{dr} s^{d-2}.$$

Therefore by applying the Discrete Reifenberg Theorem 9.2 to the ball $B_{2\bar{r}}(X)$ and the Radon measure $\bar{\mu}$, we obtain

$$\mu_{\bar{r}}(B_{2\bar{r}}(X)) = \omega_{d-2} \sum_{Y \in \text{spt } \bar{\mu}} (\tilde{r}_Y)^{d-2} \leq \omega_{d-2} C_{dr}(2\bar{r})^{d-2} \leq C'_V \cdot (\bar{r})^{d-2}.$$

This finishes the proof of the claim (10.16). This in turn finishes the proof of the packing estimate (10.14) with $C_V = c'_d C'_V$, where c'_d is some dimensional constant to account for a covering of $B_{2r_*}(0)$ by almost-disjoint balls of radius $r_*/6$.

□

By Lemma 10.1 and a purely geometric argument, we get

Lemma 10.24. *Let $R, \Lambda, \epsilon > 0$ be fixed. There exist $\delta, r_c > 0$ such that the following holds. For any $(u, D) \in \mathfrak{H}(R, \Lambda)$, any radii r_0, r_* satisfying $0 < r_0 < r_* \leq r_c$, and any subset $\mathcal{S} \subset \tilde{C}_{r_0}(u) \cap \mathcal{N}(u) \cap B_{r_c \tilde{\theta}(r_c)}(\partial D)$, let $\Lambda_* := \sup_{X \in \mathcal{S} \cap B_{2r_*}(0)} N_X(r_*)$. There exists a finite covering of $\mathcal{S} \cap B_{r_*}(0)$ such that*

$$\mathcal{S} \cap B_{r_*}(0) \subset \bigcup_{X \in \mathcal{C}} B_{r_X}(X), \quad \text{with } r_X \geq r_0;$$

$$B_{r_X/5}(X) \cap B_{r_{X'}/5}(X') = \emptyset, \quad \text{for every } X, X' \in \mathcal{C} \text{ distinct}; \quad (10.25)$$

$$\sum_{X \in \mathcal{C}} r_X^{d-2} \leq C_1^2 C_V \cdot r_*^{d-2}. \quad (10.26)$$

Here both $C_1, C_V > 0$ are dimensional constants. Moreover, for each center point $X \in \mathcal{C}$, one of the following holds:

- (i) terminal ball: $r_X = r_0$;
- (ii) definite frequency drop: $r_X > r_0$, and $N_Y(r_X/10) < \Lambda_* - \delta$ for any $Y \in \mathcal{S} \cap B_{2r_X}(X)$.

Proof. Since the proof is a standard argument, we only sketch the idea here and in particular, state how we choose the value ρ . It is clear that we only need to improve on balls $B_{r_X}(X)$ which fall the alternative (ii) of Lemma 10.1. Firstly, we cover $\mathcal{S} \cap B_{r_X}(X) \setminus B_{\rho r_X}(F_X)$ by balls of radius ρr_X , then each ball falls into alternative (iii) in Lemma 10.1, i.e. we have a δ -frequency drop. Secondly we cover the set $\mathcal{S} \cap B_{r_X}(X) \cap B_{\rho r_X}(F_X)$ by balls of radius $\hat{r}_Y := \rho r_X$:

$$\mathcal{S} \cap B_{r_X}(X) \cap B_{\rho r_X}(F_X) \subset \bigcup_{Y \in \mathcal{C}_s(X)} B_{\hat{r}_Y}(Y), \quad \text{with } Y \in \mathcal{S} \cap B_{r_X}(X) \cap B_{\rho r_X}(F_X) \subset \mathcal{S} \cap B_{\frac{6\rho r_X}{5}}(V_X)$$

where V_X is a $(d-3)$ -dimensional affine subspace. Moreover

$$B_{\hat{r}_Y/5}(Y) \cap B_{\hat{r}_{Y'}/5}(Y') = \emptyset, \quad \text{for every } Y, Y' \in \mathcal{C}_s(X) \text{ distinct.} \quad (10.27)$$

Hence

$$\#\mathcal{C}_s(X) \leq C_d \frac{\left(\frac{6\rho r_X}{5}\right)^3 \cdot (r_X)^{d-3}}{\left(\frac{\rho r_X}{5}\right)^d} \leq C'_d \cdot \rho^{3-d}.$$

Therefore

$$\sum_{Y \in \mathcal{C}_s(X)} (\hat{r}_Y)^{d-2} = (\rho r_X)^{d-2} \cdot \#\mathcal{C}_s(X) \leq C'_d \cdot \rho (r_X)^{d-2}.$$

By choosing ρ so that

$$C'_d \cdot \rho < 1, \quad (10.28)$$

this subdivision do not increase the total packing measure, and we repeat this argument until all the balls are either terminal (i.e. it has radius r_0) or have definite frequency drop. Eventually, we get the packing estimate (10.26) with constant $C_1 \cdot C_1 C_V = C_1^2 C_V$, where C_V is the constant in Lemma 10.1 and $C_1 = 10^d \rho^{-2}$ for the ρ already chosen. $\square$

We summarize the order in which we choose the constants. We always fix the dimension d , the constants $R, \Lambda > 0$ and $\epsilon > 0$.

- We determine the value of the purely dimensional constant C_V in Lemma 10.1, and the value C_b in Theorem 9.7 (recall that its value is independent of δ_{in});
- we fix the value of ρ to satisfy (10.28) in Lemma 10.24;
- with ρ and C_b fixed, we determine the values of C_1 and $\tau_c, \delta_0, \delta_{in}$ in Lemma 10.1, by (10.7), (10.20), (10.22) and (10.23), respectively;
- with ρ and δ_0 fixed, we determine the values of r_m, δ and β in Proposition 7.3 (using $\tau = \rho/10$);
- with ρ and δ_{in} fixed, we determine the value of r_{in} in Lemma 6.17, and determine the value of r_b in Theorem 9.7;
- with β fixed, by the proof of (7.6) as well as Lemma 9.50, we determine the value of α_0 in the definition of $C_r(u)$ in (2.14);
- finally, with r_m, r_{in}, r_b, τ_c chosen, we determine the value of r_c by (10.6) and (10.21).

11. PROOF OF THEOREM 1.1

The proof of the main theorem 1.1 follows easily from the covering lemma (Lemma 10.24) in Section 10. Let $\mathcal{S} := \tilde{C}_{r_0}(u) \cap \mathcal{N}(u) \cap B_{r_c \tilde{\theta}(r_c)}(\partial D)$, $X_0 \in \overline{D} \cap B_{\frac{R}{10}}(0)$, and

$$\Lambda_* := \sup_{X \in \mathcal{S} \cap B_{2r_*}(X_0)} N_X(r_*).$$

By Lemma 10.24, we can cover $\mathcal{S} \cap B_{r_*}(X_0)$ by a collection of balls $\{B_{r_X}(X)\}_{X \in \mathcal{C}^{(1)}}$, such that

- $X \in \mathcal{S} \cap B_{r_*}(X_0)$ and $B_{r_X/5}(X) \cap B_{r_{X'}/5}(X') = \emptyset$ for every $X, X' \in \mathcal{C}^{(1)}$;
- either $r_X = r_0$, or $r_X \in (r_0, r_*]$ and

$$N_Y(r_X/10) < \Lambda_* - \delta, \quad \text{for every } Y \in \mathcal{S} \cap B_{2r_X}(X); \quad (11.1)$$

• and

$$\sum_{X \in C^{(1)}} r_X^{d-2} \leq C_1^2 C_V \cdot r_*^{d-2}. \quad (11.2)$$

For each ball $B_{r_X}(X)$ in the covering such that $r_X > r_0$, we first cover $\mathcal{S} \cap B_{r_X}(X)$ by finitely many balls centered at $Y \in \mathcal{S} \cap B_{r_X}(X)$ with radius $r_Y = r_X/10$, and label the collection of centers $C_0^{(1)}(X)$. Clearly

$$\sum_{Y \in C_0^{(1)}(X)} r_Y^{d-2} \leq \#(C(X)) \cdot \left(\frac{r_X}{10}\right)^{d-2} \leq C_d r_X^{d-2}. \quad (11.3)$$

For each $Z \in \mathcal{S} \cap B_{2r_Y}(Y) \subset \mathcal{S} \cap B_{2r_X}(X)$, by (11.1) we have

$$N_Z(r_Y) = N_Z(r_X/10) < \Lambda_* - \delta.$$

Hence

$$\sup_{Z \in \mathcal{S} \cap B_{2r_Y}(Y)} N_Z(r_Y) \leq \Lambda_* - \delta.$$

We apply Lemma 10.24 again, with $\Lambda_* - \delta$ in place of Λ_* , to find a covering of the ball $B_{r_Y}(Y)$

$$\mathcal{S} \cap B_{r_Y}(Y) \subset \bigcup_{Z \in C^{(2)}(Y)} B_{r_Z}(Z),$$

The covering satisfies either $r_Z = r_0$, or $r_Z > r_0$ and

$$N_W(r_Z/10) < (\Lambda_* - \delta) - \delta, \quad \text{for every } W \in \mathcal{S} \cap B_{2r_Z}(Z); \quad (11.4)$$

and we have the packing estimate

$$\sum_{Z \in C^{(2)}(Y)} r_Z^{d-2} \leq C_1^2 C_V \cdot r_Y^{d-2}. \quad (11.5)$$

Denote

$$C^{(2)} := \left\{ Z \in C^{(2)}(Y) : Y \in C_0^{(1)}(X) \text{ and } X \in C^{(1)} \right\}.$$

Combining (11.5), (11.3), and (11.2), we get

$$\begin{aligned} \sum_{Z \in C^{(2)}} r_Z^{d-2} &\leq \sum_{X \in C^{(1)}} \sum_{Y \in C_0^{(1)}(X)} \sum_{Z \in C^{(2)}(Y)} r_Z^{d-2} \leq C_1^2 C_V \cdot \sum_{X \in C^{(1)}} \sum_{Y \in C_0^{(1)}(X)} r_Y^{d-2} \\ &\leq C_d C_1^2 C_V \cdot \sum_{X \in C^{(1)}} r_X^{d-2} \\ &\leq C_d (C_1^2 C_V)^2 r_*^{d-2}. \end{aligned}$$

For each $j = 1, \dots, j_* := \left\lceil \frac{\Lambda_*}{\delta} \right\rceil - 1$, we apply the above argument inductively to balls $B_{r_Z}(Z)$ with $r_Z > r_0$, with uniform frequency bound $\Lambda_* - j\delta$. This induction stops after j_* many steps, since for $j = j_* + 1$ the alternative (ii) of Lemma 10.24 can not happen because

$$\Lambda_* - (j_* + 1)\delta = \Lambda_* - \left\lceil \frac{\Lambda_*}{\delta} \right\rceil \delta \leq 0.$$

Moreover, for each j we have the packing estimate

$$\sum_{Z \in C^{(j+1)}} r_Z^{d-2} \leq C_d^j (C_1^2 C_V)^{j+1} r_*^{d-2}.$$

Finally let

$$C := \bigcup_{j=1}^{j_*} C^{(j)},$$

and we have the packing estimate

$$\begin{aligned} \sum_{Z \in C} r_Z^{d-2} &\leq \sum_{j=0}^{j_*} \sum_{Z \in C^{(j+1)}} r_Z^{d-2} \leq \sum_{j=0}^{j_*} C_d^j (C_1^2 C_V)^{j+1} r_*^{d-2} \\ &\leq C_d^{\left\lceil \frac{\Lambda_*}{\delta} \right\rceil} (C_1^2 C_V)^{\left\lceil \frac{\Lambda_*}{\delta} \right\rceil + 1} r_*^{d-2} \\ &\leq: C_p r_*^{d-2}, \end{aligned}$$

where the constant C_p only depends on d, Λ (since $\Lambda_* \leq C(\Lambda)$) and δ (which in turn depends on R, Λ and ϵ). We remark that since we do not know how δ depends on Λ , we can not say that C_p depends on Λ exponentially.

Since all balls in the covering have the same radius r_0 , the above packing estimate implies that

$$\#(C) r_0^{d-2} \leq \sum_{Z \in C} r_Z^{d-2} \leq C_p r_*^{d-2},$$

and thus

$$\#(C) \leq C_p \left(\frac{r_*}{r_0} \right)^{d-2}.$$

Therefore for any $X_0 \in \overline{D} \cap B_{\frac{R}{10}}(0)$ and any $r_0 < r_* \leq r_c$, we have the volume estimate

$$\left| B_{r_0} \left(\tilde{C}_{r_0}(u) \cap \mathcal{N}(u) \cap B_{r_c \tilde{\theta}(r_c)}(\partial D) \cap B_{r_*}(X_0) \right) \right| \leq \#(C) \cdot r_0^d \leq C_p r_*^{d-2} r_0^2.$$

On the other hand, for any interior point X such that $\text{dist}(X, \partial D) \geq r_c \tilde{\theta}(r_c)$, its critical scale $r_{cs}(X) \geq r_c$. In other words, the frequency function $N(X, \cdot)$ is monotone increasing on the interval $[0, r_c]$. Therefore the same proof applies to this much simpler case. (This case is similar to that of [NV0]: Even though the balls in the covering may still intersect the boundary, they are nonetheless well in the monotonic interval and should be considered *interior balls* in the application of the argument.) Finally we conclude that

$$\left| B_{r_0} \left(\tilde{C}_{r_0}(u) \cap \mathcal{N}(u) \cap B_{r_*}(X_0) \right) \right| \leq C'_p r_*^{d-2} r_0^2.$$

In particular, by the definition of $(d-2)$ -dimensional Minkowski content, we have

$$\mathcal{M}^{d-2,*} \left(\tilde{C}_{r_0}(u) \cap \mathcal{N}(u) \cap B_{r_*}(X_0) \right) \leq C r_*^{d-2},$$

and

$$\mathcal{M}^{d-2,*} \left(\tilde{C}_{r_0}(u) \cap \mathcal{N}(u) \cap B_{\frac{R}{10}}(0) \right) \leq C R^d r_c^{-2} = C(d, R, \Lambda, \epsilon).$$

Since the above bound is independent of r_0 , we conclude that the singular set satisfies the Minkowski bound

$$\mathcal{M}^{d-2,*} \left(\mathcal{S}(u) \cap B_{\frac{R}{10}}(0) \right) \leq \mathcal{M}^{d-2,*} \left(\tilde{C}_{r_0}(u) \cap \mathcal{N}(u) \cap B_{\frac{R}{10}}(0) \right) \leq C(d, R, \Lambda, \epsilon).$$

To prove the rectifiability, we apply a similar argument as in the proof of the packing estimate in Lemma 10.1. But this time we appeal to Theorem 9.5, in place of Theorem 9.2, to the $(d-2)$ -measurable set $\tilde{C}_{r_0}(u) \cap \mathcal{N}(u) \cap B_{r_c \tilde{\theta}(r_c)}(\partial D) \cap B_{r_*}(X_0)$. The rectifiability of $\mathcal{S}(u)$ follows from (2.16) and the countable additivity of rectifiable sets.

REFERENCES

- [AE] V. Adolfsson and L. Escauriaza, *$C^{1,\alpha}$ domains and unique continuation at the boundary*. Commun. Pur. Appl. Math. Volume 50 Issue 10 (1997), 935-969. 3, 4, 11, 18, 19
- [AEK] V. Adolfsson, L. Escauriaza and C.E. Kenig, *Convex domains and unique continuation at the boundary*. Rev. Mat. Iberoamericana 11 (1995), 519-525. 3
- [AKS] N. Aronszajn, A. Krzywicki and J. Szarski, *A unique continuation theorem for the exterior differential forms on Riemannian manifolds*. Ark. Mat. 4 (1962), 417-453. 3, 11
- [BW] J. Bourgain and T. Wolff, *A remark on gradients of harmonic functions in dimension ≥ 3* . Colloq. Math. 60-61 (1990), 253-260. 3
- [CK] L. A. Caffarelli and C. E. Kenig, *Gradient estimates for variable coefficient parabolic equations and singular perturbation problems*. Am. J. Math. Volume 120, Number 2 (1998), 391-439. 25, 26, 35
- [CN] J. Cheeger and A. Naber, *Regularity of Einstein manifolds and the codimension 4 conjecture*. Ann. Math. Volume 182, Issue 3 (2015), 1093-1165. 4
- [CNV] J. Cheeger, A. Naber, and D. Valtorta, *Critical sets of elliptic equations*. Commun. Pur. Appl. Math. 68 (2015) no. 2, 0173-0209.
- [DMSV] C. De Lellis, A. Marchese, E. Spadaro, and D. Valtorta, *Rectifiability and upper Minkowski bounds for singularities of harmonic Q -valued maps*. Comment. Math. Helv. 93 (2018) no. 4, 737-779. 5
- [DT] G. David and T. Toro, *Reifenberg parameterizations for sets with holes*. Mem. Amer. Math. Soc. 215 (2012), vi+102. 56
- [DEK] H. Dong, L. Escauriaza, and S. Kim, *On C^1 , C^2 , and weak type-(1, 1) estimates for linear elliptic operators: part II*. Math. Ann. 370 (2018), 447-489. 7
- [EE] M. Engelstein and N. Edelen, *Quantitative stratification for some free-boundary problems*. Trans. Amer. Math. Soc. 371 (2019), 2043-2072. 4
- [ENV] N.S. Edelen, A. Naber and D. Valtorta, *Quantitative Reifenberg Theorem for Measures*. arXiv:1612.08052 56
- [FS] M. Focardi and E. Spadaro, *On the measure and the structure of the free boundary of the lower dimensional obstacle problem*. Arch. Rational Mech. Anal. 230 (2018), 125-184. 4, 5
- [GL] N. Garofalo and F. Lin, *Monotonicity properties of variational integrals. A_p weights and unique continuation*. Indiana Univ. Math. J. 35 (1986), 245-268. 3, 11
- [Han] Q. Han, *Singular sets of solutions to elliptic equations*. Indiana Univ. Math. J. 43 (1994), 983-1002. 4, 8
- [Kaz] J. Kazdan, *Unique continuation in geometry*. Comm. Pure Appl. Math. 41 (1988), no. 5, 667-681. 3
- [Ken] C.E. Kenig, *Lecture Notes from Spring School 94': Potential Theory and Analysis*. Paskey nad Jizerou (Czech Republic). Lectures by Carlos E. Kenig. Notes by Michael Korey. 3
- [KW] C.E. Kenig and W. Wang, *A note on boundary unique continuation for harmonic functions in non-smooth domains*. Potential Anal. 8 (1998), 143-147. 3
- [KN] I. Kukavika and K. Nyström, *Unique continuation on the boundary for Dini domains*. Proc. Amer. Math. Soc. 126 (1998), 441-446. 3

- [Lin] F. Lin, *Nodal sets of solutions of elliptic and parabolic equations*. Commun. Pur. Appl. Math. **45** (1991), 287-308. [3](#), [4](#)
- [Log1] A. Logunov, *Nodal sets of Laplace eigenfunctions: polynomial upper estimates of the Hausdorff measure*. Ann. Math. **187** (2018), 221-239. [3](#)
- [Log2] A. Logunov, *Nodal sets of Laplace eigenfunctions: proof of Nadirashvili's conjecture and of the lower bound in Yau's conjecture*. Ann. Math. **187** (2018), 241-262. [3](#)
- [LM] A. Logunov and E. Malinnikova, *Nodal sets of Laplace eigenfunctions: estimates of the Hausdorff measure in dimension 2 and 3*. Oper. Theory Adv. Appl. **261** (2018), 333-344. [3](#)
- [Mat] P. Mattila, *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability*. Cambridge Studies in Advanced Mathematics **44**, Cambridge University Press, Cambridge (1995). [7](#)
- [McC] S. McCurdy, *Unique continuation on convex domains*. arXiv:1907.02640. [4](#)
- [NV0] A. Naber and D. Valtorta, *Volume estimates on the critical sets of solutions to elliptic PDEs*. Commun. Pur. Appl. Math. Volume 50 Issue 10 (2017), 1835-1897. [4](#), [77](#)
- [NV1] A. Naber and D. Valtorta, *Rectifiable-Reifenberg and the regularity of stationary and minimizing harmonic maps*. Ann. Math. (2) **185** (2017), 131-227. [4](#), [56](#)
- [NV2] A. Naber and D. Valtorta, *The singular structure and regularity of stationary and minimizing varifolds*. arXiv:1505.03428. [4](#)
- [NV3] A. Naber and D. Valtorta, *Stratification for the singular set of approximate harmonic maps*. Math. Z. **290** (2018), 1415-1455. [4](#), [56](#)
- [SW] E. J. P. G. Schmidt and N. Weck, *On the boundary behavior of solutions to elliptic and parabolic equations - with applications to boundary control for parabolic equations*. SIAM J. Control Optim. **16**(4) (1978), 593-598. [2](#), [3](#)
- [Tol] X. Tolsa, *Unique continuation at the boundary for harmonic functions in C^1 domains and Lipschitz domains with small constant*. arXiv:2004.10721. [3](#), [5](#), [56](#)
- [Tor] T. Toro, *Geometric conditions and existence of bi-Lipschitz parameterizations*, Duke Math. J. **77** (1995), 193-227.
- [Wec] N. Weck, *Über das Prinzip der eindeutigen Fortsetzbarkeit in der Kontrolltheorie*. Optimization and Optimal Control, Lecture Notes in Mathematics **477**, Springer-Verlag, Berlin, 1975, 276-284. [2](#), [3](#)
- [Wol] T. Wolff, *Counterexamples with harmonic gradient in $\mathbb{R}^d$* . Princeton Math Ser. **42**, Princeton University Press (1995), 321-384. [3](#)

CARLOS KENIG, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, IL 60637, USA

Email address: cek@math.uchicago.edu

ZIHUI ZHAO, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, IL 60637, USA

Email address: zhaozh@uchicago.edu