Hybridizable discontinuous Galerkin methods for the coupled Stokes–Biot problem

Aycil Cesmelioglu[a,1], Jeonghun J. Lee[b,2,], Sander Rhebergen[c,3]

[a]*Department of Mathematics and Statistics, Oakland University, Michigan, USA*
[b]*Department of Mathematics, Baylor University, Texas, USA*
[c]*Department of Applied Mathematics, University of Waterloo, Canada*

Abstract

We present and analyze a hybridizable discontinuous Galerkin (HDG) finite element method for the coupled Stokes–Biot problem. Of particular interest is that the discrete velocities and displacement are $H(\text{div})$-conforming and satisfy the compressibility equations pointwise on the elements. Furthermore, in the incompressible limit, the discretization is strongly conservative. We prove well-posedness of the discretization and, after combining the HDG method with backward Euler time stepping, present a priori error estimates that demonstrate that the method is free of volumetric locking. Numerical examples further demonstrate optimal rates of convergence in the L^2-norm for all unknowns and that the discretization is locking-free.

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1. Introduction

Many applications in environmental and biomedical engineering require the modeling of the interaction between a free fluid and a deformable porous media that is saturated by fluid. Such problems can be modeled by the coupled Stokes–Biot problem, a model first proposed by Showalter [1], in which the Stokes equations are coupled to Biot's consolidation model [2, 3, 4]. At the interface the two flows are coupled by enforcing mass conservation, continuity of the normal stress, a balance of forces across the interface, and the Beavers–Joseph–Saffman interface condition [5, 6]. Existence and uniqueness of weak solutions to this problem were proven in [7].

Various finite element methods have been proposed for the Stokes–Biot problem. To better describe these methods we use u^s and p^s to denote the fluid velocity and pressure in the Stokes region and u^b, z, and p^p to denote the displacement, Darcy velocity, and pore pressure in the Biot region. Badia et al. [8] introduced the first finite element method for this problem

Email addresses: `cesmelio@oakland.edu` (Aycil Cesmelioglu), `jeonghun_lee@baylor.edu` (Jeonghun J. Lee), `srheberg@uwaterloo.ca` (Sander Rhebergen)
[1]`https://orcid.org/0000-0001-8057-6349`
[2]`https://orcid.org/0000-0001-5201-8526`
[3]`https://orcid.org/0000-0001-6036-0356`

considering Lagrange finite elements for (u^s, p^s) and Lagrange and $H(\text{div})$ mixed methods for (u^b, z, p^p). Residual-based stabilization techniques were used in both subdomains to stabilize this conforming finite element method. A Lagrange multiplier method to weakly impose mass conservation across the interface was studied in Ambartsumyan et al. [9]. They used conforming stable methods for the Stokes unknowns (u^s, p^s) [10, 11, 12, 13], a Lagrange finite element for u^b and stable mixed finite elements for (z, p^p) [10, 14, 15]. To avoid poroelastic locking when using a conforming stable finite element method of the Stokes–Biot problem Cesmelioglu and Chidyagwai [16] studied the use of a heuristic stabilization technique. A numerical study with similar methods was done in [17] for a Stokes–Biot model with fluid entrance resistance. A finite element method based on the total pressure formulation of the Biot model [18, 19] was studied by Ruiz-Baier et al. [20]. Eliminating the Darcy velocity z, they consider stable Stokes elements for (u^s, p^s) and (u^b, p^b), where p^b is the total pressure, and a piecewise continuous and polynomial space for p^p. They furthermore show the existence of a unique weak solution to this formulation of the Stokes–Biot problem. Finally, let us mention that a stress tensor based approach using a weakly symmetric mixed method for poroelasticity [21] and a conforming stable mixed method for Stokes was studied in [22, 23], and that partitioned time discretization methods, focusing on efficient time stepping and stability of partitioned schemes, are studied in [24, 25, 26].

In this paper we propose a hybridizable discontinuous Galerkin (HDG) method for the coupled Stokes–Biot problem. The HDG method uses hybridization to improve the computational efficiency of traditional discontinuous Galerkin (DG) methods [27]: degrees-of-freedom (dofs) local to an element are eliminated resulting in a global problem for dofs associated only with mesh facet unknowns. The HDG method we propose here combines the HDG method of [28, 29] for the Stokes equations and the HDG method studied in [30] for the Biot model. (See also [31] for a similar HDG method and [32] for an alternative approach for the coupled Stokes–Darcy problem.) Our HDG method is constructed such that the discrete fluid velocities and displacement are H(div)-conforming. In addition, unlike existing finite element methods for the Stokes–Biot problem in the literature, the (semi-)discrete solution satisfies the compressibility equations and the mass equation in the poroelastic domain pointwise on each element of the mesh (where the latter holds provided the source/sink term lies in the pore pressure space). These properties imply that the discretization is strongly conservative [33] in the incompressible limit. In addition to these exact approximation properties, we present an a priori error analysis for the time-dependent coupled Stokes–Biot problem that avoids using Grönwall's inequality. A consequence of the latter is that the space-time error estimates do not grow exponentially in time.

The remainder of this paper is organized as follows. In sections 2 and 3 we present the coupled Stokes–Biot model and its HDG discretization. We proceed by proving consistency and well-posedness of the HDG method in section 4 and present an a priori error analysis of the discretization in section 5. Numerical experiments in section 6 verify our analysis and conclusions are drawn in section 7.

2. The coupled Stokes–Biot equations

Let $\Omega \subset \mathbb{R}^d$ be a polygonal (if $d = 2$) or polyhedral (if $d = 3$) domain that is partitioned into two non-overlapping domains Ω^s and Ω^b. The Stokes equations on Ω^s describe flow of an incompressible fluid while a deformable porous structure on Ω^b is modelled by a quasi-static poroelasticity model. We will assume that the interface between the two subdomains, $\Gamma^I = \overline{\partial\Omega^s} \cap \overline{\partial\Omega^b}$ is polygonal. We denote by n and n^j the unit outward normal to, respectively, Ω and

Ω^j $(j = s, b)$. On the interface $n = n^s = -n^b$. Furthermore, let us denote by $J = (0, T]$ the time interval of interest.

In the poroelastic part of the domain we consider two different partitions of the boundary $\Gamma^b = \partial\Omega \cap \partial\Omega^b$. The first partition is $\Gamma^b = \Gamma_P^b \cup \Gamma_F^b$ with $\Gamma_P^b \cap \Gamma_F^b = \emptyset$ and $|\Gamma_P^b| > 0$, while the second partition is given by $\Gamma^b = \Gamma_D^b \cup \Gamma_N^b$ where $\Gamma_D^b \cap \Gamma_N^b = \emptyset$, $|\Gamma_D^b| > 0$, and $|\Gamma_N^b| > 0$. In the fluid domain we partition the boundary $\Gamma^s = \partial\Omega \cap \partial\Omega^s$ as $\Gamma^s = \Gamma_D^s \cup \Gamma_N^s$ with $\Gamma_D^s \cap \Gamma_N^s = \emptyset$, $|\Gamma_D^s| > 0$, and $|\Gamma_N^s| > 0$.

We denote body forces by $f^s : \Omega^s \times J \to \mathbb{R}^d$ and $f^b : \Omega^b \times J \to \mathbb{R}^d$ and the source/sink term by $g^b : \Omega^b \times J \to \mathbb{R}$. Furthermore, we denote by $\mu^s > 0$ the (constant) dynamic viscosity of the fluid, the Biot–Willis constant and the specific storage coefficient are denoted by, respectively, $0 < \alpha < 1$ and $c_0 \geq 0$, the positive permeability constant is denoted by κ, and the Lamé constants are denoted by μ^b and λ. Note that Young's modulus of elasticity E and Poisson's ratio ν are related to the Lamé constant by $E = (3\lambda + 2\mu^b)\mu^b/(\lambda + \mu^b)$ and $\nu = \lambda/(2(\lambda + \mu^b))$.

Using the total pressure formulation of Biot's consolidation model [18, 19], we can then state the coupled Stokes–Biot problem as: Find the fluid velocity $u^s : \Omega^s \times J \to \mathbb{R}^d$, the fluid pressure $p^s : \Omega^s \times J \to \mathbb{R}$, the solid displacement $u^b : \Omega^b \times J \to \mathbb{R}^d$, the pore pressure $p^p : \Omega^b \times J \to \mathbb{R}$, the total pressure $p^b : \Omega^b \times J \to \mathbb{R}$, and the Darcy velocity $z : \Omega^b \times J \to \mathbb{R}^d$ such that

$$-\nabla \cdot \sigma^j = f^j \qquad \text{in } \Omega^j \times J, \quad j = s, b, \tag{1a}$$

$$-\nabla \cdot u^s = 0 \qquad \text{in } \Omega^s \times J, \tag{1b}$$

$$-\nabla \cdot u^b + \lambda^{-1}(\alpha p^p - p^b) = 0 \qquad \text{in } \Omega^b \times J, \tag{1c}$$

$$c_0 \partial_t p^p + \alpha\lambda^{-1}(\alpha\partial_t p^p - \partial_t p^b) + \nabla \cdot z = g^b \qquad \text{in } \Omega^b \times J, \tag{1d}$$

$$\kappa^{-1} z + \nabla p^p = 0 \qquad \text{in } \Omega^b \times J, \tag{1e}$$

and such that

$$u^s \cdot n = \left(\partial_t u^b + z\right) \cdot n \qquad \text{on } \Gamma_I \times J, \tag{2a}$$

$$\sigma^s n = \sigma^b n \qquad \text{on } \Gamma_I \times J, \tag{2b}$$

$$-(\sigma^s n) \cdot n = p^p \qquad \text{on } \Gamma_I \times J, \tag{2c}$$

$$-2\mu^s\,(\varepsilon(u^s)n)^t = \gamma(\mu^s/\kappa)^{1/2}(u^s - \partial_t u^b)^t \qquad \text{on } \Gamma_I \times J. \tag{2d}$$

Here $\sigma^j := 2\mu^j \varepsilon(u^j) - p^j \mathbb{I}$ $(j = s, b)$, $\varepsilon(u) := \left(\nabla u + (\nabla u)^T\right)/2$, $(w)^t := w - (w \cdot n)n$, and eq. (2d) is the Beavers–Joseph–Saffman condition [5, 6] in which $\gamma > 0$ is an experimentally determined dimensionless constant. To close the system we impose the boundary conditions

$$u^j = 0 \qquad \text{on } \Gamma_D^j \times J, \quad j = s, b, \tag{3a}$$

$$\sigma^j n = 0 \qquad \text{on } \Gamma_N^j \times J, \quad j = s, b, \tag{3b}$$

$$p^p = 0 \qquad \text{on } \Gamma_P^b \times J, \tag{3c}$$

$$z \cdot n = 0 \qquad \text{on } \Gamma_F^b \times J, \tag{3d}$$

and initial conditions $p^p(x, 0) = p_0^p(x)$ in Ω^b and $u^b(x, 0) = u_0^b(x)$ in Ω^b.

For notational convenience it will be useful to define functions u and p on the whole domain Ω which are such that $u|_{\Omega^j} = u^j$ and $p|_{\Omega^j} = p^j$ for $j = s, b$.

In the a priori error analysis in section 5 we will assume that there exists a ν_* such that $0 < \nu_* \leq \nu < 0.5$ on Ω^b implying that $C_* \mu^b \leq \lambda$ with $C_* = 2\nu_*/(1 - 2\nu_*)$.

3. Discretizing the Stokes–Biot problem

3.1. Notation

Let $\mathcal{T}^j$ denote a shape-regular triangulation of Ω^j, $j = s, b$. We will assume that $\mathcal{T}^s$ and $\mathcal{T}^b$ match at the interface and define $\mathcal{T} := \mathcal{T}^s \cup \mathcal{T}^b$. For any element $K \in \mathcal{T}$, h_K denotes its diameter and $h := \max_{K\in\mathcal{T}} h_K$ defines the meshsize of the triangulation. We denote the set of all facets by $\mathcal{F}$, the set of all facets in $\bar{\Omega}^j$ by $\mathcal{F}^j$, $j = s, b$, the set of all facets in Ω^j by $\mathcal{F}^j_{int}$, and the set of all facets that lie on Γ_I, Γ^j_N, Γ^b_F, and Γ^b_P by $\mathcal{F}_I$, $\mathcal{F}^j_N$, $\mathcal{F}^b_F$, and $\mathcal{F}^b_P$, respectively. Finally, we set $\Gamma^j_0 = \cup_{F\in\mathcal{F}^j} F$, $j = s, b$.

Various approximation spaces are required to define the HDG discretization of the Stokes–Biot problem eqs. (1) to (3). These approximation spaces are discontinuous Galerkin (DG) spaces defined on Ω and Ω^j ($j = s, b$):

$$V^j_h := \left\{v_h \in [L^2(\Omega^j)]^d : \ v_h \in [P_k(K)]^d, \ \forall\, K \in \mathcal{T}^j\right\}, \quad j = s, b, \tag{4a}$$

$$Q^j_h := \left\{q_h \in L^2(\Omega^j) : \ q_h \in P_{k-1}(K), \ \forall\, K \in \mathcal{T}^j\right\}, \quad j = s, b, \tag{4b}$$

$$V_h := \left\{v_h \in [L^2(\Omega)]^d : \ v_h \in [P_k(K)]^d, \ \forall K \in \mathcal{T}\right\}, \tag{4c}$$

$$Q_h := \left\{q_h \in L^2(\Omega) : \ q_h \in P_{k-1}(K), \ \forall\, K \in \mathcal{T}\right\}. \tag{4d}$$

Note that for functions $u_h \in V_h$ and $p_h \in Q_h$ we have, respectively, that $u_h|_{\Omega^j} = u^j_h \in V^j_h$ and $p_h|_{\Omega^j} = p^j_h \in Q^j_h$ for $j = s, b$. The HDG discretization also requires the following facet DG spaces that are defined on Γ^j_0 ($j = s, b$):

$$\bar{V}^j_h := \left\{\bar{v}_h \in [L^2(\Gamma^j_0)]^d : \ \bar{v}_h \in [P_k(F)]^d \ \forall\, F \in \mathcal{F}^j, \ \bar{v}_h = 0 \text{ on } \Gamma^j_D\right\}, \tag{5a}$$

$$\bar{Q}^j_h := \left\{\bar{q}_h \in L^2(\Gamma^j_0) : \ \bar{q}_h \in P_k(F)\ \forall\, F \in \mathcal{F}^j\right\}, \quad j = s, b, \tag{5b}$$

$$\bar{Q}^{b0}_h := \{\bar{q}_h \in \bar{Q}^b_h : \bar{q}_h = 0 \text{ on } \Gamma^b_P\}. \tag{5c}$$

For notational purposes, we group cell and facet unknowns as follows:

$$\begin{aligned}
\boldsymbol{v}_h = (v_h, \bar{v}^s_h, \bar{v}^b_h) &\in \boldsymbol{V}_h := V_h \times \bar{V}^s_h \times \bar{V}^b_h, \\
\boldsymbol{q}_h = (q_h, \bar{q}^s_h, \bar{q}^b_h) &\in \boldsymbol{Q}_h := Q_h \times \bar{Q}^s_h \times \bar{Q}^b_h, \\
\boldsymbol{q}^p_h = (q^p_h, \bar{q}^p_h) &\in \boldsymbol{Q}^{b0}_h := Q^b_h \times \bar{Q}^{b0}_h, \\
(\boldsymbol{v}_h, \boldsymbol{q}_h, w_h, \boldsymbol{q}^p_h) &\in \boldsymbol{X}_h := \boldsymbol{V}_h \times \boldsymbol{Q}_h \times V^b_h \times \boldsymbol{Q}^{b0}_h,
\end{aligned}$$

and for $j = s, b$,

$$\boldsymbol{v}^j_h = (v^j_h, \bar{v}^j_h) \in \boldsymbol{V}^j_h := V^j_h \times \bar{V}^j_h, \qquad \boldsymbol{q}^j_h = (q^j_h, \bar{q}^j_h) \in \boldsymbol{Q}^j_h := Q^j_h \times \bar{Q}^j_h.$$

Next, let us define

$$\begin{aligned}
V^j &:= \left\{v \in [H^1(\Omega^j)]^d \ : \ v|_{\Gamma^j_D} = 0\right\} \cap H^2(\Omega^j)^d, \quad j = s, b, \\
Q^j &:= H^1(\Omega^j), \quad j = s, b, \\
Z &:= \left\{v \in [H^1(\Omega^b)]^d \ : \ v \cdot n|_{\Gamma^b_F} = 0\right\}, \\
Q^{b0} &:= \left\{q \in H^1(\Omega^b) \ : \ q|_{\Gamma^b_P} = 0\right\} \cap H^2(\Omega^b),
\end{aligned}$$

let $\bar{V}^j$ and $\bar{Q}^j$ be the trace spaces of, respectively, V^j and Q^j restricted to Γ_0^j ($j = s, b$), and let $\bar{Q}^{b0}$ be the trace space of Q^{b0} restricted to Γ_0^b. Where no confusion can occur we will write u^j restricted to Γ_0^j as u^j instead of $u^j|_{\Gamma_0^j}$, and similarly for the other unknowns.

Following the notation also used for the discrete spaces, we write $\boldsymbol{V}^j := V^j \times \bar{V}^j$ and $\boldsymbol{Q}^j := Q^j \times \bar{Q}^j$ for $j = s, b$. Furthermore, $\boldsymbol{Q}^{b0} := Q^{b0} \times \bar{Q}^{b0}$. Extended function spaces are defined as:

$$
\begin{aligned}
&\boldsymbol{V}^j(h) := \boldsymbol{V}_h^j + \boldsymbol{V}^j, && \boldsymbol{Q}^j(h) := \boldsymbol{Q}_h^j + \boldsymbol{Q}^j, && j = s, d,\\
&\boldsymbol{Q}^{b0}(h) := \boldsymbol{Q}_h^{b0} + \boldsymbol{Q}^{b0}, && Z(h) := V_h^b + Z,
\end{aligned}
$$

We will use standard innerproduct notation: for scalar functions p and q on an element K, $(p, q)_K = \int_K pq \,\mathrm{d}x$; for vector functions p and q, $(p, q)_K = \int_K p \cdot q \,\mathrm{d}x$; and for matrix functions p and q, $(p, q)_K = \int_K p : q \,\mathrm{d}x$. Let D be the $(d-1)$-dimensional boundary ∂K or facet $F \subset \partial K$ of an element K. We then write $\langle p, q\rangle_D = \int_D p \odot q \,\mathrm{d}s$, where $\odot$ is multiplication if p and q are scalar functions, the dot-product if p and q are vector functions, and the dyadic product if p and q are matrix functions. We furthermore define $(p, q)_{\Omega^j} := \sum_{K\in\mathcal{T}^j}(p, q)_K$ and $\langle p, q\rangle_{\partial\mathcal{T}^j} := \sum_{K\in\mathcal{T}^j}\langle p, q\rangle_{\partial K}$ for $j = s, b$ and $(p, q)_\Omega := \sum_{K\in\mathcal{T}}(p, q)_K$ and $\langle p, q\rangle_{\partial\mathcal{T}} := \sum_{K\in\mathcal{T}}\langle p, q\rangle_{\partial K}$, while on the interface Γ_I we define $\langle p, q\rangle_{\Gamma_I} := \sum_{F\in\mathcal{F}_I}\langle p, q\rangle_F$. At this point it will be useful to also define $\|q\|_K := (q, q)_K^{1/2}$, $\|q\|_{\partial K} := \langle q, q\rangle_{\partial K}^{1/2}$, $\|q\|_F := \langle q, q\rangle_F^{1/2}$, $\|q\|_{\Gamma_I} := \langle q, q\rangle_{\Gamma_I}^{1/2}$, and $\|q\|_{\Omega^j} := (q, q)_{\Omega^j}^{1/2}$ for $j = s, b$.

The following bilinear forms are used in the following sections to define the HDG method (where $j = s, b$):

$$
\begin{aligned}
a_h^j(\boldsymbol{u}^j, \boldsymbol{v}^j) :=& (2\mu^j\varepsilon(u), \varepsilon(v))_{\Omega^j} + \sum_{K\in\mathcal{T}^j}\langle 2\beta^j\mu^j h_K^{-1}(u - \bar{u}^j), (v - \bar{v}^j)\rangle_{\partial K}\\
&- \langle 2\mu^j\varepsilon(u)n^j, (v - \bar{v}^j)\rangle_{\partial\mathcal{T}^j} - \langle 2\mu^j\varepsilon(v)n^j, (u - \bar{u}^j)\rangle_{\partial\mathcal{T}^j},\\
b_h^j(\boldsymbol{q}^j, \boldsymbol{v}^j) :=& -(q, \nabla\cdot v)_{\Omega^j} + \langle \bar{q}^j, (v - \bar{v}^j)\cdot n^j\rangle_{\partial\mathcal{T}^j},\\
c_h((p, r), q) :=& (\lambda^{-1}(\alpha p - r), q)_{\Omega^b},\\
a_h^I((\bar{u}^s, \bar{u}^b), (\bar{v}^s, \bar{v}^b)) :=& \langle \gamma(\mu^s/\kappa)^{1/2}(\bar{u}^s - \bar{u}^b)^t, (\bar{v}^s - \bar{v}^b)^t\rangle_{\Gamma_I},\\
b_h^I(\bar{q}^p, (\bar{v}^s, \bar{v}^b)) :=& \langle \bar{q}^p, (\bar{v}^s - \bar{v}^b)\cdot n\rangle_{\Gamma_I},
\end{aligned}
$$

where $a_h^j(\cdot,\cdot)$ is defined on $\boldsymbol{V}^j(h)\times\boldsymbol{V}^j(h)$, $b_h^s(\cdot,\cdot)$ is defined on $\boldsymbol{Q}^s(h)\times\boldsymbol{V}^s(h)$, $b_h^b(\cdot,\cdot)$ is defined both on $\boldsymbol{Q}^b(h)\times\boldsymbol{V}^b(h)$ and $\boldsymbol{Q}^{b0}(h)\times(Z(h)\times\{0\})$, $c_h((\cdot,\cdot),\cdot)$ is defined both on $(\boldsymbol{Q}^{b0}(h)\times\boldsymbol{Q}^b(h))\times\boldsymbol{Q}^{b0}(h)$ and $(\boldsymbol{Q}^{b0}(h)\times\boldsymbol{Q}^b(h))\times\boldsymbol{Q}^b(h)$, $a_h^I((\cdot,\cdot),(\cdot,\cdot))$ is defined on $((\bar{V}^s + \bar{V}_h^s)\times(\bar{V}^b + \bar{V}_h^b))\times((\bar{V}^s + \bar{V}_h^s)\times(\bar{V}^b + \bar{V}_h^b))$, and $b_h^I(\cdot,(\cdot,\cdot))$ is defined on $(\bar{Q}^{b0} + \bar{Q}_h^{b0})\times((\bar{V}^s + \bar{V}_h^s)\times(\bar{V}^b + \bar{V}_h^b))$. Furthermore, we write

$$
a_h(\boldsymbol{u}, \boldsymbol{v}) := a_h^s(\boldsymbol{u}^s, \boldsymbol{v}^s) + a_h^b(\boldsymbol{u}^b, \boldsymbol{v}^b), \qquad b_h(\boldsymbol{q}, \boldsymbol{v}) := b_h^s(\boldsymbol{q}^s, \boldsymbol{v}^s) + b_h^b(\boldsymbol{q}^b, \boldsymbol{v}^b).
$$

Let us end this section by noting that the penalty parameters β^j in the definition of a_h^j need to be chosen sufficiently large for a_h^j to be coercive, see [31, Lemma 2].

3.2. The HDG method

Let $f|_{\Omega^j} = f^j$ for $j = s, b$. We propose the following semi-discrete HDG method for the coupled Stokes and Biot problem eqs. (1) to (3): Find $(\boldsymbol{u}_h(t), \boldsymbol{p}_h(t), z_h(t), \boldsymbol{p}_h^p(t)) \in \boldsymbol{X}_h$, for $t \in J$, such that

$$
\begin{aligned}
a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + a_h^I((\bar{u}_h^s, \partial_t \bar{u}_h^b), (\bar{v}_h^s, \bar{v}_h^b)) + b_h(\boldsymbol{p}_h, \boldsymbol{v}_h) + b_h^I(\bar{p}_h^p, (\bar{v}_h^s, \bar{v}_h^b)) &= (f, v_h)_\Omega, && \text{(6a)}\\
b_h(\boldsymbol{q}_h, \boldsymbol{u}_h) + c_h((p_h^p, p_h^b), q_h^b) &= 0, && \text{(6b)}\\
(c_0 \partial_t p_h^p, q_h^p)_{\Omega^b} + c_h((\partial_t p_h^p, \partial_t p_h^b), \alpha q_h^p) - b_h^b(\boldsymbol{q}_h^p, (z_h, 0)) - b_h^I(\bar{q}_h^p, (\bar{u}_h^s, \partial_t \bar{u}_h^b)) &= (g^b, q_h^p)_{\Omega^b}, && \text{(6c)}\\
(\kappa^{-1} z_h, w_h)_{\Omega^b} + b_h^b(\boldsymbol{p}_h^p, (w_h, 0)) &= 0, && \text{(6d)}
\end{aligned}
$$

for all $(\boldsymbol{v}_h, \boldsymbol{q}_h, w_h, \boldsymbol{q}_h^p) \in \boldsymbol{X}_h$.

Let us partition the time interval J using the time levels $0 = t^0 < t^1 < \ldots < t^N = T$ where, for $n = 0, \ldots, N$, $t^n = n\Delta t$ with $\Delta t > 0$ the time step. Functions g and g^j (with $j = s, b$) evaluated at time level t^i are denoted by, respectively, g^i and $g^{j,i}$. We furthermore denote the first order discrete time derivative as $d_t g^n := (g^n - g^{n-1})/\Delta t$ for $n = 1, \ldots, N$. (It will be clear from context whether n denotes a time-level or a normal vector.) Applying Backward Euler time-stepping to eq. (6) we obtain the fully-discrete method: Find $(\boldsymbol{u}_h^{n+1}, \boldsymbol{p}_h^{n+1}, z_h^{n+1}, \boldsymbol{p}_h^{p,n+1}) \in \boldsymbol{X}_h$ such that

$$
\begin{aligned}
& a_h(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) + a_h^I((\bar{u}_h^{s,n+1}, d_t \bar{u}_h^{b,n+1}), (\bar{v}_h^s, \bar{v}_h^b)) && \text{(7a)}\\
& \qquad + b_h(\boldsymbol{p}_h^{n+1}, \boldsymbol{v}_h) + b_h^I(\bar{p}_h^{p,n+1}, (\bar{v}_h^s, \bar{v}_h^b)) = (f^{n+1}, v_h)_\Omega, \\
& b_h(\boldsymbol{q}_h, \boldsymbol{u}_h^{n+1}) + c_h((p_h^{p,n+1}, p_h^{b,n+1}), q_h^b) = 0, && \text{(7b)}\\
& (c_0 d_t p_h^{p,n+1}, q_h^p)_{\Omega^b} + c_h((d_t p_h^{p,n+1}, d_t p_h^{b,n+1}), \alpha q_h^p) - b_h^b(\boldsymbol{q}_h^p, (z_h^{n+1}, 0)) && \text{(7c)}\\
& \qquad - b_h^I(\bar{q}_h^p, (\bar{u}_h^{s,n+1}, d_t \bar{u}_h^{b,n+1})) = (g^{b,n+1}, q_h^p)_{\Omega^b}, \\
& (\kappa^{-1} z_h^{n+1}, w_h)_{\Omega^b} + b_h^b(\boldsymbol{p}_h^{p,n+1}, (w_h, 0)) = 0, && \text{(7d)}
\end{aligned}
$$

for all $(\boldsymbol{v}_h, \boldsymbol{q}_h, w_h, \boldsymbol{q}_h^p) \in \boldsymbol{X}_h$.

Lemma 1. *The following properties of the solution to eq.* (7) *hold:*

$$
\begin{aligned}
[\![u_h^{j,n+1} \cdot n]\!] &= 0 && \forall x \in F, && \forall F \in \mathcal{F}_{int}^j \cup \mathcal{F}_D^j, && \text{(8a)}\\
u_h^{j,n+1} \cdot n &= \bar{u}_h^{j,n+1} \cdot n && \forall x \in F, && \forall F \in \mathcal{F}_N^j \cup \mathcal{F}_I, && \text{(8b)}\\
[\![z_h^{n+1} \cdot n]\!] &= 0 && \forall x \in F, && \forall F \in \mathcal{F}_{int}^b \cup \mathcal{F}_F^b, && \text{(8c)}\\
u_h^{s,n+1} \cdot n &= (z_h^{n+1} + d_t u_h^{b,n+1}) \cdot n && \forall x \in F, && \forall F \in \mathcal{F}_I, && \text{(8d)}\\
\nabla \cdot u_h^{s,n+1} &= 0 && \forall x \in K, && \forall K \in \mathcal{T}^s, && \text{(8e)}\\
\nabla \cdot u_h^{b,n+1} &= \lambda^{-1}(\alpha p_h^{p,n+1} - p_h^{b,n+1}) && \forall x \in K, && \forall K \in \mathcal{T}^b, && \text{(8f)}\\
\nabla \cdot z_h^{n+1} &= \Pi_Q^b g^b - c_0 d_t p_h^{p,n+1} && && && \text{(8g)}\\
&\quad - \alpha\lambda^{-1}(\alpha d_t p_h^{p,n+1} - d_t p_h^{b,n+1}) && \forall x \in K, && \forall K \in \mathcal{T}^b,
\end{aligned}
$$

where Π_Q^b is the L^2-projection operator onto Q_h^b and where $[\![\cdot]\!]$ is the usual jump operator.

Proof. Setting $q_h = 0$ in eq. (7b), we find for all $\bar{q}_h^j \in \bar{Q}_h^j$ with $j = s, b$ that

$$
\begin{aligned}
0 &= \sum_{K \in \mathcal{T}^j} \langle \bar{q}_h^j, (u_h^{j,n+1} - \bar{u}_h^{j,n+1}) \cdot n^j \rangle_{\partial K} \\
&= \sum_{F \in \mathcal{F}_{int}^j \cup \mathcal{F}_D^j} \langle \bar{q}_h^j, [\![u_h^{j,n+1} \cdot n^j]\!] \rangle_F + \sum_{F \in (\mathcal{F}_N^j \cup \mathcal{F}_I)} \langle \bar{q}_h^j, (u_h^{j,n+1} - \bar{u}_h^{j,n+1}) \cdot n^j \rangle_F .
\end{aligned}
$$

Equations (8a) and (8b) follow noting that $(u_h^{j,n+1} \cdot n)|_F \in P_k(F)$ and $(\bar{u}_h^{j,n+1} \cdot n)|_F \in P_k(F)$. Setting now $q_h^p = 0$ in eq. (7c), we find for all $\bar{q}_h^p \in \bar{Q}_h^{b0}$ that

$$\begin{aligned}
0 &= \sum_{K \in \mathcal{T}^b} \langle \bar{q}_h^p, z_h^{n+1} \cdot n^b \rangle_{\partial K} + \langle \bar{q}_h^p, (\bar{u}_h^{s,n+1} - d_t \bar{u}_h^{b,n+1}) \cdot n \rangle_{\Gamma_I} \\
&= \sum_{F \in \mathcal{F}_{int}^b} \langle \bar{q}_h^p, [\![z_h^{n+1} \cdot n^b]\!] \rangle_F + \sum_{F \in \mathcal{F}_F^b} \langle \bar{q}_h^p, z_h^{n+1} \cdot n^b \rangle_F + \langle \bar{q}_h^p, (\bar{u}_h^{s,n+1} - (z_h^{n+1} + d_t \bar{u}_h^{b,n+1})) \cdot n \rangle_{\Gamma_I}
\end{aligned}$$

where the second equality is because $n = -n^b$.

Equation (8c) follows since $(z_h^{n+1} \cdot n)|_F \in P_k(F)$, eq. (8d) is an immediate consequence of eq. (8b) and since $((\bar{u}_h^{s,n+1} - (z_h^{n+1} + d_t \bar{u}_h^{b,n+1})) \cdot n)|_F \in P_k(F)$, while eq. (8e) follows after setting $\boldsymbol{q}_h^b = \boldsymbol{0}$ and $\bar{q}_h^s = 0$ in eq. (7b) and noting that $\nabla \cdot u_h^s \in P_{k-1}(K)$. Setting $\boldsymbol{q}_h^s = \boldsymbol{0}$, $\bar{q}_h^b = 0$, and $q_h^b = -\nabla \cdot u_h^{b,n+1} + \lambda^{-1}(\alpha p_h^{p,n+1} - p_h^{b,n+1})$ in eq. (7b) results in eq. (8f). Finally, eq. (8g) follows by setting $\bar{q}_h^p = 0$ and $q_h^p = c_0 d_t p_h^{p,n+1} + \alpha\lambda^{-1}(\alpha d_t p_h^{p,n+1} - d_t p_h^{b,n+1}) + \nabla \cdot z_h^{n+1} - \Pi_Q^b g^b$ in eq. (7c). □

Lemma 1 demonstrates that $u_h^{s,n+1}$, $u_h^{b,n+1}$, and z_h^{n+1} are $H(\text{div})$-conforming and that the compressibility equations eqs. (1b) and (1c) are satisfied pointwise on the elements by the numerical solution. For the semi-discrete method eq. (6), eq. (8g) can be replaced by

$$c_0 \partial_t p_h^p + \alpha\lambda^{-1}(\alpha \partial_t p_h^p - \partial_t p_h^b) + \nabla \cdot z_h = \Pi_Q g^b \qquad \forall x \in K, \forall K \in \mathcal{T}_h^b, \forall t \in J,$$

which states that mass is conserved pointwise on the elements if $g^b \in Q_h^b$. In the incompressible limit, i.e., $\lambda \to \infty$ and $c_0 \to 0$, the HDG method is strongly conservative [33].

4. Consistency, stability, existence and uniqueness

This section is devoted to proving consistency and stability of the semi-discrete HDG method eq. (6) and existence and uniqueness of a solution to the fully-discrete HDG method eq. (7). We start with consistency.

Lemma 2 (Consistency). *Let (u, p, z, p^p) be a solution to the coupled Stokes–Biot problem eqs.* (1) *to* (3). *Let $\boldsymbol{u} := (u, u|_{\Gamma_0^s}, u|_{\Gamma_0^b})$, $\boldsymbol{p} := (p, p|_{\Gamma_0^s}, u|_{\Gamma_0^b})$, and $\boldsymbol{p}^p := (p^p, p^p|_{\Gamma_0^b})$. Then $(\boldsymbol{u}, \boldsymbol{p}, z, \boldsymbol{p}^p)$ satisifes the semi-discrete problem eq.* (6).

Proof. Integrating by parts and using the smoothness of u^j and single-valuedness of $\bar{v}_h^j$ $(j = s, b)$,

$$\begin{aligned}
a_h(\boldsymbol{u}, \boldsymbol{v}_h) = &\sum_{j=s,b} \Big(-(2\mu^j(\nabla \cdot \varepsilon(u^j)), v_h^j)_{\Omega^j} + \langle 2\mu^j \varepsilon(u^j) n^j, \bar{v}_h^j \rangle_{\Gamma_I} \Big) \\
&+ \langle 2\mu^s \varepsilon(u^s) n^s, \bar{v}_h^s \rangle_{\Gamma_N^s} + \langle 2\mu^b \varepsilon(u^b) n^b, \bar{v}_h^b \rangle_{\Gamma_N^b}.
\end{aligned}$$

Similarly,

$$b_h(\boldsymbol{p}, \boldsymbol{v}_h) = (\nabla p, v_h)_\Omega - \langle p^s, \bar{v}_h^s \cdot n^s \rangle_{\Gamma_I} - \langle p^b, \bar{v}_h^b \cdot n^b \rangle_{\Gamma_I} - \langle p^s, \bar{v}_h^s \cdot n^s \rangle_{\Gamma_N^s} - \langle p^b, \bar{v}_h^b \cdot n^b \rangle_{\Gamma_N^b}.$$

Hence by eqs. (1a) and (3b),

$$\begin{aligned}
a_h(\boldsymbol{u}, \boldsymbol{v}_h) + b_h(\boldsymbol{p}, \boldsymbol{v}_h) = &\langle 2\mu^s \varepsilon(u^s) n^s, \bar{v}_h^s \rangle_{\Gamma_I} + \langle 2\mu^b \varepsilon(u^b) n^b, \bar{v}_h^b \rangle_{\Gamma_I} \\
&- \langle p^s, \bar{v}_h^s \cdot n^s \rangle_{\Gamma_I} - \langle p^b, \bar{v}_h^b \cdot n^b \rangle_{\Gamma_I} + (f^s, v_h^s)_{\Omega^s} + (f^b, v_h^b)_{\Omega^b}.
\end{aligned} \tag{9}$$

By eqs. (2a) to (2d), on Γ_I

$$\begin{aligned} 2\mu^j\varepsilon(u^j)n^j &= (n^j\cdot 2\mu^j\varepsilon(u^j)n^j)n^j + (2\mu^j\varepsilon(u^j)n^j)^t \\ &= p^jn^j - p^pn^j - \zeta^j\gamma(\mu^s/\kappa)^{1/2}(u^s - \partial_t u^b)^t, \end{aligned} \tag{10}$$

where $\zeta^s = 1$ and $\zeta^b = -1$. Combining eq. (9) with eq. (10) and using that $n = n^s = -n^b$,

$$\begin{aligned} a_h(\boldsymbol{u}, \boldsymbol{v}_h) + b_h(\boldsymbol{p}, \boldsymbol{v}_h) = &- \langle\gamma(\mu^s/\kappa)^{1/2}(u^s - \partial_t u^b)^t, (\bar{v}_h^s - \bar{v}_h^b)^t\rangle_{\Gamma_I} \\ &- \langle p^p, (\bar{v}_h^s - \bar{v}_h^b)\cdot n^s\rangle_{\Gamma_I} + (f, v_h)_\Omega, \end{aligned}$$

which, after rearranging, results in

$$a_h(\boldsymbol{u}, \boldsymbol{v}_h) + b_h(\boldsymbol{p}, \boldsymbol{v}_h) + a_h^I((u^s, \partial_t u^b), (\bar{v}_h^s, \bar{v}_h^b)) + b_h^I(p^p, (\bar{v}_h^s, \bar{v}_h^b)) = (f, v_h)_\Omega. \tag{11}$$

Next, by eqs. (1b) and (1c),

$$b_h(\boldsymbol{q}_h, \boldsymbol{u}) + c_h((p^p, p^b), q_h^b) = -(q_h, \nabla\cdot u)_\Omega + (q_h^b, \lambda^{-1}(\alpha p^p - p^b))_{\Omega^b} = 0. \tag{12}$$

Integration by parts and using eq. (1e),

$$(\kappa^{-1}z, w_h)_{\Omega^b} + b_h^b(\boldsymbol{p}^p, (w_h, 0)) = (\kappa^{-1}z + \nabla p^p, w_h)_{\Omega^b} = 0. \tag{13}$$

Finally, using eq. (3d), $\bar{q}_h^b = 0$ on Γ_P^b, $n^b = -n$, eq. (1d) and eq. (2a),

$$\begin{aligned} b_h^b(\boldsymbol{q}_h^p, (z, 0)) &= -(q_h^p, \nabla\cdot z)_{\Omega^b} + \langle\bar{q}_h^p, z\cdot n^b\rangle_{\Gamma_I} \\ &= (q_h^p, c_0\partial_t p^p + \lambda^{-1}\alpha(\alpha\partial_t p^p - \partial_t p^b) - g^b)_{\Omega^b} - \langle\bar{q}_h^p, (u^s - \partial_t u^b)\cdot n\rangle_{\Gamma_I}, \end{aligned}$$

which, after some rearranging can be written as

$$(c_0\partial_t p^p, q_h^p)_{\Omega^b} + c_h((\partial_t p^p, \partial_t p^b), \alpha q_h^p) - b_h^b(\boldsymbol{q}_h^p, (z, 0)) - b_h^I(\bar{q}_h^p, (u^s, \partial_t u^b)) = (g^b, q_h^p)_{\Omega^b}. \tag{14}$$

The result follows after comparing eqs. (11) to (14) to eq. (6). $\square$

Before demonstrating stability of the semi-discrete problem eq. (6) and well-posedness of the fully-discrete problem eq. (7) we first introduce some preliminary notation and results. We start by defining the following norms on $\boldsymbol{V}^j(h)$ and $\boldsymbol{Q}^j(h)$:

$$\begin{aligned} |||\boldsymbol{v}^j|||_{v,j}^2 &:= \sum_{K\in\mathcal{T}^j}\left(\|\varepsilon(v^j)\|_K^2 + h_K^{-1}\|v^j - \bar{v}^j\|_{\partial K}^2\right) && \forall \boldsymbol{v}^j \in \boldsymbol{V}^j(h), && j = s, b, \\ |||\boldsymbol{v}^j|||_{v',j}^2 &:= |||\boldsymbol{v}^j|||_{v,j}^2 + \sum_{K\in\mathcal{T}^j} h_K^2|v^j|_{2,K}^2 && \forall \boldsymbol{v}^j \in \boldsymbol{V}^j(h), && j = s, b, \\ |||\boldsymbol{q}|||_{q,j}^2 &:= \|q\|_{\Omega^j}^2 + \sum_{K\in\mathcal{T}^j} h_K\|\bar{q}^j\|_{\partial K}^2 && \forall \boldsymbol{q} \in \boldsymbol{Q}^j(h), && j = s, b. \end{aligned}$$

For functions $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $\boldsymbol{q}_h \in \boldsymbol{Q}_h$ it will be useful to also define

$$\begin{aligned} |||\boldsymbol{v}_h|||_v^2 &:= |||\boldsymbol{v}_h^s|||_{v,s}^2 + |||\boldsymbol{v}_h^b|||_{v,b}^2 + \|(\bar{v}_h^s - \bar{v}_h^b)^t\|_{\Gamma_I}^2, \\ |||\boldsymbol{q}_h|||_q^2 &:= |||\boldsymbol{q}_h^s|||_{q,s}^2 + |||\boldsymbol{q}_h^b|||_{q,b}^2. \end{aligned}$$

In what follows, $C > 0$ will denote a constant that is independent of h and Δt. A consequence of [31, Lemmas 2 and 3] are the following inequalities:

$$a_h^j(\boldsymbol{u}, \boldsymbol{v}) \le C\mu^j |||\boldsymbol{u}|||_{v',j} |||\boldsymbol{v}|||_{v',j} \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}^j(h), \tag{15a}$$

$$a_h^j(\boldsymbol{v}_h^j, \boldsymbol{v}_h^j) \ge C\mu^j |||\boldsymbol{v}_h^j|||_{v,j}^2 \qquad \forall \boldsymbol{v}_h^j \in \boldsymbol{V}_h^j. \tag{15b}$$

Due to the equivalence between $|||\cdot|||_{v',j}$ and $|||\cdot|||_{v,j}$ on $\boldsymbol{V}_h^j$, $|||\boldsymbol{u}|||_{v',j}$ in eq. (15a) can be replaced with $|||\boldsymbol{u}|||_{v,j}$ if $\boldsymbol{u}$ belongs to $\boldsymbol{V}_h^j$ (and similarly if $\boldsymbol{v}$ belongs to $\boldsymbol{V}_h^j$). Note that, as typical of interior penalty methods, that eq. (15b) only holds for a large enough β^j.

By the Cauchy–Schwarz and Korn's inequalities, we have the following boundedness result for b_h^j:

$$\begin{aligned} b_h^j(\boldsymbol{p}^j, \boldsymbol{v}^j) &\le \|p^j\|_{\Omega^j} \|\nabla v^j\|_{\Omega^j} + \Big(\sum_{K\in\mathcal{T}^j} h_K \|\bar{p}^j\|_{\partial K}^2\Big)^{1/2} \Big(\sum_{K\in\mathcal{T}^j} h_K^{-1} \|v^j - \bar{v}^j\|_{\partial K}^2\Big)^{1/2} \\ &\le C\Big(\|p^j\|_{\Omega^j}^2 + \sum_{K\in\mathcal{T}^j} h_K \|\bar{p}^j\|_{\partial K}^2\Big)^{1/2} \Big(\|\varepsilon(v^j)\|_{\Omega^j}^2 + \sum_{K\in\mathcal{T}^j} h_K^{-1} \|v^j - \bar{v}^j\|_{\partial K}^2\Big)^{1/2} \\ &\le C|||\boldsymbol{p}^j|||_{q,j} |||\boldsymbol{v}^j|||_{v,j} \quad \forall \boldsymbol{p}^j \in \boldsymbol{Q}_h^j \quad \forall \boldsymbol{v}^j \in \boldsymbol{V}^j(h). \end{aligned} \tag{16}$$

Next, we discuss various inf-sup conditions on b_h and b_h^j that are fundamental in our proofs. First, for $j = s, b$

$$\inf_{\boldsymbol{0}\ne\boldsymbol{q}_h\in\boldsymbol{Q}_h^j,\boldsymbol{q}_h} \sup_{\boldsymbol{0}\ne\boldsymbol{v}_h\in\widetilde{\boldsymbol{V}}_h^j} \frac{b_h^j(\boldsymbol{q}_h, \boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||_{v,j} |||\boldsymbol{q}_h|||_{q,j}} \ge C, \tag{17}$$

where $\widetilde{\boldsymbol{V}}_h^j := \left\{\boldsymbol{v}_h \in \boldsymbol{V}_h^j \ : \ \bar{v}_h|_{\Gamma_I} = 0\right\}$ is a subspace of $\boldsymbol{V}_h^j$. The proof of eq. (17) is given in appendix Appendix A. Let $\widehat{\boldsymbol{V}}_h := \left\{\boldsymbol{v}_h \in \boldsymbol{V}_h \ : \ \bar{v}_h^s \cdot n = \bar{v}_h^b \cdot n \text{ on } \Gamma_I\right\}$. Then we also have

$$\inf_{\boldsymbol{0}\ne\boldsymbol{q}_h\in\boldsymbol{Q}_h} \sup_{\boldsymbol{0}\ne\boldsymbol{v}_h\in\widehat{\boldsymbol{V}}_h} \frac{b_h(\boldsymbol{q}_h, \boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||_v |||\boldsymbol{q}_h|||_q} \ge C, \tag{18}$$

which was proven in Appendix B. In [30, Appendix A] we proved

$$\inf_{\boldsymbol{0}\ne\boldsymbol{q}_h^p\in\boldsymbol{Q}_h^{b,0}} \sup_{0\ne w_h\in V_h^b} \frac{b_h^b(\boldsymbol{q}_h^p, (w_h, 0))}{\|w_h\|_{\Omega^b} |||\boldsymbol{q}_h|||_{q,b}} \ge C. \tag{19}$$

We now proceed with proving stability of the semi-discrete HDG scheme eq. (6).

Theorem 1 (Stability). *Let $f^b \in W^{1,1}(J; L^2(\Omega^b))$, $f^s \in L^2(J; L^2(\Omega^s))$, and $g^b \in L^2(J; L^2(\Omega^b))$ and suppose that $(\boldsymbol{u}_h, \boldsymbol{p}_h, z_h, \boldsymbol{p}_h^p) \in C^1(J; \boldsymbol{X}_h)$ is a solution to eq. (6). Then*

$$\begin{aligned} X(t) \le X(0) + C\Big[&(\mu^s)^{-1/2}\|f^s(s)\|_{L^2(0,t;L^2(\Omega^s))} + \kappa^{-1/2}\|g^b(s)\|_{L^2(0,t;L^2(\Omega^b))} \\ &+ (\mu^b)^{-1/2}\Big(\|\partial_t f^b(s)\|_{L^1(0,t;L^2(\Omega^b))}\,\mathrm{d}s + \max_{0\le s\le t}\|f^b(s)\|_{\Omega^b}\Big)\Big], \end{aligned} \tag{20}$$

$$\begin{aligned} \Big(\int_0^t Y(s)^2\,\mathrm{d}s\Big)^{1/2} \le C\Big[&X(0) + (\mu^s)^{-1/2}\|f^s\|_{L^2(0,t;L^2(\Omega^s))} + \kappa^{-1/2}\|g^b\|_{L^2(0,t;L^2(\Omega^b))} \\ &+ (\mu^b)^{-1/2}\Big(\|\partial_t f^b(s)\|_{L^1(0,t;L^2(\Omega^b))}\,\mathrm{d}s + \max_{0\le s\le t}\|f^b(s)\|_{\Omega^b}\Big)\Big], \end{aligned} \tag{21}$$

where

$$
\begin{aligned}
X(t)^2 &:= a_h^b(\boldsymbol{u}_h^b(t), \boldsymbol{u}_h^b(t)) + c_0 \|p_h^p(t)\|_{\Omega^b}^2 + \lambda^{-1} \|\alpha p_h^p(t) - p_h^b(t)\|_{\Omega^b}^2, \\
Y(t)^2 &:= a_h^s(\boldsymbol{u}_h^s(t), \boldsymbol{u}_h^s(t)) + \gamma(\mu^s/\kappa)^{1/2} \|(\bar{u}_h^s(t) - \partial_t \bar{u}_h^b(t))^t\|_{\Gamma_I}^2 + \kappa^{-1}\|z_h(t)\|_{\Omega^b}^2,
\end{aligned} \tag{22}
$$

and $C > 0$ is a constant independent of $t > 0$.

Proof. After differentiating in time the Biot part of eq. (6b), we let $\boldsymbol{v}_h^s = \boldsymbol{u}_h^s$, $\boldsymbol{v}_h^b = \partial_t \boldsymbol{u}_h^b$, $\boldsymbol{q}_h = -\boldsymbol{p}_h$, $\boldsymbol{q}_h^p = \boldsymbol{p}_h^p$, and $w_h = z_h$ in eq. (6) and add the resulting equations. This yields

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Big[a_h^b(\boldsymbol{u}_h^b, \boldsymbol{u}_h^b) + c_0 \|p_h^p\|_{\Omega^b}{}^2 + \lambda^{-1}\|\alpha p_h^p - p_h^b\|_{\Omega^b}^2\Big] + \Big[a_h^s(\boldsymbol{u}_h^s, \boldsymbol{u}_h^s) + \gamma(\mu^s/\kappa)^{1/2}\|(\bar{u}_h^s - \partial_t \bar{u}_h^b)^t\|_{\Gamma_I}^2 \\
+ \kappa^{-1}\|z_h\|_{\Omega^b}^2\Big] = (f^s, u_h^s)_{\Omega^s} + (f^b, \partial_t u_h^b)_{\Omega^b} + (g^b, p_h^p)_{\Omega^b}.
\end{aligned} \tag{23}
$$

Observe that by the discrete Korn's inequality, coercivity of a_h^j, $j = s, b$ eq. (15b), and the inf-sup condition eq. (19), the following inequalities hold:

$$
\begin{aligned}
\|u_h^s\|_{\Omega^s} &\le C|\!|\!|\boldsymbol{u}_h^s|\!|\!|_{v,s} \le C(\mu^s)^{-1/2} a_h^s(\boldsymbol{u}_h^s, \boldsymbol{u}_h^s)^{1/2} \le C(\mu^s)^{-1/2} Y(t), \\
\|u_h^b\|_{\Omega^b} &\le C|\!|\!|\boldsymbol{u}_h^b|\!|\!|_{v,b} \le C(\mu^b)^{-1/2} a_h^b(\boldsymbol{u}_h^b, \boldsymbol{u}_h^b)^{1/2} \le C(\mu^b)^{-1/2} X(t), \\
\|p_h^p\| &\le |\!|\!|\boldsymbol{p}_h^p|\!|\!|_{q,b} \le C \sup_{0 \ne w_h \in V_h^b} \frac{(\kappa^{-1} z_h, w_h)_{\Omega^b}}{\|w_h\|_{\Omega^b}} \le C\kappa^{-1}\|z_h\|_{\Omega^b} \le C\kappa^{-1/2} Y(t).
\end{aligned}
$$

Integrating eq. (23) from 0 to t, $t \ge 0$, and using the above inequalites and eq. (22) in combination with the Cauchy–Schwarz inequality, we obtain:

$$
\begin{aligned}
&\frac{1}{2}\Big(X(t)^2 - X(0)^2\Big) + \int_0^t Y(s)^2 \,\mathrm{d}s \\
&\le C(\mu^s)^{-1/2} \int_0^t \|f^s(s)\|_{\Omega^s} Y(s) \,\mathrm{d}s + C\kappa^{-1/2} \int_0^t \|g^b(s)\|_{\Omega^b} Y(s) \,\mathrm{d}s \\
&\quad + C(\mu^b)^{-1/2}\Big(\int_0^t \|\partial_t f^b(s)\|_{\Omega^b} X(s) \,\mathrm{d}s + \|f^b(t)\|_{\Omega^b} X(t) + \|f^b(0)\|_{\Omega^b} X(0)\Big),
\end{aligned} \tag{24}
$$

where to obtain the third term on the right hand side of eq. (24) we applied integration by parts before applying the Cauchy–Schwarz inequality. Let us first prove eqs. (20) and (21) under the assumption

$$
\max_{0 \le s \le t} X(s) = X(t) > 0. \tag{25}
$$

This assumption and Young's inequality allow us to rewrite eq. (24) as

$$
\begin{aligned}
&X(t)^2 + \int_0^t Y(s)^2 \,\mathrm{d}s \\
&\qquad \le X(0)^2 + C\alpha(t)^2 + C(\mu^b)^{-1/2}\Big(\int_0^t \|\partial_t f^b(s)\|_{\Omega^b} \,\mathrm{d}s + \|f^b(t)\|_{\Omega^b} + \|f^b(0)\|_{\Omega^b}\Big) X(t)
\end{aligned} \tag{26}
$$

where $\alpha(t) := \Big((\mu^s)^{-1} \int_0^t \|f^s(s)\|_{\Omega^s}^2 \,\mathrm{d}s + \kappa^{-1} \int_0^t \|g^b(s)\|_{\Omega^b}^2 \,\mathrm{d}s\Big)^{1/2}$. Here we remark that the constant in Young's inequality is independent of t.

If $\alpha(t) > X(t)$, inequality eq. (20) is obvious. Assume therefore that $\alpha(t) \le X(t)$. Recalling that $X(0) \le X(t)$ by eq. (25), and dividing both sides of eq. (26) by $X(t) > 0$, we obtain

$$X(t) \le X(0) + C\alpha(t) + C(\mu^b)^{-1/2}\Big(\int_0^t \|\partial_t f^b(s)\|_{\Omega^b}\, \mathrm{d}s + \|f^b(t)\|_{\Omega^b} + \|f^b(0)\|_{\Omega^b}\Big), \tag{27}$$

implying eq. (20). Equation (21) follows from eqs. (20) and (26).

If assumption eq. (25) is not true, there exists $0 \le \tilde{t} < t$ such that

$$\max_{0\le s\le t} X(s) = X(\tilde{t}) > 0. \tag{28}$$

Integrating eq. (23) from 0 to $\tilde{t}$, using the same steps as above that were used to find eq. (20), but restricting ourselves to the time interval $(0,\tilde{t})$, and using eq. (28) instead of eq. (25), we find

$$\begin{aligned} X(\tilde{t}) \le X(0) + C\Big[&(\mu^s)^{-1/2}\|f^s(s)\|_{L^2(0,\tilde{t};L^2(\Omega^s))} + \kappa^{-1/2}\|g^b(s)\|_{L^2(0,\tilde{t};L^2(\Omega^b))} \\ &+ (\mu^b)^{-1/2}\Big(\|\partial_t f^b(s)\|_{L^1(0,\tilde{t};L^2(\Omega^b))}\, \mathrm{d}s + \max_{0\le s\le \tilde{t}}\|f^b(s)\|_{\Omega^b}\Big)\Big]. \end{aligned} \tag{29}$$

Then, eq. (20) holds because $X(t) < X(\tilde{t})$ by assumption eq. (28) and $\tilde{t} < t$. Finally, to prove eq. (21), note that eq. (24) holds for any t, so a crude inequality by Young's inequality and the assumption eq. (28) give

$$\begin{aligned} \int_0^t Y(s)^2\, \mathrm{d}s \le{} & X(0)^2 + C\Big((\mu^s)^{-1}\int_0^t \|f^s(s)\|^2_{\Omega^s}\, \mathrm{d}s + \kappa^{-1}\int_0^t \|g^b(s)\|^2_{\Omega^b}\, \mathrm{d}s\Big) \\ & + C(\mu^b)^{-1/2}\Big(\int_0^t \|\partial_t f^b(s)\|_{\Omega^b}\, \mathrm{d}s + \max_{0\le s\le t}\|f^b(s)\|_{\Omega^b}\Big)X(\tilde{t}) \\ \le{} & X(0)^2 + C\Big((\mu^s)^{-1}\int_0^t \|f^s(s)\|^2_{\Omega^s}\, \mathrm{d}s + \kappa^{-1}\int_0^t \|g^b(s)\|^2_{\Omega^b}\, \mathrm{d}s\Big) \\ & + C^2(\mu^b)^{-1}\Big(\int_0^t \|\partial_t f^b(s)\|_{\Omega^b}\, \mathrm{d}s + \max_{0\le s\le t}\|f^b(s)\|_{\Omega^b}\Big)^2 + \frac{1}{4}X(\tilde{t})^2. \end{aligned}$$

Equation (21) now follows by combining this result with eq. (29) and using that $\tilde{t} < t$. □

Well-posedness of the fully discrete method is now given by the following lemma.

Lemma 3 (Existence and uniqueness). *There exists a unique solution to the fully-discrete HDG method eq.* (7).

Proof. Setting the right hand sides of eq. (7) to zero, $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n, z_h^n, \boldsymbol{p}_h^{p,n}) = (\boldsymbol{0}, \boldsymbol{0}, 0, \boldsymbol{0})$, and adding the resulting equations, we obtain

$$\begin{aligned} &a_h(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) + a_h^I((\bar{u}_h^{s,n+1}, \tfrac{1}{\Delta t}\bar{u}_h^{b,n+1}), (\bar{v}_h^s, \bar{v}_h^b)) + b_h(\boldsymbol{p}_h^{n+1}, \boldsymbol{v}_h) \\ &+ b_h^I(\bar{p}_h^{p,n+1}, (\bar{v}_h^s, \bar{v}_h^b)) + b_h(\boldsymbol{q}_h, \boldsymbol{u}_h^{n+1}) + c_h((p_h^{p,n+1}, p_h^{b,n+1}), q_h^b) \\ &+ \tfrac{1}{\Delta t}(c_0 p_h^{p,n+1}, q_h^p)_{\Omega^b} + \tfrac{1}{\Delta t}c_h((p_h^{p,n+1}, p_h^{b,n+1}), \alpha q_h^p) - b_h^b(\boldsymbol{q}_h^p, (z_h^{n+1}, 0)) \\ &- b_h^I(\bar{q}_h^p, (\bar{u}_h^{s,n+1}, \tfrac{1}{\Delta t}\bar{u}_h^{b,n+1})) + (\kappa^{-1}z_h^{n+1}, w_h)_{\Omega^b} + b_h^b(\boldsymbol{p}_h^{p,n+1}, (w_h, 0)) = 0, \end{aligned} \tag{30}$$

which holds for all $(\boldsymbol{v}_h, \boldsymbol{q}_h, w_h, \boldsymbol{q}_h^p) \in \boldsymbol{X}_h$. Choosing now $\boldsymbol{v}_h^s = \boldsymbol{u}_h^{s,n+1}$, $\boldsymbol{v}_h^b = \frac{1}{\Delta t}\boldsymbol{u}_h^{b,n+1}$, $\boldsymbol{q}_h^s = -\boldsymbol{p}_h^{s,n+1}$, $\boldsymbol{q}_h^b = -\frac{1}{\Delta t}\boldsymbol{p}_h^{b,n+1}$, $w_h = z_h^{n+1}$, and $\boldsymbol{q}_h^p = \boldsymbol{p}_h^{p,n+1}$, we find

$$
\begin{aligned}
&a_h^s(\boldsymbol{u}_h^{s,n+1}, \boldsymbol{u}_h^{s,n+1}) + \tfrac{1}{\Delta t} a_h^b(\boldsymbol{u}_h^{b,n+1}, \boldsymbol{u}_h^{b,n+1}) + \gamma(\mu^s/\kappa)^{1/2} \left\| (\bar{u}_h^{s,n+1} - \frac{1}{\Delta t}\bar{u}_h^{b,n+1})^t \right\|_{\Gamma_I}^2 \\
&+ \tfrac{c_0}{\Delta t} \|p_h^{p,n+1}\|_{\Omega^b}^2 + \tfrac{\lambda^{-1}}{\Delta t} \|\alpha p_h^{p,n+1} - p_h^{b,n+1}\|_{\Omega^b}^2 + \kappa^{-1} \|z_h^{n+1}\|_{\Omega^b}^2 = 0.
\end{aligned}
$$

Coercivity of a_h^j eq. (15b) and positivity of γ, μ^s, κ, λ, α, and Δt and nonnegativity of c_0 imply that $\boldsymbol{u}_h^{n+1}$ and z_h^{n+1} are zero. Substituting now $z_h^{n+1} = 0$ and setting $\boldsymbol{v}_h = \boldsymbol{0}$, $\boldsymbol{q}_h = \boldsymbol{0}$, and $\boldsymbol{q}_h^p = \boldsymbol{0}$ in eq. (30), we obtain $b_h^b(\boldsymbol{p}_h^{p,n+1}, (w_h, 0)) = 0$ for all $w_h \in V_h^b$. It follows from eq. (19) that $\boldsymbol{p}_h^{p,n+1} = \boldsymbol{0}$. Next, substituting $\boldsymbol{u}_h^{n+1} = \boldsymbol{0}$, choosing $\bar{v}_h^s = \bar{v}_h^b = 0$ on Γ_I, and choosing $w_h = 0$, $\boldsymbol{q}_h = \boldsymbol{0}$, $\boldsymbol{q}_h^p = \boldsymbol{0}$ in eq. (30), we obtain $b_h^s(\boldsymbol{p}_h^{s,n+1}, \boldsymbol{v}_h^s) = 0$ for all $\boldsymbol{v}_h^s \in \widetilde{\boldsymbol{V}}_h^s$ and $b_h^b(\boldsymbol{p}_h^{b,n+1}, \boldsymbol{v}_h^b) = 0$ for all $\boldsymbol{v}_h^b \in \widetilde{\boldsymbol{V}}_h^b$. It follows now from eq. (17) that $\boldsymbol{p}_h^{j,n+1} = \boldsymbol{0}$ for $j = s, b$. □

5. A priori error analysis

We now prove convergence results for the HDG method. Throughout this section the superscript j will refer to s and b. For this we first introduce suitable interpolation/projection operators. For vector valued functions we define $\Pi_V : [H^1(\Omega^b)]^d \to V_h^b$ to be the Brezzi–Douglas–Marini (BDM) interpolation operator [10, Section III.3] and $\boldsymbol{\Pi}_V^{\mathrm{ell},j} := (\Pi_V^{\mathrm{ell},j}, \bar{\Pi}_V^{\mathrm{ell},j}) : [H^1(\Omega^j)]^d \to \boldsymbol{V}_h^j$ to be the elliptic interpolation operator defined as

$$
a_h^j(\boldsymbol{\Pi}_V^{\mathrm{ell},j} u, \boldsymbol{v}_h) = a_h^j((u, u), \boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^j.
$$

For scalar functions we denote by Π_Q^j, $\bar{\Pi}_Q^j$, and $\bar{\Pi}_{Q^0}^b$ the L^2-projection operators onto, respectively, Q_h^j, $\bar{Q}_h^j$, and $\bar{Q}_h^{b0}$. We next introduce the following notation for the errors:

$$
\begin{aligned}
u^j - u_h^j &= (u^j - \Pi_V^{\mathrm{ell},j} u^j) - (u_h^j - \Pi_V^{\mathrm{ell},j} u^j) &&= e_{u^j}^I - e_{u^j}^h, && \text{(31a)}\\
u^j|_{\Gamma_0^j} - \bar{u}_h^j &= (u^j|_{\Gamma_0^j} - \bar{\Pi}_V^{\mathrm{ell},j} u^j) - (\bar{u}_h^j - \bar{\Pi}_V^{\mathrm{ell},j} u^j) &&= \bar{e}_{u^j}^I - \bar{e}_{u^j}^h, && \text{(31b)}\\
z - z_h &= (z - \Pi_V z) - (z_h - \Pi_V z) &&= e_z^I - e_z^h, && \text{(31c)}\\
p^j - p_h^j &= (p^j - \Pi_Q^j p^j) - (p_h^j - \Pi_Q^j p^j) &&= e_{p^j}^I - e_{p^j}^h, && \text{(31d)}\\
p^j|_{\Gamma_0^j} - \bar{p}_h^j &= (p^j|_{\Gamma_0^j} - \bar{\Pi}_Q^j p^j) - (\bar{p}_h^j - \bar{\Pi}_Q^j p^j) &&= \bar{e}_{p^j}^I - \bar{e}_{p^j}^h, && \text{(31e)}\\
p^p - p_h^p &= (p^p - \Pi_Q^b p^p) - (p_h^p - \Pi_Q^b p^p) &&= e_{p^p}^I - e_{p^p}^h, && \text{(31f)}\\
p^p|_{\Gamma_0^b} - \bar{p}_h^p &= (p^p|_{\Gamma_0^b} - \bar{\Pi}_{Q^0}^b p^p) - (\bar{p}_h^p - \bar{\Pi}_{Q^0}^b p^p) &&= \bar{e}_{p^p}^I - \bar{e}_{p^p}^h. && \text{(31g)}
\end{aligned}
$$

We also define e_u^I and e_p^I such that $e_u^I|_{\Omega^j} = e_{u^j}^I$ and $e_p^I|_{\Omega^j} = e_{p^j}^I$. Similarly, e_u^h and e_p^h are defined such that $e_u^h|_{\Omega^j} = e_{u^j}^h$ and $e_p^h|_{\Omega^j} = e_{p^j}^h$.

We then write

$$
\boldsymbol{e}_u^I = (e_u^I, \bar{e}_{u^s}^I, \bar{e}_{u^b}^I), \quad \boldsymbol{e}_u^h = (e_u^h, \bar{e}_{u^s}^h, \bar{e}_{u^b}^h), \quad \boldsymbol{e}_p^I = (e_p^I, \bar{e}_{p^s}^I, \bar{e}_{p^b}^I), \quad \boldsymbol{e}_p^h = (e_p^h, \bar{e}_{p^s}^h, \bar{e}_{p^b}^h).
$$

It will furthermore be convenient to introduce the following notation:

$$\boldsymbol{e}^I_{u^j} = (e^I_{u^j}, \bar{e}^I_{u^j}), \qquad \boldsymbol{e}^I_{p^j} = (e^I_{p^j}, \bar{e}^I_{p^j}), \qquad \boldsymbol{e}^I_{p^p} = (e^I_{p^p}, \bar{e}^I_{p^p}),$$
$$\boldsymbol{e}^h_{u^j} = (e^h_{u^j}, \bar{e}^h_{u^j}), \qquad \boldsymbol{e}^h_{p^j} = (e^h_{p^j}, \bar{e}^h_{p^j}), \qquad \boldsymbol{e}^h_{p^p} = (e^h_{p^p}, \bar{e}^h_{p^p}).$$

The following interpolation estimates hold:

$$\|e^I_z\|_{m,K} \le C h_K^{\ell-m} \|z\|_{\ell,K}, \quad m = 0,1,2, \quad \max\{m,1\} \le \ell \le k+1, \tag{32a}$$
$$a^j_h(\boldsymbol{e}^I_{u^j}, \boldsymbol{e}^I_{u^j})^{\frac{1}{2}} \le C\sqrt{\mu^j} h^{\ell-1} \|u^j\|_{\ell,\Omega^j}, \quad 1 \le \ell \le k+1, \quad j = s,b, \tag{32b}$$

where eq. (32a) is the usual BDM interpolation estimate [34, Lemma 7] and eq. (32b) follows from standard a priori error estimate theory for second order elliptic equations.

To initialize eq. (7), we set

$$u_h^{b,0} = \Pi_V^{\text{ell},j} u^b(0), \quad \bar{u}_h^{b,0} = \bar{\Pi}_V^{\text{ell},b} u^b(0), \quad p_h^{p,0} = \Pi_Q^b p^p(0). \tag{33}$$

The initial total pressure is set by $p_h^{b,0} = \alpha p_h^{p,0} - \lambda \nabla \cdot u_h^{b,0}$.

Theorem 2 (Error equation). *Let $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n, z_h^n, \boldsymbol{p}_h^{p,n})$, for $n = 1, \dots, N$, be the solution to eq. (7) with initial conditions eq. (33). Let (u, p, z, p^p) be a solution to the coupled Stokes–Biot problem eqs. (1) to (3) on time interval $J = (0, T]$ and let $\boldsymbol{u} := (u, u|_{\Gamma_0^s}, u|_{\Gamma_0^b})$, $\boldsymbol{p} := (p, p|_{\Gamma_0^s}, p|_{\Gamma_0^b})$, and $\boldsymbol{p}^p := (p^p, p^p|_{\Gamma_0^b})$. Then, with the exact solution evaluated at $t = t^{n+1}$ (and the superscript $n+1$ suppressed for notational convenience),*

$$\begin{aligned} &a_h(\boldsymbol{e}^h_u, \boldsymbol{v}_h) + a^I_h((\bar{e}^h_{u^s}, d_t\bar{e}^h_{u^b}), (\bar{v}^s_h, \bar{v}^b_h)) + b_h(\boldsymbol{e}^h_p, \boldsymbol{v}_h) + b^I_h(\bar{e}^h_{p^p}, (\bar{v}^s_h, \bar{v}^b_h)) \\ &\quad = a^I_h((0, \partial_t u^b - d_t u^b), (\bar{v}^s_h, \bar{v}^b_h)) + a^I_h((\bar{e}^I_{u^s}, d_t\bar{e}^I_{u^b}), (\bar{v}^s_h, \bar{v}^b_h)), \end{aligned} \tag{34a}$$
$$(\kappa^{-1} e^h_z, w_h)_{\Omega^b} + b^b_h(\boldsymbol{e}^h_{p^p}, (w_h, 0)) = (\kappa^{-1} e^I_z, w_h)_{\Omega^b}, \tag{34b}$$
$$\begin{aligned} &(c_0 d_t e^h_{p^p}, q^p_h)_{\Omega^b} + c_h((d_t e^h_{p^p}, d_t e^h_{p^b}), \alpha q^p_h) - b^b_h(\boldsymbol{q}^p_h, (e^h_z, 0)) - b^I_h(\bar{q}^p_h, (\bar{e}^h_{u^s}, d_t\bar{e}^h_{u^b})) \\ &\quad = (c_0(\partial_t p^p - d_t p^p), q^p_h)_{\Omega^b} + c_h((\partial_t p^p - d_t p^p, \partial_t p^b - d_t p^b), \alpha q^p_h) - b^I_h(\bar{q}^p_h, (0, \partial_t u^b - d_t u^b)), \end{aligned} \tag{34c}$$
$$b^b_h(\boldsymbol{q}^b_h, d_t \boldsymbol{e}^h_{u^b}) + c_h((d_t e^h_{p^p}, d_t e^b_{p^b}), q^b_h) = b^b_h(\boldsymbol{q}^b_h, d_t \boldsymbol{e}^I_{u^b}), \tag{34d}$$
$$b^s_h(\boldsymbol{q}^s_h, \boldsymbol{e}^h_{u^s}) = b^s_h(\boldsymbol{q}^s_h, \boldsymbol{e}^I_{u^s}). \tag{34e}$$

Proof. By eq. (7) and lemma 2,

$$\begin{aligned} &a_h(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}_h) + a^I_h((\bar{u}^s_h - u^s, d_t\bar{u}^b_h - \partial_t u^b), (\bar{v}^s_h, \bar{v}^b_h)) \\ &\qquad + b_h(\boldsymbol{p}_h - \boldsymbol{p}, \boldsymbol{v}_h) + b^I_h(\bar{p}^p_h - p^p, (\bar{v}^s_h, \bar{v}^b_h)) = 0, \end{aligned} \tag{35a}$$
$$b_h(\boldsymbol{q}_h, \boldsymbol{u}_h - \boldsymbol{u}) + c_h((p^p_h - p^p, p^b_h - p^b), q^b_h) = 0, \tag{35b}$$
$$\begin{aligned} &(c_0 d_t p^p_h - c_0 \partial_t p^p, q^p_h)_{\Omega^b} + c_h((d_t p^p_h - \partial_t p^p, d_t p^b_h - \partial_t p^b), \alpha q^p_h) \\ &\qquad - b^b_h(\boldsymbol{q}^p_h, (z_h - z, 0)) - b^I_h(\bar{q}^p_h, (\bar{u}^s_h - u^s, d_t\bar{u}^b_h - \partial_t u^b)) = 0, \end{aligned} \tag{35c}$$
$$(\kappa^{-1}(z_h - z), w_h)_{\Omega^b} + b^b_h(\boldsymbol{p}^p_h - \boldsymbol{p}^p, (w_h, 0)) = 0. \tag{35d}$$

We split eq. (35b) into its Stokes and Biot parts as:

$$b^b_h(\boldsymbol{q}^b_h, d_t(\boldsymbol{u}^b_h - \boldsymbol{u}^b)) + c_h((d_t p^p_h - d_t p^p, d_t p^b_h - d_t p^b), q^b_h) = 0, \tag{36a}$$
$$b^s_h(\boldsymbol{q}^s_h, \boldsymbol{u}^s_h - \boldsymbol{u}^s) = 0, \tag{36b}$$

where we applied d_t to the first equation. Noting that $\partial_t u^b - d_t u^b = \partial_t u^b - d_t \bar{\Pi}_V^{\text{ell},b} u^b - d_t \bar{e}_{u^b}^I$, $\partial_t p^b - d_t p^b = \partial_t p^b - d_t \Pi_Q^b p^b - d_t e_{p^b}^I$, and $\partial_t p^p - d_t p^p = \partial_t p^p - d_t \Pi_Q^b p^p - d_t e_{p^p}^I$, combining eq. (35) and eq. (36), and using the error splitting according to eq. (31) yields

$$
\begin{aligned}
&a_h(\boldsymbol{e}_u^h, \boldsymbol{v}_h) + a_h^I((\bar{e}_{u^s}^h, d_t \bar{e}_{u^b}^h), (\bar{v}_h^s, \bar{v}_h^b)) + b_h(\boldsymbol{e}_p^h, \boldsymbol{v}_h) + b_h^I(\bar{e}_{p^p}^h, (\bar{v}_h^s, \bar{v}_h^b)) && \text{(37a)}\\
&\quad = a_h(\boldsymbol{e}_u^I, \boldsymbol{v}_h) + b_h(\boldsymbol{e}_p^I, \boldsymbol{v}_h) + a_h^I((\bar{e}_{u^s}^I, \partial_t u^b - d_t \bar{\Pi}_V^{\text{ell},b} u^b), (\bar{v}_h^s, \bar{v}_h^b)) + b_h^I(\bar{e}_{p^p}^I, (\bar{v}_h^s, \bar{v}_h^b)),\\
&(\kappa^{-1} e_z^h, w_h)_{\Omega^b} + b_h^b(\boldsymbol{e}_{p^p}^h, (w_h, 0)) = (\kappa^{-1} e_z^I, w_h)_{\Omega^b} + b_h^b(\boldsymbol{e}_{p^p}^I, (w_h, 0)), && \text{(37b)}\\
&(c_0 d_t e_{p^p}^h, q_h^p)_{\Omega^b} + c_h((d_t e_{p^p}^h, d_t e_{p^b}^h), \alpha q_h^p) - b_h^b(\boldsymbol{q}_h^p, (e_z^h, 0)) - b_h^I(\bar{q}_h^p, (\bar{e}_{u^s}^h, d_t \bar{e}_{u^b}^h)) && \text{(37c)}\\
&\quad = (c_0(\partial_t p^p - d_t \Pi_Q^b p^p), q_h^p)_{\Omega^b} + c_h((\partial_t p^p - d_t \Pi_Q^b p^p, \partial_t p^b - d_t \Pi_Q^b p_h^b), \alpha q_h^p)\\
&\qquad - b_h^b(\boldsymbol{q}_h^p, (e_z^I, 0)) - b_h^I(\bar{q}_h^p, (\bar{e}_{u^s}^I, \partial_t u^b - d_t \bar{\Pi}_V^{\text{ell,b}} u^b)),\\
&b_h^b(\boldsymbol{q}_h^b, d_t \boldsymbol{e}_{u^b}^h) + c_h((d_t e_{p^p}^h, d_t e_{p^b}^h), q_h^b) = b_h^b(\boldsymbol{q}_h^b, d_t \boldsymbol{e}_{u^b}^I) + c_h((d_t e_{p^p}^I, d_t e_{p^b}^I), q_h^b), && \text{(37d)}\\
&b_h^s(\boldsymbol{q}_h^s, \boldsymbol{e}_{u^s}^h) = b_h^s(\boldsymbol{q}_h^s, \boldsymbol{e}_{u^s}^I). && \text{(37e)}
\end{aligned}
$$

Then, by definition of the chosen interpolation/projection operators, the terms $a_h(\boldsymbol{e}_u^I, \boldsymbol{v}_h)$, $b_h(\boldsymbol{e}_p^I, \boldsymbol{v}_h)$, $b_h^I(\bar{e}_{p^p}^I, (\bar{v}_h^s, \bar{v}_h^b))$, $b_h^b(\boldsymbol{e}_{p^p}^I, (w_h, 0))$, $b_h^b(\boldsymbol{q}_h^p, (e_z^I, 0))$, $c_h((d_t e_{p^p}^I, d_t e_{p^b}^I), q_h^b)$, $(c_0 d_t e_{p^p}^I, q_h^p)_{\Omega^b}$, $c_h((d_t e_{p^p}^I, d_t e_{p^b}^I), \alpha q_h^p)$, and $b_h^I(\bar{q}_h^p, (\bar{e}_{u^s}^I, d_t \bar{e}_{u^b}^I))$ vanish. The result follows. □

The following lemma will be used in the proof of the a priori error estimates of theorem 3.

Lemma 4. *Let $\bar{e}_{p^p}^h$, z, and z^h be defined as in theorem 2. There exists a constant $C > 0$ such that:*

$$
\|\bar{e}_{p^p}^h\|_{\Gamma_I} \le C\kappa^{-1} \|z - z_h\|_{\Omega^b}. \tag{38}
$$

Proof. We start by defining

$$
\|q_h\|_{1,h}^2 := \sum_{K \in \mathcal{T}^b} \|\nabla q_h\|_K^2 + \sum_{F \in \mathcal{F}^b \setminus (\mathcal{F}_I \cup \mathcal{F}_F^b)} h_F^{-1} \|[\![q_h]\!]\|_F^2, \quad \forall q_h \in Q_h^b.
$$

Since $|\Gamma_P^b| > 0$, $\|q_h\|_{1,h}$ is a norm on Q_h^b. From the trace inequality of broken functions [35, Theorem 4.4] and discrete Poincaré inequality [36] we have that for all $q_h \in Q_h^b$,

$$
\|q_h\|_{\Gamma_I} \le C \left(\|q_h\|_{L^1(\Omega^b)} + \|q_h\|_{1,h} \right) \le C \|q^h\|_{1,h}. \tag{39}
$$

Let us now consider eq. (34b). We find that for all $w_h \in V_h^b$

$$
\begin{aligned}
0 &= (\kappa^{-1}(z - z_h), w_h)_{\Omega^b} - b_h^b(\boldsymbol{e}_{p^p}^h, (w_h, 0))\\
&= (\kappa^{-1}(z - z_h), w_h)_{\Omega^b} + (e_{p^p}^h, \nabla \cdot w_h)_{\Omega^b} - \langle \bar{e}_{p^p}^h, w_h \cdot n^b \rangle_{\partial\mathcal{T}^b \setminus \Gamma_P^b},
\end{aligned} \tag{40}
$$

where the second equality holds because $\bar{e}_{p^p}^h = 0$ on Γ_P^b (since $p^p = 0$, $\bar{p}_h^p = 0$, and $\bar{\Pi}_{Q^0}^b p^p = 0$ on Γ_P^b).

Step 1. Choose $w_h \in V_h^b \cap H(\text{div}; \Omega^b)$ such that $w_h \cdot n = \bar{e}_{p^p}^h$ on $F \in \mathcal{F}_I$ and such that the remaining moments for the degrees of freedom of a BDM element vanish. By a standard scaling argument we observe that

$$
\|w_h\|_{\Omega^b} + \Big(\sum_{F \in \mathcal{F}^b} h_F \|w_h \cdot n\|_F^2 \Big)^{1/2} \le C h^{1/2} \|\bar{e}_{p^p}^h\|_{\Gamma_I}.
$$

Substituting the above defined w_h in eq. (40), integrating by parts and using the Cauchy–Schwarz inequality and eq. (39):

$$\|\bar{e}^h_{p^p}\|^2_{\Gamma_I} = (\kappa^{-1}(z - z_h), w_h)_{\Omega^b} + \langle e^h_{p^p}, \bar{e}^h_{p^p}\rangle_{\Gamma_I} \le C\left(\kappa^{-1}\|z - z_h\|_{\Omega^b} + \|e^h_{p^p}\|_{1,h}\right)\|\bar{e}^h_{p^p}\|_{\Gamma_I}. \tag{41}$$

Step 2. By (the proof of) [37, Lemma 2.1], there exists a $w^0_h \in V^b_h \cap H(\mathrm{div}; \Omega^b)$ such that

$$\begin{aligned}
\langle w^0_h \cdot n, r\rangle_F &= h_F^{-1}\langle [\![e^h_{p^p}]\!], r\rangle_F && \forall r \in P_{k-1}(F), \quad \forall F \in \mathcal{F}^b \setminus (\mathcal{F}_I \cup \mathcal{F}^b_F),\\
(w^0_h, r)_T &= -(\nabla e^h_{p^p}, r)_T && \forall r \in [P_{k-2}(T)]^2, \quad \forall T \in \mathcal{T}^b,\\
w^0_h \cdot n &= 0 \text{ on } F && \forall F \in \mathcal{F}_I \cup \mathcal{F}^b_F.
\end{aligned}$$

Additionaly, w^0_h satisfies

$$(\nabla \cdot w^0_h, e^h_{p^p})_{\Omega^b} = \|e^h_{p^p}\|^2_{1,h}, \qquad \|w^0_h\|_{\Omega^b} \le C\|e^h_{p^p}\|_{1,h}.$$

Substituting w^0_h in eq. (40), we obtain:

$$\|e^h_{p^p}\|^2_{1,h} = -(\kappa^{-1}(z - z_h), w^0_h)_{\Omega^b} \le C\kappa^{-1}\|z - z_h\|_{\Omega^b}\|e^h_{p^p}\|_{1,h}. \tag{42}$$

The result follows after combining eq. (41) and eq. (42). □

The following theorem provides an a priori error estimate.

Theorem 3 (A priori error estimate). *Let (u, p, z, p^p) be the solution to the coupled Stokes–Biot problem eqs. (1) to (3) on the time interval $J = (0, T]$ such that $u^s \in C^0(J; H^{\ell+1}(\Omega^s))$, $\partial_t u^b \in L^2(J; H^{\ell+1}(\Omega^b))$, $\partial_{tt} u^b \in L^2(J; H^1(\Omega^b))$, $p^s \in C^0(J; L^2(\Omega^s))$, $p^p, p^b \in W^{2,1}(J; L^2(\Omega^b))$, and $z \in C^0(J; H^\ell(\Omega^b))$. Let $(\boldsymbol{u}^n_h, \boldsymbol{p}^n_h, z^n_h, \boldsymbol{p}^{p,n}_h) \in \boldsymbol{X}_h$, for $n = 1, \ldots, N$, be the solution to eq. (7) with initial conditions eq. (33). The following hold:*

$$\begin{aligned}
&(\mu^b)^{1/2}|\!|\!|\boldsymbol{e}^{h,n}_{u^b}|\!|\!|_{v,b} + c_0^{1/2}\|e^{h,n}_{p^p}\|_{\Omega^b} + \lambda^{-1/2}\|\alpha e^{h,n}_{p^p} - e^{h,n}_{p^b}\|_{\Omega^b} + (\mu^s)^{1/2}\Big(\Delta t\sum_{i=1}^n |\!|\!|\boldsymbol{e}^{h,i}_{u^s}|\!|\!|^2_{v,s}\Big)^{1/2} && \text{(43a)}\\
&\quad + \kappa^{-1/2}\Big(\Delta t\sum_{i=1}^n \|e^{h,i}_z\|^2_{\Omega^b}\Big)^{1/2} + \gamma^{1/2}(\mu^s/\kappa)^{1/4}\Big(\Delta t\sum_{i=1}^n \|(\bar{e}^{h,i}_{u^s} - d_t\bar{e}^{h,i}_{u^b})^t\|^2_{\Gamma_I}\Big)^{1/2} \le \bar{C}_1\Delta t + \bar{C}_2 h^\ell,\\
&(\mu^b)^{-1/2}|\!|\!|\boldsymbol{e}^{h,n}_{p^b}|\!|\!|_{q,b} \le \bar{C}_1\Delta t + \bar{C}_2 h^\ell, && \text{(43b)}\\
&(\mu^s)^{-1/2}\Big(\Delta t\sum_{i=1}^n |\!|\!|\boldsymbol{e}^{h,i}_{p^s}|\!|\!|^2_{q,s}\Big)^{1/2} \le \bar{C}_1\Delta t + \bar{C}_2 h^\ell, && \text{(43c)}
\end{aligned}$$

where the constants are defined as

$$\begin{aligned}
\bar{C}_1 =& C\Bigg[\max(c_0^{1/2}, \lambda^{-1/2}\alpha)\|\partial_{tt}p^p\|_{L^1(J;L^2(\Omega^b))} + \lambda^{-1/2}\|\partial_{tt}p^b\|_{L^1(J;L^2(\Omega^b))}\\
&+ \max(\gamma^{1/2}(\mu^s/\kappa)^{1/4}, \kappa^{-1/2})\|\partial_{tt}u^b\|_{L^2(J;L^2(\Gamma_I))}\Bigg],\\
\bar{C}_2 =& C\Bigg[\max\left((\mu^b)^{1/2}, (\mu^s + \gamma(\mu^s/\kappa)^{1/2})^{1/2}\right)\|\partial_t u^b\|_{L^2(J;H^{\ell+1}(\Omega^b))} + \kappa^{-1/2}T^{1/2}\|z\|_{C^0(J;H^\ell(\Omega^b))}\\
&+ \max\left((\mu^b)^{1/2}T, \left(\mu^s + \gamma(\mu^s/\kappa)^{1/2}\right)^{1/2}T^{1/2}\right)\|u^s\|_{C^0(J;H^{\ell+1}(\Omega^s))}\Bigg].
\end{aligned}$$

Proof. Choose $\boldsymbol{v}_h^s = \boldsymbol{e}_{u^s}^h$, $\boldsymbol{v}_h^b = d_t\boldsymbol{e}_{u^b}^h$, $\boldsymbol{q}_h^s = -\boldsymbol{e}_{p^s}^h$, $\boldsymbol{q}_h^b = -\boldsymbol{e}_{p^b}^h$, $\boldsymbol{q}_h^p = \boldsymbol{e}_{p^p}^h$, $w_h = e_z^h$ in eq. (34) to find (we again suppress the superscript $n+1$):

$$\begin{aligned}
a_h^s(\boldsymbol{e}_{u^s}^h, \boldsymbol{e}_{u^s}^h) &+ a_h^b(\boldsymbol{e}_{u^b}^h, d_t\boldsymbol{e}_{u^b}^h) + a_h^I((\bar{e}_{u^s}^h, d_t\bar{e}_{u^b}^h), (\bar{e}_{u^s}^h, d_t\bar{e}_{u^b}^h)) \\
&+ (\kappa^{-1}e_z^h, e_z^h)_{\Omega^b} + (c_0 d_t e_{p^p}^h, e_{p^p}^h)_{\Omega^b} + c_h((d_t e_{p^p}^h, d_t e_{p^b}^h), \alpha e_{p^p}^h - e_{p^b}^h) \\
= a_h^I&((0, \partial_t u^b - d_t u^b), (\bar{e}_{u^s}^h, d_t\bar{e}_{u^b}^h)) + a_h^I((\bar{e}_{u^s}^I, d_t\bar{e}_{u^b}^I), (\bar{e}_{u^s}^h, d_t\bar{e}_{u^b}^h)) + (\kappa^{-1}e_z^I, e_z^h)_{\Omega^b} \\
&+ (c_0(\partial_t p^p - d_t p^p), e_{p^p}^h)_{\Omega^b} + c_h((\partial_t p^p - d_t p^p, \partial_t p^b - d_t p^b), \alpha e_{p^p}^h) \\
&- b_h^I(\bar{e}_{p^p}^h, (0, \partial_t u^b - d_t u^b)) - b_h^s(\boldsymbol{e}_{p^s}^h, \boldsymbol{e}_{u^s}^I) - b_h^b(\boldsymbol{e}_{p^b}^h, d_t\boldsymbol{e}_{u^b}^I).
\end{aligned}$$

Using the algebraic inequality $a(a-b) \geq (a^2 - b^2)/2$ and multiplying the resulting equation by Δt, we obtain:

$$\begin{aligned}
\Delta t a_h^s(\boldsymbol{e}_{u^s}^h, \boldsymbol{e}_{u^s}^h) &+ \gamma(\mu^s/\kappa)^{1/2}\Delta t \,\|(\bar{e}_{u^s}^h - d_t\bar{e}_{u^b}^h)^t\|_{\Gamma_I}^2 + \kappa^{-1}\Delta t\,\|e_z^h\|_{\Omega^b}^2 \\
&+ \tfrac{1}{2}\left(a_h^b(\boldsymbol{e}_{u^b}^h, \boldsymbol{e}_{u^b}^h) - a_h^b(\boldsymbol{e}_{u^b}^{h,n}, \boldsymbol{e}_{u^b}^{h,n})\right) + \tfrac{c_0}{2}\left(\|e_{p^p}^h\|_{\Omega^b}^2 - \|e_{p^p}^{h,n}\|_{\Omega^b}^2\right) \\
&+ \tfrac{\lambda^{-1}}{2}\left(\|\alpha e_{p^p}^h - e_{p^b}^h\|_{\Omega^b}^2 - \|\alpha e_{p^p}^{h,n} - e_{p^b}^{h,n}\|_{\Omega^b}^2\right) \\
\leq &\Delta t a_h^I((0, \partial_t u^b - d_t u^b), (\bar{e}_{u^s}^h, d_t\bar{e}_{u^b}^h)) + \Delta t a_h^I((\bar{e}_{u^s}^I, d_t\bar{e}_{u^b}^I), (\bar{e}_{u^s}^h, d_t\bar{e}_{u^b}^h)) \\
&+ \Delta t(\kappa^{-1}e_z^I, e_z^h)_{\Omega^b} + c_0\Delta t(\partial_t p^p - d_t p^p, e_{p^p}^h)_{\Omega^b} \\
&+ \Delta t c_h((\partial_t p^p - d_t p^p, \partial_t p^b - d_t p^b), \alpha e_{p^p}^h) \\
&- \Delta t b_h^I(\bar{e}_{p^p}^h, (0, \partial_t u^b - d_t u^b)) - \Delta t b_h^s(\boldsymbol{e}_{p^s}^h, \boldsymbol{e}_{u^s}^I) - \Delta t b_h^b(\boldsymbol{e}_{p^b}^h, d_t\boldsymbol{e}_{u^b}^I) \\
=: &I_1^{n+1} + \ldots + I_8^{n+1}.
\end{aligned} \tag{44}$$

Defining

$$\begin{aligned}
A_i^2 &:= \tfrac{1}{2}a_h^b(\boldsymbol{e}_{u^b}^{h,i}, \boldsymbol{e}_{u^b}^{h,i}) + \tfrac{c_0}{2}\,\|e_{p^p}^{h,i}\|_{\Omega^b}^2 + \tfrac{\lambda^{-1}}{2}\,\|\alpha e_{p^p}^{h,i} - e_{p^b}^{h,i}\|_{\Omega^b}^2 \\
B_i^2 &:= \tfrac{1}{2}\Delta t a_h^s(\boldsymbol{e}_{u^s}^{h,i}, \boldsymbol{e}_{u^s}^{h,i}) + \tfrac{\kappa^{-1}}{2}\Delta t\,\|e_z^{h,i}\|_{\Omega^b}^2 + \tfrac{\gamma(\mu^s/\kappa)^{1/2}}{2}\Delta t\,\|(\bar{e}_{u^s}^{h,i} - d_t\bar{e}_{u^b}^{h,i})^t\|_{\Gamma_I}^2,
\end{aligned}$$

and by eq. (33), we can write eq. (44) as

$$A_{n+1}^2 + 2B_{n+1}^2 \leq A_n^2 + \sum_{k=1}^{8} I_k^{n+1}, \qquad A_0 = 0. \tag{45}$$

We will now bound each of the terms I_k^{n+1}, $k = 1, \ldots, 8$.

First, by the Cauchy–Schwarz and Young's inequalities, we find that for any $\delta > 0$

$$\begin{aligned}
I_1^{n+1} &\leq \Delta t\gamma(\mu^s/\kappa)^{1/2}\,\|(\partial_t u^b - d_t u^b)^t\|_{\Gamma_I}\,\|(\bar{e}_{u^s}^h - d_t\bar{e}_{u^b}^h)^t\|_{\Gamma_I} \\
&\leq \Delta t\frac{\gamma(\mu^s/\kappa)^{1/2}}{2\delta}\,\|(\partial_t u^b - d_t u^b)^t\|_{\Gamma_I}^2 + \delta\Delta t\frac{\gamma(\mu^s/\kappa)^{1/2}}{2}\,\|(\bar{e}_{u^s}^h - d_t\bar{e}_{u^b}^h)^t\|_{\Gamma_I}^2 \\
&\leq \Delta t\frac{\gamma(\mu^s/\kappa)^{1/2}}{2\delta}\,\|(\partial_t u^b - d_t u^b)^t\|_{\Gamma_I}^2 + \delta B_{n+1}^2.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
I_2^{n+1} &\le \Delta t \frac{\gamma(\mu^s/\kappa)^{1/2}}{2\delta} \|(\bar{e}^I_{u^s} - d_t\bar{e}^I_{u^b})^t\|^2_{\Gamma_I} + \delta B^2_{n+1}, \\
I_3^{n+1} &\le \Delta t \frac{\kappa^{-1}}{2\delta} \|e^I_z\|^2_{\Omega^b} + \delta B^2_{n+1},
\end{aligned}
$$

while application of the Cauchy–Schwarz inequality results in

$$
I_4^{n+1} \le c_0 \Delta t \|\partial_t p^p - d_t p^p\|_{\Omega^b} \|e^h_{p^p}\|_{\Omega^b} \le (2c_0)^{1/2} \Delta t \|\partial_t p^p - d_t p^p\|_{\Omega^b} A_{n+1}.
$$

For I_5^{n+1} we find, using the Cauchy–Schwarz and triangle inequalities and $C_*\mu^b \le \lambda$, that

$$
\begin{aligned}
I_5^{n+1} &\le \Delta t \lambda^{-1} \|\alpha(\partial_t p^p - d_t p^p) - (\partial_t p^b - d_t p^b)\|_{\Omega^b} \|\alpha e^h_{p^p}\|_{\Omega^b} \\
&\le \Delta t \lambda^{-1} \left(\|\alpha(\partial_t p^p - d_t p^p)\|_{\Omega^b} + \|\partial_t p^b - d_t p^b\|_{\Omega^b} \right) \left(\|\alpha e^h_{p^p} - e^h_{p^b}\|_{\Omega^b} + \|e^h_{p^b}\|_{\Omega^b} \right) \\
&\le \Delta t \lambda^{-1/2} \left(\|\alpha(\partial_t p^p - d_t p^p)\|_{\Omega^b} + \|\partial_t p^b - d_t p^b\|_{\Omega^b} \right) \left(\lambda^{-1/2} \|\alpha e^h_{p^p} - e^h_{p^b}\|_{\Omega^b} + (C_*\mu^b)^{-1/2} \|e^h_{p^b}\|_{\Omega^b} \right).
\end{aligned}
$$

To bound this further, let $\boldsymbol{v}^s_h = \boldsymbol{0}$ and $\boldsymbol{v}^b_h \in \widetilde{V}^b_h$ in eq. (34a) to find that $b^b_h(\boldsymbol{e}^h_{p^b}, \boldsymbol{v}^b_h) = -a^b_h(\boldsymbol{e}^h_{u^b}, \boldsymbol{v}^b_h)$. By eqs. (15a), (15b) and (17),

$$
|\!|\!|\boldsymbol{e}^h_{p^b}|\!|\!|_{q,b} \le C \sup_{\boldsymbol{0} \ne \boldsymbol{v}_h \in \widetilde{V}^b_h} \frac{b^b_h(\boldsymbol{e}^h_{p^b}, \boldsymbol{v}_h)}{|\!|\!|\boldsymbol{v}_h|\!|\!|_{v,b}} \le C\mu^b |\!|\!|\boldsymbol{e}^h_{u^b}|\!|\!|_{v,b} \le C(\mu^b)^{1/2} a^b_h(\boldsymbol{e}^h_{u^b}, \boldsymbol{e}^h_{u^b})^{1/2}. \tag{46}
$$

In other words, $(\mu^b)^{-1/2} \|e^h_{p^b}\|_{\Omega^b} \le C a^b_h(\boldsymbol{e}^h_{u^b}, \boldsymbol{e}^h_{u^b})^{1/2} \le C A_{n+1}$ so that

$$
I_5^{n+1} \le C \Delta t \lambda^{-1/2} \left(\alpha \|\partial_t p^p - d_t p^p\|_{\Omega^b} + \|\partial_t p^b - d_t p^b\|_{\Omega^b} \right) A_{n+1}.
$$

To bound I_6^{n+1} we use lemma 4 and Cauchy–Schwarz, triangle, and Young's inequalities:

$$
\begin{aligned}
I_6^{n+1} =& \Delta t \langle \bar{e}^h_{p^p}, (\partial_t u^b - d_t u^b) \cdot n \rangle_{\Gamma_I} \le C\kappa^{-1} \Delta t \|z - z_h\|_{\Omega^b} \|d_t u^b - \partial_t u^b\|_{\Gamma_I} \\
\le& C\kappa^{-1} \Delta t \|e^h_z\|_{\Omega^b} \|d_t u^b - \partial_t u^b\|_{\Gamma_I} + C\kappa^{-1} \Delta t \|e^I_z\|_{\Omega^b} \|d_t u^b - \partial_t u^b\|_{\Gamma_I} \\
\le& \delta B^2_{n+1} + \frac{C\kappa^{-1}}{2\delta} \Delta t \|d_t u^b - \partial_t u^b\|^2_{\Gamma_I} + \frac{C\kappa^{-1}}{2} \Delta t \|e^I_z\|^2_{\Omega^b} + \frac{C\kappa^{-1}}{2} \Delta t \|d_t u^b - \partial_t u^b\|^2_{\Gamma_I}.
\end{aligned}
$$

To bound I_7^{n+1} and I_8^{n+1} we first bound $|\!|\!|\boldsymbol{e}^h_p|\!|\!|_q$. Observe from eqs. (15a) and (34a), and the Cauchy–Schwarz inequality that for all $\boldsymbol{v}_h \in \widehat{\boldsymbol{V}}_h$, since $b^I_h(\bar{e}^h_{p^p}, (\bar{v}^s_h, \bar{v}^b_h)) = 0$,

$$
\begin{aligned}
b_h(\boldsymbol{e}^h_p, \boldsymbol{v}_h) =& -a_h(\boldsymbol{e}^h_u, \boldsymbol{v}_h) - a^I_h((\bar{e}^h_{u^s}, d_t\bar{e}^h_{u^b}), (\bar{v}^s_h, \bar{v}^b_h)) \\
&+ a^I_h((0, \partial_t u^b - d_t u^b), (\bar{v}^s_h, \bar{v}^b_h)) + a^I_h((\bar{e}^I_{u^s}, d_t\bar{e}^I_{u^b}), (\bar{v}^s_h, \bar{v}^b_h)) \\
\le& C\left(\mu^s |\!|\!|\boldsymbol{e}^h_{u^s}|\!|\!|_{v,s} |\!|\!|\boldsymbol{v}^s_h|\!|\!|_{v,s} + \mu^b |\!|\!|\boldsymbol{e}^h_{u^b}|\!|\!|_{v,b} |\!|\!|\boldsymbol{v}^b_h|\!|\!|_{v,b} \right) \\
&+ \gamma(u^s/\kappa)^{1/2} \|(\bar{e}^h_{u^s} - d_t\bar{e}^h_{u^b})^t\|_{\Gamma_I} |\!|\!|\boldsymbol{v}_h|\!|\!|_v \\
&+ \gamma(u^s/\kappa)^{1/2} \left(\|(\partial_t u^b - d_t u^b)^t\|_{\Gamma_I} + \|(\bar{e}^I_{u^s} - d_t\bar{e}^I_{u^b})^t\|_{\Gamma_I} \right) |\!|\!|\boldsymbol{v}_h|\!|\!|_v.
\end{aligned}
$$

By eqs. (15b) and (18) we then find that

$$
\begin{aligned}
\|\!|\boldsymbol{e}_p^h|\!\|_q \leq & C\Big\{\mu^s \|\!|\boldsymbol{e}_{u^s}^h|\!\|_{v,s} + \mu^b \|\!|\boldsymbol{e}_{u^b}^h|\!\|_{v,b} + \gamma(u^s/\kappa)^{1/2} \|(\bar{e}_{u^s}^h - d_t \bar{e}_{u^b}^h)^t\|_{\Gamma_I} \\
& + \gamma(u^s/\kappa)^{1/2} \left(\|(\partial_t u^b - d_t u^b)^t\|_{\Gamma_I} + \|(\bar{e}_{u^s}^I - d_t \bar{e}_{u^b}^I)^t\|_{\Gamma_I} \right) \Big\} \\
\leq & C\Big\{(\Delta t)^{-1/2} \left[(\mu^s)^{1/2} + \gamma^{1/2}(\mu^s/\kappa)^{1/4}\right] B_{n+1} + (\mu^b)^{1/2} A_{n+1} \\
& + \gamma(u^s/\kappa)^{1/2} \left(\|(\partial_t u^b - d_t u^b)^t\|_{\Gamma_I} + \|(\bar{e}_{u^s}^I - d_t \bar{e}_{u^b}^I)^t\|_{\Gamma_I} \right) \Big\}.
\end{aligned}
\tag{47}
$$

Therefore, using eq. (16),

$$
\begin{aligned}
I_7^{n+1} \leq & C\Delta t \|\!|\boldsymbol{e}_{p^s}^h|\!\|_{q,s} \|\!|\boldsymbol{e}_{u^s}^I|\!\|_{v,s} \\
\leq & C\Delta t \left\{(\Delta t)^{-1/2} \left[(\mu^s)^{1/2} + \gamma^{1/2}(\mu^s/\kappa)^{1/4}\right] B_{n+1} + (\mu^b)^{1/2} A_{n+1}\right\} \|\!|\boldsymbol{e}_{u^s}^I|\!\|_{v,s} \\
& + C\Delta t \left\{\gamma(u^s/\kappa)^{1/2} \left(\|(\partial_t u^b - d_t u^b)^t\|_{\Gamma_I} + \|(\bar{e}_{u^s}^I - d_t \bar{e}_{u^b}^I)^t\|_{\Gamma_I} \right)\right\} \|\!|\boldsymbol{e}_{u^s}^I|\!\|_{v,s} \\
=: & I_{71} + I_{72}.
\end{aligned}
$$

Young's inequality is used to bound I_{71}:

$$
\begin{aligned}
I_{71} = & C(\Delta t)^{1/2} \left[(\mu^s)^{1/2} + \gamma^{1/2}(\mu^s/\kappa)^{1/4}\right] B_{n+1} \|\!|\boldsymbol{e}_{u^s}^I|\!\|_{v,s} + C\Delta t (\mu^b)^{1/2} A_{n+1} \|\!|\boldsymbol{e}_{u^s}^I|\!\|_{v,s} \\
\leq & \delta B_{n+1}^2 + C\Delta t \left(\mu^s + \gamma(\mu^s/\kappa)^{1/2}\right) \|\!|\boldsymbol{e}_{u^s}^I|\!\|_{v,s}^2 + C\Delta t (\mu^b)^{1/2} A_{n+1} \|\!|\boldsymbol{e}_{u^s}^I|\!\|_{v,s}.
\end{aligned}
$$

Similarly, we find that

$$
\begin{aligned}
I_8^{n+1} \leq & C\Delta t \|\!|\boldsymbol{e}_{p^b}^h|\!\|_{q,b} \|\!| d_t \boldsymbol{e}_{u^b}^I|\!\|_{v,b} \\
\leq & C\Delta t \left\{(\Delta t)^{-1/2} \left[(\mu^s)^{1/2} + \gamma^{1/2}(\mu^s/\kappa)^{1/4}\right] B_{n+1} + (\mu^b)^{1/2} A_{n+1}\right\} \|\!| d_t \boldsymbol{e}_{u^b}^I|\!\|_{v,b} \\
& + C\Delta t \left\{\gamma(u^s/\kappa)^{1/2} \left(\|(\partial_t u^b - d_t u^b)^t\|_{\Gamma_I} + \|(\bar{e}_{u^s}^I - d_t \bar{e}_{u^b}^I)^t\|_{\Gamma_I} \right)\right\} \|\!| d_t \boldsymbol{e}_{u^b}^I|\!\|_{v,b} \\
=: & I_{81} + I_{82},
\end{aligned}
$$

and

$$
I_{81} \leq \delta B_{n+1}^2 + C\Delta t \left(\mu^s + \gamma(\mu^s/\kappa)^{1/2}\right) \|\!| d_t \boldsymbol{e}_{u^b}^I|\!\|_{v,b}^2 + C\Delta t (\mu^b)^{1/2} A_{n+1} \|\!| d_t \boldsymbol{e}_{u^b}^I|\!\|_{v,b}.
$$

Adding up the various bounds for I_k^{n+1}, $k = 1, \ldots, 8$, we find

$$
\sum_{k=1}^{8} I_k^{n+1} \leq 6\delta B_{n+1}^2 + E_{n+1} A_{n+1} + D_{n+1}, \tag{48}
$$

where

$$
\begin{aligned}
E_i = C\Delta t \Big[& c_0^{1/2} \|\partial_t p^{p,i} - d_t p^{p,i}\|_{\Omega^b} + \lambda^{-1/2} \alpha \|\partial_t p^{p,i} - d_t p^{p,i}\|_{\Omega^b} \\
& + \lambda^{-1/2} \|\partial_t p^{b,i} - d_t p^{b,i}\|_{\Omega^b} + (\mu^b)^{1/2} \|\!|\boldsymbol{e}_{u^s}^{I,i}|\!\|_{v,s} + (\mu^b)^{1/2} \|\!| d_t \boldsymbol{e}_{u^b}^{I,i}|\!\|_{v,b} \Big],
\end{aligned}
$$

and

$$
\begin{aligned}
D_i = C\Delta t\Big[&\gamma(\mu^s/\kappa)^{1/2}\,\|(\partial_t u^{b,i} - d_t u^{b,i})^t\|_{\Gamma_I}^2 + \gamma(\mu^s/\kappa)^{1/2} 2\delta\,\|(\bar{e}_{u^s}^{I,i} - d_t \bar{e}_{u^b}^{I,i})^t\|_{\Gamma_I}^2 \\
&+ \kappa^{-1}\,\|e_z^{I,i}\|_{\Omega^b}^2 + \kappa^{-1}\,\|d_t u^{b,i} - \partial_t u^{b,i}\|_{\Gamma_I}^2 + \Big(\mu^s + \gamma(\mu^s/\kappa)^{1/2}\Big)\,\|\!|\boldsymbol{e}_{u^s}^{I,i}|\!\|_{v,s}^2 \\
&+ \gamma(\mu^s/\kappa)^{1/2}\Big(\|(\partial_t u^{b,i} - d_t u^{b,i})^t\|_{\Gamma_I} + \|(\bar{e}_{u^s}^{I,i} - d_t \bar{e}_{u^b}^{I,i})^t\|_{\Gamma_I}\Big)\,\|\!|\boldsymbol{e}_{u^s}^{I,i}|\!\|_{v,s} \\
&+ \gamma(\mu^s/\kappa)^{1/2}\Big(\|(\partial_t u^{b,i} - d_t u^{b,i})^t\|_{\Gamma_I} + \|(\bar{e}_{u^s}^{I,i} - d_t \bar{e}_{u^b}^{I,i})^t\|_{\Gamma_I}\Big)\,\|\!|d_t\boldsymbol{e}_{u^b}^{I,i}|\!\|_{v,b} \\
&+ \Big(\mu^s + \gamma(\mu^s/\kappa)^{1/2}\Big)\,\|\!|d_t\boldsymbol{e}_{u^b}^{I,i}|\!\|_{v,b}^2\Big].
\end{aligned}
$$

Combining eqs. (45) and (48) (with $\delta = 1/6$) and summing over the time levels, we obtain

$$
A_n^2 + \sum_{i=1}^n B_i^2 \le \sum_{i=1}^n E_i A_i + \sum_{i=1}^n D_i.
$$

By [30, Lemma 4.1],

$$
A_n + \Big(\sum_{i=1}^n B_i^2\Big)^{1/2} \le C\Big(\sum_{i=1}^n E_i + \Big(\sum_{i=1}^n D_i\Big)^{1/2}\Big). \tag{49}
$$

We will now bound the two sums on the right hand side separately. For this we require the following inequalities (that can be proven by Taylor series expansions) [38, Lemma 3.2], [30, (4.5a)]:

$$
\sum_{i=1}^n \Delta t\|\partial_t u^{b,i} - d_t u^{b,i}\|_{\Gamma_I}^2 \le C(\Delta t)^2\|\partial_{tt} u^b\|_{L^2(J;L^2(\Gamma_I))}, \tag{50a}
$$

$$
\sum_{i=1}^n \Delta t\|\partial_t \psi^i - d_t \psi^i\|_{\Omega^b} \le \Delta t\|\partial_{tt}\psi\|_{L^1(J;L^2(\Omega^b))}, \qquad \psi = p^b, p^p, \tag{50b}
$$

$$
\Delta t\sum_{i=1}^n \|d_t u^{b,i}\|_{\ell+1,\Omega^j} \le C\|\partial_t u^b\|_{L^1(J;H^{\ell+1}(\Omega^b))}, \tag{50c}
$$

$$
\Delta t\sum_{i=1}^n \|d_t u^{b,i}\|_{\ell+1,\Omega^b}^2 \le C\|\partial_t u^b\|_{L^2(J;H^{\ell+1}(\Omega^b))}^2, \tag{50d}
$$

$$
\Delta t\sum_{i=1}^n \|d_t \bar{e}_{u^b}^{I,i}\|_{\Gamma_I}^2 \le Ch^{2\ell}\|\partial_t u^b\|_{L^2(J;H^{\ell+1}(\Omega^b))}^2, \tag{50e}
$$

and the inequalities (that can be proven using eqs. (15b), (32b), (50c) and (50d))

$$\begin{aligned}
\Delta t \sum_{i=1}^{n} |\!|\!| d_t \boldsymbol{e}_{u^b}^{I,i} |\!|\!|_{v,b} &\le C\Delta t (\mu^b)^{-1/2} \sum_{i=1}^{n} a_h^b(d_t \boldsymbol{e}_{u^b}^{I,i}, d_t \boldsymbol{e}_{u^b}^{I,i})^{1/2} \\
&\le C\Delta t \sum_{i=1}^{n} h^\ell \, \|d_t u^{b,i}\|_{\ell+1,\Omega^b} \le C h^\ell \, \|\partial_t u^b\|_{L^1(J;H^{\ell+1}(\Omega^b))} , \\
\Delta t \sum_{i=1}^{n} |\!|\!| d_t \boldsymbol{e}_{u^b}^{I,i} |\!|\!|_{v,b}^2 &\le C\Delta t (\mu^b)^{-1} \sum_{i=1}^{n} a_h^b(d_t \boldsymbol{e}_{u^b}^{I,i}, d_t \boldsymbol{e}_{u^b}^{I,i}) \\
&\le C\Delta t \sum_{i=1}^{n} h^{2\ell} \, \|d_t u^{b,i}\|_{\ell+1,\Omega^b}^2 \le C h^{2\ell} \, \|\partial_t u^b\|_{L^2(J;H^{\ell+1}(\Omega^b))}^2 , \\
\Delta t \sum_{i=1}^{n} |\!|\!| \boldsymbol{e}_{u^s}^{I,i} |\!|\!|_{v,s} &\le C\Delta t (\mu^s)^{-1/2} \sum_{i=1}^{n} a_h^s(\boldsymbol{e}_{u^s}^{I,i}, \boldsymbol{e}_{u^s}^{I,i})^{1/2} \\
&\le C\Delta t \sum_{i=1}^{n} h^\ell \, \|u^{s,i}\|_{\ell+1,\Omega^s} \le C T h^\ell \, \|u^s\|_{C^0(J;H^{\ell+1}(\Omega^s))} .
\end{aligned}$$

Similarly, we have by eqs. (32a) and (32b),

$$\begin{aligned}
\Delta t \sum_{i=1}^{n} \|e_z^{I,i}\|_{\Omega^b}^2 &\le C T h^{2\ell} \, \|z\|_{C^0(J;H^\ell(\Omega^b))}^2 , \\
\Delta t \sum_{i=1}^{n} \|\bar{e}_{u^s}^{I,i}\|_{\Gamma_I}^2 &\le C T h^{2\ell} \, \|u^s\|_{C^0(J;H^{\ell+1}(\Omega^s))}^2 .
\end{aligned}$$

We now find:

$$\begin{aligned}
\sum_{i=1}^{n} E_i \le & C\Delta t \Big[c_0^{1/2} \, \|\partial_{tt} p^p\|_{L^1(J;L^2(\Omega^b))} + \lambda^{-1/2} \alpha \, \|\partial_{tt} p^p\|_{L^1(J;L^2(\Omega^b))} + \lambda^{-1/2} \, \|\partial_{tt} p^b\|_{L^1(J;L^2(\Omega^b))} \Big] \\
& + C h^\ell \Big[(\mu^b)^{1/2} T \, \|u^s\|_{C^0(J;H^{\ell+1}(\Omega^s))} + (\mu^b)^{1/2} \, \|\partial_t u^b\|_{L^2(J;H^{\ell+1}(\Omega^b))} \Big],
\end{aligned}$$

and, after applying Young's inequality to D_i,

$$\begin{aligned}
\Big(\sum_{i=1}^{n} D_i \Big)^{1/2} \le & C\Delta t \Big[\gamma^{1/2} (\mu^s/\kappa)^{1/4} \, \|\partial_{tt} u^b\|_{L^2(J;L^2(\Gamma_I))} + \kappa^{-1/2} \, \|\partial_{tt} u^b\|_{L^2(J;L^2(\Gamma_I))} \Big] \\
& + C h^\ell \Big[\kappa^{-1/2} T^{1/2} \, \|z\|_{C^0(J;H^\ell(\Omega^b))} + \Big(\mu^s + \gamma (\mu^s/\kappa)^{1/2} \Big)^{1/2} T^{1/2} \, \|u^s\|_{C^0(J;H^{\ell+1}(\Omega^s))} \\
& \qquad + \Big(\mu^s + \gamma (\mu^s/\kappa)^{1/2} \Big)^{1/2} \, \|\partial_t u^b\|_{L^2(J;H^{\ell+1}(\Omega^b))} \Big],
\end{aligned}$$

From eq. (49) we then find:

$$
\begin{aligned}
&A_n + \Big(\sum_{i=1}^{n} B_i^2\Big)^{1/2} \\
&\le C\Delta t\Bigg[\max(c_0^{1/2}, \lambda^{-1/2}\alpha)\, \|\partial_{tt} p^p\|_{L^1(J;L^2(\Omega^b))} + \lambda^{-1/2}\, \|\partial_{tt} p^b\|_{L^1(J;L^2(\Omega^b))} \\
&\qquad + \max(\gamma^{1/2}(\mu^s/\kappa)^{1/4}, \kappa^{-1/2})\, \|\partial_{tt} u^b\|_{L^2(J;L^2(\Gamma_I))}\Bigg] \\
&+ Ch^\ell\Bigg[\max\Big((\mu^b)^{1/2}, (\mu^s + \gamma(\mu^s/\kappa)^{1/2})^{1/2}\Big)\, \|\partial_t u^b\|_{L^2(J;H^{\ell+1}(\Omega^b))} + \kappa^{-1/2}T^{1/2}\, \|z\|_{C^0(J;H^\ell(\Omega^b))} \\
&\qquad + \max\Big((\mu^b)^{1/2}T, \Big(\mu^s + \gamma(\mu^s/\kappa)^{1/2}\Big)^{1/2} T^{1/2}\Big)\, \|u^s\|_{C^0(J;H^{\ell+1}(\Omega^s))}\Bigg].
\end{aligned}
$$

Equation (43a) now follows by definition of A_i and B_i and the coercivity of a_h^s and a_h^b eq. (15b) while eq. (43b) follows from eq. (46) and noting that $a_h^b(\boldsymbol{e}_{u^b}^{h,n}, \boldsymbol{e}_{u^b}^{h,n})^{1/2} \le A_n$. Finally, eq. (43c) follows using similar steps as used to find eq. (43b): let $\boldsymbol{v}_h^b = \boldsymbol{0}$ and $\boldsymbol{v}_h^s \in \widetilde{V}_h^s$ in eq. (34a) to find that $b_h^s(\boldsymbol{e}_{p^s}^{h,i}, \boldsymbol{v}_h^s) = -a_h^s(\boldsymbol{e}_{u^s}^{h,i}, \boldsymbol{v}_h^s)$. By eqs. (15a), (15b) and (17),

$$
C_1 \|\!|\!| \boldsymbol{e}_{p^s}^{h,i} \|\!|\!|_{q,s} \le \sup_{\boldsymbol{0} \ne \boldsymbol{v}_h \in \widetilde{V}_h^s} \frac{b_h^s(\boldsymbol{e}_{p^s}^{h,i}, \boldsymbol{v}_h)}{\|\!|\!| \boldsymbol{v}_h \|\!|\!|_{v,s}} \le C_2 \mu^s \|\!|\!| \boldsymbol{e}_{u^s}^{h,i} \|\!|\!|_{v,s} \le C_3 (\mu^s)^{1/2} a_h^s(\boldsymbol{e}_{u^s}^{h,i}, \boldsymbol{e}_{u^s}^{h,i})^{1/2}.
$$

The result follows noting that $(\mu^s)^{-1}\Delta t \|\!|\!| \boldsymbol{e}_{p^s}^{h,i} \|\!|\!|_{q,s}^2 \le C B_i^2$. □

The main result of this section, an a priori error estimate for the solution to the HDG method that is robust in the limits $\lambda \to \infty$ and $c_0 \to 0$, is now a consequence of theorem 3.

Corollary 1. *Let (u, p, z, p^p) be the solution to the coupled Stokes–Biot problem eqs.* (1) *to* (3) *on time interval $J = (0, T]$. In addition to the regularity assumptions used in theorem 3 we further assume that $p^s \in C^0(J; H^\ell(\Omega^s))$. Let $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n, z_h^n, \boldsymbol{p}_h^{p,n}) \in \boldsymbol{X}_h$, for $n = 1, \dots, N$, be the solution to eq.* (7) *with initial conditions eq.* (33). *Define $\boldsymbol{u}^j = (u^j, \gamma_{V^j}(u^j))$. Then*

$$
\begin{aligned}
&(\mu^b)^{1/2} \|\!|\!| \boldsymbol{u}^{b,n} - \boldsymbol{u}_h^{b,n} \|\!|\!|_{v,b} + c_0^{1/2}\, \|p^{p,n} - p_h^{p,n}\|_{\Omega^b} + \lambda^{-1/2}\, \|\alpha(p^{p,n} - p_h^{p,n}) - (p^{b,n} - p_h^{b,n})\|_{\Omega^b} \\
&\quad + (\mu^s)^{1/2}\Big(\Delta t \sum_{i=1}^{n} \|\!|\!| \boldsymbol{u}^{s,i} - \boldsymbol{u}_h^{s,i} \|\!|\!|_{v,s}^2\Big)^{1/2} + \kappa^{-1/2}\Big(\Delta t \sum_{i=1}^{n} \|z^i - z_h^i\|_{\Omega^b}^2\Big)^{1/2} \\
&\quad + \gamma^{1/2}(\mu^s/\kappa)^{1/4}\Big(\Delta t \sum_{i=1}^{n} \|((u_s^i - \bar{u}_h^{s,i}) - d_t(u_b^i - \bar{u}_h^{b,i}))^t\|_{\Gamma_I}^2\Big)^{1/2} \le \bar{C}_1 \Delta t + \tilde{C}_2 h^\ell,
\end{aligned}
\tag{51a}
$$

$$
(\mu^b)^{-1/2} \|\!|\!| \boldsymbol{p}^n - \boldsymbol{p}_h^{b,n} \|\!|\!|_{q,b} \le \bar{C}_1 \Delta t + (\bar{C}_2 + C(\mu^b)^{-1/2}\, \|p^b\|_{C^0(J;H^\ell(\Omega^b))}) h^\ell,
\tag{51b}
$$

$$
(\mu^s)^{-1/2}\Big(\Delta t \sum_{i=1}^{n} \|\!|\!| \boldsymbol{p}^{s,i} - \boldsymbol{p}_h^{s,i} \|\!|\!|_{q,s}^2\Big)^{1/2} \le 2\bar{C}_1 \Delta t + \widehat{C}_2 h^\ell,
\tag{51c}
$$

where $\bar{C}_1$ and $\bar{C}_2$ are the constants defined in theorem 3 and

$$\begin{aligned}
\tilde{C}_2 =& \bar{C}_2 + C\Big[(\mu^b)^{1/2}\|u^b\|_{C^0(J;H^{\ell+1}(\Omega^b))} + \max((\mu^s)^{1/2}, \gamma^{1/2}(\mu^s/\kappa)^{1/4})T^{1/2}\|u^s\|_{C^0(J;H^{\ell+1}(\Omega^s))} \\
&+ \gamma^{1/2}(\mu^s/\kappa)^{1/4}\|\partial_t u^b\|_{L^2(J;H^{\ell+1}(\Omega^b)} + \max(c_0^{1/2}, \lambda^{-1/2}\alpha)\|p^p\|_{C^0(J;H^{\ell}(\Omega^b))} \\
&+ \lambda^{-1/2}\|p^b\|_{C^0(J;H^{\ell}(\Omega^b))} + \kappa^{-1/2}T^{1/2}\|z\|_{C^0(J;H^{\ell}(\Omega^b))}\Big], \\
\widehat{C_2} =& \left(4\bar{C}_2^2 + C(\mu^s)^{-1}T\,\|p^s\|^2_{C^0(J;H^{\ell}(\Omega^s))}\right)^{1/2}.
\end{aligned}$$

Proof. Equation (51a) is a direct consequence of the triangle inequality, eq. (43a), and the following estimates:

$$\begin{aligned}
|||\boldsymbol{e}^{I,n}_{u^b}|||_{v,b} &\le Ch^{\ell}\|u^b\|_{C^0(J;H^{\ell+1}(\Omega^b))}, \\
\|e^{I,n}_{p^p}\|_{\Omega^b} &\le Ch^{\ell}\|p^p\|_{C^0(J;H^{\ell}(\Omega^b))}, \\
\|\alpha e^{I,n}_{p^p} - e^{I,n}_{p^b}\|_{\Omega^b} &\le Ch^{\ell}(\alpha\|p^p\|_{C^0(J;H^{\ell}(\Omega^b))} + \|p^b\|_{C^0(J;H^{\ell}(\Omega^b))}), \\
\left(\Delta t\sum_{i=1}^{n}|||\boldsymbol{e}^{I,i}_{u^s}|||^2_{v,s}\right)^{1/2} &\le CT^{1/2}h^{\ell}\|u^s\|_{C^0(J;H^{\ell+1}(\Omega^s))}, \\
\left(\Delta t\sum_{i=1}^{n}\|e^{I,i}_{z}\|^2_{\Omega^b}\right)^{1/2} &\le CT^{1/2}h^{\ell}\|z\|_{C^0(J;H^{\ell}(\Omega^b))}, \\
\left(\Delta t\sum_{i=1}^{n}\|\bar{e}^{I,i}_{u^s} - d_t\bar{e}^{I,i}_{u^b}\|^2_{\Gamma_I}\right)^{1/2} &\le Ch^{\ell}\Big(T^{1/2}\|u^s\|_{C^0(J;H^{\ell+1}(\Omega^s))} + \|\partial_t u^b\|_{L^2(J;H^{\ell+1}(\Omega^b))}\Big).
\end{aligned}$$

Next, note that

$$|||\boldsymbol{e}^{I,i}_{p^j}|||_{q,j} \le Ch^{\ell}\,\|p^{j,i}\|_{\ell,\Omega^j}, \qquad j = s, b. \tag{52}$$

Equation (51b) follows by a triangle inequality, eq. (43b), and eq. (52). Finally, to show eq. (51c) we note that, by a triangle inequality, eq. (52), and Young's inequality

$$\begin{aligned}
(\mu^s)^{-1}\Delta t\sum_{i=1}^{n}|||\boldsymbol{p}^{s,i} - \boldsymbol{p}^{s,i}_h|||^2_{q,s} &\le 2\left[(\bar{C}_1\Delta t + \bar{C}_2 h^{\ell})^2 + C(\mu^s)^{-1}h^{2\ell}\Delta t\sum_{i=1}^{n}\|p^{s,i}\|^2_{\ell,\Omega^s}\right] \\
&\le 2\left[(\bar{C}_1\Delta t + \bar{C}_2 h^{\ell})^2 + C(\mu^s)^{-1}Th^{2\ell}\,\|p^s\|^2_{C^0(J;H^{\ell}(\Omega^s))}\right] \\
&\le 2\left[2\bar{C}_1^2\Delta t^2 + (2\bar{C}_2^2 + C(\mu^s)^{-1}T\,\|p^s\|^2_{C^0(J;H^{\ell}(\Omega^s))})h^{2\ell}\right],
\end{aligned}$$

so that the result follows. □

Remark 1. We briefly sketch the proof of stability and obtaining an error estimate for the divergence of the Darcy velocity. Starting with the semi-discrete problem, we have, equivalent to eq. (8g),

$$\nabla\cdot z_h = \Pi^b_Q g^b - c_0\partial_t p^p_h - \alpha\lambda^{-1}(\alpha\partial_t p^p_h - \partial_t p^b_h) \qquad \forall x \in K, \quad \forall K \in \mathcal{T}_h.$$

Therefore, stability of $\nabla\cdot z_h$ requires control of $\partial_t p_h^p$ and $\alpha\partial_t p_h^p - \partial_t p_h^b$. To achieve this, differentiate all equations in eq. (6) with respect to time and choose as test functions $\boldsymbol{v}_h^s = \partial_t \boldsymbol{u}_h^s$, $\boldsymbol{v}_h^b = \partial_t^2 \boldsymbol{u}_h^b$, $\boldsymbol{q}_h = -\partial_t \boldsymbol{p}_h$, $\boldsymbol{q}_h^p = \partial_t \boldsymbol{p}_h^p$, and $w_h = \partial_t z_h$. Similar to eq. (23) we find:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Big[a_h^b(\partial_t \boldsymbol{u}_h^b, \partial_t \boldsymbol{u}_h^b) + c_0\|\partial_t p_h^p\|_{\Omega^b}^2 + \lambda^{-1}\|\alpha\partial_t p_h^p - \partial_t p_h^b\|_{\Omega^b}^2\Big] \\
+ \Big[a_h^s(\partial_t \boldsymbol{u}_h^s, \partial_t \boldsymbol{u}_h^s) + \gamma(\mu^s/\kappa)^{1/2}\|\partial_t(\bar{u}_h^s - \partial_t \bar{u}_h^b)^t\|_{\Gamma_I}^2 \\
+ \kappa^{-1}\|\partial_t z_h\|_{\Omega^b}^2\Big] = (\partial_t f^s, \partial_t u_h^s)_{\Omega^s} + (\partial_t f^b, \partial_t^2 u_h^b)_{\Omega^b} + (\partial_t g^b, \partial_t p_h^p)_{\Omega^b}.
\end{aligned}
$$

Following the same steps as in the proof of theorem 1 then results in control of $\sup_{0\le t\le T}[c_0\|\partial_t p_h^p(t)\|_{\Omega^b}^2 + \lambda^{-1}\|\alpha\partial_t p_h^p - \partial_t p_h^b\|_{\Omega^b}]$. The same idea, albeit more tedious, can be used for the fully discrete case to obtain control of $d_t p_h^{p,n+1}$ and $\alpha d_t p_h^{p,n+1} - d_t p_h^{b,n+1}$. Lastly, we point out that when deriving an error estimate for $\nabla\cdot z_h$, the quantities that we want to control are $\|\partial_t p^{p,n+1} - d_t p_h^{p,n+1}\|_{\Omega^b}$ and $\alpha\lambda^{-1/2}\|(\alpha\partial_t p^{p,n+1} - \partial_t p^{b,n+1}) - (\alpha d_t p_h^{p,n+1} - d_t p_h^{b,n+1})\|_{\Omega^b}$. Control of these terms can be obtained using the error equations. Since f^s, f^b, g^b, and their derivatives vanish from the error equations, no additional regularity assumptions are needed for the source terms.

6. Numerical examples

We present some numerical examples using the fully discrete HDG method eq. (7) to find approximate solutions to the coupled Stokes and Biot problem eqs. (1) to (3). All examples have been implemented using Netgen/NGSolve [39, 40].

6.1. Stationary test case

In this first test case we consider the following stationary problem:

$$
\begin{aligned}
-\nabla\cdot\sigma^j &= f^j && \text{in } \Omega^j, \quad j = s, b, && \text{(53a)}\\
-\nabla\cdot u^s &= 0 && \text{in } \Omega^s, && \text{(53b)}\\
-\nabla\cdot u^b + \lambda^{-1}(\alpha p^p - p^b) &= 0 && \text{in } \Omega^b, && \text{(53c)}\\
c_0\tau p^p + \alpha\tau\lambda^{-1}(\alpha p^p - p^b) + \nabla\cdot z &= g^b && \text{in } \Omega^b, && \text{(53d)}\\
\kappa^{-1}z + \nabla p^p &= 0 && \text{in } \Omega^b, && \text{(53e)}
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
u^j &= U^j && \text{on } \Gamma_D^j, \quad j = s, b, && \text{(54a)}\\
\sigma^j n &= S^j && \text{on } \Gamma_N^j, \quad j = s, b, && \text{(54b)}\\
p^p &= P^p && \text{on } \Gamma_P^b, && \text{(54c)}\\
z\cdot n &= Z && \text{on } \Gamma_F^b, && \text{(54d)}
\end{aligned}
$$

and interface conditions

$$
\begin{aligned}
u^s\cdot n &= \left(\tau u^b + z\right)\cdot n + M^u && \text{on } \Gamma_I, && \text{(55a)}\\
\sigma^s n &= \sigma^b n + M^s && \text{on } \Gamma_I, && \text{(55b)}\\
-(\sigma^s n)\cdot n &= p^p + M^p && \text{on } \Gamma_I, && \text{(55c)}\\
-2\mu^s(\varepsilon(u^s)n)^t &= \gamma(\mu^s/\kappa)^{1/2}(u^s - \partial_t u^b)^t + M^e && \text{on } \Gamma_I. && \text{(55d)}
\end{aligned}
$$

We consider the unit square domain $\Omega = (0,1)^2$ partitioned as: $\overline{\Omega} = \overline{\Omega}^s \cup \overline{\Omega}^b$ with $\overline{\Omega}^s = [0,1] \times [1/2,1]$ and $\overline{\Omega}^b = [0,1] \times [0,1/2]$. We set $\Gamma_D^s = \{x \in \Gamma^s : x_1 = 0 \text{ or } x_2 = 1\}$, $\Gamma_N^s = \{x \in \Gamma^s : x_1 = 1\}$, $\Gamma_P^b = \Gamma_D^b = \{x \in \Gamma^b : x_1 = 0 \text{ or } x_2 = 0\}$, and $\Gamma_N^b = \Gamma_F^b = \{x \in \Gamma^b : x_1 = 1\}$. The source terms f^s, f^b, and g^b, the boundary data U^s, U^b, S^s, S^b, P^p, and Z, and the interface data M^u, M^s, M^p, and M^e are chosen such that the exact solution is given by:

$$
\begin{aligned}
u^s &= \begin{bmatrix} \pi x_1 \cos(\pi x_1 x_2) + 1 \\ -\pi x_2 \cos(\pi x_1 x_2) + 2x_1 \end{bmatrix}, & p^s &= \sin(3x_1)\cos(4x_2), \\
u^b &= \begin{bmatrix} \cos(4x_1)\cos(3x_2) \\ \sin(5x_1)\cos(2x_2) \end{bmatrix}, & p^p &= \sin(3x_1 x_2),
\end{aligned}
$$

Note that $p^b = -\lambda \nabla \cdot u^b + \alpha p^p$ and $z = -\kappa \nabla p^p$.

We choose the following parameters: $\mu^s = 10^{-2}$, $\mu^b = 10^{-3}$, $\alpha = 0.2$, $\lambda = 10^2$, $\kappa = 10^{-2}$, $c_0 = 10^{-2}$, $\gamma = 0.3$, $\tau = 10^{-2}$. We choose the interior penalty parameters as $\beta^s = \beta^b = 8k^2$, where k is the polynomial degree.

We present the errors in the L^2-norm and rates of convergence for all unknowns in table 1 for polynomial degrees $k = 1$, $k = 2$, and $k = 3$. We observe optimal rates of convergence for all unknowns.

6.2. *Time-dependent test case*

We now consider the time-dependent problem eq. (1) with boundary and interface conditions given by, respectively,

$$
\begin{aligned}
u^j &= U^j && \text{on } \Gamma_D^j \times J, \quad j = s, b, \\
\sigma^j n &= S^j && \text{on } \Gamma_N^j \times J, \quad j = s, b, \\
p^p &= P^p && \text{on } \Gamma_P^b \times J, \\
z \cdot n &= Z && \text{on } \Gamma_F^b \times J,
\end{aligned}
$$

and

$$
\begin{aligned}
u^s \cdot n &= \left(\partial_t u^b + z\right) \cdot n + M^u && \text{on } \Gamma_I \times J, \\
\sigma^s n &= \sigma^b n + M^s && \text{on } \Gamma_I \times J, \\
-(\sigma^s n) \cdot n &= p^p + M^p && \text{on } \Gamma_I \times J, \\
-2\mu^s (\varepsilon(u^s) n)^t &= \gamma(\mu^s/\kappa)^{1/2}(u^s - \partial_t u^b)^t + M^e && \text{on } \Gamma_I \times J.
\end{aligned}
$$

We consider the same domain and partitioning of the boundary as in section 6.1. The source terms f^s, f^b, and g^b, the boundary data U^s, U^b, S^s, S^b, P^p, and Z, and the interface data M^u, M^s, M^p, and M^e are chosen such that the exact solution is given by:

$$
\begin{aligned}
u^s &= \begin{bmatrix} \pi x_1 \cos(\pi(x_1 x_2 - t)) + 1 \\ -\pi x_2 \cos(\pi(x_1 x_2 - t)) + 2x_1 \end{bmatrix}, & p^s &= \sin(3x_1)\cos(4(x_2 - t)), \\
u^b &= \begin{bmatrix} \sin(10\pi t)\cos(4(x_1 - t))\cos(3x_2) \\ \sin(10\pi t)\sin(5x_1)\cos(2(x_2 - t)) \end{bmatrix}, & p^p &= \sin(3(x_1 x_2 - t)),
\end{aligned}
$$

Note that $p^b = -\lambda \nabla \cdot u^b + \alpha p^p$ and $z = -\kappa \nabla p^p$.

Cells	$\|u_h^s - u^s\|_{\Omega^s}$	Rate	$\left\|\nabla \cdot u_h^s\right\|_{\Omega^s}$	Rate	$\|z_h - z\|_{\Omega^b}$	Rate	$\|\nabla \cdot (z_h - z)\|_{\Omega^b}$	Rate
$k = 1$								
152	1.9e-02	-	2.4e-15	-	7.1e-04	-	2.3e-03	-
608	4.5e-03	2.1	6.4e-15	-	2.7e-04	1.4	1.3e-03	0.8
2432	1.1e-03	2.0	1.2e-14	-	8.3e-05	1.7	6.5e-04	1.0
9728	2.8e-04	2.0	2.5e-14	-	1.9e-05	2.2	3.3e-04	1.0
38912	6.9e-05	2.0	4.8e-14	-	3.6e-06	2.4	1.6e-04	1.0
$k = 2$								
152	1.6e-03	-	1.3e-14	-	4.4e-05	-	1.0e-04	-
608	2.0e-04	3.0	3.4e-14	-	5.9e-06	2.9	3.3e-05	1.6
2432	2.5e-05	3.0	6.9e-14	-	7.8e-07	2.9	8.3e-06	2.0
9728	3.1e-06	3.0	1.4e-13	-	8.3e-08	3.2	2.1e-06	2.0
38912	3.9e-07	3.0	2.8e-13	-	7.7e-09	3.4	5.2e-07	2.0
$k = 3$								
152	7.6e-05	-	1.8e-13	-	6.6e-06	-	5.1e-06	-
608	4.2e-06	4.2	3.7e-13	-	2.9e-07	4.5	1.2e-06	2.0
2432	2.6e-07	4.0	7.8e-13	-	1.7e-08	4.1	1.6e-07	3.0
9728	1.6e-08	4.0	1.6e-12	-	8.9e-10	4.3	2.0e-08	3.0
38912	1.0e-09	4.0	3.1e-12	-	4.1e-11	4.4	2.4e-09	3.0
Cells	$\|u_h^b - u^b\|_{\Omega^b}$	Rate	$\left\|p_h^s - p^s\right\|_{\Omega^s}$	Rate	$\|p_h^b - p^b\|_{\Omega^b}$	Rate	$\|p_h^p - p^p\|_{\Omega^b}$	Rate
$k = 1$								
152	2.3e-01	-	4.3e-02	-	2.1e+01	-	2.8e-02	-
608	4.4e-02	2.4	2.3e-02	0.9	1.2e+01	0.9	1.5e-02	0.9
2432	1.1e-02	2.0	1.1e-02	1.0	5.9e+00	1.0	7.7e-03	1.0
9728	2.7e-03	2.0	5.7e-03	1.0	2.9e+00	1.0	3.9e-03	1.0
38912	6.8e-04	2.0	2.8e-03	1.0	1.5e+00	1.0	1.9e-03	1.0
$k = 2$								
152	6.8e-03	-	4.0e-03	-	1.4e+00	-	1.2e-03	-
608	2.7e-04	4.7	1.1e-03	1.9	5.8e-01	1.3	3.9e-04	1.6
2432	2.9e-05	3.2	2.6e-04	2.0	1.4e-01	2.0	9.7e-05	2.0
9728	3.3e-06	3.1	6.6e-05	2.0	3.6e-02	2.0	2.4e-05	2.0
38912	3.9e-07	3.1	1.6e-05	2.0	9.0e-03	2.0	6.0e-06	2.0
$k = 3$								
152	2.3e-04	-	2.0e-04	-	7.1e-02	-	3.7e-05	-
608	6.6e-06	5.2	3.1e-05	2.6	1.6e-02	2.2	8.1e-06	2.2
2432	3.8e-07	4.1	3.9e-06	3.0	2.0e-03	3.0	1.0e-06	3.0
9728	2.3e-08	4.0	4.9e-07	3.0	2.5e-04	3.0	1.3e-07	3.0
38912	1.6e-09	3.9	6.1e-08	3.0	3.1e-05	3.0	1.6e-08	3.0

Table 1: Errors and rates of convergence for different polynomial degrees k for the test case described in section 6.1.

Cells	$\|u_h^s - u^s\|_{\Omega^s}$	Rate	$\left\|\nabla \cdot u_h^s\right\|_{\Omega^s}$		$\|z_h - z\|_{\Omega^b}$	Rate	$\|\nabla \cdot (z_h - z)\|_{\Omega^b}$	Rate
$k = 1$								
152	2.5e-02	-	2.7e-15		4.3e-03	-	4.0e-02	-
570	4.3e-03	2.5	3.4e-15		7.7e-04	2.5	6.8e-03	2.5
2346	9.2e-04	2.2	6.1e-15		1.0e-04	2.9	1.1e-03	2.7
9520	2.1e-04	2.2	1.2e-14		1.4e-05	2.9	3.2e-04	1.7
37540	5.0e-05	2.0	2.2e-14		2.5e-06	2.5	1.5e-04	1.1
$k = 2$								
152	1.7e-03	-	1.3e-14		4.3e-03	-	3.9e-02	-
570	1.4e-04	3.6	2.8e-14		7.7e-04	2.5	6.7e-03	2.5
2346	1.2e-05	3.5	5.7e-14		1.0e-04	2.9	8.8e-04	2.9
9520	1.4e-06	3.1	1.1e-13		1.3e-05	2.9	1.2e-04	2.9
37540	1.7e-07	3.1	2.2e-13		1.7e-06	3.0	1.5e-05	3.0
Cells	$\|u_h^b - u^b\|_{\Omega^b}$	Rate	$\left\|p_h^s - p^s\right\|_{\Omega^s}$	Rate	$\|p_h^b - p^b\|_{\Omega^b}$	Rate	$\|p_h^p - p^p\|_{\Omega^b}$	Rate
$k = 1$								
152	2.8e-03	-	4.3e-02	-	6.6e+00	-	6.0e-02	-
570	7.2e-04	2.0	2.2e-02	1.0	3.4e+00	1.0	1.7e-02	1.8
2346	1.7e-04	2.1	1.1e-02	1.0	1.7e+00	1.0	7.4e-03	1.2
9520	4.1e-05	2.1	5.3e-03	1.0	8.2e-01	1.0	3.6e-03	1.0
37540	1.1e-05	2.0	2.7e-03	1.0	4.1e-01	1.0	1.8e-03	1.0
$k = 2$								
152	1.9e-04	-	4.0e-03	-	4.4e-01	-	5.4e-02	-
570	2.5e-05	3.0	9.2e-04	2.1	1.0e-01	2.1	9.7e-03	2.5
2346	2.8e-06	3.1	2.1e-04	2.1	2.6e-02	2.0	1.3e-03	2.9
9520	3.4e-07	3.0	5.2e-05	2.1	6.2e-03	2.0	1.7e-04	2.9
37540	4.4e-08	3.0	1.3e-05	2.0	1.6e-03	2.0	2.2e-05	2.9

Table 2: Errors and rates of convergence for polynomial degrees $k = 1$ and $k = 2$ for the test case described in section 6.2.

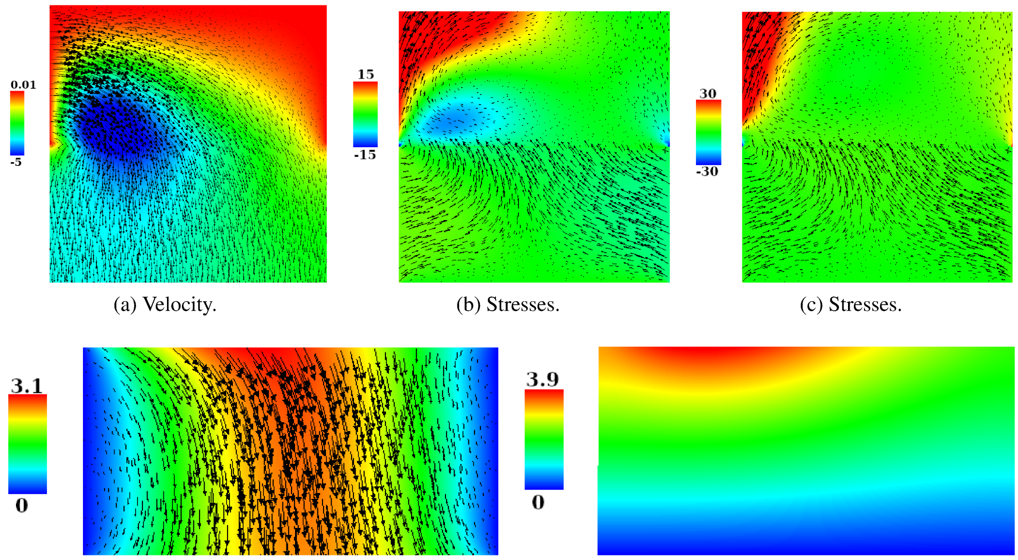
We choose the following parameters: $\mu^s = 10^{-2}$, $\mu^b = 10^{-3}$, $\alpha = 0.2$, $\lambda = 10^2$, $\kappa = 10^{-2}$, $c_0 = 10^{-2}$, and $\gamma = 0.3$. We choose the interior penalty parameters as $\beta^s = \beta^b = 8k^2$, where k is the polynomial degree. To avoid needing to take very small time steps, we implement the two-step Backward Differentiation Formulae (BDF2) time-stepping method. We choose the time step $\Delta t = \frac{1}{10}h^{3/2}$ and consider the time interval $J = [0, 0.01]$.

We present the errors in the L^2-norm and rates of convergence for all unknowns in table 2 for polynomial degrees $k = 1$ and $k = 2$. We observe optimal rates of convergence for all unknowns.

6.3. *Coupling of surface/subsurface flow*

In this final example we consider an example proposed in [23, Section 8.2]. For this we consider the domain $\Omega = (0, 2) \times (-1, 1)$ with $\Omega^s = (0, 2) \times (0, 1)$ and $\Omega^b = (0, 2) \times (-1, 0)$ and the time interval $J = (0, T)$ with $T = 3$. The body forces, source/sink terms, and initial conditions are set as $f^s = 0$, $f^b = 0$, $g^b = 0$, $p_0 = 0$, and $u_0 = 0$. We consider three parameter sets: (1) $(\kappa, c_0, \lambda, \mu^b) = (1, 1, 1, 1)$; (2) $(\kappa, c_0, \lambda, \mu^b) = (10^{-4}, 10^{-4}, 10^6, 1)$; and (3) $(\kappa, c_0, \lambda, \mu^b) = (10^{-4}, 10^{-4}, 10^6, 10^6)$. The remaining parameters are chosen as $\mu^s = 1$, $\alpha = 1$, and $\gamma = 1$.

Let $\Gamma_D^s = \partial\Omega^s \cap \partial\Omega$, $\Gamma_N^b = \Gamma_P^b = \{x \in \partial\Omega^b \,:\, x_2 = -1\}$, and $\Gamma_D^b = \Gamma_F^b = \partial\Omega^b \setminus (\Gamma_I \cup \Gamma_N^b)$. We



(a) Velocity. (b) Stresses. (c) Stresses.

(d) Displacement. (e) Pore pressure.

Figure 1: Test case from section 6.3 with parameter set 1, $(\kappa, c_0, \lambda, \mu^b) = (1, 1, 1, 1)$. Top left: u^s and $z + \partial_t u^b$ (arrows) and u^s_2 and $z_2 + \partial_t u^b_2$ (color). Top middle: $-(\sigma^s_{12}, \sigma^s_{22})^T$ and $-(\sigma^b_{12}, \sigma^b_{22})^T$ (arrows) and $-\sigma^s_{12}$ and $-\sigma^b_{12}$ (color). Top right: $-(\sigma^s_{12}, \sigma^s_{22})^T$ and $-(\sigma^b_{12}, \sigma^b_{22})^T$ (arrows) and $-\sigma^s_{22}$ and $-\sigma^b_{22}$ (color). Bottom left: u^b (arrows) and $|u^b|$ (color). Bottom right: p^p.

impose the following boundary conditions:

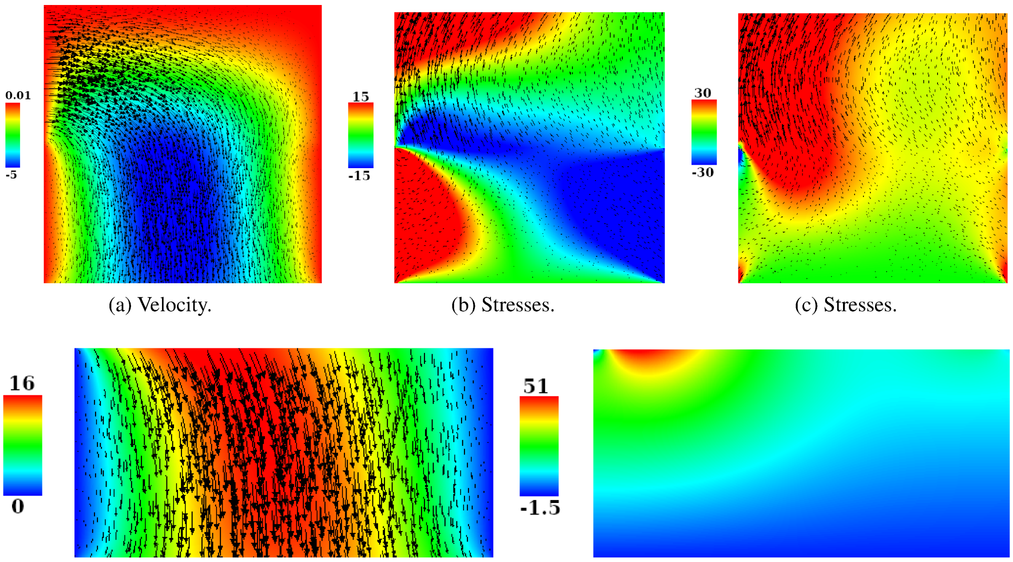
$$
\begin{aligned}
&u^s = \left[-20x_2(x_2 - 1)(2 - x_1), 0\right]^T, && \text{on } \Gamma^s_D,\\
&u^b = 0, \quad z \cdot n = 0, && \text{on } \Gamma^b_D = \Gamma^b_F,\\
&p^p = 0, \quad \sigma^b n = 0, && \text{on } \Gamma^b_N = \Gamma^b_P.
\end{aligned}
$$

We compute the solution on an unstructured simplicial mesh consisting of 9508 elements, using $k = 2$, and a time step of $\Delta t = 0.06$.

We plot the solution obtained with the three different parameter sets in figs. 1 to 3. The results compare well to those obtained by the locking-free method of [23]; the solution does not exhibit locking or oscillations despite Poisson ratio $\nu = 0.4999995$ (for parameter set 2) and despite modeling a very stiff poroelastic medium (parameter set 3). Furthermore, we observe from figs. 1a, 2a and 3a that the second component of the velocity is continuous across the interface, i.e., mass is conserved at the interface. From figs. 1b, 2b and 3b and from figs. 1c, 2c and 3c we observe that $-\sigma^s_{12} = -\sigma^b_{12}$ and $-\sigma^s_{22} = -\sigma^b_{22}$ implying conservation of momentum on the interface.

7. Conclusions

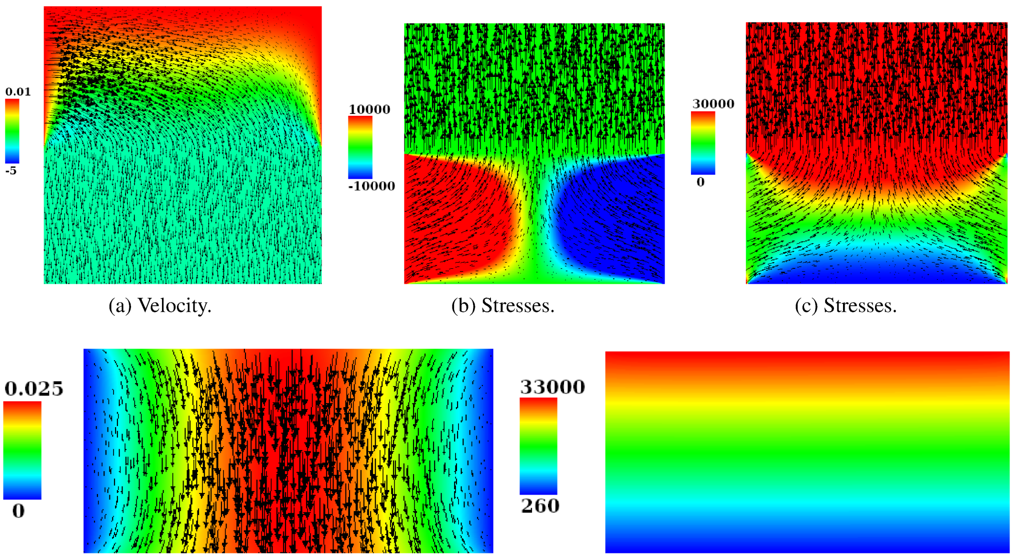
We introduced an HDG method for the coupled Stokes–Biot problem that is provably robust in the incompressible limit, $\lambda \to \infty$ and $c_0 \to 0$. Consistency was shown for the semi-discrete case while well-posedness and a priori error estimates were determined after combining the



(a) Velocity. (b) Stresses. (c) Stresses.

(d) Displacement. (e) Pore pressure.

Figure 2: Test case from section 6.3 with parameter set 2, $(\kappa, c_0, \lambda, \mu^b) = (10^{-4}, 10^{-4}, 10^6, 1)$. Top left: u^s and $z + \partial_t u^b$ (arrows) and u_2^s and $z_2 + \partial_t u_2^b$ (color). Top middle: $-(\sigma_{12}^s, \sigma_{22}^s)^T$ and $-(\sigma_{12}^b, \sigma_{22}^b)^T$ (arrows) and $-\sigma_{12}^s$ and $-\sigma_{12}^b$ (color). Top right: $-(\sigma_{12}^s, \sigma_{22}^s)^T$ and $-(\sigma_{12}^b, \sigma_{22}^b)^T$ (arrows) and $-\sigma_{22}^s$ and $-\sigma_{22}^b$ (color). Bottom left: u^b (arrows) and $|u^b|$ (color). Bottom right: p^p.



(a) Velocity. (b) Stresses. (c) Stresses.

(d) Displacement. (e) Pore pressure.

Figure 3: Test case from section 6.3 with parameter set 3, $(\kappa, c_0, \lambda, \mu^b) = (10^{-4}, 10^{-4}, 10^6, 10^6)$. Top left: u^s and $z + \partial_t u^b$ (arrows) and u_2^s and $z_2 + \partial_t u_2^b$ (color). Top middle: $-(\sigma_{12}^s, \sigma_{22}^s)^T$ and $-(\sigma_{12}^b, \sigma_{22}^b)^T$ (arrows) and $-\sigma_{12}^s$ and $-\sigma_{12}^b$ (color). Top right: $-(\sigma_{12}^s, \sigma_{22}^s)^T$ and $-(\sigma_{12}^b, \sigma_{22}^b)^T$ (arrows) and $-\sigma_{22}^s$ and $-\sigma_{22}^b$ (color). Bottom left: u^b (arrows) and $|u^b|$ (color). Bottom right: p^p.

HDG method with backward Euler time-stepping. Furthermore, we showed that the discrete velocities and displacement are $H(\text{div})$-conforming and that the compressibility equations are satisfied pointwise by the numerical solution on the elements. Mass is conserved pointwise on the elements for the semi-discrete problem (up to the error of the L^2-projection of the source/sink term into the discrete pore pressure space). Finally, numerical examples demonstrate optimal rates of convergence for all unknowns in the L^2-norm and that the numerical method is locking-free.

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Appendix A. Proof of the inf-sup condition eq. (17)

The inf-sup condition was proven for the case of homogeneous Dirichlet boundary conditions on the whole boundary of the domain in [28, Lemma 4.4] and [41, Lemma 1] for the HDG method and [42, Lemma 8] for a variation of the HDG method. Here we generalize these proofs to the case where homogeneous Dirichlet boundary conditions are only posed on part of the domain boundary. The proof proceeds in three steps.

Step 1. Let $\Pi_V^j : [H^1(\Omega^j)]^d \to V_h^j$ be the BDM interpolation operator [10, Section III.3] and define the norm $\|v_h^j\|_{1,h,j}$ as

$$\|v_h^j\|_{1,h,j}^2 := \|\varepsilon(v_h^j)\|_{\Omega^j}^2 + \sum_{F\in\mathcal{F}^j} h_F^{-1} \|[\![v_h^j]\!]\|_F^2 ,$$

where h_F is the radius of the face F. Note that

$$|\!|\!|(v_h, \{v_h\})|\!|\!|_{v,j}^2 = \sum_{K\in\mathcal{T}^j} \left(\|\varepsilon(v_h)\|_K^2 + h_K^{-1} \|v_h - \{v_h\}\|_{\partial K}^2 \right).$$

On boundary facets $\{v_h\} = v_h$ and so

$$\begin{aligned}
\sum_{K\in\mathcal{T}^j} h_K^{-1} \|(v_h - \{v_h\})\|_{\partial K}^2 &= \sum_{F\in\mathcal{F}_{int}^j} \left(h_{K^+}^{-1} \|(v_h^+ - \{v_h\})\|_F^2 + h_{K^-}^{-1} \|(v_h^- - \{v_h\})\|_F^2 \right) \\
&= \sum_{F\in\mathcal{F}_{int}^j} \left(\tfrac{1}{4} h_{K^+}^{-1} \|v_h^+ - v_h^-\|_F^2 + \tfrac{1}{4} h_{K^-}^{-1} \|v_h^- - v_h^+\|_F^2 \right) \\
&= \sum_{F\in\mathcal{F}_{int}^j} \tfrac{1}{4} (h_{K^+}^{-1} + h_{K^-}^{-1}) \|v_h^+ - v_h^-\|_F^2 \\
&\leq C \sum_{F\in\mathcal{F}_{int}^j} h_F^{-1} \|[\![v_h]\!]\|_F^2 ,
\end{aligned}$$

where, assuming shape regularity of the mesh, the inequality is by equivalence of h_F, h_{K^+}, and h_{K^-} where F is a facet shared by K^+ and K^-. Throughout this proof $C > 0$ is a generic constant independent of h. This shows that $|\!|\!|(v_h, \{v_h\})|\!|\!|_{v,j} \leq C \|v_h^j\|_{1,h,j}$. Then, for all $v \in [H^1(\Omega^j)]^d$,

$$|\!|\!|(\Pi_V^j v, \{\Pi_V^j v\})|\!|\!|_{v,j} \leq C \|\Pi_V^j v\|_{1,h,j} \leq C \|v\|_{1,\Omega^j}^2 , \tag{A.1}$$

where the second inequality was shown in the proof of [34, Proposition 10]. We next define

$$\begin{aligned}
[H_{ID,j}^1(\Omega^j)]^d &:= \left\{ v \in [H^1(\Omega^j)]^d \;:\; v = 0 \text{ on } \Gamma_I \cup \Gamma_D^j \right\}, \\
\tilde{\boldsymbol{V}}_h^j &:= \left\{ \boldsymbol{v}_h \in \widetilde{\boldsymbol{V}}_h^j \;:\; v_h \in H(\text{div};\Omega^j),\ \bar{v}_h^j \cdot n = v_h^j \cdot n \text{ on } \partial\Omega^j \right\}.
\end{aligned}$$

Given $|\Gamma_N^j| > 0$, it was shown in [43, Lemma B.1] that there exists a constant $C > 0$ such that for all $q_h \in Q_h^j$ there is a $v_{q_h} \in [H^1_{ID,j}(\Omega^j)]^d$ that satisfies

$$-\nabla \cdot v_{q_h} = q_h \qquad C \|v_{q_h}\|_{1,\Omega^j} \le \left\|q_h\right\|_{\Omega^j}. \tag{A.2}$$

By eqs. (A.1) and (A.2) we note that

$$|||(\Pi_V^j v_{q_h}, \{\Pi_V^j v_{q_h}\})|||_{v,j} \le C \|v_{q_h}\|_{1,\Omega^j} \le C\left\|q_h\right\|_{\Omega^j}.$$

Note also that $(\Pi_V^j v_{q_h}, \{\Pi_V^j v_{q_h}\}) \in \tilde{\tilde{\boldsymbol{V}}}_h^j$. We therefore find that

$$\sup_{0\neq\boldsymbol{v}_h\in\tilde{\tilde{\boldsymbol{V}}}_h^j} \frac{b_h^j((q_h^j,0),\boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||_{v,j}} \ge \frac{b_h^j((q_h^j,0),(\Pi_V^j v_{q_h}, \{\Pi_V^j v_{q_h}\}))}{|||(\Pi_V^j v_{q_h}, \{\Pi_V^j v_{q_h}\})|||_{v,j}} \ge C\,\|q_h^j\|_{\Omega^j}.$$

Step 2. Noting that $(v_h, 0) \in \widetilde{\boldsymbol{V}}_h^j$, there exists a C such that

$$\sup_{\boldsymbol{0}\neq\boldsymbol{v}_h\in\widetilde{\boldsymbol{V}}_h^j} \frac{b_h^j((0,\bar{q}_h),\boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||_{v,j}} \ge \sup_{0\neq v_h\in V_h^j} \frac{b_h^j((0,\bar{q}_h),(v_h,0))}{|||(v_h,0)|||_{v,j}} \ge C|||(0,\bar{q}_h)|||_{q,j} \quad \forall \bar{q}_h \in \bar{Q}_h^j, \tag{A.3}$$

where the second inequality was shown in the proof of [41, Lemma 3]. (Although [41, Lemma 3] assumed quasi-conformity, the result can be extended to shape-regular meshes.)

Step 3. Define $b_1^j((q_h^j,0),\boldsymbol{v}_h) = b_h^j((q_h^j,0),\boldsymbol{v}_h)$ and $b_2^j((0,\bar{q}_h),\boldsymbol{v}_h) = b_h^j((0,\bar{q}_h),\boldsymbol{v}_h)$. Noting that

$$\tilde{\tilde{\boldsymbol{V}}}_h^j = \left\{\boldsymbol{v}_h \in \widetilde{\boldsymbol{V}}_h^j \ : \ b_2^j((0,\bar{q}_h),\boldsymbol{v}_h) = 0\ \forall \bar{q}_h \in \bar{Q}_h^j\right\},$$

the result follows after applying [44, Theorem 3.1].

Appendix B. Proof of the inf-sup condition eq. (18)

The proof is similar to that given in appendix Appendix A. It is given here for completeness. The proof again proceeds in three steps.

Step 1. Let $\Pi_V : [H^1(\Omega)]^d \to V_h$ be the BDM interpolation operator [10, Section III.3] and define the norm $\|v_h\|_{1,h}$ as

$$\|v_h\|_{1,h}^2 := \|\varepsilon(v_h)\|_\Omega^2 + \sum_{F\in\mathcal{F}} h_F^{-1}\,\|[\![v_h]\!]\|_F^2,$$

where h_F is the radius of the face F. As in appendix Appendix A we have that for all $v \in [H^1(\Omega)]^d$,

$$|||(\Pi_V v, \{\Pi_V v\}, \{\Pi_V v\})|||_v \le C\,\|\Pi_V v\|_{1,h} \le C\,\|v\|_{1,\Omega}, \tag{B.1}$$

where, in this proof, $C > 0$ is a generic constant independent of h. Let $\Gamma_D = \Gamma_D^s \cup \Gamma_D^b$ and $\Gamma_N = \Gamma_N^s \cup \Gamma_N^b$ and define

$$[H_D^1(\Omega)]^d := \left\{v \in [H^1(\Omega)]^d \ : \ v = 0 \text{ on } \Gamma_D\right\},$$
$$\tilde{\tilde{\boldsymbol{V}}}_h := \left\{\boldsymbol{v}_h \in \widehat{\boldsymbol{V}}_h \ : \ v_h \in H(\text{div};\Omega),\ \bar{v}_h^s \cdot n = \bar{v}_h^b \cdot n = v_h \cdot n \text{ on } \partial\Omega \cup \Gamma_I\right\}.$$

Given $|\Gamma_N| > 0$ there exists a constant $C > 0$ such that for all $q_h \in Q_h$ there is a $v_{q_h} \in [H_D^1(\Omega)]^d$ that satisfies (see [43, Lemma B.1])

$$-\nabla \cdot v_{q_h} = q_h \qquad C \|v_{q_h}\|_{1,\Omega} \le \left\|q_h\right\|_{\Omega}. \tag{B.2}$$

By eqs. (B.1) and (B.2),

$$|||(\Pi_V v_{q_h}, \{\Pi_V v_{q_h}\}, \{\Pi_V v_{q_h}\})|||_v \le C \|v_{q_h}\|_{1,\Omega} \le C\left\|q_h\right\|_{\Omega}.$$

Since $(\Pi_V v_{q_h}, \{\Pi_V v_{q_h}\}, \{\Pi_V v_{q_h}\}) \in \tilde{\tilde{\boldsymbol{V}}}_h$ we obtain

$$\sup_{0 \neq \boldsymbol{v}_h \in \tilde{\tilde{\boldsymbol{V}}}_h} \frac{b_h((q_h, 0), \boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||_v} \ge \frac{b_h((q_h, 0), (\Pi_V v_{q_h}, \{\Pi_V v_{q_h}\}, \{\Pi_V v_{q_h}\}))}{|||(\Pi_V v_{q_h}, \{\Pi_V v_{q_h}\}, \{\Pi_V v_{q_h}\})|||_v} \ge C \|q_h\|_{\Omega}.$$

Step 2. Noting that $(v_h, 0, 0) \in \widehat{\boldsymbol{V}}_h$, there exists a C such that

$$\sup_{\boldsymbol{0} \neq \boldsymbol{v}_h \in \widehat{\boldsymbol{V}}_h} \frac{b_h((0, \bar{q}_h), \boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||_v} \ge \sup_{0 \neq v_h \in V_h} \frac{b_h((0, \bar{q}_h), (v_h, 0, 0))}{|||(v_h, 0, 0)|||_v} \ge C|||(0, \bar{q}_h)|||_q \quad \forall \bar{q}_h \in \bar{Q}_h, \tag{B.3}$$

where the second inequality was shown in the proof of [41, Lemma 3].

Step 3. Define $b_1((q_h, 0), \boldsymbol{v}_h) = b_h((q_h, 0), \boldsymbol{v}_h)$ and $b_2((0, \bar{q}_h), \boldsymbol{v}_h) = b_h((0, \bar{q}_h), \boldsymbol{v}_h)$. Noting that

$$\begin{aligned}
\tilde{\tilde{\boldsymbol{V}}}_h &= \left\{\boldsymbol{v}_h \in \widehat{\boldsymbol{V}}_h \;:\; v_h^j \in H(\text{div}; \Omega^j),\ \bar{v}_h^j \cdot n = v_h^j \cdot n \text{ on } \partial\Omega \cup \Gamma_I,\ j = s, b\right\}. \\
&= \left\{\boldsymbol{v}_h \in \widehat{\boldsymbol{V}}_h \;:\; b_2((0, \bar{q}_h), \boldsymbol{v}_h) = 0\ \forall \bar{q}_h \in \bar{Q}_h\right\},
\end{aligned}$$

the result follows after applying [44, Theorem 3.1].