

COMPUTING NODAL DEFICIENCY WITH A REFINED DIRICHLET-TO-NEUMANN MAP

G. BERKOLAIKO, G. COX, B. HELFFER, AND M.P. SUNDQVIST

ABSTRACT. Recent work of the authors and their collaborators has uncovered fundamental connections between the Dirichlet-to-Neumann map, the spectral flow of a certain family of self-adjoint operators, and the nodal deficiency of a Laplacian eigenfunction (or an analogous deficiency associated to a non-bipartite equipartition). Using a refined construction of the Dirichlet-to-Neumann map, we strengthen all of these results, in particular getting improved bounds on the nodal deficiency of degenerate eigenfunctions. Our framework is very general, allowing for non-bipartite partitions, non-simple eigenvalues, and non-smooth nodal sets. Consequently, the results can be used in the general study of spectral minimal partitions, not just nodal partitions of generic Laplacian eigenfunctions.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded set, with piecewise C^2 boundary, and suppose φ_* is an eigenfunction of the Dirichlet Laplacian $-\Delta$ on Ω , with eigenvalue λ_* . We denote by Γ the nodal set of φ_* ,

$$\Gamma = \overline{\{x \in \Omega : \varphi_*(x) = 0\}},$$

and by $k(\varphi_*)$ the number of nodal domains of φ_* , i.e. the number of connected components of the set $\{x \in \Omega : \varphi_*(x) \neq 0\}$. We also let $\ell(\varphi_*) = \min\{m : \lambda_m = \lambda_*\}$ denote the minimal label of the eigenvalue λ_* , where $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ are the ordered Dirichlet eigenvalues of Ω , repeated according to their multiplicity. The Courant nodal domain theorem states that $k(\varphi_*) \leq \ell(\varphi_*)$, or equivalently, that the nodal deficiency $\delta(\varphi_*) := \ell(\varphi_*) - k(\varphi_*)$ is nonnegative.

Letting $\text{DN}(\Gamma, \lambda_*)$ denote the two-sided Dirichlet-to-Neumann map on Γ , which will be defined below, we now state a special case of our main result.

Theorem 1.1. *The eigenfunction φ_* has nodal deficiency*

$$\delta(\varphi_*) = \text{Mor DN}(\Gamma, \lambda_*), \tag{1.1}$$

and the corresponding eigenvalue λ_ has multiplicity*

$$\dim \ker(\Delta + \lambda_*) = \dim \ker \text{DN}(\Gamma, \lambda_*) + 1. \tag{1.2}$$

The symbol Mor denotes the *Morse index*, i.e. the number of negative eigenvalues of the operator $\text{DN}(\Gamma, \lambda_*)$, which is self-adjoint and lower semi-bounded. A similar formula for the nodal deficiency appeared in [12]; see also [8]. The version of the Dirichlet-to-Neumann map appearing in the above theorem is more involved than the one used in [8, 12], but consequently gives us a stronger result, as we now explain.

We denote the nodal domains of φ_* by D_1, \dots, D_k . When defining the Dirichlet-to-Neumann map, one must take into account that λ_* is a Dirichlet eigenvalue on each D_i . Introducing the notation $\Gamma_i = \overline{\partial D_i} \cap \overline{\Omega}$, we define the closed subspace

$$S = \left\{ g \in L^2(\Gamma) : \int_{\Gamma_i} g_i \frac{\partial \varphi_{*,i}}{\partial \nu_i} = 0, \ i = 1, \dots, k \right\} \tag{1.3}$$

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of $L^2(\Gamma)$, where g_i denotes the restriction of g to Γ_i , $\varphi_{*,i}$ is the restriction of φ_* to D_i , and ν_i is the outward unit normal to D_i . For sufficiently smooth $g \in S$, each boundary value problem

$$\begin{cases} -\Delta u_i = \lambda_* u_i & \text{in } D_i, \\ u_i = g_i & \text{on } \partial D_i \cap \Omega, \\ u_i = 0 & \text{on } \partial D_i \cap \partial\Omega, \end{cases} \quad (1.4)$$

has a solution u_i^g . Defining a function $\gamma_N u^g$ on Γ by

$$\gamma_N u^g|_{\Gamma_i \cap \Gamma_j} = \frac{\partial u_i^g}{\partial \nu_i} + \frac{\partial u_j^g}{\partial \nu_j} \quad (1.5)$$

for all $i \neq j$, we let

$$\text{DN}(\Gamma, \lambda_*)g = \Pi_S(\gamma_N u^g), \quad (1.6)$$

where Π_S denotes the $L^2(\Gamma)$ -orthogonal projection onto the subspace S .

The solution to the problem (1.4) is non-unique, but the choice of particular solution u_i^g is irrelevant for the definition on account of the projection in (1.6). In Theorem 3.1 we use this freedom to give an equivalent formulation of the Dirichlet-to-Neumann map that does not involve Π_S .

The earlier works [8, 12] avoided the difficulty of defining the Dirichlet-to-Neumann map at a Dirichlet eigenvalue by evaluating the quantities in Theorem 1.1 at $\lambda_* + \varepsilon$, with a small positive ε . The resulting expression for the nodal deficiency was

$$\delta(\varphi_*) = \text{Mor DN}(\Gamma, \lambda_* + \varepsilon) + 1 - \dim \ker(\Delta + \lambda_*). \quad (1.7)$$

Unlike (1.1), which immediately implies $\delta(\varphi_*) \geq 0$, the equality (1.7) only yields the same conclusion if we know that λ_* is simple, or have additional information about the spectrum of $\text{DN}(\Gamma, \lambda_* + \varepsilon)$. Therefore, we obtain a more useful result by computing the Dirichlet-to-Neumann map at λ_* instead of $\lambda_* + \varepsilon$.

An even stronger motivation for eliminating the ε -perturbation is that the unperturbed operator $\text{DN}(\Gamma, \lambda_*)$ appears naturally as the Hessian of the energy functional on the space of generic equipartitions [7]. The minima of this functional are spectral minimal partitions, as defined in [17], which are often non-bipartite (unlike the decompositions of Ω into nodal domains of an eigenfunction φ_* , mentioned above). One of the simplest examples of a non-bipartite partition is the so-called Mercedes star partition, which is an (unproven but natural) candidate for the minimal 3-partition of the disk; see [11] and references therein. The main result of this paper, Theorem 1.7, is a generalization of Theorem 1.1 to partitions that are not necessarily bipartite, but have certain criticality properties that make them prime candidates for being minimal.

We first recall¹ that a k -partition of Ω is a family $\mathcal{D} = \{D_i\}_{i=1}^k$ of mutually disjoint, open, connected subsets of Ω , with $\overline{\Omega} = \overline{D_1} \cup \cdots \cup \overline{D_k}$. We say that the subdomains D_i and D_j are *neighbors* if $\text{Int}(\overline{D_i} \cup \overline{D_j}) \neq D_i \cup D_j$. We also recall that \mathcal{D} is *bipartite* if we can color the partition with two colors in such a way that any two neighbors have different colors. Defining the *boundary set* of the partition to be

$$\Gamma := \bigcup_i (\partial D_i \cap \Omega), \quad (1.8)$$

we next impose a suitable regularity assumption on \mathcal{D} .

Definition 1.2. A partition \mathcal{D} is said to be *weakly regular* if its boundary set Γ satisfies:

- (i) Except for finitely many critical points $\{x_\ell\} \subset \Gamma \cap \Omega$, Γ is locally diffeomorphic to a regular curve. In a neighborhood of each x_ℓ , Γ is a union of $\nu_\ell \geq 3$ smooth half-curves with one end at x_ℓ .
- (ii) $\Gamma \cap \partial\Omega$ consists of a finite set of boundary points $\{z_m\}$. In a neighborhood of each z_m , Γ is a union of ρ_m distinct smooth half-curves with one end at z_m .

¹Here we are following the convention of [18]; in [11, 17] such a \mathcal{D} is called a *strong partition*.

- (iii) The half-curves meeting at each x_ℓ and z_m are pairwise transversal to one another, and to $\partial\Omega$.

The subdomains are only allowed to have corners at points where at least three subdomains meet, or on $\partial\Omega$. However, the definition still allows for partitions where a subdomain D_i is a neighbor of itself, as shown in Figure 1.1. To rule out such examples, we say that a partition \mathcal{D} is *two-sided*² if $\text{Int}(\overline{D_i}) = D_i$ for each i . For the rest of the paper we will only consider two-sided, weakly regular partitions. This is a reasonable hypothesis, as it is satisfied by nodal partitions, and more generally by spectral minimal partitions [10, 17].

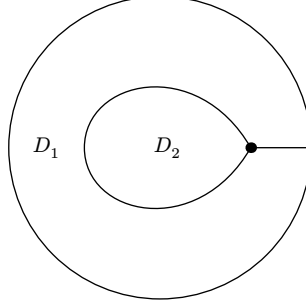


FIGURE 1.1. A partition of a disk that is not two-sided, since D_1 neighbors itself.

For a two-sided, weakly regular partition each D_i is a Lipschitz domain, so we can define trace operators and solve boundary value problems in a standard way. Without the transversality condition (iii) the D_i may have cusps, and the analysis becomes much more difficult; see, for instance [5]. If the partition is not two-sided, then some D_i lies on both sides of its boundary. In this case it is possible to define separate trace operators on each side of the common boundary; we do not do this here, but refer to [15, Section 1.7] for an example of this construction.

To extend the notion of a “nodal partition” to partitions that are not necessarily bipartite, it is convenient to introduce a generalization of the Laplacian. The construction involves a choice of signed weight functions, which will also be used to define a generalized two-sided Dirichlet-to-Neumann map on the partition boundary set.

Definition 1.3. Given a two-sided, weakly regular partition $\mathcal{D} = \{D_i\}$, let

$$\Gamma_i := \partial D_i \cap \Omega. \quad (1.9)$$

We say that functions $\chi_i: \Gamma_i \rightarrow \{\pm 1\}$ are *valid weights* if they are constructed as follows. Given an orientation of each ∂D_i , and an orientation of each smooth component of Γ , we define χ_i on each smooth component of Γ_i to be $+1$ if the orientation of ∂D_i agrees with the orientation of the corresponding smooth component of Γ , and equal to -1 otherwise.

In Figure 1.2 we illustrate the construction of a valid set of weights, and also give an example of a non-valid choice of weights. Note that χ_i is constant on each smooth segment of Γ_i ; the value at the corner points is irrelevant. According to Definition 1.3, there are two ways χ_i can change sign on Γ_i : 1) it can change sign at a corner; or 2) it can take different signs on different connected components. It is easily shown that a partition is bipartite if and only if the weights $\chi_i \equiv 1$ are valid, cf. [7, Lemma 9], and so non-constant weights are essential for the study of non-bipartite partitions.

Remark 1.4. An equivalent construction of valid weights can be given in terms of a co-orientation of each ∂D_i and each smooth component of Γ . Along each ∂D_i we choose a vector field V_i that is equal to either ν_i or $-\nu_i$. Choosing a vector field V that is a smooth unit normal to each smooth component of Γ , we set $\chi_i = V \cdot V_i$. A special case of this construction appeared in [7], where V_i was chosen to be the outward unit normal ν_i , in which case $\chi_i = -\chi_j$ whenever D_i

²In [11] such partitions are said to be *nice*. We prefer the term two-sided, as it conveys the fact that each smooth component of Γ is contained in the boundary of two distinct subdomains.

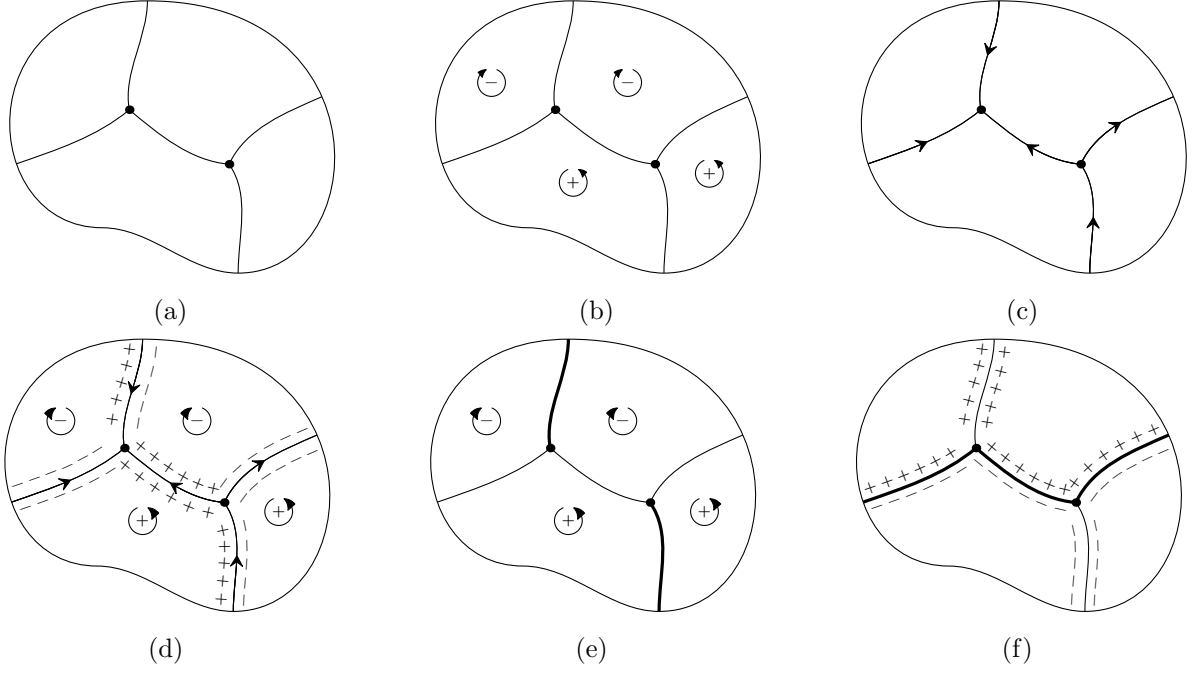


FIGURE 1.2. A partition (a), with a choice of orientation for the boundary ∂D_i of each subdomain (b), and an orientation of each smooth part of the boundary set Γ (c). In (d) we show the corresponding valid weights χ_i , and in (e) we show the resulting cut, as described in Appendix A. In (f) we display a non-valid choice of weights, i.e. functions $\chi_i: \Gamma_i \rightarrow \{\pm 1\}$ that are not induced by any choice of orientations.

and D_j are neighbors. The extra flexibility in the present construction will be useful below, in our discussion of χ -nodality.

Valid weights have a natural geometric interpretation in terms of the cutting construction in [18, Section 4], where one removes a portion Γ^* of the nodal set from the domain Ω in such a way that the resulting partition of $\Omega \setminus \Gamma^*$ is bipartite; see Appendix A for details.

We now introduce a weighted version of the Laplacian, $-\Delta^\chi$, corresponding to the bilinear form defined on the domain $\text{dom}(t^\chi)$ consisting of $u \in L^2(\Omega)$ such that

$$\begin{cases} u_i := u|_{D_i} \in H^1(D_i), & (1.10) \end{cases}$$

$$\begin{cases} u_i = 0 \text{ on } \partial D_i \cap \partial\Omega, & (1.11) \end{cases}$$

$$\begin{cases} \chi_i u_i = \chi_j u_j \text{ on } \Gamma_i \cap \Gamma_j \text{ for all } i, j = 1, \dots, k, & (1.12) \end{cases}$$

and given by

$$t^\chi(u, v) = \sum_{i=1}^k \int_{D_i} \nabla u_i \cdot \nabla v_i. \quad (1.13)$$

The Laplacians Δ^χ for different valid weights will be shown in Proposition 2.6 to be unitarily equivalent. As a consequence, if the partition is bipartite, then Δ^χ is unitarily equivalent to the Dirichlet Laplacian on Ω . Furthermore, the nodal sets of the eigenfunctions of Δ^χ are independent of χ , justifying the following definition.

Definition 1.5. A two-sided, weakly regular partition \mathcal{D} is said to be χ -nodal if it is the nodal partition for some eigenfunction of Δ^χ . The *defect* of a χ -nodal k -partition is defined to be

$$\delta(\mathcal{D}) = \ell(\mathcal{D}) - k, \quad (1.14)$$

where $\ell(\mathcal{D})$ denotes the minimal label of λ_* in the spectrum of $-\Delta^\chi$.

In Section 2.3 we will show that a partition is χ -nodal if and only if it satisfies the strong pair compatibility condition [18].

Definition 1.6. A two-sided, weakly regular partition \mathcal{D} is said to satisfy the *strong pair compatibility condition* (SPCC) if there exists a choice of positive ground states $\{u_i\}_{i=1}^k$ for the Dirichlet Laplacians on D_i such that, for any pair of neighbors D_i and D_j , the function u_{ij} defined by

$$u_{ij}|_{D_i} = u_i, \quad u_{ij}|_{D_j} = -u_j, \quad (1.15)$$

is an eigenfunction of the Dirichlet Laplacian on $\text{Int}(\overline{D_i \cup D_j})$.

We stress that the choice of the ground states in the definition (which is merely a choice of normalization on each D_i) is global — it can not change from one pair of neighbors to another. This distinguishes SPCC from the *weak pair compatibility condition*³ (WPCC) also appearing in the literature; see Appendix A. It is immediate that nodal partitions satisfy the SPCC. We also mention that for a smooth partition, where the set $\{x_\ell\} \subset \Gamma \cap \Omega$ of singular points is empty, the SPCC is equivalent to being a critical point of the energy functional on the set of equipartitions; see [9].

Finally, we will define a χ -weighted version of the two-sided Dirichlet-to-Neumann map, denoted $\text{DN}(\Gamma, \lambda_*, \chi)$. The full definition, given in Section 3, is rather delicate because λ_* is a Dirichlet eigenvalue and Γ has corners. We just mention here that, similar to the Laplacian Δ^χ , the Dirichlet-to-Neumann maps defined with different valid $\{\chi_i\}$ are unitarily equivalent; the precise nature of the equivalence is clarified in Theorem 3.1. If each χ_i is constant, $\text{DN}(\Gamma, \lambda_*, \chi)$ reduces to the operator $\text{DN}(\Gamma, \lambda_*)$ already described in (1.6).

The main result of this paper is the following.

Theorem 1.7. *A two-sided, weakly regular partition \mathcal{D} satisfies the SPCC if and only if it is χ -nodal, in which case it has defect*

$$\delta(\mathcal{D}) = \text{Mor DN}(\Gamma, \lambda_*, \chi), \quad (1.16)$$

and the corresponding eigenvalue λ_ of $-\Delta^\chi$ has multiplicity*

$$\dim \ker(\Delta^\chi + \lambda_*) = \dim \ker \text{DN}(\Gamma, \lambda_*, \chi) + 1. \quad (1.17)$$

The quantities in (1.16) and (1.17) are independent of χ . In particular, different valid weights $\{\chi_i\}$ may be used in defining the Laplacian $-\Delta^\chi$ and the Dirichlet-to-Neumann map $\text{DN}(\Gamma, \lambda_*, \chi)$.

Remark 1.8. Theorem 1.7 contains Theorem 1.1 as a special case, and hence is an improvement over the results of [8, 12], as described above. Similarly, for non-bipartite partitions, it refines [18, Theorem 4.1].

Remark 1.9. In higher dimensions the nodal sets of eigenfunctions can be more complicated, and the analysis of corner domains is significantly more involved (see, for instance [13]), so we restrict our attention to the planar case. The conclusion of Theorem 1.7 immediately extends to higher dimensions if the nodal set $\cup \Gamma_i$ is a smoothly embedded hypersurface.

Outline. In Section 2 we give some preliminary analysis, describing Sobolev spaces on the boundary set Γ , weighted Dirichlet and Neumann traces, and the weighted Laplacian Δ^χ . We also show that a partition is χ -nodal if and only if it satisfies the SPCC, and prove some delicate regularity results. In Section 3 we define the weighted, two-sided Dirichlet-to-Neumann operator $\text{DN}(\Gamma, \lambda_*, \chi)$ and establish its fundamental properties. In Section 4 we prove Theorem 1.7 by studying the spectral flow of an analytic family of self-adjoint operators. In Section 5 we illustrate our results by applying them to partitions of the circle.

In Appendix A we discuss the strong and weak pair compatibility conditions, and the connection between our χ weights and the cutting construction of [18]. Finally, in Appendix B we describe an alternate, more explicit construction of the canonical solution to a boundary value problem that arises in our construction of the Dirichlet-to-Neumann map.

³In earlier papers, for instance [16], WPCC is simply referred to as the *pair compatibility condition* (PCC).

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2. PRELIMINARY ANALYSIS

In this section we provide some background for our construction of the Dirichlet-to-Neumann map, in particular defining Sobolev spaces on the boundary set Γ , weighted Dirichlet and Neumann traces, and the weighted Laplacian. We also establish that SPCC is equivalent to χ -nodality.

2.1. Sobolev spaces on the boundary set. Recall that $\Gamma_i = \overline{\partial D_i} \cap \overline{\Omega}$. Since $|\chi_i| \equiv 1$ on Γ_i , we have

$$\begin{aligned} g \in L^2(\Gamma) &\iff g_i := g|_{\Gamma_i} \in L^2(\Gamma_i) \text{ for each } i \\ &\iff \chi_i g_i \in L^2(\Gamma_i) \text{ for each } i. \end{aligned}$$

The situation for $H^{1/2}$ is more complicated. If Γ has intersections then it is not a Lipschitz manifold, and the space $H^{1/2}(\Gamma)$ cannot be defined in the usual way; cf. [19]. Moreover, on each subdomain the conditions $g_i \in H^{1/2}(\Gamma_i)$ and $\chi_i g_i \in H^{1/2}(\Gamma_i)$ need not be equivalent, due to the possible discontinuities of χ_i at the corner points. We thus define the space

$$H_\chi^{1/2}(\Gamma) := \{g \in L^2(\Gamma) : \mathcal{E}_i(\chi_i g_i) \in H^{1/2}(\partial D_i), i = 1, \dots, k\}, \quad (2.1)$$

where $\mathcal{E}_i : L^2(\Gamma_i) \rightarrow L^2(\partial D_i)$ is the extension by zero to the rest of ∂D_i , i.e.

$$\mathcal{E}_i(\chi_i g_i) := \begin{cases} \chi_i g_i & \text{on } \Gamma_i, \\ 0 & \text{on } \partial D_i \setminus \Gamma_i. \end{cases}$$

The condition $\mathcal{E}_i(\chi_i g_i) \in H^{1/2}(\partial D_i)$ is more restrictive than $\chi_i g_i \in H^{1/2}(\Gamma_i)$ if $\partial D_i \cap \partial \Omega \neq \emptyset$. For instance, if $\chi_i g_i$ is a nonzero constant on Γ_i , its extension by zero will not be an element of $H^{1/2}(\partial D_i)$. A necessary and sufficient condition for $\mathcal{E}_i(\chi_i g_i) \in H^{1/2}(\partial D_i)$ will be recalled below, in Lemma 2.12. We define the norm

$$\|g\|_{H_\chi^{1/2}(\Gamma)}^2 := \sum_{i=1}^k \|\mathcal{E}_i(\chi_i g_i)\|_{H^{1/2}(\partial D_i)}^2, \quad (2.2)$$

and let $H_\chi^{-1/2}(\Gamma)$ denote the dual space to $H_\chi^{1/2}(\Gamma)$.

We next define a weighted Dirichlet trace (i.e. restriction to the nodal set) operator. A natural domain for this operator is the set $\text{dom}(t^\chi)$ that was defined above in (1.10)–(1.12), equipped with the norm $\|u\|_{\text{dom}(t^\chi)}^2 = \sum_i \|u_i\|_{H^1(D_i)}^2$.

Lemma 2.1. *The trace map*

$$\gamma_D^\chi : \text{dom}(t^\chi) \longrightarrow H_\chi^{1/2}(\Gamma) \quad (2.3)$$

defined by $(\gamma_D^\chi u)|_{\Gamma_i} = \chi_i u_i|_{\Gamma_i}$ is bounded, and has a bounded right inverse.

Proof. For each D_i there is a bounded trace operator $H^1(D_i) \rightarrow H^{1/2}(\partial D_i)$. We thus let $(\gamma_D^\chi u)|_{\Gamma_i} = \chi_i u_i|_{\Gamma_i}$ for each i ; the condition $\chi_i u_i = \chi_j u_j$ guarantees that $\gamma_D^\chi u$ is a well-defined function for any $u \in \text{dom}(t^\chi)$. Moreover, for each i we have $\chi_i(\gamma_D^\chi u)|_{\Gamma_i} = u_i|_{\Gamma_i}$, and hence

$$\mathcal{E}_i(\chi_i(\gamma_D^\chi u)|_{\Gamma_i}) = u_i|_{\partial D_i}$$

because $u_i = 0$ on $\partial D_i \cap \partial \Omega$. Since $u_i|_{\partial D_i} \in H^{1/2}(\partial D_i)$, it follows from (2.1) that $\gamma_D^\chi u \in H_\chi^{1/2}(\Gamma)$, with

$$\|\gamma_D^\chi u\|_{H_\chi^{1/2}(\Gamma)}^2 = \sum_{i=1}^k \|u_i|_{\partial D_i}\|_{H^{1/2}(\partial D_i)}^2 \leq C \sum_{i=1}^k \|u_i\|_{H^1(D_i)}^2,$$

as was to be shown.

To construct a right inverse, we first recall that for each i the trace map $H^1(D_i) \rightarrow H^{1/2}(\partial D_i)$ has a bounded right inverse, $\Upsilon_i: H^{1/2}(\partial D_i) \rightarrow H^1(D_i)$. Let $g \in H_\chi^{1/2}(\Gamma)$, so that $\mathcal{E}_i(\chi_i g_i) \in H^{1/2}(\partial D_i)$, and define $u_i = \Upsilon_i(\mathcal{E}_i(\chi_i g_i)) \in H^1(D_i)$. The corresponding function $u \in L^2(\Omega)$, defined by $u|_{D_i} = u_i$ for each i , is contained in $\text{dom}(t^\chi)$, since

$$\chi_i u_i|_{\Gamma_i \cap \Gamma_j} = \chi_i(\chi_i g_i) = g_i = g_j = \chi_j u_j|_{\Gamma_i \cap \Gamma_j}$$

for all i, j . Moreover, we have

$$\|u\|_{\text{dom}(t^\chi)}^2 = \sum_i \|u_i\|_{H^1(D_i)}^2 \leq C \sum_i \|\mathcal{E}_i(\chi_i g_i)\|_{H^{1/2}(\partial D_i)}^2 = C \|g\|_{H_\chi^{1/2}(\Gamma)}^2,$$

and so $\Upsilon g = u$ defines a bounded right inverse $\Upsilon: H_\chi^{1/2}(\Gamma) \rightarrow \text{dom}(t^\chi)$. \square

We next define a weighted, two-sided version of the normal derivative that will appear naturally in our construction of the Dirichlet-to-Neumann map.

Lemma 2.2. *If $u \in L^2(\Omega)$, with $u_i \in H^1(D_i)$ and $\Delta u_i \in L^2(D_i)$ for each i , then there exists a unique $\gamma_N^\chi u \in H_\chi^{-1/2}(\Gamma)$ such that*

$$\langle \gamma_N^\chi u, \gamma_D^\chi v \rangle_* = \sum_{i=1}^k \int_{D_i} (\nabla u_i \cdot \nabla v_i + (\Delta u_i) v_i) \quad (2.4)$$

for all $v \in \text{dom}(t^\chi)$, where $\langle \cdot, \cdot \rangle_*$ denotes the dual pairing between $H_\chi^{1/2}(\Gamma)$ and $H_\chi^{-1/2}(\Gamma)$.

Note that the definition of $\gamma_N^\chi u$ does not require any consistency conditions on the boundary values of u along Γ . That is, we do not require $u \in \text{dom}(t^\chi)$.

Proof. The construction is almost identical to that of [19, Lemma 4.3]. Letting $\Upsilon: H_\chi^{1/2}(\Gamma) \rightarrow \text{dom}(t^\chi)$ denote a bounded right inverse to γ_D^χ , as in Lemma 2.1, we define $\gamma_N^\chi u \in H_\chi^{-1/2}(\Gamma)$ by its action on arbitrary $g \in H_\chi^{1/2}(\Gamma)$, namely

$$\langle \gamma_N^\chi u, g \rangle_* := \sum_{i=1}^k \int_{D_i} (\nabla u_i \cdot \nabla (\Upsilon g)_i + (\Delta u_i) (\Upsilon g)_i).$$

It is easily verified that this has all the required properties. \square

Remark 2.3. If, in addition to the hypotheses of Lemma 2.2, $u_i|_{\partial D_i} \in H^1(\partial D_i)$ for each i , then [19, Theorem 4.24] implies $\partial_{\nu_i} u_i \in L^2(\partial D_i)$, and so Green's formula can be written as

$$\int_{D_i} (\nabla u_i \cdot \nabla v_i + (\Delta u_i) v_i) = \int_{\partial D_i} \frac{\partial u_i}{\partial \nu_i} v_i = \int_{\Gamma_i} \left(\chi_i \frac{\partial u_i}{\partial \nu_i} \right) (\chi_i v_i).$$

That is, the dual pairing of $\partial_{\nu_i} u_i \in H^{-1/2}(\partial D_i)$ and $v_i|_{\partial D_i} \in H^{1/2}(\partial D_i)$ is given by their $L^2(\partial D_i)$ inner product. Summing over i and comparing with (2.4), we find that

$$\gamma_N^\chi u|_{\Gamma_i \cap \Gamma_j} = \chi_i \frac{\partial u_i}{\partial \nu_i} + \chi_j \frac{\partial u_j}{\partial \nu_j} \in L^2(\Gamma_i \cap \Gamma_j) \quad (2.5)$$

for any neighbors D_i and D_j .

2.2. The sign-weighted Laplacian. In this section we describe the self-adjoint operator Δ^χ and its dependence on χ .

Definition 2.4. We say that two sets of valid weights $\{\chi_i\}$ and $\{\tilde{\chi}_i\}$ are *edge equivalent* if for each $i \neq j$ we have $\tilde{\chi}_i \tilde{\chi}_j = \chi_i \chi_j$ on $\Gamma_i \cap \Gamma_j$, and *domain equivalent* if for each i we have either $\tilde{\chi}_i \equiv \chi_i$ or $\tilde{\chi}_i \equiv -\chi_i$.

In Figure 2.1, the weights in (a) and (b) are edge equivalent.

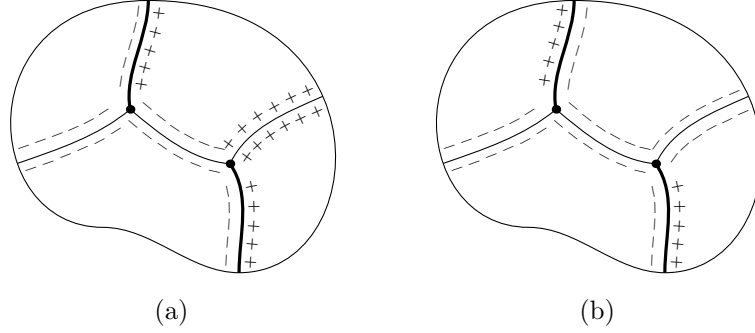


FIGURE 2.1. The weights in (a) and (b) are edge equivalent but not domain equivalent.

Remark 2.5. In terms of Definition 1.3, edge equivalence corresponds to only changing the orientations of the smooth components of Γ , while domain equivalence corresponds to only changing the orientations of the ∂D_i . It is thus clear that for any valid sets of weights χ and $\tilde{\chi}$ there is a valid weight $\hat{\chi}$ such that χ is edge equivalent to $\hat{\chi}$ and $\hat{\chi}$ is domain equivalent to $\tilde{\chi}$.

Recall that $-\Delta^\chi$ corresponds to the bilinear form t^χ defined in (1.13), with $\text{dom}(t^\chi)$ given by (1.10)–(1.12). The following proposition summarizes its basic properties.

Proposition 2.6. *If \mathcal{D} is a two-sided, weakly regular partition and $\{\chi_i\}$ are valid weights, then Δ^χ is a self-adjoint operator on $L^2(\Omega)$, with domain*

$$\text{dom}(\Delta^\chi) = \{u \in \text{dom}(t^\chi) : \Delta u_i \in L^2(D_i) \text{ for each } i \text{ and } \gamma_N^\chi u = 0\}. \quad (2.6)$$

For any other set of valid weights $\{\tilde{\chi}_i\}$ we have:

- (1) If χ and $\tilde{\chi}$ are edge equivalent, then $\Delta^\chi = \Delta^{\tilde{\chi}}$;
- (2) If χ and $\tilde{\chi}$ are domain equivalent, then Δ^χ is unitarily equivalent to $\Delta^{\tilde{\chi}}$.

Consequently, Δ^χ and $\Delta^{\tilde{\chi}}$ are unitarily equivalent for any choices of valid weights, and so the property of being χ -nodal is independent of the choice of a valid χ .

Proof. It is easily seen that t^χ is a closed, semi-bounded bilinear form, with dense domain in $L^2(\Omega)$. It thus generates a semi-bounded self-adjoint operator, which we denote $-\Delta^\chi$, with domain

$$\begin{aligned} \text{dom}(\Delta^\chi) = \{u \in \text{dom}(t^\chi) : \text{there exists } f \in L^2(\Omega) \text{ such that} \\ t^\chi(u, v) = \langle f, v \rangle_{L^2(\Omega)} \text{ for all } v \in \text{dom}(t^\chi)\}. \end{aligned} \quad (2.7)$$

For any such u we have $-\Delta^\chi u = f$.

To prove (2.6), we first assume that $u \in \text{dom}(\Delta^\chi)$, as described in (2.7). If $v_i \in H_0^1(D_i)$ for some i , then its extension by zero to the rest of Ω is contained in $\text{dom}(t^\chi)$. Denoting this extension by v , we get from (1.13) and (2.7) that

$$\int_{D_i} f v_i = t^\chi(u, v) = \int_{D_i} \nabla u_i \cdot \nabla v_i.$$

Since v_i was an arbitrary function in $H_0^1(D_i)$, this means $\Delta u_i = f_i \in L^2(D_i)$ in a distributional sense. This holds for each i , so it follows from Lemma 2.2 that $\gamma_N^\chi u$ is defined, and satisfies

$$\langle \gamma_N^\chi u, \gamma_D^\chi v \rangle_* = t^\chi(u, v) - \langle f, v \rangle_{L^2(\Omega)} = 0$$

for all $v \in \text{dom}(t^\chi)$. Since γ_D^χ is surjective, this implies $\gamma_N^\chi u = 0$.

On the other hand, suppose $u \in \text{dom}(t^\chi)$ satisfies $\Delta u_i \in L^2(D_i)$ for each i and $\gamma_N^\chi u = 0$. Lemma 2.2 then implies

$$t^\chi(u, v) = - \sum_{i=1}^k \int_{D_i} (\Delta u_i) v_i = \langle f, v \rangle_{L^2(\Omega)}$$

for all $v \in \text{dom}(t^\chi)$, where $f \in L^2(\Omega)$ is defined by $f_i = -\Delta u_i$ for each i . Using (2.7), this gives $u \in \text{dom}(\Delta^\chi)$ and completes the proof.

Finally, we describe the dependence of the operator Δ^χ on the weights $\{\chi_i\}$. The first claim follows immediately from the definitions. If $\tilde{\chi}_i \tilde{\chi}_j = \chi_i \chi_j$ on $\Gamma_i \cap \Gamma_j$, then $\chi_i u_i = \chi_j u_j$ is equivalent to $\tilde{\chi}_i u_i = \tilde{\chi}_j u_j$, hence $\text{dom}(t^{\tilde{\chi}}) = \text{dom}(t^\chi)$ and the result follows.

For the second claim, consider the unitary map $U: L^2(\Omega) \rightarrow L^2(\Omega)$ defined by

$$(Uv)|_{D_i} = \begin{cases} v & \text{if } \tilde{\chi}_i \equiv \chi_i, \\ -v & \text{if } \tilde{\chi}_i \equiv -\chi_i. \end{cases}$$

This sends $\text{dom}(t^\chi)$ to $\text{dom}(t^{\tilde{\chi}})$, with $t^{\tilde{\chi}}(Uv, Uw) = t^\chi(v, w)$ for all $v, w \in \text{dom}(t^\chi)$, which implies $\Delta^\chi = U^{-1} \Delta^{\tilde{\chi}} U$ and completes the proof.

These two equivalences combined with Remark 2.5 shows that Δ^χ is unitarily equivalent to any other $\Delta^{\tilde{\chi}}$ with a valid $\tilde{\chi}$. The unitary map U does not affect the nodal set, therefore a partition \mathcal{D} is χ -nodal either for all valid choices of χ or for none. \square

2.3. Pair compatibility and χ -nodal partitions. Next, we discuss the connection between the strong pair compatibility condition and the χ -nodal condition.

Proposition 2.7. *A two-sided, weakly regular partition \mathcal{D} is χ -nodal if and only if it satisfies the SPCC.*

Proof. First suppose \mathcal{D} is χ -nodal, so it is the nodal set of some eigenfunction φ_* of Δ^χ . Since $\varphi_{*,i}|_{\partial D_i} = 0$ is contained in $H^1(\partial D_i)$, we can use Remark 2.3 and Proposition 2.6 to get

$$0 = \gamma_N^\chi \varphi_*|_{\Gamma_i \cap \Gamma_j} = \chi_i \frac{\partial \varphi_{*,i}}{\partial \nu_i} + \chi_j \frac{\partial \varphi_{*,j}}{\partial \nu_j} \quad (2.8)$$

for any neighbors D_i and D_j .

We now let $\eta_i = \text{sgn } \varphi_{*,i}$. For each D_i the function $u_i := \eta_i \varphi_{*,i}$ is a positive ground state, and the transmission condition (2.8) becomes

$$\eta_i \chi_i \frac{\partial u_i}{\partial \nu_i} + \eta_j \chi_j \frac{\partial u_j}{\partial \nu_j} = 0. \quad (2.9)$$

Since $\partial_{\nu_i} u_i$ and $\partial_{\nu_j} u_j$ are both negative, we conclude that $\eta_i \chi_i = -\eta_j \chi_j$, yielding $\partial_{\nu_i} u_i = \partial_{\nu_j} u_j$ on $\Gamma_i \cap \Gamma_j$. It follows that u_{ij} , as defined in (1.15), is a Dirichlet eigenfunction on $\text{Int}(\overline{D_i \cup D_j})$, hence \mathcal{D} satisfies the SPCC.

Conversely, suppose \mathcal{D} satisfies the SPCC. This means there exist positive ground states u_i for the Dirichlet Laplacian on D_i such u_{ij} is a Dirichlet eigenfunction on $\text{Int}(\overline{D_i \cup D_j})$ whenever D_i and D_j are neighbors. This implies $\partial_{\nu_i} u_i = \partial_{\nu_j} u_j$ on $\Gamma_i \cap \Gamma_j$.

Now define valid weights by choosing the same orientation for all D_i . This implies that $\chi_i = -\chi_j$ on $\Gamma_i \cap \Gamma_j$ (the orientation of the segments of Γ is irrelevant). It follows that the function u defined by $u|_{D_i} = u_i$ satisfies the transmission condition (2.8) and hence is an eigenfunction of Δ^χ . \square

Remark 2.8. The χ weights chosen in the final step of the proof coincide with those used in [7]; see Remark 1.4. In Appendix A we will see that this χ corresponds to the so-called “maximal cut” of the boundary set.

It is well known that nodal partitions and spectral minimal partitions have the *equal angle property*: at a singular point, the half-curves meet with equal angle [17]. This is also true of χ -nodal partitions.

Corollary 2.9. *If $\mathcal{D} = \{D_i\}$ is χ -nodal, then it satisfies the equal angle property.*

Proof. Since \mathcal{D} satisfies the SPCC, the result follows from applying [17, Theorem 2.6] to each pair of neighboring domains. \square

It is an immediate consequence of the equal angle property that each D_i has convex corners, a fact we will use in Proposition 2.10 to conclude H^2 regularity of Dirichlet eigenfunctions.

2.4. Regularity properties of the Dirichlet kernel. Let Δ_∞^χ be the Laplacian in Ω with Dirichlet boundary conditions imposed on $\partial\Omega \cup \Gamma$. More precisely, it is the Laplacian with the domain

$$\begin{aligned} \text{dom}(\Delta_\infty^\chi) &= \{u \in \text{dom}(t^\chi) : \Delta u_i \in L^2(D_i) \text{ for each } i \text{ and } \gamma_D^\chi u = 0\} \\ &= \{u \in L^2(\Omega) : u_i \in H_0^1(D_i) \text{ and } \Delta u_i \in L^2(D_i) \text{ for each } i\}. \end{aligned}$$

The reason for the subscript ∞ will become apparent in Section 4. For now, we would like to understand the properties of the eigenspace of $-\Delta_\infty^\chi$ corresponding to the eigenvalue λ_* . The main result of this section is the following.

Proposition 2.10. *Let \mathcal{D} be a χ -nodal k -partition and φ_* be the eigenfunction of Δ^χ with boundary set Γ . The subspace*

$$\Phi := \ker(\Delta_\infty^\chi + \lambda_*) \tag{2.10}$$

has the following properties:

- (1) $\dim \Phi = k$;
- (2) $\ker \gamma_N^\chi|_\Phi = \text{span}\{\varphi_*\}$;
- (3) for any $\varphi \in \Phi$, $\varphi|_{D_i} \in H^2(D_i) \cap H_0^1(D_i)$;
- (4) $\gamma_N^\chi(\Phi) \subset H_\chi^{1/2}(\Gamma)$.

Proof. It follows immediately that for each i the restriction $\varphi_{*,i} \in H_0^1(D_i)$ of φ_* satisfies the eigenvalue equation $\Delta \varphi_{*,i} + \lambda_* \varphi_{*,i} = 0$ in a distributional sense. Moreover, it does not change sign and is therefore the ground state of the Dirichlet Laplacian on D_i .

Extending each $\varphi_{*,i}$ by zero outside its domain, we obtain k linearly independent eigenfunctions $\widetilde{\varphi_{*,i}}$ of $-\Delta_\infty^\chi$ corresponding to the eigenvalue λ_* . Conversely, for any $\varphi \in \Phi$, its restriction φ_i is a λ_* -eigenfunction of the Dirichlet Laplacian on D_i (if non-zero), and therefore must be proportional to the ground state. We conclude that $\dim \Phi = k$.

From Proposition 2.6 we get $\gamma_N^\chi \varphi_* = 0$. Let $\psi \in \Phi$ be another function such that $\gamma_N^\chi \psi = 0$. Since the restriction of ψ to (say) subdomain D_1 is a multiple of its ground state, there is a linear combination of φ_* and ψ which identically vanishes on D_1 . By a straightforward extension of the unique continuation principle to Δ^χ , this linear combination is zero everywhere and therefore ψ is a multiple of φ_* .

Next, Corollary 2.9 implies that each D_i has piecewise smooth boundary with convex corners, so it follows from [15, Remark 3.2.4.6] that $\varphi_i \in H^2(D_i)$ for any $\varphi \in \Phi$.

Finally, let

$$E_i: L^2(\Gamma_i) \rightarrow L^2(\Gamma) \tag{2.11}$$

denote extension by zero. We have

$$\gamma_N^\chi \widetilde{\varphi_{*,i}} = E_i \left(\chi_i \frac{\partial \varphi_{*,i}}{\partial \nu_i} \right), \tag{2.12}$$

and therefore the claim $\gamma_N^\chi(\Phi) \subset H_\chi^{1/2}(\Gamma)$ follows from the next proposition applied to $\varphi_{*,i}$. \square

Proposition 2.11. *If $u \in H^2(D_i) \cap H_0^1(D_i)$, then $\partial_{\nu_i} u \in H^{1/2}(\Gamma_i)$, $\chi_i \partial_{\nu_i} u \in H^{1/2}(\Gamma_i)$ and $E_i(\chi_i \partial_{\nu_i} u) \in H_\chi^{1/2}(\Gamma)$.*

The assumption that u vanishes on the boundary is essential. If D_i has corners, then the unit normal ν_i is discontinuous there, and for a general function in $H^2(D_i)$, or even $C^\infty(\overline{D_i})$, there is no guarantee that $\partial_{\nu_i} u \in H^{1/2}(\partial D_i)$. A simple example is $u(x, y) = x$ on the unit square; its normal derivative is piecewise constant, but is not contained in $H^{1/2}$.

Localizing around a single corner and performing a suitable change of variables, it suffices to prove the result for the model domain $D = \mathbb{R}_+ \times \mathbb{R}_+$, which has boundary $\partial D = (\mathbb{R}_+ \times \{0\}) \cup (\{0\} \times \mathbb{R}_+)$. We first recall some preliminary results on boundary Sobolev spaces.

Lemma 2.12. [15, Theorem 1.5.2.3] *Given $f_1, g_1 \in H^{1/2}(\mathbb{R}_+)$, the composite function*

$$h = \begin{cases} f_1, & \text{on } \mathbb{R}_+ \times \{0\} \\ g_1, & \text{on } \{0\} \times \mathbb{R}_+ \end{cases}$$

is in $H^{1/2}(\partial D)$ if and only if

$$\int_0^1 |f_1(t) - g_1(t)|^2 \frac{dt}{t} < \infty. \quad (2.13)$$

In particular, the conclusion $h \in H^{1/2}(\partial D)$ holds for any f_1 and g_1 satisfying the stronger condition

$$\int_0^1 |f_1(t)|^2 \frac{dt}{t} + \int_0^1 |g_1(t)|^2 \frac{dt}{t} < \infty. \quad (2.14)$$

We also need to know the image of the trace map on each smooth component of the boundary.

Lemma 2.13. [15, Theorem 1.5.2.4] *The trace map*

$$\begin{aligned} H^2(\mathbb{R}_+ \times \mathbb{R}_+) &\rightarrow H^{3/2}(\mathbb{R}_+) \times H^{1/2}(\mathbb{R}_+) \times H^{3/2}(\mathbb{R}_+) \times H^{1/2}(\mathbb{R}_+) \\ u(x, y) &\mapsto \left(u(x, 0), \frac{\partial u}{\partial y}(x, 0), u(0, y), \frac{\partial u}{\partial x}(0, y) \right) \end{aligned} \quad (2.15)$$

is continuous, with image consisting of all (f_0, f_1, g_0, g_1) that satisfy the compatibility conditions

$$f_0(0) = g_0(0) \quad (2.16)$$

and

$$\int_0^1 \frac{|f'_0(t) - g_1(t)|^2}{t} dt + \int_0^1 \frac{|f_1(t) - g'_0(t)|^2}{t} dt < \infty. \quad (2.17)$$

While the above two lemmas are both if and only if statements, we do not require their full strength in the following proof. It is enough to know that (2.14) is a *sufficient* condition for h to be in $H^{1/2}(\partial D)$, and (2.17) is a *necessary* condition for (f_0, f_1, g_0, g_1) to be in the image of the trace map defined in (2.15).

Proof of Proposition 2.11. As mentioned above, it suffices to prove the result for the model domain $D = \mathbb{R}_+ \times \mathbb{R}_+$. If $u \in H^2(D) \cap H_0^1(D)$, then its corresponding traces

$$(f_0(x), f_1(x), g_0(y), g_1(y)) = \left(u(x, 0), \frac{\partial u}{\partial y}(x, 0), u(0, y), \frac{\partial u}{\partial x}(0, y) \right)$$

satisfy $f_0(x) = 0$ and $g_0(y) = 0$ for all $x, y > 0$, so Lemma 2.13 implies

$$\int_0^1 \left| \frac{\partial u}{\partial x}(0, t) \right|^2 \frac{dt}{t} + \int_0^1 \left| \frac{\partial u}{\partial y}(t, 0) \right|^2 \frac{dt}{t} < \infty.$$

It then follows from Lemma 2.12 that the function

$$h = \begin{cases} \frac{\partial u}{\partial y}(x, 0), & \text{on } \mathbb{R}_+ \times \{0\} \\ \frac{\partial u}{\partial x}(0, y), & \text{on } \{0\} \times \mathbb{R}_+ \end{cases}$$

is in $H^{1/2}(\partial D)$. Since $h = -\partial_\nu u$, this proves the first part of the proposition.

Since the weight χ is constant (either +1 or -1) on each axis, we obtain

$$\int_0^1 \left| \chi \frac{\partial u}{\partial x}(0, t) \right|^2 \frac{dt}{t} = \int_0^1 \left| \frac{\partial u}{\partial x}(0, t) \right|^2 \frac{dt}{t} < \infty, \quad (2.18)$$

and similarly for the integral involving $\partial u / \partial y$, which implies $\chi \partial_\nu u \in H^{1/2}(\partial D)$.

This completes our analysis on the model domain D , where we have shown that $\chi_i \partial_{\nu_i} u \in H^{1/2}(\Gamma_i)$. Finally, we consider the extension $E_i(\chi_i \partial_{\nu_i} u)$ to the rest of Γ . Fixing another domain

D_j , we must show that $\chi_j E_i(\chi_i \partial_{\nu_i} u) \in H^{1/2}(\Gamma_j)$. On each smooth segment of Γ_j , this function is given by $\chi_j \chi_i \partial_{\nu_i} u$ (if Γ_i intersects Γ_j nontrivially) and 0 otherwise. Either way, it follows from (2.18) that the finiteness condition (2.14) holds, and so $\chi_j E_i(\chi_i \partial_{\nu_i} u) \in H^{1/2}(\Gamma_j)$, as was to be shown. \square

3. DEFINING THE WEIGHTED DIRICHLET-TO-NEUMANN OPERATOR

In this section we construct the weighted, two-sided Dirichlet-to-Neumann operator $\text{DN}(\Gamma, \lambda_*, \chi)$ for a χ -nodal partition with eigenvalue λ_* . In Section 3.1 we give a definition using the standard theory of self-adjoint operators and coercive bilinear forms; the details of this construction are then provided in Sections 3.2 and 3.3.

As mentioned in the introduction, the construction is rather involved because λ_* is a Dirichlet eigenvalue on each D_i . In this case one can also view the Dirichlet-to-Neumann map as a multi-valued operator (or linear relation); this approach is described in [2, 4, 6]. Another difficulty is that Γ has corners. While the Dirichlet-to-Neumann map can be defined on domains with minimal boundary regularity (see [1]), our results require delicate regularity properties, as in Proposition 2.10, that are not available in that case.

3.1. Definition via bilinear forms. We define $\text{DN}(\Gamma, \lambda_*, \chi)$ as the self-adjoint operator corresponding to a bilinear form on the closed subspace

$$S_\chi := \left\{ g \in L^2(\Gamma) : \int_{\Gamma_i} \chi_i g_i \frac{\partial \varphi_{*,i}}{\partial \nu_i} = 0, \ i = 1, \dots, k \right\} \quad (3.1)$$

of $L^2(\Gamma)$, where $\varphi_{*,i}$ denotes the restriction of φ_* to D_i , and we recall that $\Gamma_i = \overline{\partial D_i} \cap \Gamma$.

Let $g \in H_\chi^{1/2}(\Gamma) \cap S_\chi$. For each i , the problem

$$\begin{cases} -\Delta u_i = \lambda_* u_i & \text{in } D_i, \\ u_i = \chi_i g_i & \text{on } \Gamma_i, \\ u_i = 0 & \text{on } \partial D_i \setminus \Gamma_i, \end{cases} \quad (3.2)$$

has a unique solution $u_i^g \in H^1(D_i)$ that satisfies the orthogonality condition

$$\int_{D_i} u_i^g \varphi_{*,i} = 0. \quad (3.3)$$

Using these solutions, we define the symmetric bilinear form

$$a(g, h) := \sum_i \int_{D_i} (\nabla u_i^g \cdot \nabla u_i^h - \lambda_* u_i^g u_i^h), \quad \text{dom}(a) = H_\chi^{1/2}(\Gamma) \cap S_\chi. \quad (3.4)$$

It follows from Lemma 2.2 that the two-sided normal derivative $\gamma_N^\chi u^g \in H_\chi^{-1/2}(\Gamma)$ is defined.

The main result of this section is the following.

Theorem 3.1. *Let $\mathcal{D} = \{D_i\}$ be a χ -nodal partition.*

- (1) *The bilinear form defined in (3.4) generates a self-adjoint operator $\text{DN}(\Gamma, \lambda_*, \chi)$, which has domain*

$$\text{dom}(\text{DN}(\Gamma, \lambda_*, \chi)) = \{g \in H_\chi^{1/2}(\Gamma) \cap S_\chi : \gamma_N^\chi u^g \in L^2(\Gamma)\}, \quad (3.5)$$

and is given by

$$\text{DN}(\Gamma, \lambda_*, \chi)g = \Pi_\chi(\gamma_N^\chi u^g), \quad (3.6)$$

where Π_χ is the $L^2(\Gamma)$ -orthogonal projection onto S_χ .

- (2) *For each $g \in \text{dom}(\text{DN}(\Gamma, \lambda_*, \chi))$, there exists a function $\tilde{u} \in \text{dom}(t^\chi)$ such that $\gamma_N^\chi \tilde{u} \in S_\chi$ and \tilde{u}_i solves (3.2) for each i , hence*

$$\text{DN}(\Gamma, \lambda_*, \chi)g = \gamma_N^\chi \tilde{u}. \quad (3.7)$$

If we additionally require $\int_\Omega \tilde{u} \varphi_ = 0$, then \tilde{u} is unique.*

- (3) *For any other set of valid weights $\{\tilde{\chi}_i\}$, we have:*

(a) *If χ and $\tilde{\chi}$ are edge equivalent, then $\text{DN}(\Gamma, \lambda_*, \chi)$ is unitarily equivalent to $\text{DN}(\Gamma, \lambda_*, \tilde{\chi})$;*

(b) If χ and $\tilde{\chi}$ are domain equivalent, then $\text{DN}(\Gamma, \lambda_*, \chi) = \text{DN}(\Gamma, \lambda_*, \tilde{\chi})$.
Consequently, $\text{DN}(\Gamma, \lambda_*, \chi)$ and $\text{DN}(\Gamma, \lambda_*, \tilde{\chi})$ are unitarily equivalent for any two valid sets of weights χ and $\tilde{\chi}$.

Remark 3.2. If $u \in \text{dom}(t^\chi)$ and u_i solves (3.2) for each i , it must be of the form $u = u^g + \varphi$ for some $\varphi \in \Phi$, by Proposition 2.10. Since $\gamma_N^\chi \varphi \in S_\chi^\perp$, we have $\Pi_\chi(\gamma_N^\chi u^g) = \Pi_\chi(\gamma_N^\chi u)$, meaning u^g can be replaced by any other solution to (3.2). The distinguished solution u_i^g has nice analytic properties, which we will use in Lemma 3.6 to prove that a is semi-bounded and closed. On the other hand, $\gamma_N^\chi u^g$ may not be in the subspace S_χ , so we need to apply the orthogonal projection Π_χ in (3.6). By choosing different solutions to (3.2) we can eliminate this projection, as in (3.7).

3.2. The subspace S_χ . We start by discussing some useful properties of the subspace S_χ defined in (3.1). Recall that Φ is the kernel of the Dirichlet Laplacian Δ_∞^χ , as described in Proposition 2.10.

Lemma 3.3. *The subspace $S_\chi \subset L^2(\Gamma)$ can be written as*

$$S_\chi = \left\{ g \in L^2(\Gamma) : \langle \gamma_N^\chi \varphi, g \rangle_{L^2(\Gamma)} = 0 \text{ for all } \varphi \in \Phi \right\} = (\gamma_N^\chi(\Phi))^\perp. \quad (3.8)$$

Therefore, it is a closed subspace of codimension $k - 1$.

Proof. Formula (3.8) is a direct consequence of the properties of the space Φ and equation (2.12), since

$$\int_{\Gamma_i} \chi_i g_i \frac{\partial \varphi_{*,i}}{\partial \nu_i} = \int_{\Gamma} g \gamma_N^\chi \widetilde{\varphi_{*,i}}.$$

From Proposition 2.10 we have

$$\dim \gamma_N^\chi(\Phi) = \dim \Phi - \dim \ker(\gamma_N^\chi|_\Phi) = k - 1.$$

Since $S_\chi^\perp = \gamma_N^\chi(\Phi)$, this completes the proof. \square

Lemma 3.4. *The set $H_\chi^{1/2}(\Gamma) \cap S_\chi$ is dense in S_χ .*

Proof. We first claim that $H_\chi^{1/2}(\Gamma)$ is dense in $L^2(\Gamma)$. Fix $g \in L^2(\Gamma)$ and let $\varepsilon > 0$. Letting $\tilde{\Gamma}$ denote the smooth part of Γ , which is diffeomorphic to a finite number of open intervals, we can find a function \tilde{g} on $\tilde{\Gamma}$ such that $\|g - \tilde{g}\|_{L^2(\Gamma)} < \varepsilon$ and $\tilde{g} \in C_0^\infty(\tilde{\Gamma})$. Since the weights χ_i are constant on each component of $\tilde{\Gamma}$, it follows that $\chi_i \tilde{g} \in C_0^\infty(\tilde{\Gamma} \cap \Gamma_i)$, and hence $\chi_i \tilde{g} \in H^{1/2}(\Gamma_i)$, for each i . This implies $\tilde{g} \in H_\chi^{1/2}(\Gamma)$ and thus proves the claim.

Now let $g \in S_\chi$, $\varepsilon > 0$, and choose $\tilde{g} \in H_\chi^{1/2}(\Gamma)$ as above. Lemma 3.3 implies $(I - \Pi_\chi)\tilde{g} \in \gamma_N^\chi(\Phi)$, which in turn belongs to $H_\chi^{1/2}(\Gamma)$ by Proposition 2.11. Therefore,

$$\Pi_\chi \tilde{g} = \tilde{g} - (I - \Pi_\chi)\tilde{g} \in H_\chi^{1/2}(\Gamma) \cap S_\chi. \quad (3.9)$$

We now use the fact that $\Pi_\chi g = g$ to obtain

$$\|g - \Pi_\chi \tilde{g}\|_{S_\chi} = \|\Pi_\chi(g - \tilde{g})\|_{S_\chi} \leq \|g - \tilde{g}\|_{L^2(\Gamma)} < \varepsilon,$$

as was to be shown. \square

Finally, we describe the set of functionals in $H_\chi^{-1/2}(\Gamma) = H_\chi^{1/2}(\Gamma)^*$ that vanish on $H_\chi^{1/2}(\Gamma) \cap S_\chi$. This will be used below, in the proof of Theorem 3.1, when we describe the domain of the Dirichlet-to-Neumann map.

Lemma 3.5. *If $\tau \in H_\chi^{-1/2}(\Gamma)$ and $\tau(g) = 0$ for all $g \in H_\chi^{1/2}(\Gamma) \cap S_\chi$, then there exists a function $h \in \gamma_N^\chi(\Phi) = S_\chi^\perp$ such that $\tau(g) = \langle g, h \rangle_{L^2(\Gamma)}$ for all $g \in H_\chi^{1/2}(\Gamma)$.*

Proof. From Lemma 3.4 we have the $L^2(\Gamma)$ -orthogonal decomposition

$$H_\chi^{1/2}(\Gamma) = \left(H_\chi^{1/2}(\Gamma) \cap S_\chi \right) \oplus \gamma_N^\chi(\Phi),$$

therefore any functional that vanishes on $H_\chi^{1/2}(\Gamma) \cap S_\chi$ is a functional on $\gamma_N^\chi(\Phi)$ extended by zero. Since $\gamma_N^\chi(\Phi)$ is finite dimensional, a functional $\hat{\tau}: \gamma_N^\chi(\Phi) \rightarrow \mathbb{R}$ is continuous with respect

to any choice of norm. In particular, it is continuous with respect to the $L^2(\Gamma)$ norm, so there exists $h \in \gamma_N^\chi(\Phi)$ such that $\hat{\tau}(g) = \langle g, h \rangle_{L^2(\Gamma)}$ for all $g \in \gamma_N^\chi(\Phi)$. \square

3.3. Proof of Theorem 3.1. From Lemma 3.4 we know that the symmetric bilinear form a is densely defined. The next step is to show that it is semi-bounded and closed. This is an immediate consequence of the completeness of $H_\chi^{1/2}(\Gamma)$ and the following inequalities; see, for instance, [20, Section 11.2].

Lemma 3.6. *There exist constants $C, c > 0$ and $m \in \mathbb{R}$ such that*

$$|a(g, h)| \leq C \|g\|_{H_\chi^{1/2}(\Gamma)} \|h\|_{H_\chi^{1/2}(\Gamma)} \quad (3.10)$$

and

$$a(g, g) \geq c \|g\|_{H_\chi^{1/2}(\Gamma)}^2 + m \|g\|_{L^2(\Gamma)}^2 \quad (3.11)$$

for all $g, h \in H_\chi^{1/2}(\Gamma) \cap S_\chi$.

In the proof we let C, c denote positive constants, and m a real constant, whose meaning may change from line to line.

Proof. For each i the unique solution u_i^g to (3.2) and (3.3) satisfies a uniform estimate

$$\|u_i^g\|_{H^1(D_i)} \leq C \|\mathcal{E}_i(\chi_i g_i)\|_{H^{1/2}(\partial D_i)}.$$

Recalling the definition of the $H_\chi^{1/2}(\Gamma)$ norm in (2.2), it follows that

$$|a(g, h)| \leq C \|g\|_{H_\chi^{1/2}(\Gamma)} \|h\|_{H_\chi^{1/2}(\Gamma)}$$

for all $g, h \in H_\chi^{1/2}(\Gamma) \cap S_\chi$.

On the other hand, a standard compactness argument (see [3, Lemma 2.3]) shows that for any $\varepsilon > 0$ there exists a constant $K(\varepsilon) > 0$ such that

$$\|u_i\|_{L^2(D_i)}^2 \leq \varepsilon \|\nabla u_i\|_{L^2(D_i)}^2 + K(\varepsilon) \|u_i|_{\Gamma_i}\|_{L^2(\Gamma_i)}^2 \quad (3.12)$$

for all u_i in the set

$$\left\{ u_i \in H^1(D_i) : \Delta u_i + \lambda_* u_i = 0, \int_{D_i} u_i \varphi_{*,i} = 0, u_i|_{\partial D_i \cap \partial \Omega} = 0 \right\}.$$

In particular, the estimate (3.12) holds for each u_i^g . It then follows, exactly as in [3, Proposition 3.3], that

$$\begin{aligned} \int_{D_i} (|\nabla u_i^g|^2 - \lambda_* |u_i^g|^2) &\geq \frac{1}{2} \|u_i^g\|_{H^1(D_i)}^2 + m \|g_i\|_{L^2(\Gamma_i)}^2 \\ &\geq c \|\mathcal{E}_i(\chi_i g_i)\|_{H^{1/2}(\partial D_i)}^2 + m \|g_i\|_{L^2(\Gamma_i)}^2 \end{aligned}$$

for each i , with constants $c > 0$ and $m \in \mathbb{R}$, and hence

$$a(g, g) \geq c \|g\|_{H_\chi^{1/2}(\Gamma)}^2 + m \|g\|_{L^2(\Gamma)}^2$$

for all $g \in H_\chi^{1/2}(\Gamma) \cap S_\chi$. \square

We are now ready to prove the main result.

Proof of Theorem 3.1. From Lemmas 3.4 and 3.6 we know that the symmetric bilinear form a is densely defined, lower semi-bounded and closed, so it generates a self-adjoint operator on S_χ , which we denote by A for brevity. Its domain is given by

$$\begin{aligned} \text{dom}(A) &= \{g \in H_\chi^{1/2}(\Gamma) \cap S_\chi : \text{there exists } f \in S_\chi \text{ such that} \\ &\quad a(g, h) = \langle f, h \rangle_{L^2(\Gamma)} \text{ for all } h \in H_\chi^{1/2}(\Gamma) \cap S_\chi\}, \end{aligned} \quad (3.13)$$

and $Ag = f$ for any such g .

We now characterize the domain of A . First suppose $g \in \text{dom}(A)$, and let $f = Ag \in S_\chi$. Using Lemma 2.2 and the definition of a in (3.4), we get

$$a(g, h) = \langle \gamma_N^\chi u^g, h \rangle_* \quad (3.14)$$

for all $h \in H_\chi^{1/2}(\Gamma) \cap S_\chi$. On the other hand, (3.13) implies

$$a(g, h) = \langle f, h \rangle_{L^2(\Gamma)} = \int_\Gamma fh,$$

so we find that

$$\gamma_N^\chi u^g - f \in H_\chi^{-1/2}(\Gamma)$$

vanishes on $H_\chi^{1/2}(\Gamma) \cap S_\chi$. From Lemma 3.5 we get $\gamma_N^\chi u^g - f \in \gamma_N^\chi(\Phi) = S_\chi^\perp$, and hence $\gamma_N^\chi u^g \in L^2(\Gamma)$. Since $f \in S_\chi$, it follows that $f = \Pi_\chi(\gamma_N^\chi u^g)$.

Conversely, if $g \in H_\chi^{1/2}(\Gamma) \cap S_\chi$ and $\gamma_N^\chi u^g \in L^2(\Gamma)$, we have

$$\int_\Gamma (\Pi_\chi(\gamma_N^\chi u^g))h = \int_\Gamma (\gamma_N^\chi u^g)h = a(g, h)$$

for all $h \in H_\chi^{1/2}(\Gamma) \cap S_\chi$. According to (3.13), this implies $g \in \text{dom}(A)$, with $Ag = \Pi_\chi(\gamma_N^\chi u^g)$.

Next, we prove the existence of \tilde{u} . Since Π_χ is the orthogonal projection onto $S_\chi = \gamma_N^\chi(\Phi)^\perp$, we have

$$Ag = \Pi_\chi(\gamma_N^\chi u^g) = \gamma_N^\chi u^g - \gamma_N^\chi \varphi$$

for some $\varphi \in \Phi$. Setting $\tilde{u} = u^g - \varphi$, we obtain $Ag = \gamma_N^\chi \tilde{u}$, as required. If \hat{u} is another function in $\text{dom}(t^\chi)$ such that $Ag = \gamma_N^\chi \hat{u}$ and \hat{u}_i solves (3.2) for each i , then $\hat{u} - \tilde{u} \in \Phi$ and also $\hat{u} - \tilde{u} \in \ker \gamma_N^\chi$. By Proposition 2.10, $\hat{u} - \tilde{u}$ is a multiple of φ_* , and so requiring \tilde{u} to be orthogonal to φ_* determines it uniquely.

Finally, we establish the dependence on the weights. If χ and $\tilde{\chi}$ are edge equivalent, the desired unitary transformation is multiplication by $\tilde{\chi}_i/\chi_i$ on Γ_i . The edge equivalence ensures this is well-defined, since $\tilde{\chi}_i/\chi_i = \tilde{\chi}_j/\chi_j$ on $\Gamma_i \cap \Gamma_j$. The result when χ and $\tilde{\chi}$ are domain equivalent follows immediately from the definition. \square

4. THE SPECTRAL FLOW: PROOF OF THEOREM 1.7

To prove our main theorem we study the spectral flow of a family of self-adjoint operators. This idea was pioneered by Friedlander in [14], though our approach is closer to that of [3, 4]. To characterize the negative eigenvalues of $\text{DN}(\Gamma, \lambda_*, \chi)$ it is fruitful to study the family of operators $-\Delta_\sigma^\chi$, $0 \leq \sigma < \infty$, induced by the symmetric bilinear form

$$t_\sigma^\chi(u, v) = \sum_{i=1}^k \int_{D_i} \nabla u_i \cdot \nabla v_i + \sigma \int_\Gamma uv, \quad \text{dom}(t_\sigma^\chi) = \text{dom}(t^\chi), \quad (4.1)$$

where $\text{dom}(t^\chi)$ was defined in (1.10)–(1.12). As in Proposition 2.6, it can be shown that each Δ_σ^χ is self-adjoint, with domain

$$\text{dom}(\Delta_\sigma^\chi) = \{u \in \text{dom}(t^\chi) : \Delta u_i \in L^2(D_i) \text{ for each } i \text{ and } \gamma_N^\chi u + \sigma \gamma_D^\chi u = 0\}. \quad (4.2)$$

It can be easily seen that the eigenfunction φ_* of Δ^χ that vanishes on the set Γ is an eigenfunction of Δ_σ^χ for all σ . We can therefore consider the reduced operator $\hat{\Delta}_\sigma^\chi$, which is simply Δ_σ^χ restricted to $\text{span}\{\varphi_*\}^\perp$. We recall (see Section 2.4) that Δ_∞^χ is the Laplacian on Ω with Dirichlet boundary conditions imposed on $\partial\Omega \cup \Gamma$.

Proposition 4.1. *For each $\sigma \in [0, \infty)$ the linear mapping*

$$T : \ker(\Delta_\sigma^\chi + \lambda_*) \longrightarrow \ker(\text{DN}(\Gamma, \lambda_*, \chi) + \sigma), \quad Tu = \gamma_D^\chi u, \quad (4.3)$$

is surjective, and its kernel is spanned by φ_ , hence*

$$\dim \ker(\Delta_\sigma^\chi + \lambda_*) - 1 = \dim \ker(\text{DN}(\Gamma, \lambda_*, \chi) + \sigma).$$

Equivalently, in terms of the reduced operator, the restriction of T to $\ker(\widehat{\Delta}_\sigma^\chi + \lambda_*)$ is bijective and

$$\dim \ker(\widehat{\Delta}_\sigma^\chi + \lambda_*) = \dim \ker(\mathrm{DN}(\Gamma, \lambda_*, \chi) + \sigma).$$

Proof. We first show that T is well-defined. Assume that u is an eigenfunction of $-\Delta_\sigma^\chi$ with eigenvalue λ_* . From (4.2) we see that u satisfies the transmission condition $\gamma_N^\chi u + \sigma \gamma_D^\chi u = 0$ on Γ . On each D_i we can use Green's second identity to conclude that

$$0 = \int_{\Gamma_i} u_i \frac{\partial \varphi_{*,i}}{\partial \nu_i} = \int_{\Gamma_i} \chi_i (\chi_i u_i) \frac{\partial \varphi_{*,i}}{\partial \nu_i}.$$

This means that the Dirichlet trace $\gamma_D^\chi u \in H_\chi^{1/2}(\Gamma)$ belongs to the subspace S_χ defined in (3.1). Moreover, since $\gamma_N^\chi u = -\sigma \gamma_D^\chi u$ is contained in $L^2(\Gamma)$, we see from (3.5) that $\gamma_D^\chi u$ belongs to the domain of $\mathrm{DN}(\Gamma, \lambda_*, \chi)$, with

$$\mathrm{DN}(\Gamma, \lambda_*, \chi) \gamma_D^\chi u = \Pi_\chi(\gamma_N^\chi u) = -\sigma \Pi_\chi(\gamma_D^\chi u) = -\sigma \gamma_D^\chi u. \quad (4.4)$$

This means $\gamma_D^\chi u \in \ker(\mathrm{DN}(\Gamma, \lambda_*, \chi) + \sigma)$, so T is well-defined.

We next show that T is surjective. Let $g \in \ker(\mathrm{DN}(\Gamma, \lambda_*, \chi) + \sigma)$ be given. From the second part of Theorem 3.1, we know that there exists $\tilde{u}_i \in H^1(D_i)$ satisfying the equation $\Delta \tilde{u}_i + \lambda_* \tilde{u}_i = 0$ and the boundary conditions $\gamma_D^\chi \tilde{u} = g$, such that

$$\gamma_N^\chi \tilde{u} = \mathrm{DN}(\Gamma, \lambda_*, \chi) g = -\sigma g. \quad (4.5)$$

This is precisely the transmission condition $\gamma_N^\chi \tilde{u} + \sigma \gamma_D^\chi \tilde{u} = 0$, so we conclude from (4.2) that $\tilde{u} \in \mathrm{dom}(\Delta_\sigma^\chi)$ and hence $\tilde{u} \in \ker(\Delta_\sigma^\chi + \lambda_*)$. Since $T\tilde{u} = \gamma_D^\chi \tilde{u} = g$, this proves surjectivity.

It remains to prove that the kernel of T is spanned by φ_* . From Proposition 2.10 we know that $\varphi_* \in \ker(\Delta_\sigma^\chi + \lambda_*)$ for all σ and $\gamma_D^\chi \varphi_* = 0$, hence $\varphi_* \in \ker T$. Finally, suppose that u is any function in $\ker T$. This means $u \in \ker(\Delta_\sigma^\chi + \lambda_*)$ and $\gamma_D^\chi u = 0$, hence $\gamma_N^\chi u = 0$ by the transmission condition, so it follows from Proposition 2.10 that u is proportional to φ_* . \square

Remark 4.2. In the above proof, in particular (4.4), we see that if $u \in \ker(\Delta_\sigma^\chi + \lambda_*)$, so that $\gamma_D^\chi u$ is an eigenfunction of $\mathrm{DN}(\Gamma, \lambda_*, \chi)$, then $\gamma_N^\chi u \in S_\chi$, and hence $\Pi_\chi(\gamma_N^\chi u) = \gamma_N^\chi u$. In other words, it is the particular solution \tilde{u} whose existence is guaranteed by the second part of Theorem 3.1.

We are now ready to prove our main result.

Proof of Theorem 1.7. The equality (1.17) follows from Proposition 4.1 with $\sigma = 0$. To prove (1.16) we consider the spectral flow for the reduced operator family $-\widehat{\Delta}_\sigma^\chi$ defined above. Since this is an analytic family of self-adjoint operators for $0 \leq \sigma < \infty$, we can arrange the eigenvalues into analytic branches $\{\gamma_m(\sigma)\}$ such that:

- (1) $\{\gamma_m(0)\}$ are the ordered eigenvalues of $-\widehat{\Delta}_0^\chi$, repeated according to multiplicity;
- (2) each function $\sigma \mapsto \gamma_m(\sigma)$ is non-decreasing;
- (3) as $\sigma \rightarrow \infty$, the $\gamma_m(\sigma)$ converge to the eigenvalues of $-\widehat{\Delta}_\infty^\chi$.

The first statement is simply our convention for labelling the branches, the second follows from the monotonicity of the quadratic form $t_\sigma^\chi(u, u)$ from (4.1), and the third can be proved using the method of [3, Theorem 2.5].

At $\sigma = 0$ the operator $\widehat{\Delta}_0^\chi$ has $\ell - 1$ eigenvalues below λ_* . On the other hand, at $\sigma = \infty$ the first eigenvalue of $-\widehat{\Delta}_\infty^\chi$ is λ_* , with multiplicity k (one for each nodal domain). This means the first eigenvalue of the reduced operator $-\widehat{\Delta}_\infty^\chi$ is also λ_* , but with multiplicity $k - 1$.

Therefore, of the first $\ell - 1$ eigenvalue curves, precisely $k - 1$ converge to λ_* , while the remaining $\ell - k$ converges to strictly larger values, and hence intersect λ_* at some finite value of σ . In other words,

$$\ell - k = \#\{m : \gamma_m(\sigma) = \lambda_* \text{ for some } \sigma \in (0, \infty)\}.$$

From Proposition 4.1 we know that λ_* is an eigenvalue of $-\widehat{\Delta}_\sigma^\chi$ if and only if $-\sigma$ is an eigenvalue of $\mathrm{DN}(\Gamma, \lambda_*, \chi)$, with the same multiplicity, and hence

$$\#\{m : \gamma_m(\sigma) = \lambda_* \text{ for some } \sigma \in (0, \infty)\} = \mathrm{Mor} \mathrm{DN}(\Gamma, \lambda_*, \chi)$$

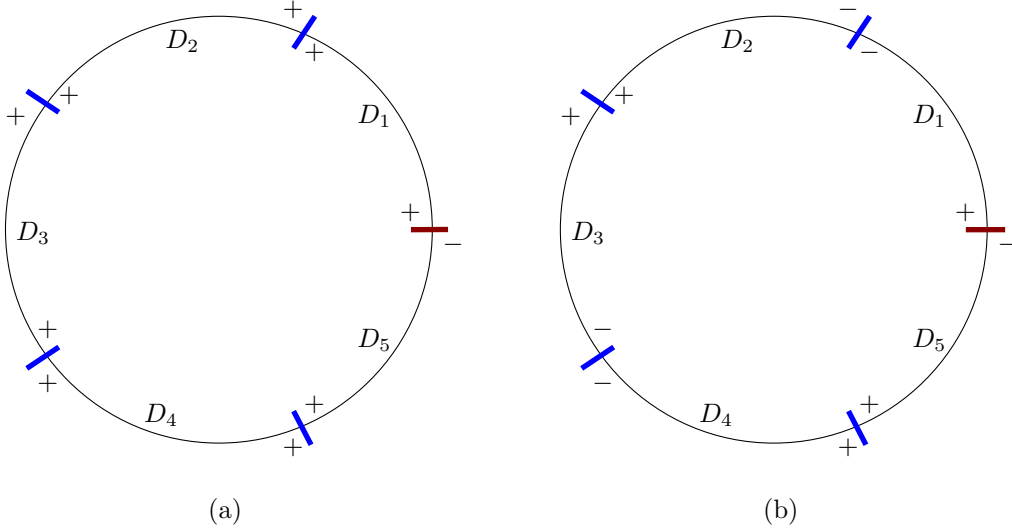


FIGURE 5.1. Two choices of valid χ for an odd partition of a circle, as in Section 5. The value of χ_i is indicated by a + or - next to the corresponding side of the partition boundary. The choices two choices are edge-equivalent (Definition 2.4). One way to see they are valid is to note that they define the same “valid cut” of Γ , which is highlighted in red; see Definition A.4 and Proposition A.6.

is the number of negative eigenvalues of $\text{DN}(\Gamma, \lambda_*, \chi)$, counted with multiplicity. \square

5. EQUIPARTITIONS OF THE UNIT CIRCLE REVISITED

Here we analyze equipartitions of the circle, calculating explicitly the different terms in Theorem 1.7. The same example was previously considered in [18], but with the Dirichlet-to-Neumann map evaluated at $\lambda_* + \varepsilon$, as described in the introduction. Also, in [18] the magnetic point of view was used, with the operator $T = -(\frac{d}{d\theta} - \frac{i}{2})^2$. We use here the equivalent presentation with cuts, replacing T by $-\Delta^\chi$.

Letting $\mathcal{D} = \{D_i\}_{i=1}^k$ be a k -equipartition of the circle, we will show that \mathcal{D} is χ -nodal, corresponding to a Δ^χ eigenvalue of multiplicity two, with defect $\delta(\mathcal{D}) = 0$. Comparing with Theorem 1.7, we should thus have

$$\text{Mor DN}(\Gamma, \lambda_*, \chi) = 0, \quad \dim \ker \text{DN}(\Gamma, \lambda_*, \chi) = 1. \quad (5.1)$$

Indeed, we find that $\text{DN}(\Gamma, \lambda_*, \chi)$ is identically zero on the space S_χ , which is one dimensional, confirming (5.1).

Remark 5.1. Recall that $S_\chi \subset L^2(\Gamma)$ has codimension $k - 1$. A k -partition of the circle has k boundary points, so $L^2(\Gamma) \cong \mathbb{R}^k$ and hence S_χ is one dimensional. A k partition of an interval, however, has only $k - 1$ boundary points, and so S_χ is zero dimensional. In this case the nullity and Morse index of DN must be zero, so Theorem 1.7 says that the partition has zero deficiency and corresponds to a simple eigenvalue, thus reproducing the Sturm oscillation theorem.

We view the circle as $[0, 2\pi]$ with the endpoints identified. We choose as division points $\theta_i = 2\pi i/k$ for $0 \leq i \leq k$, naturally identifying θ_0 and θ_k . The partition thus consists of the subintervals

$$D_i = (\theta_{i-1}, \theta_i), \quad 1 \leq i \leq k,$$

and the boundary set is given by $\Gamma = \{\theta_i\}_{i=0}^{k-1}$. We next define the weight functions χ_i , which in our case are functions on $\partial D_i = \{\theta_{i-1}, \theta_i\}$ with values in $\{\pm 1\}$.

If k is even we are in the bipartite case, and we can choose $\chi_i \equiv 1$ for each i , in which case Δ^χ is the Laplacian. We therefore only consider odd k , and introduce a single cut at $\theta = 0$, as was done in [18]. As weight functions we take $\chi_i \equiv 1$ for $0 \leq i \leq k - 1$, and for $i = k$ we take $\chi_k(\theta_{k-1}) = 1$ and $\chi_k(\theta_k) = -1$, see Figure 5.1(a).

For the operator $-\Delta^\chi$ we recall from (1.12) the compatibility condition $\chi_i u_i = \chi_j u_j$ on the common boundaries of D_i and D_j , which here says that functions u in the domain of $-\Delta^\chi$ should be continuous at each θ_i except the cut, where $u(0) = -u(2\pi)$. We recall also from (2.6) the transmission conditions $\chi_i \partial_{\nu_i} u_i + \chi_j \partial_{\nu_j} u_j = 0$. The outward normal derivative ∂_{ν_i} is $-\partial_\theta$ at the left end-point and ∂_θ at the right end-point, so functions in the domain of $-\Delta^\chi$ should be differentiable at each θ_i except the cut, where $u'(0) = -u'(2\pi)$. In summary, we have

$$\text{dom}(\Delta^\chi) = \{u \in H^2(0, 2\pi) : u(0) = -u(2\pi), u'(0) = -u'(2\pi)\}. \quad (5.2)$$

This operator is known as the *anti-periodic Hill operator* or the magnetic Laplace operator on a circle with flux $1/2$.

The spectrum of $-\Delta^\chi$ consists of eigenvalues $\lambda = (j/2)^2$, where j is positive and odd. Each eigenspace is two dimensional, spanned by $\sin(j\theta/2)$ and $\cos(j\theta/2)$. The partition \mathcal{D} is χ -nodal since it is generated by the eigenfunction $\varphi_*(\theta) = \sin(k\theta/2)$. The minimal label of the corresponding eigenvalue $\lambda_* = (k/2)^2$ is $\ell(\mathcal{D}) = k$ and thus $\delta(\mathcal{D}) = 0$, as claimed above.

We now turn to the Dirichlet-to-Neumann operator, for which we use a different valid choice of weights⁴, letting $\hat{\chi}_i(\theta_{i-1}) = \cos(k\theta_{i-1}/2) = (-1)^{i-1}$ and $\hat{\chi}_i(\theta_i) = \cos(k\theta_i/2) = (-1)^i$ for all $1 \leq i \leq k$; see Figure 5.1(b). The condition for the boundary data $g = (g_0, g_1, \dots, g_{k-1}) \in \mathbb{R}^k$ to be in the subspace $S_{\hat{\chi}}$ defined in (3.1) is $g_{i-1} - g_i = 0$, yielding $g = h(1, 1, \dots, 1)^t$, $h \in \mathbb{R}$. For this choice of $\{\hat{\chi}_i\}$, the boundary value problem (3.2) becomes

$$-u_i'' = \lambda_* u_i \text{ in } D_i, \quad u_i(\theta_{i-1}) = h \cos(k\theta_{i-1}/2), \quad u_i(\theta_i) = h \cos(k\theta_i/2), \quad (5.3)$$

with the general solution

$$u_i(\theta) = h \cos(k\theta/2) + c_i \sin(k\theta/2),$$

where c_i is an arbitrary constant. According to Remark 3.2, we can calculate the Dirichlet-to-Neumann map using *any* solution to the boundary value problem, so we choose $c_i = 0$. It follows immediately that

$$\frac{\partial u_i}{\partial \nu_i}(\theta_{i-1}) = \frac{\partial u_i}{\partial \nu_i}(\theta_i) = 0$$

for each i , hence the two-sided normal derivative $\gamma_N^\chi u$ vanishes on Γ , and

$$\text{DN}(\Gamma, \lambda_*, \chi)h = \Pi_\chi(\gamma_N^\chi u) = 0, \quad (5.4)$$

as expected.

APPENDIX A. WEIGHTS, CUTS AND PAIR COMPATIBILITY CONDITIONS

In this section we elaborate on some of our constructions and their connection to previous literature. In Section A.1 we discuss the relationship between the strong pair compatibility condition in Definition 1.6, and the weak pair compatibility that appeared in earlier works, such as [16], where it was simply referred to as the *pair compatibility condition*. In Section A.2 we describe the cutting construction of [18], and explain how it is related to the valid weights introduced in Definition 1.3.

A.1. Weak vs strong pair compatibility conditions. The strong pair compatibility condition (SPCC) was already described in Definition 1.6, which we repeat here for convenience.

Definition A.1. A two-sided, weakly regular partition \mathcal{D} is said to satisfy the *strong pair compatibility condition* (SPCC) if there exists a choice of positive ground states $\{u_i\}_{i=1}^k$ for the Dirichlet Laplacians on D_i such that, for any pair of neighbors D_i, D_j , the function u_{ij} defined by

$$u_{ij}|_{D_i} = u_i, \quad u_{ij}|_{D_j} = -u_j, \quad (A.1)$$

is an eigenfunction of the Dirichlet Laplacian on $\text{Int}(\overline{D_i \cup D_j})$.

⁴The two choices of weights are edge equivalent, therefore the Laplacian $\Delta^{\hat{\chi}}$ is identical to Δ^χ .

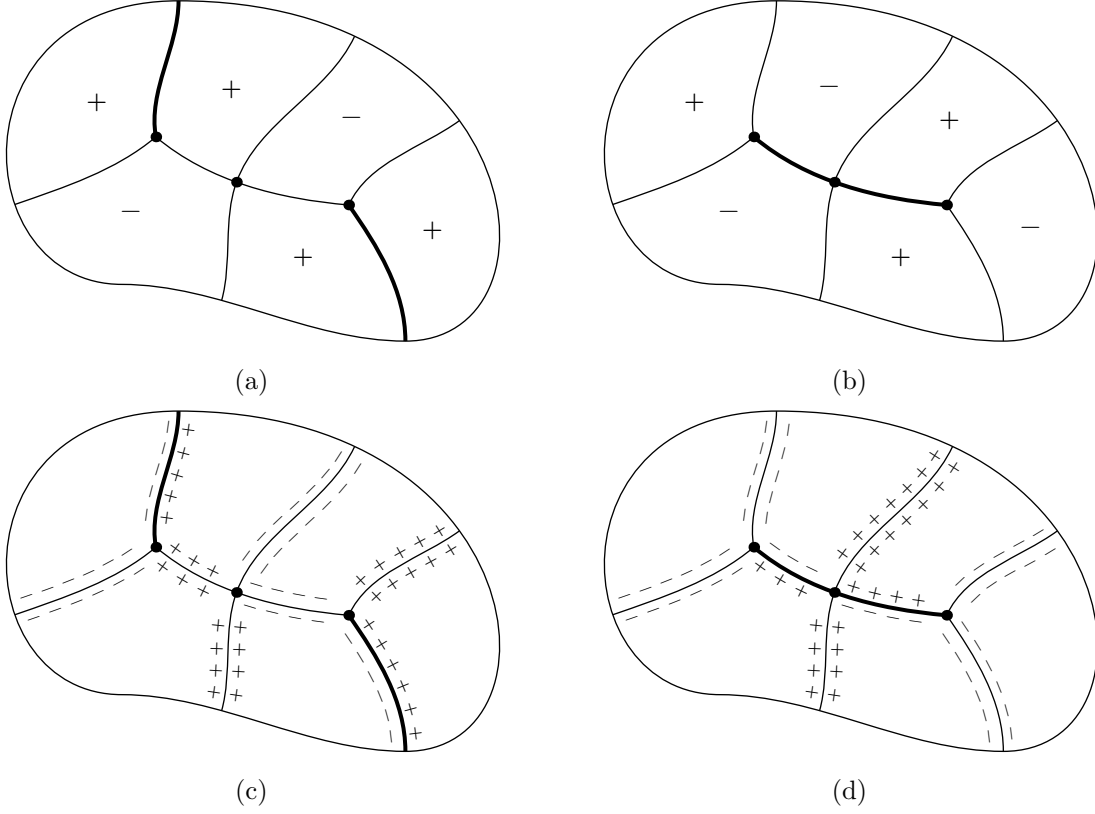


FIGURE A.1. Two valid cuts of the same partition are shown in (a) and (b), with the thick line denoting Γ^* and the \pm signs indicating the chosen orientations of each D_i . In (c) and (d) we show possible choices of $\{\chi_i\}$ for each of these cuts.

Nodal partitions obviously satisfy the SPCC. The same is true of spectral minimal partitions (see [17]), and in Proposition 2.7 we showed that a partition satisfies the SPCC if and only if it is χ -nodal. A partition satisfying the SPCC is necessarily an *equipartition*, in the sense that the ground state energy (the smallest eigenvalue of the Dirichlet Laplacian) on each D_i is the same. We denote this common value by $\lambda(\mathcal{D})$.

We next recall the weak pair compatibility condition.

Definition A.2. A two-sided, weakly regular equipartition \mathcal{D} is said to satisfy the *weak pair compatibility condition* (WPCC) if for each pair of neighbors D_i, D_j , there exists an eigenfunction of the Dirichlet Laplacian on $\text{Int}(\overline{D_i} \cup \overline{D_j})$ with eigenvalue $\lambda(\mathcal{D})$ and nodal set $\partial D_i \cap \partial D_j$.

Remark A.3. By [17, Theorem 2.6] applied to each pair of neighbors, it follows that partitions that satisfy the WPCC also have the equal angle property; cf. Corollary 2.9.

It is obvious that SPCC implies WPCC. When Ω is simply connected, a bipartite equipartition satisfying WPCC is nodal, and hence satisfies SPCC, by [16, Theorem 1.3]. If Ω is not simply connected, however, it is possible to find an equipartition (for a Schrödinger operator with C^∞ potential) that satisfies WPCC but not SPCC, as shown in [16, Section 7].

A.2. Weights and cuts. Assuming throughout that \mathcal{D} is a two-sided, weakly regular partition, with nodal set Γ , we first decompose the smooth part of Γ into disjoint open curves, labeled $\{C_a\}$, so that $\Gamma = \overline{\cup_a C_a}$. Since \mathcal{D} is two-sided, each C_a is contained in $\Gamma_i \cap \Gamma_j$ for some $i \neq j$. Without loss of generality we can assume $i < j$, and we denote these labels by $i(a)$ and $j(a)$.

Definition A.4. A subset $\mathcal{C} \subset \{C_a\}$ is called a *valid cut* of the partition \mathcal{D} if there exists a choice of orientations on the subdomains $\{D_i\}$ such that $C_a \in \mathcal{C}$ if and only if $D_{i(a)}$ and $D_{j(a)}$ have the same orientation.

It is sometimes convenient to identify a subset $\mathcal{C} = \{C_{a_1}, \dots, C_{a_p}\} \subset \{C_a\}$ with the corresponding closed subset

$$\Gamma^* := \overline{C_{a_1} \cup \dots \cup C_{a_p}} \quad (\text{A.2})$$

of Γ . We mention that $\mathcal{C} \subset \{C_a\}$ is a valid cut if $\Gamma \setminus \Gamma^*$ is a \mathbb{Z}_2 -homological 1-cycle of Ω (viewed as a cell complex) relative to the boundary $\partial\Omega$. It is immediate that the empty set is a valid cut of \mathcal{D} if and only if \mathcal{D} is bipartite.

The *maximal cut* $\mathcal{C} = \{C_a\}$, for which $\Gamma^* = \Gamma$, is always valid—it corresponds to all subdomains having the same orientation. However, usually one is interested in cuts that are as small as possible. We thus say that a cut is *minimal* if $\Omega \setminus \Gamma^*$ is connected.

Proposition A.5. [18, Prop 4.2] *There exists a minimal valid cut $\mathcal{C} \subset \{C_a\}$.*

Finally, we describe how valid cuts are related to the valid weights $\{\chi_i\}$ in Definition 1.3. Given a set of valid weights $\{\chi_i\}$, we obtain a valid cut \mathcal{C} by declaring that $C_a \in \mathcal{C}$ if and only if $\chi_{i(a)} = -\chi_{j(a)}$. That is, the cut set Γ^* is the union of all $\Gamma_i \cap \Gamma_j$ along which $\chi_i = -\chi_j$. More precisely, we have the following.

Proposition A.6. *Valid cuts are in one-to-one correspondence with edge-equivalence classes of valid weights.*

Proof. Given a valid cut, i.e. a choice of orientation for each D_i , we get an induced orientation on each ∂D_i . Choosing an orientation on each smooth component of Γ , we obtain a valid set of weights $\{\chi_i\}$ with the property that $\chi_i = -\chi_j$ if and only if $\partial D_i \cap \partial D_j$ is in the cut set Γ^* . Changing the orientation on any smooth part of Γ will give a different, but edge equivalent, set of weights (recall Definition 2.4), so we get a map from valid cuts to edge-equivalence classes of valid weights. Conversely, a set of valid weights gives an orientation on each D_i , and hence a valid cut. It is easily seen that edge-equivalent weights generate the same cut. \square

Remark A.7. The proof of Proposition A.6 suggests an equivalent way to define valid cuts and weights: a cut Γ^* is valid if a generic closed path in Ω intersects $\Gamma \setminus \Gamma^*$ an even number of times, and a choice of weights $\{\chi_i\}$ is valid if the set $\{C_a : \chi_{i(a)} = -\chi_{j(a)}\}$ defines a valid cut. This alternative definition is not as constructive as Definition 1.3, but it has the advantage of not depending on the manifold structure of Ω , and is thus more convenient for considering partitions on metric graphs.

Remark A.8. Another way of viewing the constructions in this paper is to introduce Aharonov–Bohm operators, as in [18]. Given a set of weights χ that generates a minimal valid cut, the corresponding Δ^χ is equivalent to a certain Aharonov–Bohm operator, with Aharonov–Bohm solenoids with flux π placed at the singular points x_ℓ of Γ for which ν_ℓ is odd (recall Definition 1.2).

APPENDIX B. EXPLICIT CONSTRUCTION OF THE CANONICAL SOLUTION TO (3.2)

In this section we give an alternate, more explicit proof of the second claim in Theorem 3.1 regarding the existence of a “canonical solution” $\tilde{u} \in \text{dom}(t^\chi)$ such that $\gamma_N^\chi \tilde{u} \in S_\chi$ and \tilde{u}_i solves (3.2) for each i . To do this we write the condition $\gamma_N^\chi \tilde{u} \in S_\chi$ as a finite system of linear equations and then, by analyzing the corresponding matrix, prove that a solution always exists.

Fix $g \in \text{dom}(A)$. For each i , the general solution of (3.2) is given by

$$u_i = u_i^g + c_i \varphi_{*,i} \quad (\text{B.1})$$

for some $c_i \in \mathbb{R}$. Since $g \in \text{dom}(A)$, we know from (3.5) that the two-sided normal derivative $\gamma_N^\chi u$ is a function in $L^2(\Gamma)$, and is given by $\chi_i \partial_{\nu_i} u_i + \chi_j \partial_{\nu_j} u_j$ on $\Gamma_i \cap \Gamma_j$. This will be an element of the subspace S_χ if and only if

$$I_i := \int_{\Gamma_i} \chi_i (\gamma_N^\chi u) \partial_{\nu_i} \varphi_{*,i} = 0 \quad (\text{B.2})$$

for each i . Since each point in the smooth part of Γ_i is contained in precisely one other Γ_j , we can rewrite this integral as

$$\begin{aligned} I_i &= \sum_{j \neq i} \int_{\Gamma_i \cap \Gamma_j} (\partial_{\nu_i} u_i + \chi_{ij} \partial_{\nu_j} u_j) \partial_{\nu_i} \varphi_{*,i} \\ &= \sum_{j \neq i} \int_{\Gamma_i \cap \Gamma_j} (\partial_{\nu_i} u_i^g + \chi_{ij} \partial_{\nu_j} u_j^g) \partial_{\nu_i} \varphi_{*,i} + \sum_{j \neq i} \int_{\Gamma_i \cap \Gamma_j} (c_i \partial_{\nu_i} \varphi_{*,i} + \chi_{ij} c_j \partial_{\nu_j} \varphi_{*,j}) \partial_{\nu_i} \varphi_{*,i}, \end{aligned} \quad (\text{B.3})$$

where we have denoted $\chi_{ij} = \chi_i \chi_j$ for convenience. Let us introduce the notations

$$\alpha_{i,i} = 0, \quad \alpha_{i,j} = \int_{\Gamma_i \cap \Gamma_j} |\partial_{\nu_i} \varphi_{*,i}|^2, \quad i \neq j.$$

It follows from (2.8) that $|\partial_{\nu_i} \varphi_{*,i}| = |\partial_{\nu_j} \varphi_{*,j}|$ on $\Gamma_i \cap \Gamma_j$, and so $\alpha_{i,j} = \alpha_{j,i}$ for all i, j . We similarly get

$$\int_{\Gamma_i \cap \Gamma_j} \chi_{ij} (\partial_{\nu_j} \varphi_{*,j}) (\partial_{\nu_i} \varphi_{*,i}) = -\alpha_{i,j},$$

We then define

$$d_i = - \sum_{j \neq i} \int_{\Gamma_i \cap \Gamma_j} (\partial_{\nu_i} u_i^g + \chi_{ij} \partial_{\nu_j} u_j^g) \partial_{\nu_i} \varphi_{*,i} \quad (\text{B.4})$$

so the equation (B.3) becomes

$$\sum_{j \neq i} (c_i - c_j) \alpha_{i,j} = d_i. \quad (\text{B.5})$$

We write the resulting system of equations in matrix form as

$$\underbrace{\begin{bmatrix} \sum_j \alpha_{1,j} & -\alpha_{1,2} & \cdots & -\alpha_{1,k} \\ -\alpha_{2,1} & \sum_j \alpha_{2,j} & \cdots & -\alpha_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{k,1} & -\alpha_{k,2} & \cdots & \sum_j \alpha_{k,j} \end{bmatrix}}_{=:A} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_k \end{bmatrix}, \quad (\text{B.6})$$

and observe that the vector $(c_1, c_2, \dots, c_k)^t = (1, 1, \dots, 1)^t$ lies in the kernel of the matrix A .

Without loss of generality, we can label the domains $\{D_i\}$ in the partition inductively so that D_{i+1} is a neighbor of at least one of D_1, \dots, D_i , with D_1 arbitrary. For the numbers $\alpha_{i,j}$, this means that

$$\begin{aligned} \alpha_{1,2} &> 0, \\ \alpha_{1,3} + \alpha_{2,3} &> 0, \\ \alpha_{1,4} + \alpha_{2,4} + \alpha_{3,4} &> 0, \\ &\vdots \\ \alpha_{1,k} + \alpha_{2,k} + \cdots + \alpha_{k-1,k} &> 0. \end{aligned} \quad (\text{B.7})$$

Lemma B.1. *Let A be the symmetric $k \times k$ matrix in (B.6) and assume that the inequalities in (B.7) hold. Then $\ker A$ is spanned by $(1, 1, \dots, 1)^t$.*

Proof. Consider the quadratic form $q[\mathbf{c}] = \langle A\mathbf{c}, \mathbf{c} \rangle$ corresponding to the matrix A above, where $\mathbf{c} = (c_1, c_2, \dots, c_k)^t$. From (B.5) we find that the quadratic form $q[\mathbf{c}]$ can be written as

$$\begin{aligned} &\alpha_{1,2}(c_1 - c_2)^2 \\ &+ \alpha_{1,3}(c_1 - c_3)^2 + \alpha_{2,3}(c_2 - c_3)^2 \\ &+ \alpha_{1,4}(c_1 - c_4)^2 + \alpha_{2,4}(c_2 - c_4)^2 + \alpha_{3,4}(c_3 - c_4)^2 \\ &+ \cdots \\ &+ \alpha_{1,k}(c_1 - c_k)^2 + \alpha_{2,k}(c_2 - c_k)^2 + \cdots + \alpha_{k-1,k}(c_{k-1} - c_k)^2. \end{aligned}$$

Since $\alpha_{i,j} \geq 0$ for all i, j , we see that q (and hence A) is non-negative. It remains to identify the kernel. Assume that $q[\mathbf{c}] = 0$ for some \mathbf{c} . Then, reading from the top line above, we conclude that $c_2 = c_1$ since $\alpha_{1,2} > 0$. Inserting $c_1 = c_2$, we conclude from the next row that $c_3 = c_2$ since $\alpha_{1,3} + \alpha_{2,3} > 0$. Continuing in this manner, we conclude that $c_k = c_{k-1} = \dots = c_2 = c_1$. This means that the kernel of A is spanned by the vector $(1, 1, \dots, 1)^t$. \square

Finally, from (B.4) we observe that $\sum d_i$ contains a term

$$- \int_{\Gamma_i \cap \Gamma_j} \left\{ (\partial_{\nu_i} u_i^g + \chi_{ij} \partial_{\nu_j} u_j^g) \partial_{\nu_i} \varphi_{*,i} + (\partial_{\nu_j} u_j^g + \chi_{ij} \partial_{\nu_i} u_i^g) \partial_{\nu_j} \varphi_{*,j} \right\}$$

for each pair of neighboring domains, and by (2.8) each of the integrands vanishes. This means $\sum d_i = 0$, so the vector $\mathbf{d} = (d_1, d_2, \dots, d_k)^t$ is orthogonal to the kernel of A . Thus, the system (B.6) will always be solvable, by the Fredholm alternative for symmetric matrices.

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DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843-3368, USA
Email address: `gberkolaiko@tamu.edu`

DEPARTMENT OF MATHEMATICS AND STATISTICS, MEMORIAL UNIVERSITY OF NEWFOUNDLAND, ST. JOHN'S,
NL A1C 5S7, CANADA
Email address: `gcox@mun.ca`

LABORATOIRE DE MATHÉMATIQUES JEAN LERAY, UNIVERSITÉ DE NANTES, 44 322 NANTES, FRANCE
Email address: `Bernard.Helffer@univ-nantes.fr`

LUND UNIVERSITY, DEPARTMENT OF MATHEMATICAL SCIENCES, BOX 118, 221 00 LUND, SWEDEN
Email address: `mikael.persson.sundqvist@math.lth.se`