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Calculus of Variations

Principal spectral theory in multigroup age-structured models with nonlocal diffusion

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Abstract

In modeling the population dynamics of biological species and the transmission dynamics of infectious diseases, age-structure and nonlocal diffusion are two important components since individuals need to be mature enough to move and they disperse and interact each other nonlocally. In this paper we study the principal spectral theory of age-structured models with nonlocal diffusion within a population of multigroups. First, we provide a criterion on the existence of the principal eigenvalue by using the theory of positive resolvent operators with their perturbations. Then we define the generalized principal eigenvalue and use it to investigate the influence of diffusion rate on the principal eigenvalue. Next we establish the strong maximum principle for age-structured nonlocal diffusion operators. Finally, as an example we apply our established theory to an age-structured cooperative system with nonlocal diffusion.

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Contents

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1 Introduction

For scalar linear and nonlinear age-structured equations with nonlocal diffusion, recently we [14, 24–26] developed some basic theories including the semigroup of linear operators, asymptotic behavior, spectral theory, asynchronous exponential growth, strong maximum principle, global dynamics, etc. In this paper, we continue to study the existence of the principal eigenvalue, asymptotic behavior of the generalized principal eigenvalue with respect to the diffusion rate, and global dynamics of multigroup age-structured models with nonlocal diffusion and cooperative type nonlinearity. More precisely, we are interested in the eigenvalue problem corresponding to the following multigroup age-structured models with nonlocal diffusion:

$$\begin{cases} \partial_a u_i(a,x) = \frac{D_i}{\gamma_i^{m_i}} \left[\int_\Omega J_{\gamma_i}(x-y)u_i(a,y)dy - u_i(a,x)\right] \\ \qquad\qquad -\mu_i(a,x)u_i(a,x), \quad a \in (0,a^+), x \in \overline{\Omega}, \\ u_i(0,x) = \sum_{j=1}^M \int_0^{a^+} \beta_{ij}(a,x)u_j(a,x)da, \quad x \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where $u_i(a,x)$, $i = 1,\dots,M$, denotes the density of individuals that belong to the ith group at age a and location $x \in \overline{\Omega}$ and M denotes the number of groups in a population; $a^+ < \infty$ represents the maximum age and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $D_i > 0$, $i = 1,\dots,M$, are the diffusion rates, $\gamma_i > 0, i = 1,\dots,M$, represent the diffusion ranges, and $m_i \in [0,2), i = 1,\dots,M$, are the cost parameters with $J_{\gamma_i}(x) = \frac{1}{\gamma_i^N} J\left(\frac{x}{\gamma_i}\right), i = 1,\dots,M$, for $x \in \mathbb{R}^N$. The diffusion kernel J satisfies the following assumption.

Assumption 1.1 The kernel $J \in C(\mathbb{R}^N)$ is nonnegative and supported in $B(0,r)$ for some $r > 0$, and satisfies $J(0) > 0$ and $\int_{\mathbb{R}^N} J(x)dx = 1$, where $B(0,r) \subset \mathbb{R}^N$ is the open ball centered at 0 with radius r.

We point out that the nonlocal diffusion operator in (1.1) corresponds to zero Dirichlet boundary condition. Next we provide the following assumptions on the transmission rates β_{ij} and the death rates μ_i for all $i, j = 1, \dots, M$:

Assumption 1.2
- $\beta_{ij} \in C(\mathbb{R}^N, L^\infty_+(0, a^+))$;
- $\mu_i \in C(\mathbb{R}^N, L^\infty_{\text{loc},+}[0, a^+))$;
- $\min_{1\le i\le M} \int_0^{a^+} \underline{\mu}_i(a)da = \infty$ and $\tilde{\mu} := \min_{1\le i\le M}\{\inf_{[0,a^+]\times\overline{\Omega}} \mu_i(a,x)\} > 0$, where

$$\underline{\mu}_i(a) := \min_{x\in\overline{\Omega}} \mu_i(a,x), \ \ \overline{\mu}_i(a) := \max_{x\in\overline{\Omega}} \mu_i(a,x),$$
$$\underline{\beta}_{ij}(a) := \min_{x\in\overline{\Omega}} \beta_{ij}(a,x), \ \ \overline{\beta}_{ij}(a) := \max_{x\in\overline{\Omega}} \beta_{ij}(a,x).$$

- Moreover, assume that for any $(a, x) \in (0, a^+) \times \overline{\Omega}$, the matrix

$$\beta(a,x) = \left(\beta_{ij}(a,x)\right)_{(i,j)=1,\dots,M} \text{ is either irreducible or primitive.}$$

The last assumption ensures that the spectral radii are the principal eigenvalues of these matrices. Throughout the paper, we will denote spectral radius of a matrix or a linear operator T by $r(T)$.

The motivation for studying the principal spectral theory is to investigate the global dynamics of the following nonlinear cooperative age-structured models with nonlocal diffusion:

$$\begin{cases} \partial_t u_i(t,a,x) + \partial_a u_i(t,a,x) = D_i\left[\int_\Omega J(x-y)u_i(t,a,y)dy - u_i(t,a,x)\right] \\ \qquad\qquad\qquad\qquad\qquad -\mu_i(a,x)u_i(t,a,x), \quad t>0, a\in(0,a^+), x\in\overline{\Omega}, \\ u_i(t,0,x) = f_i\left(\sum_{j=1}^M \int_0^{a^+} \beta_{ij}(a,x)u_j(t,a,x)da\right), \quad t>0, x\in\overline{\Omega}, \\ u_i(0,a,x) = u_{i0}(a,x), \quad (a,x)\in(0,a^+)\times\overline{\Omega}, \end{cases} \tag{1.2}$$

where $u_i(t, a, x)$ for $i = 1, \dots, M$ denotes the density of population at time t, age a and position x, J is a dispersal kernel and f is a cooperative type nonlinearity describing the cooperative rate of the population. Such equations appear naturally in describing some ecological problems when in addition to the dispersion of the individuals in the environment, the birth and death of these individuals are also modeled, see Fife [19], García-Melián and Rossi [21], Hutson et al. [22], Medlock and Kot [36], and Murray [37]. It could be used to characterize the spatio-temporal dynamics of biological species and transmission dynamics of infectious diseases in which the age structure of the population is a very important factor and the dispersal is in long distance. We mention again that the nonlocal diffusion operator in (1.2) corresponds to zero Dirichlet boundary condition, which indicates that the region outside their habitat, $\mathbb{R}^N \setminus \overline{\Omega}$, is hostile that the population cannot survive there, see Hutson et al. [22]. Next we provide the assumptions on f in the following.

Assumption 1.3 We assume that $f = (f_1, \dots, f_M)$ satisfies the following conditions,

(i) $f_i \in C^1(\mathbb{R}_+)$;
(ii) $f_i'(y) > 0$ for all $y \in [0, \infty)$;
(iii) $f_i(0) \equiv 0$ and $\frac{f_i(y)}{y}$ is decreasing with respect to $y \in [0, \infty)$;
(iv) There exists $L > 0$ such that $f_i(y) \le L$ for all $y \in \mathbb{R}_+$.

In Assumption 1.3, (i) assumes the good regularity of f; (ii) guarantees the cooperativity of system (1.2); (iii) implies that the nonlinearity is sub-homogeneous; (iv) guarantees that solutions of (1.2) will remain uniformly bounded for all times. In applications, $f(y)$ can correspond to some classical nonlinearity, for example each f_i, $1 \le i \le M$ can be chosen as follows, $f_i(y) = \frac{y}{1+Ay}$, $y \ge 0$, with $A > 0$ being a constant or $f_i(y) = 1 - e^{-y}$, $y \ge 0$.

Before proceeding, we briefly recall the principal spectral theory in cooperative reaction diffusion and nonlocal diffusion systems including autonomous and time-periodic systems. For reaction diffusion, Dancer [13] investigated the principal eigenvalue of a linear cooperating elliptic system with small diffusion. Later Lam and Lou [29] analyzed the asymptotic behavior of the principal eigenvalue for cooperative elliptic systems. Recently Bai and He [2] generalized the results in [29] to cooperative periodic-parabolic systems. The above results mainly focused on the asymptotic behavior of the principal eigenvalue with a small diffusion. Most recently Zhang and Zhao [52] studied the case of a large diffusion and also obtained the asymptotic behavior of the basic reproduction ratio in reaction-diffusion systems. For nonlocal diffusion, Bao and Shen [3] first provided a criteria for the existence of principal eigenvalues of linear time periodic cooperative systems. Liang et al. [31] studied the principal eigenvalue for periodic nonlocal dispersal systems with time delay. In the reaction diffusion case, most researchers established the existence of principal eigenvalues by employing Krein-Rutman theorem due to the compactness of solution maps for second order elliptic operators and further investigated the asymptotic behavior with respect to diffusion coefficient by using variational structure for the autonomous case and sup-inf characterization method for the time-periodic case. While in the nonlocal diffusion case, due to the lack of compactness of solution maps one needs to use different methods such as generalized Krein-Rutman theorem [16, 38], see Coville [10, 12] and the references cited therein, or perturbation of positive operators [8], see Rawal and Shen [40], Shen and Xie [42] and the references cited therein, to obtain the existence of generalized principal eigenvalues. Further, combining these two methods, Shen and Vo [43] and Su et al. [44] discussed the asymptotic behavior of generalized principal eigenvalues in the time-periodic case by employing the idea from Berestycki et al. [5]. There are also many other studies on the analysis of (generalized) principal eigenvalues for nonlocal diffusion equations in different situations including cooperative systems, see Liang and Zhou [32], Li et al. [30] and the references cited therein.

To the best of our knowledge, there is little literature on the principal eigenvalue in age-structured cooperative models with nonlocal diffusion. Ducrot et al. [15] obtained the principal eigenvalue in investigating the existence of traveling wave solutions of multigroup age-structured epidemic models; however, they considered the Laplace diffusion and spatial variable independent coefficients. The purpose of this paper is to first investigate the existence of the principal eigenvalue of multigroup age-structured models with nonlocal diffusion and then study the asymptotic behavior of the principal eigenvalue in both small and large diffusions. We will extend the idea in our previous paper [14] for the scalar case to cooperative systems. More concretely, we will choose an extended function space to include the integral boundary condition, see the definition of $\mathcal{A}$ in (2.11), which is different from the previous studies. The reason behind is that there is a ∂_a term and an integral boundary condition in the equation (1.1), which prevent us to use directly the results of autonomous and time-periodic cases in nonlocal diffusion operators. Next, we will introduce the theory of resolvent positive operators with their perturbations by Arendt [1] and Thieme [46, 47] to investigate the existence of principal eigenvalue, which is similar to Bürger's idea [8] for perturbations of positive operators and generalized Krein-Rutman theorem [16, 38]. Last, we follow the idea of Berestycki et al. [6] to define the generalized principal eigenvalue and use it to study the asymptotic behavior with respect to diffusion. We would like to mention that only the Dirichlet boundary condition is considered here, but the theory can be applied to Neumann case as well, see [25], where we studied a scalar age-structured model with nonlocal diffusion of Neumann type.

The paper is organized as follows. In Sect. 2, we introduce our working operators and function spaces. In Sect. 3, we first analyze the spectral bound $s(\mathcal{B}_1+\mathcal{C})$ of $\mathcal{B}_1+\mathcal{C}$, which corresponds to age-structured models without nonlocal diffusion, and compare it with the spectral bound $s(\mathcal{A})$ of $\mathcal{A}$ defined in (2.11), which corresponds to age-structured models with nonlocal diffusion, and then obtain a non-strict size relation between $s(\mathcal{B}_1+\mathcal{C})$ and $s(\mathcal{A})$. In Sect. 4, we find an easily verifiable and sufficient condition for $s(\mathcal{A})$ being the principal eigenvalue. In Sect. 5, we study the effects of diffusion rate and diffusion range on the generalized principal eigenvalue of $\mathcal{A}$ and discuss the continuous dependence of the principal eigenvalue on the transmission and death rates β and μ. In Sect. 6, we give the strong maximum principle which is of fundamental importance and independent interest. In Sect. 7, we apply our established theory to the age-structured cooperative system with nonlocal diffusion (1.2) and analyze the existence, uniqueness and stability via the magnitude of spectral bound of the linearized operator. Moreover, we investigate the asymptotic properties of the nontrivial equilibrium of (1.2) with respect to the diffusion rate and diffusion range. In "Appendix", we first establish the spectral theory when $\mu(a,x)\equiv\mu(a)$ and $\beta(a,x)\equiv\beta(a)$ for problem (1.1) and then recall the theory of resolvent positive operators with their perturbations and Perron–Frobenius theory.

Finally, we want to mention that the assumptions that J has a compact support and Ω is bounded can be relaxed. For the principal spectral theory, we only need Ω to be bounded without requiring that J has a compact support. Moreover, the boundedness of Ω seems necessary due to the lack of Harnack's inequality for such parabolic problems, see Shen and Vo [43]. However, in order to study the limiting properties of principal eigenvalues, J is needed to be compactly supported for Taylor expansion later. In addition, the condition that Ω is bounded can even be removed if one only defines the generalized principal eigenvalue, see Berestycki [5]. Here to give a unified presentation of the results, we assumed both of them.

Besides, we consider here a general form of boundary condition on a; i.e. the second equation in (1.1). In fact, the matrix β could be diagonal representing the birth rates of each group i. Otherwise, β_{ij} can represent the transmission rate from group j to group i, see Ducrot [15] for a multigroup age-structured epidemic model.

2 Notations

In this section, we will introduce our notations and some preparatory results. We denote by X and X_+ respectively the Banach space $X=C(\overline{\Omega})$ and its positive cone or the Banach space $X=L^1(\Omega)$ and its positive cone. Here recall that $\Omega\subset\mathbb{R}^N$ is a given bounded domain. Recall that for both cases X_+ is a normal and generating cone. In addition, we denote by I the identity operator.

Then we define the following function spaces

$$\mathcal{X}=X^M\times L^1((0,a^+),X^M),\ \ \mathcal{X}_0=\{0_{X^M}\}\times L^1((0,a^+),X^M),$$

endowed with the product norms and the positive cones:

$$\begin{aligned}&\mathcal{X}^+=X_+^M\times L_+^1((0,a^+),X^M)=X_+^M\times\{u\in L^1((0,a^+),X^M):u(a,\cdot)\in X_+^M,\ \text{a.e. in }(0,a^+)\},\\&\mathcal{X}_0^+=\mathcal{X}^+\cap\mathcal{X}_0.\end{aligned}$$

We define the norm in X^M as follows, $\|u\|_{X^M}:=\max_{1\le i\le M}\|u_i\|_X$. We also define the linear positive and bounded operator $K\in\mathcal{L}(X)$ by

$$[K\varphi](\cdot) = \int_\Omega J(\cdot - y)\varphi(y)dy, \ \forall \varphi \in X. \tag{2.1}$$

Note that one has by Assumption 1.1

$$\|K\|_{\mathcal{L}(X)} \le \begin{cases} \sup_{y\in\Omega} \int_\Omega J(x-y)dx & \text{if } X = L^1(\Omega) \\ \sup_{x\in\overline{\Omega}} \int_\Omega J(x-y)dy & \text{if } X = C(\overline{\Omega}) \end{cases} \le \int_{\mathbb{R}^N} J(z)dz = 1. \tag{2.2}$$

In addition, we define the linear positive and bounded operator $\mathcal{K} \in \mathcal{L}(X^M)$ by

$$\begin{aligned} &[\mathcal{K}\varphi](\cdot) \\ &= \operatorname{diag}\left\{\int_\Omega J(\cdot - y)\varphi_1(y)dy, \dots, \int_\Omega J(\cdot - y)\varphi_M(y)dy\right\}, \ \forall \varphi = (\varphi_1, \dots, \varphi_M) \in X^M. \end{aligned} \tag{2.3}$$

2.1 Evolution Family Without Diffusion

We consider the following problem posed in X for $0 \le \tau \le a < a^+$ and $i = 1, \dots, M$:

$$\begin{cases} \partial_a v_i(a) = -\mu_i(a, \cdot)v_i(a), \ \tau < a < a^+, \\ v_i(\tau) = \eta_i \in X. \end{cases} \tag{2.4}$$

This problem generates an evolution family on X^M, denoted by $\Pi = \operatorname{diag}\{\Pi_1, \dots, \Pi_M\}$ that is explicitly given for $0 \le \tau \le a < a^+$ and $\eta = (\eta_1, \dots, \eta_M) \in X^M$ by

$$\begin{aligned} &\Pi_i(\tau, a)\eta_i = \pi_i(\tau, a, \cdot)\eta_i \\ &\text{with } \pi_i(\tau, a, x) := \exp\left(-\int_\tau^a \mu_i(s, x)ds\right) \text{ for } 0 \le \tau \le a < a^+ \text{ and } x \in \overline{\Omega}. \end{aligned} \tag{2.5}$$

Observe that one has

$$\|\Pi(\tau, a)\|_{\mathcal{L}(X^M)} \le \max_{i=1,\dots,M} \exp\left(-\int_\tau^a \underline{\mu}_i(s)ds\right) \le e^{-\widetilde{\mu}(a-\tau)} \le 1, \ \ \forall\, 0 \le \tau \le a < a^+. \tag{2.6}$$

We also define the following family of bounded linear operators $\{W_\lambda\}_{\lambda > -\widetilde{\mu}} \subset \mathcal{L}(\mathcal{X}, \mathcal{X}_0)$ for $(\eta, g) \in \mathcal{X}$ by

$$\begin{aligned} &W_\lambda(\eta, g) = (0_{X^M}, h) \\ &\text{with } h(a) = e^{-\lambda a}\Pi(0, a)\eta + \int_0^a e^{-\lambda(a-s)}\Pi(s, a)g(s)ds. \end{aligned} \tag{2.7}$$

We will show that this provides a family of positive pseudoresolvents. To this aim, one can make some computations to obtain

$$\begin{aligned} W_\nu W_\lambda(\eta, g) &= \int_0^a e^{-\nu(a-s)}\Pi(s, a)e^{-\lambda s}\Pi(0, s)\eta ds \\ &\quad + \int_0^a e^{-\nu(a-s)}\Pi(s, a)\int_0^s e^{-\lambda(s-\tau)}\Pi(\tau, s)g(\tau)d\tau ds \\ &= \int_0^a e^{-\nu a}e^{-(\lambda-\nu)s}ds\Pi(0, a)\eta + \int_0^a \int_0^s e^{\lambda\tau - \nu a}e^{-(\lambda-\nu)s}\Pi(\tau, a)g(\tau)d\tau ds. \end{aligned}$$

Hence for $\nu \neq \lambda$, we have

$$W_\nu W_\lambda(\eta, g) = \frac{1}{\nu-\lambda}\left(e^{-\lambda a} - e^{-\nu a}\right)\Pi(0,a)\eta + \frac{1}{\nu-\lambda}\left(e^{-(\lambda-\nu)a} - e^{-(\lambda-\mu)\tau}\right)$$
$$\int_0^a e^{\lambda\tau-\nu a}\Pi(\tau,a)g(\tau)d\tau$$
$$= \frac{1}{\nu-\lambda}\left(W_\lambda - W_\nu\right)(\eta, g).$$

Moreover, one see (for example Magal and Ruan [33, Lemma 3.8.3]) that for all $\lambda > -\widetilde{\mu}$,

$$W_\lambda(\eta, g) = 0_{\mathcal{X}} \text{ only occurs if } \eta = 0_{X^M}, g = 0_{L^1((0,a^+),X^M)}$$

and

$$\lim_{\lambda\to\infty} \lambda W_\lambda(0_{X^M}, g) = (0_{X^M}, g),\ \forall (0_{X^M}, g) \in \mathcal{X}_0.$$

Moreover, for any $\lambda > -\widetilde{\mu}$, one has

$$\|W_\lambda\|_{\mathcal{L}(\mathcal{X},\mathcal{X}_0)} \leq \frac{1}{\lambda+\widetilde{\mu}}.$$

Thus by Pazy [39, Section 1.9] there exists a unique closed Hille–Yosida operator B_1 in $\mathcal{X}$ such that

$$(\lambda I - B_1)^{-1} = W_\lambda \text{ for all } \lambda > -\widetilde{\mu}.$$

Recalling (2.1) we also define the bounded linear operator $\mathcal{B}_2 \in \mathcal{L}(\mathcal{X}_0)$ given by

$$\mathcal{B}_2(0_{X^M}, g) = \left(0_{X^M}, \mathcal{D}\mathcal{K}g(\cdot)\right),\ \forall (0, g) \in \mathcal{X},\ \text{with } \mathcal{D} := \text{diag}\{D_1, \ldots, D_M\}.$$

2.2 Evolution Family With Diffusion

Consider now the following evolution equation for $\eta_i \in X$ and $0 \leq \tau \leq a < a^+$ and $i = 1, \ldots, M$:

$$\begin{cases} \partial_a u_i(a) = D_i(K-I)u_i(a) - \mu_i(a,\cdot)u_i(a),\ \tau < a < a^+, \\ u_i(\tau) = \eta_i \in X. \end{cases} \tag{2.8}$$

Define the evolution family

$$\{\mathcal{U}(\tau,a)\}_{0\leq\tau\leq a<a^+} = \text{diag}\left\{\mathcal{U}_1(\tau,a), \ldots, \mathcal{U}_M(\tau,a)\right\}_{0\leq\tau\leq a<a^+},$$

where $\mathcal{U}_i$ is associated with (2.8). Using the constant of variation formula $\mathcal{U}_i$ becomes for all $0 \leq \tau \leq a < a^+$ the solution of the equation

$$\mathcal{U}_i(\tau,a) = e^{-D_i(a-\tau)}\Pi_i(\tau,a) + D_i\int_\tau^a e^{-D_i(a-l)}\Pi_i(l,a)K\,\mathcal{U}_i(\tau,l)dl. \tag{2.9}$$

Note that the right hand side of (2.8) is linear and bounded with respect to u, thus the existence and uniqueness of $\{\mathcal{U}_i(\tau,a)\}_{0\leq\tau\leq a<a^+}$ can be obtained from the general semigroup theory (see [39]). Next let us prove that $\{\mathcal{U}_i(\tau,a)\}_{0\leq\tau\leq a<a^+}$ is exponentially bounded for each $i = 1, \ldots, M$.

To this aim fix $\phi = (\phi_1, \ldots, \phi_M) \in X^M$, $\tau \in [0, a^+)$ and set $u_i(a) = \mathcal{U}_i(\tau, a)\phi_i$. Then one has

$$\|u_i(a)\|_X \le e^{-(D_i+\widetilde{\mu})(a-\tau)}\|\phi_i\|_X + D_i\|K\|_{\mathcal{L}(X)}\int_\tau^a e^{-(D_i+\widetilde{\mu})(a-l)}\|u_i(l)\|_X dl.$$

Next Gronwall's inequality yields

$$\|u_i(a)\|_X\, e^{(D_i+\widetilde{\mu})(a-\tau)} \le \|\phi_i\|_X\, e^{D_i\|K\|_{\mathcal{L}(X)}(a-\tau)},$$

which implies due to (2.2) that

$$\|\mathcal{U}_i(\tau, a)\|_{\mathcal{L}(X)} \le e^{-\widetilde{\mu}(a-\tau)}.$$

As a consequence $\{\mathcal{U}(\tau, a)\}_{0\le\tau\le a<a^+}$ is positive and exponentially bounded in X^M and satisfies

$$\|\mathcal{U}(a, a+t)\|_{\mathcal{L}(X^M)} \le e^{-\widetilde{\mu}t},\ \forall t \ge 0, 0 \le a < a^+ - t. \tag{2.10}$$

Now we define the family of bounded linear operators $\{R_\lambda\}_{\lambda>-\widetilde{\mu}} \subset \mathcal{L}(\mathcal{X}, \mathcal{X}_0)$ as follows:

$$\begin{aligned} &R_\lambda(\eta, g) = (0_{X^M}, h) \\ &\text{with } h(a) = e^{-\lambda a}\mathcal{U}(0, a)\eta + \int_0^a e^{-\lambda(a-s)}\mathcal{U}(s, a)g(s)ds. \end{aligned}$$

Moreover, for any $\lambda > -\widetilde{\mu}$, one has

$$\|R_\lambda\|_{\mathcal{L}(\mathcal{X},\mathcal{X}_0)} \le \frac{1}{\lambda+\widetilde{\mu}}.$$

Then by the same procedure as the case without diffusion, we can prove that this provides a family of positive pseudoresolvents. Thus again by Pazy [39, Section 1.9] there exists a unique closed Hille–Yosida operator $\mathcal{B}$ in $\mathcal{X}$ such that

$$(\lambda I - \mathcal{B})^{-1} = R_\lambda \text{ for all } \lambda > -\widetilde{\mu}.$$

Next we define the part of $\mathcal{B}$ in $\mathcal{X}_0$, denoted by $\mathcal{B}_0$. That is,

$$\mathcal{B}_0 x = \mathcal{B}x,\ \forall x \in D(\mathcal{B}_0),\ \text{with } D(\mathcal{B}_0) := \{x \in D(\mathcal{B}) : \mathcal{B}x \in \mathcal{X}_0\}.$$

Note that $\mathcal{B}_0$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators on $\mathcal{X}_0$, denoted by $\{T_{\mathcal{B}_0}(t)\}_{t\ge0}$. Moreover, it satisfies the following estimate

$$\left\|T_{\mathcal{B}_0}(t)\right\|_{\mathcal{L}(\mathcal{X}_0)} \le e^{-\widetilde{\mu}t},\ \forall t \ge 0.$$

Observe now that we have $B_1 + \mathcal{B}_2 - \mathcal{D}I = \mathcal{B}$. From now on for the sake of convenience, we denote $\mathcal{B}_1 := B_1 - \mathcal{D}I$.

On the other hand, we define $\mathcal{C} \in \mathcal{L}(\mathcal{X}_0, \mathcal{X})$ by

$$\mathcal{C}(0_{X^M}, h) = \left(\int_0^{a^+}\beta(a, \cdot)h(a)da,\ 0_{L^1((0,a^+),X^M)}\right),\ (0_{X^M}, h) \in \mathcal{X}_0,$$

and $\mathcal{A} : dom(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ by

$$\begin{cases} dom(\mathcal{A}) = dom(\mathcal{B}) \subset \mathcal{X}_0, \\ \mathcal{A} = \mathcal{B} + \mathcal{C}. \end{cases} \tag{2.11}$$

This shows that $\mathcal{A}$ is not densely defined in $\mathcal{X}$. In addition, we will later use the matrix norm defined as follows:

$$\|\overline{\beta}\|_{\mathcal{L}([L^\infty(0,a^+)]^M)} := \max_{1\le i\le M} \sum_{j=1}^{M} \|\overline{\beta}_{ij}\|_{L^\infty(0,a^+)}.$$

Remark 2.1 In addition, for each fixed $x \in \overline{\Omega}$, following the above procedures, one can obtain the age-structured operator, denoted by $\mathcal{B}_1^x + \mathcal{C}^x$ defined on $\mathbb{R}^M \times L^1((0,a^+),\mathbb{R}^M)$.

Now define the map $F : \mathcal{X}_0 \to \mathcal{X}$ by

$$F\left(0_{X^M},\psi\right) = \left(f\left(\int_0^{a^+}\beta(a,\cdot)\psi(a)da\right),\ 0_{L^1((0,a^+),X^M)}\right),\quad f = \mathrm{diag}\{f_1,\dots,f_M\}.$$

Then by identifying $U(t) = \left(0_{X^M}, u(t)\right)$, one can write down problem (1.2) as the following abstract Cauchy problem:

$$\begin{cases} \frac{dU}{dt} = \mathcal{B}U + F(U), \\ U(0) = U_0, \end{cases} \quad \text{with } U_0 = \left(0_{X^M}, u_0\right). \tag{2.12}$$

As mentioned before, we will study the principal spectral theory of the linearized problem corresponding to (2.12), that is the principal spectral theory of $\mathcal{A} = \mathcal{B} + f'(0)\mathcal{C}$. For the sake of convenience, we first ignore the term $f'(0)$ before investigating the global dynamics of (1.2), see Sect. 7. Here $f'(0) = \mathrm{diag}\{f_1'(0),\dots,f_M'(0)\}$.

Finally, let us introduce briefly our idea in establishing the existence of principal eigenvalue. Observe that if $\alpha \in \rho(\mathcal{B}_1 + \mathcal{C})$, then the existence of nontrivial solutions of

$$\mathcal{A}u = (\mathcal{B}_2 + \mathcal{B}_1 + \mathcal{C})u = \alpha u$$

in $\mathcal{X}_0$ is equivalent to the existence of nontrivial solutions of

$$\mathcal{B}_2(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}v = v$$

in $\mathcal{X}$, where I is the identity operator. Next on one hand, we will prove that $\mathcal{A}$ is a positive and compact perturbation of $\mathcal{B}_1 + \mathcal{C}$ (see "Appendix" for precise definitions). On the the hand, we will provide an easily verifiable and general sufficient condition for $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$. Finally we can apply the theory of resolvent positive operators with their perturbations to study the existence of principal eigenvalue of our problem (1.1).

Before ending with this section, we would like to emphasize that when we use notations $<,\ \le,\ =,\ >$, and $\ge$, they could indicate the order in X or in X^M and in $\mathbb{R}$ or $\mathbb{R}^M$ depending on the context. For the sake of convenience, we will also omit subscripts of the zero elements in the function spaces.

3 Preliminaries

In this section we present some necessary propositions and lemmas 1) to figure out the existence of the spectral bounds of $\mathcal{B}_1 + \mathcal{C}$ and $\mathcal{A}$ which correspond to the evolution families without diffusion $\{\Pi(\tau,a)\}_{0\le\tau\le a<a^+}$ and with diffusion $\{\mathcal{U}(\tau,a)\}_{0\le\tau\le a<a^+}$ respectively; 2) to show that $\mathcal{A}$ is a positive and compact perturbation of $\mathcal{B}_1 + \mathcal{C}$. For convenience, we consider the kernel J without scaling, but the theory is valid for the kernels with scaling. We emphasize that the following results hold for both $X = C(\overline{\Omega})$ and $X = L^1(\Omega)$ if we do not indicate what X is exactly.

First we provide the following additional assumption throughout this section.

Assumption 3.1 Define

$$H_\alpha := \int_0^{a^+} \underline{\beta}(a) e^{-(\alpha+\mathcal{D})a} \overline{\Pi}(0,a)da,$$

where

$$\overline{\Pi}(\gamma, a) = \text{diag}\{e^{-\int_\gamma^a \overline{\mu}_1(s)ds}, \dots, e^{-\int_\gamma^a \overline{\mu}_M(s)ds}\}. \tag{3.1}$$

Assume that there exists $\alpha_0 \in \mathbb{R}$ such that $r(H_{\alpha_0}) > 1$.

3.1 Characterization of $s(\mathcal{B}_1 + \mathcal{C})$

Now recalling that the functions

$$\{\pi(\tau, a, x)\}_{0 \le \tau \le a < a^+, x \in \overline{\Omega}} = \text{diag}\{\pi_1(\tau, a, x), \dots, \pi_M(\tau, a, x)\}_{0 \le \tau \le a < a^+, x \in \overline{\Omega}}$$

are defined in (2.5), we define for $\alpha \in \mathbb{R}$ a continuous function $G_\alpha : \overline{\Omega} \to \mathcal{L}(\mathbb{R}^M)$, where $\mathcal{L}(\mathbb{R}^M)$ denotes all $M \times M$ matrices in $\mathbb{R}$

$$G_\alpha(x) = \int_0^{a^+} \beta(a, x) e^{-(\alpha+\mathcal{D})a} \pi(0, a, x)da, \ \forall x \in \overline{\Omega}. \tag{3.2}$$

We also consider for $\alpha \in \mathbb{R}$ a multiplication operator $\mathcal{G}_\alpha \in \mathcal{L}(X^M)$ given by

$$[\mathcal{G}_\alpha g](x) = G_\alpha(x) g(x), \ g \in X^M. \tag{3.3}$$

Then the following proposition holds.

Proposition 3.2 *Let Assumption 3.1 hold. Then there exists $\alpha^{**} \in (\alpha_0, \infty)$ satisfying the equation*

$$\max_{x \in \overline{\Omega}} r\left(G_{\alpha^{**}}(x)\right) = \max_{x \in \overline{\Omega}} r\left(\int_0^{a^+} \beta(a, x) e^{-(\alpha^{**}+\mathcal{D})a} \pi(0, a, x)da\right) = 1. \tag{3.4}$$

Moreover, $\mathcal{B}_1 + \mathcal{C}$ is a resolvent positive operator with $s(\mathcal{B}_1 + \mathcal{C}) = \alpha^{**}$ and

$$r\left(\mathcal{G}_{\alpha^{**}}\right) = r\left(\int_0^{a^+} \beta(a, \cdot) e^{-(\alpha^{**}+\mathcal{D})a} \Pi(0, a)da\right) = 1. \tag{3.5}$$

Proof Observe that the operator $\alpha I - \mathcal{B}_1 - \mathcal{C}$ is invertible if and only if the operator $I - \mathcal{C}(\alpha I - \mathcal{B}_1)^{-1}$ is invertible. In that case, we have

$$(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1} = (\alpha I - \mathcal{B}_1)^{-1}\left[I - \mathcal{C}(\alpha I - \mathcal{B}_1)^{-1}\right]^{-1}.$$

We now compute the inverse of $I - \mathcal{C}(\alpha I - \mathcal{B}_1)^{-1}$. To this aim choose $\alpha \in \rho(\mathcal{B}_1)$ and consider

$$(\widehat{\kappa}, \widehat{\varphi}) = \left[I - \mathcal{C}(\alpha I - \mathcal{B}_1)^{-1}\right](\kappa, \varphi).$$

First we define

$$(0, \phi) = (\alpha I - \mathcal{B}_1)^{-1}(\kappa, \varphi).$$

It follows that

$$\widehat{\varphi} = \varphi \text{ and } \widehat{\kappa} = \kappa - \int_0^{a^+} \beta(a, \cdot)\phi(a)da.$$

Next recall from (2.7) that one has

$$\phi(a) = e^{-(\alpha+\mathcal{D})a}\Pi(0, a)\kappa + \int_0^a e^{-(\alpha+\mathcal{D})(a-s)}\Pi(s, a)\varphi(s)ds.$$

It follows that

$$\begin{aligned}\kappa &- \int_0^{a^+} \beta(s, \cdot)e^{-(\alpha+\mathcal{D})s}\Pi(0, s)\kappa ds \\ &= \int_0^{a^+} \beta(s, x) \int_0^s e^{-(\alpha+\mathcal{D})(s-\tau)}\Pi(\tau, s)\widehat{\varphi}(\tau)d\tau ds + \widehat{\kappa},\end{aligned}$$

which is equivalent to

$$(I - \mathcal{G}_\alpha)\kappa = \int_0^{a^+} \beta(s, \cdot) \int_0^s e^{-(\alpha+\mathcal{D})(s-\tau)}\Pi(\tau, s)\widehat{\varphi}(\tau)d\tau ds + \widehat{\kappa}, \tag{3.6}$$

where $\mathcal{G}_\alpha$ is defined in (3.3). Thus if $1 \in \rho(\mathcal{G}_\alpha)$, then

$$\kappa = (I - \mathcal{G}_\alpha)^{-1}\left[\int_0^{a^+} \beta(s, \cdot) \int_0^s e^{-(\alpha+\mathcal{D})(s-\tau)}\Pi(\tau, s)\widehat{\varphi}(\tau)d\tau ds + \widehat{\kappa}\right], \tag{3.7}$$

which implies that

$$\begin{aligned}(\kappa, \varphi) &= \left[I - \mathcal{C}(\alpha I - \mathcal{B}_1)^{-1}\right]^{-1}(\widehat{\kappa}, \widehat{\varphi}) \\ &= \left((I - \mathcal{G}_\alpha)^{-1}\left[\int_0^{a^+} \beta(s, \cdot) \int_0^s e^{-(\alpha+\mathcal{D})(s-\tau)}\Pi(\tau, s)\widehat{\varphi}(\tau)d\tau ds + \widehat{\kappa}\right], \widehat{\varphi}\right).\end{aligned} \tag{3.8}$$

It follows that $\alpha \in \rho(\mathcal{B}_1 + \mathcal{C})$ and thus $(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}$ exists. Now we have shown that

$$\alpha \in \rho(\mathcal{B}_1 + \mathcal{C}) \cap \mathbb{R} \Leftrightarrow 1 \in \rho(\mathcal{G}_\alpha),$$

thus the problem is inverted into finding such α satisfying $1 \in \rho(\mathcal{G}_\alpha)$.

By assumptions on β and μ, we have

$$\mathcal{G}_\alpha g \geq \int_0^{a^+} \underline{\beta}(a)e^{-(\alpha+\mathcal{D})a}\overline{\Pi}(0, a)dag = H_\alpha g, \ g \in X^M, \tag{3.9}$$

Then one has from (3.9) that $\mathcal{G}_\alpha \geq H_\alpha$ in the sense of positive operators (actually H_α is a matrix function of α). Since $\underline{\beta}(a)$ is irreducible or primitive, H_α is also irreducible or primitive. Thus Perron–Frobenius theorem (see Proposition A.10 in "Appendix") applies and provides that the spectral radius $r(H_\alpha)$ is the principal eigenvalue of H_α. Moreover, observing that $r(H_\alpha)$ is continuous and decreasing with respect to α and satisfies

$$\lim_{\alpha\to\infty} r(H_\alpha) = 0, \ r(H_{\alpha_0}) > 1,$$

then there is a unique $\alpha^* \in (\alpha_0, \infty)$ such that

$$r(H_{\alpha^*}) = r\left(\int_0^{a^+} \underline{\beta}(a)e^{-(\alpha^*+\mathcal{D})a}\overline{\Pi}(0, a)da\right) = 1.$$

Now by the theory of positive operators (see [35]), we have immediately that $r(\mathcal{G}_{\alpha^*}) \geq r(H_{\alpha^*}) = 1$. Again observing that $r(\mathcal{G}_\alpha)$ is also a strictly decreasing continuous function with respect to α (or see the proof in Proposition 3.3), it follows that there exists a unique $\alpha^{**} \in \mathbb{R}$ satisfying $r(\mathcal{G}_{\alpha^{**}}) = 1$. Note that for any $\alpha \in \mathbb{R}$, when $\alpha > \alpha^{**}$ we have $r(\mathcal{G}_\alpha) < r(\mathcal{G}_{\alpha^{**}}) = 1$, $(I - \mathcal{G}_\alpha)^{-1}$ exists. It follows that $\alpha \in \rho(\mathcal{B}_1 + \mathcal{C})$ when $\alpha > \alpha^{**}$, which implies that $\rho(\mathcal{B}_1 + \mathcal{C})$ contains a ray (α^{**}, ∞). Further, $(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}$ is obviously a positive operator by (3.8) for all $\alpha > \alpha^{**}$. Thus $\mathcal{B}_1 + \mathcal{C}$ is a resolvent positive operator.

Moreover, α^{**} is larger than any other real spectral value in $\sigma(\mathcal{B}_1 + \mathcal{C})$. It follows that $\alpha^{**} = s_{\mathbb{R}}(\mathcal{B}_1 + \mathcal{C})$, where $s_{\mathbb{R}}(A) := \sup\{\lambda \in \mathbb{R}; \lambda \in \sigma(A)\}$. Now we have known that $\mathcal{B}_1 + \mathcal{C}$ is a resolvent positive operator. But since $\mathcal{X}$ is a Banach space with a normal and generating cone $\mathcal{X}^+$ and $s(\mathcal{B}_1 + \mathcal{C}) \geq \alpha^{**} > -\infty$ due to $\alpha^{**} \in \sigma(\mathcal{B}_1 + \mathcal{C})$, we can conclude from Theorem A.5 that $s(\mathcal{B}_1 + \mathcal{C}) = s_{\mathbb{R}}(\mathcal{B}_1 + \mathcal{C}) = \alpha^{**}$.

Next note that $\mathcal{G}_\alpha$ is actually a positive multiplication operator in X^M. We can obtain the spectral radius $r(\mathcal{G}_\alpha)$ of $\mathcal{G}_\alpha$ (see [31, Proposition 2.7]) via

$$r(\mathcal{G}_\alpha) = \max_{x \in \overline{\Omega}} r(G_\alpha(x)) = \max_{x \in \overline{\Omega}} r\left(\int_0^{a^+} \beta(a, x) e^{-(\alpha + \mathcal{D})a} \pi(0, a, x) da\right).$$

Thus α^{**} satisfies (3.4). □

3.2 Characterization of $s(\mathcal{A})$

Next we will prove that $\mathcal{A}$ is resolvent positive and provide a precise characterization of its spectral bound $s(\mathcal{A})$. Recall that $\{\mathcal{U}(\tau, a)\}_{0 \leq \tau \leq a < a^+}$ is defined in (2.9) and let us define for $\lambda \in \mathbb{R}$ the operator $\mathcal{M}_\lambda \in \mathcal{L}(X^M)$ by

$$\mathcal{M}_\lambda \phi = \int_0^{a^+} \beta(a, \cdot) e^{-\lambda a} \mathcal{U}(0, a) \phi \, da, \ \forall \phi \in X^M. \tag{3.10}$$

Then the following proposition holds.

Proposition 3.3 *There exists $\lambda_0 \in \mathbb{R}$ such that*

$$r(\mathcal{M}_{\lambda_0}) = r\left(\int_0^{a^+} \beta(a, \cdot) e^{-\lambda_0 a} \mathcal{U}(0, a) \, da\right) = 1. \tag{3.11}$$

Moreover, the operator $\mathcal{A}$ is resolvent positive and its spectral bound satisfies $s(\mathcal{A}) = \lambda_0$.

Proof Consider the resolvent equation

$$(0, \phi) = (\lambda I - \mathcal{A})^{-1}(\zeta, \varphi), \ \forall (\zeta, \varphi) \in \mathcal{X}, \ \lambda \in \rho(\mathcal{A}),$$

following the same procedure in Proposition 3.2, we can obtain

$$\begin{aligned} &[(\lambda I - \mathcal{A})^{-1}(\zeta, \varphi)](a, \cdot) \\ &= \Big(0, \quad e^{-\lambda a} \mathcal{U}(0, a)(I - \mathcal{M}_\lambda)^{-1} \left[\int_0^{a^+} \beta(s, \cdot) \int_0^s e^{-\lambda(s-\tau)} \mathcal{U}(\tau, s) \varphi(\tau) d\tau ds + \zeta\right] \\ &\quad + \int_0^a e^{-\lambda(a-\tau)} \mathcal{U}(\tau, a) \varphi(\tau) d\tau\Big). \end{aligned} \tag{3.12}$$

It follows that $\lambda \in \rho(\mathcal{A}) \cap \mathbb{R} \Leftrightarrow 1 \in \rho(\mathcal{M}_\lambda)$. Now define an operator $\mathcal{C}_\lambda : X^M \to X^M$ for $\lambda \in \mathbb{C}$ by

$$\mathcal{C}_\lambda \phi := \int_0^{a^+} \underline{\beta}(a) e^{-\lambda a} \overline{\Pi}(0, a) e^{\mathcal{D}(\mathcal{K}-I)a} \phi da, \ \forall \phi \in X^M,$$

where $\overline{\Pi}(0, a)$ is defined in (3.1) and $\{e^{\mathcal{D}(\mathcal{K}-I)a}\}_{a\geq 0}$ denotes the strongly continuous semigroup generated by the bounded operator $\mathcal{D}(\mathcal{K} - I)$. We can see from the assumptions on β and μ that $\mathcal{M}_\lambda \geq \mathcal{C}_\lambda$ in the positive operator sense.

Now we claim that $r(\mathcal{M}_\lambda)$ is decreasing and log-convex (and thus continuous) with respect to the parameter $\lambda \in \mathbb{R}$.

Claim 3.4 *$r(\mathcal{M}_\lambda)$ is decreasing and log-convex with respect to $\lambda \in \mathbb{R}$.*

For now let us assume that the claim is true. On the other hand, from Theorem A.2-(iv) in the "Appendix", there exists a unique simple real value ξ_0 such that $r(\mathcal{C}_{\xi_0}) = 1$. Therefore, by the theory of positive operators,

$$r(\mathcal{M}_{\xi_0}) \geq r(\mathcal{C}_{\xi_0}) = 1.$$

Moreover, $\lim_{\lambda\to\infty} r(\mathcal{M}_\lambda) = 0$. Since $r(\mathcal{M}_\lambda)$ is continuous and decreasing with respect to λ by Claim 3.4, there exists a real $\lambda_0 \geq \xi_0$ such that $r(\mathcal{M}_{\lambda_0}) = 1$.

Next let us prove that λ_0 is unique. To this aim, assume that there is $\lambda_\varsigma < \lambda_\mu$ such that $r(\mathcal{M}_{\lambda_\varsigma}) = r(\mathcal{M}_{\lambda_\mu}) = 1$. Since $\lambda \to r(\mathcal{M}_\lambda)$ is decreasing and log convex, it follows that $r(\mathcal{M}_\lambda) = 1$ for all $\lambda \geq \lambda_\varsigma$. This contradicts the fact that $r(\mathcal{M}_\lambda) \to 0$ as $\lambda \to \infty$. Thus there is a unique $\lambda_0 \in \mathbb{R}$ such that $r(\mathcal{M}_{\lambda_0}) = 1$. This is equivalent to the uniqueness of λ_0. Moreover, we have shown that the mapping $\lambda \to r(\mathcal{M}_\lambda)$ is strictly decreasing on the interval $(-\infty, \infty)$.

In addition, since $\mathcal{M}_\lambda$ is positive, $1 = r(\mathcal{M}_{\lambda_0}) \in \sigma(\mathcal{M}_{\lambda_0}) \neq \emptyset$, which implies that $\lambda_0 \in \sigma(\mathcal{A})$, thus $\sigma(\mathcal{A}) \neq \emptyset$. At last, the conclusion that $s(\mathcal{A}) = \lambda_0$ follows by the same argument in Proposition 3.2, we omit it. Moreover, $\mathcal{A}$ is resolvent positive since $\rho(\mathcal{A})$ contains a ray (λ_0, ∞) and $(\lambda I - \mathcal{A})^{-1}$ is positive for all $\lambda > \lambda_0$ by (3.12). □

Now let us prove the above claim.

Proof of Claim 3.4 We use the generalized Kingman's theorem from Kato [27] to show it. First we claim that $\lambda \to \mathcal{M}_\lambda$ is completely monotonic. Then $\lambda \to r(\mathcal{M}_\lambda)$ is decreasing and super-convex by Thieme [47, Theorem 2.5] and hence log-convex. By the definition from Thieme [47], an infinitely often differentiable function $f : (\Delta, \infty) \to Z_+$ is said to be *completely monotonic* if

$$(-1)^n f^{(n)}(\lambda) \in Z_+, \ \forall \lambda > \Delta, n \in \mathbb{N},$$

where Z_+ is a normal and generating cone of an ordered Banach space Z and (Δ, ∞) is the domain of f. A family $\{F_\lambda\}_{\lambda>a}$ of positive operators on Z is said to be *completely monotonic* if $f(\lambda) = F_\lambda x$ is completely monotonic for every $x \in Z_+$. For our case, $\mathcal{M}_\lambda$ is indeed infinitely often differentiable with respect to $\lambda \in \mathbb{R}$ and

$$(-1)^n \mathcal{M}_\lambda^{(n)} \phi = \int_0^{a^+} \beta(a, \cdot) a^n e^{-\lambda a} \mathcal{U}(0, a) \phi da \in X_+^M, \ \lambda \in \mathbb{R}, n \in \mathbb{N}, \phi \in X_+^M.$$

Thus, our result follows. □

Remark 3.5 From the above Proposition 3.3, we can obtain that $s(\mathcal{A}) \geq s(\mathcal{B}_1 + \mathcal{C})$ since $\mathcal{A}$ is resolvent positive. Theorem A.6 applies and provides that case (i) was ruled out. But we cannot obtain the strict relation, i.e. $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$, because α^{**} and λ_0 are obtained by taking the spectral radius of the operators equal to 1 (see Propositions 3.2 and 3.3) where a limit process occurs in which the strict relation may not be preserved. However, if $r(\mathcal{G}_\alpha)$ and $r(\mathcal{M}_\lambda)$ are eigenvalues of $\mathcal{G}_\alpha$ and $\mathcal{M}_\lambda$ respectively, we could obtain the strict relation, see Marek [35, Theorem 4.3] which is the Frobenius theory for positive operators.

3.3 A Special Case: $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$

Next, we give a special case where $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$ holds.

Proposition 3.6 *Assume that $\mu(a, x) \equiv \mu(a)$, $\beta(a, x) \equiv \beta(a)$ and $D_i = D$ for all $i = 1, \ldots, M$, then one has $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$.*

Proof Note that when $\mu(a, x) \equiv \mu(a)$ and $\beta(a, x) \equiv \beta(a)$, $s(\mathcal{B}_1 + \mathcal{C}) = \alpha^{**}$ and $s(\mathcal{A}) = \lambda_0$ satisfies the following equations

$$r\left(\int_0^{a^+} \beta(a) e^{-\alpha^{**} a} e^{-Da} e^{-\int_0^a \mu(s)ds} da\right) = 1 \tag{3.13}$$

and

$$r(\mathcal{M}_{\lambda_0}) = r\left(\int_0^{a^+} \beta(a) e^{-\lambda_0 a} e^{-\int_0^a \mu(s)ds} e^{D(\mathcal{K}-I)a} da\right) = 1, \tag{3.14}$$

respectively. It is known from García-Melián and Rossi [21, Theorem 2.1] that the operator $-L$ defined by

$$L\varphi := \int_\Omega J(\cdot - y)\varphi(y)dy - \varphi, \quad \varphi \in C(\overline{\Omega}),$$

has a principal and simple eigenvalue $0 < \theta_0 < 1$ associated with a positive eigenfunction φ_0. It follows that $-D\mathcal{K} + DI$ has a principal eigenvalue $D\theta_0$ associated with an eigenfunction $\varphi = \{\varphi_0, \ldots, \varphi_0\}$ in the sense that each component of eigenfunctions is positive and θ_0 is isolated. Note that θ_0 is not simple any more. Further, from Theorem A.2 we have shown that $\mathcal{M}_{\kappa_0 - D\theta_0}$ has an eigenvalue associated to 1 with a positive eigenfunction $\Phi_0 \in X^M$ and

$$r(\mathcal{M}_{\kappa_0 - D\theta_0}) = 1, \tag{3.15}$$

where κ_0 is the principal eigenvalue of the multigroup age-structured operator; i.e. κ_0 satisfies the following characteristic equation

$$r\left(\int_0^{a^+} \beta(a) e^{-\kappa_0 a} e^{-\int_0^a \mu(s)ds} da\right) = 1. \tag{3.16}$$

Now comparing (3.13) with (3.16) and (3.15) with (3.14), we have $\alpha^{**} = \kappa_0 - D$ while $\lambda_0 = \kappa_0 - D\theta_0$. It is obvious that $\lambda_0 > \alpha^{**}$, which implies that $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$. □

3.4 A Key Proposition

Next we give a key proposition on the solvability of the following equation, which is important in studying the effects of diffusion rate on the principal eigenvalue later. Consider the problem

$$\begin{cases} \partial_a u_i(a,x) = -(\alpha + D_i)u_i(a,x) - \mu_i(a,x)u_i(a,x), & (a,x) \in (0,a^+) \times \overline{\Omega}, \\ u_i(0,x) = \sum_{j=1}^M \int_0^{a^+} \beta_{ij}(a,x)u_j(a,x)da, & x \in \overline{\Omega}. \end{cases} \tag{3.17}$$

Proposition 3.7 *Let Assumption 3.1 hold. Then there exists a continuous function $x \to \alpha(x) : \overline{\Omega} \to \mathbb{R}$ such that for any $x \in \overline{\Omega}$, equation (3.17) with $\alpha = \alpha(x)$ has a positive solution $a \to u(a,x) = (u_1(a,x), \dots, u_M(a,x)) \in W^{1,1}((0,a^+), \mathbb{R}^M)$ and*

$$r\left(\int_0^{a^+} \beta(a,x)e^{-(\alpha(x)+\mathcal{D})a}\pi(0,a,x)da\right) = 1, \ \forall x \in \overline{\Omega}. \tag{3.18}$$

Moreover, $\alpha(x) \le \alpha^{**}$ for all $x \in \overline{\Omega}$, where α^{**} is defined in (3.4).

Proof Solving (3.17) explicitly, we obtain a formal positive solution

$$u_i(a,x) = e^{-(\alpha+D_i)a}\pi_i(0,a,x)u_i(0,x)$$

provided $u(0,x) = (u_1(0,x), \dots, u_M(0,x)) > 0$. Then plugging it into the integral initial condition we get that

$$\sum_{j=1}^M \int_0^{a^+} \beta_{ij}(a,x)e^{-(\alpha+D_j)a}\pi_j(0,a,x)u_j(0,x)da = u_i(0,x).$$

Now define

$$G(\alpha,x) := G_\alpha(x) = \int_0^{a^+} \beta(a,x)e^{-(\alpha+\mathcal{D})a}\pi(0,a,x)da.$$

Observe that $G : \mathbb{R} \times \overline{\Omega} \to \mathcal{L}(\mathbb{R}^M)$ is continuously differentiable with respect to α and continuous with respect to x respectively due to the assumptions on β and μ, where $\mathcal{L}(\mathbb{R}^M)$ denotes all $M \times M$ matrices in $\mathbb{R}$. Moreover, for any $x \in \overline{\Omega}$, one has by Assumption 3.1 that

$$\lim_{\alpha\to\infty} r\left(G(\alpha,x)\right) = 0, \ \ r\left(G(\alpha_0,x)\right) \ge r(H_{\alpha_0}) > 1. \tag{3.19}$$

Thus for any $x \in \overline{\Omega}$, thanks to the monotonicity of G with respect to α, there always exists a unique $\alpha(x)$ such that (3.18) hold.

Next let us prove that α is continuous. Observe that

$$\frac{\partial G(\alpha,x)}{\partial\alpha} = -\int_0^{a^+} \beta(a,x)ae^{-(\alpha+\mathcal{D})a}\Pi(0,a,x)da, \ \forall x \in \overline{\Omega}. \tag{3.20}$$

Thus now for any $x \in \overline{\Omega}$, $-\frac{\partial G}{\partial\alpha}(\alpha,x)$ is irreducible or primitive and nonnegative. Since the spectral radius of an irreducible and nonnegative matrix is coming from a simple root of the corresponding characteristic polynomial, we have by implicit function theorem (applying to the characteristic polynomial) that for any $x \in \overline{\Omega}$, $r(G(\cdot,x))$ is smooth in $\mathbb{R}$. Moreover, for any $x \in \overline{\Omega}$, $G(\cdot,x)$ is decreasing in the matrix sense, which implies by Perron–Frobenius theorem (see Proposition A.10) that

$$\frac{\partial r(G(\alpha,x))}{\partial\alpha} < 0.$$

The continuity of α comes again from implicit function theorem (applying to the spectral radius). In addiction, one has that $\alpha(x) \leq \alpha^{**}$ by (3.4) since $\alpha^{**} = \max_{x\in\overline{\Omega}} \alpha(x)$ due to the monotonicity of G_α with respect to α. Thus the proposition is proved. □

3.5 Compact Perturbation

In this subsection, we will show that $\mathcal{A}$ is a compact and positive perturbation of $\mathcal{B}_1 + \mathcal{C}$.

Proposition 3.8 *For any real number $\alpha > \alpha^{**}$, $\mathcal{B}_2(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}$ is a compact operator in $\mathcal{X}$.*

Proof We only prove the result in the case $X = C(\overline{\Omega})$, since

$$L^1((0,a^+),[C(\overline{\Omega})]^M) \subset L^1((0,a^+),[L^1(\Omega)]^M).$$

Let us choose a sequence $\{(\eta_n,\psi_n)\}_{n\in\mathbb{N}} \subset \mathcal{X}$ satisfying

$$\|(\eta_n,\psi_n)\|_{\mathcal{X}} := \|\psi_n\|_{L^1((0,a^+),X^M)} + \|\eta_n\|_{X^M} \leq 1, \text{ for any } n \in \mathbb{N}.$$

By (3.8) we have for Re$\alpha > \alpha^{**}$ that

$$\mathcal{B}_2(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}(\eta_n,\psi_n) = (0,\phi_n) = (0, \mathcal{D}\mathcal{K}g_{1n} + \mathcal{D}\mathcal{K}g_{2n}),$$

where

$$\begin{aligned} g_{1n}(a) &= e^{-(\alpha+\mathcal{D})a}\Pi(0,a)(I-\mathcal{G}_\alpha)^{-1} \\ &\qquad \Big[\int_0^{a^+}\beta(s,\cdot)\int_0^s e^{-(\alpha+\mathcal{D})(s-\tau)}\Pi(\tau,s)\psi_n(\tau)d\tau ds + \eta_n\Big], \\ g_{2n}(a) &= \int_0^a e^{-(\alpha+\mathcal{D})(a-\tau)}\Pi(\tau,a)\psi_n(\tau)d\tau. \end{aligned} \tag{3.21}$$

Note that g_{1n} and g_{2n} are continuous with respect to $a \in [0,a^+]$, so is ϕ_n. In the following context, we denote $D_{\max} := \max_{1\leq i\leq M} D_i$ and $D_{\min} := \min_{1\leq i\leq M} D_i$.

First observe that when $\alpha > \alpha^{**}$, one has

$$\begin{aligned} \|g(\eta_n,\psi_n)\|_{X^M} :&= \left\|(I-\mathcal{G}_\alpha)^{-1}\Big[\int_0^{a^+}\beta(s,\cdot)\int_0^s e^{-(\alpha+\mathcal{D})(s-\tau)}\Pi(\tau,s)\psi_n(\tau)d\tau ds + \eta_n\Big]\right\|_{X^M} \\ &\leq C_\alpha\Big[\int_0^{a^+}\|\overline{\beta}\|_{\mathcal{L}([L^\infty(0,a^+)]^M)} \\ &\qquad \int_0^s e^{-(\alpha+D_{\min})(s-\tau)}\|\Pi(\tau,s)\|_{\mathcal{L}(X^M)}\|\psi_n(\tau)\|_{X^M}d\tau ds + \|\eta_n\|_{X^M}\Big] \\ &\leq C_\alpha\Big[\|\overline{\beta}\|_{\mathcal{L}([L^\infty(0,a^+)]^M)}\int_0^{a^+}\|\psi_n(\tau)\|_{X^M}d\tau \\ &\qquad \int_\tau^{a^+} e^{-(\alpha+D_{\min}+\widetilde{\mu})(s-\tau)}ds + \|\eta_n\|_{X^M}\Big] \\ &\leq C_\alpha\left[\frac{\|\overline{\beta}\|_{\mathcal{L}([L^\infty(0,a^+)]^M)}}{\alpha+D_{\min}+\widetilde{\mu}}\|\psi_n\|_{L^1((0,a^+),X^M)} + \|\eta_n\|_{X^M}\right] \\ &\leq C_\alpha\left[\frac{\|\overline{\beta}\|_{\mathcal{L}([L^\infty(0,a^+)]^M)}}{\alpha+D_{\min}+\widetilde{\mu}} + 1\right] =: \widetilde{C}_\alpha, \end{aligned} \tag{3.22}$$

where we used the fact that $\left\|(I-\mathcal{G}_\alpha)^{-1}\right\|_{\mathcal{L}(X^M)} \le C_\alpha$ with $C_\alpha > 0$ being a constant, due to $\alpha > \alpha^{**}$. Here $\widetilde{C}_\alpha > 0$ is another constant.

Next, one has

$$\|g_{1n}\|_{L^1((0,a^+),X^M)} \le \widetilde{C}_\alpha \int_0^{a^+} e^{-(\alpha+D_{\min}+\widetilde{\mu})a} da \le \frac{\widetilde{C}_\alpha}{\alpha+D_{\min}+\widetilde{\mu}},$$
$$\|g_{2n}\|_{L^1((0,a^+),X^M)} \le \int_0^{a^+} \|\psi_n(\tau)\|_{X^M}\, d\tau \int_\tau^{a^+} e^{-(\alpha+D_{\min}+\widetilde{\mu})(a-\tau)} da \le \frac{1}{\alpha+D_{\min}+\widetilde{\mu}}.$$

It follows from (2.2) that for any $n \in \mathbb{N}$,

$$\|\phi_n\|_{L^1((0,a^+),X^M)} \le \frac{D_{\max}}{\alpha+D_{\min}+\widetilde{\mu}} \left[\widetilde{C}_\alpha + 1\right]. \tag{3.23}$$

Moreover, thanks to the presence of the continuous kernel J, one can obtain that the functions $\{\phi_n\}_{n\in\mathbb{N}}$ are equicontinuous with respect to $x \in \overline{\Omega}$. It follows by Arzela-Ascoli theorem that $\{\phi_n(a,\cdot)\}_{n\in\mathbb{N}}$ is compact in $X^M = C(\overline{\Omega}, \mathbb{R}^M)$ for any $a \in [0, a^+]$.

Next let us show that g_{1n} and g_{2n} are equi-integrable respect to a. Observe by (3.21) that for any $n \in \mathbb{N}$ and $l > 0$, one has (note that $e^{-(a+\mathcal{D})(a-\tau)}$ and $\pi(\tau, a, \cdot)$ are commuted since they are diagonal matrices)

$$\begin{aligned}
&|g_{2n}(a+l) - g_{2n}(a)| \\
&\le \int_a^{a+l} e^{-(\alpha+\mathcal{D})(a+l-\tau)} \pi(\tau, a+l, \cdot)\psi_n(\tau)d\tau \\
&\quad + \int_0^a \left[e^{-(\alpha+\mathcal{D})(a+l-\tau)}\pi(\tau, a+l, \cdot) - e^{-(\alpha+\mathcal{D})(a-\tau)}\pi(\tau, a, \cdot)\right]\psi_n(\tau)d\tau \\
&\le \int_a^{a+l} e^{-(\alpha+\mathcal{D})(a+l-\tau)} \pi(\tau, a+l, \cdot)\psi_n(\tau)d\tau \\
&\quad + \int_0^a e^{-(\alpha+\mathcal{D})(a-\tau)}\pi(\tau, a+l, \cdot)\left[I - e^{-(\alpha+\mathcal{D})l}\right]\psi_n(\tau)d\tau \\
&\quad + \int_0^a e^{-(\alpha+\mathcal{D})(a-\tau)}\pi(\tau, a, \cdot)\left[I - \pi(a, a+l, \cdot)\right]\psi_n(\tau)d\tau \\
&\le \int_a^{a+l} e^{-(\alpha+\mathcal{D}+\widetilde{\mu})(a+l-\tau)}\psi_n(\tau)d\tau \\
&\quad + \int_0^a e^{-(\alpha+\mathcal{D}+\widetilde{\mu})(a-\tau)} e^{-\widetilde{\mu}l}\left[I - e^{-(\alpha+\mathcal{D})l}\right]\psi_n(\tau)d\tau \\
&\quad + \int_0^a e^{-(\alpha+\mathcal{D}+\widetilde{\mu})(a-\tau)}\left[I - e^{-\widetilde{\mu}l} I\right]\psi_n(\tau)d\tau.
\end{aligned}$$

It follows by setting $k = \alpha + D_{\min} + \widetilde{\mu}$ that

$$\begin{aligned}
&\int_0^{a^+} \|g_{2n}(a+l) - g_{2n}(a)\|_{X^M} da \\
&\le \int_0^{a^+} \int_a^{a+l} e^{-k(a+l-\tau)} \|\psi_n(\tau)\|_{X^M} d\tau da \\
&\quad + \int_0^{a^+} \int_0^a e^{-k(a-\tau)} e^{-\widetilde{\mu}l}\left[1 - e^{-(\alpha+D_{\max})l}\right] \|\psi_n(\tau)\|_{X^M} d\tau da
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{a^+}\int_0^{a} e^{-k(a-\tau)}\left[1-e^{-\widetilde{\mu}l}\right]\|\psi_n(\tau)\|_{X^M}\,d\tau da\\
&:= \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3.
\end{aligned}
$$

Via integration by parts, one has

$$
\begin{aligned}
\mathrm{I}_2 &\le e^{-\widetilde{\mu}l}\left[1-e^{-(\alpha+D_{\max})l}\right]\int_0^{a^+}\int_\tau^{a^+} e^{-k(a-\tau)}da\,\|\psi_n(\tau)\|_{X^M}\,d\tau\\
&\le \frac{1}{k}\left[1-e^{-(\alpha+D_{\max})l}\right]\|\psi_n\|_{L^1((0,a^+),X^M)} \overset{l\to 0}{\longrightarrow} 0,\ \text{uniformly in } n\in\mathbb{N}.
\end{aligned}
$$

Similarly, one also obtain $\mathrm{I}_3 \to 0$ as $l \to 0$ uniformly in $n \in \mathbb{N}$.

Next let us deal with I_1. To this aim, we split it into two cases: $0 \le a \le \tau \le a+l \le a^+$ and $0 \le a \le \tau \le a^+ \le a+l$.

Case $0 \le a \le \tau \le a+l \le a^+$. Via integration by parts, one has

$$
\mathrm{I}_1 \le \int_0^{a^+}\int_{\tau-l}^{\tau} e^{-k(a+l-\tau)}da\,\|\psi_n(\tau)\|_{X^M}\,d\tau \overset{l\to 0}{\longrightarrow} 0,\ \text{uniformly in } n\in\mathbb{N}.
$$

Case $0 \le a \le \tau \le a^+ \le a+l$. Via integration by parts, one has

$$
\begin{aligned}
\mathrm{I}_1 &\le \int_0^{a+l}\int_a^{a+l} e^{-k(a+l-\tau)}\,\|\psi_n(\tau)\|_{X^M}\,d\tau da\\
&\le \int_0^{a+l}\int_{\tau-l}^{\tau} e^{-k(a+l-\tau)}da\,\|\psi_n(\tau)\|_{X^M}\,d\tau \overset{l\to 0}{\longrightarrow} 0,\ \text{uniformly in } n\in\mathbb{N}.
\end{aligned}
$$

In summary, we have shown that $\int_0^{a^+}\|g_{2n}(a+l)-g_{2n}(a)\|_{X^M}\,da \to 0$ as $l \to 0$ uniformly in $n \in \mathbb{N}$. Similarly, one can show by (3.22) that $\int_0^{a^+}\|g_{1n}(a+l)-g_{1n}(a)\|_{X^M}\,da \to 0$ as $l \to 0$ uniformly in $n \in \mathbb{N}$. It follows that $\int_0^{a^+}\|\phi_n(a+l)-\phi_n(a)\|_{X^M}\,da \to 0$ as $l \to 0$ uniformly in $n \in \mathbb{N}$. Combining with (3.23), $\{a \to \phi_n(a)\}_{n\in\mathbb{N}}$ is compact in $L^1((0,a^+),X^M)$. Thus for any $a \in [0,a^+]$ there exists a limit $\phi(a) \in X^M$ such that, up to a subsequence, $\phi_n \to \phi$ in $L^1((0,a^+),X^M)$. Hence the operator $\mathcal{B}_2(\alpha I-\mathcal{B}_1-\mathcal{C})^{-1}$ is compact on $\mathcal{X}$. □

Corollary 3.9 *The operator $\mathcal{B}_2$ is a compact perturbator of $\mathcal{B}_1+\mathcal{C}$ and the operator $\mathcal{A} = \mathcal{B}_1+\mathcal{B}_2+\mathcal{C}$ a compact perturbation of $\mathcal{B}_1+\mathcal{C}$.*

Proof $(\alpha I-\mathcal{B}_1-\mathcal{C})^{-1}\mathcal{B}_2(\alpha I-\mathcal{B}_1-\mathcal{C})^{-1}$ is compact for some $\alpha > s(\mathcal{B}_1+\mathcal{C})$ since $\mathcal{B}_2(\alpha I-\mathcal{B}_1-\mathcal{C})^{-1}$ is compact by Proposition 3.8. □

4 Principal Spectral Theory

In this section, we state and prove the main results on the existence of principal eigenvalues. We will assume the existence of $s(\mathcal{B}_1+\mathcal{C})$ throughout this section. First we provide a sufficient condition to make the spectral bound $s(\mathcal{A})$ become the principal eigenvalue.

4.1 Principal Eigenvalue

Theorem 4.1 *Assume $s(\mathcal{A}) > s(\mathcal{B}_1+\mathcal{C})$, then $s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$.*

Proof Denote

$$\mathcal{F}_\lambda = \mathcal{B}_2(\lambda I - \mathcal{B}_1 - \mathcal{C})^{-1}, \quad \mathrm{Re}\lambda > \alpha^{**}. \tag{4.1}$$

Note that $\mathcal{A} = \mathcal{B}_1 + \mathcal{C} + \mathcal{B}_2$ is a compact perturbation of $\mathcal{B}_1 + \mathcal{C}$ by Corollary 3.9. We will use Theorem A.9 to prove the conclusion. First, we know that $\mathcal{A}$ is resolvent positive by Proposition 3.3. It follows that case (i) in Theorem A.6 will be ruled out. Secondly, by the assumption $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$ we know that only case (iii) in Theorem A.6 will happen, otherwise $s(\mathcal{A}) = s(\mathcal{B}_1 + \mathcal{C})$ which is a contradiction if case (ii) in Theorem A.6 would happen. Hence, there exists $\lambda_2 > \lambda_1 > s(\mathcal{B}_1 + \mathcal{C})$ such that $r(\mathcal{F}_{\lambda_1}) \geq 1 > r(\mathcal{F}_{\lambda_2})$. Now the hypothesis in Theorem A.9 holds, then $s(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with a positive eigenfunction, has finite algebraic multiplicity, and is a pole of the resolvent of $\mathcal{A}$. It follows that $s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$. □

Combining the above theorem with Proposition 3.6, one can immediately obtain the following conclusion.

Corollary 4.2 *Assume $\mu(a, x) \equiv \mu(a)$ and $\beta(a, x) \equiv \beta(a)$ and in addition, $D_i = D$ for all $1 \leq i \leq M$, then $s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$.*

Next, we give a sufficient and necessary condition to reach $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$.

Corollary 4.3 *The inequality $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$ holds if and only if there is $\lambda^* > s(\mathcal{B}_1 + \mathcal{C})$ such that $r(\mathcal{F}_{\lambda^*}) \geq 1$, where $\mathcal{F}_\lambda$ is defined in (4.1).*

Proof If there exists $\lambda^* > s(\mathcal{B}_1 + \mathcal{C})$ such that $r(\mathcal{F}_{\lambda^*}) \geq 1$, then case (iii) in Theorem A.6 will happen which implies that $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$, because we can always find ϑ large enough such that $r(\mathcal{F}_\vartheta) < 1$ regarding to (3.21). Conversely, if $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$, by the same argument in Theorem 4.1, we have the desired result. □

Note that Theorem 4.1 is valid for both $X = L^1(\Omega)$ and $X = C(\overline{\Omega})$, as long as $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$. Next we will show that $s(\mathcal{A})$ is also algebraically simple under the additional assumption on β. Once it is true, the eigenfunctions in $X = L^1(\Omega)$ and $X = C(\overline{\Omega})$ respectively associated with $s(\mathcal{A})$ are the same, due to $C(\overline{\Omega}) \subset L^1(\Omega)$.

Assumption 4.4 There exists $a_1 \in [0, a^+)$ such that $\underline{\beta}_{ii}(a) > 0$ a.e. $[a_1, a^+)$ for all $1 \leq i \leq M$.

Remark 4.5 Before proceeding, let us make some comments on Assumption 4.4. It is motivated by Engel and Nagel [17, Theorem 4.4] to show that the semigroup generated by the age-structured operator is irreducible. In our situation, we will prove a similar property, which is called conditionally strictly positive (see Definition A.8 in "Appendix"), under this assumption. In addition, if one would like to relate this assumption to cooperativity, this assumption can be relaxed to $\underline{\beta}_{im(i)}(a) > 0$ a.e. in $[a_1, a^+)$ for some $1 \leq i \leq M$ and all $m(i) \in \{1, 2, \ldots, M\}$ are different to each other.

Theorem 4.6 *Let Assumption 4.4 hold and assume $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$, then $s(\mathcal{A})$ is the algebraically simple principal eigenvalue of $\mathcal{A}$.*

Proof We will show that all positive nonzero fixed points of $\mathcal{F}_\lambda$ are conditionally strictly positive (see Definition A.8 in "Appendix"), and then employ Theorem A.9 again to conclude the result.

First observe that $\mathcal{F}_\lambda$ maps $\mathcal{X}$ into $\mathcal{X}_0$, then we introduce the restriction of $\mathcal{F}_\lambda$ to $\mathcal{X}_0$ and the associated operator L_λ, $\lambda > \alpha^{**}$ in $L^1((0, a^+), X^M)$, see (3.21),

$$\begin{aligned} [L_\lambda \psi](a, x) = \mathcal{D} \int_\Omega \mathcal{J}(x-y) e^{-(\lambda+\mathcal{D})a} \pi(0, a, y)[(I - G_\lambda)^{-1} \tilde{g}\psi](y) dy \\ + \mathcal{D} \int_\Omega \mathcal{J}(x-y) \int_0^a e^{-(\lambda+\mathcal{D})(a-\gamma)} \pi(\gamma, a, y) \psi(\gamma, y) d\gamma dy, \end{aligned} \tag{4.2}$$

where $\mathcal{J} = \operatorname{diag}\{J, \ldots, J\}$ and $\tilde{g} : L^1((0, a^+), X^M) \to X^M$ is given by

$$[\tilde{g}\psi](y) := \int_0^{a^+} \beta(s, y) \int_0^s e^{-(\lambda+\mathcal{D})(s-\gamma)} \pi(\gamma, s, y) \psi(\gamma, y) d\gamma ds.$$

We use L_λ both for the operator in $L^1((0, a^+), X^M)$ and the operator in $\mathcal{X}_0 = \{0\} \times L^1((0, a^+), X^M)$. Observe that $(a, x) \to [L_\lambda \psi](a, x)$ is continuous. Thus L_λ is strictly positive in the sense that for $\psi \in L^1_+((0, a^+), X^M)$ being a fixed point of L_λ, if there exists some point $(a_0, x_0) \in [0, a^+) \times \overline{\Omega}$ such that $[L_\lambda \psi](a_0, x_0) = 0_{\mathbb{R}^M}$, then $\psi \equiv 0_{\mathbb{R}^M}$ in $[0, a^+) \times \overline{\Omega}$.

In fact, $[L_\lambda \psi](a_0, x_0) = 0_{\mathbb{R}^M}$ implies that

$$\mathcal{D} \int_\Omega \mathcal{J}(x_0 - y) e^{-(\lambda+\mathcal{D})a_0} \pi(0, a_0, y)[(I - G_\lambda)^{-1} \tilde{g}\psi](y) dy = 0_{\mathbb{R}^M},$$

which follows by the positivity of $\int_\Omega \mathcal{J}(x_0 - y) dy$ and $(I - G_\lambda)^{-1} = \sum_{n=0}^\infty G_\lambda^n$, $\lambda > \alpha^{**}$, along with exponential functions that

$$\int_0^{a^+} \beta(s, y) \int_0^s \psi(\gamma, y) d\gamma ds = 0_{\mathbb{R}^M} \text{ for all } y \in B(x_0, r). \tag{4.3}$$

Now denote

$$H(s, y) := \int_s^{a^+} \beta(\sigma, y) d\sigma.$$

Then (4.3) can be transformed by using integration by parts into

$$\begin{aligned} 0_{\mathbb{R}^M} &= \int_0^{a^+} \beta(s, y) \int_0^s \psi(\gamma, y) d\gamma ds \\ &= -H(s, y) \int_0^s \psi(\gamma, y) d\gamma \Big|_{s=0}^{s=a^+} + \int_0^{a^+} H(s, y) \psi(s, y) ds \\ &= \int_0^{a^+} H(s, y) \psi(s, y) ds, \text{ for all } y \in B(x_0, r). \end{aligned}$$

But by Assumption 4.4, one has $H(s, y) \geq \int_s^{a^+} \underline{\beta}(\sigma) d\sigma$ and thus all diagonal elements of $H(s, y)$ are positive for all $(s, y) \in [0, a^+) \times \overline{\Omega}$. This will give us $\psi \equiv 0_{\mathbb{R}^M}$ in $[0, a^+) \times B(x_0, r)$.

Next, by remembering that ψ is a fixed point of L_λ and considering the second term of (4.2), we first ignore the exponential terms due to their positivity, then iterate L_λ for $n-$times to obtain

$$
\begin{aligned}
0_{\mathbb{R}^M} &= [L_\lambda \psi](a_0, x_0) = [L_\lambda^n \psi](a_0, x_0) \\
&\geq \mathcal{D}^n \int_\Omega \cdots \int_\Omega \prod_{m=1}^{n} \left[\mathcal{J}(x_{m-1} - x_m) \int_0^{a_{m-1}} e^{-(\lambda+\mathcal{D})(a_{m-1}-a_m)} \pi(a_m, a_{m-1}, x_m) da_m \right] \\
&\quad \psi(a_n, x_n) dx_n \cdots dx_1.
\end{aligned}
$$

It follows that $\psi(\cdot, x) \equiv 0_{\mathbb{R}^M}$ in $B(x_0, nr) \cap \overline{\Omega}$. Now when n is sufficiently large, $B(x_0, nr) \cap \overline{\Omega}$ will cover $\overline{\Omega}$, thus $\psi \equiv 0_{\mathbb{R}^M}$ in $[0, a^+) \times \overline{\Omega}$. Thus L_λ is strictly positive.

Now for any positive nonzero fixed point of L_λ, denoted by $\psi \in L^1_+((0, a^+), X^M)$, and any $\psi^* \in L^\infty_+((0, a^+), (X^M)^*)$ with $L^*_\lambda \psi^* \neq 0$, where X^* denotes the dual space of X, one has

$$
\langle \psi, \psi^* \rangle = \langle L_\lambda \psi, \psi^* \rangle > 0.
$$

It follows that all positive nonzero fixed points of L_λ are conditionally strictly positive and so is $\mathcal{F}_\lambda$. □

4.2 Criteria

Since the condition $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$ is hard to check, it is expected to find an easily verifiable and general sufficient condition for $\lambda_1(\mathcal{A})$ being the principal eigenvalue of $\mathcal{A}$ for the sake of applications. This leads us to our main theorem on the existence of principal eigenvalue of $\mathcal{A}$ in this section.

Before proceeding, we first provide another assumption on β to make sure that the principal eigenfunction ϕ can attain its positive maximum and minimum in $[0, a_2] \times \overline{\Omega}$ for some $a_2 \in (0, a^+)$.

Assumption 4.7 We assume that $\beta \equiv 0_{\mathcal{L}(\mathbb{R}^M)}$ in $[a_2, a^+) \times \overline{\Omega}$ for some $a_2 \in (0, a^+)$.

We would like to mention that above assumption is somehow reasonable for applications. It means that the birth rate or transmission rate of the population becomes zero when they reach very large ages.

Now, let us rewrite the function space $\mathcal{X}$ as follows:

$$
\mathcal{X} = X^M \times L^1((0, a^+), X^M) = X^M \times \Big(L^1((0, a_2), X^M) \times L^1((a_2, a^+), X^M) \Big)
$$

with a function $\psi \in L^1((0, a^+), X^M)$ mapped into $(\psi|_{(0,a_2)}, \psi|_{(a_2,a^+)}) \in L^1((0, a_2), X^M) \times L^1((a_2, a^+), X^M)$. Define the operator $\widehat{\mathcal{B}}$ in $X^M \times L^1((0, a_2), X^M)$ by

$$
\begin{aligned}
\widehat{\mathcal{B}}(0, \psi) &= (-\psi(0),\ -\partial_a \psi + \mathcal{D}[\mathcal{K} - I]\psi(a) - \mu(a, \cdot)\phi(a)), \\
dom(\widehat{\mathcal{B}}) &= \{0_{X^M}\} \times W^{1,1}((0, a_2), X^M).
\end{aligned}
$$

Note that $\widehat{\mathcal{B}}$ is a closed operator under Assumption 4.7 and $\mu(a, x) = \mathrm{diag}\{\mu_1(a, x), \ldots, \mu_M(a, x)\}$. Moreover, define the operator $\widehat{\mathcal{C}}$ as follows:

$$
\widehat{\mathcal{C}}(0, h) = \left(\int_0^{a_2} \beta(a, \cdot) h(a) da,\ 0 \right),\ dom(\widehat{\mathcal{C}}) = \{0_{X^M}\} \times L^1((0, a_2), X^M).
$$

Define the operator $\widehat{\mathcal{A}} := \widehat{\mathcal{B}} + \widehat{\mathcal{C}}$ with $dom(\widehat{\mathcal{A}}) = \{0_{X^M}\} \times W^{1,1}((0, a_2), X^M)$.

Next let us show $\sigma(\widehat{\mathcal{A}}) \cap \mathbb{R} = \sigma(\mathcal{A}) \cap \mathbb{R}$. To do so, it suffices to show $\rho(\widehat{\mathcal{A}}) \cap \mathbb{R} = \rho(\mathcal{A}) \cap \mathbb{R}$. Recalling the argument in Proposition 3.3, it says that

$$
\lambda \in \rho(\mathcal{A}) \cap \mathbb{R} \Leftrightarrow 1 \in \rho(\mathcal{M}_\lambda).
$$

Similarly, Proposition 3.3 with $a^+ = a_2$ applies to $\widehat{\mathcal{A}}$ to get

$$\lambda \in \rho(\widehat{\mathcal{A}}) \cap \mathbb{R} \Leftrightarrow 1 \in \rho(\widehat{\mathcal{M}_\lambda}),$$

where $\widehat{\mathcal{M}_\lambda} \in \mathcal{L}(X^M)$ is defined by

$$\widehat{\mathcal{M}_\lambda}\phi = \int_0^{a_2} \beta(a,\cdot)e^{-\lambda a}\mathcal{U}(0,a)\phi\, da, \;\; \forall \phi \in X^M.$$

But it is true that under Assumption 4.7, the operator $\mathcal{M}_\lambda = \widehat{\mathcal{M}_\lambda}$. It follows that $\sigma(\widehat{\mathcal{A}}) \cap \mathbb{R} = \sigma(\mathcal{A}) \cap \mathbb{R}$, thus we can study the principal spectral theory of $\widehat{\mathcal{A}}$ instead of $\mathcal{A}$ in the following, provided Assumption 4.7 holds. Further, in order to not introduce too many notations, we still denote $\mathcal{A}$ and $\mathcal{B}$ under Assumption 4.7.

Remark 4.8 Under Assumption 4.7, Assumption 4.4, if needed, can be modified as that there exists a_1 such that $\underline{\beta}_{ii}(a) > 0$ a.e $[a_1, a_2]$ for all $1 \le i \le M$. In summary, if Assumptions 4.4 and 4.7 are both satisfied, then there exists $0 < a_1 < a_2 < a^+$ such that $\beta((a_2, a^+), \cdot) \equiv 0$: 1) to guarantee some strictly positivity of the principal eigenfunction and $\beta_{ii}([a_1, a_2], \cdot) > 0$ for $1 \le i \le M$: 2) to guarantee some irreducibility which implies that the principal eigenvalue is simple. In addition, the above assumptions are also valid for $a^+ = \infty$.

Now we provide the second criteria under Assumption 4.7.

Theorem 4.9 *Let Assumption 4.7 hold. Assume that*

$$x \to \frac{1}{\alpha^{**} - \alpha(x)} \notin L^1_{loc}(\overline{\Omega}), \tag{4.4}$$

and that for each $x \in \overline{\Omega}$, $\mathcal{B}_1^x + \mathcal{C}^x$ possesses a positive eigenvector $\phi(x)$ corresponding to $\alpha(x)$, then $s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$. Here $\alpha(x)$ is defined in Proposition 3.7 and $\mathcal{B}_1^x + \mathcal{C}^x$ is defined in Remark 2.1.

Proof The idea of the proof below came from Liang et al. [31, Lemma 3.8] or see Bao and Shen [3, Proposition 3.1]. For completeness and the reader's convenience, we provide a detailed and modified proof.

By assumption, for any $x \in \overline{\Omega}$, $\phi(\cdot, x) := [\phi(x)](\cdot) = [[\phi_1(x)], \dots, [\phi_M(x)]]\,(\cdot)$ as the principal eigenfunction of $\mathcal{B}_1^x + \mathcal{C}^x$ is belonging to $W^{1,1}((0,a_2),\mathbb{R}^M)$. We will prove that the eigenfunction $\phi(\cdot, x)$ is continuous for all $x \in \overline{\Omega}$.

To this aim, let us first write down the equation that ϕ satisfies,

$$\begin{cases} \partial_a\phi(a,x) = -(\mathcal{D} + \mu(a,x))\phi(a,x) - \alpha(x)\phi(a,x), & a \in (0,a_2), \\ \phi(0,x) = \int_0^{a_2} \beta(a,x)\phi(a,x)da. \end{cases} \tag{4.5}$$

Let us choose a sequence $x_n \to x_0$ as $n \to \infty$ and consider the sequence $\phi(\cdot, x_n)$ with normalization $\|\phi(\cdot, x_n)\|_{L^1((0,a_2),\mathbb{R}^M)} = 1$ for all $x_n \in \overline{\Omega}$. Observing the first equation of (4.5), one has

$$\|\partial_a\phi(\cdot, x_n)\|_{L^1((0,a_2),\mathbb{R}^M)} \le C,$$

where $C > 0$ is a constant varying according to the context and independent of n. It follows that $\phi(\cdot, x_n) \in W^{1,1}((0,a_2),\mathbb{R}^M) \subset L^\infty((0,a_2),\mathbb{R}^M)$ and $\|\phi(\cdot, x_n)\|_{L^\infty((0,a_2),\mathbb{R}^M)} \le C$. Again by the first equation of (4.5), one has

$$\|\partial_a\phi(\cdot, x_n)\|_{L^\infty((0,a_2),\mathbb{R}^M)} \le C.$$

Thus we have $\|\phi(\cdot, x_n)\|_{W^{1,\infty}((0,a_2),\mathbb{R}^M)} \le C$. By the compact Sobolev embedding, we can find a limit, denoted by $\widehat{\phi}(\cdot)$, up to a subsequence such that

$$\phi(\cdot, x_n) \to \widehat{\phi}(\cdot) \text{ uniformly on } [0, a_2].$$

Since $x \to \mu(\cdot, x) \in C(\overline{\Omega}, [L^\infty(0, a_2)]^M)$, one has $\mu(a, x_n) \to \mu(a, x_0)$ in $[L^\infty(0, a_2)]^M$, and thus $\mu(a, x_n)\phi(a, x_n) \to \mu(a, x_0)\widehat{\phi}(a)$ in $[L^\infty(0, a_2)]^M$. Applying the same argument to β and passing to the limit on (4.5), one obtains

$$\begin{cases} \partial_a \widehat{\phi}(a) = -(\mathcal{D} + \mu(a, x_0))\widehat{\phi}(a) - \alpha(x_0)\widehat{\phi}(a), & a \in (0, a_2), \\ \widehat{\phi}(0) = \int_0^{a_2} \beta(a, x_0)\widehat{\phi}(a)da. \end{cases} \tag{4.6}$$

Observe that $\widehat{\phi}$ is the principal eigenfunction of the operator $\mathcal{B}_1^{x_0} + \mathcal{C}^{x_0}$ corresponding to $\alpha(x_0)$. Moreover, $\|\widehat{\phi}\|_{L^1((0,a_2),\mathbb{R}^M)} = 1$. Thanks to the simplicity of the principal eigenvalue, we have $\widehat{\phi}(a) = \phi(a, x_0)$. Thus the eigenfunction $\phi(\cdot, x)$ is continuous for all $x \in \overline{\Omega}$. We normalize ϕ such that

$$\max_{0 \le i \le M, (a,x) \in [0,a_2] \times \overline{\Omega}} \phi_i(a, x) = 1.$$

According to Assumption 1.1 on the kernel J, there exist $r > 0$ and $c_0 > 0$ such that $J(x - y) > c_0$ for all $x, y \in \overline{\Omega}$ with $|x - y| < r$. Next let

$$c_1 = \min_{1 \le i \le M, (a,x) \in [0,a_2] \times \overline{\Omega}} \phi_i(a, x).$$

Due to Assumption 4.7, $c_1 > 0$ holds. Since $(\zeta - \alpha)^{-1} \notin L^1_{loc}(\overline{\Omega})$, we can choose $\zeta > \alpha^{**}$, some $\delta > 0$ and $x_1 \in \Omega$ such that $B(x_1, \delta) \subset B(x_1, 2\delta) \subset \Omega$,

$$\int_{B(x_1,\delta)} \frac{1}{\zeta - \alpha(x)} dx \ge 2(D_{\min} c_0 c_1)^{-1}$$

and $3\delta < r$, where $B(x, r)$ is the ball centered at x with radius r and $D_{\min} = \min_{1 \le i \le M}\{D_i\}$. Let $p(x)$ be a continuous function on $\overline{\Omega}$ defined by

$$p(x) = \begin{cases} 1, & x \in B(x_1, \delta), \\ 0, & x \in \overline{\Omega} \setminus B(x_1, 2\delta) \end{cases} \tag{4.7}$$

with $p(x) \le 1$ for all $x \in \overline{\Omega}$ and $[\widetilde{\phi}(x)](a) = \widetilde{\phi}(a, x) := p(x)[\phi(x)](a), \forall (a, x) \in [0, a_2] \times \overline{\Omega}$. It then follows that for any $(a, x) \in [0, a_2] \times \overline{\Omega} \setminus B(x_1, 2\delta)$ and $1 \le i \le M$, we have

$$\int_\Omega J(x - y) \frac{dy}{\zeta - \alpha(y)} \widetilde{\phi}_i(a, y) \ge 0.$$

For any $(a, x) \in [0, a_2] \times B(x_1, 2\delta)$ and $1 \le i \le M$, we see that

$$\begin{aligned} &\int_\Omega J(x - y) \frac{dy}{\zeta - \alpha(y)} \widetilde{\phi}_i(a, y) \\ &\ge \int_{B(x_1,\delta)} J(x - y) \frac{dy}{\zeta - \alpha(y)} [\phi_i(y)](a) \\ &\ge 2c_0 c_1 (D_{\min} c_0 c_1)^{-1} \ge 2D_{\min}^{-1} \widetilde{\phi}_i(a, x). \end{aligned} \tag{4.8}$$

Note that

$$\begin{aligned}[(\zeta I - \mathcal{B}_1 - \mathcal{C})^{-1}(0, \widetilde{\phi})](x) &= (\zeta I - \mathcal{B}_1^x - \mathcal{C}^x)^{-1}(0, [\widetilde{\phi}(x)]) \\ &= (\zeta - \alpha(x))^{-1}(0, [\widetilde{\phi}(x)])\end{aligned} \tag{4.9}$$

for all $x \in \overline{\Omega}$. It then follows that

$$\mathcal{F}_\zeta(0, \widetilde{\phi}) = \mathcal{B}_2(\zeta I - \mathcal{B}_1 - \mathcal{C})^{-1}(0, \widetilde{\phi}) \geq 2(0, \widetilde{\phi}) > (0, \widetilde{\phi}). \tag{4.10}$$

Thus, there exists $\zeta > s(\mathcal{B}_1 + \mathcal{C})$ such that $r(\mathcal{F}_\zeta) > 1$. Then by Corollary 4.3, it follows that $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$ which implies the desired result by Theorem 4.1. □

Remark 4.10 Such a non-locally integrable condition (4.4) is comparable with the one in the nonlocal diffusion problem, see Coville [10] and Shen and Vo [43].

4.3 Relation Between $\mathcal{M}_\lambda$ and $\mathcal{A}$

We next give a proposition to characterize the relation between the eigenvalues of $\mathcal{M}_\lambda$ to those of $\mathcal{A} = \mathcal{B} + \mathcal{C}$, also see Kang and Ruan [24] or Walker [49].

Proposition 4.11 *Under Assumption 4.7, let $\lambda \in \mathbb{C}$ and $m \in \mathbb{N} \setminus \{0\}$. Then $\lambda \in \sigma_p(\mathcal{A})$ with geometric multiplicity m if and only if $1 \in \sigma_p(\mathcal{M}_\lambda)$ with geometric multiplicity m, where $\sigma_p(A)$ denotes the point spectrum of A.*

Proof Let $\lambda \in \mathbb{C}$. Suppose that $\lambda \in \sigma_p(\mathcal{A})$ has geometric multiplicity m so that there are m linearly independent elements

$$\left(0, \phi_1\right), \ldots, \left(0, \phi_m\right) \in dom(\mathcal{A}) \text{ with } (\lambda - \mathcal{A})\left(0, \phi_j\right) = (0, 0) \text{ for } j = 1, \ldots, m.$$

Then by solving the above eigenvalue problem explicitly, we get

$$\phi_j(a) = e^{-\lambda a}\mathcal{U}(0, a)\phi_j(0) \text{ with } \phi_j(0) = \mathcal{M}_\lambda \phi_j(0).$$

Hence, $\phi_1(0), \ldots, \phi_m(0)$ are necessarily linearly independent eigenvectors of $\mathcal{M}_\lambda$ corresponding to the eigenvalue 1.

Now suppose that $1 \in \sigma_p(\mathcal{M}_\lambda)$ has geometric multiplicity m so that there are linearly independent $\psi_1, \ldots, \psi_m \in X^M$ with $\mathcal{M}_\lambda \psi_j = \psi_j$ for $j = 1, \ldots, m$. Put $(0, \phi_j) = \left(0, e^{-\lambda a}\mathcal{U}(0, a)\psi_j\right) \in dom(\mathcal{A})$ and note that for $j = 1, \ldots, m$, we have

$$\partial_a \phi_j + \lambda \phi_j - \mathcal{D}[\mathcal{L} - I]\phi_j + \mu \phi_j = 0, \int_0^{a_2} \beta(a, \cdot)\phi_j(a)da = \mathcal{M}_\lambda \psi_j = \psi_j = \phi_j(0),$$

which is equivalent to

$$\mathcal{A}(0, \phi_j) = \lambda(0, \phi_j) \text{ and } (0, \phi_j) \in dom(\mathcal{A}).$$

Thus $\lambda \in \sigma_p(\mathcal{A})$. If $\alpha_1, \ldots, \alpha_m$ are any scalars, the unique solvability of the Cauchy problem

$$\partial_a \phi + \lambda \phi - \mathcal{D}[\mathcal{L} - I]\phi + \mu \phi = 0, \ \phi(0, \cdot) = \sum_{j=1}^m \alpha_j \psi_j$$

ensures that $(0, \phi_1), \ldots, (0, \phi_m)$ are linearly independent. Hence, the result is desired. □

5 Limiting Properties

In this section we study the effects of diffusion rate and diffusion range characterized by γ on the spectral bound $s(\mathcal{A})$ of $\mathcal{A}$ respectively. Remembering in the previous section, we have shown that under Assumption 4.7, the eigenvalue problem to $\mathcal{A}$ on $[0, a^+)$ is equivalent to the one on $[0, a_2]$ and further the principal eigenfunction associated with $s(\mathcal{A})$ is positive in $[0, a_2]$.

Thus in the following context, we will let Assumption 4.7 hold throughout the whole section. Before proceeding, let us first clarify the strict positivity in X. If $f > 0$ in $X = C(\overline{\Omega})$, it means that $f(x) > 0$ for all $x \in \overline{\Omega}$, if $f > 0$ in $X = L^1(\Omega)$, it means that $f(x) > 0$ a.e. in Ω. Following Berestyki et al. [5, 6], we introduce the following definition.

Definition 5.1 Define the *generalized principal eigenvalue* by

$$\begin{cases} \lambda_p(\mathcal{A}) := \sup\{\lambda \in \mathbb{R} : \\ \exists\, \phi \in W^{1,1}((0, a_2), X^M) \text{ s.t. } \phi > 0 \text{ and} (-\mathcal{A} + \lambda)(0, \phi) \leq (0, 0) \text{ in } [0, a_2]\}, \\ \lambda'_p(\mathcal{A}) := \inf\{\lambda \in \mathbb{R} : \\ \exists\, \phi \in W^{1,1}((0, a_2), X^M) \text{ s.t. } \phi > 0 \text{ and} (-\mathcal{A} + \lambda)(0, \phi) \geq (0, 0) \text{ in } [0, a_2]\}, \end{cases} \tag{5.1}$$

Note that the sets in Definition 5.1 are nonempty, see the proof of Theorem 5.3 in the following. As mentioned before, such ideas are widely used to prove the existence and asymptotic behavior of principal eigenvalues with respect to diffusion rate, see Coville [10], Li et al. [30] and Su et al. [45] for nonlocal diffusion equations, Shen and Vo [43] and Su et al. [44] for time periodic nonlocal diffusion equations. As Shen and Vo [43] highlighted for the time periodic case, we remark that the parabolic-type operators $\mathcal{A}$ containing ∂_a is not self-adjoint, and thus we lack the usual $L^2(\Omega)$ variational formula for the principal eigenvalue $\lambda_1(\mathcal{A})$. The generalized principal eigenvalue of $\lambda_p(\mathcal{A})$, $\lambda'_p(\mathcal{A})$ defined in (5.1) remedy the situation and play crucial roles in the following text.

5.1 Without Kernel Scaling

In this subsection first we study the diffusion without kernel scaling and have the following result.

Proposition 5.2 *Let Assumption 4.7 hold and assume that $\lambda_1(\mathcal{A})$ is the eigenvalue of $\mathcal{A}$ associated with $(0, \phi_1)$ with $\phi_1 > 0_{\mathbb{R}^M}$, then $\lambda_1(\mathcal{A}) = \lambda_p(\mathcal{A}) = \lambda'_p(\mathcal{A})$.*

Proof First, we prove that $\lambda_1 = \lambda_p$. Since $\lambda_1(\mathcal{A})$ is the eigenvalue of $\mathcal{A}$ associated with $(0, \phi_1) \in dom(\mathcal{A})$, that is

$$\mathcal{A}(0, \phi_1) - \lambda_1(0, \phi_1) = (0, 0) \text{ in } [0, a_2]; \tag{5.2}$$

and since $\phi_1 > 0$ in $[0, a_2]$, we have $\lambda_1 \leq \lambda_p$. Suppose by contradiction that $\lambda_1 < \lambda_p$. From the definition of λ_p, there are $\lambda \in (\lambda_1, \lambda_p)$ and $(0, \phi) \in dom(\mathcal{A})$ such that

$$-\mathcal{A}(0, \phi) + \lambda(0, \phi) \leq (0, 0) \text{ in } [0, a_2];$$

that is,

$$\begin{cases} \partial_a \phi(a) - \mathcal{D}[\mathcal{K} - I]\phi + \mu(a, \cdot)\phi + \lambda\phi \leq 0, \\ \phi(0) - \int_0^{a_2} \beta(a, \cdot)\phi(a)da \leq 0. \end{cases} \tag{5.3}$$

Now solving the first inequality in (5.3), we obtain

$$\phi(a) \le e^{-\lambda a}\mathcal{U}(0,a)\phi(0).$$

Plugging it into the second inequality in (5.3), we have

$$\phi(0) \le \int_0^{a_2} \beta(a,\cdot)e^{-\lambda a}\mathcal{U}(0,a)\phi(0)da. \tag{5.4}$$

It follows that $\mathcal{M}_\lambda\phi(0) \ge \phi(0)$, which implies that $r(\mathcal{M}_\lambda) \ge 1$. But we know that λ_1 is the eigenvalue of $\mathcal{A}$, then by Proposition 4.11 we have $r(\mathcal{M}_{\lambda_1}) = 1$. Since $\lambda \to r(\mathcal{M}_\lambda)$ is decreasing by Claim 3.4, one has $\lambda_1 \ge \lambda$. This contradiction leads to $\lambda_1 = \lambda_p$.

Next, we prove $\lambda_1 = \lambda'_p$. Obviously, $\lambda_1 \ge \lambda'_p$. Assume that $\lambda_1 > \lambda'_p$. There are $\tilde{\lambda} \in (\lambda'_p, \lambda_1)$ and $(0, \tilde{\phi}) \in dom(\mathcal{A})$ with $\tilde{\phi} > 0$ in $[0, a_2]$ such that $-\mathcal{A}(0,\tilde{\phi}) + \tilde{\lambda}(0,\tilde{\phi}) \ge (0,0)$. By reversing the above inequalities, we have the desired conclusion via a similar argument as above. □

Now we give the main theorem in this section about the effects of diffusion rate on $s(\mathcal{A})$. In the next result, we write $s^D(\mathcal{A})$ for $s(\mathcal{A})$ to highlight the dependence on $D = (D_1, \ldots, D_M)$.

Theorem 5.3 *Let Assumption 4.7 hold and assume that $s^D(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$, then the function $D \to s^D(\mathcal{A})$ is continuous on $(0,\infty)^M$ and satisfies*

$$s^D(\mathcal{A}) \to \begin{cases} s(\mathcal{B}_1^0 + \mathcal{C}) & \text{as } D \to 0^+_{\mathbb{R}^M}, \\ -\infty & \text{as } D \to \infty_{\mathbb{R}^M}, \end{cases} \tag{5.5}$$

where

$$\mathcal{B}_1^0(0,f) := (-f(0,\cdot),\ -\partial_a f - \mu f),\ \ (0,f) \in dom(\mathcal{A}).$$

Proof Since $s^D(\mathcal{A})$ is a simple eigenvalue, the continuity of $D \to s^D(\mathcal{A})$ follows from the similar argument in Theorem 4.9, or see Kato [28, Section IV. 3.5] for the classical perturbation theory.

For the limits, we first claim that for every $\epsilon > 0$, there exists $D_\epsilon > 0$ such that

$$s^D(\mathcal{A}) \le s(\mathcal{B}_1^0 + \mathcal{C}) + \epsilon,\ \forall D \in (0, D_\epsilon). \tag{5.6}$$

Here $D \in (0, D_\epsilon)$ means $D_i \in (0, D_{i\epsilon})$ for $1 \le i \le M$. Denote $\vartheta = s(\mathcal{B}_1^0 + \mathcal{C})$. Consider the equation (4.5) with $\mathcal{D} = 0_{\mathcal{L}(\mathbb{R}^M)}$ for each $i = 1, \ldots, M$, which is written as follows,

$$\begin{cases} \partial_a\phi_i(a,x) = -(\alpha(x) + \mu_i(a,x))\phi_i(a,x), & (a,x) \in (0,a_2) \times \overline{\Omega}, \\ \phi_i(0,x) = \sum_{j=1}^M \int_0^{a_2} \beta_{ij}(a,x)\phi_j(a,x)da, & x \in \overline{\Omega}. \end{cases} \tag{5.7}$$

By Proposition 3.7, we know that for each $x \in \overline{\Omega}$, (5.7) has a positive solution $\phi \in W^{1,1}((0,a_2),\mathbb{R}^M)$, which is the principal eigenvector of $G_\alpha(x)$ associated with 1. Moreover, by the argument in Theorem 4.9, $\phi(\cdot, x)$ is also continuous in $x \in \overline{\Omega}$. Thus $\phi \in W^{1,1}((0,a_2),[C(\overline{\Omega})]^M)$, $(0,\phi) \in dom(\mathcal{A})$ and $\phi > 0$ in $[0,a_2]$. Further, it is easy to check that for each $i = 1, \ldots, M$,

$$
\begin{aligned}
&-[\mathcal{A}(0,\phi)]_i+(\vartheta+\epsilon)[(0,\phi)]_i\\
&=\Bigg(\phi_i(0,x)-\sum_{j=1}^{M}\int_0^{a_2}\beta_{ij}(a,x)\phi_j(a,x)da,\\
&\qquad \partial_a\phi_i(a,x)-D_i\left[\int_\Omega J(x-y)\phi_i(a,y)dy-\phi_i(a,x)\right]+\mu_i(a,x)\phi+(\vartheta+\epsilon)\phi_i\Bigg).
\end{aligned}
$$

Since $\min_{[0,a_2]\times\overline{\Omega}}\phi_i>0$ and $\max_{[0,a_2]\times\overline{\Omega}}\phi_i<\infty$ for each $i=1,\ldots,M$, it is straightforward to check that for each $\epsilon>0$, there exists $D_{i\epsilon}>0$ such that for each $D_i\in(0,D_{i\epsilon})$, there holds

$$
\begin{aligned}
&\partial_a\phi_i(a,x)-D_i\left[\int_\Omega J(x-y)\phi_i(a,y)dy-\phi_i(a,x)\right]+\mu_i(a,x)\phi_i+(\vartheta+\epsilon)\phi_i\\
&=-D_i\left[\int_\Omega J(x-y)\phi_i(a,y)dy-\phi_i(a,x)\right]+(\vartheta-\alpha(x))\phi_i+\epsilon\phi_i\\
&\geq -D_i\left[\int_\Omega J(x-y)\phi_i(a,y)dy-\phi_i(a,x)\right]+\epsilon\phi_i\\
&\geq 0, \qquad (5.8)
\end{aligned}
$$

where we used $\vartheta\geq\alpha(x)$ from Proposition 3.7 where $D_i=0$. It then follows that $-\mathcal{A}(0,\phi)+(\vartheta+\epsilon)(0,\phi)\geq(0,0)$ which, by the definition of $\lambda_p'(\mathcal{A})$, implies that $s^D(\mathcal{A})=\lambda_p'(\mathcal{A})\leq s(\mathcal{B}_1^0+\mathcal{C})+\epsilon$.

Note from Proposition 3.2 that $s(\mathcal{B}_1^0+C)=\alpha_1$ which satisfies

$$
\max_{x\in\overline{\Omega}} r\left(\int_0^{a_2}\beta(a,x)e^{-\alpha_1 a}\pi(0,a,x)da\right)=1.
$$

While $s(\mathcal{B}_1+\mathcal{C})=\alpha^{**}$ by Proposition 3.2 which satisfies

$$
\max_{x\in\overline{\Omega}} r\left(\int_0^{a_2}\beta(a,x)e^{-(\alpha^{**}+\mathcal{D})a}\pi(0,a,x)da\right)=1.
$$

Since (β_{ij}) is irreducible or primitive, it implies that the spectral radius of the matrix

$$
\int_0^{a_2}\beta(a,x)e^{-(\alpha^{**}+\mathcal{D})a}\pi(0,a,x)da
$$

is monotone with respect to the matrix, see Proposition A.10-(iv). It follows that $\alpha_1-D_{\max}\leq\alpha^{**}$, where $D_{\max}=\max_{1\leq i\leq M}\{D_i\}$. Thus By Remark 3.5, we find that

$$
s^D(\mathcal{A})\geq s(\mathcal{B}_1+\mathcal{C})\geq s^D(\mathcal{B}_1^0+\mathcal{C})-D_{\max}.
$$

It follows that

$$
\liminf_{D\to 0^+_{\mathbb{R}^M}} s^D(\mathcal{A})\geq s(\mathcal{B}_1^0+\mathcal{C}). \tag{5.9}
$$

Setting $D\to 0^+_{\mathbb{R}^M}$, we find that

$$
s(\mathcal{B}_1^0+\mathcal{C})\leq\liminf_{D\to 0^+_{\mathbb{R}^M}} s^D(\mathcal{A})\leq\limsup_{D\to 0^+_{\mathbb{R}^M}} s^D(\mathcal{A})\leq s(\mathcal{B}_1^0+\mathcal{C})+\epsilon,\ \ \forall\epsilon>0,
$$

which leads to $s^D(\mathcal{A})\to s(\mathcal{B}_1^0+\mathcal{C})$ as $D\to 0^+_{\mathbb{R}^M}$.

Finally, to show that $s^D(\mathcal{A}) \to -\infty$ as $D \to \infty_{\mathbb{R}^M}$, we consider the operator $K - I$. It is known again from García-Melián and Rossi [21, Theorem 2.1] that the principal eigenvalue of $-K + I$ exists and is positive. Let $\theta_0 > 0$ be the principal eigenvalue of $-K + I$ and φ_0 be an associated positive eigenfunction. Let $(\lambda^1, \Psi^1(a))$ be the principal eigenpair of the age-structured operator, that is, they satisfies the following equation

$$\begin{cases} \partial_a \Psi^1(a) = -(\lambda^1 + \underline{\mu}(a))\Psi^1(a), \\ \Psi^1(0) = \int_0^{a_2} \overline{\beta}(a)\Psi^1(a)da, \end{cases}$$

where λ^1 satisfies

$$r\left(\int_0^{a_2} \overline{\beta}(a)e^{-\lambda^1 a}e^{-\int_0^a \underline{\mu}(s)ds}da\right) = 1.$$

Note that $\Psi^1(a) = (\Psi_1^1(a), \ldots, \Psi_M^1(a))$ is positive. Now let $\lambda_D = -D_{\min}\theta_0 + \lambda^1$ and

$$\Psi(a, x) = \varphi_0(x)\Psi^1(a) = \left(\varphi_0(x)\Psi_1^1(a), \ldots, \varphi_0(x)\Psi_M^1(a)\right),$$

where $D_{\min} = \min_{1 \le i \le M}\{D_i\}$. It is obvious that $(0, \Psi) \in dom(\mathcal{A})$ with $\Psi > 0$ in $[0, a_2]$ and we see that for each $i = 1, \ldots, M$,

$$\begin{aligned} &-[\mathcal{A}(0, \Psi)]_i + \lambda_D[(0, \Psi)]_i \\ &= \Bigg(\Psi_i(0, x) - \sum_{j=1}^M \int_0^{a_2} \beta_{ij}(a, x)\Psi_j(a, x)da, \\ &\quad \partial_a \Psi_i(a, x) - D_i\left[\int_\Omega J(x - y)\Psi_i(a, y)dy - \Psi_i(a, x)\right] + \mu_i(a, x)\Psi_i + \lambda_D\Psi_i\Bigg) \\ &:= \left(\mathrm{I}_1, \mathrm{I}_2\right), \end{aligned} \tag{5.10}$$

where

$$\begin{aligned} \mathrm{I}_2 &= \frac{\partial \Psi_i^1(a)}{\partial a}\varphi_0(x) - D_i\left[\int_\Omega J(x - y)\varphi_0(y)dy - \varphi_0(x)\right]\Psi_i^1(a) \\ &\quad + \mu_i(a, x)\Psi_i^1(a)\varphi_0(x) + (-D_{\min}\theta_0 + \lambda^1)\varphi_0(x)\Psi_i^1(a) \\ &\ge \left(\partial_a\Psi_i^1(a) + \underline{\mu}_i(a)\Psi_i^1(a) + \lambda^1\Psi_i^1(a)\right)\varphi_0(x) + D_i\theta_0\varphi_0(x)\Psi_i^1(a) - D_{\min}\theta_0\varphi_0(x)\Psi_i^1(a) \\ &\ge 0 \end{aligned} \tag{5.11}$$

and

$$\mathrm{I}_1 = \sum_{j=1}^M \int_0^{a_2} \overline{\beta}_{ij}(a)\Psi_j^1(a)da\varphi_0(x) - \sum_{j=1}^M \int_0^{a_2} \beta_{ij}(a, x)\Psi_j^1(a)\varphi_0(x)da \ge 0. \tag{5.12}$$

Thus, $(\lambda_D, (0, \Psi))$ is a test pair for $\lambda_p'(\mathcal{A})$. It follows that $s^D(\mathcal{A}) = \lambda_p'(\mathcal{A}) \le \lambda_D$. Setting $D \to \infty_{\mathbb{R}^M}$, we reach at $s^D(\mathcal{A}) \to -\infty$ as $D \to \infty_{\mathbb{R}^M}$. □

Remark 5.4 From Proposition 3.2, we know that $s(\mathcal{B}_1^0 + \mathcal{C})$ equals the value α_1 which satisfies

$$\max_{x \in \overline{\Omega}} r\left(\int_0^{a_2} \beta(a, x)e^{-\alpha_1 a}\pi(0, a, x)da\right) = 1.$$

Theorem 5.5 *Let Assumption 4.7 hold and assume that* $\mu(a,x)=\mu^1(a)+\mu^2(x)$, $\beta(a,x)\equiv\beta(a)$ *and* $\mu_i^2(x)\equiv\mu(x)$, $D_i=D$ *for all* $i=1,\dots,M$, *suppose that* J *is symmetric, i.e.* $J(x)=J(-x)$ *and, in addition, the operator*

$$v\to D\left[\int_\Omega J(\cdot-y)v(y)-v\right]-\mu v : C(\overline{\Omega})\to C(\overline{\Omega})$$

admits a principal eigenvalue, then $D\to s^D(\mathcal{A})$ *is strictly decreasing.*

Proof We write $\mathcal{A}=\mathcal{T}+L$, where

$$L(0,v)=\left(0,\ D\left[\int_\Omega J(\cdot-y)v(y)dy-v\right]-\mu v\right),\ \ v\in C(\overline{\Omega})$$
$$\mathcal{T}(0,\phi)=\left(-\phi(0)+\int_0^{a_2}\beta(a)\phi(a)da,\ -\phi'-\mu^1\phi\right),\ \ \phi\in W^{1,1}((0,a_2),\mathbb{R}^M).$$

Let $(\lambda_1^D(L),(0,v_1))$ be the principal eigenpair of $-L$. Then by the same argument as in Shen and Vo [43, Theorem C(2)], we have that $D\to\lambda_1^D(L)$ is strictly increasing. Now let $(\lambda_1(\mathcal{T}),(0,\phi_1))$ be the principal eigenpair of $\mathcal{T}$. It follows that $s^D(\mathcal{A})=-\lambda_1^D(L)+\lambda_1(\mathcal{T})$ is the principal eigenvalue of $\mathcal{A}$ with the principal eigenfunction $(0,v_1\phi_1)$. As $D\to\lambda_1^D(L)$ is strictly increasing, so $D\to s^D(\mathcal{A})$ is strictly decreasing. □

5.2 With Kernel Scaling

In this subsection we study the effects of diffusion rate and diffusion range on the principal eigenvalue. Define $K_{\gamma_i,\Omega}$ for $1\le i\le M$ as follows:

$$[K_{\gamma_i,\Omega}f](\cdot)=\int_\Omega J_{\gamma_i}(\cdot-y)f(y)dy,\ \ f\in X. \tag{5.13}$$

Here the kernek J satisfies the scaling $J_{\gamma_i}(x)=\frac{1}{\gamma_i^N}J\left(\frac{x}{\gamma_i}\right)$ for $x\in\mathbb{R}^N$, where $\gamma=(\gamma_1,\dots,\gamma_M)>0$ represents the diffusion range. Now we introduce the nonlocal diffusion operator $\frac{D_i}{\gamma_i^{m_i}}[K_{\gamma_i,\Omega}-I]$, where $m=(m_1,\dots,m_M)\in[[0,2)]^M$ is the cost parameter.

Compared with the non-scaled case, the scaled kernel will provide us many more results, in particular, when we study the global dynamics of (1.2), see Sect. 7. For example, when $m\in[(0,2)]^M$, both small and large diffusion ranges are favored provided $s(\mathcal{B}_1^0+\mathcal{C})>0$. In the meanwhile, such situations will bring us additional difficulties since more parameters are involved and thus more delicate inequalities are needed to obtain the desired results, for example, the decaying rates of generalized principal eigenfunctions in terms of γ, see the proof of Theorem 5.7 for more details. We mention that such analysis for a scalar nonlocal equation was developed by Shen and Vo [43] and borrowed here for us to deal with our equation coupled with age structure.

Write $\mathcal{A}_{\gamma,m,\Omega}=\mathcal{B}_{\gamma,m,\Omega}+\mathcal{C}$ for $\mathcal{A}=\mathcal{B}+\mathcal{C}$ to highlight the dependence on γ, m and Ω and further denote $\mathcal{B}^\mu_{\gamma,m,\Omega}$, $\mathcal{C}^\beta$ for $\mathcal{B}$, $\mathcal{C}$ to represent the dependence on μ and β respectively. We mainly employ the idea from Shen and Vo [43, Theorem D] to prove the following results.

Proposition 5.6 *Let* $m\ge 0$, $\gamma>0$. *We have the following statements.*

(i) $s(\mathcal{B}_{\gamma,m,\Omega}+\mathcal{C}^\beta)$ *is non-decreasing with respect to* β *and* $s(\mathcal{B}^\mu_{\gamma,m,\Omega}+\mathcal{C})$ *is non-increasing with respect to* μ;

(ii) *Let the assumptions in Theorem 4.9 hold, where D_i is changed into $\frac{D_i}{\gamma^m}$ for $1 \le i \le M$, then $s(\mathcal{A}_{\gamma,m,\Omega})$ is the principal eigenvalue of $\mathcal{A}_{\gamma,m,\Omega}$. Assume that $\lambda_1(\mathcal{A}_{\gamma,m,\Omega})$ is the eigenvalue of $\mathcal{A}_{\gamma,m,\Omega}$ associated with $\phi \in W^{1,1}((0,a_2),[C(\overline{\Omega})]^M)$ satisfying $\phi > 0$ in $[0,a_2]$, then*

$$\lambda_1(\mathcal{A}_{\gamma,m,\Omega}) = \lambda_p(\mathcal{A}_{\gamma,m,\Omega}) = \lambda_p'(\mathcal{A}_{\gamma,m,\Omega});$$

(iii) *Moreover, $\lambda_p(\mathcal{B}^{\mu}_{\gamma,m,\Omega}+\mathcal{C})$ is Lipschitz continuous with respect to μ in $C(\overline{\Omega},[L^\infty_+(0,a_2)]^M)$. More precisely,*

$$|\lambda_p(\mathcal{B}^{\mu^1}_{\gamma,m,\Omega}+\mathcal{C}) - \lambda_p(\mathcal{B}^{\mu^2}_{\gamma,m,\Omega}+\mathcal{C})| \le \|\mu^1-\mu^2\|_{C(\overline{\Omega},[L^\infty_+(0,a_2)]^M)}$$

for any $\mu^1,\mu^2 \in C(\overline{\Omega},[L^\infty_+(0,a_2)]^M)$;

(iv) *If $\Omega_1 \subset \Omega_2$, then $\lambda_p'(\mathcal{A}_{\gamma,m,\Omega_1}) \le \lambda_p'(\mathcal{A}_{\gamma,m,\Omega_2})$. Assume that in addition $X = C(\overline{\Omega})$, $s(\mathcal{A}_{\gamma,m,\Omega_1})$ and $s(\mathcal{A}_{\gamma,m,\Omega_2})$ are principal eigenvalues of $\mathcal{A}_{\gamma,m,\Omega_1}$ and $\mathcal{A}_{\gamma,m,\Omega_2}$ respectively, then*

$$|\lambda_p'(\mathcal{A}_{\gamma,m,\Omega_1}) - \lambda_p'(\mathcal{A}_{\gamma,m,\Omega_2})| \le C_0|\Omega_2 \setminus \Omega_1|,$$

where $C_0 > 0$ depends on $a, \gamma, D, m, J_\gamma$ and Ω_2;

(v) *Assume that $s(\mathcal{A}_{\gamma,m,\Omega})$ is the principal eigenvalue of $\mathcal{A}_{\gamma,m,\Omega}$, then the function $\gamma \to s(\mathcal{A}_{\gamma,m,\Omega})$ is continuous.*

Proof First note that Proposition 5.2 holds for $\mathcal{A}_{\gamma,m,\Omega}$, thus (ii) follows.

For (i), if $(\beta^1_{ij}) \ge (\beta^2_{ij})$ for $i,j = 1,\dots,M$, it follows that $\mathcal{M}_\lambda(\beta^1) \ge \mathcal{M}_\lambda(\beta^2)$ in the positive operator sense which implies that $r(\mathcal{M}_\lambda(\beta^1)) \ge r(\mathcal{M}_\lambda(\beta^2))$. Thus by Proposition 3.3, we have $s(\mathcal{B}_{\gamma,m,\Omega}+\mathcal{C}^{\beta^1}) \ge s(\mathcal{B}_{\gamma,m,\Omega}+\mathcal{C}^{\beta^2})$ by the monotonicity of $r(\mathcal{M}_\lambda)$ with respect to λ. Similarly, when $\mu^1_i \ge \mu^2_i$ for $i = 1,\dots,M$, since $\mathcal{U}(0,a)$ is positive in X, we have $\mathcal{U}_{\mu^1}(0,a) \le \mathcal{U}_{\mu^2}(0,a)$ in the positive operator sense, which implies that $\mathcal{M}_\lambda(\mu^1) \le \mathcal{M}_\lambda(\mu^2)$. Then it follows that $r(\mathcal{M}_\lambda(\mu^1)) \le r(\mathcal{M}_\lambda(\mu^2))$, hence $s(\mathcal{B}^{\mu^1}_{\gamma,m,\Omega}+\mathcal{C}) \le s(\mathcal{B}^{\mu^2}_{\gamma,m,\Omega}+\mathcal{C})$ by the above argument.

To prove (iii) and (iv), we can use the same argument as in Shen and Vo [43, Proposition 6.1] by fixing the first component of $-\mathcal{A}_{\gamma,m,\Omega}(0,\phi)+\lambda(0,\phi) = (0,0)$; i.e. keeping the integral condition $\int_0^{a_2}\beta(a,\cdot)\phi(a)da = \phi(0)$ hold. In order to illustrate, we prove (iii) and omit (iv) (note that the reversed relation in (iv) compared with [43, Proposition 6.1(4)]). Let us fix $\lambda < \lambda_p(\mathcal{B}^{\mu^1}_{\gamma,m,\Omega}+\mathcal{C})$. By Definition 5.1, there exists $(0,\phi) \in D(\mathcal{A}_{\gamma,m,\Omega})$ with $\phi > 0$ such that for each $i = 1,\dots,M$,

$$-[\mathcal{B}^{\mu_1}_{\gamma,m,\Omega}(0,\phi)]_i - [\mathcal{C}(0,\phi)]_i + [\lambda(0,\phi)]_i \le (0,0), \quad \text{in } [0,a_2].$$

Clearly,

$$\begin{aligned}
(0,0) &\ge -[\mathcal{B}^{\mu_1}_{\gamma,m,\Omega}(0,\phi)]_i - [\mathcal{C}(0,\phi)]_i + \lambda[(0,\phi)]_i \\
&= \left(\phi_i(0,x) - \sum_{j=1}^M \int_0^{a_2} \beta_{ij}(a,x)\phi_j(a,x)da, \quad \frac{\partial\phi_i}{\partial a} - \frac{D_i}{\gamma_i^{m_i}}\left[K_{\gamma_i,\Omega}-I\right]\phi_i + \mu^1_i\phi_i + \lambda\phi_i\right) \\
&= \left(\phi_i(0,x) - \sum_{j=1}^M \int_0^{a_2} \beta_{ij}(a,x)\phi_j(a,x)da, \quad \frac{\partial\phi_i}{\partial a} - \frac{D_i}{\gamma_i^{m_i}}\left[K_{\gamma_i,\Omega}-I\right]\phi_i \right. \\
&\quad \left. +[\mu^2_i+\mu^1_i-\mu^2_i]\phi_i + \lambda\phi_i\right)
\end{aligned}$$

$$\geq \left(\phi_i(0,x) - \sum_{j=1}^{M}\int_0^{a_2} \beta_{ij}(a,x)\phi_j(a,x)da,\right.$$
$$\left.\frac{\partial\phi_i}{\partial a} - \frac{D_i}{\gamma_i^{m_i}}\left[K_{\gamma_i,\Omega} - I\right]\phi_i + \mu_i^2\phi_i + \lambda\phi_i - \left\|\mu^1-\mu^2\right\|_{C(\overline{\Omega},[L_+^\infty(0,a_2)]^M)}\phi_i\right),$$

Again by Definition 5.1,

$$\lambda - \left\|\mu^1-\mu^2\right\|_{C(\overline{\Omega},[L_+^\infty(0,a_2)]^M)} \leq \lambda_p(\mathcal{B}^{\mu^2}_{\gamma,m,\Omega} + \mathcal{C}).$$

Since this holds for any $\lambda < \lambda_p(\mathcal{B}^{\mu^1}_{\gamma,m,\Omega} + \mathcal{C})$, we arrive at

$$\lambda_p(\mathcal{B}^{\mu^1}_{\gamma,m,\Omega} + \mathcal{C}) - \lambda_p(\mathcal{B}^{\mu^2}_{\gamma,m,\Omega} + \mathcal{C}) \leq \left\|\mu^1-\mu^2\right\|_{C(\overline{\Omega},[L_+^\infty(0,a_2)]^M)}.$$

Switching the roles of μ^1 and μ^2, we find that

$$\lambda_p(\mathcal{B}^{\mu^2}_{\gamma,m,\Omega} + \mathcal{C}) - \lambda_p(\mathcal{B}^{\mu^1}_{\gamma,m,\Omega} + \mathcal{C}) \leq \left\|\mu^1-\mu^2\right\|_{C(\overline{\Omega},[L_+^\infty(0,a_2)]^M)}.$$

Thus the result follows.

For (v) we can use the same argument in proving the continuity of $D \to s^D(\mathcal{A})$ in Theorem 5.3 and omit it here. □

Theorem 5.7 *Let Assumption 4.7 hold and assume that $s(\mathcal{A}_{\gamma,m,\Omega})$ is the principal eigenvalue of $\mathcal{A}_{\gamma,m,\Omega}$, then*

(i) *As $\gamma \to \infty_{\mathbb{R}^M}$, there holds*

$$s(\mathcal{A}_{\gamma,m,\Omega}) \to \begin{cases} s(\mathcal{B}_1^0 + \mathcal{C}) - D, & m = 0_{\mathbb{R}^M},\ D_i \equiv D,\ 1 \leq i \leq M, \\ s(\mathcal{B}_1^0 + \mathcal{C}), & m > 0_{\mathbb{R}^M}; \end{cases} \tag{5.14}$$

(ii) *Suppose that in addition, J is symmetric, i.e. $J(x) = J(-x)$ and $\mu_i \in C^2(\mathbb{R}^N, L_+^\infty(0,a_2))$ and $\beta_{ij} \in C^2(\mathbb{R}^N, L_+^\infty(0,a_2))$ for all $1 \leq i, j \leq M$. As $\gamma \to 0^+_{\mathbb{R}^M}$ there holds*

$$s(\mathcal{A}_{\gamma,m,\Omega}) \to s(\mathcal{B}_1^0 + \mathcal{C}),\ \forall m \in [[0,2)]^M,$$

where

$$\mathcal{B}_1^0(0,f) := (-f(0,\cdot),\ -\partial_a f - \mu f),\ (0,f) \in dom(\mathcal{A}).$$

Proof (i) We first prove the result in the case $m > 0_{\mathbb{R}^M}$. Note from Remark 5.4 that $s(\mathcal{B}_1^0 + \mathcal{C}) = \alpha_1$ which satisfies

$$\max_{x\in\overline{\Omega}} r\left(\int_0^{a_2} \beta_{ij}(a,x)e^{-\alpha_1 a}\pi_j(0,a,x)da\right) = 1.$$

While $s(\mathcal{B}_1 + \mathcal{C}) = \alpha^{**}$ by Proposition 3.2 which satisfies

$$\max_{x\in\overline{\Omega}} r\left(\int_0^{a_2} \beta_{ij}(a,x)e^{-(\alpha^{**}+\frac{D_j}{\gamma_j^{m_j}})a}\pi_j(0,a,x)da\right) = 1.$$

It follows that $\alpha_1 - \left(\frac{D_j}{\gamma_j^{m_j}}\right)_{\max} \leq \alpha^{**}$, where $\left(\frac{D_j}{\gamma_j^{m_j}}\right)_{\max} = \max_{1\leq j\leq M}\left\{\frac{D_j}{\gamma_j^{m_j}}\right\}$. Thus By Remark 3.5, we find that

$$s(\mathcal{A}_{\gamma,m,\Omega}) \geq s(\mathcal{B}_1 + \mathcal{C}) \geq s(\mathcal{B}_1^0 + \mathcal{C}) - \left(\frac{D_j}{\gamma_j^{m_j}}\right)_{\max}.$$

Hence, we have

$$\liminf_{\gamma\to\infty_{\mathbb{R}^M}} \lambda_1(\mathcal{A}_{\gamma,m,\Omega}) \geq s(\mathcal{B}_1^0 + \mathcal{C}). \tag{5.15}$$

Let us still consider equation (5.7) with a positive solution $\phi(a, x) \in W^{1,1}((0, a_2), [C(\overline{\Omega})]^M)$ and $\vartheta = s(\mathcal{B}_1^0 + \mathcal{C})$. For any $\epsilon > 0$, we see that for each $(a, x) \in [0, a_2] \times \overline{\Omega}$ and for each $i = 1, \ldots, M$,

$$\begin{aligned}
&-[\mathcal{A}_{\gamma,m,\Omega}(0,\phi)]_i + (\vartheta+\epsilon)[(0,\phi)]_i \\
&= \Bigg(\phi_i(0,x) - \sum_{j=1}^M \int_0^{a_2} \beta_{ij}(a,x)\phi_j(a,x)da, \\
&\quad \partial_a\phi_i(a,x) - \frac{D_i}{\gamma_i^{m_i}}\left[\int_\Omega J_{\gamma_i}(x-y)\phi_i(a,y)dy - \phi_i(a,x)\right] + \mu_i(a,x)\phi_i + (\vartheta+\epsilon)\phi_i\Bigg)
\end{aligned}$$

and

$$\begin{aligned}
&\partial_a\phi_i(a,x) - \frac{D_i}{\gamma_i^{m_i}}\left[\int_\Omega J_{\gamma_i}(x-y)\phi_i(a,y)dy - \phi_i(a,x)\right] + \mu_i(a,x)\phi_i + (\vartheta+\epsilon)\phi_i \\
&= -\frac{D_i}{\gamma_i^{m_i}}\left[\int_\Omega J_{\gamma_i}(x-y)\phi_i(a,y)dy - \phi_i(a,x)\right] + \epsilon\phi_i + (\vartheta - \alpha(x))\phi_i \\
&\geq -\frac{D_i}{\gamma_i^{m_i}}\left[\int_\Omega J_{\gamma_i}(x-y)\phi_i(a,y)dy - \phi_i(a,x)\right] + \epsilon\phi_i.
\end{aligned} \tag{5.16}$$

Since $\min_{[0,a_2]\times\overline{\Omega}} \phi_i > 0$, $\max_{[0,a_2]\times\overline{\Omega}} \phi_i < \infty$ and

$$\left\| \frac{D_i}{\gamma_i^{m_i}}\left[\int_\Omega J_{\gamma_i}(\cdot - y)\phi_i(a,y)dy - \phi_i(a,\cdot)\right]\right\|_{C(\overline{\Omega})} \to 0 \text{ as } \gamma_i \to \infty,$$

there is $\gamma_{i\epsilon} > 0$ such that (5.16)≥ 0 for all $\gamma_i \geq \gamma_{i\epsilon}$ and for all $i = 1, \ldots, M$. It then follows that $-\mathcal{A}_{\gamma,m,\Omega}(0,\phi) + (\vartheta+\epsilon)(0,\phi) \geq (0,0)$, which by the definition of $\lambda_p'(\mathcal{A}_{\gamma,m,\Omega})$ implies that

$$s(\mathcal{A}_{\gamma,m,\Omega}) = \lambda_p'(\mathcal{A}_{\gamma,m,\Omega}) \leq s(\mathcal{B}_1^0 + \mathcal{C}) + \epsilon.$$

The arbitrariness of ϵ then yields (i) with $m > 0_{\mathbb{R}^M}$.

Now we prove the result in the cases $m = 0_{\mathbb{R}^M}$ and $D_i \equiv D$ for all $1 \leq i \leq M$. Remark 3.5 ensures that $\lambda_1(\mathcal{A}_{\gamma,m,\Omega}) \geq s(\mathcal{B}_1 + \mathcal{C}) = s(\mathcal{B}_1^0 + \mathcal{C}) - D$. It remains to show that

$$\limsup_{\gamma\to\infty_{\mathbb{R}^M}} \lambda_1(\mathcal{A}_{\gamma,m,\Omega}) \leq s(\mathcal{B}_1^0 + \mathcal{C}) - D. \tag{5.17}$$

Let ϕ be the solution of (5.7) as above. For any $\epsilon > 0$, we have that for each $(a, x) \in [0, a_2]\times\overline{\Omega}$ and for all $i = 1, \ldots, M$,

$$
\begin{aligned}
&-[\mathcal{A}_{\gamma,0,\Omega}(0,\phi)]_i+(\vartheta+\epsilon)[(0,\phi)]_i\\
&=\Bigg(\phi_i(0,x)-\sum_{j=1}^M\int_0^{a_2}\beta_{ij}(a,x)\phi_j(a,x)da,\\
&\qquad \partial_a\phi_i(a,x)-D\left[\int_\Omega J_{\gamma_i}(x-y)\phi_i(a,y)dy-\phi_i(a,x)\right]+\mu_i(a,x)\phi_i+(\vartheta+\epsilon)\phi_i\Bigg),
\end{aligned}
$$

where

$$
\begin{aligned}
&\partial_a\phi_i(a,x)-D\left[\int_\Omega J_{\gamma_i}(x-y)\phi_i(a,y)dy-\phi_i(a,x)\right]+\mu_i(a,x)\phi_i+(\vartheta+\epsilon)\phi_i\\
&\geq -D\left[\int_\Omega J_{\gamma_i}(x-y)\phi_i(a,y)dy-\phi_i(a,x)\right]+\epsilon\phi_i+(\vartheta-\alpha(x))\phi_i\\
&\geq -D\left[\int_\Omega J_{\gamma_i}(x-y)\phi_i(a,y)dy-\phi_i(a,x)\right]+\epsilon\phi_i. \qquad (5.18)
\end{aligned}
$$

Hence for $\epsilon>0$, there holds for all $(a,x)\in[0,a_2]\times\overline{\Omega}$

$$
-[\mathcal{A}_{\gamma,0,\Omega}(0,\phi)]_i+(\vartheta+\epsilon-D)[(0,\phi)]_i\geq\left(0,-D\int_\Omega J_{\gamma_i}(x-y)\phi_i(a,y)dy+\epsilon\phi_i\right).
$$

As $\left\|\int_\Omega J_{\gamma_i}(\cdot-y)\phi_i(a,y)dy\right\|_{C(\overline{\Omega})}\to 0$ when $\gamma_i\to\infty$, we can follow the arguments in the case $m>0_{\mathbb{R}^M}$ to conclude (5.17).

(ii) Let $\phi=(\phi_1,\dots,\phi_M)$ be the solution of (5.7). Due to the regularities of μ_i and β_{ij} with respect to x, by Proposition 3.2 and implicit function theorem, we have $\alpha\in C^2(\overline{\Omega})$ and $\phi\in W^{1,1}((0,a_2),[C^2(\overline{\Omega})]^M)$ (see [14] for more details). Let $\tilde{\phi}\in W^{1,1}((0,a_2),[C^2(\mathbb{R}^N)]^M)$ be positive and satisfy $\tilde{\phi}(a,x)=\phi(a,x)$ for $(a,x)\in[0,a_2]\times\overline{\Omega}$. For any $\epsilon>0$, similar argument as in (5.16) leads to for each $i=1,\dots,M$ that

$$
\partial_a\phi_i(a,x)-\frac{D_i}{\gamma_i^{m_i}}\left[\int_\Omega J_{\gamma_i}(x-y)\phi_i(a,y)dy-\phi_i(a,x)\right]+\mu_i(a,x)\phi_i+(\vartheta+\epsilon)\phi_i \qquad (5.19)
$$

$$
\begin{aligned}
&\geq-\frac{D_i}{\gamma_i^{m_i}}\left[\int_\Omega J_{\gamma_i}(x-y)\phi_i(a,y)dy-\phi_i(a,x)\right]+\epsilon\phi_i\\
&\geq-\frac{D_i}{\gamma_i^{m_i}}\left[\int_{\mathbb{R}^N} J_{\gamma_i}(x-y)\tilde{\phi}_i(a,y)dy-\tilde{\phi}_i(a,x)\right]+\epsilon\phi_i\\
&=-\frac{D_i}{\gamma_i^{m_i}}\left[\int_{\mathbb{R}^N} J(z)\tilde{\phi}_i(a,x+\gamma_iz)dz-\tilde{\phi}_i(a,x)\right]+\epsilon\phi_i,\ (a,x)\in[0,a_2]\times\overline{\Omega}. \quad (5.20)
\end{aligned}
$$

Then by Taylor expansion (see the same argument as in Shen and Vo [43, Theorem D(2)]) dealing with the estimates of

$$
\frac{D_i}{\gamma_i^{m_i}}\left[\int_{\mathbb{R}^N} J(z)\tilde{\phi}_i(a,x+\gamma_iz)dz-\tilde{\phi}_i(a,x)\right],
$$

we can show that (5.20)≥ 0 in $[0,a_2]\times\overline{\Omega}$ for sufficiently small γ_i and $i=1,\dots,M$. It follows that

$$
\begin{aligned}
\mathcal{A}_{\gamma,m,\Omega}(0,\phi)+(\vartheta+\epsilon)(0,\phi)\geq(0,0)\text{ in }[0,a_2]\times\overline{\Omega}\\
\text{for }(0,\dots,0)<(\gamma_1,\dots,\gamma_M)\ll(1,\dots,1),
\end{aligned}
$$

which implies

$$\limsup_{\gamma\to 0^+_{\mathbb{R}^M}} \lambda_1(\mathcal{A}_{\gamma,m,\Omega}) = \limsup_{\gamma\to 0^+_{\mathbb{R}^M}} \lambda'_p(\mathcal{A}_{\gamma,m,\Omega}) \le s(\mathcal{B}_1^0+\mathcal{C}).$$

Now we show the reverse inequality; i.e.,

$$\liminf_{\gamma\to 0^+_{\mathbb{R}^M}} \lambda_1(\mathcal{A}_{\gamma,m,\Omega}) \ge s(\mathcal{B}_1^0+\mathcal{C}). \tag{5.21}$$

For any $\epsilon > 0$, there exists an open ball $B_\epsilon \subset \Omega$ of radius ϵ such that $\alpha(x)+\epsilon \ge s(\mathcal{B}_1^0+\mathcal{C}) := \vartheta$ in B_ϵ, where $\alpha(x)$ is from Proposition 3.7 for $\mathcal{D} = 0$ and $s(\mathcal{B}_1^0+\mathcal{C})$ corresponding the value α_1 in Remark 5.4. Let $\tilde{\phi}_{i\epsilon} \in W^{1,1}((0,a_2), C^2(\mathbb{R}^N))$ be nonnegative and satisfy for each $i = 1,\dots,M$ that

$$\begin{aligned}&\tilde{\phi}_{i\epsilon} = \phi_i \text{ in } [0,a_2]\times\overline{B}_\epsilon, \quad \tilde{\phi}_{i\epsilon} = 0 \text{ in } [0,a_2]\times(\mathbb{R}^N\setminus B_{2\epsilon})\\ &\text{and} \sup_{[0,a_2]\times\mathbb{R}^N} \tilde{\phi}_{i\epsilon} \le \sup_{[0,a_2]\times\mathbb{R}^N} \phi_i = 1.\end{aligned}$$

Set $\tilde{\phi} = (\tilde{\phi}_{1\epsilon},\dots,\tilde{\phi}_{M\epsilon})$. Then we have for $(a,x)\in[0,a_2]\times\overline{B}_\epsilon$ that

$$-[\mathcal{A}_{\gamma,m,B_\epsilon}(0,\phi)]_i + \left(\vartheta - \epsilon - \frac{1}{|\ln\epsilon|}\right)[(0,\phi)]_i := (\mathrm{I}_3,\mathrm{I}_4),$$

where

$$\mathrm{I}_3 = \phi_i(0,x) - \sum_{j=1}^M \int_0^{a_2} \beta_{ij}(a,x)\phi_j(a,x)da = 0$$

and

$$\begin{aligned}\mathrm{I}_4 &= \partial_a\phi_i(a,x) - \frac{D_i}{\gamma_i^{m_i}}\left[\int_{B_\epsilon} J_{\gamma_i}(x-y)\phi_i(a,y)dy - \phi_i(a,x)\right]\\ &\quad + \left[\mu_i(a,x)+\vartheta-\epsilon-\frac{1}{|\ln\epsilon|}\right]\phi_i(a,x)\\ &= -\frac{D_i}{\gamma_i^{m_i}}\left[\int_{B_\epsilon} J_{\gamma_i}(x-y)\phi_i(a,y)dy - \phi_i(a,x)\right] + \left[-\alpha(x)+\vartheta-\epsilon-\frac{1}{|\ln\epsilon|}\right]\phi_i(a,x)\\ &\le -\frac{D_i}{\gamma_i^{m_i}}\left[\int_{B_\epsilon} J_{\gamma_i}(x-y)\phi_i(a,y)dy - \phi_i(a,x)\right] - \frac{\phi_i(a,x)}{|\ln\epsilon|}\\ &= -\frac{D_i}{\gamma_i^{m_i}}\left[\int_{\mathbb{R}^N} J_{\gamma_i}(x-y)\tilde{\phi}_{i\epsilon}(a,y)dy - \tilde{\phi}_{i\epsilon}(a,x) - \int_{B_{2\epsilon}\setminus B_\epsilon} J_{\gamma_i}(x-y)\tilde{\phi}_{i\epsilon}(a,y)dy\right]\\ &\quad - \frac{\phi_i(a,x)}{|\ln\epsilon|}.\end{aligned}$$

Still based on Taylor expansion (see the same argument as in Shen and Vo [43, Theorem D(2)]) dealing with the estimate of

$$\frac{D_i}{\gamma_i^{m_i}}\left[\int_{\mathbb{R}^N} J_{\gamma_i}(x-y)\tilde{\phi}_{i\epsilon}(a,y)dy - \tilde{\phi}_{i\epsilon}(a,x) - \int_{B_{2\epsilon}\setminus B_\epsilon} J_{\gamma_i}(x-y)\tilde{\phi}_{i\epsilon}(a,y)dy\right],$$

we have for each $i = 1, \dots, M$, by choosing $\gamma_i > 0$ such that $\gamma_i^{k_i} = \epsilon$ with $k_i = \frac{m_i+2N}{N}$, that

$$-[\mathcal{A}_{\gamma,m,B_\epsilon}(0,\phi)]_i + \left(\vartheta - \epsilon - \frac{1}{|\ln \epsilon|}\right)[(0,\phi)]_i \le (0,0) \text{ in } [0,a_2] \times B_\epsilon, \ 0 < \epsilon \ll 1.$$

It then follows from the generalized principal eigenvalue and Proposition 5.2 that

$$\lambda_1(\mathcal{A}_{\gamma,m,B_\epsilon}) = \lambda_p(\mathcal{A}_{\gamma,m,B_\epsilon}) \ge s(\mathcal{B}_1^0 + \mathcal{C}) - \epsilon - \frac{1}{|\ln \epsilon|}, \quad 0 < \epsilon \ll 1.$$

By Proposition 5.6-(iii), $\lambda_1(\mathcal{A}_{\gamma,m,\Omega}) \ge \lambda_1(\mathcal{A}_{\gamma,m,B_\epsilon})$, which yields that

$$\lambda_1(\mathcal{A}_{\gamma,m,\Omega}) \ge s(\mathcal{B}_1^0 + \mathcal{C}) - \epsilon - \frac{1}{|\ln \epsilon|}, \quad 0 < \epsilon \ll 1.$$

Letting $\gamma \to 0^+_{\mathbb{R}^M}$, we have (5.21). Thus the result is desired. □

Remark 5.8 (1) Note that when $\beta(a,x) \equiv \beta(a)$ and $\mu(a,x) \equiv \mu(a)$, the age-structure and nonlocal diffusion can be decoupled, then the spectrum of $\mathcal{A}$ is quite clear, see "Appendix". Thus the limiting properties of the principal eigenvalue of $\mathcal{A}$ is fully and only determined by the one of nonlocal diffusion, and we omit the case.

(2) Note that we did not discuss the case when $m = 2_{\mathbb{R}^M}$ and $\gamma \to 0^+_{\mathbb{R}^M}$. We conjecture that the principal eigenvalue for scalar age-structured models with nonlocal diffusion converges to the one for scalar age-structured models with Laplace diffusion. Actually, without age-structure, the autonomous nonlocal diffusion operator has an L^2 variational structure which can be used to show the convergence, see Berestycki et al. [5] and Su et al. [45]. While for the time-periodic nonlocal diffusion operator, Shen and Xie [41, 42] used the idea of solution mappings to show the convergence, where they employed the spectral mapping theorem which is not valid in our case since we have a first order differential operator ∂_a that is unbounded. However, when we add a nonlocal boundary condition to the transmission rate β, it can be proved that the semigroup generated by solutions is eventually compact so that the spectral mapping theorem holds. Thus we can use it to show the desired convergence, see Kang and Ruan [23].

6 Strong Maximum Principle

In this section by using the sign of spectral bound $s(\mathcal{A})$ we establish the strong maximum principle under the case without kernel scaling, which is of fundamental importance and independent interest.

Definition 6.1 (*Strong Maximum Principle*) We say that $\mathcal{A}$ admits the *strong maximum principle* if for any function $(0,u) \in dom(\mathcal{A})$ satisfying

$$\begin{cases} \mathcal{A}(0,u) \le (0,0) & \text{in } [0,a_2] \times \Omega, \\ (0,u) \ge (0,0) & \text{in } [0,a_2] \times \partial\Omega, \end{cases} \tag{6.1}$$

there must hold $u > 0_{\mathbb{R}^M}$ in $[0,a_2] \times \Omega$ unless $u \equiv 0_{\mathbb{R}^M}$ in $[0,a_2] \times \Omega$.

Theorem 6.2 *Assume that there exists* $0 \le a_1 < a_2 < a^+$ *such that* $\beta \equiv 0_{\mathcal{L}(\mathbb{R}^M)}$ *on* $[a_2, a^+) \times \overline{\Omega}$ *and* $\underline{\beta}_{ij} > 0$ *on* $[a_1, a_2)$ *for all* $1 \le i \ne j \le M$. *In addition, assume that* $\lambda_1(\mathcal{A})$ *is the principal eigenvalue of* $\mathcal{A}$, *then* $\mathcal{A}$ *admits the strong maximum principle if and only if* $\lambda_1(\mathcal{A}) < 0$.

Proof If $\lambda_1 := \lambda_1(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$ associated with an eigenfunction $\phi \in W^{1,1}((0,a_2),[C(\overline{\Omega})]^M)$ with $\phi > 0_{\mathbb{R}^M}$, then

$$\mathcal{A}(0,\phi) - \lambda_1(0,\phi) = (0,0);$$

that is, for each $i = 1,\dots,M$,

$$\begin{cases} -\partial_a\phi_i + D_i\left[\int_\Omega J(x-y)\phi_i(a,y)dy - \phi_i(a,x)\right] - \mu_i(a,x)\phi_i - \lambda_1\phi_i = 0, \\ \phi_i(0,x) - \sum_{j=1}^M \int_0^{a_2} \beta_{ij}(a,x)\phi_j(a,x)da = 0. \end{cases} \tag{6.2}$$

For the sufficiency, that is $\lambda_1 < 0$ implies the strong maximum principle, let $(0,u) \in dom(\mathcal{A})$ be nonzero and satisfy (6.1). Assume by contradiction that there exists $(a_0,x_0) \in [0,a_2]\times\Omega$ such that $u_j(a_0,x_0) = \min_{[0,a_2]\times\Omega} u_j \le 0$ for some $j \in \{1,\dots,M\}$. Then consider the set

$$\Gamma := \{\epsilon \in \mathbb{R} : u_i + \epsilon\phi_i \ge 0 \ \text{ in } \ [0,a_2]\times\Omega, \ \text{ for each } i = 1,\dots,M\}.$$

Denote by $\epsilon_0 = \min\Gamma$ and $\psi = u + \epsilon_0\phi$. It is clear that $\epsilon_0 \ge 0$ by the assumption $u_j(a_0,x_0) \le 0$ and that $\psi \ge 0$. Now if $\epsilon_0 > 0$, by simple computations, we have for each $i = 1,\dots,M$ that

$$\begin{cases} \partial_a\psi_i - D_i\left[\int_\Omega J(x-y)\psi_i(a,y)dy - \psi_i(a,x)\right] + \mu_i(a,x)\psi_i \\ \qquad \ge -\epsilon_0\lambda_1\phi_i > 0, \quad (a,x) \in (0,a_2]\times\overline{\Omega}, \\ \psi_i(0,x) \ge \sum_{j=1}^M \int_0^{a_2} \beta_{ij}(a,x)\psi_j(a,x)da, \quad x \in \overline{\Omega}. \end{cases} \tag{6.3}$$

That is,

$$\begin{cases} \partial_a\psi_i > D_i\left[\int_\Omega J(x-y)\psi_i(a,y)dy - \psi_i(a,x)\right] - \mu_i(a,x)\psi_i, & (a,x) \in (0,a_2]\times\overline{\Omega}, \\ \psi_i(0,x) \ge \sum_{j=1}^M \int_0^{a_2} \beta_{ij}(a,x)\psi_j(a,x)da, & x \in \overline{\Omega}. \end{cases} \tag{6.4}$$

It follows from the first inequality in (6.4) that $\psi(a,\cdot) > \mathcal{U}(0,a)\psi(0,\cdot) \ge 0_{\mathbb{R}^M}$ in $(0,a_2]\times\Omega$. Plugging it into the second inequality, we have $\psi(0,x) > 0_{\mathbb{R}^M}$ by Assumption 4.4 which implies that ψ is strictly positive in $[0,a_2]\times\Omega$. This contradicts the fact that ϵ_0 is the infimum of Γ.

If $\epsilon_0 = 0$, it follows that $u_j(a_0,x_0) = 0$ and thus $u_j \ge 0$.

Case $a_0 > 0$. Recalling again the constant of variation formula (2.9), one has

$$u_j(a,x) \ge e^{-D_j a}\pi_j(0,a,x)u(0,x) + D_j\int_0^a e^{-D_j(a-l)}\pi_j(l,a,x)[Ku_j](l,x)dl. \tag{6.5}$$

Considering the above inequality at (a_0,x_0), it follows that for any $l \in [0,a_0]$, one has $[Ku_j](l,x_0) = 0$ and thus $u_j(l,x_1) = 0$ for all $x_1 \in B(x_0,r)$. Next consider (6.5) at (l,x_1), one has $u_j(l,x_2) = 0$ for all $x_2 \in B(x_1,r)$. Then continue this process as we did in Theorem 4.6, we get $u_j(l,\cdot) \equiv 0$ in $\overline{\Omega}\cap B(x_0,nr)$ with some $n \in \mathbb{N}$ large enough for all $l \in [0,a_0]$. On the other hand, by the nonlocal equation, the solution starting at $u_j(a_0,\cdot) \equiv 0$ will be zero, i.e. $u_j(l,\cdot) \equiv 0$ when $l > a_0$, which implies $u_j \equiv 0$. Now consider the following equation

$$\sum_{i=1}^M \int_0^{a_2} \beta_{ji}(a,x)u_i(a,x)da \le u_j(0,x) = 0, \ \forall x \in \overline{\Omega},$$

the assumption on β implies that for all $i \ne j$, $u_i(a,x) = 0$ in $[a_1,a_2]\times\overline{\Omega}$. Then consider the equation (6.5) for u_i with $i \ne j$ at $(\widetilde{a},x)$ for some $\widetilde{a} \in (a_1,a_2]$, one can by the above argument to obtain $u_i \equiv 0$ for all $i \neq j$. Thus $u \equiv 0$, which contradicts the fact that u is nonzero.

Case $a_0 = 0$. One has $u_j(0, x_0) = 0$, then the integral boundary condition implies

$$\sum_{i=1}^{M} \int_0^{a_2} \beta_{ji}(a, x_0) u_i(a, x_0) da \le u_j(0, x_0) = 0$$

which shows $u_i(\cdot, x_0) \equiv 0$ in $[a_1, a_2]$ for $i \neq j$. Then we can choose a point $\widetilde{a} \in (a_1, a_2]$. Considering the equation (6.5) for u_i with $i \neq j$ at $(\widetilde{a}, x_0)$, we have the same contradiction as above. Hence $u > 0$ in $[0, a_2] \times \Omega$, which concludes the desired result.

For the necessity, that is, strong maximum principle implies $\lambda_1 < 0$, the proof of each component is similar to that of Shen and Vo [43, Theorem F] and is omitted here. □

7 Applications

In this section, we apply the theory established in the previous sections to the age-structured cooperative model with nonlocal diffusion, i.e. (1.2). Let Assumptions 1.1, 1.2 and 1.3 hold. In addition, we also let Assumptions 4.4 and 4.7 hold, which is rewritten as follows,

Assumption 7.1 There exist a_1 and a_2 with $0 \le a_1 < a_2 < a^+$ such that $\beta \equiv 0_{\mathcal{L}(\mathbb{R}^M)}$ on $[a_2, a^+) \times \overline{\Omega}$ and $\underline{\beta}_{ii} > 0$ on $[a_1, a_2)$ for all $1 \le i \le M$.

Recall that if Assumption 7.1 holds, then the principal eigenfunction $\phi(\cdot, x)$ is continuous with respect to x by the simplicity of principal eigenvalue. For the sake of simplicity, we will not repeat Assumptions 1.1, 1.2 and 1.3 and 7.1 in this section.

7.1 Comparison Principle

Let us first consider the kernel without scaling and write down the equation that the equilibrium satisfies

$$\begin{cases} \frac{\partial u_i(a,x)}{\partial a} = D_i \left[\int_\Omega J(x-y) u_i(a,y) dy - u_i(a,x)\right] \\ \qquad\qquad -\mu_i(a,x) u_i(a,x), \quad (a,x) \in (0, a_2] \times \overline{\Omega}, \\ u_i(0,x) = f_i\left(\sum_{j=1}^{M} \int_0^{a_2} \beta_{ij}(a,x) u_j(a,x) da\right), \quad x \in \overline{\Omega}, \end{cases} \tag{7.1}$$

where $i = 1, \ldots, M$. We denote $f(u) = \text{diag}\{f_1(u_1), \ldots, f_M(u_M)\}$.

Definition 7.2 $u \in W^{1,1}((0, a_2), [C(\overline{\Omega})]^M)$ is called a *super-solution (resp. sub-solution)* of (7.1) if = are replaced by $\ge$ (resp. $\le$) in the two equations of (7.1).

Now let us prove the comparison principle for (7.1).

Lemma 7.3 *Let* $0_{\mathbb{R}^M} < u \in W^{1,1}((0, a_2), [C(\overline{\Omega})]^M)$ *be a sub-solution of* (7.1) *and* $0_{\mathbb{R}^M} < v \in W^{1,1}((0, a_2), [C(\overline{\Omega})]^M)$ *be a super-solution of* (7.1). *Then* $u \le v$ *in* $[0, a_2] \times \overline{\Omega}$.

Proof Let $\alpha_* := \sup\{\alpha > 0 : \alpha u \le v \text{ in } [0, a_2] \times \overline{\Omega}\}$. By assumptions on u and v, the number α_* is well defined and positive. If $\alpha_* \ge 1$, then we are done. So we assume that $\alpha_* < 1$.

Set $w := v - \alpha_* u$, then $w \ge 0$. Further, set

$$a_0 := \min\{a \in [0, a_2] : \exists x \in \overline{\Omega}, i \in \{1, \ldots, M\}, \text{ s.t. } w_i(a_0, x) = 0\}.$$

Such a_0 exists due to the definition of α_*. It follows that there exists $x_0 \in \overline{\Omega}$ such that $w_i(a_0, x_0) = 0$.

If $a_0 \in (0, a_2]$, observe that w_i satisfies the following equation,

$$\partial_a w_i(a,x) \geq D_i \left[\int_\Omega J(x-y) w_i(a,y)dy - w_i(a,x) \right] \\ -\mu_i(a,x)w_i(a,x), \ (a,x) \in (0,a_2] \times \overline{\Omega}.$$

Recalling the constant of variation formula (2.9), one has

$$w_i(a,x) \geq e^{-D_i a}\pi_i(0,a,x)w_i(0,x) + D_i \int_0^a e^{-D_i(a-l)}\pi_i(l,a,x)[Kw_i](l,x)dl. \quad (7.2)$$

Considering the above inequality at (a_0, x_0), we have a contradiction, since by the definition of a_0, $w_i(a,x) > 0$ for all $(a,x) \in [0,a_0) \times \overline{\Omega}$ implies the right hand side of (7.2) is positive.

If $a_0 = 0$, one has $w_i(0, x_0) = 0$. Thanks to Assumption 7.1 on β, one has

$$\sum_{j=1}^M \int_0^{a_2} \beta_{ij}(a,x_0)u_j(a,x_0)da > 0.$$

On the other hand by Assumption 1.3-(iii) on f, one has that $w_i(0, x_0)$ satisfies

$$\begin{aligned} w_i(0,x_0) &= v_i(0,x_0) - \alpha_* u_i(0,x_0) \\ &\geq f_i\left(\sum_{j=1}^M \int_0^{a_2} \beta_{ij}(a,x_0)v_j(a,x_0)da\right) - \alpha_* f_i\left(\sum_{j=1}^M \int_0^{a_2} \beta_{ij}(a,x_0)u_j(a,x_0)da\right) \\ &> f_i\left(\sum_{j=1}^M \int_0^{a_2} \beta_{ij}(a,x_0)v_j(a,x_0)da\right) - f_i\left(\sum_{j=1}^M \int_0^{a_2} \beta_{ij}(a,x_0)\alpha_* u_j(a,x_0)da\right) \\ &\geq 0, \end{aligned}$$

where we used the Assumption 1.3-(iii) and $\alpha_* < 1$. It is a contradiction with $w_i(0, x_0) = 0$. Thus $\alpha_* \geq 1$ and the proof is complete. □

7.2 Existence and Uniqueness of Positive Equilibrium

Next let us define the linearized operator $\mathcal{A}^L$ which is obtained by linearizing (7.1) at $u = 0$:

$$\mathcal{A}^L(0,\phi) := \left(-\phi(0,\cdot) + f'(0)\int_0^{a_2} \beta(a,\cdot)\phi(a,\cdot)da, \ -\partial_a\phi + \mathcal{D}(\mathcal{K} - I)\phi - \mu\phi\right), \\ (0,\phi) \in dom(\mathcal{A}^L), \quad (7.3)$$

where $dom(\mathcal{A}^L) = \{0\} \times W^{1,1}((0,a_2), [C(\overline{\Omega})]^M)$ and denote the spectral bound of $\mathcal{A}^L$ by λ_1^L, where $f'(0) = \text{diag}\{f_1'(0), \ldots, f_M'(0)\}$. Recall from Proposition 3.3 that λ_1^L satisfies

$$r\left(f'(0)\int_0^{a_2} \beta(a,\cdot)e^{-\lambda_1^L a}\mathcal{U}(0,a)da\right) = 1.$$

Theorem 7.4 *Assume $\lambda_1^L > 0$, then there exists at least one positive nontrivial solution $u^*(a,x)$ of (7.1) belonging to $W^{1,1}((0,a_2), [L^1(\Omega)]^M)$.*

Proof **1. Construction of super/sub-solutions.** Set $\overline{u}_i \equiv L$ for all $1 \le i \le M$, where L is from the Assumption 1.3-(iv). Let us verify that $\overline{u}(a,x)$ is indeed a super-solution of (7.1).

$$\begin{aligned}
&\partial_a \overline{u}_i(a,x) - D\left[\int_\Omega J(x-y)\overline{u}_i(a,y)dy - \overline{u}_i(a,x)\right] + \mu_i(a,x)\overline{u}_i(a,x)\\
&= DL\left[1 - \int_\Omega J(x-y)dy\right] + \mu_i(a,x)L \ge 0,\ \text{for all } 1 \le i \le M. \qquad (7.4)
\end{aligned}$$

Further, for all $1 \le i \le M$

$$\overline{u}_i(0,x) = L \ge f_i\left(\sum_{j=1}^{M}\int_0^{a_2}\beta_{ij}(a,x)\overline{u}_j(a,x)da\right)$$

Next, we construct a sub-solution of (7.1) motivated by Coville [10, Theorem 1.6]. For any $\delta > 0$ sufficiently small, we can find a small constant $\epsilon = \epsilon(\delta) > 0$ such that $f(u) \ge (f'(0) - \delta I)u$ for $0_{\mathbb{R}^M} < u \le \epsilon_{\mathbb{R}^M}$. Such ϵ can be achieved due to Assumption 1.1 on f.

Then we consider the following linear equation

$$\begin{cases}\partial_a\phi(a,x) = -(\mathcal{D} + \mu(a,x))\phi(a,x) - \alpha\phi(a,x),\ a \in (0,a_2),\\ \phi(0,x) = (f'(0) - \delta I)\int_0^{a_2}\beta(a,x)\phi(a,x)da.\end{cases} \qquad (7.5)$$

Then by Proposition 3.7, there exists a continuous function $x \to \alpha(x) : \mathbb{R}^n \to \mathbb{R}$ such that for any $x \in \mathbb{R}^n$, equation (7.5) with $\alpha = \alpha(x)$ has a positive solution $a \to \phi(a,x) \in W^{1,1}((0,a_2),\mathbb{R}^M)$. Denote $\alpha^{**} = \max_{x\in\overline{\Omega}}\alpha(x)$. From the definition of α^{**} there exists a sequence of points $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in \overline{\Omega}$ and $|\alpha^{**} - \alpha(x_n)| \le \frac{1}{n}$. Thus, by the continuity of $\alpha(x)$, for each n there exists $\eta_n > 0$ such that for all $x \in B_{\eta_n}(x_n)$ we have $|\alpha^{**} - \alpha(x)| \le \frac{2}{n}$.

Now we consider a sequence of real numbers $\{\epsilon_n\}_{n\in\mathbb{N}}$ which converges to zero such that $\epsilon_n \le \frac{\eta_n}{2}$. Next let $\{\chi_n\}_{n\in\mathbb{N}}$ be the following sequence of cut-off functions: $\chi_n(x) := \chi(\frac{|x-x_n|}{\epsilon_n})$ where χ is a smooth function such that $0 \le \chi \le 1$, $\chi(x) = 0$ for $|x| \ge 2$ and $\chi(x) = 1$ for $|x| \le 1$.

Finally, let us consider the following sequence of continuous functions $\{\alpha_n\}_{n\in\mathbb{N}}$ defined by $\alpha_n(x) := \sup\{\alpha(x), \alpha^{**}\chi_n(x)\}$. Observe that by construction the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ is such that $\|\alpha - \alpha_n\|_{C(\overline{\Omega})} \to 0$.

By construction, for each n, the function α_n satisfies $\max_{x\in\overline{\Omega}}\alpha_n = \alpha^{**}$ and $\alpha_n \equiv \alpha^{**}$ in $B_{\frac{\epsilon_n}{2}}(x_n)$. Therefore, the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfies $\frac{1}{\alpha^{**}-\alpha_n} \notin L^1_{loc}(\Omega)$. Next set

$$\mu_n(a,x) = \mu(a,x) - \alpha_n(x)I + \alpha(x)I$$

and consider the equation (7.5) with μ being replaced by μ_n. Then it can be checked that

$$r\left((f'(0) - \delta I)\int_0^{a_2}\beta(a,x)e^{-(\mathcal{D}+\alpha_n(x))a}e^{-\int_0^a\mu_n(s,x)ds}da\right) = 1.$$

It follows that α_n is a continuous function such that for any $x \in \mathbb{R}^n$, equation (7.5) with μ being replaced by μ_n and with $\alpha = \alpha_n(x)$, has a positive solution $a \to \phi_n(a,x) \in W^{1,1}((0,a_2),\mathbb{R}^M)$. Hence by Theorem 4.9, there exists a principal eigenpair (λ_1^n, ϕ_n) of the eigenvalue problem:

$$\begin{cases}\partial_a\phi(a,x) = \mathcal{D}\left[\int_\Omega \mathcal{J}(x-y)\phi(a,y)dy - \phi(a,x)\right]\\ \qquad\qquad -\mu_n(a,x)\phi(a,x) - \lambda\phi(a,x),\quad (a,x) \in (0,a_2)\times\overline{\Omega},\\ \phi(0,x) = (f'(0) - \delta I)\int_0^{a_2}\beta(a,x)\phi(a,x)da,\quad x \in \overline{\Omega}\end{cases}$$

such that $0 < \phi_n \in W^{1,1}((0, a_2), [C(\overline{\Omega})]^M)$.

Using the fact that $\|\mu - \mu_n\|_{C(\overline{\Omega},[L^\infty(0,a_2)]^M)} \to 0$ as $n \to \infty$, from Proposition 5.6 it follows that for n big enough, say $n \geq n_0$, we have

$$\lambda_1^n > \frac{\lambda_1^L}{2} > 0.$$

Moreover, by choosing n_0 bigger if necessary, we achieve for $n \geq n_0$ that

$$\lambda_1^n - \|\mu - \mu_n\|_{C(\overline{\Omega},[L^\infty(0,a_2)]^M)} \geq \frac{\lambda_1^L}{4} > 0.$$

Now for $n \geq n_0$ fixed and $\psi = \epsilon_1 \phi_n$ with $\epsilon_1 > 0$ small enough such that $\int_0^{a_2} \beta(a, x)\psi(a, x)da \leq \epsilon_{\mathbb{R}^M}$, we have

$$\begin{cases} \partial_a \psi(a, x) - \mathcal{D}\left[\int_\Omega \mathcal{J}(x - y)\psi(a, y)dy - \psi(a, x)\right] + \mu(a, x)\psi(a, x) \\ \qquad = -(\mu_n(a, x) - \mu(a, x) + \lambda_1^n)\psi \leq 0, \\ \psi(0, x) = (f'(0) - \delta I) \int_0^{a_2} \beta(a, x)\psi(a, x)da \leq f\left(\int_0^{a_2} \beta(a, x)\psi(a, x)da\right), \end{cases}$$

where we used the fact that $f(u) \geq (f'(0) - \delta I)u$ for $0_{\mathbb{R}^M} < u \leq \epsilon_{\mathbb{R}^M}$. It implies that for $\epsilon_1 > 0$ sufficiently small and n large enough, $\epsilon_1 \phi_n$ is a sub-solution of (7.1). From now on, we fix a n large enough and denote $\underline{u} = \epsilon_1 \phi_n$.

2. Existence via iterative scheme. Now it is clear that we can choose ϵ small enough such that $\underline{u} \leq \overline{u}$. Then by a basic iterative scheme we obtain the existence of a positive nontrivial solution u of (7.1). For the completeness, we provide the iterative scheme in the following.

Let u_n for $n \geq 1$ be the solution of the following linear problem

$$\begin{cases} \partial_a u_n(a, x) = \mathcal{D}\left[\int_\Omega \mathcal{J}(x - y)u_n(a, y)dy - u_n(a, x)\right] \\ \qquad -\mu(a, x)u_n(a, x), \quad (a, x) \in (0, a_2) \times \overline{\Omega}, \\ u_n(0, x) = f\left(\int_0^{a_2} \beta(a, x)u_{n-1}(a, x)da\right), \quad x \in \overline{\Omega}, \end{cases} \tag{7.6}$$

where $u_0 = \underline{u}$. First note that u_n is well defined and is belonging to $W^{1,1}((0, a_2), [L^1(\Omega)]^M)$. Then we will show that u_n is increasing and that

$$\underline{u} \leq u_1 \leq u_2 \leq \cdots \leq \overline{u}. \tag{7.7}$$

Indeed, taking $w := u_1 - \underline{u}$ and $v := \overline{u} - u_1$, by Assumption 1.3-(ii) of f, they satisfy respectively

$$\begin{cases} \partial_a w(a, x) \geq \mathcal{D}\left[\int_\Omega \mathcal{J}(x - y)w(a, y)dy - w(a, x)\right] \\ \qquad -\mu(a, x)w(a, x), \quad (a, x) \in (0, a_2) \times \overline{\Omega}, \\ w(0, x) \geq 0, \quad x \in \overline{\Omega} \end{cases}$$

and

$$\begin{cases} \partial_a v(a, x) \geq \mathcal{D}\left[\int_\Omega \mathcal{J}(x - y)v(a, y)dy - v(a, x)\right] \\ \qquad -\mu(a, x)v(a, x), \quad (a, x) \in (0, a_2) \times \overline{\Omega}, \\ v(0, x) \geq 0, \quad x \in \overline{\Omega}. \end{cases}$$

Using comparison principle of nonlocal diffusion equations, we conclude that $w \geq 0$ and $v \geq 0$, that is $\underline{u} \leq u_1 \leq \overline{u}$. Now by induction, we can obtain the desired result (7.7).

Next for $(a, x) \in [0, a_2] \times \overline{\Omega}$ a.e., $u_n(a, x)$ has a limit, denoted by $u^*(a, x)$, that is $u_n(a, x) \to u^*(a, x)$ in $[0, a_2] \times \overline{\Omega}$ a.e. and thus by the continuity of f we have that for any $x \in \overline{\Omega}$,

$$f\left(\int_0^{a_2} \beta(a, x)u_n(a, x)da\right) \xrightarrow{n\to\infty} f\left(\int_0^{a_2} \beta(a, x)u^*(a, x)da\right),$$

which implies that $u^*(0, x) = f\left(\int_0^{a_2} \beta(a, x)u^*(a, x)da\right)$. In addition, one has

$$\begin{aligned} p_n(a, x) &:= \mathcal{D}\left[\int_\Omega \mathcal{J}(x-y)u_n(a, y)dy - u_n(a, x)\right] - \mu(a, x)u_n(a, x) \\ &\xrightarrow{n\to\infty} \mathcal{D}\left[\int_\Omega \mathcal{J}(x-y)u^*(a, y)dy - u^*(a, x)\right] - \mu(a, x)u^*(a, x) := p(a, x), \end{aligned}$$

a.e. in $[0, a_2] \times \overline{\Omega}$. Hence, for any $x \in \overline{\Omega}$ and $[\eta, \xi] \subset [0, a_2]$, one has

$$u_n(\xi, x) - u_n(\eta, x) = \int_\eta^\xi p_n(a, x)da,$$

which implies that

$$u^*(\xi, x) - u^*(\eta, x) = \int_\eta^\xi p(a, x)da.$$

It follows that $u^* \in W^{1,1}((0, a_2), [L^1(\Omega)]^M)$ satisfies the equation (7.1) with $\partial_a u^* = p$ a.e. in $(0, a_2) \times \Omega$. Further, $u^*(\cdot, x)$ is continuous in $[0, a_2]$ for a.e. $x \in \Omega$. Thus, one has $u_n(0, x) \to u^*(0, x)$ as $n \to \infty$ in Ω, which implies that

$$u^*(0, x) = f\left(\int_0^{a_2} \beta(a, x)u^*(a, x)da\right).$$

Thus the proof is complete. □

Next we investigate the uniqueness of u^*. Before proceeding, we first study the regularity of u^* with respect to x. We make the following additional assumption.

Assumption 7.5 Assume that $F(x, u) := u - G_0(x)f(u)$ is strictly monotone with respect to $u \in \mathbb{R}_+^M$ for any $x \in \overline{\Omega}$, where $G_0(x)$ is defined in (3.2) with $\alpha = 0$ and a^+ replaced by a_2.

Assumption 7.5 with $G_0(x) = I$ is widely used to obtain the regularity of solutions of nonlocal diffusion problems, see Bates et al. [4] and Berestycki and Rodríguez [7].

Now let us revisit the problem (7.1). Solving the first equation of (7.1), one obtains

$$u(a, x) = e^{-\mathcal{D}a}\pi(0, a, x)u(0, x) + \mathcal{D}\int_0^a e^{-\mathcal{D}(a-l)}\pi(l, a, x)[\mathcal{K}u](l, x)dl.$$

Then plugging the above equality into the boundary condition, one has

$$\begin{aligned} \widetilde{u}(x) := \int_0^{a_2} \beta(a, x)u(a, x)da &= \int_0^{a_2} \beta(a, x)e^{-\mathcal{D}a}\pi(0, a, x)u(0, x)da \\ &+ \mathcal{D}\int_0^{a_2} \beta(a, x)\int_0^a e^{-\mathcal{D}(a-l)}\pi(l, a, x)[\mathcal{K}u](l, x)dlda \\ &=: G_0(x)f(\widetilde{u}(x)) + H(x), \end{aligned} \tag{7.8}$$

where

$$H(x) = \mathcal{D}\int_0^{a_2} \beta(a,x)\int_0^a e^{-\mathcal{D}(a-l)}\pi(l,a,x)[\mathcal{K}u](l,x)dlda$$

is continuous, due to $\mathcal{K}u \in W^{1,1}((0,a_2),[C(\overline{\Omega})]^M)$ for any $u \in W^{1,1}((0,a_2),[L^1(\Omega)]^M)$, by Assumption 1.1 on J. Now under Assumption 7.5, for any $x \in \overline{\Omega}$, one has $\widetilde{u}(x) = F^{-1}(x, H(x))$, where F^{-1} denotes the inverse of F with respect to u for any fixed $x \in \overline{\Omega}$. Thus $\widetilde{u}$ is continuous. It follows that $u(0,\cdot)$ is continuous and so is $u(a,\cdot)$.

Theorem 7.6 *Under Assumption 7.5, the positive equilibrium u^* is unique.*

Proof We prove the uniqueness by using the sliding argument. Let u and v be two positive bounded solutions of (7.1). Since they are bounded and strictly positive, the following quantity is well defined:

$$\kappa^* := \inf\{\kappa > 0 : \kappa u \geq v \text{ in } [0,a_2]\times\overline{\Omega}\}.$$

We claim that $\kappa^* \leq 1$. Indeed, assume by contradiction that $\kappa^* > 1$. We consider the following nonlocal problem

$$\partial_a w = \mathcal{D}\left[\int_\Omega \mathcal{J}(x-y)w(a,y)dy - w(a,x)\right] - \mu(a,x)w(a,x),\ \ (a,x)\in(0,a_2)\times\overline{\Omega}. \tag{7.9}$$

By Bao and Shen [3, Proposition 2.1] and Assumption 1.1 on J, solutions of equation (7.9) have strong monotone property; i.e., for $\phi,\psi \in [C_+(\overline{\Omega})]^M$ with $\phi \geq \psi, \phi \not\equiv \psi$, $w(a,x;\phi) \gg w(a,x;\psi)$, $a > 0$ at which both $w(a,x;\phi)$ and $w(a,x;\psi)$ exist, where w is the solution of (7.9). Here the notation $\gg$ means that if $f_1 \gg f_2$ with $f_i = (f_{i1},\ldots,f_{iM})$ for $i = 1,2$ in $[C_+(\overline{\Omega})]^M$, then $f_{1j}(x) > f_{2j}(x)$ for all $x \in \overline{\Omega}$ and $1 \leq j \leq M$.

On one hand, from the integral boundary condition with Assumption 7.1 on β, we have due to $\kappa^* > 1$ and assumptions of f that

$$\begin{aligned}\kappa^* u_0 := \kappa^* u(0,x;u_0) &= \kappa^* f\left(\int_0^{a_2}\beta(a,x)u(a,x)da\right)\\ &> f\left(\int_0^{a_2}\beta(a,x)\kappa^* u(a,x)da\right)\\ &\geq f\left(\int_0^{a_2}\beta(a,x)v(a,x)da\right)\\ &= v(0,x;v_0) =: v_0.\end{aligned}$$

It follows from the strong monotone property that

$$w(a,x;\kappa^* u_0) \gg w(a,x;v_0). \tag{7.10}$$

On the other hand, let $\phi(a,x) = \kappa^* w(a,x;u_0)$. Then $\phi(0,x) = \kappa^* u_0$ and

$$\partial_a\phi = \mathcal{D}\left[\int_\Omega \mathcal{J}(x-y)\phi(a,y)dy - \phi(a,x)\right] - \mu(a,x)\phi(a,x),\ \ (a,x)\in(0,a_2)\times\overline{\Omega}.$$

By the uniqueness of solutions for nonlocal diffusion equations, we have

$$\kappa^* w(a,x;u_0) = w(a,x;\kappa^* u_0) \tag{7.11}$$

Now combing (7.10) and (7.11), we have

$$\kappa^* u(a,x) = \kappa^* w(a,x;u_0) \gg w(a,x;v_0) = v(a,x),$$

which is a contradiction with the definition of κ^*. We conclude that $u \geq v$. Now switch u and v in the above argument, we also have $v \geq u$, which shows the uniqueness of the solution. □

7.3 Stability

In this subsection we will show the global stability of the positive equilibrium u^* obtained in Theorem 7.4. First the existence of a solution $u(t,a,x)$ for (1.2) defined for all time $t \geq 0$ follows from a standard semigroup method by writing equation (1.2) as an abstract Cauchy problem (2.12), which is shown in the following,

$$\begin{cases} \frac{dU}{dt} = \mathcal{B}U + F(U), \\ U(0) = U_0, \end{cases} \quad \text{with } U_0 = \big(0, u_0\big). \tag{7.12}$$

and based on the Lipshcitz assumption on f, see Thieme [46, 48] or Magal and Ruan [33]. Next, thanks to the definition of $\mathcal{B}$, there holds that $\mathcal{B}$ is resolvent positive. Moreover, F is monotone due to Assumption 1.3-(ii) on f, i.e. $0 \leq U \leq V \Rightarrow 0 \leq F(U) \leq F(V)$. Thus by Magal et al. [34, Theorem 4.5], we can conclude that weak comparison principle holds for (7.12), which is written as follows,

Lemma 7.7 (Weak Comparison Principle) *Assume that $\mathcal{B}$ is resolvent positive and F is monotone. In addition, $U_0 \in \mathcal{X}_0$ and $U_0 \geq 0_{\mathcal{X}_0}$ but $U_0 \not\equiv 0_{\mathcal{X}_0}$, then the mild solution to (7.12), $U(t) \geq 0_{\mathcal{X}_0}$ for any $t \geq 0$.*

It follows that weak comparison principle also holds for (1.2). Now we give the strong comparison principle for (1.2).

Lemma 7.8 (Strong Comparison Principle) *Assume that $u_0(a,x) \geq 0_{\mathbb{R}^M}$ but $u_0(a,x) \not\equiv 0_{\mathbb{R}^M}$ in $[0,a_2] \times \overline{\Omega}$, then the solution to (1.2), $u(t,a,x) > 0_{\mathbb{R}^M}$ for any $t > 0$ in $[0,a_2] \times \overline{\Omega}$.*

Proof Solving the problem (1.2) along the characteristic line $a - t = c$, where $c \in \mathbb{R}$, we now derive the formula for a solution to (1.2). For fixed $c \in \mathbb{R}$, we set $w(t) = u(t, t+c)$ for $t \in [\max(-c, 0), \infty)$. With $a = t + c$ one obtains for $t \in [\max(-c, 0), \infty)$ the equation

$$\partial_t w(t) = \mathcal{D}[\mathcal{K} - I]w - \mu(t+c, \cdot)w. \tag{7.13}$$

We first study the case $c \geq 0$. Clearly, $w(0) = u(0,c) = u(0,a-t) = u_0(a-t)$. Considering the equation (7.13) with initial data $w(0) \geq 0_{\mathbb{R}^M}$ and $w(0) \not\equiv 0_{\mathbb{R}^M}$, we have $w(t) > 0_{\mathbb{R}^M}$ for $t > 0$ by the strong comparison principle of the nonlocal diffusion problem, due to $J(0) > 0$ in Assumption 1.1. It follows that $u(t,a) > 0_{\mathbb{R}^M}$ for $a \geq t$. On the other hand, integrating (7.13) from 0 to t, one obtains

$$w(t) = \mathcal{U}(c, t+c)w(0).$$

and

$$u(t,a) = \mathcal{U}(a-t, a)u_0(a-t).$$

Next we consider the case $c < 0$. Integrating (7.13) from $-c$ to t, one gets

$$w(t) = \mathcal{U}(0, t+c)w(-c).$$

and

$$u(t,a) = \mathcal{U}(0,a)u(t-a,0).$$

Thus now the solution to (1.2) reads as follows,

$$u(t,a) = \begin{cases} \mathcal{U}(a-t,a)u_0(a-t), & a \geq t, \\ \mathcal{U}(0,a)u(t-a,0), & a < t. \end{cases} \tag{7.14}$$

Next we plug the explicit formula (7.14) into $u(t,0)$ to obtain

$$u(t,0) = f\Big(\int_0^t \chi(a)\beta(a,\cdot)\mathcal{U}(0,a)u(t-a,0)da \\ + \int_t^{a_2} \chi(a)\beta(a,\cdot)\mathcal{U}(a-t,a)u_0(a-t)da\Big), \tag{7.15}$$

where $\chi(a)$ is a cutoff function satisfying $\chi(a) = 1$ when $a \in (0, a_2)$ otherwise $\chi(a) = 0$. Now we separate two cases.

Case 1. If $t < a_2$, (7.15) is written as follows,

$$u(t,0) = f\left(\int_0^t \beta(a,\cdot)\mathcal{U}(0,a)u(t-a,0)da + \int_t^{a_2} \beta(a,\cdot)\mathcal{U}(a-t,a)u_0(a-t)da\right). \tag{7.16}$$

Since $u(t,a) = \mathcal{U}(a-t,a)u_0(a-t) > 0_{\mathbb{R}^M}$ for $a \geq t$ and $\beta_{ii}(a,\cdot) \geq \underline{\beta}_{ii}(a) > 0$ a.e. in $[a_1, a_2]$ by Assumption 7.1 on β, the second term in the right hand of (7.16) must be positive. It follows by Assumption 1.3 on f, we have $u(t,0) > 0_{\mathbb{R}^M}$. Thus $u(t,a) > 0_{\mathbb{R}^M}$ for $a < t$ via (7.14).

Case 2. If $t \geq a_2$, (7.15) is written as follows,

$$u(t,0) = f\left(\int_0^{a_2} \beta(a,\cdot)\mathcal{U}(0,a)u(t-a,0)da\right). \tag{7.17}$$

Let us claim that $u(t,0,x) := [u(t,0)](x) > 0_{\mathbb{R}^M}$ in $[a_2,\infty)\times\overline{\Omega}$. By contradiction, suppose that there exist $i \in \{1,\ldots,M\}$ and $(t_0,x_0) \in [a_2,\infty)\times\overline{\Omega}$ such that $u_i(t_0,0,x_0) = 0$. By Assumption 1.3 on f, one obtains

$$\begin{aligned} 0 &= \sum_{j=1}^M \int_0^{a_2} \beta_{ij}(a,x_0)\mathcal{U}_j(0,a)u_j(t_0-a,0,x_0)da \\ &\geq \int_0^{a_2} \beta_{ii}(a,x_0)e^{-\int_0^a (D_i+\overline{\mu}_i(s))ds}e^{D_iKa}u_i(t_0-a,0,x_0)da, \end{aligned}$$

where we used the fact that $e^{-\int_0^a (D_i+\overline{\mu}_i(s))ds}$ and e^{D_iKa} are communicated. By Assumption 7.1 on β, one has $\beta_{ii}(a,x_0) \geq \underline{\beta}_{ii}(a) > 0$ a.e. in $[a_1,a_2]$, then we can find one point $b_0 \in [a_2-\epsilon, a_2]$ such that $e^{D_iKa}u_i(t_0-b_0,0,x_0) = 0$, where $\epsilon > 0$ small enough satisfying $a_1 \leq a_2-\epsilon$. By definition, one has

$$e^{D_iKa}u_i(t_0-b_0,0,x_0) = \sum_{n=0}^\infty \frac{(D_ia)^n}{n!}K^{*n} * u_i(t_0-b_0,0,x_0),$$

where K^{*n} denotes the $n-$fold convolution of K, that is $K^{*n} = K * \cdots * K$, n times. It follows that for each $n \in \mathbb{N}$,

$$K^{*n} * u_i(t_0-b_0,0,x_0) = 0.$$

However, by Assumption 1.1 on J, one has $J > 0$ in $B(0, r)$, which implies that

$$u_i(t_0 - b_0, 0, x) = 0, \text{ for all } x \in B(x_0, nr) \cap \overline{\Omega}.$$

When n is large enough, $B(x_0, nr) \cap \overline{\Omega}$ covers $\overline{\Omega}$, and thus $u_i(t_0 - b_0, 0, \cdot) \equiv 0$ in $\overline{\Omega}$.

Next replace t_0 by $t_0 - b_0$ in (7.15). If $t_0 - b_0$ falls in $[0, a_2)$, by the argument as Case 1, one has $u(t_0 - b_0, 0) > 0_{\mathbb{R}^M}$, which is a contradiction. Hence, $t_0 - b_0$ must fall in $[a_2, \infty)$. Then by the same argument as Case 2, one can find $b_1 \in [a_2 - \epsilon, a_2]$, such that $u_i(t_0 - b_0 - b_1, 0) = 0$. Now doing the above process by induction, one can find a sequence $\{b_i\}_{i\geq 0}$ such that $u_i(t_0 - \sum_{i=0}^{M} b_i, 0) = 0$ for any $M \geq 0$. But we know every b_i is in $[a_2 - \epsilon, a_2]$, then there always exists a minimal $M_0 > 0$, such that $t_0 - \sum_{i=0}^{M_0} b_i < a_2$. Then by Case 1, one has $u_i(t_0 - \sum_{i=0}^{M_0} b_i, 0) > 0$.

Now consider the $i-$th equation of (7.17) at $t = t_0 - \sum_{i=0}^{M_0-1} b_i$, which is larger than or equal to a_2, we get a contradiction, since now the left hand side of (7.17) equals to zero, while the right hand side of (7.17) is larger than zero.

In summary, we cannot have $i \in \{1, \ldots, M\}$ and $(t, x) \in (0, \infty) \times \overline{\Omega}$ such that $u_i(t, 0, x) = 0$, which implies $u(t, 0, x) > 0_{\mathbb{R}^M}$ and thus $u(t, a) > 0_{\mathbb{R}^M}$ by (7.14). Hence the proof is complete. □

Now we provide the following global stability result.

Theorem 7.9 (Stability) *Let Assumption 7.5 hold. Assume $\lambda_1^L > 0$, then the nontrivial equilibrium u^* is stable in the sense of $u(t, a, x) \to u^*(a, x)$ pointwise as $t \to \infty$, where $u(t, a, x)$ is a solution of (1.2) with initial data $u_0(a, x) \geq 0_{\mathbb{R}^M}$ but $u(a, x) \not\equiv 0_{\mathbb{R}^M}$ in $[0, a_2] \times \overline{\Omega}$.*

Proof If $u_0(a, x) \geq 0_{\mathbb{R}^M}$ but $u(a, x) \not\equiv 0_{\mathbb{R}^M}$ in $[0, a_2] \times \overline{\Omega}$, using strong comparison principle (Lemma 7.8), there exists a positive constant δ such that $u(1, a, x) > \delta_{\mathbb{R}^M}$ in $[0, a_2] \times \overline{\Omega}$. Since $\lambda_1^L > 0$, we can still allow $\epsilon \underline{u}$ defined in Theorem 7.4 to be a sub-solution of (7.1) for ϵ small enough. Since $u(1, a, x) \geq \delta_{\mathbb{R}^M}$ and $\underline{u}$ is bounded, by choosing ϵ smaller if necessary we also achieve that $\epsilon \underline{u} \leq u(1, a, x)$. Now let us denote $\underline{U}(t, a, x)$ the solution of (1.2) with initial data $\epsilon \underline{u}$. By weak comparison principle (Lemma 7.7), $\underline{U}(t, a, x) \geq \epsilon \underline{u}(a, x)$ for all $t \geq 0$. Given $s \geq 0$, let $z^s(t, a, x) := \underline{U}(t + s, a, x) - \underline{U}(t, a, x)$, which satisfies $z^s(0, a, x) \geq 0_{\mathbb{R}^M}$ by the above argument and

$$\begin{cases} \frac{dU}{dt} = \mathcal{B}U + GU, \\ U(0) = U_0, \end{cases} \quad \text{with } U = \left(0, z^s\right), \tag{7.18}$$

on $(0, \infty) \times [0, a_2] \times \overline{\Omega}$ for some function G on $(0, \infty) \times [0, a_2] \times \Omega$ with $\|G\|_{L^\infty} \leq \left\|F'\right\|_{L^\infty}$. The weak comparison principle (Lemma 7.7) then implies that $z^s \geq 0_{\mathbb{R}^M}$ for all $s \geq 0$, which follows that $\underline{U}(t, a, x)$ is a non-decreasing function of the time and $\underline{U}(t, a, x) \leq u(t+1, a, x)$.

On the other hand, L which is defined in the proof in Theorem 7.4 is a super-solution of (7.1) and u_0 is bounded, we also have $u(t, a, x) \leq \overline{U}(t, a, x)$ if necessary choosing L large enough, where $\overline{U}(t, a, x)$ denotes the solution of (1.2) with initial data $\overline{U}(0, a, x) = L_{\mathbb{R}^M} \geq u_0$. A similar argument as above using the comparison principle shows that $\overline{U}$ is a non-increasing function of t. Thus we have for all time $t \geq 0$ that

$$\epsilon \underline{u} \leq \underline{U}(t, a, x) \leq u(t + 1, a, x) \leq \overline{U}(t + 1, a, x).$$

Since $\underline{U}(t, a, x)$ (respectively $\overline{U}(t, a, x)$) is a uniformly bounded monotonic function of t, $\underline{U}$ (resp. $\overline{U}$) converges pointwise to $\underline{p}$ (resp. $\overline{p}$) which is a solution of (7.1). From $\underline{U} \neq 0_{\mathbb{R}^M}$, using the uniqueness of a non-trivial solution of (7.1), we deduce that $\underline{p} \equiv \overline{p} = u^* \neq 0_{\mathbb{R}^M}$ and therefore, $u(t, a, x) \to u^*$ pointwise in $[0, a_2] \times \overline{\Omega}$. □

7.4 Global Dynamics in Terms of Diffusion Rate and Diffusion Range

In the following we give a similar result on the global dynamics of (1.2) by using the values of diffusion rate $D = (D_1, \dots, D_M)$ and diffusion range $\gamma = (\gamma_1, \dots, \gamma_M)$ without and with kernel scaling, respectively. Before that, we introduce a notation $\ll$ which means that if $x_{\mathbb{R}^M} \ll y_{\mathbb{R}^M}$, then x_i is much smaller than y_i for all $1 \le i \le M$.

Theorem 7.10 *Let Assumption 7.5 hold. Assume that $s(\mathcal{A}^L)$ coincides the principal eigenvalue of $\mathcal{A}^L$ defined in (7.3), then equation (1.2) admits a unique positive equilibrium $u^* \in [C([0, a_2] \times \overline{\Omega})]^M$ that is stable for each $0_{\mathbb{R}^M} < D \ll 1_{\mathbb{R}^M}$ if $s(\mathcal{B}_1^0 + \mathcal{C}) > 0$, where $s(\mathcal{B}_1^0 + \mathcal{C}) = \alpha_2$ and α_2 satisfies*

$$\max_{x \in \overline{\Omega}} r \left(f'(0) \int_0^{a_2} \beta(a, x) e^{-\alpha_2 a} \pi(0, a, x) da \right) = 1. \tag{7.19}$$

Proof Note that $\mathcal{A}^L$ defined in (7.3) also satisfies all the properties of $\mathcal{A}$ discussed in Sect. 5. Then by Theorem 5.3, $s^D(\mathcal{A}^L) > 0$ for all $0 < D \ll 1$ if $s(\mathcal{B}_1^0 + \mathcal{C}) > 0$. Thus the result follows from Theorem 7.4, Theorem 7.6 and Theorem 7.9. □

Theorem 7.11 *Let Assumption 7.5 hold. Assume that $s(\mathcal{A}^L)$ coincides the principal eigenvalue of $\mathcal{A}^L$ defined in (7.3), then we have the following results.*

(i) *For each $m > 0_{\mathbb{R}^M}$, assume $s(\mathcal{B}_1^0 + \mathcal{C}) = \alpha_2 > 0$, then there exists $1_{\mathbb{R}^M} \ll \gamma^1 < \infty_{\mathbb{R}^M}$ such that for each $\gamma > \gamma_1$ equation (1.2) with kernel scaling defined in (5.13) admits a unique stable positive equilibrium $u^* \in [C([0, a_2] \times \overline{\Omega})]^M$;*

(ii) *Suppose that J is symmetric, i.e. $J(x) = J(-x)$, $\mu_i \in C^2(\mathbb{R}^N, L_+^\infty(0, a_2))$ and $\beta_{ij} \in C^2(\mathbb{R}^N, L_+^\infty(0, a_2))$ for all $1 \le i, j \le M$. For each $m \in [[0, 2)]^M$, assume $s(\mathcal{B}_1^0 + \mathcal{C}) = \alpha_2 > 0$, then there exists $0_{\mathbb{R}^M} < \gamma_2 \ll 1_{\mathbb{R}^M}$ such that for each $0_{\mathbb{R}^M} < \gamma < \gamma_2$ equation (1.2) with kernel scaling defined in (5.13) admits a unique stable positive equilibrium $u^* \in [C([0, a_2] \times \overline{\Omega})]^M$.*

Proof It follows from Theorems 5.7, 7.4, 7.6 and 7.9. □

At the end of this section, we investigate the asymptotic behavior of the equilibrium u^* in terms of D without kernel scaling and in terms of γ with kernel scaling respectively. In order to highlight the dependence of u^* on D or γ, we denote u^* by u_D^* or u_γ^*. Before proceeding, we first give a lemma on the solution of (7.1) without nonlocal diffusion; that is,

$$\begin{cases} \partial_a v(a, x) = -\mu(a, x) v(a, x), & (a, x) \in (0, a_2) \times \overline{\Omega}, \\ v(0, x) = f\left(\int_0^{a_2} \beta(a, x) v(a, x) da \right), & x \in \overline{\Omega}. \end{cases} \tag{7.20}$$

Lemma 7.12 *Assume*

$$\min_{x \in \overline{\Omega}} r \left(f'(0) \int_0^{a_2} \beta(a, x) \pi(0, a, x) da \right) > 1, \tag{7.21}$$

then the equation (7.20) has a unique positive solution, denoted by $v^*(a, x)$, which is belonging to $W^{1,1}((0, a_2), [C(\overline{\Omega})]^M)$.

Proof First note that (7.21) implies that for any $x \in \overline{\Omega}$,

$$r\left(f'(0)\int_0^{a_2}\beta(a,x)\pi(0,a,x)da\right) > 1.$$

Then one can always find an element $0_{\mathbb{R}^M} < v \in X^M$ again by the argument as Theorem 7.4 such that

$$(f'(0)-\delta)\int_0^{a_2}\beta(a,x)\pi(0,a,x)dav \geq v,$$

provided $\delta > 0$ is sufficiently small. Now we fix $x \in \overline{\Omega}$. We see that $\underline{v}(a,x) := \epsilon\pi(0,a,x)v$ is a sub-solution of (7.20) when (7.21) holds by taking $\epsilon > 0$ sufficiently small. Meanwhile, $\overline{v} := L_{\mathbb{R}^M}$ for L sufficiently large is also a super-solution of (7.20). Now it is clear that we can choose $\epsilon > 0$ and $L > 0$ such that $\underline{v} \leq \overline{v}$. Then by a basic iterative scheme as in Theorem 7.4 we obtain the existence of a positive nontrivial solution $v^*(\cdot,x) \in W^{1,1}(0,a_2)$ of (7.20) for any $x \in \overline{\Omega}$. Next we can use the sliding argument again as we did in Theorem 7.6 to show that $v^*(\cdot,x)$ is unique. At last, noting that $v^*(\cdot,x) \in [\epsilon e^{-\int_0^{a_2}\overline{\mu}(s)ds}, L]$, the continuity of v^* in x comes from a similar argument as Theorem 4.9, we omit them here. □

Theorem 7.13 *Let Assumption 7.5 hold. Assume that $s(\mathcal{A}^L)$ coincides the principal eigenvalue of $\mathcal{A}^L$ defined in (7.3), and in addition, assume (7.21) holds, and v^* is from Lemma 7.12, we have the following asymptotic results:*

(i) *Assume $u_D^*(a,x)$ is given by Theorem 7.10, then*

$$\lim_{D\to 0^+_{\mathbb{R}^M}} u_D^*(a,x) = v^*(a,x), \text{ uniformly in } (a,x) \in [0,a_2]\times\overline{\Omega}; \tag{7.22}$$

(ii) *Assume $u_\gamma^*(a,x)$ is given by Theorem 7.11, $m \in [[0,2)]^M$ and J is symmetric, i.e. $J(x) = J(-x)$, then*

$$\lim_{\gamma\to 0^+_{\mathbb{R}^M}} u_\gamma^*(a,x) = v^*(a,x), \text{ uniformly in } (a,x) \in [0,a_2]\times\overline{\Omega}; \tag{7.23}$$

(iii) *Assume $u_\gamma^*(a,x)$ is given by Theorem 7.11 and $m > 0_{\mathbb{R}^M}$, then*

$$\lim_{\gamma\to \infty_{\mathbb{R}^M}} u_\gamma^*(a,x) = v^*(a,x), \text{ uniformly in } (a,x) \in [0,a_2]\times\overline{\Omega}. \tag{7.24}$$

Proof We first show (iii). It suffices to show that for each $0 < \delta \ll 1$, there exists $\gamma_\delta > 0_{\mathbb{R}^M}$ such that for each $\gamma \in (0_{\mathbb{R}^M}, \gamma_\delta)$ there holds

$$(1-\delta)v^*(a,x) \leq u_\gamma^*(a,x) \leq (1+\delta)v^*(a,x),\ (a,x)\in[0,a_2]\times\overline{\Omega}.$$

We here outline the proof of the upper bound and the lower bound follows from similar arguments. Denote $v := (1+\delta)v^*$ and define $F_{\gamma_i} : C([0,a_2]\times\overline{\Omega}) \to C([0,a_2]\times\overline{\Omega})$ as follows,

$$F_{\gamma_i}(v_i) := \frac{D_i}{{\gamma_i}^{m_i}}\left[\int_\Omega J_{\gamma_i}(x-y)v_i(a,y)dy - v_i(a,x)\right],\ v_i \in C([0,a_2]\times\overline{\Omega}).$$

By the same argument in Theorem 5.7-(iii), one can show for each $v_i \in C([0,a_2]\times\overline{\Omega})$

$$F_{\gamma_i}(v_i) = \mathcal{O}({\gamma_i}^{-m_i}),\ \text{as } \gamma_i \to \infty \text{ uniformly in } (a,x)\in[0,a_2]\times\overline{\Omega}. \tag{7.25}$$

On the other hand, thanks to the non-negativeness of β and v^*, one has

$$\sum_{j=1}^{M}\int_0^{a_2}\beta_{ij}(a,x_0)v_j^*(a,x_0)da>0.$$

Since for each $(a,x)\in[0,a_2]\times\overline{\Omega}$,

$$\begin{aligned}
&f_i\left(\sum_{j=1}^{M}\int_0^{a_2}\beta_{ij}(a,x)v_j(a,x)da\right)-(1+\delta)f_i\left(\sum_{j=1}^{M}\int_0^{a_2}\beta_{ij}(a,x)v_j^*(a,x)da\right)\\
&=\sum_{j=1}^{M}\int_0^{a_2}\beta_{ij}(a,x)v_j(a,x)da\left[\frac{f_i\left(\sum_{j=1}^{M}\int_0^{a_2}\beta_{ij}(a,x)v_j(a,x)da\right)}{\sum_{j=1}^{M}\int_0^{a_2}\beta_{ij}(a,x)v_j(a,x)da}\right.\\
&\quad\left.-\frac{f_i\left(\sum_{j=1}^{M}\int_0^{a_2}\beta_{ij}(a,x)v_j^*(a,x)da\right)}{\sum_{j=1}^{M}\int_0^{a_2}\beta_{ij}(a,x)v_j^*(a,x)da}\right]<0.
\end{aligned}$$

where we use Assumption 1.3-(iii), there exists a sufficiently small positive constant $c=c(\delta)$, which satisfies $c(\delta)\to 0$ as $\delta\to 0$, such that

$$\begin{aligned}
&\sup_{[0,a_2]\times\overline{\Omega}}\left[f_i\left(\sum_{j=1}^{M}\int_0^{a_2}\beta_{ij}(a,x)v_j(a,x)da\right)\right.\\
&\qquad\left.-(1+\delta)f_i\left(\sum_{j=1}^{M}\int_0^{a_2}\beta_{ij}(a,x)v_j^*(a,x)da\right)\right]\\
&\le -c<0.
\end{aligned}\tag{7.26}$$

It implies that for any $\delta>0$, we can find $\gamma_i(\delta)>0$ such that $|F_{\gamma_i}(v_i)|\le c(\delta)$ for each $\gamma_i\in(\gamma_i(\delta),\infty)$. Set $\gamma(\delta)=\max_{1\le i\le M}\gamma_i(\delta)$.

Now fix this $\gamma(\delta)$, let us show that for each $\gamma\in(\gamma(\delta),\infty_{\mathbb{R}^M})$, there holds $u_\gamma^*(a,x)\le v(a,x)$ for all $(a,x)\in[0,a_2]\times\overline{\Omega}$. To do that, fix any $\gamma\in(\gamma(\delta),\infty_{\mathbb{R}^M})$ and define

$$\alpha_*:=\sup\{\alpha>0:\alpha u_\gamma^*(a,x)\le v(a,x)\text{ in }[0,a_2]\times\overline{\Omega}\}.$$

Since $\min_{[0,a_2]\times\overline{\Omega}}u_\gamma^*>0_{\mathbb{R}^M}$ and $v(a,x)$ is bounded, the number α_* is well defined and positive. Due to the continuity of $v(a,x)$ and $u_\gamma^*(a,x)$, there holds $v(a,x)\ge\alpha_*u_\gamma^*(a,x)$ for all $(a,x)\in[0,a_2]\times\overline{\Omega}$.

Clearly, if $\alpha_*\ge 1$, then we are done. So we assume that $\alpha_*<1$. Set $w:=v-\alpha_*u_\gamma^*$, then $w\ge 0$. Further, set $a_0:=\min\{a\in[0,a_2]:\exists x\in\overline{\Omega}, i\in\{1,\dots,M\},\text{ s.t. } w_i(a_0,x)=0\}$. Such a_0 exists due to the definition of α_*. It follows that there exists $x_0\in\overline{\Omega}$ such that $w_i(a_0,x_0)=0$.

If $a_0=0$, that is, $w_i(0,x_0)=0$. One has by (7.26)

$$
\begin{aligned}
v_i(0,x) &= (1+\delta)f)i\left(\sum_{j=1}^{M}\int_0^{a_2}\beta_{ij}(a,x)v_i^*(a,x)da\right)\\
&> (1+\delta)f_i\left(\sum_{j=1}^{M}\int_0^{a_2}\beta_{ij}(a,x)v_j^*(a,x)da\right)+f_i\left(\sum_{j=1}^{M}\int_0^{a_2}\beta_{ij}(a,x)v_j(a,x)da\right)\\
&\quad -(1+\delta)f_i\left(\sum_{j=1}^{M}\int_0^{a_2}\beta_{ij}(a,x)v_j^*(a,x)da\right)+\frac{c}{2}\\
&= f_i\left(\sum_{j=1}^{M}\int_0^{a_2}\beta_{ij}(a,x)v_j(a,x)da\right)+\frac{c}{2}.
\end{aligned}
$$

Thus $w_i(0,x_0)$ satisfies

$$
\begin{aligned}
w_i(0,x_0) &= v_i(0,x_0)-\alpha_* u^*_{\gamma\, i}(0,x_0)\\
&> f_i\left(\sum_{j=1}^{M}\int_0^{a_2}\beta_{ij}(a,x_0)v_j(a,x_0)da\right)\\
&\qquad +\frac{c}{2}-\alpha_* f_i\left(\sum_{j=1}^{M}\int_0^{a_2}\beta_{ij}(a,x_0)u^*_{\gamma\, j}(a,x_0)da\right)\\
&> f_i\left(\sum_{j=1}^{M}\int_0^{a_2}\beta_{ij}(a,x_0)v_j(a,x_0)da\right)\\
&\qquad +\frac{c}{2}-f_i\left(\sum_{j=1}^{M}\int_0^{a_2}\beta_{ij}(a,x_0)\alpha_* u^*_{\gamma\, j}(a,x_0)da\right)\\
&\geq \frac{c}{2},
\end{aligned}
\tag{7.27}
$$

where we used the Assumption 1.3-(iii) and $\alpha_* < 1$. It is a contradiction with $w_i(0,x_0)=0$.

If $a_0 \in (0,a_2]$, observe that w_i satisfies

$$
\begin{aligned}
\partial_a w_i(a,x) &= \frac{D_i}{{\gamma_i}^{m_i}}\left[\int_\Omega J_{\gamma_i}(x-y)w_i(a,y)dy-w_i(a,x)\right]-\mu_i(a,x)w_i(a,x)\\
&\quad -\frac{D_i}{{\gamma_i}^{m_i}}\left[\int_\Omega J_{\gamma_i}(x-y)v_i(a,y)dy-v_i(a,x)\right]\\
&= \frac{D_i}{{\gamma_i}^{m_i}}\left[\int_\Omega J_{\gamma_i}(x-y)w_i(a,y)dy-w_i(a,x)\right]\\
&\quad -\mu_i(a,x)w_i(a,x)-F_{\gamma_i}(v_i),
\end{aligned}
\tag{7.28}
$$

Again by the constant of variation formula (2.9), one has

$$
\begin{aligned}
w_i(a,x) &= e^{-\frac{D_i}{{\gamma_i}^{m_i}}a}\pi_i(0,a,x)w_i(0,x)+\frac{D_i}{{\gamma_i}^{m_i}}\int_0^a e^{-\frac{D_i}{{\gamma_i}^{m_i}}(a-l)}\pi_i(l,a,x)[K_{\gamma_i,\Omega}w_i](l,x)dl\\
&\quad +\int_0^a e^{-\frac{D_i}{{\gamma_i}^{m_i}}(a-l)}\pi_i(l,a,x)[F_{\gamma_i}(v_i)](l,x)dl.
\end{aligned}
\tag{7.29}
$$

Recall that $w_i(0, x) > \frac{c(\delta)}{2}$ by (7.27) which is independent in γ_i and D_i. Now considering the above inequality (7.29) at (a_0, x_0), $w_i(a, x) > 0$ for all $(a, x) \in [0, a_0) \times \overline{\Omega}$ implies

$$e^{-\frac{D_i}{\gamma_i^{m_i}}a}\pi_i(0, a, x)w_i(0, x) \geq \frac{c(\delta)}{4}e^{-\int_0^{a_2}\overline{\mu}_i(s)ds},$$
$$\frac{D_i}{\gamma_i^{m_i}}\int_0^a e^{-\frac{D_i}{\gamma_i^{m_i}}(a-l)}\pi_i(l, a, x)[K_{\gamma_i,\Omega}w_i](l, x)dl = \mathcal{O}(\gamma_i^{-m_i}).$$

These inequalities combining with (7.25) (up to increase γ_i if necessary) implies the right hand side of (7.29) is positive. But the left hand side $w_i(a_0, x_0) = 0$ induces a contradiction. Thus $\alpha_* \geq 1$ and the proof is complete.

For (ii) note by the argument in Theorem 5.7-(ii) that

$$F_{\gamma_i}(v_i) = \frac{D_i}{\gamma_i^{m_i}}\left[\int_\Omega J_{\gamma_i}(x-y)v_i(a, y)dy - v_i(a, x)\right] = \mathcal{O}(\gamma_i^{2-m_i}) \text{ as } \gamma_i \to 0^+$$

uniformly in $(a, x) \in [0, a_2] \times \overline{\Omega}$. Then we revisit (7.29). Observe

$$\frac{D_i}{\gamma_i^{m_i}}K_{\gamma_i,\Omega}w_i = \mathcal{O}(\gamma_i^{-m_i}) \text{ as } \gamma_i \to 0^+$$

uniformly in $(a, x) \in [0, a_2] \times \overline{\Omega}$. It follows that when $0 < \gamma_i \ll 1$,

$$\frac{D_i}{\gamma_i^{m_i}}\int_0^a e^{-\frac{D_i}{\gamma_i^{m_i}}(a-l)}\pi_i(l, a, x)[K_{\gamma_i,\Omega}w_i](l, x)dl$$
$$-\int_0^a e^{-\frac{D_i}{\gamma_i^{m_i}}(a-l)}\pi_i(l, a, x)[F_{\gamma_i}(v_i)](l, x)dl > 0.$$

Then the remaining proof is the same with (iii).

For (i) we follow the lines as in the proof of (iii) except that we need to set $\gamma = 1_{\mathbb{R}^M}$ and replace the limit

$$F_{\gamma_i}(v_i) = \frac{D_i}{\gamma_i^{m_i}}\left[\int_\Omega J_{\gamma_i}(x-y)v_i(a, y)dy - v_i(a, x)\right] \to 0 \text{ as } \gamma_i \to 0^+$$

uniformly in $(a, x) \in [0, a_2] \times \overline{\Omega}$ by the following limit

$$F_1(v_i) = D_i\left[\int_\Omega J(x-y)v_i(a, y)dy - v_i(a, x)\right] \to 0 \text{ as } D_i \to 0^+$$

uniformly in $(a, x) \in [0, a_2] \times \overline{\Omega}$ and (7.29) is replaced by the following equality

$$w_i(a, x) = e^{-D_i a}\pi_i(0, a, x)w(0, x) + D_i\int_0^a e^{-D_i(a-l)}\pi_i(l, a, x)[Kw_i](l, x)dl$$
$$-\int_0^a e^{-D_i(a-l)}\pi_i(l, a, x)[F_1(v_i)](l, x)dl.$$

□

8 Discussions

Age-structured models with nonlocal diffusion could be used to characterize the spatio-temporal transmission dynamics of infectious diseases in which the age structure of hosts is a very important factor and the disease spreads from places to places which are not geographically connected via the long distance traveling of hosts. There are very few theoretical studies on the dynamics of such equations due to the lack of methods and techniques in treating them. In this paper, we studied the spectrum theory for multigroup age-structured models with nonlocal diffusion. First we gave a sufficient and easily verifiable condition on the existence of principal eigenvalue by using the theory of resolvent positive operators with their perturbations. Then we used the generalized principal eigenvalue to characterize the principal eigenvalue and applied it to discuss the effects of diffusion rate and diffusion range on the principal eigenvalue. Next we established the strong maximum principle for such age-structured models with nonlocal diffusion. Finally we investigated the existence, uniqueness and stability of such equations with cooperative type of nonlinearity.

Here we assumed that the diffusion kernels are the same for each component. We expect to study the effects of different kernels for different components on the principal eigenvalue and in particular the dynamics of such systems in the future. In addition, we expect that the results on the principal eigenvalue and the construction of sub- and super-solutions can be applied to study traveling wave solutions and spreading speeds of multigroup age-structured models with nonlocal diffusion and we leave this for future consideration.

Finally, we believe that our results can be applied to age-structured models with nonlocal diffusion of Neumann type, see Kang and Ruan [25] where we applied such a theory to a scalar age-structured equation with nonlocal diffusion of Neumann boundary conditions and found that the principal eigenvalue converges to that of the equation without diffusion derived from the spatial average of reaction terms. Further, similar results for cooperative systems with Neumann boundary conditions can be found in Zhang and Zhao [52].

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Data Availability There is no data associated with this paper.

A Appendix

A.1 Transmission and Death Rates Independent of *x*

Now we introduce the eigenvalues and eigenfunctions of the nonlocal problem with Dirichlet boundary condition, which are denoted by $(\theta_i, \varphi_i)_{i\geq 0}$, in the domain $\Omega \subset \mathbb{R}^N$; that is,

$$\begin{cases} -L\varphi := -(J * \varphi_i - \varphi_i)(x) = \theta_i \varphi_i(x), \; x \in \Omega \\ \varphi_i(x) = 0, \; x \in \mathbb{R}^N \setminus \Omega \end{cases} \tag{A.1}$$

with

$$\int_\Omega \varphi_i^2(x)dx = 1, \; i \geq 0. \tag{A.2}$$

Note that the eigenfunctions φ_i of (A.1) satisfy $\varphi_i = 0$ in $\mathbb{R}^N \backslash \Omega$, the integral in the convolution term can indeed be confined in Ω. Therefore, we define the operator

$$[Ku](x) = \int_\Omega J(x-y)u(y)dy, \ u \in L^2(\Omega).$$

Now observe that θ is an eigenvalue of (A.1)-(A.2) if and only if $\hat{\theta} = 1 - \theta$ is an eigenvalue of K in $L^2(\Omega)$. It is easy to see that K is compact and self-adjoint in $L^2(\Omega)$. Hence, by the classical spectral theorem, there exists an orthonormal basis consisting of eigenvectors of K with corresponding eigenvalues $\{\hat{\theta}_n\} \subset \mathbb{R}$ and $\hat{\theta}_n \to 0$. Furthermore, we are interested in the existence of a principal eigenvalue, that is an eigenvalue associated to a nonnegative eigenfunction. We state a result related to the principal eigenvalue (see [11, 21, 22]).

Theorem A.1 [21] *Problem* (A.1)–(A.2) *admits an eigenvalue θ_0 associated to a positive eigenfunction $\varphi_0 \in C(\overline{\Omega})$. Moreover, it is simple and unique and satisfies $0 < \theta_0 < 1$. Furthermore, θ_0 can be variationally characterized as*

$$\theta_0 = 1 - \left(\sup_{u \in L^2(\Omega), \|u\|_{L^2(\Omega)}=1} \int_\Omega \left(\int_\Omega J(x-y)u(y)dy \right)^2 dx \right)^{1/2}. \tag{A.3}$$

For other eigenvalues we can arrange them as $0 < \theta_0 < \theta_1 \le \theta_2 \le \cdots \to 1$. Next we introduce an operator for the system of nonlocal diffusion with Dirichlet boundary condition

$$-\mathcal{L}u = \text{diag}\{-Lu_1, \dots, -Lu_M\}, \ u = (u_1, \dots, u_M) \in W := L^2(\Omega, \mathbb{R}^M). \tag{A.4}$$

Then it is easy to see that $-\mathcal{L}$ has the same eigenvalues as the ones of $-L$. Moreover, the eigenvalues of $-\mathcal{L}$ can be still arranged in the following way:

$$0 < \theta_0 < \theta_1 \le \theta_2 \le \cdots.$$

Here we would like to emphasize that θ_0 is a principal eigenvalue of $\mathcal{L}$ associated with a positive eigenfunction $(\varphi_0, \dots, \varphi_0)$ in the sense that each component of the eigenfunction is positive and θ_0 is isolated. Note that θ_0 is not simple, since there are linearly independent positive eigenfunctions, for example $(2\varphi_0, \varphi_0, \dots, \varphi_0)$. But for convenience, we are only interested in $(\varphi_0, \dots, \varphi_0)$.

Now we denote the usual population operator without diffusion by $\widetilde{B}$ defined in $V := L^2((0, a^+), \mathbb{R}^M)$:

$$[\widetilde{B}\eta_i](a) = -\frac{\partial \eta_i(a)}{\partial a} - \overline{\mu}_i(a)\eta_i(a), \ \forall \eta \in dom(B), \tag{A.5}$$

$$dom(\widetilde{B}) = \left\{ \eta(a) | \eta, B\eta \in L^2(0, a^+), \ \eta_i(0) = \sum_{j=1}^M \int_0^{a^+} \underline{\beta}_{ij}(a)\eta_j(a)da \right\} \tag{A.6}$$

and $\{\kappa_j\}_{j \ge 0}$ be the eigenvalues of $\widetilde{B}$, i.e., the solution of the following equation

$$\begin{aligned} \partial_a w_i(a,x) \ge D_i \left[\int_\Omega J(x-y)w_i(a,y)dy - w_i(a,x) \right] \\ -\mu_i(a,x)w_i(a,x), \quad (a,x) \in (0, a_2] \times \overline{\Omega}. \end{aligned}$$

while the principal eigenvalue, denoted by κ_0, satisfies

$$r\left(\int_0^{a^+} \underline{\beta}(a) e^{-\kappa_0 a} \pi(a) da \right) = 1,$$

where det denotes the determinant and

$$\pi(a) := \mathrm{diag}\{\pi_1(a), \dots, \pi_M(a)\} = \mathrm{diag}\{e^{-\int_0^a \overline{\mu}_1(\rho)d\rho}, \dots, e^{-\int_0^a \overline{\mu}_M(\rho)d\rho}\}.$$

Arrange κ in the following way (see [50]):

$$\kappa_0 > \mathrm{Re}\kappa_1 \geq \mathrm{Re}\kappa_2 \geq \cdots .$$

Introduce the state space $E := L^2((0, a^+) \times \Omega, \mathbb{R}^M)$ with the usual norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$ and define an operator $\widetilde{A} : E \to E$ by

$$\begin{aligned} &[\widetilde{A}\phi]_i(a, x) = (J * \phi_i - \phi_i)(a, x) - \frac{\partial \phi_i(a, x)}{\partial a} - \overline{\mu}_i(a)\phi_i(a, x), \ \forall \phi \in dom(\widetilde{A}), \\ &dom(\widetilde{A}) = \Big\{\phi(a, x)\big|\phi, A\phi \in E, \phi|_{\mathbb{R}^N \setminus \Omega} = 0, \phi_i(0, x) = \sum_{j=1}^M \int_0^{a^+} \underline{\beta}_{ij}(a)\phi_j(a, x)da\Big\}. \end{aligned} \tag{A.7}$$

Next let us solve the resolvent equation

$$(\xi I - \widetilde{A})\phi = \psi, \ \forall \psi \in E.$$

If for any $i, j \geq 0, \xi + \theta_i \neq \kappa_j$, then define

$$\phi_\psi(a, x) = \sum_{i=0}^{\infty} \big((\xi + \theta_i)I - \widetilde{B}\big)^{-1} \langle \psi(a, \cdot), \Phi_i \rangle_W \circ \Phi_i(x),$$

where

$$\begin{aligned} &\langle \psi(a, \cdot), \Phi_i \rangle_W \\ &\quad = \left(\int_\Omega \psi_1(a, x)\varphi_i(x)dx, \dots, \int_\Omega \psi_M(a, x)\varphi_i(x)dx\right), \ \Phi_i = (\varphi_i, \dots, \varphi_i) \in W, \end{aligned}$$

and

$$u \circ \Phi_i = (u_1\varphi_i, \dots, u_M\varphi_i), \ u \in V.$$

Since $\widetilde{B}$ is the infinitesimal generator of a bounded strongly continuous semigroup, there exist constants $O > 0$ and $\omega \in \mathbb{R}$ such that

$$\big\|(\xi I - \widetilde{B})^{-1}\big\|_{\mathcal{L}(V)} \leq \frac{O}{\mathrm{Re}\xi - \omega}, \ \forall \mathrm{Re}\xi > \omega.$$

Recall that $\theta_i > 0$ for all i, then $\mathrm{Re}(\xi + \theta_i) > \omega$ for all $i > 0$ provided $\mathrm{Re}\xi > \omega$,

$$\begin{aligned} &\sum_{i=0}^{\infty} \left\| \big((\xi + \theta_i)I - \widetilde{B}\big)^{-1} \langle \psi(a, \cdot), \Phi_i \rangle_W \right\|_V^2 \\ &\quad \leq \left[\frac{O}{\mathrm{Re}(\xi + \theta_0) - \omega}\right]^2 \sum_{i=0}^{\infty} \|\langle \psi(a, \cdot), \Phi_i \rangle_W\|_V^2 \\ &\quad \leq \left[\frac{O}{\mathrm{Re}(\xi + \theta_0) - \omega}\right]^2 \|\psi\|_E^2 < \infty. \end{aligned} \tag{A.8}$$

Thus, $\phi_\psi(a,x)$ is well defined. Moreover, for any $n>0$,

$$(\xi I-\widetilde{A})\sum_{i=0}^{n}\big((\xi+\theta_i)I-\widetilde{B}\big)^{-1}\langle\psi(a,\cdot),\Phi_i\rangle_W\circ\Phi_i(x)$$
$$=\sum_{i=0}^{n}\langle\psi(a,\cdot),\Phi_i\rangle_W\circ\Phi_i(x)\to\psi(a,x)\text{ in }E\text{ as }n\to\infty.$$

Since $\widetilde{B}$ and $\mathcal{L}$ are both closed operators on E, so is $\widetilde{A}$. Hence $(\xi I-\widetilde{A})\phi_\psi=\psi$, i.e. $\phi_\psi(a,x)$ is a solution of the resolvent equation. Now choose $\phi\in dom(\widetilde{A})$, we have

$$\begin{aligned}\langle\widetilde{A}\phi,\phi\rangle_E&=\sum_{i=1}^{M}\int_{(0,a^+)\times\Omega}-\frac{\partial\phi_i(a,x)}{\partial a}\phi_i(a,x)dadx-\int_{(0,a^+)\times\Omega}\overline{\mu}_i(a)|\phi_i(a,x)|^2dadx\\&\quad+\int_{(0,a^+)\times\Omega}(J*\phi_i(a,x)-\phi_i(a,x))\phi_i(a,x)dadx\\&\le\sum_{i=1}^{M}\frac{1}{2}\int_\Omega|\phi_i(0,x)|^2dx\\&=\sum_{i=1}^{M}\sum_{j=1}^{M}\frac{1}{2}\int_\Omega\left[\int_0^{a^+}\underline{\beta}_{ij}(a)\phi_j(a,x)da\right]^2dx\\&\le\sum_{i=1}^{M}\sum_{j=1}^{M}\frac{1}{2}\int_\Omega\left[\int_0^{a^+}\underline{\beta}_{ij}^2(a)da\right]\left[\int_0^{a^+}\phi_j^2(a,x)da\right]dx\\&\le\frac{1}{2}\left\|\underline{\beta}\right\|_{\mathcal{L}(V)}^2\|\phi\|_E^2,\end{aligned}\tag{A.9}$$

where $\left\|\underline{\beta}\right\|_{\mathcal{L}(V)}:=\max_{1\le j\le M}\sum_{i=1}^{M}\left\|\underline{\beta}_{ij}\right\|_{L^2(0,a^+)}^2$ and we used the symmetry of J

$$\begin{aligned}&\int_{(0,a^+)\times\Omega}(J*\phi_i(a,x)-\phi_i(a,x))\phi_i(a,x)dadx\\&\le\int_{(0,a^+)}\int_\Omega\int_\Omega J(x-y)(\phi_i(a,y)-\phi_i(a,x))\phi_i(a,x)dydxda\\&=-\frac{1}{2}\int_{(0,a^+)}\int_\Omega\int_\Omega J(x-y)(\phi_i(a,y)-\phi_i(a,x))^2dydxda\le 0.\end{aligned}\tag{A.10}$$

It follows that for all sufficiently large ξ, $\widetilde{A}-\xi I$ is a dissipative operator on E.

On the other hand, it can be shown that ϕ is the unique solution of the resolvent equation by the uniqueness resolvent solution of age-structured models with orthonormal basis in W. Thus $\xi\in\rho(\widetilde{A})$, the resolvent set of $\widetilde{A}$, and

$$(\xi I-\widetilde{A})^{-1}\psi=\sum_{i=0}^{\infty}\big((\xi+\theta_i)I-\widetilde{B}\big)^{-1}\langle\psi(a,\cdot),\Phi_i\rangle_W\circ\Phi_i(x).\tag{A.11}$$

It yields that $\mathcal{R}(\xi I-\widetilde{A})$, the range of $\xi I-\widetilde{A}$ is equal to the whole space E, and by (A.9), $\widetilde{A}-\xi I$ is dissipative when ξ is sufficiently large, it follows from Pazy [39, Chapter I, Theorem 4.6] that $dom(\widetilde{A}-\xi I)$ is dense and $\overline{dom(\widetilde{A}-\xi I)}=E$, so does $dom(\widetilde{A})$ and $\overline{dom(\widetilde{A})}=E$.

Moreover, from (A.8) we have

$$\left\| (\xi I - \widetilde{A})^{-1} \right\|_E \leq \frac{O}{\mathrm{Re}(\xi + \theta_0) - \omega}.$$

Hille–Yosida theorem implies that $\widetilde{A}$ is an infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$. (In fact, one can conclude the same result by using Lumer–Phillips theorem in Pazy [39].)

If there are some i, j such that $\xi + \theta_i = \kappa_j$, then

$$\Phi_i(a, x) = e^{-(\xi+\theta_i)a}\pi(a)\mathit{\Phi}_i(x)$$

satisfies $(\xi I - \widetilde{A})\Phi_i = 0$; i.e., $\xi \in \sigma_p(\widetilde{A})$, the point spectrum of A. Furthermore, if $(\xi I - \widetilde{A})\Theta = 0$, expanding the known initial function $\Theta(0, x)$ as

$$\Theta(0, x) = \left(\sum_{i=0}^{\infty} \alpha_{1i}\varphi_i(x), \ldots, \sum_{i=0}^{\infty} \alpha_{Mi}\varphi_i(x) \right) \text{ in } W,$$

then we have

$$\Theta(a, x) = \left(\sum_{i=0}^{\infty} \alpha_{1i} e^{-(\xi+\theta_i)a}\pi_1(a)\varphi_i(x), \ldots, \sum_{i=0}^{\infty} \alpha_{Mi} e^{-(\xi+\theta_i)a}\pi_M(a)\varphi_i(x) \right).$$

In view of the initial condition

$$\Theta(0, x) = \int_0^{a^+} \underline{\beta}(a)\Theta(a, x)da,$$

we get for each $i = 1, \ldots, M$, either $\alpha_{1i} = 0, \ldots, \alpha_{Mi} = 0$ or $1 \in \sigma_P \left(\int_0^{a^+} \underline{\beta}(a)e^{-(\xi+\theta_i)a}\pi(a) da \right)$, which implies that $\kappa_j = \xi + \theta_i$ is an eigenvalue of $\widetilde{B}$ by the theory of age-structured models. Hence, $\xi = \kappa_j - \theta_i$ is an eigenvalue of $\widetilde{A}$. In particular, for $\xi_0 = \kappa_0 - \theta_0$, which is the principal eigenvalue of $\widetilde{A}$, $(\xi I - \widetilde{A})\Phi = 0$ has one positive linear solution, which is

$$\Phi_{\xi_0}(a, x) = e^{-\kappa_0 a}\pi(a)\mathit{\Phi}_0(x). \tag{A.12}$$

Define an operator

$$\mathcal{C}_\xi = \int_0^{a^+} \underline{\beta}(a)e^{-\xi a}\pi(a)e^{\mathcal{L}a}da.$$

It is easy to see that $\mathcal{C}_\xi$ is a positive and self-adjoint operator in W, since $\mathcal{L}$ is self-adjoint, and that $\mathit{\Phi}_0(x)$ is the eigenfunction of the eigenvalue 1 of $\mathcal{C}_{\xi_0}$. Thus, $r(\mathcal{C}_{\xi_0}) \geq 1$.

In addition, note that $\{\varphi_i\}_{i\geq 0}$ are indeed in $C_b(\Omega)$ due to the fact that J is continuous and $e^{La} : C_b(\Omega) \to C_b(\Omega)$ is an e^{-a} contraction mapping, where e^{-a} is the Kuratowski measure of noncompactness in the metric space $(C_b(\Omega), d)$, see Fang and Zhao [18], where $C_b(\Omega)$ represents the space of continuous bounded functions in Ω, and for any $u, v \in C_b(\Omega)$, $d(u, v) := \sum_{k=1}^{\infty} \frac{1}{2^k} \max_{-k\leq x\leq k} |u(x)-v(x)|$. It follows by Perron–Frobenius Theorem that

$$\begin{aligned} r_e(\mathcal{C}_{\xi_0}) \leq \left\|\mathcal{C}_{\xi_0}\right\|_e &\leq r \left(\int_0^{a^+} \underline{\beta}(a)e^{-\xi_0 a}\pi(a) \left\| e^{\mathcal{L}a} \right\|_e da \right) \\ &\leq r \left(\int_0^{a^+} \underline{\beta}(a)e^{-\kappa_0 a}\pi(a)\mathrm{diag}\{e^{-(1-\theta_0)a}, \ldots, e^{-(1-\theta_0)a}\} \right) da \end{aligned}$$

$$
\begin{aligned}
&< r\left(\int_0^{a^+} \underline{\beta}(a)e^{-\kappa_0 a}\pi(a)da\right)\\
&= 1,
\end{aligned}
\tag{A.13}
$$

where $r_e(A)$ and $\|A\|_e$ represent the essential spectral radius and essential norm of operator A in E, respectively. Now suppose that $r(\mathcal{C}_{\xi_0}) > 1$, for the sake of contraction, we then see from the generalized Krein-Rutman theorem (see [38] or [51]) that $r(\mathcal{C}_{\xi_0})$ is an eigenvalue of $\mathcal{C}_{\xi_0}$ corresponding to a positive eigenvector $\psi \in W$. It follows that

$$
r(\mathcal{C}_{\xi_0})\langle\psi, \Phi_0\rangle_W = \langle\mathcal{C}_{\xi_0}\psi, \Phi_0\rangle_W = \langle\psi, \mathcal{C}_{\xi_0}\Phi_0\rangle_W = \langle\psi, \Phi_0\rangle_W,
$$

which implies that $r(\mathcal{C}_{\xi_0}) = 1$ since $\langle\psi, \Phi_0\rangle > 0$. This is a contradiction. Thus $r(\mathcal{C}_{\xi_0}) = 1$.

In summary, we have the following theorem.

Theorem A.2 *The following statements are valid.*

(i) *The operator $\widetilde{A}$ defined in (A.7) generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on E;*
(ii) $\sigma(\widetilde{A}) = \sigma_P(\widetilde{A}) = \{\kappa_i - \theta_j\}_{i,j=0}^{\infty}$;
(iii) *The operator $\widetilde{A}$ has a real principal eigenvalue ξ_0 corresponding to the eigenfunction Φ_{ξ_0} defined in (A.12); that is, ξ_0 is greater than any real part of eigenvalues of A;*
(iv) *For the operator $\mathcal{C}_{\xi_0}$, 1 is an eigenvalue with an eigenfunction $\Phi_0(x)$. Furthermore, $r(\mathcal{C}_{\xi_0}) = 1$.*

The proofs of (i)-(iii) are similar to those in Chan and Guo [9, Theorem 1] or Kang and Ruan [26, Theorem 2.2]. We omit them here. The proof of (iv) is shown in the above argument.

A.2 Resolvent Positive Operators

In this section we recall the theory of resolvent positive operators, the readers can refer to Thieme [47, 48] for details. A linear operator $A : Z_1 \to Z$, defined on a linear subspace Z_1 of Z, is said to be *positive* if $Ax \in Z_+$ for all $x \in Z_1 \cap Z_+$ and A is not the 0 operator, where Z_+ is a closed convex cone that is normal and generating.

Definition A.3 A closed operator A in Z is said to be *resolvent positive* if the resolvent set of A, $\rho(A)$, contains a ray (ω, ∞) and $(\lambda I - A)^{-1}$ is a positive operator (i.e. maps Z_+ into Z_+) for all $\lambda > \omega$.

Definition A.4 We define the *spectral bound* of a closed operator A as

$$
s(A) = \sup\{\mathrm{Re}\lambda \in \mathbb{R}; \lambda \in \sigma(A)\},
$$

the *real spectral bound* of A as

$$
s_{\mathbb{R}}(A) = \sup\{\lambda \in \mathbb{R}; \lambda \in \sigma(A)\},
$$

and the *spectral radius* of A as

$$
r(A) = \sup\{|\lambda|; \lambda \in \sigma(A)\}.
$$

If B is a resolvent positive operator and $C : dom(B) \to Z$ is a positive linear operator, then $A = B + C$ is called a *positive perturbation* of B. If $B + C$ is a positive perturbation of B and $\lambda > s(B)$, then $C(\lambda I - B)^{-1}$ is automatically bounded (without C being necessarily closed). This is a consequence of Z_+ being normal and generating.

Theorem A.5 [47, Theorem 3.5] *Let the cone Z_+ be normal and generating and A be a resolvent positive operator in Z. Then $s(A) = s_{\mathbb{R}}(A) < \infty$ and $s(A) \in \sigma(A)$ whenever $s(A) > -\infty$. Moreover, there is a constant $c > 0$ such that*

$$\left\|(\lambda I - A)^{-1}\right\| \leq c \left\|(\mathrm{Re}\lambda I - A)^{-1}\right\| \text{ whenever } \mathrm{Re}\lambda > s(A).$$

Now define

$$F_\lambda = C(\lambda I - B)^{-1},\ \lambda > s(B). \tag{A.14}$$

Theorem A.6 [48, Theorem 3.6] *Let Z be an ordered Banach space with normal and generating cone Z_+ and let $A = B + C$ be a positive perturbation of B. Then $r(F_\lambda)$ is a decreasing convex function of $\lambda > s(B)$, and exactly one of the following three cases holds:*

(i) *if $r(F_\lambda) \geq 1$ for all $\lambda > s(B)$, then A is not resolvent positive;*
(ii) *if $r(F_\lambda) < 1$ for all $\lambda > s(B)$, then A is resolvent positive and $s(A) = s(B)$;*
(iii) *if there exists $\nu > \lambda > s(B)$ such that $r(F_\nu) < 1 \leq r(F_\lambda)$, then A is resolvent-positive and $s(B) < s(A) < \infty$; further $s = s(A)$ is characterized by $r(F_s) = 1$.*

Definition A.7 The operator $C : dom(B) \to Z$ is called a *compact perturbator* of B and $A = B + C$ a *compact perturbation* of B if

$$(\lambda I - B)^{-1} F_\lambda : \overline{dom(B)} \to \overline{dom(B)} \text{ is compact for some } \lambda > s(B)$$

and

$$(\lambda I - B)^{-1}(F_\lambda)^2 : Z \to Z \text{ is compact for some } \lambda > s(B).$$

C is called an *essentially compact perturbator* of B and $A = B + C$ an *essentially compact perturbation* of B if there is some $n \in \mathbb{N}$ such that $(\lambda I - B)^{-1} F_\lambda^n$ is compact for all $\lambda > s(B)$.

Definition A.8 Let F_λ be a positive resolvent output family for B. A vector $x \in X_+$ is said to be *conditionally strictly positive* if the following holds:

If $x^* \in Z_+^*$ and $F_\lambda^* x^* \neq 0$ for some (and then for all) $\lambda > s(B)$, then $\langle x, x^* \rangle > 0$.

Similarly a functional $x^* \in Z_+^*$ is said to be *conditionally strictly positive* if the following holds:

If $x \in Z_+$ and $F_\lambda x \neq 0$ for some (and then for all) $\lambda > s(B)$, then $\langle x, x^* \rangle > 0$.

Theorem A.9 [47, Theorems 4.7 and 4.9] *Assume that C is an essentially compact perturbator of B. Moreover assume that there exists $\lambda_2 > \lambda_1 > s(B)$ such that $r(F_{\lambda_1}) \geq 1 > r(F_{\lambda_2})$. Then $s(B) < s(A) < \infty$ and the following statements hold:*

(i) *$s(A)$ is an eigenvalue of A associated with positive eigenvectors of A and A^*, has finite algebraic multiplicity, and is a pole of the resolvent of A. If C is a compact perturbator of B, then all spectral values λ of A with $\mathrm{Re}\lambda \in (s(B), s(A)]$ are poles of the resolvent of A and eigenvalues of A with finite algebraic multiplicity;*
(ii) *1 is an eigenvalue of $F_{s(A)}$ and is associated with an eigenvector $w \in Z$ of $F_{s(A)}$ such that $(\lambda I - B)^{-1} w \in Z_+$ and with an eigenvector $v^* \in Z_+^*$ of $F_{s(A)}^*$. Actually $s(A)$ is the largest $\lambda \in \mathbb{R}$ for which 1 is an eigenvalue of F_λ.*

Moreover, if Z is a Banach lattice and there exists a fixed point of F_s^* in Z_+^* that is conditionally strictly positive, then the following statements hold:

(iii) *$s = s(A)$ is associated with a positive eigenvector v of A such that $w = (s(A)I - B)v$ is a positive fixed point of $F_{s(A)}$;*
(iv) *s is the only eigenvalue of A associated with a positive eigenvector.*

Finally we assume in addition that all positive non-zero fixed points of F_s are conditionally strictly positive. Then the following holds:

(v) *$s = s(A)$ is a first order pole of the resolvent of A.*
(vi) *The eigenspace of A associated with $s(A)$ is one-dimensional and spanned by a positive eigenvector v of A. The eigenspace of A^* associated with $s(A)$ is also spanned by a positive eigenvector v^*.*

A.3 Perron–Frobenius Theory

In this section we recall Perron–Frobenius theory, the interested readers can refer to Marek [35] for more details.

Proposition A.10 *If A is a nonnegative and irreducible matrix, then*

(i) *the spectral radius $r(A)$ is a simple eigenvalue of A, i.e. from $(A - r(A)I)^p y = 0$ it follows that $Ay = r(A)y$ and if $Ay_1 = r(A)y_1$, $y_1 \neq 0$, then there exists a constant c such that $y = cy_1$;*
(ii) *corresponding to $r(A)$ there exists one eigenvector x_0 with all positive components;*
(iii) *corresponding to $r(A)$ there exists one eigenvector of the adjoint matrix A' with all positive components;*
(iv) *if $A \geq B$ and $A \neq B$, then $r(A) > r(B)$.*

Moreover, if A is a nonnegative and primitive matrix, the results in the above proposition hold and in addition, the spectral radius $r(A)$ is a dominant eigenvalue of A, i.e. it is strictly larger than the modulus of any other eigenvalue λ of A.

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