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Odd-Dimensional GKM-Manifolds of Non-Negative Curvature

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Let M be a closed, odd GKM_3 manifold of non-negative sectional curvature. We show that in this situation one can associate an ordinary abstract GKM_3 graph to M and prove that if this graph is orientable, then both the equivariant and the ordinary rational cohomology of M split off the cohomology of an odd-dimensional sphere.

1. Introduction

A long-standing problem in Riemannian geometry is the classification of positively and non-negatively curved manifolds. One characteristic shared by many of the known examples is a high degree of symmetry. The Grove Symmetry Program suggests we attempt the classification of such manifolds with the additional hypothesis of "large" symmetries. The eventual goal of this program is to be able to eliminate the hypothesis of symmetries entirely.

A natural first step is to consider the case of abelian symmetries. For the case of positive curvature, results due to Grove and Searle [20], Rong [32] and Fang and Rong [10], and Wilking [37] give us a classification up to diffeomorphism, homeomorphism, or rational homotopy equivalence for a T^k-action, provided k equals $\lfloor (n+1)/2 \rfloor$, $\lfloor (n-1)/2 \rfloor$, or is greater than or equal to $\lfloor (n+4)/4 \rfloor$, respectively. For non-negative curvature, an

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equivariant diffeomorphism classification for dimensions less than or equal to nine for T^k-actions with $k = \lfloor 2n/3 \rfloor$ follows from the work of Galaz-García and Searle [13], Galaz-García and Kerin [12], and Escher and Searle [9]. An equivariant diffeomorphism classification for dimensions less than or equal to 6 for T^k-actions with $k = \lfloor 2n/3 \rfloor - 1$ follows from the work of Kleiner [27], Searle and Yang [33], Galaz-García [11], Galaz-García and Kerin [12], Grove and Wilking [21], Galaz-García and Searle [14], and Escher and Searle [8]. Note that all of these results rely heavily on the existence of fixed point sets of "small" codimension.

From this point of view, one may consider GKM_k manifolds as occupying the other end of the spectrum. Note that these are $2n$-dimensional manifolds with a torus action of rank $\leq n$. A consequence of the GKM_k condition is that for a T^m-action on M^{2n}, for all $l \leq \min(k, m) - 1$, there exist codimension l torus subgroups of T^m fixing $2l$-dimensional submanifolds of M, which we denote by N^{2l}. Furthermore, with the induced T^l-action, each N^{2l} is a torus manifold, that is an orientable, even-dimensional manifold such that T^l has non-empty fixed point set. GKM_k manifolds of both positive and non-negative curvature were studied by Goertsches and Wiemeler in [17, 18], respectively, where they showed that the GKM_3, respectively GKM_4, condition allows them to classify such manifolds up to real, respectively, rational cohomology type. The notion of GKM manifold was extended to odd dimensions by He [26]. We call such manifolds *odd GKM manifolds*. Inspired by the work of [17, 18] and [26], we consider the case of odd GKM_3 manifolds of positive and non-negative curvature.

Our main result concerns odd GKM_3 manifolds of non-negative sectional curvature. Throughout this article, we only consider cohomology with rational coefficients, and note that an abstract GKM_3 graph is *orientable* provided its top cohomology class is non-trivial.

Main Theorem 1.1. *Let M^{2n+1} be a closed, non-negatively curved odd GKM_3 manifold, $\bar{\Gamma}_M$ the GKM_3 graph of M, and k the number of floating edges at a vertex of $\bar{\Gamma}_M$. Suppose that (Γ, α, ∇), the abstract, even-dimensional GKM_3 graph obtained from $\bar{\Gamma}_M$, is orientable. Then $H^*(M^{2n+1})$ splits off the cohomology ring of an odd dimensional sphere, that is,*

$$H^*(M) \cong H^*(\Gamma, \alpha, \nabla) \otimes H^*(S^{2k+1}).$$

In the process of proving Theorem 1.1, we also obtain a similar result for the equivariant cohomology of M, see Theorem 5.4. For the definition of floating edges of a GKM graph of an odd-dimensional GKM manifold, see Definition 2.32.

In Theorem 1.1, the abstract, even-dimensional GKM_3 graph, Γ, obtained from the odd GKM_3 graph associated to M^{2n+1}, has two-dimensional faces that contain at most four vertices. The rational cohomology of a GKM_k manifold is completely determined by its corresponding vertex-edge graph, Γ. Unfortunately, there is no general classification of even-dimensional GKM_3 graphs whose two-dimensional faces contain at most four vertices. That is, it is as yet unknown whether every such GKM_3 graph corresponds to a closed, non-negatively curved GKM_3 manifold. However, if there are no quadrangles as two-dimensional faces of Γ, that is, if no two-dimensional face in the odd-dimensional GKM graph of M is of the form (4) in Theorem 4.3, then by the main result of [17], $H^*(\Gamma, \alpha, \nabla)$ is isomorphic to the real cohomology ring of a compact rank one symmetric space (CROSS). Since the result in [17] was obtained via a classification of all possible GKM_3 graphs and by applying the GKM theorem, the result also holds for rational coefficients. We obtain the following theorem.

Theorem 1.2. *Let M^{2n+1} be a closed, non-negatively curved, odd GKM_3 manifold. Suppose that the two-dimensional faces in $\bar{\Gamma}_M$, the odd-dimensional GKM graph of M, are not of the form* (4) *in Theorem* 4.3 *and that* (Γ, α, ∇), *the abstract, even-dimensional GKM_3 graph obtained from $\bar{\Gamma}_M$, is orientable. Then $H^*(M^{2n+1})$ is the tensor product of the rational cohomology ring of an odd dimensional sphere, and a simply-connected CROSS, that is,*

$$H^*(M) \cong H^*(N^{2n-2k}) \otimes H^*(S^{2k+1}),$$

where N^{2n-2k} is a simply-connected CROSS.

Using Theorem 1.4 of [18], we see that if we assume that our manifold is GKM_4, then the cohomology ring of the manifold splits as that of an odd-dimensional sphere and a finite quotient of a non-negatively curved torus manifold.

Theorem 1.3. *Let M^{2n+1} be a closed, non-negatively curved odd GKM_4 manifold. Suppose that* (Γ, α, ∇), *the abstract, even-dimensional GKM_4 graph obtained from $\bar{\Gamma}_M$, the odd-dimensional GKM graph of M, is orientable. Then $H^*(M^{2n+1})$ is the tensor product of the cohomology ring of an odd dimensional sphere, and the cohomology ring of a (quotient of) a torus manifold, that is,*

$$H^*(M) \cong H^*(N^{2n-2k}/G) \otimes H^*(S^{2k+1}),$$

where N is a simply-connected, non-negatively curved torus manifold and G is a finite group acting isometrically (and orientably) on N.

To date, only odd GKM graphs without signs have been treated in the literature. Adding the restriction that the manifold admits a T-invariant almost contact structure, that is, an almost contact structure invariant under a torus action, allows us to talk about odd GKM graphs with signs. Using Theorem 1.5 of [18], we obtain the following result.

Theorem 1.4. *Let M^{2n+1} be a closed, non-negatively manifold with an odd GKM$_4$ T-action, which admits a T-invariant, alternating, almost contact structure. Suppose that (Γ, α, ∇), the abstract, even-dimensional GKM$_4$ graph obtained from $\bar{\Gamma}_M$, the odd-dimensional GKM graph of M, is orientable. Then the rational cohomology ring of M is isomorphic to the tensor product of the rational cohomology ring of an odd-dimensional sphere and the rational cohomology ring of a generalized Bott manifold.*

See Definition 6.4 for a definition of an *alternating* almost contact structure. Finally, we also obtain a full rational cohomology classification for positively curved odd GKM$_3$ manifolds, as follows.

Theorem 1.5. *Let M^{2n+1} be a closed, positively curved, odd GKM$_3$ manifold. Then M^{2n+1} has the rational cohomology ring of S^{2n+1}.*

We note that of the known examples of odd-dimensional manifolds of positive curvature, the only ones that are rational homology spheres, but not diffeomorphic to spheres, are the so-called Berger spaces, $B^7 = SO(5)/SO(3)$ and $B^{13} = SU(5)/(Sp(2) \times \mathbb{Z}_2 S^1)$. These two manifolds admit both an invariant transitive and an invariant cohomogeneity one action, but the corresponding maximal torus does not satisfy the requirements to be odd GKM$_3$, since the maximal torus in both cases does not have fixed points.

Finally, we point out that the positive and non-negative curvature hypotheses in our results serve to restrict the number of vertices in the two-dimensional faces of the odd GKM$_k$ graph associated to the manifold. Thus, all of the above theorems can be reframed in a curvature-free setting by simply assuming the corresponding restrictions on these graphs.

1.1. Organization

The paper is organized as follows. We include basic notation and preliminary material in Section 2. In Section 3, we classify the universal covers of closed, non-negatively curved 5-dimensional GKM manifolds. In Section 4, we classify the corresponding graphs of these 5-manifolds. In Section 5, we prove Theorems 1.1, 1.2, 1.3, and 1.5. In Section 6, we give the proof of Theorem 1.4.

2. Preliminaries

In this section we gather basic results and facts about transformation groups, equivariant cohomology, even- and odd-dimensional GKM and GKM_k theory, as well as results concerning G-invariant manifolds of non-negative sectional curvature.

2.1. Transformation groups

Let G be a compact Lie group acting on a smooth manifold M. We denote by $G_x = \{g \in G \mid gx = x\}$ the *isotropy group* at $x \in M$ and by $G(x) = \{gx \mid g \in G\}$ the *orbit* of x. Note that $G(x)$ is homeomorphic to G/G_x since G is compact. We denote the orbit space of the G-action by M/G and note that if M admits a lower sectional curvature bound and the G-action is isometric, then M/G is an Alexandrov space admitting the same lower curvature bound. We denote the fixed point set of M by G as either M^G or $\mathrm{Fix}(M; G)$, using whichever may be more convenient.

One measurement for the size of a transformation group $G \times M \to M$ is the dimension of its orbit space M/G, also called the *cohomogeneity* of the action. This dimension is clearly constrained by the dimension of the fixed point set M^G of G in M. In fact, $\dim(M/G) \geq \dim(M^G) + 1$ for any non-trivial, non-transitive action. In light of this, the *fixed point cohomogeneity* of an action, denoted by $\mathrm{cohomfix}(M; G)$, is defined by

$$\mathrm{cohomfix}(M; G) = \dim(M/G) - \dim(M^G) - 1 \geq 0.$$

A manifold with fixed point cohomogeneity 0 is also called a *G-fixed point homogeneous manifold*.

We now recall Theorem I.9.1 of Bredon [3], which characterizes how to lift a group action to a covering space.

Theorem 2.1. [3] *Let G be a connected Lie group acting effectively on a connected, locally path-connected space X and let X' be any covering space of X. Then there is a covering group G' of G with an effective action of G' on X' covering the given action. Moreover, G' and its action on X' are unique.*

The kernel of $G' \to G$ is a subgroup of the group of deck transformations of $X' \to X$. In particular, if $X' \to X$ has finitely many sheets, then so does $G' \to G$.

If G has a fixed point in X, then $G' = G$ and $\mathrm{Fix}(X'; G)$ *is the full inverse image of* $\mathrm{Fix}(X; G)$.

The presence of a G-action on a manifold M induces a topological stratification of M. In particular, when the group is a torus, that is, $G = T$, a special name is given to the strata of the T-action.

Definition 2.2 (**The k-skeleton of** M). *For $G = T$, a torus, we define the k-skeleton of M to be*

$$M_k = \{p \in M \mid \dim(T(p)) \leq k\}.$$

We then obtain a T-invariant topological stratification of M as follows: $M_0 \subset M_1 \subset \cdots \subset M_{\dim(T)} = M$ *on M, where the* 0*-skeleton M_0 is exactly the fixed point set M^T.*

2.2. Equivariant cohomology

We begin by providing some basic information about equivariant cohomology and equivariantly formal manifolds for torus actions.

Definition 2.3 (**Equivariant Cohomology**). *Given an action of a torus T on a compact manifold M, the equivariant cohomology of the action is defined as*

$$H_T^*(M) = H^*(M \times_T ET),$$

where $ET \to BT$ is the classifying bundle of T and ET is a contractible space on which T acts freely.

The equivariant cohomology has the natural structure of an $H^*(BT)$-algebra, via the projection $M \times_T ET \to BT$. Note that $H^*(BT)$ is isomorphic to the ring of rational polynomials on the Lie algebra $\mathfrak{t}$, in the following sense. Denoting the rational points in $\mathfrak{t}$, that is, the tensor product of the integer lattice in $\mathfrak{t}$ with $\mathbb{Q}$, by $\mathfrak{t}_{\mathbb{Q}}$, then $H^*(BT)$, a rational polynomial ring in $\dim(T)$ variables, is isomorphic to $S(\mathfrak{t}_{\mathbb{Q}}^*)$, the symmetric algebra over $\mathfrak{t}_{\mathbb{Q}}^*$.

Given an action of a torus T on M, we may compare $H_T^*(M)$ with $H_T^*(M^T)$ using the Borel Localization Theorem (see, e.g., Corollary 3.1.8 in Allday and Puppe [1]).

Theorem 2.4 (Borel Localization Theorem). *The restriction map*

$$H_T^*(M) \to H_T^*(M^T)$$

is an $H^*(BT)$-module isomorphism modulo $H^*(BT)$-torsion.

Using this localization theorem, it is clear that if $H_T^*(M)$ is actually a free $H^*(BT)$-module, one can hope for a stronger relation between the manifold M and its fixed point set M^T. This motivates the following definition.

Definition 2.5 (**Equivariantly Formal**). *We say that an action of a torus T on M is equivariantly formal if $H_T^*(M)$ is a free $H^*(BT) \cong S(\mathfrak{t}_{\mathbb{Q}}^*)$-module.*

As a compact manifold has finite-dimensional cohomology, it follows from Corollary 4.2.3 of [1] that equivariant formality is equivalent to the degeneration of the Leray–Serre spectral sequence of the Borel fibration $M \hookrightarrow M \times_T ET \to BT$ at the E_2-term. Moreover, the following are some well-known and important properties of equivariantly formal actions.

Proposition 2.6. *An action of a torus T on a compact manifold M with $H^{\text{odd}}(M) = \{0\}$ is equivariantly formal. The converse implication is true provided that the T-fixed point set M^T is finite.*

The first statement of Proposition 2.6 follows because the spectral sequence degenerates at the E_2-term, if $H^{\text{odd}}(M) = \{0\}$. The second statement is a consequence of the Borel Localization Theorem, as then $H_T^*(M) \cong H^*(BT) \otimes H^*(M)$ injects into the module $H_T^*(M^T) \cong H^*(BT) \otimes H^*(M^T)$, which vanishes in odd degrees. The next proposition is Theorem 3.10.4 in [1].

Proposition 2.7. [1] *For any action of a torus T on a compact manifold M, we have $\dim H^*(M^T) \leq \dim H^*(M)$. Equality holds if and only if the action is equivariantly formal.*

The Leray–Hirsch theorem implies that for equivariantly formal actions the ordinary cohomology ring is encoded in the equivariant cohomology algebra.

Proposition 2.8. *For an equivariantly formal action of a torus T on a compact manifold M the natural map $H_T^*(M) \to H^*(M)$ is surjective and induces an isomorphism of $\mathbb{Q}$-algebras*

$$\frac{H_T^*(M)}{S^+(\mathfrak{t}_{\mathbb{Q}}^*) \cdot H_T^*(M)} \cong H^*(M),$$

where $S^+(\mathfrak{t}_{\mathbb{Q}}^*)$ denotes the ideal in $S(\mathfrak{t}_{\mathbb{Q}}^*)$ generated by $\mathbb{Q}$-valued polynomials of positive degree.

For equivariantly formal actions, the Borel Localization Theorem 2.4 gives an embedding of $H_T^*(M)$ into $H_T^*(M^T)$. The image of this embedding can be described as follows, combining Lemma 2.3 of Chang and Skjelbred [5] for the first isomorphism with the version of the same lemma given in Theorem 11.51 of Guillemin and Sternberg [22] for the second isomorphism.

Chang Skjelbred Lemma 2.9. [5], [22] *If a T-action on M is equivariantly formal, then the equivariant cohomology $H_T^*(M)$ only depends on the fixed point set M^T and the* 1*-skeleton M_1:*

$$H_T^*(M) \cong \operatorname{im}\left(H_T^*(M_1) \to H_T^*(M^T)\right) \cong \bigcap \operatorname{im}(H_T^*(M^K) \to H_T^*(M^T)),$$

*where the intersection is taken over all corank-*1 *subtori K of T.*

Moreover, equivariant formality is *inherited* by subtori of the T-action. More precisely, we have the following well-known proposition and corollary, see, for example, [26].

Proposition 2.10. *If a T-action on M is equivariantly formal, then for any subtorus K of T, both the K-subaction on M and the induced T/K-action on M^K are equivariantly formal.*

Corollary 2.11. *If a T-action on M is equivariantly formal, then for any subtorus K of T, every connected component of M^K has T-fixed points.*

2.3. Even-dimensional GKM theory

The class of manifolds, now referred to as *GKM manifolds*, was first introduced in the seminal work of Goresky, Kottwitz, and MacPherson [19] to study the relation between equivariant cohomology and ordinary cohomology and is named for these authors. The GKM Theorem 2.23 (cf. Theorem 1.2.2 of [19]) states that over an appropriate coefficient ring R, the equivariant cohomology ring of a GKM manifold M can be computed via its 1-skeleton and the isotropy information of the GKM torus action. Motivated by their work, the concepts of GKM manifold and GKM graph were introduced by Guillemin and Zara in [23] to build a bridge between the topology and the combinatorics of these spaces. We begin with the formal definition of a GKM manifold.

Definition 2.12 (**GKM Torus Action and Manifold**). *We say that the effective action of a torus $T^l = T$, $l \leq n$, on an orientable, compact, connected manifold M^{2n} is GKM, and M^{2n} is called a GKM manifold, if*

(1) *the fixed point set M^T of the action is finite;*
(2) *for every $p \in M^T$, the weights $\alpha_{i,p} \in \mathfrak{t}_{\mathbb{Q}}{}^*/\{\pm 1\}, i = 1, \ldots, n$, of the isotropy representation of T on T_pM are pairwise linearly independent; and*
(3) *the T-action is equivariantly formal.*

We note that the original definition in [23] does not require the manifold to be either orientable or the T-action to be equivariantly formal, rather, both are assumed (sometimes implicitly) as separate hypotheses in their theorems. We include both hypotheses in our definition, since both are included in the definition of a GKM manifold in most of the recent literature (see, e.g., Goertsches and Wiemeler [17, 18] and Kuroki [29]).

By Proposition 2.6, Conditions 1 and 3 imply the vanishing of the odd-dimensional rational cohomology groups of M. Condition 2 is equivalent to the condition that M_1, the 1-skeleton of M, consists of a disjoint union of T-invariant, orientable submanifolds, each of which is either fixed point free, or an embedded S^2. By Corollary 2.11, Condition 3 implies that M_1 consists entirely of T-invariant embedded 2-spheres. Moreover, Conditions 1 and 3 combined with Proposition 2.7 tell us that each such S^2 contains exactly two T-fixed points.

Definition 2.13 (**GKM$_k$ Torus Action and Manifold**). *We say that the effective action of a torus T on an orientable, compact, connected manifold M^{2n} is GKM$_k$, and we call M^{2n} a GKM$_k$ manifold, if*

(1) *M^{2n} is GKM, and*
(2) *for each $p \in M^T$ any set of k weights, $\alpha_{i,p} \in \mathfrak{t}_{\mathbb{Q}}{}^*/\{\pm 1\}, i = 1, \ldots, n$, of the isotropy representation of T on T_pM is linearly independent.*

Remark 2.14. *A GKM$_k$ manifold is GKM$_l$ for all $2 \leq l \leq k$, and a GKM$_2$ manifold is a GKM manifold.*

By convention, the T^1-action on S^2 is considered to be a GKM$_2$ manifold. Note also that the linear independence is well-defined for elements that are only defined up to sign. Condition 2 is equivalent to M_{k-1}, the $(k-1)$-skeleton, being a union of T-invariant submanifolds, which are either fixed point free or are $(2k-2)$-dimensional and have T-fixed points. Observe that by Corollary 2.11, the definition implies that M_{k-1} consists entirely of $(2k-2)$-dimensional T-invariant submanifolds.

Once we have established some conventions, we define the notion of an *abstract* GKM_k graph, which consists of a triple: a graph, an axial function, and a connection. However, in Section 4, we will be working with *geometric* GKM_k graphs, that is, graphs that correspond to the graph of a GKM_k manifold. We note that an abstract GKM_k graph may not correspond to the graph of a GKM_k manifold and that the graph described in our Main Theorem 1.1 is abstract.

We employ the following conventions when speaking about abstract graphs. Given an abstract graph Γ, we denote by

- $E(\Gamma)$ its set of oriented edges; and by
- $V(\Gamma)$ its set of vertices.

We always assume that both the edge and vertex sets are finite, and we allow multiple edges between vertices. For an edge $e \in E(\Gamma)$ we denote by

- $\bar{e}$ the edge with opposite orientation;
- $i(e)$ its initial vertex; and
- $t(e)$ its terminal vertex.

We assume that for any edge e, $i(e) \neq t(e)$, that is, an edge cannot connect a vertex to itself. For a vertex $v \in V(\Gamma)$ the set of edges emanating from v is denoted by E_v.

Before we can define an abstract GKM_k graph, we define a connection, ∇, on a graph Γ, as in [23].

Definition 2.15 (**Connection**). *A connection on a graph Γ is a collection ∇ of maps $\nabla_e : E_{i(e)} \to E_{t(e)}$, for each $e \in E(\Gamma)$, such that*

(1) $\nabla_e(e) = \bar{e}$, *and*

(2) $\nabla_{\bar{e}} = (\nabla_e)^{-1}$.

We are now in a position to define an *abstract GKM_k graph*. We note that abstract GKM graphs were originally defined in [23], and called *abstract one-skeleta* there. However, in the subsequent literature, they have been referred to as abstract GKM graphs, so we do so as well.

Definition 2.16 (**Abstract GKM$_k$ graph**). *Let $k \geq 2$. Then an abstract GKM$_k$ graph, (Γ, α, ∇), or simply (Γ, α) when $k \geq 3$, consists of a graph Γ, an axial function $\alpha : E(\Gamma) \to \mathfrak{t}^*_{\mathbb{Q}}/\{\pm 1\}$, and a connection ∇, such that*

(1) *the underlying graph Γ is n-valent;*

(2) *for any $e \in E(\Gamma)$, $\alpha(\bar{e}) = \alpha(e)$; and*

(3) *for any $v \in V(\Gamma)$ and any set of k distinct edges $e_1, \ldots, e_k \in E_v$, the elements $\alpha(e_1), \ldots, \alpha(e_k)$ are linearly independent;*

(4) *for any $v \in V(\Gamma)$ and any pair of distinct edges $e, f \in E_v$, we have that*

$$\alpha(\nabla_e f) = \pm\alpha(f) + c_{e,f}\alpha(e),$$

for some constant $c = c_{e,f} \in \mathbb{Z}$, depending on e and f.

Remark 2.17. *The abstract GKM$_k$ graph for $k \geq 3$ consists of a triple, but since the connection is canonical for $k \geq 3$ [23], there is no need to list it in this case.*

We also define a variant of the abstract GKM$_k$ graph, which is signed, motivated by the fact that one often assumes that a GKM manifold admits an invariant almost complex structure.

Definition 2.18 (**Abstract Signed GKM$_k$ graph**). *Let $k \geq 2$. Then an abstract signed GKM$_k$ graph is a GKM$_k$ graph where $\pm$ is replaced by $+$. That is, the following modifications are made to Definition 2.16.*

(1) *The axial function now takes values in an R-module V, that is, $\alpha : E(\Gamma) \to V$.*

(2) *In Part (4) is modified to be*

$$\alpha(\nabla_e f) = \alpha(f) + c_{e,f}\alpha(e).$$

In order to motivate the previous definitions, we now describe the geometric graph obtained from a GKM$_k$ manifold, M^{2n}.

Definition 2.19 (**Geometric GKM$_k$ graph**). *Let M^{2n} be a GKM$_k$ manifold. Then the geometric GKM$_k$ graph of M is an abstract GKM$_k$ graph, $(\Gamma_M, \alpha_M, \nabla_M)$ or simply (Γ_M, α_M) for $k \geq 3$, where Γ_M and α_M are defined as follows.*

(1) *Γ_M is the quotient by the torus action of the 1-skeleton, M_1/T, considered as a graph.*

(2) *The axial function* $\alpha_M : E(\Gamma_M) \to \mathfrak{t}^*_{\mathbb{Q}}/\{\pm 1\}$ *is defined on each edge, e, of* Γ_M, *as the corresponding weight of the isotropy representation, considered as an element of* $\mathfrak{t}^*_{\mathbb{Q}}/\{\pm 1\}$.

Remark 2.20. *With this definition* $V(\Gamma_M)$ *corresponds to the (isolated) fixed points of the torus action and* $E(\Gamma_M)$ *corresponds to the 2-spheres fixed by a codimension one subtorus of* T, *containing two isolated fixed points of* T.

Equivalently, given a GKM_k action of a torus T on an orientable manifold M^{2n}, the geometric graph of this action is constructed as follows: we have one vertex for each fixed point, every invariant 2-sphere contains exactly two fixed points, and we associate to it one edge connecting the corresponding vertices. The axial function associates to each edge, that is, invariant 2-sphere, the corresponding weight of the isotropy representation.

Remark 2.21. *Given a* GKM_k*-manifold* M^{2n}, *for every choice of* $l \leq k-1$ *edges* $e_1, \ldots, e_l$ *at a vertex* p, *we can define a* $2l$*-dimensional* T*-invariant submanifold* N^{2l} *of* M, *that is itself* GKM_l *with the induced torus action as follows. Let* $\mathfrak{h} := \bigcap_{i=1}^{l} \ker \alpha_M(e_i)$, *consider the subtorus* $H \subset T$ *with Lie algebra* $\mathfrak{h}$, *and define* N *to be the connected component of* M^H *containing the fixed point* p. *The* GKM_k *condition implies that* N *is* $2l$*-dimensional, as its tangent space at* p *is precisely the sum of those weight spaces whose weights vanish on* $\mathfrak{h}$. *We then see that in the above definition of* Γ_M, *the graph of the* GKM_k *manifold* M, *the quotient of the* 1*-skeleton of* N *by* T *is a subgraph* $\Gamma_N \subset \Gamma_M$, *and we call this subgraph a face. In the special case when* M^{2n} *is a quasitoric manifold, the quotient* M/T *is an* n*-dimensional simple polytope,* P^n. *In this special case,* Γ_M *is the* 1*-skeleton of* P^n, *and each face of* P^n *corresponds to some* N^{2l}/T, *whose* 1*-skeleton is* Γ_N.

We now explicitly describe the connection, ∇_M, in the GKM_3 case, as it is easier to describe than in the GKM case, and moreover we are only concerned with the GKM_3 case in this article. In this setting, given any two distinct edges e, f emanating from a vertex v, there is a unique two-dimensional face, F, containing e and f. Let $e' \neq \bar{e}$ be the unique edge in F, such that $i(e') = t(e)$. Setting $(\nabla_M)_e f = e'$, it follows from [18] that $(\Gamma_M, \alpha_M, \nabla_M)$ satisfies the conditions for an abstract GKM_3 graph. In particular, we see that the connection allows us to slide edges along edges inside the corresponding two-dimensional face of the graph.

This leads us to define the concept of an *abstract face* of an abstract GKM$_k$ graph. Let (Γ, α, ∇) be an abstract GKM$_k$ graph, then we say that Γ' is an *l-dimensional face* of (Γ, α, ∇) if it is an l-valent subgraph, invariant under ∇.

We are now in a position to define the equivariant cohomology of an abstract GKM graph, which was first done in Section 1.7 of [23] and was denoted simply by $H(\Gamma, \alpha)$. Since we also define the cohomology of an abstract GKM graph below in Definition 2.24, to avoid confusion, we denote the equivariant cohomology of an abstract GKM graph by $H_T(\Gamma, \alpha)$.

Definition 2.22 (**Equivariant Cohomology of an Abstract GKM Graph**). *The equivariant cohomology of an abstract GKM graph (Γ, α, ∇) is defined as*

$$H_T^*(\Gamma, \alpha) = \{(f_v)_{v \in V(\Gamma)} \in \bigoplus_{v \in V(\Gamma)} S(\mathfrak{t}_{\mathbb{Q}}^*) \mid \alpha(e) \textit{ divides } f_{i(e)} - f_{t(e)} \textit{ for all } e \in E(\Gamma)\},$$

where the generators of $S(\mathfrak{t}_{\mathbb{Q}}^*)$ are assigned degree 2. It is naturally an $S(\mathfrak{t}_{\mathbb{Q}}^*)$-algebra.

We now recall Theorem 1.2.2, also known as the GKM Theorem, in [19] here.

Theorem 2.23 (**GKM Theorem**). [19] *For a GKM action of a torus T on M, with GKM graph Γ_M, the injection $H_T^*(M) \to H_T^*(M^T) = \bigoplus_{p \in M^T} S(\mathfrak{t}_{\mathbb{Q}}^*)$ has as image exactly $H_T^*(\Gamma_M, \alpha_M)$. Thus, $H_T^*(M) \cong H_T^*(\Gamma_M, \alpha_M)$ as $S(\mathfrak{t}_{\mathbb{Q}}^*)$-algebras.*

Motivated by Proposition 2.8 we define the cohomology of an abstract GKM graph as follows.

Definition 2.24 (**Cohomology of an abstract GKM Graph**). *The cohomology of an abstract GKM graph (Γ, α, ∇) is defined as*

$$H^*(\Gamma, \alpha) = \frac{H_T^*(\Gamma, \alpha)}{S^+(\mathfrak{t}_{\mathbb{Q}}^*) \cdot H_T^*(\Gamma, \alpha)}.$$

Remark 2.25. *Thus, for a GKM action of a torus T on M, we have $H^*(M) \cong H^*(\Gamma_M)$ by Theorem 2.23 and Proposition 2.8.*

2.4. Results for GKM$_k$ manifolds in positive and non-negative curvature

We state here two results for GKM$_k$ manifolds of positive and non-negative sectional curvature that we need for the proofs of Theorem 1.2 and 1.3. The first is a classification result for positively curved GKM$_3$ manifolds from [17].

Theorem 2.26. [17] *Let M be a closed, positively curved, orientable GKM_3 Riemannian manifold. Then M has the real cohomology ring of a compact rank one symmetric space.*

The second is a classification result for non-negatively curved GKM_4 manifolds from [18].

Theorem 2.27. [18] *Let M be a non-negatively curved GKM_4 manifold. Then*

$$H^*(M) \cong H^*(\widetilde{M}/G),$$

where $\widetilde{M}$ is a simply-connected, non-negatively curved torus manifold and G is a finite group acting isometrically on $\widetilde{M}$.

2.5. Odd-dimensional GKM theory

GKM theory was generalized to torus actions on odd-dimensional manifolds with one-dimensional fixed point set in [26]. The odd GKM condition is an odd-dimensional analogue of the even-dimensional GKM condition. We describe the corresponding theory here.

Definition 2.28 (Odd GKM Torus Action and Manifold). *We say that the action of a torus T on an orientable, compact, connected manifold M^{2n+1} is odd GKM if*

(1) *the fixed point set M^T of the action is a finite union of circles;*
(2) *for every $p \in M^T$, the weights $\alpha_{i,p} \in \mathfrak{t}_{\mathbb{Q}}{}^*/\{\pm 1\}, i = 1, \ldots, n$, of the isotropy representation of T on T_pM are pairwise linearly independent; and*
(3) *it is equivariantly formal.*

Remark 2.29. *Condition 2 is equivalent to requiring the 1-skeleton*

$$M_1 = \{p \in M \mid \dim(T \cdot p) \leq 1\}$$

to be a finite union of three-dimensional T-invariant submanifolds.

Recall that for GKM manifolds, Condition 2 of Definition 2.12 implies that up to diffeomorphism, there is only one two-dimensional T-invariant submanifold, S^2, and that $S^2 \cap M^T$ consists of exactly two isolated points. In contrast, here there are an infinite number of possible T-invariant 3-manifolds that could occur, see Section 4 of [26] for a complete classification of such 3-manifolds.

Note that in [26], it is neither assumed that the action is equivariantly formal nor that M is orientable. We include these conditions because we do so in the even-dimensional setting, and as mentioned earlier, both are often part of the definition of a GKM action.

Remark 2.30. *One should note that in* [16] *a different generalization of GKM theory to odd dimensions was introduced for the so-called Cohen–Macaulay actions. The GKM-type actions considered there are more general than those of* [26], *as they do not necessarily have fixed points. That is, using the stratification induced by the M_k skeleta, one only has that $M_l \neq \emptyset$ for some $l \geq 0$, rather than $M_0 = M^T \neq \emptyset$. On the other hand, the definition from* [16] *is also more restrictive in terms of the stratification of the k-skeleta. Namely, given N, a connected component of $M_{l+1} \setminus M_l$, with $M_l \neq \emptyset$, N contains exactly two components of M_l. By contrast, in the definition given in* [26], *the number of such components is greater than or equal to* 1.

As in classical GKM theory one can associate to an odd GKM manifold a *geometric odd GKM graph*. For equivariantly formal actions, one can compute the equivariant, as well as the ordinary rational, cohomology of the manifold from this graph.

To encode the structure of the 1-skeleton in a graph, two types of vertices are defined in [26]. We will decorate our graphs with a bar to distinguish them from even-dimensional GKM graphs.

Definition 2.31 (**Vertex Types**). *The graph, $\bar{\Gamma}_M$, of an odd GKM manifold M has two types of vertices*:

(1) *One circle for each circle in the fixed point set; and*

(2) *One square for each invariant three-dimensional submanifold in M_1, the* 1*-skeleton of M.*

We denote the set of circles by $V_\circ$ and the set of squares by $V_\square$.

We also have restrictions on how edges are formed and distinguish between two particular types.

Definition 2.32 (**Edge and Edge Types**). *We connect a circle to a square by an edge if the fixed circle is contained in the corresponding three-dimensional submanifold.*

Further, at any circle in the graph, $\bar{\Gamma}_M$, we distinguish between the following edge types:

(1) *a floating edge, that is, an edge connecting to a square of valence* 1, *and*

(2) *a grounded edge, that is, an edge connecting to a square of valence* ≥ 2.

We further distinguish the following important subsets of $V_\circ$ and $V_\square$, namely, we denote by $V_\circ(s)$ the set of circles connected to $s \in V_\square$, and by $V_\square(c)$ the set of squares connected to $c \in V_\circ$.

In analogy with how weights are assigned to the GKM$_k$ graph, we now assign weights to square vertices rather than edges.

Definition 2.33 (**Weight Function**). *The weight function* $\alpha_M : V_\square \to \mathfrak{t}^*_{\mathbb{Q}}/\{\pm 1\}$ *assigns to each square s of $\bar{\Gamma}_M$, a weight, $\alpha_M(s)$, which is the weight of the isotropy representation at any fixed circle in the three-dimensional submanifold corresponding to s, considered as an element of* $\mathfrak{t}^*_{\mathbb{Q}}/\{\pm 1\}$.

Note that by definition, any edge connects a circle to a square.

We can also introduce a notion of connection on such a graph, as in the even-dimensional setting. The only difference is that we do not specify a single edge along which we transport, but two circles in the same three-dimensional component of the 1-skeleton.

Definition 2.34. *A connection on the graph, $\bar{\Gamma}_M$, of an odd-dimensional GKM manifold M is a collection of maps $(\bar{\nabla}_M)_{c_1,c_2,s_0} : V_\square(c_1) \to V_\square(c_2)$, for every $s_0 \in V_\square$ and $c_1, c_2 \in V_\circ(s_0)$, satisfying the following conditions*:

(1) $(\bar{\nabla}_M)_{c_1,c_2,s_0}(s_0) = s_0$

(2) $(\bar{\nabla}_M)_{c_2,c_1,s_0} = (\bar{\nabla}_M)^{-1}_{c_1,c_2,s_0}$

(3) *For every $s \in V_\square(c_1)$, there exists a constant $c \in \mathbb{Z}$ such that*

$$\bar{\alpha}_M((\bar{\nabla}_M)_{c_1,c_2,s_0}(s)) = \pm\bar{\alpha}_M(s) + c\,\bar{\alpha}_M(s_0).$$

The following proposition guarantees the existence of a connection on the graph of every odd-dimensional GKM manifold. The proof is completely analogous to the proof of Proposition 2.3 in [18], or the proof on page 5 of [23].

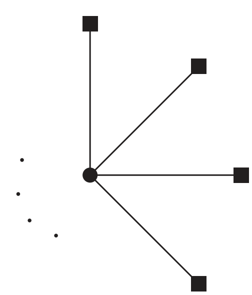
Proposition 2.35. *There exists a connection on the graph, $\bar{\Gamma}_M$, of every odd-dimensional GKM manifold.*

With these notions, we now define a *geometric odd GKM graph*.

Definition 2.36 (**Geometric Odd GKM Graph**). *Let M^{2n+1} be an odd GKM manifold. We define a geometric odd GKM graph, $(\bar{\Gamma}_M, \bar{\alpha}_M, \bar{\nabla}_M)$, where $\bar{\Gamma}_M$ is the graph obtained from the 1-skeleton of M, with vertices, edges and weight function, $\bar{\alpha}_M$, and a connection, $\bar{\nabla}_M$, as described above.*

Remark 2.37. *For odd GKM graphs whose squares have valence 1 or 2 only, we will use the following notational shortcut. Namely, we will denote each 2-valent square $s \in V_\square$, by s_{ij} where $c_i, c_j \in V_\circ(s)$ are the unique circle vertices connecting to s. In particular, in analogy with the orientation assigned to edges in the even-dimensional case, this allows us to assign an orientation to a 2-valent square, namely we let $\bar{s}_{ji} = s_{ij}$.*

Example 2.38. *The geometric odd GKM graph of a $(2n+1)$-dimensional sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ with the standard T^n-action induced by the standard representation on n of the $n+1$ summands is a pinwheel with n edges terminating in squares, corresponding to fixed 3-spheres, as follows*:



In the following proposition we collect a few properties of odd GKM graphs.

Proposition 2.39. *Let M^{2n+1} be an odd GKM manifold, then the following hold*:

(1) *The geometric odd GKM graph is connected*

(2) *Each circle in the geometric odd GKM graph has valence n.*

(3) *The total Betti number $b(M^{2n+1})$ is equal to $2m$ for some $m \in \mathbb{Z}^+$, and there are exactly m circles in the graph. Moreover, each square in the odd GKM graph has valence bounded between 1 and m.*

Proof. We include a proof for the sake of completeness, noting that the proof follows along the same lines as in the even-dimensional case. To prove Part 1, we note the Chang–Skjelbred Lemma 2.9 states that for an equivariantly formal action, the image of the injective map $H_T^*(M) \to H_T^*(M^T)$ is the same as the image of the map $H_T^*(M_1) \to H_T^*(M^T)$. As M is connected, it follows that the image of $H^0(M_1) \cong H_T^0(M_1) \to H_T^0(M^T) \cong H^0(M^T)$ is one-dimensional, which implies that M_1 is connected. This is equivalent to the GKM graph being connected.

To prove Part 2, recall that any edge emanating from a circle corresponds to a weight of the isotropy representation at that circle. Because the codimension of this circle is $2n$, and T acts on the normal space of this circle without fixed vectors, there are precisely n such weights.

To prove Part 3, it follows by Proposition 2.7 that the equivariant formality of the action is equivalent to the equality of total Betti numbers $\dim H^*(M) = \dim H^*(M^T)$. Because any circle in M^T contributes 2 to the total Betti number of M^T, and since any square must contain a circle fixed by T and can contain at most m circles, the result follows. ■

We are now in a position to define an odd GKM$_k$ manifold.

Definition 2.40 (**Odd GKM$_k$ manifolds**). *An odd-dimensional GKM manifold is called odd GKM$_k$, for $k \geq 2$, if the following hold.*

(1) *M is odd-dimensional GKM, and;*
(2) *At any fixed circle, any k weights of the isotropy representation are linearly independent.*

Thus, odd GKM manifolds are the same as odd GKM$_2$ manifolds. In the same way that a geometric odd GKM graph is associated to an odd GKM manifold, we obtain an odd geometric GKM$_k$ graph from an odd GKM$_k$ manifold. Note that for geometric odd GKM$_3$ graphs, for every $s_0 \in V_\square$, given $c_1, c_2 \in V_\circ(s_0)$, the condition $\bar{\alpha}_M((\bar{\nabla}_M)_{c_1,c_2,s_0}(s)) = \pm\bar{\alpha}_M(s) + c\bar{\alpha}_M(s_0)$ alone uniquely determines the square $(\bar{\nabla}_M)_{c_1,c_2,s_0}(s)$, for all $s \in V_\square(c_1)$, that is, the connection is unique. Since the connection for a geometric odd GKM$_k$ graph is canonical for $k \geq 3$, we will denote such graphs simply by $(\bar{\Gamma}_M, \bar{\alpha}_M)$.

For a GKM$_k$ manifold M, and any $k-1$ weights at a fixed circle, there is a unique $(2k-1)$-dimensional submanifold fixed by a codimension $k-1$ subtorus generated by the intersection of the kernels of the $k-1$ weights. We will denote this submanifold by N_T^{2k-1}. As in the even-dimensional case, we make the following definition of a face.

Definition 2.41 (**Face**). *We call the subgraph of the GKM$_k$ graph of M corresponding to N_T^{2k-1} a $(k-1)$-face of the graph.*

Theorem 4.6 in [26] tells us how the GKM graph encodes the equivariant cohomology, whose relevant content we recall here. We choose an orientation on every component of M^T, which then allows us to identify its cohomology canonically with $H^*(S^1) = \mathbb{Q}[\theta]/(\theta^2)$. The inclusion $M^T \to M$ induces an injection

$$H_T^*(M) \longrightarrow H_T^*(M^T) = \bigoplus_{C \in V_\circ} S(\mathfrak{t}_{\mathbb{Q}}^*) \otimes H^*(S^1),$$

and the image of this map is described by the following divisibility relations. For any $s \in V_\square$, let N_s be the three-dimensional connected submanifold fixed by the subtorus with Lie algebra $\ker \alpha(s)$. Then, for $c_1, \ldots, c_l \in V_\circ$, the circles contained in N_s, we have

$$(P_c + Q_c\theta)_{c \in V_\circ} \in \bigoplus_{c \in V_\circ} S(\mathfrak{t}_{\mathbb{Q}}^*) \otimes H^*(S^1),$$

where $P_c, Q_c \in S(\mathfrak{t}_{\mathbb{Q}}^*)$, satisfies

$$P_{c_1} \equiv \cdots \equiv P_{c_l} \mod \alpha(s) \tag{2.1}$$

and

$$\sum_{i=1}^{l} \pm Q_{c_i} \equiv 0 \mod \alpha(s). \tag{2.2}$$

Here, the $\pm$ signs in the sum are determined as follows. Recall that for a closed manifold M, the fixed point sets of torus actions are closed submanifolds that are orientable if M is. Thus N_s is orientable, and so the orbit space N_s/T is orientable as a topological manifold (with boundary), as well. The circles c_i are boundary components of N_s/T, and if the pre-chosen orientation on each c_i coincides with the induced boundary orientation, with respect to any orientation of N_s/T, then the sign of Q_{c_i} is $+$, and if not, then its sign is $-$.

Remark 2.42. *It is not possible in general to consistently orient all components of M^T in such a way that for all N_s we find an orientation on N_s/T with the property that the circles in N_s carry the induced boundary orientation. Consider, for example, $S^1 \times \mathbb{C}P^2$ with the standard T^2 product action that is trivial on the first factor.*

2.6. Geometric results in the presence of a lower curvature bound

We now recall some general results about G-manifolds with non-negative curvature which we use throughout.

We recall the classification of closed, non-negatively curved, three-dimensional T^1-fixed point homogeneous manifolds due to Galaz-García [11].

Theorem 2.43. [11] *Let M^3 be a closed, non-negatively curved T^1-fixed point homogeneous Riemannian manifold. Then M is diffeomorphic to one of S^3, $L_{p,q}$, $S^2 \times S^1$, $S^2 \tilde{\times} S^1$, the non-trivial S^2-bundle over S^1, $\mathbb{R}P^2 \times S^1$, or $\mathbb{R}P^3 \# \mathbb{R}P^3$.*

Moreover, an analysis of the isometric circle action yields the following.

(1) *If M^3 has total Betti number equal to 2, the isometric circle action fixes one circle; and*
(2) *If M^3 has total Betti number equal to 4, the isometric circle action fixes two circles.*

Remark 2.44. *We make the following two observations.*

(1) *By Proposition 2.7, it follows that non-negatively curved T^1-fixed point homogeneous 3-manifolds are equivariantly formal.*
(2) *The only orientable manifold on this list that is not a rational cohomology sphere is $S^2 \times S^1$. Moreover, $S^2 \times S^1$ is the only manifold on this list with total Betti number equal to 4.*

The following theorem by Spindeler, [35], gives a characterization of non-negatively curved G-fixed point homogeneous manifolds.

Theorem 2.45. [35] *Assume that G acts fixed point homogeneously on a closed, non-negatively curved Riemannian manifold M. Let F be a fixed point component of maximal dimension. Then there exists a smooth submanifold N of M, without boundary, such that M is diffeomorphic to the normal disk bundles $D(F)$ and $D(N)$ of F and N glued together along their common boundaries*

$$M = D(F) \cup_\partial D(N).$$

Further, N is G-invariant and all points of $M \setminus \{F \cup N\}$ belong to principal G-orbits.

Remark 2.46. *In fact N is actually $\mathrm{Isom}_F(M)$-invariant, where $\mathrm{Isom}_F(M)$ is the subgroup of isometries of M leaving F invariant, by Lemma 3.30 in* [35].

Finally, we recall the following Splitting Theorem due to Cheeger and Gromoll [6].

Theorem 2.47. [6] *Let M be a compact manifold of non-negative Ricci curvature. Then $\pi_1(M)$ contains a finite normal subgroup Ψ such that $\pi_1(M)/\Psi$ is a finite group extended by $\mathbb{Z}^k$, and $\widetilde{M}$, the universal covering of M, splits isometrically as $\overline{M} \times \mathbb{R}^k$, where $\overline{M}$ is compact.*

3. Closed, Non-Negatively Curved 5-Dimensional GKM Manifolds

Observe that a closed $(2n+1)$-dimensional, equivariantly formal manifold admitting an effective, isometric T^n-action is GKM. In this section we prove Theorem 3.1, which classifies the universal covers of closed, non-negatively curved, 5-dimensional, equivariantly formal manifolds admitting an effective, isometric T^2-action and hence the universal covers of closed, non-negatively curved, 5-dimensional GKM manifolds. Theorem 3.1 will facilitate the proof of Theorem 4.3, where we classify the geometric graphs of closed, non-negatively curved, 5-dimensional GKM manifolds.

We also note that as $M^T \neq \emptyset$ by equivariant formality, some $S^1 \subset T^2$ fixes a codimension 2 submanifold in M^5, that is, the T^2-action on M is S^1-fixed point homogeneous. Recall that by Theorem 2.45 if M^5 admits an isometric T^2-action that is S^1-fixed point homogeneous, then we may decompose M^5 as a union of disk bundles, that is,

$$M^5 = D^2(F^3) \cup_E D(N), \tag{3.1}$$

where F^3 is the codimension two fixed point set of some circle subgroup of T, E is the common boundary of the disk bundles, and by Remark 2.46, N is a T^2-invariant submanifold. Moreover, F^3 is itself S^1-fixed point homogeneous by Proposition 2.10.

Theorem 3.1. *Let M^5 be a closed, non-negatively curved, equivariantly formal 5-dimensional manifold admitting an isometric T^2-action. Then $\mathrm{rk}(H_1(M^5;\mathbb{Z})) \leq 1$ and we may classify the corresponding universal cover, $\widetilde{M}^5$, as follows.*

(1) *For $\mathrm{rk}(H_1(M^5;\mathbb{Z})) = 0$, $\widetilde{M}^5$ is diffeomorphic to one of S^5, $S^3 \times S^2$, or $S^3 \tilde{\times} S^2$, the non-trivial S^3 bundle over S^2;*

(2) *For $\mathrm{rk}(H_1(M^5;\mathbb{Z})) = 1$, $\widetilde{M}^5$ is diffeomorphic to $\mathbb{R} \times M^4$, where M^4 is one of S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, or $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$.*

Combining the homeomorphism classification due to Rong [32] and the diffeomorphism classification results of Smale [34] and Barden [2], we have the following theorem for the positive curvature case.

Theorem 3.2. [32] *Let M^5 be a closed, simply-connected, positively curved 5-dimensional manifold admitting an isometric T^2-action. Then M^5 is diffeomorphic to S^5.*

Since positively curved manifolds have finite fundamental group, the following corollary is immediate, allowing us to classify the universal covers of positively curved 5-dimensional GKM manifolds as follows.

Corollary 3.3. *Let M^5 be a closed, positively curved, equivariantly formal 5-dimensional manifold admitting an isometric T^2-action. Then the universal cover of M^5 is diffeomorphic to S^5.*

In order to prove Theorem 3.1, we first need to prove Proposition 3.4 and Lemmas 3.5 and 3.6, which follow.

Proposition 3.4. *Let M^5 be a closed, orientable, equivariantly formal, non-negatively curved 5-dimensional Riemannian manifold with an isometric T^2-action. Then the following hold for $b(M^5)$, the total Betti number of M^5.*

(1) $2 \leq b(M^5) \leq 8$; *and*
(2) *If* $6 \leq b(M^5) \leq 8$, *then in the decomposition of M^5 in Display 3.1, F^3 is diffeomorphic to $S^2 \times S^1$.*

Before we begin the proof, we need the following result concerning the dimension of the submanifold at maximal distance from the codimension two fixed point set in M^{2n+1}.

Lemma 3.5. *Suppose M^{2n+1} is an S^1-fixed point homogeneous closed, orientable manifold of non-negative curvature. Let F be a codimension two fixed point set component of the S^1-action and suppose N is the submanifold given in the disk bundle decomposition of Theorem 2.45. Then if $N \cap \mathrm{Fix}(M; S^1) \neq \emptyset$,* $\mathrm{codim}(N)$ *is even.*

Proof. Recall that by Theorem 2.45 all singularities of the S^1-action are contained in F and N, and N is S^1-invariant. Let $N \cap \mathrm{Fix}(M; S^1) \neq \emptyset$. Then any connected component A of $\mathrm{Fix}(M; S^1)$ that is not contained in F must be contained in N and is of even codimension in M. Since N is S^1-invariant, A is also of even codimension in N and the result follows. ■

We are now in a position to prove Proposition 3.4.

Proof of Proposition 3.4. Recall that the hypothesis of equivariant formality guarantees that the T^2-action on M is S^1-fixed point homogeneous and the manifold decomposes as in Display 3.1. Moreover, by Proposition 2.7 the total Betti number of $\mathrm{Fix}(M^5; S^1)$ equals that of M^5. Suppose first that N does not contain any S^1-fixed points, that is, $\mathrm{Fix}(M^5; S^1) = F^3$. Since F^3 is itself non-negatively curved and S^1-fixed point homogeneous with respect to some other subcircle of T^2, the proposition is proven, as by Theorem 2.43, the total Betti number of F^3 is either 2 or 4.

We now assume that there are S^1-fixed points in N. This implies by Lemma 3.5 that N is of dimension one or three only. If N is one-dimensional, it follows from the classification of 1-manifolds that $N = S^1$, and so, the total Betti number of M is either 4 or 6.

Assume then that N is three-dimensional. If N^3 is fixed by some circle subgroup of T^2, it is an orientable, totally geodesic submanifold of M, and thus non-negatively curved. Since the T^2-action is effective and equivariantly formal, by Corollary 2.11, this implies that N^3 is S^1-fixed point homogeneous for some $S^1 \subset T^2$ and so N^3 is one of the manifolds classified in Theorem 2.43. Therefore, the total Betti number of N^3 is 2 or 4. Then by Proposition 2.7, the total Betti number of the S^1-fixed point set in N^3 is also bounded from above by 4. It follows that the total Betti number of M is bounded by 8, and if F^3 is a rational cohomology sphere, then it is bounded by 6.

Suppose then that N^3 is not fixed by any circle subgroup of T^2. Then the T^2-action on N^3 is of cohomogeneity one, and since there are S^1-fixed points, it follows by the classification of T^2 cohomogeneity one 3-manifolds in Mostert [30], and Neumann [31], that N^3 must be one of S^3, $L_{p,q}$, $S^2 \times S^1$, $\mathbb{R}P^2 \times S^1$, or $S^2 \tilde{\times} S^1$. Note that in all these cases, the total Betti number of N^3 is bounded between 2 and 4. Thus, it follows that the total Betti number of M is bounded by 8, and, again, if F^3 is a rational cohomology sphere, it is bounded by 6. This proves Part (1).

We now prove Part (2). We assume that $6 \leq b(M^5) \leq 8$ and that F^3 is not diffeomorphic to $S^2 \times S^1$, to derive a contradiction. Then by Theorem 2.43, F^3 is a rational (co)homology sphere, and so $b(F^3) = 2$. As we saw above, this means that $b(M^5) \leq 6$ and so $b(M^5) = 6$. Since $b(M^5) \leq b(F^3) + b(N)$ this gives us that $b(N) = 4$. So N must be $S^2 \times S^1$. Since E is the total space of the circle bundle inside the disk bundle $D^2(F^3) \to F^3$, it follows from the Gysin sequence (see Switzer [36]) that E has the same rational homology as $S^1 \times S^3$. Consider now the Mayer–Vietoris sequence of the decomposition (3.1)

$$H_5(M^5) \longrightarrow \mathbb{Q} \longrightarrow 0 \longrightarrow H_4(M^5) \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}^2 \longrightarrow H_3(M^5)$$
$$\longrightarrow 0 \longrightarrow \mathbb{Q} \longrightarrow H_2(M^5) \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q} \longrightarrow H_1(M^5) \longrightarrow 0.$$

Poincaré duality and the fact that $b(M^5) = 6$ gives us that $b_2 = b_3$ and hence $b_1 = b_4 = 2 - b_2$. Exactness at $H_2(M^5)$ implies that $b_2 = b_3 \in \{1, 2\}$. However, it is clear that exactness is violated for $b_2 = b_3 = 1$ or $b_2 = b_3 = 2$. ■

With this information, we now obtain the following lemma.

Lemma 3.6. *Let M^5 be a closed, orientable, equivariantly formal, non-negatively curved 5-dimensional Riemannian manifold with an isometric T^2-action. Then*

$$\mathrm{rk}(H_1(M^5)) \leq 1.$$

Proof. We assume that $\mathrm{rk}(H_1(M^5)) = k \geq 2$, in order to derive a contradiction. This assumption implies that the total Betti number of M^5 must be greater than or equal to 6. Recall again that M^5 decomposes as in Display 3.1, with F^3 the codimension two fixed point set of S^1. Therefore, $F^3 = S^2 \times S^1$ by Proposition 3.4, and so $b(F^3) = 4$. Further, since all singularities of the S^1-action are contained in F and N, and $b(M^T) = b(M^5) \geq 6$, by Proposition 2.7, then $M^T \cap N \neq \emptyset$. Then Lemma 3.5 gives us that N is of dimension 1 or 3. In fact, we will show that $\dim(N) = 3$.

Recall that E is a sphere bundle over both F and N. Since M is orientable, it follows from the disk bundle decomposition of M that E is also orientable. We first assume $\dim(N) = 1$, to derive a contradiction. Then E is an orientable S^3 bundle over S^1, that is, $E = S^3 \times S^1$, and so $H_3(E) \cong \mathbb{Q}$. However, it follows from the homology Mayer–Vietoris sequence that $H_4(M) \cong \mathbb{Q}^k \hookrightarrow H_3(E)$, $k \geq 2$, giving us the desired contradiction.

Hence $\dim(N) = 3$. Since $M^T \cap N \neq \emptyset$, the T^2-action on N is S^1-ineffective for some $S^1 \subset T^2$, and N is S^1-fixed point homogeneous for some other $S^1 \subset T^2$. So N is orientable and non-negatively curved. By Theorem 2.43, N is one of S^3, $L_{p,q}$, $S^2 \times S^1$ or $\mathbb{R}P^3\#\mathbb{R}P^3$. However, if N is one of S^3, $L_{p,q}$, or $\mathbb{R}P^3\#\mathbb{R}P^3$, then by the Gysin sequence with rational coefficients, $H_1(E) \cong \mathbb{Q}$. Poincaré duality then gives us that $H_3(E) \cong \mathbb{Q}$, which is a contradiction, as in the case of $\dim(N) = 1$.

Thus both F and N are diffeomorphic to $S^2 \times S^1$. So E is a principal S^1 bundle over $S^2 \times S^1$ and from the homology Mayer–Vietoris sequence of M, we have $H_4(M^5) \cong \mathbb{Q}^k \hookrightarrow H_3(E), k \geq 2$. Using the Gysin sequence with rational coefficients corresponding to the fibration $S^1 \hookrightarrow E \to S^2 \times S^1$, we see that $H_1(E) \cong \mathbb{Q}$ or $\mathbb{Q}^2$. Poincaré duality then implies that $k = 2$, so $b_1 = b_4 = 2$. Since $b_0 = b_5 = 1$, it then follows from the homology Mayer–Vietoris sequence of M^5 that $b_2 = b_3 \geq 2$, which implies $b(M^5) \geq 10$. But Proposition 3.4 guarantees that the total Betti number is bounded above by 8, a contradiction. ■

Proof of Theorem 3.1. By Lemma 3.6, $\mathrm{rk}(H_1(M^5)) \leq 1$ and by Theorem 2.47, $\widetilde{M}^5$ is either a closed, simply-connected, non-negatively curved manifold, or it splits isometrically as $\mathbb{R}^1 \times M^4$, where M^4 is a closed, simply-connected, non-negatively curved 4-manifold. The proof of Part (1) then follows directly from the work of Galaz-García and Spindeler [15] (cf. [7]). The proof of Part (2) follows by noting that since the T^2-action on M^5 has non-empty fixed point set, we may apply Theorem 2.1 to lift the T^2-action to $\widetilde{M}^5$. By Theorem 1 of Hano [25], the isometry group of $\mathbb{R}^1 \times M^4$ splits as the product of the isometry groups of $\mathbb{R}^1$ and of M^4. Since T^2 is a compact Lie group, this implies that the T^2-action on the $\mathbb{R}^1$ factor is trivial and fixes isolated points in M^4. In particular, M^4 is then a non-negatively curved torus manifold. The classification of 4-dimensional, non-negatively curved torus manifolds up to diffeomorphism follows from the work of Kleiner [27], Searle and Yang [33], and Galaz-García [11]. ■

4. The Classification of Graphs Corresponding to Closed, Non-Negatively Curved 5-Dimensional Odd GKM Manifolds

Our goal in this section is to classify the graphs corresponding to closed, non-negatively curved 5-dimensional odd GKM manifolds. We first prove that the lower curvature bound imposes severe restrictions on odd GKM graphs.

Proposition 4.1. *Let M^{2n+1} be a non-negatively curved odd GKM manifold. Then each square in the graph, $\bar{\Gamma}_M$, has valence one or two.*

Proof. The GKM condition in combination with the assumption of non-negative curvature guarantees us that the squares correspond to three-dimensional components of fixed point sets of codimension one subtori, which themselves are T^1-fixed point homogeneous, non-negatively curved, three-dimensional manifolds. The result then follows directly from Theorem 2.43. ■

The corresponding result in positive curvature follows directly from the classification of positively curved 3-manifolds due to Hamilton [24].

Proposition 4.2. *Let M^{2n+1} be a positively curved odd GKM manifold, then each square in the graph, $\bar{\Gamma}_M$, has valence one.*

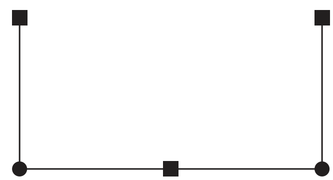
In the following theorem, we obtain a classification of the graphs of closed, non-negatively curved, 5-dimensional odd GKM manifolds.

Theorem 4.3. *Let M^5 be a non-negatively curved, 5-dimensional odd GKM manifold. Then its graph, $\bar{\Gamma}_{M^5}$, is one of the following possibilities, enumerated according to the total Betti number of M^5.*

(1) *For total Betti number equal to* 2, *we obtain a circle with two edges terminating in squares*

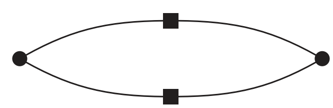
.

(2) *For total Betti number equal to* 4, *we have the following two possibilities*



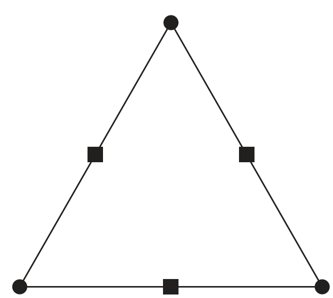
,

and



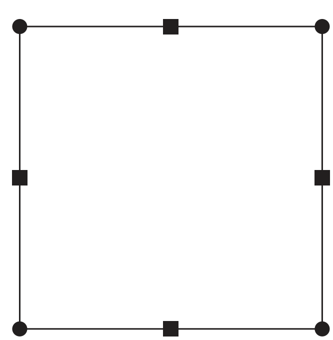
.

(3) *For total Betti number equal to* 6, *we obtain a closed circuit in the form of a triangle*



.

(4) *For total Betti number equal to* 8, *we obtain a closed circuit in the form of a quadrangle*



.

Proof. By Proposition 2.39, we have that the graph is connected, each circle is 2-valent, and the number of fixed circles equals half the total Betti number.

We showed in Proposition 3.4 that the total Betti number for such 5-manifolds is bounded between 2 and 8. In the case where it is 2, M^5 is a rational cohomology sphere, the T-fixed point set is a single circle, and the graph is necessarily of the described form. If the total Betti number is 4, the connectedness of the graph implies directly that it is of one of the two given shapes.

For the case of total Betti number 6 or 8, Proposition 3.4 tells us that any F^3 is diffeomorphic to $S^2 \times S^1$ in the decomposition of M^5 in Display 3.1, which has total Betti number 4. This in turn implies by Proposition 2.43 that every square in the graph has valence 2. The connectedness of the graph then directly implies the claim. ■

Example 4.4. *The standard examples for Theorem 4.3 are the T^2-actions on S^5, $S^2 \times S^3$, $S^4 \times S^1$, $\mathbb{C}P^2 \times S^1$, and $S^2 \times S^2 \times S^1$, respectively.*

Note that for positive curvature, using Proposition 4.2, it follows that only the first graph in Theorem 4.3 occurs and we immediately obtain the following theorem.

Theorem 4.5. *The unique graph, $\bar{\Gamma}_M$, corresponding to the positively curved 5-dimensional GKM manifolds is a circle with two edges terminating in squares*

 .

5. Proof of the Main Theorem 1.1

In this section we prove the Main Theorem 1.1, which we then use in Subsection 5.3 to prove Theorems 1.2, 1.3, and 1.5.

5.1. Proof of the Main Theorem 1.1

Let M^{2n+1} be a closed, non-negatively curved odd GKM_3 manifold. As shown in Proposition 4.1, any square in the graph, $\bar{\Gamma}_M$, has valence one or two.

We now show how to construct an abstract (even-dimensional) GKM_3 graph, (Γ, α), from a geometric odd GKM_3 graph, $(\bar{\Gamma}_M, \bar{\alpha}_M)$, for which the squares in $\bar{\Gamma}_M$ are only of valence one or two. We define (Γ, α) to be *the graph obtained from the geometric odd GKM_3 graph* $(\bar{\Gamma}_M, \bar{\alpha}_M)$, by

(1) replacing each circle by a vertex;

(2) replacing each 2-valent square, s, in 2 grounded edges by a single edge, e;

(3) labeling the edge e obtained from s by the weight of the square, $\bar{\alpha}_M(s)$, that is, defining $\alpha(e) := \bar{\alpha}_M(s)$ to be the axial function; and

(4) deleting all floating edges, together with their squares.

In order to facilitate discussion of the new graph, Γ, we denote the application, $\pi : \bar{\Gamma}_M \to \Gamma$, of the first three of these changes to $\bar{\Gamma}_M$, respectively, as follows:

(1) $\pi(c) = v$, where $c \in V_\circ(\bar{\Gamma}_M)$ and v is its image in Γ;

(2) $\pi(s) = e$, for $s \in V^2_\square$, where s is the square connected to c_1 and c_2, and e is its image in Γ, with $i(e) = \pi(c_1)$ and $t(e) = \pi(c_2)$.

(3) $\alpha(\pi(s)) := \bar{\alpha}_M(s)$, for $s \in V^2_\square$.

Before we show that (Γ, α) is an abstract GKM_3 graph, we illustrate the process of obtaining these graphs in the following examples.

Example 5.1. *Applying this construction to the odd-dimensional graphs in Theorem 4.3, we see that we obtain the following graphs.*

(1) *The image of the graph in Part* (1) *is a vertex*:

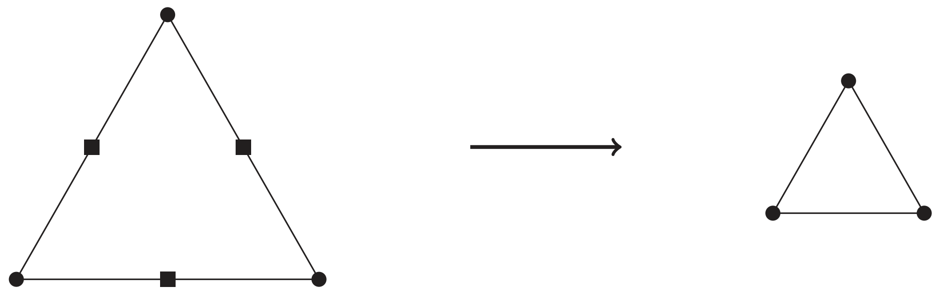

.

(2) *The images of the graphs in Part* (2) *are an interval and a lune, respectively*:

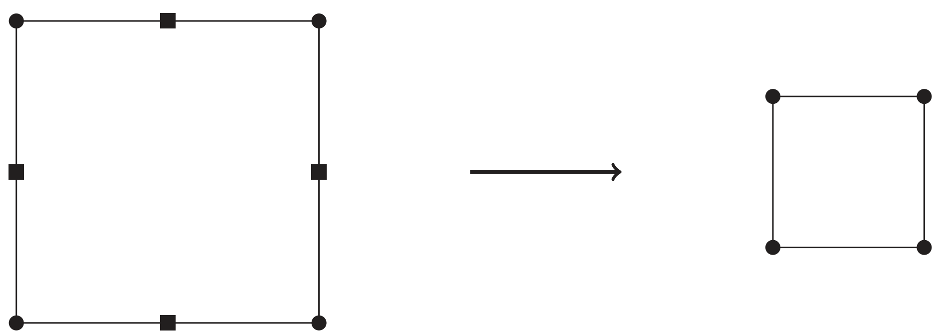

,


.

(3) *The image of the graph in Part* (3) *is a triangle*:



(4) *The image of the graph in Part* (4) *is a quadrangle*:



Lemma 5.2. *Let* $(\bar{\Gamma}_M, \bar{\alpha}_M)$ *be the odd* GKM_3 *graph corresponding to the closed, non-negatively curved* GKM_3 *manifold*, M^{2n+1}. *Let* (Γ, α) *be the graph obtained from* $(\bar{\Gamma}_M, \bar{\alpha}_M)$. *Then* (Γ, α) *admits a connection*, ∇, *such that* (Γ, α, ∇) *is an abstract* GKM_3 *graph*.

Proof. Our goal is to show that Γ is an abstract GKM_3 graph. Recall that by Definition 2.16, an abstract GKM_3 graph consists of a triple (Γ, α, ∇) such that Γ satisfies Properties 1 and 2 of Definition 2.16, the axial function α satisfies Property 4 of Definition 2.16, and that a connection, ∇, exists and satisfies Property 3 of Definition 2.16.

We begin by showing that Γ satisfies Properties 1 and 2 of Definition 2.16. To prove Property 1, we first claim that the graph Γ obtained from $\bar{\Gamma}_M$ is m-valent, for some $m \le n$. By Proposition 2.39, the odd-dimensional graph $\bar{\Gamma}_M$ is connected. So, to prove this claim it suffices to show that given any two circles, $c_1, c_2 \in V_\circ(\bar{\Gamma}_M)$, joined by a square, s_0, the number of floating edges emanating from c_1 and c_2 is the same. Let $e, f \in E_{c_1}(\bar{\Gamma}_M)$, such that e contains the square s_0, and f is a floating edge with square s. Observe that because of the GKM_3 condition, e and f uniquely determine a two-dimensional face of $\bar{\Gamma}_M$. Moreover, due to the linear dependence condition on the connection, it follows that $(\bar{\nabla}_M)_{c_1,c_2,s_0}(s)$ is a square belonging to the same two-dimensional face as s. Now, Theorem 4.3 tells us that for odd GKM_3 graphs corresponding to odd GKM_3 manifolds of non-negative curvature, there is only one graph of a two-dimensional face that has a floating edge. So, this graph has two floating edges, two circle vertices, and one grounded edge. This then implies that the connection sends a 1-valent square to a 1-valent square. Since the graphs of all the other two-dimensional faces have no floating edges, it follows that the connection sends a 2-valent square to a 2-valent square. We may conclude that at every circle in $\bar{\Gamma}_M$ there are exactly the same number of floating edges, and thus all vertices in the graph Γ have the same valency, and Property 1 holds. Note that the weights of the graph Γ are still 3-independent, and so Property 2 holds.

By the definition of the axial function $\alpha(e) := \bar{\alpha}_M(s)$, where $\pi(s) = e$, we see that Property 4 of Definition 2.16 also holds.

We now need to show that we have a connection on Γ that satisfies Property 3. In order to do so, we observe that we still have a notion of two-dimensional faces in Γ. Namely, given two edges e,f attached to some vertex v in Γ, there is a unique two-dimensional face in the odd-dimensional graph, $\bar{\Gamma}_M$, containing the edges corresponding to e and f. Moreover, this two-dimensional face in $\bar{\Gamma}_M$ has no floating edges, since the graphs with floating edges in Theorem 4.3 do not survive to graphs with a two-dimensional face, as can be seen in Example 5.1. We claim that this gives a well-defined connection on Γ, by sliding edges along edges inside these two-dimensional faces in the usual fashion. In fact, we may directly translate the conditions in Definition 2.34 satisfied by the connection on $\bar{\Gamma}_M$, by making the following substitutions:

$$(\bar{\nabla}_M)_{c_1,c_2,s_0}(s) \mapsto \nabla_e(f),$$

where $e = \pi(s_0)$, and $f = \pi(s)$. With this definition, it is straightforward to verify that the connection satisfies both Definition 2.15 and Property 3 of Definition 2.16. ■

Remark 5.3. *Lemma 5.2 tells us that the number of floating edges in $\bar{\Gamma}_M$ is independent of the vertex. This fact is reflected in the statement of Theorem 5.4.*

We now show how to express the equivariant cohomology of the geometric odd GKM_3 graph $(\bar{\Gamma}_M, \bar{\alpha}_M)$ in terms of the equivariant cohomology of the abstract graph (Γ, α)

obtained from $(\bar{\Gamma}_M, \bar{\alpha}_M)$. Recall that for an n-valent, abstract GKM graph (Γ, α, ∇), we say it is *orientable* if $H^{2n}(\Gamma, \alpha) \neq 0$.

In particular, for odd GKM_3 graphs, we prove the following result.

Theorem 5.4. *Let M^{2n+1} be a closed, non-negatively curved odd GKM_3 manifold and $(\bar{\Gamma}_M, \bar{\alpha}_M)$ its graph. Let k be the number of floating edges at a vertex in $\bar{\Gamma}_M$, let (Γ, α) be the graph obtained from $(\bar{\Gamma}_M, \bar{\alpha}_M)$, and assume that (Γ, α) is orientable. Then we have*

$$H_T^*(M) \cong H_T^*(\Gamma, \alpha) \otimes H^*(S^{2k+1})$$

as $S(\mathfrak{t}_{\mathbb{Q}}^*)$-algebras, where the $S(\mathfrak{t}_{\mathbb{Q}}^*)$-algebra structure on $H_T^*(\Gamma, \alpha) \otimes H^*(S^{2k+1})$ is the tensor product of the standard $S(\mathfrak{t}_{\mathbb{Q}}^*)$-algebra structure on $H_T^*(\Gamma, \alpha)$ and the trivial one on $H^*(S^{2k+1})$. Therefore, we obtain

$$H^*(M) \cong H^*(\Gamma, \alpha) \otimes H^*(S^{2k+1}).$$

Proof of Theorem 5.4. By Proposition 4.1, any square in the odd GKM graph of M has valence one or two. We denote by $V_\square^1$ and $V_\square^2$ the sets of squares with valence one and two, respectively. For a square $s \in V_\square^1$ we denote the unique circle connected to s by $c(s)$, whereas for $s \in V_\square^2$ we denote the two circles by $c_1(s)$ and $c_2(s)$, with any ordering.

Then Displays (2.1) and (2.2) reduce to the following divisibility relations for P_c and Q_c:

$$P_{c_1(s)} \equiv P_{c_2(s)} \bmod \alpha(s), \text{ and } Q_{c_1(s)} \equiv \pm Q_{c_2(s)} \bmod \alpha(s) \text{ for } s \in V_\square^2, \text{ and} \tag{5.1}$$

$$Q_{c(s)} \equiv 0 \bmod \alpha(s) \text{ for } s \in V_\square^1. \tag{5.2}$$

Then the equivariant cohomology of M is given by

$$H_T^*(M) \cong \{(P_c + Q_c\theta)_{c \in V_\circ} \,|\, P_c, Q_c \text{ satisfy Relations 5.1 and 5.2}\}.$$

By comparing the divisibility relations in Displays (5.1) and (5.2) for $H_T^*(M)$ and the description of the equivariant cohomology in terms of the GKM graph in even dimensions in Display (2.1), we see that the divisibility relations imposed by grounded edges in the odd-dimensional GKM graph $\bar{\Gamma}_M$ are precisely those imposed by the edges in Γ. As a consequence, we see that we obtain an isomorphism

$$\psi : H_T^{even}(M) \longrightarrow H_T^*(\Gamma, \alpha, \nabla). \tag{5.3}$$

Moreover, we also see by Proposition 2.8 and Definition 2.24 that $H^{even}(M) = H^*(\Gamma, \alpha, \nabla)$.

For the cohomology in odd dimensions, note that the Q_c have to be divisible by all k weights of the floating edges at c, so $\deg(Q_c) \geq k$. Thus $H_T^{2l+1}(\Gamma, \alpha) = 0$ for $l < k$. On the other hand, as Γ is an orientable $(n-k)$-valent graph, we have that $H^{2(n-k)}(M) = H^{2(n-k)}(\Gamma, \alpha) \neq 0$. Poincaré duality now implies that $H^{2k+1}(M) \neq 0$. By Proposition 2.8, $H_T^*(M) \to H^*(M)$ is onto, and it follows that $H_T^{2k+1}(M) \neq 0$. So there exists a nonzero element $\omega \in H_T^{2k+1}(M)$.

Our goal is now to show that we may multiply elements of the even-dimensional cohomology by ω to obtain an $S(\mathfrak{t}_{\mathbb{Q}}^*)$-module isomorphism from $H_T^{even}(M)$ to $H_T^{odd}(M)$. Using the divisibility relations in Display (5.2), we may express any nontrivial element ω in $H_T^{2k+1}(M)$ as

$$\omega = (\omega_c \theta)_{c \in V_\circ} \in H_T^{2k+1}(M),$$

where

$$\omega_c = a_c\, \alpha(s_1(c)) \cdot \alpha(s_2(c)) \cdot \ldots \cdot \alpha(s_k(c)), \tag{5.4}$$

$a_c \in \mathbb{Q}$ depends only on the circle c, and $s_1(c), \ldots, s_k(c)$ are the squares in the k floating edges connected to c. The ω_c must also satisfy the first set of divisibility relations in Display (5.1), and since $\omega \in H_T^{2k+1}(M)$, this is equivalent to requiring $\omega_{c_1(s)} \equiv \pm\omega_{c_2(s)} \mod \alpha(s)$, for $s \in V_\square^2$.

We now want to show that $a_c \neq 0$ for all $c \in V_\circ$. We will argue by contradiction. Assume that $a_{c'} = 0$ for some c'. Since ω is non-trivial, there is a $c'' \neq c'$ such that $a_{c''} \neq 0$. The connectivity of the graph implies that there must be some $s \in V_\square^2$, such that $a_{c_1(s)} = 0$, but $a_{c_2(s)} \neq 0$. Then $0 = \omega_{c_1(s)}$ by Equation (5.4). But then the divisibility relation in (5.1) gives us that $\omega_{c_1(s)} \equiv \omega_{c_2(s)} \equiv 0 \mod \alpha(s)$, and Equation (5.4) tells us that $\alpha(s)$ is then a scalar multiple of one of the $\alpha(s_i)$, $1 \leq i \leq k$. However, Part 2 of the definition of a GKM-manifold tells us $\alpha(s)$ and $\alpha(s_i)$ are pairwise linearly independent for $1 \leq i \leq k$, a contradiction.

We now claim that $a_c \neq 0$ for all $c \in V_\circ$ implies that $\dim(H_T^{2k+1}(M)) = 1$. Suppose instead that we have two linearly independent elements $\mu, \omega \in H_T^{2k+1}(M)$, where μ_c and ω_c are as in Equation (5.4). Then for some c, let $\eta_c = \gamma\omega_c - \mu_c$, with $\gamma = a_c^\mu / a_c^\omega$. But then for this same c, $\eta_c = 0$, contradicting the fact that for any element of $H_T^{2k+1}(M)$, $a_c \neq 0$ for all $c \in V_\circ$.

It now follows that multiplication with ω defines an $S(\mathfrak{t}^*_{\mathbb{Q}})$-module injection $H_T^{even}(M) \to H_T^{odd}(M)$, that is,

$$\begin{aligned} \phi : H_T^{even}(M) &\longrightarrow H_T^{odd}(M) \\ \psi &\longmapsto \psi \cdot \omega, \end{aligned} \tag{5.5}$$

with $\omega \in H_T^{2k+1}(M)$. We claim that ϕ is an isomorphism. To do so, we must show that ϕ is onto. Let $Q = (Q_c\theta)_{c\in V_\circ} \in H_T^{odd}(M)$. For each $c \in V_\circ$, the polynomial Q_c is divisible by any $\alpha(s)$ with $c(s) = c$ for $s \in V_\square^1$, and so we can write $Q = (Q_c\theta) = (P_c\,\omega_c\theta) = P \cdot \omega$, for some $P = (P_c)$. Then to show that ϕ is onto, we have to show that $(P_c)_{c\in V_\circ} \in H_T^{even}(M)$. That is, we must verify that the P_c satisfy the divisibility relation (5.1) for all $c \in V_\circ$. Let $s' \in V_\square^2$ be arbitrary. Then $\alpha(s')$ divides both $Q_{c_1(s')} \pm Q_{c_2(s')}$ and $\omega_{c_1(s')} \pm \omega_{c_2(s')}$, where $\pm$ is taken to be the same sign in both expressions. Then we compute

$$\begin{aligned} Q_{c_1(s)} \pm Q_{c_2(s)} &= P_{c_1(s)}\omega_{c_1(s)} \pm P_{c_2(s)}\omega_{c_2(s)} \\ &= (P_{c_1(s)} - P_{c_2(s)})\,\omega_{c_1(s)} + P_{c_2(s)}(\omega_{c_1(s)} \pm \omega_{c_2(s)}), \end{aligned}$$

and the divisibility assumptions imply that $\alpha(s')$ divides $(P_{c_1(s')} - P_{c_2(s')})\,\omega_{c_1(s')}$. By the same argument used to show that $a_c \neq 0$ for all $c \in V_\circ$, $\alpha(s')$ does not divide $\omega_{c_1(s')}$, since $s' \in V_\square^2$. So $\alpha(s')$ has to divide $P_{c_1(s')} - P_{c_2(s')}$, which is precisely the divisibility relation (5.1). So, we have shown that $(P_c)_{c\in V_\circ} \in H_T^{even}(M)$. Hence, ϕ is onto and multiplication by ω defines an $S(\mathfrak{t}^*_{\mathbb{Q}})$-module isomorphism.

We can now prove that $H_T^*(M) \cong H_T^*(\Gamma, \alpha) \otimes H^*(S^{2k+1})$, by extending the isomorphism in Display (5.3) via the isomorphism in Display (5.5) to an $S(\mathfrak{t}^*_{\mathbb{Q}})$-algebra isomorphism

$$\begin{aligned} \Psi : H_T^*(\Gamma, \alpha) \otimes H^*(S^{2k+1}) &\longrightarrow H_T^*(M) \\ \gamma \otimes id + \beta \otimes \mu_{S^{2k+1}} &\longmapsto \psi(\gamma) + \phi(\psi(\beta)) = \psi(\gamma) + \psi(\beta)\omega, \end{aligned}$$

where $\gamma, \beta \in H_T^*(\Gamma, \alpha)$, and $\mu_{S^{2k+1}}$ is the volume form of S^{2k+1}.

The second statement of the theorem then follows immediately from Proposition 2.8 and Definition 2.24 because the $S(\mathfrak{t}^*_{\mathbb{Q}})$-module structure on $H^*(\Gamma, \alpha) \otimes H^*(S^{2k+1})$ only exists on the first factor. ■

Theorem 1.1 is now a direct consequence of Theorem 5.4.

5.2. Orientability of the associated graph Γ

Without the assumption of orientability of the associated graph, the conclusion of the Main Theorem 5.4 does not hold. We include an example here. Let $(S^n(1), g)$ denote the sphere of radius one with the standard metric. Consider

$$M^9 = S^2(1) \times S^2(1) \times S^2(1) \times S^3(1)$$

with the product metric and with a T^4-action given as a product of four T^1-actions, where the action on each S^2 is by rotation, and on $S^3 \subset \mathbb{C}^2$ is fixed point homogeneous, namely,

$$(e^{i\theta}, (z_1, z_2)) \mapsto (e^{i\theta} z_1, z_2).$$

Then consider the $\mathbb{Z}_2$-action on M, given by the antipodal map on each S^2 and on S^3 by

$$(z_1, z_2) \mapsto (z_1, \bar{z}_2).$$

As the $\mathbb{Z}_2$-action on M commutes with the T^4-action and is free, orientation-preserving, and by isometries, the quotient

$$N^9 = M/(\mathbb{Z}_2)$$

is a non-negatively curved, closed, orientable manifold with the induced T^4-action.

Using the transfer isomorphism (see, e.g., Theorem III.2.4 in [3]), we obtain $H^*(N) = H^*(M)^{\mathbb{Z}_2}$, and we compute the Betti numbers of N to be

$$b_i(N) = \begin{cases} 1 & i = 0, 9 \\ 3 & i = 4, 5 \\ 0 & \text{otherwise.} \end{cases}$$

Both M and N are orientable; however, the associated GKM graph of N is the quotient of a 4-dimensional cube, $\mathrm{I}^4/\mathbb{Z}_2$, which is not orientable.

We claim that the T^4-action on N is odd GKM$_4$, and hence odd GKM$_3$. First, the fixed point set of the T^4-action on N consists of exactly 4 circles. Second, the condition on the weights is satisfied, since the action of T^4 on M is odd GKM$_4$, and the condition on the weights of the isotropy representations at the fixed points of N is inherited from M. Third, we can use Proposition 2.7 to see that the T^4-action is equivariantly formal: as computed above, the sum of the Betti numbers of N is equal to 8, as is the sum of the Betti numbers of N^{T^4}. So, the claim holds. However, the cohomology ring of N does not split off the cohomology of an odd-dimensional sphere, and so if we remove the hypothesis on the orientability of the associated graph Γ, we see that the conclusion of Theorem 1.1 does not hold for N.

Note that in this example N is not simply-connected, and so we pose the following question:

Question 5.5. *If N is a closed, simply-connected, odd GKM_3 manifold, is the associated graph Γ orientable?*

5.3. Applications of Theorem 5.4

We consider some special subcases of Theorem 5.4. Firstly, if the metric on the GKM_3 manifold is positively curved, then by Theorem 3.2, every two-dimensional face of the GKM graph has only one circle. This implies that the GKM graph, $\bar{\Gamma}_M$, is the pinwheel depicted in Example 2.38, and so Γ is a single vertex. In particular, a single vertex graph is orientable. Theorem 1.5 is then immediate.

As indicated in the Introduction, the proof of Theorem 1.2 follows from the proof of Theorem 2.26.

Theorem 1.3 follows in the same way using Theorem 2.27. In order to apply Theorem 2.27, we need to verify that Γ, the GKM_4 graph obtained from the odd-dimensional GKM graph of M, $\bar{\Gamma}_M$, is a graph with *small three-dimensional faces* (see Definition 3.5 in [18]). As noted in [18], a GKM_4 graph that has two-dimensional faces with at most 4 vertices must have small three-dimensional faces. Since M has non-negative curvature, Theorem 4.3 tells us that the two-dimensional faces of $\bar{\Gamma}_M$ have at most 4 circle vertices, and hence the two-dimensional faces of Γ have at most 4 vertices. The result follows.

6. Invariant Almost Contact Structures

The goal of this section is to prove Theorem 1.4 of the Introduction. We begin by recalling the definition of an almost contact structure.

Definition 6.1 (Almost contact structure). *An almost contact structure (ϕ, ξ, η) on a $(2n+1)$-manifold M consists of a $(1,1)$-tensor field ϕ, a vector field ξ, and a differential one-form η such that*

$$\eta(\xi) = 1 \text{ and } \phi^2(X) = -X + \eta(X)\xi,$$

for any vector field X on M. Note that the vector field ξ, which is called the Reeb vector field, is uniquely determined by ϕ and η, namely at a point p it is the unique vector ξ_p such that $\phi_p(\xi_p) = 0$ and $\eta_p(\xi_p) = 1$.

Given an almost contact structure (ϕ, ξ, η) on M^{2n+1}, if $N^{2k+1} \subset M^{2n+1}$ is a submanifold such that ξ is tangent to N, and ϕ restricts to a well-defined tensor field on N, then (ϕ, ξ, η) restricts to an almost contact structure on N. In this case we call N an *almost contact* submanifold.

The following lemma may be well-known, but could not be located by the authors elsewhere in the literature. For completeness, a proof is presented here.

Lemma 6.2. *Let (ϕ, ξ, η) be an almost contact structure on a manifold M, invariant under the action of a compact Lie group G. Then every connected component of the fixed point set M^G of the action is an almost contact submanifold.*

Proof. Let N be a connected component of M^G, and recall that N is an embedded submanifold of M by work the of Kobayashi [28]. Then for every point $p \in N$, the tangent space of N is given by

$$T_pN = (T_pM)^G,$$

the set of vectors fixed by the isotropy representation of G at p. The G-invariance of ϕ and the fact that p is fixed by G then implies that ϕ maps T_pN to itself. For the same reasons, it follows that ξ is tangent to N. Thus, the connected components of M^G are almost contact submanifolds. ■

Combining Lemma 6.2 and the fact that the almost contact structure gives us a T-invariant almost complex structure on $\ker \eta_p$, we obtain the following proposition.

Proposition 6.3. *Let M^{2n+1} be an odd GKM manifold with a T-invariant almost contact structure (ϕ, ξ, η). Then the following are true.*

(1) *Every component of the fixed point set of T is an isolated, closed flow line of ξ.*
(2) *At any fixed point p of the torus action, the weights of the isotropy representation at p are well-defined elements of $\mathfrak{t}^*_{\mathbb{Q}}$.*

Proof. By the definition of an odd GKM action, every component of the fixed point set of T is an isolated circle. Lemma 6.2 then gives us that the restriction of (ϕ, ξ, η) to any of these isolated circles is an almost contact submanifold, and the first statement follows.

To prove the second statement, we note that ϕ defines a T-invariant almost complex structure on $\ker \eta_p$. Since the almost contact structure is T-invariant, we have $\eta_{tp}(dt_p(v)) = \eta_p(v)$ for all $v \in T_pM$ and $t \in T$. So at a fixed point p, $\ker(\eta_p)$ is T-invariant. This fact combined with the fact that for each point the tangent space to $\ker(\eta_p)$ has a complex structure, gives us a *complex* T-representation at p, and the second statement follows. ■

By Proposition 6.3, in the presence of a T-invariant almost contact structure on M, we have that the weights are well-defined elements of $\mathfrak{t}^*_{\mathbb{Q}}$. This then allows us to slightly modify the odd GKM graph, $\bar{\Gamma}_M$, of the T-action on M, which we call a *signed odd GKM graph*, as follows. We consider the same underlying graph, $\bar{\Gamma}_M$, but now we assign weights to edges, not to squares, that is, to each edge connecting a circle c to a square s, we assign the corresponding weight of the isotropy representation at the circle c, which is an element in $\mathfrak{t}^*_{\mathbb{Q}}$. Regarded modulo ± 1, this weight is the same as the weight assigned to the square s in the original odd-dimensional graph, $\bar{\Gamma}_M$. If, in addition, the signed weights on the edges emanating from a square sum to 0, we call such a graph *alternating*. This leads us to make the following definition.

Definition 6.4. *If the signed odd GKM graph induced from the invariant almost contact structure on the odd GKM manifold is alternating, then we say that the almost contact structure is* alternating.

The connection of a signed odd GKM graph is modified as follows (cf. Definition 2.34). Formally, if we denote the set of edges emanating from a circle c by $E(c)$, then the axial function $\bar{\alpha}_M$ is a collection of maps $E(c) \to \mathfrak{t}^*_{\mathbb{Q}}$, for all c. The connection can be regarded as a collection of maps $(\bar{\nabla}_M)_{c_1,c_2,s_0} : E(c_1) \to E(c_2)$, where $c_1, c_2 \in V_\circ(s_0)$, and it satisfies that for every edge $e \in E(c_1)$ there exists a constant $c \in \mathbb{Z}$ such that

$$\bar{\alpha}_M((\bar{\nabla}_M)_{c_1,c_2,s_0}(e)) = \bar{\alpha}_M(e) + c\bar{\alpha}_M(e_0),$$

where e_0 is an edge connecting c_1 or c_2 with s_0.

Remark 6.5. *If M^{2n} is a closed, non-negatively curved GKM$_k$ manifold admitting an invariant almost complex structure, then the associated classical GKM graph is a* signed *GKM graph (see Remark 2.18). At the beginning of Subsection 5.1 we showed how one may obtain an abstract GKM graph from an odd GKM graph whose squares have valence less than or equal to two. It follows immediately that we can obtain classical* signed *GKM graphs from* alternating *odd GKM graphs.*

We now restate a result from the proof of Lemma 5.6 in [17], noting that this result is independent of curvature.

Lemma 6.6. [17] *Let Γ be an abstract, signed GKM_3 graph. Then Γ admits no biangles.*

Recall that we denote by Σ^k the orbit space of the linear, effective action of the k-dimensional torus on S^{2k}. The following corollary to Lemma 6.6 is immediate (cf. the proof of Lemma 7.1 in [18]).

Corollary 6.7. Let Γ be an abstract, signed GKM_3 graph. Then there are no maximal simplices in Γ with the combinatorial type of Σ^k.

Before we prove Theorem 1.4, we first recall the definition of a generalized Bott manifold.

Definition 6.8 (**Generalized Bott manifold**). *We say that a manifold X is a* generalized Bott manifold *if it is the total space of an iterated $\mathbb{C}P^{n_i}$-bundle*

$$X = X_k \to X_{k-1} \to \cdots \to X_1 \to X_0 = \{pt\},$$

where each X_i is the total space of the projectivization of a Whitney sum of $n_i + 1$ complex line bundles over X_{i-1}.

Remark 6.9. *Torus manifolds over $\prod \Delta^{n_i}$, where Δ^{n_i} denotes the standard simplex of dimension n_i, admitting an invariant almost complex structure were classified in Choi, Masuda, and Suh [4]. They are all diffeomorphic to generalized Bott manifolds.*

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let M^{2n+1} be a closed, non-negatively curved, odd GKM_4 manifold, which admits an invariant almost contact structure that is alternating. By assumption, $\bar{\Gamma}_M$ is alternating, and so the abstract GKM graph obtained from it is signed. By Corollary 6.7, a signed, abstract GKM_3 graph has no maximal simplices with the combinatorial type of Σ^k. Since GKM_4 manifolds are also GKM_3, it follows that Γ contains no maximal simplices with the combinatorial type of Σ^k.

We can now argue as in Section 7 of [18] to obtain the result. We briefly outline the proof here for the sake of completeness. As noted at the end of Section 5.3, non-negative curvature and the GKM_4 condition guarantee that Γ will have small three-dimensional faces. So we may apply Theorem 3.11 of [18] to show that Γ is finitely covered by a graph, $\tilde{\Gamma}$, which is the vertex-edge graph of a finite product of simplices. One then shows that the quasitoric manifold corresponding to the graph $\tilde{\Gamma}$ admits an invariant complex structure in Theorem 7.1 of [18]. Applying Theorem 6.4 of [4] then shows us that $\tilde{\Gamma}$ is the GKM graph of a generalized Bott manifold. Finally, we use Theorem 7.5 of [18], to show that $\tilde{\Gamma} = \Gamma$. Thus, we may apply the GKM theorem to conclude that the rational cohomology ring of M is the tensor product of the rational cohomology ring of an odd-dimensional sphere and the rational cohomology ring of a generalized Bott manifold, as desired. ∎

It seems very likely that the graphs corresponding to non-negatively curved, odd GKM_3 manifolds admitting an invariant almost contact structure are alternating. We finish with the following conjecture.

Conjecture 6.10. *Let M^{2n+1} be a closed, non-negatively curved odd-dimensional GKM_3 manifold admitting an invariant almost contact structure. Then the odd GKM_3 graph corresponding to M^{2n+1} is alternating.*

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References

[1] Allday, C. and V. Puppe. *Cohomological Methods in Transformation Groups*. Cambridge: Cambridge University Press, 1993.

[2] Barden, D. "Simply connected 5-manifolds." *Ann. Math. (2)* 82 (1965): 365–85.

[3] Bredon, G. *Introduction to Compact Transformation Groups*, vol. 48. New York–London: Academic Press, 1972.

[4] Choi, S., M. Masuda, and D. Y. Suh. “Quasitoric manifolds over a product of simplices.” *Osaka J. Math.* 47 (2010): 109–29.

[5] Chang, T. and T. Skjelbred. “The topological Schur lemma and related results.” *Ann. Math. (2)* 100 (1974): 307–21.

[6] Cheeger, J. and D. Gromoll. “The splitting theorem for manifolds of nonnegative Ricci curvature.” *J. Differential Geom.* 6 (1971): 119–28.

[7] Dong, Z., C. Escher, and C. Searle. “Almost torus manifolds of non-negative curvature.” (2018): preprint arXiv:1811.01493v1.

[8] Escher, C. and C. Searle. “Non-negatively curved 6-dimensional manifolds of almost maximal symmetry rank.” *J. Geom. Anal.* 29 (2019): 1002–17.

[9] Escher, C. and C. Searle. “Torus actions, maximality, and non-negative curvature.” *J. Reine Angew. Math.* (2021): doi.org/10.1515/crelle-2021-0035.

[10] Fang, F. and X. Rong. “Homeomorphism classification of positively curved manifolds with almost maximal symmetry rank.” *Math. Ann.* 332 (2005): 81–101.

[11] Galaz-Garcia, F. “Nonnegatively curved fixed point homogeneous manifolds in low dimensions.” *Geom. Dedicata* 157 (2012): 367–96.

[12] Galaz-García, F. and M. Kerin. “Cohomogeneity two torus actions on non-negatively curved manifolds of low dimension.” *Math. Z.* 276 (2014): 133–52.

[13] Galaz-Garcia, F. and C. Searle. “Low-dimensional manifolds with non-negative curvature and maximal symmetry rank.” *Proc. Amer. Math. Soc.* 139 (2011): 2559–64.

[14] Galaz-García, F. and C. Searle. “Nonnegatively curved 5-manifolds with almost maximal symmetry rank.” *Geom. Topol.* 18 (2014): 1397–435.

[15] Galaz-García, F. and W. Spindeler. “Nonnegatively curved fixed point homogeneous 5-manifolds.” *Ann. Global Anal. Geom.* 41 (2012): 253–63. Erratum, Ann. Global Anal. Geom. 45 (2014): 151–3.

[16] Goertsches, O., H. Nozawa, and D. Töben. “Equivariant cohomology of K-contact manifolds.” *Math. Ann.* 354 (2012): 1555–82.

[17] Goertsches, O. and M. Wiemeler. “Positively curved GKM-manifolds.” *Int. Math. Res. Not.* 22 (2015): 12015–41.

[18] Goertsches, O. and M. Wiemeler. “Non-negatively curved GKM orbifolds.” *Math. Z.* (2020): preprint arXiv:1802.05871v3.

[19] Goresky, M., R. Kottwitz, and R. MacPherson. “Equivariant cohomology, Koszul duality, and the localization theorem.” *Invent. Math.* 131 (1998): 25–83.

[20] Grove, K. and C. Searle. “Positively curved manifolds with maximal symmetry rank.” *J. Pure Appl. Algebra* 91 (1994): 137–42.

[21] Grove, K. and B. Wilking. “A knot characterization and 1-connected nonnegatively curved 4-manifolds with circle symmetry.” *Geom. Topol.* 18, no. 5 (2014): 3091–110.

[22] Guillemin, V. and S. Sternberg. “Supersymmetry and Equivariant de Rham Theory.” *Mathematics, Past and Present*. Berlin: Springer, 1999.

[23] Guillemin, V. and C. Zara. “1-skeleta, Betti numbers, and equivariant cohomology.” *Duke Math. J.* 107 (2001): 283–349.

[24] Hamilton, R. “Three-manifolds with positive Ricci curvature.” *J. Differential Geom.* 17 (1982): 255–306.

[25] Hano, J. “On affine transformations of a Riemannian manifold.” *Nagoya Math. J.* 9 (1955): 99–109.

[26] He, C. “Localization of certain odd-dimensional manifolds with torus actions.” (2016): preprint arXiv:1608.04392v5.

[27] Kleiner, B. “Riemannian Four-Manifolds With Nonnegative Curvature and Continuous Symmetry.” PhD thesis, UC Berkeley, 1990.

[28] Kobayashi, S. “Fixed points of isometries.” *Nagoya Math. J.* 13 (1958): 63–8.

[29] Kuroki, S. “Introduction to GKM theory.” *Trends Math.* 11 (2009): 113–29.

[30] Mostert, P. S. “On a compact Lie group acting on a manifold.” *Ann. Math. (2)* 65 (1957): 447–55.

[31] Neumann, W. D. “3-Dimensional G-Manifolds With 2-Dimensional Orbits.” *Proc. Conf. on Transformation Groups (New Orleans, La., 1967)*. 220–2. New York: Springer, 1968.

[32] Rong, X. “Positively curved manifolds with almost maximal symmetry rank.” *Geom. Dedicata* 95 (2002): 157–82.

[33] Searle, C. and D. Yang. “On the topology of non-negatively curved simply-connected 4-manifolds with continuous symmetry.” *Duke Math. J.* 74 (1994): 547–56.

[34] Smale, S. “On the structure of 5-manifolds.” *Ann. Math. (2)* 75 (1962): 38–46.

[35] Spindeler, W. “S^1-Actions on 4-Manifolds and Fixed Point Homogeneous Manifolds of Nonnegative Curvature.” PhD thesis, Westfälische Wilhelms-Universität Münster, 2014.

[36] Switzer, R. *Algebraic Topology—Homology and Homotopy*. New York–Heidelberg: Springer, 1975.

[37] Wilking, B. “Torus actions on manifolds of positive sectional curvature.” *Acta Math.* 191 (2003): 259–97.