

Stability analysis of an age-structured epidemic model with vaccination and standard incidence rate[☆]

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ABSTRACT

Age structure of the host population is a crucial factor in the transmission and control of infectious diseases, since the risk from an infection increases along with age, different age groups interact heterogeneously, vaccination programs focus on specific age groups, and epidemiological data are reported according to ages. In this paper we consider an age-structured epidemic model of the susceptible–exposed–infectious–recovered (SEIR) type with vaccination and standard incidence rate. After establishing the well-posedness of the initial–boundary value problem, we study the existence and stability of the disease-free and endemic steady states based on the basic reproduction number $\mathcal{R}_0$. It is shown that the disease-free steady state is globally asymptotically stable if $\mathcal{R}_0 < 1$, the endemic steady state is unique if $\mathcal{R}_0 > 1$ and is locally asymptotically stable under some additional conditions. Some numerical simulations are presented to illustrate the theoretical results.

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1. Introduction

Since the risk from an infection increases along with age, different age groups interact heterogeneously, vaccination programs focus on specific age groups, and epidemiological data are reported according to ages, age structure of the host population is a crucial factor in the transmission and control of infectious diseases. This is more evident for childhood infectious diseases such chickenpox, hand–foot–mouth disease, measles, mumps, pertussis, smallpox, etc. (Hethcote [1]). Among them, measles is one of the most common acute respiratory infections in children worldwide and is one of the leading causes of death among young children globally, despite the availability of a safe and effective vaccine (WHO [2]).

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Mathematical modeling of measles epidemics started in the mid-eighteenth century with the work of Daniel Bernoulli (Dietz and Heesterbeek [3]). For a brief review of literature on modeling measles, we refer to Huang et al. [4]. Age structure has been considered as one of the most important factors that affect the outcome and consequences of the transmission dynamics of measles. In fact, age has to be understood as either the infection age or the chronological age: the former is the time elapsed since individuals became infected, while the latter is the demographic age of individuals involved in the process. Actually, the work of D. Bernoulli cited above dealt with measles epidemic based on the chronological age of the population, while the pioneer model of Kermack and McKendrick [5–7] was based on the infection age of individuals. Starting in the 1970s, researchers have noticed that the chronological age of the host population plays a crucial role in the transmission process of infectious diseases and have proposed various age-structured models to study the transmission dynamics of childhood diseases, in particular measles, see for instance, Anderson and May [8], Corey and Noymer [9], Greenhalgh [10,11], Halloran et al. [12], Hethcote [13], Huang and Rohani [14], Kang et al. [15], Manfredi and Williams [16], McLean and Anderson [17,18], Schenzle [19], and Tudor [20]. We refer to the monographs of Iannelli [21], Inaba [22], Li et al. [23], and Webb [24] for fundamental theories on age-structured epidemic models.

To calibrate the seasonal measles data from London, Liverpool, New York, and Baltimore, Earn et al. [25] proposed a susceptible–exposed–infectious–recovered (SEIR) model with mean transmission rate to explain both regular cycles and irregular epidemics as the consequences of changes in birth and vaccination rates. Huang et al. [4] used a periodic transmission rate to replace the mean transmission rate in the model of Earn et al. [25], studied the effect of vaccination and seasonality on the transmission dynamics of measles, and employed the periodic SEIR model to simulate the monthly data in China from January 2004 to December 2016. Kang et al. [15] generalized the periodic SEIR epidemic model describing measles in Huang et al. [4] to an age-structured SEIR model with periodic transmission rate, established the well-posedness of the initial–boundary value problem for the age-structured SEIR model, and discussed the existence of time periodic solutions of the model by using a fixed point theorem.

Note that only the existence of a positive periodic steady state was obtained in Kang et al. [15] and the stability remains an open issue. To study the stability of the positive endemic steady state, we drop the assumption on the periodicity of the transmission rate. More specifically, we consider an age-structured SEIR model with demographic age related parameters by dividing the population into four different groups: susceptible, exposed, infectious, and recovered. It is assumed that individuals vaccinated or recovered from the infection would obtain immunity and go to the recovered class directly. Moreover, the natural mortality of individuals, the progression rate from incubation to infection, the recovery rate and the vaccination rate against measles are all related to the age of the population. First we will discuss the existence and uniqueness of solutions of the model. Then we will study the existence and stability of the disease-free and endemic steady states. Finally, we will provide numerical simulations to illustrate our results.

The paper is organized as follows. In Section 2, an age-structured measles model with SEIR structure is presented. In Section 3, the existence and uniqueness of solutions to the initial–boundary problem associated with the model are established. In Section 4, we evaluate the basic reproduction number $\mathcal{R}_0$ and use it as a threshold to determine the existence and stability of disease-free steady state. Global stability of the disease-free steady state is also obtained. In Section 5, we study the existence and stability of the endemic steady state. In Section 6, we provide some numerical simulations to illustrate our results obtained in the previous sections. A brief discussion is given in Section 7.

2. Mathematical modeling

Assume that the population is classified into four different groups: susceptible, exposed, infectious, and recovered. Let $S(a, t)$, $E(a, t)$, $I(a, t)$, and $R(a, t)$ represent the densities of susceptible, exposed, infectious,

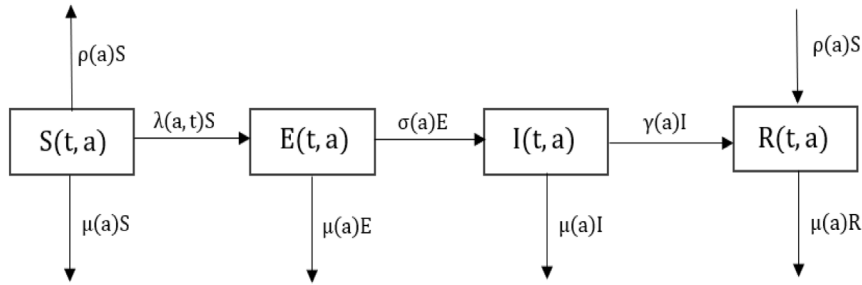


Fig. 2.1. Flowchart of measles transmission in a population.

and recovered individuals at time t with age a , respectively. We consider the following age-structured epidemic model (see the flowchart in Fig. 2.1):

$$\begin{cases} \frac{\partial S}{\partial t} + \frac{\partial S}{\partial a} = -S(t, a) \int_0^{a^+} \beta(a, a') \frac{I(t, a')}{N(t, a')} da' - (\mu(a) + \rho(a))S(t, a), \\ \frac{\partial E}{\partial t} + \frac{\partial E}{\partial a} = S(t, a) \int_0^{a^+} \beta(a, a') \frac{I(t, a')}{N(t, a')} da' - (\sigma(a) + \mu(a))E(t, a), \\ \frac{\partial I}{\partial t} + \frac{\partial I}{\partial a} = \sigma(a)E(t, a) - (\gamma(a) + \mu(a))I(t, a), \\ \frac{\partial R}{\partial t} + \frac{\partial R}{\partial a} = \rho(a)S(t, a) + \gamma(a)I(t, a) - \mu(a)R(t, a) \end{cases} \quad (2.1)$$

with boundary conditions

$$S(t, 0) = A, E(t, 0) = 0, I(t, 0) = 0, R(t, 0) = 0 \quad (2.2)$$

and initial conditions

$$S(0, a) = S_0(a), E(0, a) = E_0(a), I(0, a) = I_0(a), R(0, a) = R_0(a), \quad (2.3)$$

where $\mu(a)$ is the mortality rate, $\rho(a)$ is the vaccination rate, $\sigma(a)$ is the reciprocal of the incubation period, $\gamma(a)$ is the reciprocal of the illness period, $\beta(a, b)$ is the rate at which susceptible individuals of age a are infected by infectious individuals of age b , A is a positive constant, $S_0(a)$, $E_0(a)$, $I_0(a)$ and $R_0(a)$ are nonnegative continuous functions of a . $N(a, t) = S(t, a) + E(t, a) + I(t, a) + R(t, a)$ denotes the total population, and a^+ is the maximum age that an individual may reach, which is finite.

In this paper, we will study the existence and stability of both disease-free and endemic steady states in (2.1) (see Zou et al. [26] and Li et al. [27]). For this purpose, we make the following assumptions.

Assumption 2.1. We assume that

- (i) There is a certain initial population size and there is no immigration and migration;
- (ii) The population is homogeneously mixed and population activities are free from outside interference;
- (iii) $\mu(\cdot)$ is locally integrable and $\int_0^{a^+} \mu(a) da = +\infty$; $\rho(\cdot)$, $\sigma(\cdot)$, $\gamma(\cdot) \in L_+^\infty(0, a^+)$, $\beta(\cdot, \cdot) \in L_+^\infty((0, a^+) \times (0, a^+))$, all these functions are extended by zero outside of the interval $[0, a^+]$;
- (iv) $\beta(a, b) = k(a)\tilde{\beta}(b)$, where $k(a)$ is the age-specific (average) probability of becoming infected through contact with infectious individuals of age a and $\tilde{\beta}(b)$ denotes the age-specific per-capita contact rate.

Lemma 2.2. If the initial function $N_0(a)$ is bounded, then the total population $N(t, a)$ is ultimately bounded.

Proof. By adding the equations in model (2.1), we obtain the following equation for the total population $N(t, a)$:

$$\frac{\partial N}{\partial t} + \frac{\partial N}{\partial a} = -\mu(a)N(t, a) \quad (2.4)$$

with boundary condition

$$N(t, 0) = A \quad (2.5)$$

and initial conditions

$$N(0, a) = S_0(a) + E_0(a) + I_0(a) + R_0(a) \triangleq N_0(a). \quad (2.6)$$

Integrating Eq. (2.4) along the characteristic lines, we obtain

$$N(t, a) = \begin{cases} A\pi(a), & a < t, \\ N_0(a-t)\frac{\pi(a)}{\pi(a-t)}, & a \geq t, \end{cases} \quad (2.7)$$

where

$$\pi(a) = e^{-\int_0^a \mu(s) ds}.$$

Obviously, the total population $N(t, a)$ is ultimately bounded if $N_0(a)$ is bounded. ■

In order to simplify the initial-boundary value problem (IBVP) (2.1)–(2.3), we let

$$s(t, a) = \frac{S(t, a)}{N(t, a)}, \quad e(t, a) = \frac{E(t, a)}{N(t, a)}, \quad i(t, a) = \frac{I(t, a)}{N(t, a)}, \quad r(t, a) = \frac{R(t, a)}{N(t, a)}.$$

Then system (2.1)–(2.3) becomes

$$\begin{cases} \frac{\partial s}{\partial t} + \frac{\partial s}{\partial a} = -\lambda(t, a)s(t, a) - \rho(a)s(t, a), \\ \frac{\partial e}{\partial t} + \frac{\partial e}{\partial a} = \lambda(t, a)s(t, a) - \sigma(a)e(t, a), \\ \frac{\partial i}{\partial t} + \frac{\partial i}{\partial a} = \sigma(a)e(t, a) - \gamma(a)i(t, a), \\ \frac{\partial r}{\partial t} + \frac{\partial r}{\partial a} = \rho(a)s(t, a) + \gamma(a)i(t, a) \end{cases} \quad (2.8)$$

with boundary conditions

$$s(t, 0) = 1, \quad e(t, 0) = 0, \quad i(t, 0) = 0, \quad r(t, 0) = 0 \quad (2.9)$$

and initial conditions

$$\begin{cases} s(0, a) = \frac{S_0(a)}{N_0(a)} \triangleq s_0(a), \quad e(0, a) = \frac{E_0(a)}{N_0(a)} \triangleq e_0(a), \\ i(0, a) = \frac{I_0(a)}{N_0(a)} \triangleq i_0(a), \quad r(0, a) = \frac{R_0(a)}{N_0(a)} \triangleq r_0(a), \end{cases} \quad (2.10)$$

where

$$\lambda(t, a) = k(a) \int_0^{a^+} \tilde{\beta}(a')i(t, a') da'.$$

Since $s(t, a) + e(t, a) + i(t, a) + r(t, a) = 1$, we have $s(t, a) = 1 - e(t, a) - i(t, a) - r(t, a)$, then the IBVP (2.8)–(2.10) can be reduced to a three-equation system for (e, i, r) as follows:

$$\begin{cases} \frac{\partial e}{\partial t} + \frac{\partial e}{\partial a} = \lambda(t, a)(1 - e(t, a) - i(t, a) - r(t, a)) - \sigma(a)e(t, a), \\ \frac{\partial i}{\partial t} + \frac{\partial i}{\partial a} = \sigma(a)e(t, a) - \gamma(a)i(t, a), \\ \frac{\partial r}{\partial t} + \frac{\partial r}{\partial a} = \rho(a)(1 - e(t, a) - i(t, a) - r(t, a)) + \gamma(a)i(t, a) \end{cases} \quad (2.11)$$

with boundary conditions

$$e(t, 0) = 0, \quad i(t, 0) = 0, \quad r(t, 0) = 0 \quad (2.12)$$

and initial conditions

$$e(0, a) = e_0(a), \quad i(0, a) = i_0(a), \quad r(0, a) = r_0(a). \quad (2.13)$$

In the following we will analyze the IBVP (2.11)–(2.12).

3. Well-posedness

In order to consider the existence and uniqueness of solutions in IBVP (2.11)–(2.13), we rewrite it as an abstract Cauchy problem (Inaba [28]). Define a Banach space $\mathbf{X} = L^1(0, a^+) \times L^1(0, a^+) \times L^1(0, a^+)$ with the norm $\|\phi\| = \sum_{i=1}^3 \|\phi_i\|$ for $\phi = (\phi_1, \phi_2, \phi_3)^T \in \mathbf{X}$, where $\|\phi_i\| = \int_0^{a^+} |\phi_i| \, da$.

Let $A : D(A) \subset \mathbf{X} \rightarrow \mathbf{X}$ be a linear operator defined by

$$\begin{aligned} (A\phi)(a) &= \left(-\frac{d\phi_1(a)}{da}, -\frac{d\phi_2(a)}{da}, -\frac{d\phi_3(a)}{da} \right)^T, \\ D(A) &= \{ \phi \in \mathbf{X} \mid \phi_i \in W^{1,1}(0, a^+), \phi(0) = (0, 0, 0)^T \}, \end{aligned} \quad (3.1)$$

where $W^{1,1}(0, a^+)$ denotes the Sobolev space of all absolutely continuous functions on $(0, a^+)$. Assume that $\beta(a, a') \in L_+^\infty((0, a^+) \times (0, a^+))$ and define a nonlinear operator $F : \mathbf{X} \rightarrow \mathbf{X}$ as

$$(F\phi)(a) = \begin{pmatrix} (P\phi_2)(a)(1 - \phi_1(a) - \phi_2(a) - \phi_3(a)) - \sigma(a)\phi_1(a) \\ \sigma(a)\phi_1(a) - \gamma(a)\phi_2(a) \\ \rho(a)(1 - \phi_1(a) - \phi_2(a) - \phi_3(a)) + \gamma(a)\phi_2(a) \end{pmatrix},$$

where

$$\lambda(\cdot) = \int_0^{a^+} \beta(\cdot, a')\phi_2(a') \, da' \triangleq (P\phi_2)(\cdot) \in L^\infty(0, a^+).$$

Let $u(t) = (e(t, \cdot), i(t, \cdot), r(t, \cdot))^T \in \mathbf{X}$. Under the operation of the operator A and F defined above, the above partial differential equations (2.11)–(2.13) can be rewritten as an abstract Cauchy problem as follows

$$\frac{du(t)}{dt} = Au(t) + F(u(t)) \quad (3.2)$$

with initial conditions

$$u(0) = u_0, \quad u_0 = (e_0, i_0, r_0)^T \in \mathbf{X}.$$

We have the following lemma.

Lemma 3.1. *The linear operator A defined by (3.1) is the infinitesimal generator of a C_0 -semigroup $\{T(t)\}_{t \geq 0} = \{e^{tA}\}_{t \geq 0}$.*

Proof. Recall that the Hille–Yosida Theorem (see Magal and Ruan [29]) states that a linear operator $A : D(A) \subset \mathbf{X} \rightarrow \mathbf{X}$ is the infinitesimal generator of a C_0 -semigroup $\{T(t)\}_{t \geq 0}$ if and only if A is a Hille–Yosida operator with dense domain (i.e. $\overline{D(A)} = \mathbf{X}$), which means that there exist two constants $M \geq 1, \omega \in \mathbb{R}$, such that $(\lambda I - A)$ is a bijection from $D(A)$ to X and $(\lambda I - A)^{-1}$ is bounded from X into itself; Moreover, $(\omega, +\infty) \subset \rho(A)$, where $\rho(A)$ is the resolvent set of A , and

$$\|(\lambda I - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}, \quad \forall \lambda > \omega, \quad \forall n \geq 1.$$

Firstly, since $D(A) = \{\phi \in \mathbf{X} | \phi_i \in W^{1,1}(0, a^+), \phi(0) = (0, 0, 0)^T\}$, it is clear that $\overline{D(A)} = \mathbf{X}$. Secondly, consider the following abstract Cauchy problem

$$\begin{cases} \frac{du}{dt} = Au(t), & t \geq 0, \\ u(0) = \phi \in \mathbf{X}, \end{cases}$$

in which the linear operator A is given by

$$A\phi = -\phi', \quad \phi = (\phi_1, \phi_2, \phi_3)^T \in \mathbf{X}, \quad \|\phi\| = \sum_{i=1}^3 \int_0^{a^+} |\phi_i| da.$$

There exists $\omega = 0, M = 1$, such that $\forall \phi \in D(A), \psi \in \mathbf{X}$, for each $\lambda > 0$ and almost all $a \in (0, a^+)$, we have

$$\begin{aligned} (\lambda I - A)\phi(a) = \psi(a) &\iff \lambda\phi(a) + \phi'(a) = \psi(a) \iff \phi(a) = \int_0^a e^{-\lambda(a-s)} \psi(s) ds \\ &\iff (\lambda I - A)^{-1}\psi(a) = \phi(a) = \int_0^a e^{-\lambda(a-s)} \psi(s) ds, \end{aligned}$$

which means that $(\lambda I - A)$ is a bijection from $D(A)$ to X and

$$\begin{aligned} \|(\lambda I - A)^{-1}\psi\| &\leq \int_0^{a^+} \int_0^a e^{-\lambda(a-s)} \psi(s) ds da \\ \iff \|(\lambda I - A)^{-1}\| &\leq \int_s^{a^+} e^{-\lambda(a-s)} da \leq \frac{1}{\lambda} \\ \iff \|(\lambda I - A)^{-n}\| &= \|((\lambda I - A)^{-1})^n\| \leq \|(\lambda I - A)^{-1}\|^n \leq \frac{1}{\lambda^n}. \end{aligned}$$

By Hille–Yosida Theorem, A is the infinitesimal generator of a C_0 -semigroup $\{T(t)\}_{t \geq 0}$. ■

Lemma 3.2. *The nonlinear operator F of system (3.2) is Lipschitz continuous in $\mathbf{X}$.*

Proof. By Assumption 2.1(iii) in Section 2, this conclusion is obvious. ■

From Lemmas 3.1 and 3.2, we have the following result (see Proposition 4.16 in Webb [24]).

Lemma 3.3. *For system (3.2) and each $u_0 \in \mathbf{X}$, there exist a maximum existence interval $[0, t_0)$ and a unique continuous mild solution $t \rightarrow u(t, u_0) \in \mathbf{X}$ from $[0, t_0)$ to $\mathbf{X}$, such that*

$$u(t, u_0) = e^{tA}u_0 + \int_0^t e^{(t-s)A}F(u(s, u_0)) ds,$$

and either $t_0 = +\infty$ or $\lim_{t \rightarrow t_0-} \|u(t, u_0)\| = \infty$. Moreover, if $u_0 \in D(A)$, then $u(t, u_0) \in D(A)$ for $0 \leq t < t_0$ and the function $t \rightarrow u(t, u_0)$ is continuously differentiable and satisfies (3.2).

Let

$$\Omega = \{(e, i, r) \in \mathbf{X} | 0 \leq e + i + r \leq 1\}.$$

We have the following results.

Lemma 3.4. *There exists a number $\alpha \in (0, 1)$ such that*

$$(I + \alpha F)\Omega \subset \Omega.$$

Proof. If we define the vector $v = (v_1, v_2, v_3)^T$ by

$$(I + \alpha F)(u_1, u_2, u_3)^T = (v_1, v_2, v_3)^T, \quad (3.3)$$

then we have

$$v_1(a) + v_2(a) + v_3(a) = \alpha(Pu_2 + \rho)(1 - u_1(a) - u_2(a) - u_3(a)) + (u_1(a) + u_2(a) + u_3(a)).$$

Set $\lambda^+ = \sup \lambda$ and $\rho^+ = \sup \rho$ and choose $\alpha : \alpha \leq (\lambda^+ + \rho^+)^{-1}$. Then we have $v_1(a) + v_2(a) + v_3(a) \leq 1$. Moreover, set $\sigma^+ := \sup \sigma$, $\gamma^+ := \sup \gamma$ and choose α such that $\alpha\sigma^+ \leq 1$ and $\alpha\gamma^+ \leq 1$. Thus we have $v_1 \geq 0$ and $v_2 \geq 0$. Therefore, if we choose

$$0 < \alpha < \min \left\{ \frac{1}{\lambda^+ + \rho^+}, \frac{1}{\sigma^+}, \frac{1}{\gamma^+} \right\},$$

then we have the desired result $(I + \alpha F)\Omega \subset \Omega$. ■

Now following the method of Busenberg et al. [30], we can rewrite the Cauchy problem (3.2) as follows:

$$\frac{du(t)}{dt} = \left(A - \frac{1}{\alpha} I \right) u(t) + \frac{1}{\alpha} (I + \alpha F)u(t), \quad u(0) = u_0, \quad (3.4)$$

where α is chosen such that (3.3) holds. The mild solution of this problem is then given by the variation of constants formula:

$$u(t) = e^{-\frac{1}{\alpha}t} e^{tA} u_0 + \int_0^t e^{-\frac{1}{\alpha}(t-s)} e^{(t-s)A} [u(s) + \alpha F(u(s))] ds.$$

The mild solution defines a strongly continuous semigroup $\{U(t)\}_{t \geq 0}$ by $U(t)u_0 = u(t)$. Define an iterative sequence by

$$u^0(t) = u_0, \quad u^{n+1}(t) = e^{-\frac{1}{\alpha}t} e^{tA} u_0 + \int_0^t e^{-\frac{1}{\alpha}(t-s)} e^{(t-s)A} [u^n(s) + \alpha F(u^n(s))] ds.$$

If $u_n \in \Omega$, it follows that $e^{tA} u_0, e^{(t-s)A} [u^n(s) + \alpha F(u^n(s))] \in \Omega$. Hence, $u^{n+1} \in \Omega$ because it is the convex sum of two elements of the convex set Ω . It follows from the Lipschitz continuity that the iterative sequence $\{u_n\}$ converges uniformly to the mild solution $U(t)u_0 \in \Omega$. Thus, we have the following existence and uniqueness result.

Theorem 3.5. *The abstract Cauchy problem (3.2) has a unique global classical solution on $\mathbf{X}$ for initial data $u_0 \in \Omega \cap D(A)$.*

Therefore, we can conclude that system (2.11)–(2.13) has a unique positive global solution with respect to the positive initial data.

4. Existence and stability of the disease-free steady state

Since $s(t, a) + e(t, a) + i(t, a) + r(t, a) = 1$, we have $r(t, a) = 1 - s(t, a) - e(t, a) - i(t, a)$, then the IBVP (2.8)–(2.10) can be reduced to a three-equation system for (s, e, i) as follows:

$$\begin{cases} \frac{\partial s}{\partial t} + \frac{\partial s}{\partial a} = -\lambda(t, a)s(t, a) - \rho(a)s(t, a), \\ \frac{\partial e}{\partial t} + \frac{\partial e}{\partial a} = \lambda(a)s(t, a) - \sigma(a)e(t, a), \\ \frac{\partial i}{\partial t} + \frac{\partial i}{\partial a} = \sigma(a)e(t, a) - \gamma(a)i(t, a) \end{cases} \quad (4.1)$$

with boundary conditions

$$s(t, 0) = 1, \quad e(t, 0) = 0, \quad i(t, 0) = 0 \quad (4.2)$$

and initial conditions

$$s(0, a) = s_0(a), \quad e(0, a) = e_0(a), \quad i(0, a) = i_0(a). \quad (4.3)$$

In the following we will analyze the IBVP (4.1)–(4.3)

In this section, we investigate the existence and stability of the disease-free steady state. Since the steady state of system (4.1)–(4.3) is independent of time t , it has the form $(s(a), e(a), i(a))$ and satisfies the following time-independent system of ordinary differential equations:

$$\begin{cases} \frac{ds}{da} = -\lambda(a)s(a) - \rho(a)s(a), \\ \frac{de}{da} = \lambda(a)s(a) - \sigma(a)e(a), \\ \frac{di}{da} = \sigma(a)e(a) - \gamma(a)i(a) \end{cases} \quad (4.4)$$

with initial value conditions

$$s(0) = 1, \quad e(0) = 0, \quad i(0) = 0,$$

where

$$\lambda(a) = k(a) \int_0^{a^+} \tilde{\beta}(a') i(a') da'.$$

For the disease-free steady state $E^0 = (s^0(a), e^0(a), i^0(a))$, it is clear that $e^0(a) = i^0(a) = 0$, so $\lambda(a) = 0$ and $s^0(a)$ satisfies the following initial value problem:

$$\begin{cases} \frac{ds}{da} = -\rho(a)s(a), \\ s(0) = 1, \end{cases}$$

whose solution is $s^0(a) = e^{-\int_0^a \rho(s) ds}$, so the disease-free steady state exists and is unique.

4.1. Local stability of the disease-free steady state

To study the local stability of the disease-free steady state $E^0 = (s^0(a), 0, 0)$, it suffices to calculate the linearized system of (4.1)–(4.3) at E^0 . We first make a translation transformation as follows

$$\bar{s}(t, a) = s(t, a) - s^0(a), \quad \bar{e}(t, a) = e(t, a) - e^0(a), \quad \bar{i}(t, a) = i(t, a) - i^0(a).$$

Then system (4.1)–(4.3) becomes

$$\begin{cases} \frac{\partial \bar{s}}{\partial t} + \frac{\partial \bar{s}}{\partial a} = -\bar{\lambda}(t, a)\bar{s}(t, a) - \rho(a)\bar{s}(t, a) - \bar{\lambda}(t, a)s^0(a), \\ \frac{\partial \bar{e}}{\partial t} + \frac{\partial \bar{e}}{\partial a} = \bar{\lambda}(t, a)\bar{s}(t, a) - \sigma(a)\bar{e}(t, a) + \bar{\lambda}(t, a)s^0(a), \\ \frac{\partial \bar{i}}{\partial t} + \frac{\partial \bar{i}}{\partial a} = \sigma(a)\bar{e}(t, a) - \gamma(a)\bar{i}(t, a) \end{cases}$$

with boundary conditions

$$\bar{s}(t, 0) = 0, \quad \bar{e}(t, 0) = 0, \quad \bar{i}(t, 0) = 0,$$

where

$$\bar{\lambda}(t, a) = k(a) \int_0^{a^+} \tilde{\beta}(a') \bar{i}(t, a') da'.$$

The linearized part of above system is

$$\begin{cases} \frac{\partial \bar{s}}{\partial t} + \frac{\partial \bar{s}}{\partial a} = -\rho(a)\bar{s}(t, a) - \bar{\lambda}(t, a)s^0(a), \\ \frac{\partial \bar{e}}{\partial t} + \frac{\partial \bar{e}}{\partial a} = -\sigma(a)\bar{e}(t, a) + \bar{\lambda}(t, a)s^0(a), \\ \frac{\partial \bar{i}}{\partial t} + \frac{\partial \bar{i}}{\partial a} = \sigma(a)\bar{e}(t, a) - \gamma(a)\bar{i}(t, a) \end{cases} \quad (4.5)$$

with boundary conditions

$$\bar{s}(t, 0) = 0, \bar{e}(t, 0) = 0, \bar{i}(t, 0) = 0.$$

Now, we consider the following nonzero exponential solution of system (4.5):

$$\bar{s}(t, a) = \bar{s}(a)e^{\lambda t}, \bar{e}(t, a) = \bar{e}(a)e^{\lambda t}, \bar{i}(t, a) = \bar{i}(a)e^{\lambda t}.$$

Then $\bar{s}(a), \bar{e}(a), \bar{i}(a)$ and λ satisfy the following ordinary differential equations

$$\begin{cases} \frac{d\bar{s}}{da} = -\lambda\bar{s}(a) - \rho(a)\bar{s}(a) - \hat{\lambda}(a)s^0(a), \\ \frac{d\bar{e}}{da} = -\lambda\bar{e}(a) - \sigma(a)\bar{e}(a) + \hat{\lambda}(a)s^0(a), \\ \frac{d\bar{i}}{da} = -\lambda\bar{i}(a) + \sigma(a)\bar{e}(a) - \gamma(a)\bar{i}(a) \end{cases} \quad (4.6)$$

with initial value conditions

$$\bar{s}(0) = 0, \bar{e}(0) = 0, \bar{i}(0) = 0,$$

where

$$\hat{\lambda}(a) = k(a) \int_0^{a+} \tilde{\beta}(a')\bar{i}(a') da'.$$

Letting $\Lambda = \int_0^{a+} \tilde{\beta}(a)\bar{i}(a) da$, we have $\hat{\lambda}(a) = \Lambda k(a)$. The solutions $\bar{e}(a)$ and $\bar{i}(a)$ of system (4.6) can be solved as

$$\begin{cases} \bar{e}(a) = \Lambda \int_0^a e^{-\int_\tau^a (\lambda + \sigma(s)) ds} k(\tau)s^0(\tau) d\tau, \\ \bar{i}(a) = \Lambda \int_0^a \int_0^\eta e^{-\int_\eta^a (\lambda + \gamma(s)) ds} e^{-\int_\tau^\eta (\lambda + \sigma(s)) ds} \sigma(\eta)k(\tau)s^0(\tau) d\tau d\eta. \end{cases}$$

Substituting the expression of $\bar{i}(a)$ into Λ , we have

$$\Lambda = \int_0^{a+} \tilde{\beta}(a) \Lambda \int_0^a \int_0^\eta e^{-\int_\eta^a (\lambda + \gamma(s)) ds} e^{-\int_\tau^\eta (\lambda + \sigma(s)) ds} \sigma(\eta)k(\tau)s^0(\tau) d\tau d\eta da,$$

which yields the following characteristic equation about λ

$$1 = \int_0^{a+} \tilde{\beta}(a) \int_0^a \int_0^\eta e^{-\int_\eta^a (\lambda + \gamma(s)) ds} e^{-\int_\tau^\eta (\lambda + \sigma(s)) ds} \sigma(\eta)k(\tau)s^0(\tau) d\tau d\eta da. \quad (4.7)$$

Denoting the right-hand side of Eq. (4.7) by $F(\lambda)$; i.e.,

$$F(\lambda) \triangleq \int_0^{a+} \tilde{\beta}(a) \int_0^a \int_0^\eta e^{-\int_\eta^a (\lambda + \gamma(s)) ds} e^{-\int_\tau^\eta (\lambda + \sigma(s)) ds} \sigma(\eta)k(\tau)s^0(\tau) d\tau d\eta da,$$

we can define the basic reproduction number as $\mathcal{R}_0 = F(0)$, or explicitly as

$$\mathcal{R}_0 = \int_0^{a+} \tilde{\beta}(a) \int_0^a \int_0^\eta e^{-\int_\eta^a \gamma(s) ds} e^{-\int_\tau^\eta \sigma(s) ds} \sigma(\eta)k(\tau)s^0(\tau) d\tau d\eta da. \quad (4.8)$$

Remark 4.1. Recall that $k(a)$ is the probability of infection, $\sigma(a)$ is the reciprocal of the incubation period, $s^0(a)$ is the initial susceptible population size with age a . Thus, we can see that

$$\underbrace{\int_0^\eta \underbrace{e^{-\int_\tau^\eta \sigma(s) ds}}_{\text{probability of exposedness after incubation}} \underbrace{k(\tau)}_{\text{probability of infection}} \underbrace{s^0(\tau)}_{\text{initial susceptibles}} d\tau}_{\text{number of exposed individuals that one infected individual produces after the incubation period}}$$

represents the number of exposed individuals that one infected individual produces after the incubation period. Next note that $\gamma(a)$ is the reciprocal of the illness period so that $e^{-\int_\eta^a \gamma(s) ds}$ is the probability of exposed at η becoming infected at a , so

$$\underbrace{\int_0^a \underbrace{e^{-\int_\eta^a \gamma(s) ds}}_{\text{probability of being infected after exposure}} \underbrace{\sigma(\eta)}_{\text{rate of becoming infected}} \underbrace{\int_0^\eta e^{-\int_\tau^\eta \sigma(s) ds} k(\tau) s^0(\tau) d\tau}_{\text{number of exposed individuals after incubation period}} d\eta}_{\text{number of infected individuals that one infected individual produces after the illness period}}$$

denotes the number of infected individuals that one infected individual produces after the illness period. Finally recall that $\lambda(a) = k(a) \int_0^{a+} \tilde{\beta}(a') i(a') da'$ describes the infection force in which $\tilde{\beta}(a')$ is the infection rate by infectious individuals of age a' , we have

$$\mathcal{R}_0 = \underbrace{\int_0^{a+} \tilde{\beta}(a) \underbrace{\int_0^a e^{-\int_\eta^a \gamma(s) ds} \sigma(\eta) \underbrace{\int_0^\eta e^{-\int_\tau^\eta \sigma(s) ds} k(\tau) s^0(\tau) d\tau}_{\text{number of exposed individuals after incubation period}} d\eta}_{\text{number of infected individuals after illness period}} da}_{\text{number of infections that one infected individuals produces after the infectious period}}$$

which is the (average) number of infections that one infected individual produces after the infectious period.

Remark 4.2. Since the basic assumption of a finite maximum age a^+ holds, we know that the spectrum of the abstract Cauchy problem (3.2) consists only of the point spectrum and it coincides with the set of roots of the characteristic equation (4.7), the rigorous proof can be followed by Webb [31]. Thus, in order to know the local stability of the disease-free steady state E^0 , it is enough for us to discuss the signs of the real parts of the characteristic roots of (4.7).

For the local stability of the disease-free steady state E^0 , we have the following result.

Theorem 4.3. Under Assumption 2.1, the disease-free steady state E^0 is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.

Proof. From the expression of $F(\lambda)$, we can acquire some basic properties about $F(\lambda)$ for $\lambda \in \mathbb{R}$:

$$F'(\lambda) < 0, \lim_{\lambda \rightarrow -\infty} F(\lambda) = +\infty, \lim_{\lambda \rightarrow +\infty} F(\lambda) = 0.$$

Then there exists a unique real root λ^* such that $F(\lambda^*) = 1$ (see Fig. 4.1).

Case 1: $F(\lambda) = 1$ has only one root. In this case the root must be λ^* . If $\mathcal{R}_0 < 1$, i.e., $F(0) < 1$, then $\lambda^* < 0$ as showed in Fig. 4.1. Therefore $F(\lambda) = 1$ has a unique negative real root λ^* if $\mathcal{R}_0 < 1$.

Case 2: $F(\lambda) = 1$ has complex roots. Let $\lambda = \alpha + i\beta$ be a complex root satisfying $F(\lambda) = 1$. Then $\text{Re}F(\lambda) = 1, \text{Im}F(\lambda) = 0$ and

$$\text{Re}(e^\lambda) = \text{Re}(e^{\alpha+i\beta}) = \text{Re}[e^\alpha(\cos \beta + i \sin \beta)] = e^\alpha \cos \beta \leq e^\alpha = e^{\text{Re} \lambda}.$$

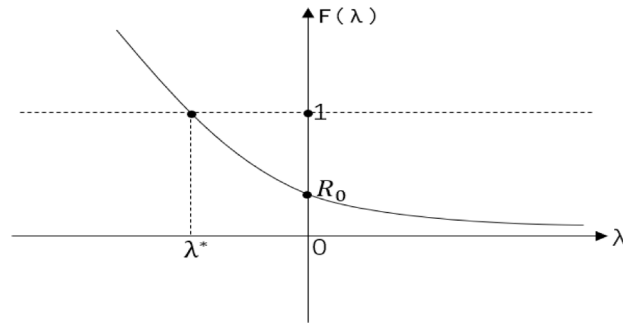


Fig. 4.1. The approximate graph of $F(\lambda)$.

From (4.7), we have

$$\begin{aligned} 1 = \operatorname{Re} F(\lambda) &= \int_0^{a^+} \tilde{\beta}(a) \int_0^a \int_0^\eta \operatorname{Re}(e^{-\lambda(a-\tau)}) e^{-\int_\eta^a \gamma(s) ds} e^{-\int_\tau^\eta \sigma(s) ds} \sigma(\eta) k(\tau) s^0(\tau) d\tau d\eta da \\ &\leq \int_0^{a^+} \tilde{\beta}(a) \int_0^a \int_0^\eta e^{-[(\operatorname{Re} \lambda)(a-\tau)]} e^{-\int_\eta^a \gamma(s) ds} e^{-\int_\tau^\eta \sigma(s) ds} \sigma(\eta) k(\tau) s^0(\tau) d\tau d\eta da \\ &= F(\operatorname{Re} \lambda). \end{aligned}$$

Hence, we have

$$1 \leq F(\operatorname{Re} \lambda) \iff F(\lambda^*) \leq F(\operatorname{Re} \lambda) \iff \operatorname{Re} \lambda \leq \lambda^* < 0,$$

which means that the complex root must have negative real part if $F(\lambda) = 1$ has a complex root.

From the above analysis we know that the root of $F(\lambda) = 1$ always has negative real part if $\mathcal{R}_0 < 1$. Thus, the disease-free steady state E^0 is locally asymptotically stable if $\mathcal{R}_0 < 1$.

Conversely, $\lambda^* > 0$ if $\mathcal{R}_0 > 1$; i.e., the disease-free steady state E^0 is unstable if $\mathcal{R}_0 > 1$. ■

4.2. Global stability of the disease-free steady state

In this section, we discuss global stability of the disease-free steady state E^0 when $\mathcal{R}_0 < 1$ and have the following result.

Theorem 4.4. Under Assumption 2.1 the disease-free steady state E^0 is globally asymptotically stable if $\mathcal{R}_0 < 1$.

Proof. We need to show that

$$\begin{cases} \lim_{t \rightarrow \infty} s(t, a) = s^0(a) = e^{-\int_0^a \rho(s) ds}, \\ \lim_{t \rightarrow \infty} e(t, a) = e^0(a) = 0, \\ \lim_{t \rightarrow \infty} i(t, a) = i^0(a) = 0. \end{cases} \quad (4.9)$$

Integrating the first equation of system (4.1) along the characteristic lines, we get

$$s(t, a) = \begin{cases} s_0(a-t) e^{-\int_0^t (\lambda(s, a-t+s) + \rho(a-t+s)) ds}, & t \leq a, \\ e^{-\int_0^a (\lambda(t-a+s, s) + \rho(s)) ds}, & t > a. \end{cases} \quad (4.10)$$

Using the same method, we can get

$$e(t, a) = \begin{cases} e_0(a-t)e^{-\int_0^t \sigma(a-t+s) ds} \\ \quad + \int_0^t e^{-\int_\tau^t \sigma(a-t+s) ds} \lambda(\tau, a-t+\tau) s(\tau, a-t+\tau) d\tau, & t \leq a, \\ \int_0^a e^{-\int_\tau^a \sigma(s) ds} \lambda(t-a+\tau, \tau) s(t-a+\tau, \tau) d\tau, & t > a \end{cases} \quad (4.11)$$

and

$$i(t, a) = \begin{cases} i_0(a-t)e^{-\int_0^t \gamma(a-t+s) ds} \\ \quad + \int_0^t e^{-\int_\tau^t \gamma(a-t+s) ds} \sigma(a-t+\tau) e(\tau, a-t+\tau) d\tau, & t \leq a, \\ \int_0^a e^{-\int_\tau^a \gamma(s) ds} \sigma(\tau) e(t-a+\tau, \tau) d\tau, & t > a. \end{cases} \quad (4.12)$$

Since we want to study the solutions when $t \rightarrow \infty$, it suffices to study $s(t, a)$, $e(t, a)$ and $i(t, a)$ when $t > a$. Substituting the expression of $e(t, a)$ into $i(t, a)$ when $t > a$, we have

$$i(t, a) = \int_0^a e^{-\int_\tau^a \gamma(s) ds} \sigma(\tau) \int_0^\tau e^{-\int_\xi^\tau \sigma(s) ds} \lambda(t-a+\xi, \xi) s(t-a+\xi, \xi) d\xi d\tau. \quad (4.13)$$

Let $\chi(t) = \int_0^{a^+} \tilde{\beta}(a) i(t, a) da$. Then $\lambda(t, a)$ can be written as

$$\lambda(t, a) = k(a)\chi(t).$$

Substituting (4.13) into the expression of $\chi(t)$, we have

$$\begin{aligned} \chi(t) &= \int_0^{a^+} \tilde{\beta}(a) \int_0^a e^{-\int_\tau^a \gamma(s) ds} \sigma(\tau) \int_0^\tau e^{-\int_\xi^\tau \sigma(s) ds} \lambda(t-a+\xi, \xi) s(t-a+\xi, \xi) d\xi d\tau da \\ &= \int_0^t \tilde{\beta}(a) \int_0^a e^{-\int_\tau^a \gamma(s) ds} \sigma(\tau) \int_0^\tau e^{-\int_\xi^\tau \sigma(s) ds} k(\xi) \chi(t-a+\xi) s(t-a+\xi, \xi) d\xi d\tau da \\ &\quad + \int_t^{a^+} \tilde{\beta}(a) i(t, a) da. \end{aligned}$$

When t is sufficiently large ($t > a^+$), the last term above is zero.

On the other hand, since $\lambda(t, a) \geq 0$, by (4.10) we know that

$$s(t-a+\xi, \xi) = e^{-\int_0^\xi (\lambda(t-a+s, s) + \rho(s)) ds} \leq e^{-\int_0^\xi \rho(s) ds} = s^0(\xi).$$

Thus

$$\begin{aligned} \chi(t) &= \int_0^t \tilde{\beta}(a) \int_0^a e^{-\int_\tau^a \gamma(s) ds} \sigma(\tau) \int_0^\tau e^{-\int_\xi^\tau \sigma(s) ds} k(\xi) \chi(t-a+\xi) s(t-a+\xi, \xi) d\xi d\tau da \\ &\leq \int_0^t \tilde{\beta}(a) \int_0^a e^{-\int_\tau^a \gamma(s) ds} \sigma(\tau) \int_0^\tau e^{-\int_\xi^\tau \sigma(s) ds} k(\xi) \chi(t-a+\xi) s^0(\xi) d\xi d\tau da. \end{aligned} \quad (4.14)$$

Switching the integrating order of the first part in (4.14), we have

$$\chi(t) \leq \int_0^t \tilde{\beta}(a) \int_0^a \int_0^\tau \chi(t-a+\xi) e^{-\int_\tau^a \gamma(s) ds} e^{-\int_\xi^\tau \sigma(s) ds} \sigma(\tau) k(\xi) s^0(\xi) d\xi d\tau da. \quad (4.15)$$

Recalling the fact in (4.8) that

$$\mathcal{R}_0 = \int_0^{a^+} \tilde{\beta}(a) \int_0^a \int_0^\eta e^{-\int_\eta^a \gamma(s) ds} e^{-\int_\tau^\eta \sigma(s) ds} \sigma(\eta) k(\tau) s^0(\tau) d\tau d\eta da$$

and taking the superior limit in both sides of (4.15), we obtain that

$$\limsup_{t \rightarrow \infty} \chi(t) \leq \mathcal{R}_0 \limsup_{t \rightarrow \infty} \chi(t).$$

When $\mathcal{R}_0 < 1$, we have

$$\limsup_{t \rightarrow \infty} \chi(t) = 0 \quad (4.16)$$

since $\chi(t) \geq 0$. From $\chi(t) = \int_0^{a+} \tilde{\beta}(a)i(t, a) da$, we obtain

$$\limsup_{t \rightarrow \infty} i(t, a) = 0 = i^0(a).$$

From (4.12), we have

$$\limsup_{t \rightarrow \infty} e(t, a) = 0 = e^0(a).$$

From (4.10) and (4.16), we obtain

$$\begin{aligned} \limsup_{t \rightarrow \infty} s(t, a) &= \limsup_{t \rightarrow \infty} e^{-\int_0^a (\lambda(t-a+s, s) + \rho(s)) ds} \\ &= \limsup_{t \rightarrow \infty} e^{-\int_0^a (k(s)\chi(t-a+s) + \rho(s)) ds} \\ &= e^{-\int_0^a \rho(s) ds} \\ &= s^0(a). \end{aligned}$$

Until now, we have proved (4.9), so the disease-free steady state E^0 is globally asymptotically stable if $\mathcal{R}_0 < 1$. ■

5. Existence and stability of the endemic steady state

In this section, we mainly discuss the existence and stability of the endemic steady state.

5.1. Existence of the endemic steady state

We first have the existence and uniqueness of the endemic steady state as follows.

Theorem 5.1. Under Assumption 2.1 system (2.11)–(2.13) has a unique positive endemic steady state $E^* = (s^*(a), e^*(a), i^*(a))$ if $\mathcal{R}_0 > 1$.

Proof. The endemic steady state $E^* = (s^*(a), e^*(a), i^*(a))$ must satisfy the following ordinary differential equations

$$\begin{cases} \frac{ds^*}{da} = -\lambda^*(a)s^*(a) - \rho(a)s^*(a), \\ \frac{de^*}{da} = \lambda^*(a)s^*(a) - \sigma(a)e^*(a), \\ \frac{di^*}{da} = \sigma(a)e^*(a) - \gamma(a)i^*(a) \end{cases} \quad (5.1)$$

with initial value conditions

$$s^*(0) = 1, \quad e^*(0) = 0, \quad i^*(0) = 0,$$

where

$$\lambda^*(a) = k(a) \int_0^{a+} \tilde{\beta}(a)i^*(a) da = k(a)\Lambda^*, \quad \Lambda^* = \int_0^{a+} \tilde{\beta}(a)i^*(a) da.$$

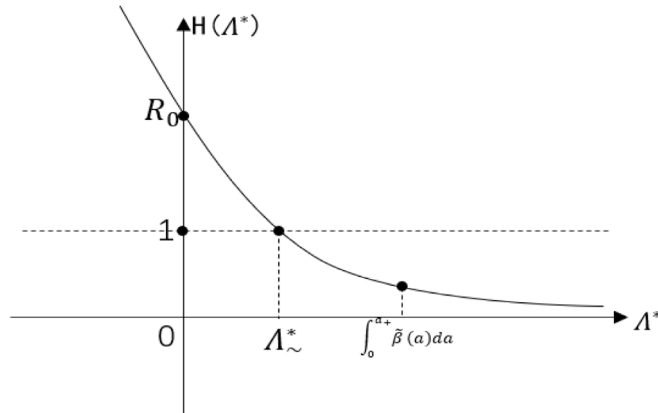


Fig. 5.1. The approximate graph of $H(\Lambda^*)$.

The solution of (5.1) is

$$\begin{cases} s^*(a) = e^{-\int_0^a (\Lambda^* k(s) + \rho(s)) ds}, \\ e^*(a) = \Lambda^* \int_0^a e^{-\int_\tau^a \sigma(s) ds - \int_0^\tau (\Lambda^* k(s) + \rho(s)) ds} k(\tau) d\tau, \\ i^*(a) = \Lambda^* \int_0^a \int_0^\eta e^{-\int_\eta^a \gamma(s) ds - \int_\tau^\eta \sigma(s) ds - \int_0^\tau (\Lambda^* k(s) + \rho(s)) ds} \sigma(\eta) k(\tau) d\tau d\eta. \end{cases} \quad (5.2)$$

Note that when $\Lambda^* = 0$, the endemic steady state $E^* = (s^*(a), e^*(a), i^*(a))$ becomes the disease-free steady state $E^0 = (s^0(a), e^0(a), i^0(a))$.

Substituting the expression of $i^*(a)$ into $\Lambda^* = \int_0^{a+} \tilde{\beta}(a) i^*(a) da$ and eliminating Λ^* since $\Lambda^* \neq 0$ for the endemic steady state, we have

$$1 = \int_0^{a+} \tilde{\beta}(a) \int_0^a \int_0^\eta e^{-\int_\eta^a \gamma(s) ds - \int_\tau^\eta \sigma(s) ds - \int_0^\tau (\Lambda^* k(s) + \rho(s)) ds} \sigma(\eta) k(\tau) d\tau d\eta da. \quad (5.3)$$

Denoting the right-hand side of (5.3) by $H(\Lambda^*)$, i.e.,

$$H(\Lambda^*) \triangleq \int_0^{a+} \tilde{\beta}(a) \int_0^a \int_0^\eta e^{-\int_\eta^a \gamma(s) ds - \int_\tau^\eta \sigma(s) ds - \int_0^\tau (\Lambda^* k(s) + \rho(s)) ds} \sigma(\eta) k(\tau) d\tau d\eta da. \quad (5.4)$$

Then there exists a unique endemic steady state if and only if there exists a unique Λ^* such that $H(\Lambda^*) = 1$ and $\Lambda^* > 0$. Actually, it is easy to acquire some basic properties about $H(\Lambda^*)$ for $\Lambda^* \in \mathbb{R}$:

$$H'(\Lambda^*) < 0, \quad \lim_{\Lambda^* \rightarrow -\infty} H(\Lambda^*) = +\infty, \quad H(0) = \mathcal{R}_0,$$

$$\lim_{\Lambda^* \rightarrow +\infty} H(\Lambda^*) = \lim_{\Lambda^* \rightarrow +\infty} \frac{1}{\Lambda^*} \int_0^{a+} \tilde{\beta}(a) i^*(a) da \leq \lim_{\Lambda^* \rightarrow +\infty} \frac{1}{\Lambda^*} \int_0^{a+} \tilde{\beta}(a) da = 0,$$

which imply that $H(\Lambda^*) = 1$ has a unique positive and real root $\Lambda_\sim^*$ if $\mathcal{R}_0 > 1$ (see Fig. 5.1). Then system (2.11)–(2.13) has a unique positive endemic steady state $E^* = (s^*(a), e^*(a), i^*(a))$, where $s^*(a), e^*(a), i^*(a)$ are given in (5.2) and $\Lambda^* = \Lambda_\sim^*$.

On the other hand,

$$H(\Lambda_\sim^*) = \frac{1}{\Lambda_\sim^*} \int_0^{a+} \tilde{\beta}(a) i^*(a) da < \frac{1}{\Lambda_\sim^*} \int_0^{a+} \tilde{\beta}(a) da,$$

which tells us that if $\Lambda_\sim^* > \int_0^{a+} \tilde{\beta}(a) da$, then $H(\Lambda_\sim^*) < 1$, this contradicts $H(\Lambda_\sim^*) = 1$. So we have $\Lambda_\sim^* \in (0, \int_0^{a+} \tilde{\beta}(a) da)$ and obtain an approximate interval for $\Lambda_\sim^*$. ■

5.2. Local stability of the endemic steady state

In order to analyze the local stability of the endemic steady state if $\mathcal{R}_0 > 1$, we also need to discuss the linearized system of (2.11)–(2.13) at E^* . Let

$$\tilde{s}(t, a) = s(t, a) - s^*(a), \quad \tilde{e}(t, a) = e(t, a) - e^*(a), \quad \tilde{i}(t, a) = i(t, a) - i^*(a).$$

Then system of (4.1)–(4.3) takes the form

$$\begin{cases} \frac{\partial \tilde{s}}{\partial t} + \frac{\partial \tilde{s}}{\partial a} = -\lambda^*(a)\tilde{s}(t, a) - \rho(a)\tilde{s}(t, a) - s^*(a)\tilde{\lambda}(t, a) - \tilde{\lambda}(t, a)\tilde{s}(t, a), \\ \frac{\partial \tilde{e}}{\partial t} + \frac{\partial \tilde{e}}{\partial a} = \lambda^*(a)\tilde{s}(t, a) - \sigma(a)\tilde{e}(t, a) + s^*(a)\tilde{\lambda}(t, a) + \tilde{\lambda}(t, a)\tilde{s}(t, a), \\ \frac{\partial \tilde{i}}{\partial t} + \frac{\partial \tilde{i}}{\partial a} = \sigma(a)\tilde{e}(t, a) - \gamma(a)\tilde{i}(t, a) \end{cases} \quad (5.5)$$

with boundary conditions

$$\tilde{s}(t, 0) = 0, \quad \tilde{e}(t, 0) = 0, \quad \tilde{i}(t, 0) = 0,$$

where

$$\lambda^*(a) = k(a)\Lambda^*, \quad \Lambda^* = \int_0^{a^+} \tilde{\beta}(a)i^*(a) \, da, \quad \tilde{\lambda}(t, a) = k(a) \int_0^{a^+} \tilde{\beta}(a)\tilde{i}(t, a) \, da.$$

The linearized part of system (5.5) is

$$\begin{cases} \frac{\partial \tilde{s}}{\partial t} + \frac{\partial \tilde{s}}{\partial a} = -\lambda^*(a)\tilde{s}(t, a) - \rho(a)\tilde{s}(t, a) - s^*(a)\tilde{\lambda}(t, a), \\ \frac{\partial \tilde{e}}{\partial t} + \frac{\partial \tilde{e}}{\partial a} = \lambda^*(a)\tilde{s}(t, a) - \sigma(a)\tilde{e}(t, a) + s^*(a)\tilde{\lambda}(t, a), \\ \frac{\partial \tilde{i}}{\partial t} + \frac{\partial \tilde{i}}{\partial a} = \sigma(a)\tilde{e}(t, a) - \gamma(a)\tilde{i}(t, a) \end{cases} \quad (5.6)$$

with boundary conditions

$$\tilde{s}(t, 0) = 0, \quad \tilde{e}(t, 0) = 0, \quad \tilde{i}(t, 0) = 0.$$

Now, we consider the following nonzero exponential solution of system (5.6)

$$\tilde{s}(t, a) = \tilde{s}(a)e^{\omega t}, \quad \tilde{e}(t, a) = \tilde{e}(a)e^{\omega t}, \quad \tilde{i}(t, a) = \tilde{i}(a)e^{\omega t}.$$

Then $\tilde{s}(a), \tilde{e}(a), \tilde{i}(a)$ and ω satisfy the following ordinary differential equations

$$\begin{cases} \frac{d\tilde{s}}{da} = -\omega\tilde{s}(a) - \Lambda^*k(a)\tilde{s}(a) - \rho(a)\tilde{s}(a) - \tilde{\Lambda}k(a)s^*(a), \\ \frac{d\tilde{e}}{da} = -\omega\tilde{e}(a) + \Lambda^*k(a)\tilde{s}(a) - \sigma(a)\tilde{e}(a) + \tilde{\Lambda}k(a)s^*(a), \\ \frac{d\tilde{i}}{da} = -\omega\tilde{i}(a) + \sigma(a)\tilde{e}(a) - \gamma(a)\tilde{i}(a) \end{cases} \quad (5.7)$$

with initial value conditions

$$\tilde{s}(0) = 0, \quad \tilde{e}(0) = 0, \quad \tilde{i}(0) = 0,$$

where

$$\tilde{\Lambda} = \int_0^{a^+} \tilde{\beta}(a)\tilde{i}(a) \, da.$$

Since $\tilde{\Lambda} \neq 0$, let $\check{s} = \frac{\tilde{s}}{\tilde{\Lambda}}, \check{e} = \frac{\tilde{e}}{\tilde{\Lambda}}, \check{i} = \frac{\tilde{i}}{\tilde{\Lambda}}$. Then system (5.7) becomes

$$\begin{cases} \frac{d\check{s}}{da} = -\omega\check{s}(a) - \Lambda_{\sim}^* k(a)\check{s}(a) - \rho(a)\check{s}(a) - k(a)s^*(a), \\ \frac{d\check{e}}{da} = -\omega\check{e}(a) + \Lambda_{\sim}^* k(a)\check{s}(a) - \sigma(a)\check{e}(a) + k(a)s^*(a), \\ \frac{d\check{i}}{da} = -\omega\check{i}(a) + \sigma(a)\check{e}(a) - \gamma(a)\check{i}(a) \end{cases} \quad (5.8)$$

with initial value conditions

$$\check{s}(0) = 0, \check{e}(0) = 0, \check{i}(0) = 0.$$

Moreover, we have

$$1 = \int_0^{a^+} \tilde{\beta}(a)\check{i}(a) da.$$

The solution of system (5.8) is

$$\begin{cases} \check{s}(a) = - \int_0^a e^{-\int_{\eta}^a (\omega + \Lambda_{\sim}^* k(s) + \rho(s)) ds} s^*(\eta) k(\eta) d\eta, \\ \check{e}(a) = \int_0^a e^{-\int_{\eta}^a (\omega + \sigma(s)) ds} (\Lambda_{\sim}^* k(\eta)\check{s}(\eta) + s^*(\eta)k(\eta)) d\eta, \\ \check{i}(a) = \int_0^a e^{-\int_{\eta}^a (\omega + \gamma(s)) ds} \sigma(\eta)\check{e}(\eta) d\eta. \end{cases}$$

Denoting $G(\omega) = \int_0^{a^+} \tilde{\beta}(a)\check{i}(a) da$, we want to show that all roots of $G(\omega) = 1$ have negative real parts if $\mathcal{R}_0 > 1$. Substituting the expressions of $\check{s}(a)$ and $\check{e}(a)$ into $G(\omega)$, we have

$$\begin{aligned} G(\omega) &= \int_0^{a^+} \tilde{\beta}(a) \int_0^a e^{-\int_{\eta}^a (\omega + \gamma(s)) ds} \sigma(\eta) \int_0^{\eta} e^{-\int_{\tau}^{\eta} (\omega + \sigma(s)) ds} (\Lambda_{\sim}^* k(\tau)\check{s}(\tau) + s^*(\tau)k(\tau)) d\tau d\eta da \\ &= \int_0^{a^+} \tilde{\beta}(a) \int_0^a e^{-\int_{\eta}^a [\omega + \gamma(s)] ds} \sigma(\eta) \int_0^{\eta} e^{-\int_{\tau}^{\eta} [\omega + \sigma(s)] ds} [s^*(\tau)k(\tau) \\ &\quad - \Lambda_{\sim}^* k(\tau) \int_0^{\tau} e^{-\int_{\xi}^{\tau} [\omega + \Lambda_{\sim}^* k(s) + \rho(s)] ds} s^*(\xi)k(\xi) d\xi] d\tau d\eta da \end{aligned} \quad (5.9)$$

and

$$\begin{aligned} G(0) &= - \int_0^{a^+} \tilde{\beta}(a) \int_0^a e^{-\int_{\eta}^a \gamma(s) ds} \sigma(\eta) \int_0^{\eta} e^{-\int_{\tau}^{\eta} \sigma(s) ds} \Lambda_{\sim}^* k(\tau) \\ &\quad \times \int_0^{\tau} e^{-\int_{\xi}^{\tau} (\Lambda_{\sim}^* k(s) + \rho(s)) ds} s^*(\xi)k(\xi) d\xi d\tau d\eta da \\ &\quad + \int_0^{a^+} \tilde{\beta}(a) \int_0^a e^{-\int_{\eta}^a \gamma(s) ds} \sigma(\eta) \int_0^{\eta} e^{-\int_{\tau}^{\eta} \sigma(s) ds} s^*(\tau)k(\tau) d\tau d\eta da. \end{aligned}$$

According to the expression of $H(\Lambda_{\sim}^*)$ in (5.4), we have

$$H(\Lambda_{\sim}^*) = \int_0^{a^+} \tilde{\beta}(a) \int_0^a e^{-\int_{\eta}^a \gamma(s) ds} \sigma(\eta) \int_0^{\eta} e^{-\int_{\tau}^{\eta} \sigma(s) ds} s^*(\tau)k(\tau) d\tau d\eta da = 1.$$

Then $G(0)$ can be transformed into

$$\begin{aligned} G(0) &= 1 - \int_0^{a^+} \tilde{\beta}(a) \int_0^a e^{-\int_{\eta}^a \gamma(s) ds} \sigma(\eta) \int_0^{\eta} e^{-\int_{\tau}^{\eta} \sigma(s) ds} \Lambda_{\sim}^* k(\tau) \\ &\quad \times \int_0^{\tau} e^{-\int_{\xi}^{\tau} (\Lambda_{\sim}^* k(s) + \rho(s)) ds} s^*(\xi)k(\xi) d\xi d\tau d\eta da \\ &< 1. \end{aligned}$$

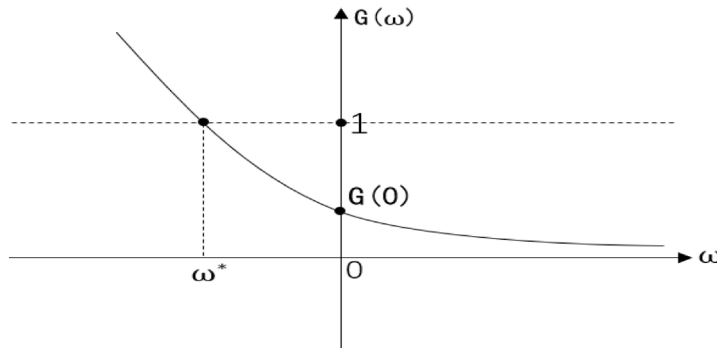


Fig. 5.2. The approximate graph of $G(\omega)$.

Observe that due to $\Lambda^*_{\sim} > 0$, one has

$$G(\omega) < \int_0^{a^+} \tilde{\beta}(a) \int_0^a e^{-\int_{\eta}^a [\omega + \gamma(s)] ds} \sigma(\eta) \int_0^{\eta} e^{-\int_{\tau}^{\eta} [\omega + \sigma(s)] ds} s^*(\tau) k(\tau) d\tau d\eta da \triangleq \mathcal{G}(\omega).$$

Since $\mathcal{G}(\omega)$ is decreasing with respect to ω , one has $G(\omega) < \mathcal{G}(0) = H(\Lambda^*_{\sim}) = 1$ for all $\text{Re } \omega > 0$. It follows that $G(\omega) = 1$ only happens in the region $\text{Re } \omega < 0$. Thus all roots of $G(\omega) = 1$ have negative real parts if $\mathcal{R}_0 > 1$. The approximate graph about $G(\omega)$ is shown in Fig. 5.2. We have the following results.

Theorem 5.2. Under Assumption 2.1, if $\mathcal{R}_0 > 1$, then the endemic steady state $E^* = (s^*(a), e^*(a), i^*(a))$ is locally asymptotically stable.

Proof. The proof is similar to that of Theorem 4.3 and is omitted here. ■

6. Numerical simulations

In this section, we provide some numerical simulations to illustrate our results obtained in the previous sections. For simplicity, we consider the case where $\mu(a) = \frac{(a-30)^2}{10^4}$ is dependent of the age variable a , $\beta(a, b) = 23.5635$, $\sigma(a) = 18.25$, $\gamma(a) = 73$, $a^+ = 80$ and $\rho(a) = \rho$ are constant. We emphasize the effect of vaccination.

In Fig. 6.1, we illustrate the change of the basic reproduction number $\mathcal{R}_0$ with the change of the vaccination rate ρ . We can see that $\mathcal{R}_0$ decreases as ρ increases. Letting $\rho = 0.2$, we can calculate $\mathcal{R}_0 = 1.5921 > 1$, Fig. 6.2 shows that total subpopulations $\int_0^{80} s(t, a) da$, $\int_0^{80} e(t, a) da$ and $\int_0^{80} i(t, a) da$ tend to the endemic steady state E^* , which confirms the local asymptotical stability of E^* when $\mathcal{R}_0 > 1$ in Theorem 5.2. In particular, Fig. 6.3 shows that the susceptible population, the exposed population and the infected population of this endemic steady state versus age a and time t , respectively, and Fig. 6.4 presents the age distribution of the infected population at time ($t = 17.3$), in which most infected individuals come from young age.

Next, we have $\mathcal{R}_0 = 0.516617 < 1$ when $\rho = 0.6$, Fig. 6.5 indicates that total subpopulations $\int_0^{80} s(t, a) da$, $\int_0^{80} e(t, a) da$ and $\int_0^{80} i(t, a) da$ tend to the disease-free steady state E_0 , which confirms the global stability of E_0 when $\mathcal{R}_0 < 1$ in Theorem 4.4. Increasing the vaccination rate (Fig. 6.1) is an essential and effective control strategy the infectious disease.

Finally, we consider the combined influence of the vaccination rate ρ and transmission rate $\beta(a, b) = \beta$. From Fig. 6.6(a), we can see that when the increasing of vaccination and the reduction of contact are

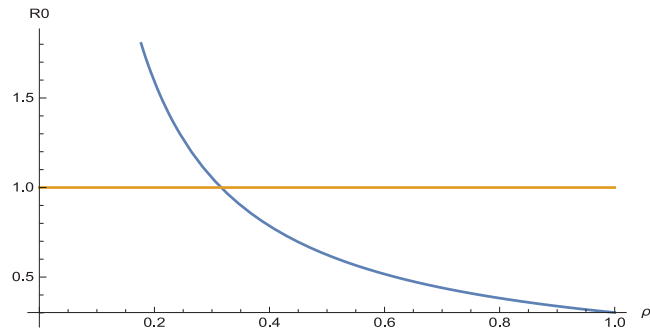


Fig. 6.1. The basic reproduction number $\mathcal{R}_0$ versus the vaccination ρ .

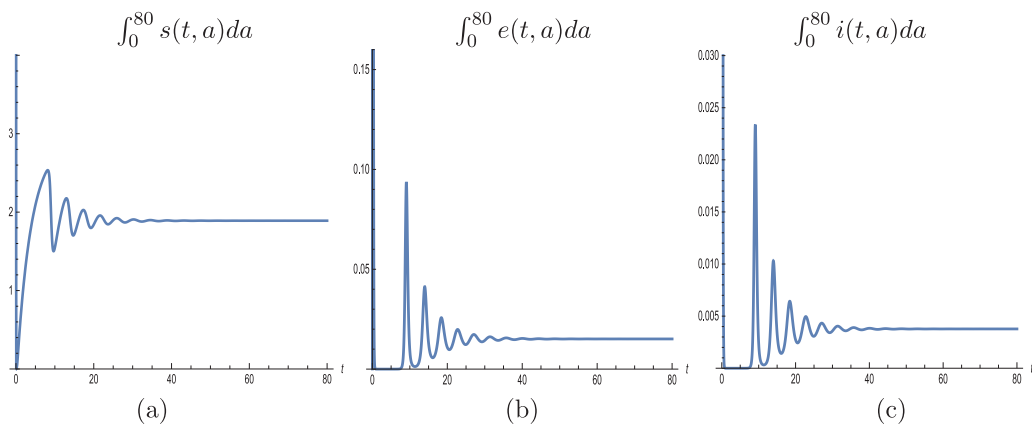


Fig. 6.2. Behavior of the model when $\mathcal{R}_0 > 1$ and the vaccination rate is $\rho = 0.2$: (a) total susceptible population $\int_0^{80} s(t, a) da$ versus time t ; (b) total exposed population $\int_0^{80} e(t, a) da$ versus time t ; (c) total infected population $\int_0^{80} i(t, a) da$ versus time t .

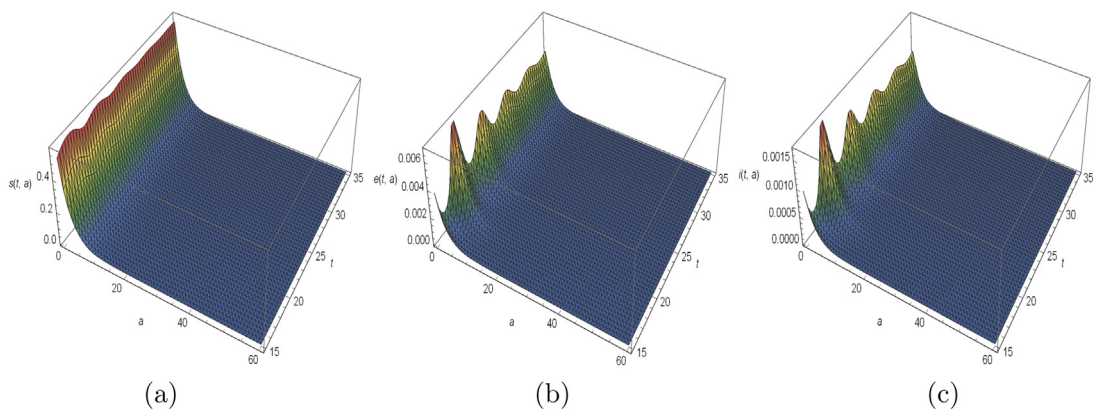


Fig. 6.3. Plots of the susceptible population $s(t, a)$, the infected population $i(t, a)$ and the exposed population $e(t, a)$ versus age a and time t (in time 15–35).

combined, controlling measles will be more effective. Fig. 6.6(b) gives the curve $\mathcal{R}_0 = 1$ that the measles will go extinct when ρ and β takes values below it. It indicates that if β is increases, then we need to increase the vaccination rate significantly to prevent measles transmission.

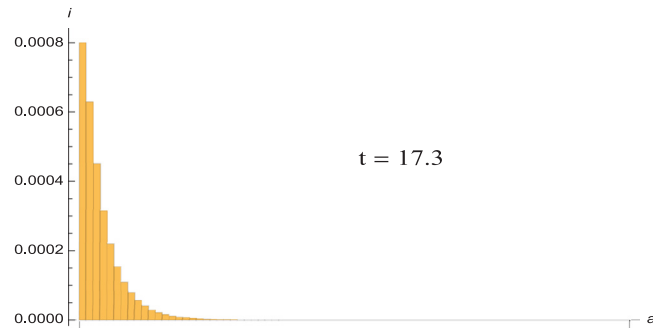


Fig. 6.4. Age distribution of the infected population at the time ($t = 17.3$).

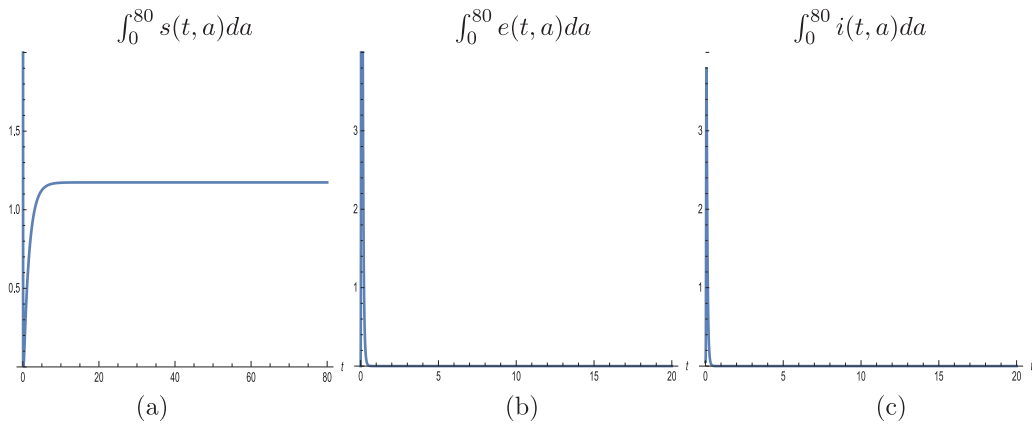


Fig. 6.5. Behavior of the model when $\mathcal{R}_0 < 1$ and the vaccination rate is $\rho = 0.6$: (a) total susceptible population $\int_0^{80} s(t, a) da$ versus time t ; (b) total exposed population $\int_0^{80} e(t, a) da$ versus time t ; (c) total infected population $\int_0^{80} i(t, a) da$ versus time t .

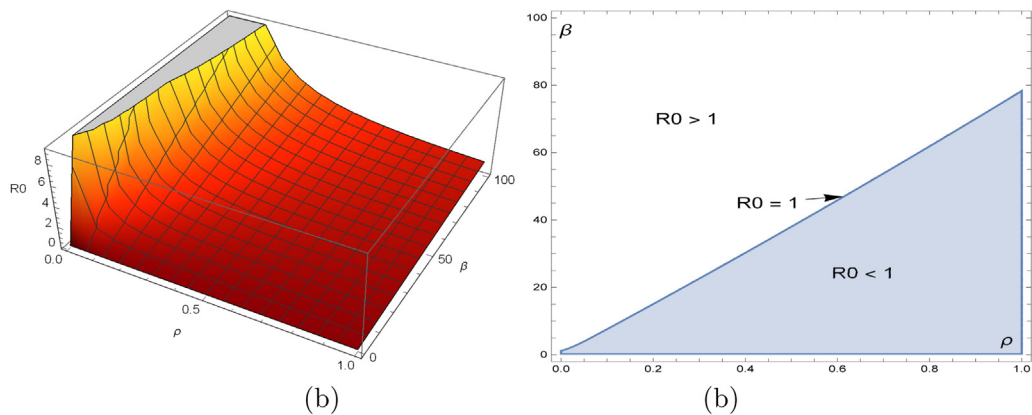


Fig. 6.6. (a) The basic reproduction number $\mathcal{R}_0$ versus ρ and β ; (b) The effects of ρ on the $\mathcal{R}_0$ as β increases.

7. Discussion

The airborne transmission route of measles does not require detailed specification of different types of contact between susceptible and infectious individuals. Moreover, infected individuals are most infectious during the prodrome period before the appearance of the rash. These facts indicate that it is reasonable to assume the mixing between susceptible and infectious individuals and use standard incidence rate to

describe the transmission. Since one of the main measures in controlling measles is to find the optimal age to vaccinate children in order to have the maximum impact on the incidence of disease-related morbidity and mortality for a given rate of vaccination coverage, age-structured epidemic models have been extensively used to study the transmission dynamics and control of measles, see Anderson and May [8], Corey and Noymer [9], Greenhalgh [10,11], Halloran et al. [12], Hethcote [13], Huang and Rohani [14], Kang et al. [15], Manfredi and Williams [16], McLean and Anderson [17,18], Schenzle [19], and Tudor [20].

Based on a periodic SEIR epidemic model describing measles (Earn et al. [25], Huang et al. [4]), Kang et al. [15] studied an age-structured SEIR model with periodic infection rate. After establishing the well-posedness of the initial-boundary value problem for the periodic age-structured SEIR model, they discussed the existence of time periodic solutions of the model by using a fixed point theorem and showed that there is also a threshold value for the existence and uniqueness of a nontrivial endemic periodic solution. Note that Kang et al. [15] were only able to establish the existence of periodic solutions in the age-structured SEIR model with periodic infection rate (see also Huang et al. [4]), while the stability of the periodic solutions remains an open issue.

In fact, stability in age-structured epidemic models is a very interesting and challenging problem (Iannelli [21], Inaba [22], Li et al. [23], and Webb [24]). To determine the stability of the steady states, in this paper we considered an age-structured epidemic model of the SEIR type with vaccination and standard incidence rate; that is, we assumed that the coefficients are not time periodic. We established the well-posedness of the initial-boundary value problem, studied the existence and stability of the disease-free and endemic steady states based on the basic reproduction number $\mathcal{R}_0$, and showed that the disease-free steady state is globally asymptotically stable if $\mathcal{R}_0 < 1$. Moreover, we proved that the endemic steady state is unique if $\mathcal{R}_0 > 1$ and is locally asymptotically stable under some additional conditions. Finally we carried out some numerical simulations to illustrate the theoretical results. Our results indicate that reduce the transmission rate and increase the vaccination rate are the most effective control and prevention measures for measles. In particular, if the transmission rate cannot be reduced further, then the vaccination rate needs to be maximized, and vice versa.

Measles is still a challenging and major public health problem worldwide and vaccination remains the most effective measure to prevent and control of measles outbreaks. Mathematical models have been used to refine vaccine policy. Currently, two doses of measles vaccination are administered for children at different ages in different countries. It will be interesting to consider the optimal problems in age-structured epidemic models (Anderson and May [8], Halloran et al. [12], Hethcote [13], McLean and Anderson [17,17]) and explore optimal vaccination ages for different countries. Also, it will be interesting to modify the age-structured model in this paper and apply to some infectious diseases, such as mumps and pertussis (Hethcote [1]).

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