



Propagation dynamics in periodic predator-prey systems with nonlocal dispersal



Shi-Liang Wu^a, Liyan Pang^a, Shigui Ruan^{b,*}

^a School of Mathematics and Statistics, Xidian University, Xi'an, Shaanxi 710071, China

^b Department of Mathematics, University of Miami, Coral Gables, FL 33146, USA

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ABSTRACT

In this paper, we study the propagation dynamics of a time-periodic predator-prey system with nonlocal dispersal. We first establish the existence and nonexistence of periodic traveling waves and then investigate the spreading properties of solutions starting from compactly supported initial conditions. Roughly speaking, we show that if predators disperse faster than the prey, then both species spread simultaneously; whereas if the prey disperses faster than predators, then there exist two separate invasion fronts, one front occurs as the prey invades open habitats, and the other front appears when predators catch up the prey. We emphasize that one needs to find some new techniques to treat nonlocal predator-prey systems due to the presence of the time dependence of nonlinearity, the lack of compactness of the nonlocal dispersal operators and the lack of the comparison principle for predator-prey systems.

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R É S U M É

Dans cet article, nous étudions la dynamique de propagation pour un système prédateur-proie périodique en temps avec une dispersion non-locale. Nous étudions d'abord l'existence et l'inexistence des ondes progressives périodiques et nous étudions ensuite la propriété de propagation des solutions à partir de conditions initiales à supports compacts. Nous montrons que si les prédateurs se dispersent plus rapidement que la proie, alors les deux espèces se sont propagées simultanément; tandis que si la proie se disperse plus vite que les prédateurs, alors il existe deux fronts d'invasion séparés, un front se produit lorsque la proie envahit les habitats ouverts et l'autre front apparaît lorsque les prédateurs rattrapent la proie. Nous soulignons qu'il faut trouver de nouvelles techniques pour traiter les systèmes prédateur-proie non-locaux en raison de la présence de la dépendance temporelle de la non-linéarité, le manque de compacité des opérateurs de dispersion non-locale et l'absence de principe de comparaison pour les systèmes prédateur-proie.

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* Corresponding author.

E-mail addresses: slwu@xidian.edu.cn (S.-L. Wu), lanhai.happy@163.com (L. Pang), ruan@math.miami.edu (S. Ruan).

1. Introduction

Classical reaction-diffusion equations with Laplacian operators are usually used to model the spatial diffusion and interaction in population dynamics (Murray [29], Okubo and Levin [32]). However, such equations are only appropriate for local dispersal of species since Laplacian operators describe the random walk only when the step size and time size are small compared with the spatial variable and time, respectively. Whereas recent field data demonstrate that biological populations disperse nonlocally (Nathan et al. [30]) and convolution operators are more suitable to characterize such nonlocal dispersal in population dynamics (Hao et al. [21], Sherratt [37], Zhao and Ruan [51,52]). On the other hand, time-periodic coefficients have been incorporated to population dynamical models to characterize seasonal effects of weather, food supply, mating habits, harvesting seasons, etc. (Cushing [11], Rinaldi et al. [35], Taylor et al. [40]).

Taking both the nonlocal dispersal and time-periodicity into account, in this paper we study time-periodic traveling wave solutions (periodic traveling waves for short) and spreading properties for the following time-periodic predator-prey system with nonlocal dispersal

$$\begin{cases} (u_1)_t = d_1(J_1 * u_1 - u_1) + u_1(r_1(t) - b_1(t)u_1 - a_1(t)u_2), \\ (u_2)_t = d_2(J_2 * u_2 - u_2) + u_2(-r_2(t) - b_2(t)u_2 + a_2(t)u_1), \end{cases} \quad (1.1)$$

where $t > 0$, $x \in \mathbb{R}$, $u_1(t, x)$ and $u_2(t, x)$ denote the density of the prey and predators at time t and location x , respectively. $d_1, d_2 > 0$ are the dispersal coefficients, $r_i(t)$, $a_i(t)$, $b_i(t) \in C^1(\mathbb{R})$ ($i = 1, 2$) are positive T -periodic functions for some constant $T > 0$, and $\bar{r}_1 = \frac{1}{T} \int_0^T r_1(t) dt > 0$. The convolution operator $J_i * v - v$ describes the nonlocal dispersal process and is defined by (Andreu et al. [2], Bates [4])

$$(J_i * v)(t, x) - v(t, x) = \int_{\mathbb{R}} J_i(x - y)v(t, y)dy - v(t, x), \quad i = 1, 2.$$

Throughout this paper, we always make the following assumption on $J_i(\cdot)$ ($i = 1, 2$):

(H₁) $J_i \in C^1(\mathbb{R})$, $J_i(-x) = J_i(x) \geq 0$, $\int_{\mathbb{R}} J_i(x)dx = 1$, and J_i has compact support with $\chi_i := \text{supp} J_i > 0$.

The corresponding spatially homogeneous system of (1.1) is

$$\begin{cases} u_1'(t) = u_1(t)(r_1(t) - b_1(t)u_1(t) - a_1(t)u_2(t)), \\ u_2'(t) = u_2(t)(-r_2(t) - b_2(t)u_2(t) + a_2(t)u_1(t)). \end{cases} \quad (1.2)$$

By Zhao [53, Theorem 3.1.2], it follows that if $\bar{r}_1 > 0$, then

$$u_1'(t) = u_1(t)(r_1(t) - b_1(t)u_1(t))$$

admits a unique positive T -periodic solution $p(t)$, which is globally asymptotically stable for all positive initial values. Moreover, $p(t)$ can be explicitly given by

$$p(t) = \frac{p_0 e^{\int_0^t r_1(s)ds}}{1 + p_0 \int_0^t e^{\int_0^s r_1(\tau)d\tau} b_1(s)ds}, \quad p_0 = \frac{e^{\int_0^T r_1(s)ds} - 1}{\int_0^T e^{\int_0^s r_1(\tau)d\tau} b_1(s)ds}. \quad (1.3)$$

Similarly, if $\overline{a_2 p - r_2} := \frac{1}{T} \int_0^T [a_2(t)p(t) - r_2(t)]dt > 0$, then

$$u_2'(t) = u_2(t)(a_2(t)p(t) - r_2(t) - b_2(t)u_2(t)) \quad (1.4)$$

has a unique positive periodic solution $q(t)$, which is also globally asymptotically stable.

In the last two decades, there are many studies devoted to periodic traveling waves for *monotone* time-periodic evolution systems. For example, Alikakos et al. [1] investigated the existence, uniqueness and stability of bistable periodic traveling fronts of a time-periodic reaction-diffusion equation, Fang and Zhao [17] developed the theory of bistable traveling fronts for monotone evolution systems, Zhao and Ruan [49,50] studied the existence, uniqueness and stability of monostable periodic traveling waves for time-periodic reaction-diffusion systems, Liang, Yi and Zhao [26] established the theory of spreading speeds and monostable traveling waves for monotone periodic semiflows. On the other hand, there are many practical models, such as predator-prey systems and epidemic systems, that are nonmonotone. Due to the lack of the comparison principle for such nonmonotone systems, it is very interesting and challenging to study the existence and properties of traveling waves. Recently, Zhang, Wang and Zhao [46,48] proposed a method based on the Schauder's fixed theorem to investigate periodic traveling waves for two time-periodic diffusive epidemic models, Zhang, Wang and Zhao [47] further considered periodic traveling waves for a time-periodic delayed reaction-diffusion model without quasi-monotonicity.

Since solution maps of the nonlocal dispersal system (1.1) lack compactness with respect to compact open topology, the method in Zhang, Wang and Zhao [46–48] is not applicable to such nonmonotone time-periodic predator-prey systems with nonlocal dispersal. To overcome this difficulty, in this paper we prove the existence of periodic traveling waves of (1.1) (see Theorem 2.12) by appealing to the asymptotic fixed point theorem (see Hale and Lopes [20], Zhao [53]) with the help of the Kuratowski measure of noncompactness (see Deimling [12]). The method used here was also applied by Li et al. [25] to a scalar time-periodic nonlocal dispersal equation with stage structure. The nonexistence of periodic traveling waves (see Theorem 2.14) is then proved by constructing an auxiliary system and using comparison argument.

Although periodic traveling waves may determine the long time behavior of (1.1) with wave-like initial values, it is very interesting and important to understand how solutions of (1.1) starting from compactly supported initial conditions evolve as time increases. Recently, there are some results on the spreading properties of nonmonotone systems which can be sandwiched by two auxiliary monotone systems, see for example Fang and Zhao [16], Hsu and Zhao [22], Li et al. [25], Wang [41], Weinberger et al. [42], and Wu et al. [44]. However, the techniques used in the above references cannot be applied to predator-prey and epidemic systems. More recently, Ducrot [13,14] and Ducrot et al. [15] considered the spreading properties of solutions for some autonomous predator-prey and epidemic systems with local diffusion. To the best of our knowledge, there has been no results on the spreading properties for time-periodic predator-prey systems with nonlocal dispersal. We would like to emphasize that the main difficulties encountered when studying (1.1) are the lack of compactness of nonlocal dispersal operators, the lack of comparison principle of the predator-prey system, and the presence of the time dependence of nonlinearity.

More precisely, in this paper, we establish the spreading properties for solutions of system (1.1) with compactly supported initial conditions. Roughly speaking, we show that if predators disperse faster than their prey, then both species spread simultaneously (see Theorem 3.3); whereas if the prey diffuses faster than predators, then there exist two separate invasion fronts, one front occurs as the prey invades open habitats, and the other front appears when predators catch up the prey (see Theorem 3.2). Although some of the proofs are inspired by Ducrot [15] for the autonomous predator-prey system with local diffusion, there are certain new ideas in this paper which are different from those in [15]. Firstly, to prove that the prey is always able to spread outside of predators' range in the case where the prey is faster than predators (i.e., Theorem 3.2 (ii)), we need to establish *a priori* estimate for solutions to the nonlocal system (1.1) (see the proof of Lemma 3.8). Secondly, due to the occurrence of time-periodicity and nonlocal dispersal, the technique in Ducrot [15] cannot be used to study the lower estimates on the spreading speed. In this paper, we will generalize the persistence theory in dynamical systems to the initial value problem of (1.1). We

mention that the multi-front propagation phenomena have been studied intensively for bistable reaction-diffusion systems and refer to Carrère [8], Fife and McLeod [18], Liu et al. [27], Poláčik [33], Zhang and Zhao [45], and the references therein.

The rest of this paper is organized as follows. In section 2, we prove the existence and nonexistence of periodic traveling waves of (1.1). Section 3 is devoted to the study of the spreading properties of (1.1) to the initial value problem.

2. Periodic traveling waves

In this section, we consider the existence and nonexistence of periodic traveling waves for system (1.1). We always assume that (H_1) and the following condition are satisfied:

$$(H_2) \quad \overline{a_2 p - r_2} > 0.$$

We first give some notations.

Notation 2.1. (i) Set $f_M := \max_{t \in [0, T]} f(t)$ and $f_m := \min_{t \in [0, T]} f(t)$ for a given function $f \in C[0, T]$ and $\mathcal{J}_i^\nu := \int_{\mathbb{R}} J_i(x) e^{\nu|x|} dx (i = 1, 2)$ for any given $\nu \in \mathbb{R}$.

(ii) Denote $\underline{d} := \min\{d_1, d_2\}$, $\bar{d} := \max\{d_1, d_2\}$, and $\bar{\mathcal{J}}^\nu := \max\{\mathcal{J}_1^\nu, \mathcal{J}_2^\nu\}$.

(iii) Let $\theta > 0$ be any given constant. Define

$$X_\theta := \left\{ \Upsilon = (\Upsilon_1, \Upsilon_2) \in C(\mathbb{R}, \mathbb{R}^2) : \sup_{x \in \mathbb{R}} |\Upsilon_i(x)| e^{-\theta|x|} < +\infty, i = 1, 2 \right\},$$

$$B_\theta^T := \left\{ w = (w_1, w_2) \in C([0, T] \times \mathbb{R}, \mathbb{R}^2) : \begin{array}{l} w_i(0, x) = w_i(T, x), \quad \forall x \in \mathbb{R}, i = 1, 2 \\ \sup_{t \in [0, T], x \in \mathbb{R}} |w_i(t, x)| e^{-\theta|x|} < +\infty \end{array} \right\}$$

equipped, respectively, with the norms

$$\|\Upsilon\|_\theta := \max\{\|\Upsilon_1\|_\theta, \|\Upsilon_2\|_\theta\} = \max\left\{ \sup_{x \in \mathbb{R}} |\Upsilon_1(x)| e^{-\theta|x|}, \sup_{x \in \mathbb{R}} |\Upsilon_2(x)| e^{-\theta|x|} \right\},$$

$$\|w\|_\theta^T := \max\{\|w_1\|_\theta^T, \|w_2\|_\theta^T\} = \max\left\{ \sup_{t \in [0, T], x \in \mathbb{R}} |w_1(t, x)| e^{-\theta|x|}, \sup_{t \in [0, T], x \in \mathbb{R}} |w_2(t, x)| e^{-\theta|x|} \right\}.$$

As usual, a solution $(u_1(t, x), u_2(t, x))$ of (1.1) is called a T -periodic traveling wave solution if $(u_1(t, x), u_2(t, x)) = (U_1(t, \xi), U_2(t, \xi))$, $\xi = x + ct$, and $(U_1(t, \xi), U_2(t, \xi)) = (U_1(t + T, \xi), U_2(t + T, \xi))$, $\forall \xi, t \in \mathbb{R}$. It is clear that the wave profile function $(U_1(t, \xi), U_2(t, \xi))$ satisfies

$$\begin{cases} (U_1)_t + c(U_1)_\xi = d_1(J_1 * U_1 - U_1) + U_1(r_1(t) - b_1(t)U_1 - a_1(t)U_2), \\ (U_2)_t + c(U_2)_\xi = d_2(J_2 * U_2 - U_2) + U_2(-r_2(t) - b_2(t)U_2 + a_2(t)U_1), \end{cases} \quad (2.1)$$

where $J_i * U_i(t, \xi) = \int_{\mathbb{R}} J_i(\xi - y) U_i(t, y) dy$, $i = 1, 2$.

In this section, we prove the existence of periodic traveling waves by applying the asymptotic fixed point theorem combined with the Kuratowski measure of noncompactness. In the following, we state some known definitions and lemmas.

Definition 2.2 (Zhao [53]). Let E be a Banach space.

(i) The *Kuratowski measure of noncompactness* in E is defined by

$$\alpha(B) = \inf\{r : B \text{ has a finite open cover of diameter } \leq r\}$$

for any bounded set B . Obviously, $\alpha(B) = 0$ if and only if $\overline{B}$ is compact. Moreover, $\alpha(\tilde{B}_1 + \tilde{B}_2) \leq \alpha(\tilde{B}_1) + \alpha(\tilde{B}_2)$ for any bounded sets $\tilde{B}_1$ and $\tilde{B}_2$.

(ii) A continuous mapping $f : E \rightarrow E$ is said to be α -condensing if it is bounded and $\alpha(f(B)) < \alpha(B)$ for any nonempty closed bounded set $B \subset E$ with $\alpha(B) > 0$; and it is *compact dissipative* if there is a bounded set B_0 in E such that B_0 attracts each compact set in E . Clearly, a compact map is α -condensing.

Lemma 2.3 (*Asymptotic fixed point theorem (Nussbaum [31])*). *Let E be a Banach space. If $f : E \rightarrow E$ is α -condensing and compact dissipative, then f has a fixed point.*

To prove that an operator is α -condensing (see Lemma 2.11), we need the following properties of the Kuratowski measure of noncompactness, see Deimling [12, Section 7.4] and Banaś[3, Lemma 5].

Lemma 2.4. *Let E be a Banach space, $I \subseteq \mathbb{R}^n$ compact and $\Gamma \subseteq C(I, E)$ bounded. For each $t \in I$, define the slice $\Gamma(t) := \{b(t) : b \in \Gamma\}$. Then*

- (i) $\alpha(\Gamma) \geq \sup_{t \in I} \alpha(\Gamma(t))$. Moreover, if Γ is equicontinuous in the sense that for any $\epsilon > 0$ there exists a $\delta = \delta(\epsilon)$ such that $\sup\{|b(t_1) - b(t_2)| : b \in \Gamma\} < \epsilon$ whenever $|t_1 - t_2| < \delta$, then $\alpha(\Gamma) = \sup_{t \in I} \alpha(\Gamma(t))$;
- (ii) If Γ is equicontinuous, then $\alpha(\int_0^t \Gamma(s) ds) \leq \int_0^t \alpha(\Gamma(s)) ds$, where $\int_0^t \Gamma(s) ds := \{\int_0^t b(s) ds : b \in \Gamma\}$.

2.1. Construction of a linear evolution system

Set

$$\Delta(\lambda, c) := d_2\left(\int_{\mathbb{R}} J_2(y) e^{-\lambda y} dy - 1\right) - c\lambda + \overline{a_2 p - r_2}, \quad \lambda, c > 0.$$

One can easily verify that there exists $c_* > 0$ such that the equation $\Delta(\lambda, c) = 0$ has two positive roots $\lambda_1 := \lambda_1(c) < \lambda_2 := \lambda_2(c)$ for $c > c_*$. Moreover, $\Delta(\lambda, c) < 0$ for $\lambda \in (\lambda_1, \lambda_2)$. Recall that $q(t)$ is the unique positive T -periodic solution of equation (1.4).

Given any $c > c_*$, let $\beta \in (0, \lambda_1)$ be small enough such that $d_1\left(\int_{\mathbb{R}} J_1(y) e^{-\beta y} dy - 1\right) - c\beta < 0$. Take $\kappa \in (0, \min\{\lambda_1, \frac{\lambda_2 - \lambda_1}{2}, \beta\})$, $\varphi(t) := \exp\left\{\int_0^t (a_2(s)p(s) - r_2(s) - \overline{a_2 p - r_2}) ds\right\}$, and $R_1 > 0$ such that

$$R_1 > \max\left\{1, \max_{t \in [0, T]} \left(\frac{\varphi(t)}{q(t)}\right)^{\frac{\beta}{\lambda_1}}, \max_{t \in [0, T]} \left(\frac{a_1(t)\varphi(t)}{p(t)b_1(t)}\right)\right\}.$$

Obviously, $\Delta(\lambda_1 + \kappa, c) < 0$. Then, choose a positive number R_2 such that

$$R_2 > \max\left\{1, R_1^{\frac{\kappa}{\beta}}, -\frac{a_{2M}p_MR_1 + b_{2M}\varphi_M}{\Delta(\lambda_1 + \kappa, c)}\right\}.$$

Based on the above choice of the numbers β, κ, R_1, R_2 , we now define two functions $\overline{U}(t, \xi) = (\overline{U}_1(t, \xi), \overline{U}_2(t, \xi))$ and $\underline{U}(t, \xi) = (\underline{U}_1(t, \xi), \underline{U}_2(t, \xi))$ as follows

$$\overline{U}_1(t, \xi) = p(t), \quad \underline{U}_1(t, \xi) = \max\{0, p(t)(1 - R_1 e^{\beta \xi})\},$$

$$\overline{U}_2(t, \xi) = \min\{e^{\lambda_1 \xi} \varphi(t), q(t)\}, \quad \underline{U}_2(t, \xi) = \max\{0, e^{\lambda_1 \xi} \varphi(t)(1 - R_2 e^{\kappa \xi})\}.$$

Lemma 2.5. *The function $\overline{U}_2(t, \xi)$ satisfies*

$$(\overline{U}_2)_t + c(\overline{U}_2)_\xi - d_2[J_2 * \overline{U}_2 - \overline{U}_2] - \overline{U}_2[a_2(t)p(t) - r_2(t) - b_2(t)\overline{U}_2] \geq 0, \quad \forall \xi \neq \frac{1}{\lambda_1} \ln \frac{q(t)}{\varphi(t)}.$$

Proof. If $\xi < \frac{1}{\lambda_1} \ln \frac{q(t)}{\varphi(t)}$, then $\overline{U}_2(t, \xi) = e^{\lambda_1 \xi} \varphi(t)$. By some calculations, we have

$$\begin{aligned} & (\overline{U}_2)_t + c(\overline{U}_2)_\xi - d_2[J_2 * \overline{U}_2 - \overline{U}_2] - \overline{U}_2[a_2(t)p(t) - r_2(t) - b_2(t)\overline{U}_2] \\ &= \varphi(t)[a_2(t)p(t) - r_2(t) - \overline{a_2 p - r_2}]e^{\lambda_1 \xi} + c\lambda_1 \varphi(t)e^{\lambda_1 \xi} \\ & \quad - d_2 \varphi(t)e^{\lambda_1 \xi} \left(\int_{\mathbb{R}} J_2(y)e^{-\lambda_1 y} dy - 1 \right) - \varphi(t)e^{\lambda_1 \xi} [a_2(t)p(t) - r_2(t) - b_2(t)\overline{U}_2] \\ & \geq \varphi(t)e^{\lambda_1 \xi} \left[-\overline{a_2 p - r_2} + c\lambda_1 - d_2 \left(\int_{\mathbb{R}} J_2(y)e^{-\lambda_1 y} dy - 1 \right) \right] = 0. \end{aligned}$$

If $\xi > \frac{1}{\lambda_1} \ln \frac{q(t)}{\varphi(t)}$, then $\overline{U}_2(t, \xi) = q(t)$. Thus

$$\begin{aligned} & (\overline{U}_2)_t + c(\overline{U}_2)_\xi - d_2[J_2 * \overline{U}_2 - \overline{U}_2] - \overline{U}_2[a_2(t)p(t) - r_2(t) - b_2(t)\overline{U}_2] \\ &= q'(t) - q(t)(a_2(t)p(t) - r_2(t) - b_2(t)q(t)) = 0. \end{aligned}$$

This completes the proof. $\square$

Lemma 2.6. *The function $\underline{U}_1(t, \xi)$ satisfies*

$$(\underline{U}_1)_t + c(\underline{U}_1)_\xi - d_1[J_1 * \underline{U}_1 - \underline{U}_1] - \underline{U}_1[r_1(t) - b_1(t)\underline{U}_1 - a_1(t)\overline{U}_2] \leq 0, \quad \forall \xi \neq \frac{1}{\beta} \ln \frac{1}{R_1}.$$

Proof. Clearly, $\frac{1}{\beta} \ln \frac{1}{R_1} < 0$. If $\xi > \frac{1}{\beta} \ln \frac{1}{R_1}$, then $\underline{U}_1(t, \xi) = 0$. Thus

$$(\underline{U}_1)_t + c(\underline{U}_1)_\xi - d_1[J_1 * \underline{U}_1 - \underline{U}_1] - \underline{U}_1[r_1(t) - b_1(t)\underline{U}_1 - a_1(t)\overline{U}_2] = 0.$$

If $\xi < \frac{1}{\beta} \ln \frac{1}{R_1}$, then $\underline{U}_1(t, \xi) = p(t)(1 - R_1 e^{\beta \xi})$. By the choice of R_1 , $\frac{1}{\beta} \ln \frac{1}{R_1} < \frac{1}{\lambda_1} \ln \frac{q(t)}{\varphi(t)}$, and hence, $\overline{U}_2(t, \xi) = e^{\lambda_1 \xi} \varphi(t)$. Noting that $p'(t) = p(t)[r_1(t) - b_1(t)p(t)]$, thus

$$\begin{aligned} & (\underline{U}_1)_t + c(\underline{U}_1)_\xi - d_1[J_1 * \underline{U}_1 - \underline{U}_1] - \underline{U}_1[r_1(t) - b_1(t)\underline{U}_1 - a_1(t)\overline{U}_2] \\ &= p'(t)(1 - R_1 e^{\beta \xi}) - c\beta R_1 p(t)e^{\beta \xi} + d_1 R_1 e^{\beta \xi} p(t) \left(\int_{\mathbb{R}} J_1(y)e^{-\beta y} dy - 1 \right) \\ & \quad - p(t)(1 - R_1 e^{\beta \xi})[r_1(t) - b_1(t)\underline{U}_1 - a_1(t)\overline{U}_2] \\ &= p'(t) - p(t)[r_1(t) - b_1(t)p(t) + b_1(t)p(t)R_1 e^{\beta \xi}] + p(t)a_1(t)\overline{U}_2 \\ & \quad - p'(t)R_1 e^{\beta \xi} - p(t)R_1 e^{\beta \xi} \left(c\beta - d_1 \left(\int_{\mathbb{R}} J_1(y)e^{-\beta y} dy - 1 \right) \right) \\ & \quad + p(t)R_1 e^{\beta \xi} [r_1(t) - p(t)b_1(t) + b_1(t)p(t)R_1 e^{\beta \xi} - a_1(t)\overline{U}_2] \\ & \leq [-p(t)b_1(t)R_1 e^{\beta \xi} + a_1(t)\overline{U}_2]p(t) + R_1 p(t)e^{\beta \xi} [R_1 b_1(t)p(t)e^{\beta \xi} - a_1(t)\overline{U}_2] \end{aligned}$$

$$\begin{aligned}
&= p(t)(R_1 e^{\beta\xi} - 1)[R_1 p(t)b_1(t)e^{\beta\xi} - a_1(t)\varphi(t)e^{\lambda_1\xi}] \\
&\leq (R_1 e^{\beta\xi} - 1)[p(t)b_1(t)R_1 - a_1(t)\varphi(t)]p(t)e^{\lambda_1\xi} \leq 0.
\end{aligned}$$

This completes the proof. $\square$

Lemma 2.7. *The function $\underline{U}_2(t, \xi)$ satisfies*

$$(\underline{U}_2)_t + c(\underline{U}_2)_\xi - d_2(J_2 * \underline{U}_2 - \underline{U}_2) - \underline{U}_2[a_2(t)\underline{U}_1 - r_2(t) - b_2(t)\underline{U}_2] \leq 0, \quad \forall \xi \neq \frac{1}{\kappa} \ln \frac{1}{R_2}.$$

Proof. Obviously, $\frac{1}{\kappa} \ln \frac{1}{R_2} < 0$. If $\xi > \frac{1}{\kappa} \ln \frac{1}{R_2}$, then $\underline{U}_2(t, \xi) = 0$. Thus

$$(\underline{U}_2)_t + c(\underline{U}_2)_\xi - d_2(J_2 * \underline{U}_2 - \underline{U}_2) - \underline{U}_2(a_2(t)\underline{U}_1 - r_2(t) - b_2(t)\underline{U}_2) = 0.$$

If $\xi < \frac{1}{\kappa} \ln \frac{1}{R_2}$, then $\underline{U}_2(t, \xi) = \varphi(t)e^{\lambda_1\xi}(1 - R_2 e^{\kappa\xi})$. Since $R_2 > R_1^{\frac{\kappa}{\beta}}$, we have $\frac{1}{\kappa} \ln \frac{1}{R_2} < \frac{1}{\beta} \ln \frac{1}{R_1}$, and hence $\underline{U}_1(t, \xi) = p(t)(1 - R_1 e^{\beta\xi})$. Noting that $0 < \kappa < \min\{\beta, \lambda_1\}$, direct computations show that

$$\begin{aligned}
&(\underline{U}_2)_t + c(\underline{U}_2)_\xi - d_2[J_2 * \underline{U}_2 - \underline{U}_2] - \underline{U}_2[a_2(t)\underline{U}_1 - r_2(t) - b_2(t)\underline{U}_2] \\
&= \varphi(t)e^{\lambda_1\xi} \left[a_2(t)p(t) - r_2(t) - \overline{a_2 p - r_2} + c\lambda_1 - d_2 \left(\int_{\mathbb{R}} J_2(y)e^{-\lambda_1 y} dy - 1 \right) \right. \\
&\quad \left. - a_2(t)p(t)(1 - R_1 e^{\beta\xi}) + r_2(t) + b_2(t)\varphi(t)e^{\lambda_1\xi} - R_2 b_2(t)\varphi(t)e^{(\lambda_1 + \kappa)\xi} \right] \\
&\quad - R_2 \varphi(t)e^{(\lambda_1 + \kappa)\xi} \left[a_2(t)p(t) - r_2(t) - \overline{a_2 p - r_2} + c(\lambda_1 + \kappa) - R_2 b_2(t)\varphi(t)e^{(\lambda_1 + \kappa)\xi} \right. \\
&\quad \left. - d_2 \left(\int_{\mathbb{R}} J_2(y)e^{-(\lambda_1 + \kappa)y} dy - 1 \right) - a_2(t)p(t)(1 - R_1 e^{\beta\xi}) + r_2(t) + b_2(t)\varphi(t)e^{\lambda_1\xi} \right] \\
&= \varphi(t)e^{\lambda_1\xi} [a_2(t)p(t)R_1 e^{\beta\xi} + b_2(t)\varphi(t)e^{\lambda_1\xi} - R_2 b_2(t)\varphi(t)e^{(\lambda_1 + \kappa)\xi}] - R_2 \varphi(t)e^{(\lambda_1 + \kappa)\xi} \\
&\quad \times [-\Delta(\lambda_1 + \kappa, c) + a_2(t)p(t)R_1 e^{\beta\xi} + b_2(t)\varphi(t)e^{\lambda_1\xi} - R_2 b_2(t)\varphi(t)e^{(\lambda_1 + \kappa)\xi}] \\
&= [a_2(t)p(t)R_1 e^{\beta\xi} + b_2(t)\varphi(t)e^{\lambda_1\xi} - R_2 b_2(t)\varphi(t)e^{(\lambda_1 + \kappa)\xi}] \varphi(t)e^{\lambda_1\xi}(1 - R_2 e^{\kappa\xi}) \\
&\quad + R_2 \varphi(t)e^{(\lambda_1 + \kappa)\xi} \Delta(\lambda_1 + \kappa, c) \\
&\leq [a_2(t)p(t)R_1 e^{\beta\xi} + b_2(t)\varphi(t)e^{\lambda_1\xi}] \varphi(t)e^{\lambda_1\xi} + R_2 \varphi(t)e^{(\lambda_1 + \kappa)\xi} \Delta(\lambda_1 + \kappa, c) \\
&= \varphi(t)e^{(\lambda_1 + \kappa)\xi} [a_2(t)p(t)R_1 e^{(\beta - \kappa)\xi} + b_2(t)\varphi(t)e^{(\lambda_1 - \kappa)\xi} + R_2 \Delta(\lambda_1 + \kappa, c)] \\
&\leq \varphi(t)e^{(\lambda_1 + \kappa)\xi} [a_{2M} p_M R_1 + b_{2M} \varphi_M + R_2 \Delta(\lambda_1 + \kappa, c)] \leq 0.
\end{aligned}$$

This completes the proof. $\square$

Take $\alpha_1 = 2b_{1M}p_M + q_M a_{1M}$ and $\alpha_2 = r_{2M} + 2q_M b_{2M}$, and define

$$\Gamma_T = \{V \in B_\theta^T : \underline{U}(t, \xi) \leq V(t, \xi) \leq \overline{U}(t, \xi)\}.$$

It is clear that Γ_T is bounded in $C([0, T] \times \mathbb{R}, \mathbb{R}^2)$ with respect to the norm $\|\cdot\|_\theta^T$. For any $V = (V_1, V_2) \in \Gamma_T$, we define $\mathcal{N}(V) := (\mathcal{N}_1(V), \mathcal{N}_2(V))$ by

$$\begin{aligned}
\mathcal{N}_1(V)(t, \xi) &= \alpha_1 V_1(t, \xi) + V_1(t, \xi)[r_1(t) - b_1(t)V_1(t, \xi) - a_1(t)V_2(t, \xi)], \\
\mathcal{N}_2(V)(t, \xi) &= \alpha_2 V_2(t, \xi) + V_2(t, \xi)[-r_2(t) - b_2(t)V_2(t, \xi) + a_2(t)V_1(t, \xi)].
\end{aligned}$$

For given $V = (V_1, V_2) \in \Gamma_T$, consider the following linear evolution system

$$\begin{cases} (U_i)_t = d_i(J_i * U_i - U_i) - c(U_i)_\xi - \alpha_i U_i + \mathcal{N}_i(V), & t \in (0, T], \xi \in \mathbb{R}, i = 1, 2, \\ (U_1(0, \xi), U_2(0, \xi)) = (U_1^0(\xi), U_2^0(\xi)) \in X_\theta, & \xi \in \mathbb{R}. \end{cases} \quad (2.2)$$

2.2. Properties of solution maps of the linear system

In this subsection, we consider the properties of solution maps to a linear system related to (2.2). Let $\mathcal{T}(t) := (\mathcal{T}_1(t), \mathcal{T}_2(t)) : X_\theta \rightarrow X_\theta$ be the solution map associated with the linear system

$$\begin{cases} (U_i)_t = \mathcal{L}_i U_i := d_i(J_i * U_i - U_i) - c(U_i)_\xi - \alpha_i U_i, & t > 0, \xi \in \mathbb{R}, i = 1, 2, \\ (U_1(0, \xi), U_2(0, \xi)) = (U_1^0(\xi), U_2^0(\xi)) \in X_\theta, & \xi \in \mathbb{R}, \end{cases} \quad (2.3)$$

where the domain of the linear operator $\mathcal{L} := (\mathcal{L}_1, \mathcal{L}_2)$ is $D(\mathcal{L}) = \{U = (U_1, U_2) \in X_\theta : U_\xi \in X_\theta\}$. According to Bates and Chen [5, Lemma 2.1], for each $t > 0$, $\mathcal{T}(t)$ is a positive operator on X_θ . We further denote $\mathcal{G}(t) := (G_1(t), G_2(t))$ by the solution map associated to

$$\begin{cases} (u_i)_t = d_i(J_i * u_i - u_i) - \alpha_i u_i, & t > 0, \xi \in \mathbb{R}, i = 1, 2, \\ (u_1(0, \xi), u_2(0, \xi)) = \psi(\xi) := (\psi_1(\xi), \psi_2(\xi)) \in X_\theta, & \xi \in \mathbb{R}. \end{cases} \quad (2.4)$$

Obviously, $\mathcal{G}(t)[\psi] = (G_1(t)[\psi_1], G_2(t)[\psi_2])$ for any $\psi = (\psi_1, \psi_2) \in X_\theta$. Define $a_0(\psi_i)(\xi) = \psi_i(\xi)$ and $a_k(\psi_i)(\xi) = \int_{\mathbb{R}} J_i(\xi - y) a_{k-1}(\psi_i)(y) dy$ for any integer $k \geq 1$. It follows from Weng and Zhao [43] that

$$G_i(t)[\psi_i](\xi) = e^{-\alpha_i t} P_i(t)[\psi_i](\xi), \quad \forall t \geq 0, \xi \in \mathbb{R}, (\psi_1, \psi_2) \in X_\theta, \quad (2.5)$$

where $(P_1(t), P_2(t)) =: \mathcal{P}(t)$ is defined by $P_i(t)[\psi_i](\xi) = e^{-d_i t} \sum_{k=0}^{\infty} \frac{(d_i t)^k}{k!} a_k(\psi_i)(\xi)$. Moreover, we can see that $\mathcal{T}(t)[\psi] = (\mathcal{T}_1(t)[\psi_1], \mathcal{T}_2(t)[\psi_2])$ for any $\psi = (\psi_1, \psi_2) \in X_\theta$, where

$$\mathcal{T}_i(t)[\psi_i](\xi) = e^{-(d_i + \alpha_i)t} \sum_{k=0}^{\infty} \frac{(d_i t)^k}{k!} a_k(\psi_i)(\xi - ct), \quad \forall t > 0, \xi \in \mathbb{R}, i = 1, 2. \quad (2.6)$$

Then, we have the following result.

Lemma 2.8. *There exists a $\theta_0 > 0$ such that for any $\theta \in (0, \theta_0]$, $\|\mathcal{T}(T)\|_\theta < 1$.*

Proof. For any $\psi = (\psi_1, \psi_2) \in X_\theta$, we get $\|a_0(\psi)\|_\theta = \|\psi\|_\theta$ and

$$|a_k(\psi_i)(\xi)| e^{-\theta|\xi|} \leq \int_{\mathbb{R}} J_i(y) |a_{k-1}(\psi_i)(\xi - y)| dy e^{-\theta|\xi|} \leq \mathcal{J}_i^\theta \|a_{k-1}(\psi_i)\|_\theta$$

for $\xi \in \mathbb{R}$, $k \geq 1$. By induction, we obtain $\|a_k(\psi_i)\|_\theta \leq \mathcal{J}_i^\theta \|a_{k-1}(\psi_i)\|_\theta \leq \cdots \leq (\mathcal{J}_i^\theta)^k \|\psi_i\|_\theta \leq (\mathcal{J}_i^\theta)^k \|\psi\|_\theta$. It then follows that

$$\begin{aligned} |\mathcal{T}_i(t)[\psi_i](\xi)| e^{-\theta|\xi|} &\leq e^{-(d_i + \alpha_i)t} \sum_{k=0}^{\infty} \frac{(d_i t)^k}{k!} |a_k(\psi_i)(\xi - ct)| e^{-\theta|\xi|} \\ &\leq e^{(\theta c - d_i - \alpha_i)t} \sum_{k=0}^{\infty} \frac{(d_i t)^k}{k!} |a_k(\psi_i)(\xi - ct)| e^{-\theta|\xi - ct|} \end{aligned}$$

$$\leq e^{(\theta c - d_i - \alpha_i)t} \sum_{k=0}^{\infty} \frac{(\mathcal{J}_i^\theta d_i t)^k}{k!} \|\psi\|_\theta = e^{(\theta c - d_i - \alpha_i + \mathcal{J}_i^\theta d_i)t} \|\psi\|_\theta,$$

where $i = 1, 2$, $\xi \in \mathbb{R}$ and $t > 0$. Thus, for any $t > 0$, we have

$$\|\mathcal{T}(t)[\psi]\|_\theta \leq \max\{e^{(\theta c - d_1 - \alpha_1 + \mathcal{J}_1^\theta d_1)t}, e^{(\theta c - d_2 - \alpha_2 + \mathcal{J}_2^\theta d_2)t}\} \|\psi\|_\theta. \quad (2.7)$$

In particular,

$$\|\mathcal{T}(T)[\psi]\|_\theta \leq \max\{e^{(\theta c - d_1 - \alpha_1 + \mathcal{J}_1^\theta d_1)T}, e^{(\theta c - d_2 - \alpha_2 + \mathcal{J}_2^\theta d_2)T}\} \|\psi\|_\theta. \quad (2.8)$$

Since $\lim_{\theta \rightarrow 0^+} e^{(\theta c - d_i - \alpha_i + \mathcal{J}_i^\theta d_i)T} = e^{-\alpha_i T} < 1$, $i = 1, 2$, we deduce that there exists a $\theta_0 > 0$ such that for any $\theta \in (0, \theta_0]$,

$$\|\mathcal{T}(T)\|_\theta = \sup_{\|\psi\|_\theta=1} \|\mathcal{T}(t)[\psi]\|_\theta \leq \max\{e^{(\theta c - d_1 - \alpha_1 + \mathcal{J}_1^\theta d_1)T}, e^{(\theta c - d_2 - \alpha_2 + \mathcal{J}_2^\theta d_2)T}\} < 1.$$

This completes the proof. $\square$

Denote $\underline{\alpha} := \min\{\alpha_1, \alpha_2\}$. Given any $\theta > 0$ and $L_i^+ > 0$ ($i = 1, 2$), define

$$\mathcal{Y}_{L^+} := \{\psi \in X_\theta : (0, 0) \leq \psi(x) \leq L^+ := (L_1^+, L_2^+), \forall x \in \mathbb{R}\}.$$

Then $\mathcal{Y}_{L^+}$ is a nonempty, closed, and convex subset of X_θ . We further prove that for each $t > 0$, $\mathcal{T}(t)$ is an α -contraction on $\mathcal{Y}_{L^+}$ for small $\theta > 0$. More precisely, we have the following result.

Lemma 2.9. *For any given $\theta > 0$, $\alpha(\mathcal{T}(t)[\mathcal{Y}_{L^+}]) \leq e^{(\theta c - \underline{d} - \underline{\alpha})t} \alpha(\mathcal{Y}_{L^+})$ for any $t > 0$.*

Proof. Note that for any $\psi = (\psi_1, \psi_2) \in \mathcal{Y}_{L^+}$, we can conclude that $0 \leq a_k(\psi_i)(x) \leq L_i^+$ for any $k \geq 1$ and $x \in \mathbb{R}$, $i = 1, 2$. Then by (2.6), we can obtain that $\mathcal{T}(t)[\psi] \in X_\theta$ with $0 \leq \mathcal{T}(t)[\psi](\cdot) \leq L^+$; i.e., $\mathcal{T}(t)$ maps $\mathcal{Y}_{L^+}$ into itself. By (2.7), for any $\psi, \phi \in \mathcal{Y}_{L^+}$, we have

$$\|\mathcal{T}(t)[\psi] - \mathcal{T}(t)[\phi]\|_\theta \leq e^{(\theta c - \underline{d} - \underline{\alpha} + \bar{\mathcal{J}}^\theta \bar{d})t} \|\psi - \phi\|_\theta. \quad (2.9)$$

Thus, $\mathcal{T}(t)$ is a continuous map on $\mathcal{Y}_{L^+}$. Set $\mathcal{T}(t) = \mathcal{T}^{(1)}(t) + \mathcal{T}^{(2)}(t) = (\mathcal{T}_1^{(1)}(t), \mathcal{T}_2^{(1)}(t)) + (\mathcal{T}_1^{(2)}(t), \mathcal{T}_2^{(2)}(t))$, where

$$\mathcal{T}_i^{(1)}(t)[\psi_i](x) = e^{-(d_i + \alpha_i)t} \psi_i(x - ct), \quad \mathcal{T}_i^{(2)}(t)[\psi_i](x) = e^{-(d_i + \alpha_i)t} \sum_{k=1}^{\infty} \frac{(d_i t)^k}{k!} a_k(\psi_i)(x - ct). \quad (2.10)$$

By an argument similar to (2.7), we can show that $\|\mathcal{T}^{(1)}(t)\psi - \mathcal{T}^{(1)}(t)\phi\|_\theta \leq e^{(\theta c - \underline{d} - \underline{\alpha})t} \|\psi - \phi\|_\theta$ holds. Thus, $\alpha(\mathcal{T}^{(1)}(t)[\mathcal{Y}_{L^+}]) \leq e^{(\theta c - \underline{d} - \underline{\alpha})t} \alpha(\mathcal{Y}_{L^+})$ for any $t > 0$.

Next, we show that $\mathcal{T}^{(2)}(t)$ is compact for each $t > 0$. For any $\psi = (\psi_1, \psi_2) \in \mathcal{Y}_{L^+}$ and $y_1, y_2 \in \mathbb{R}$, we have

$$\begin{aligned} & |\mathcal{T}_i^{(2)}(t)[\psi_i](y_1) - \mathcal{T}_i^{(2)}(t)[\psi_i](y_2)| \\ & \leq e^{-(d_i + \alpha_i)t} \sum_{k=1}^{\infty} \frac{(d_i t)^k}{k!} |a_k(\psi_i)(y_1 - ct) - a_k(\psi_i)(y_2 - ct)| \\ & = e^{-(d_i + \alpha_i)t} \sum_{k=1}^{\infty} \frac{(d_i t)^k}{k!} \left| \int_{\mathbb{R}} (J_i(y_1 - ct - y) - J_i(y_2 - ct - y)) a_{k-1}(\psi_i)(y) dy \right| \end{aligned}$$

$$\begin{aligned}
&\leq L_i^+ e^{-(d_i+\alpha_i)t} \sum_{k=1}^{\infty} \frac{(d_i t)^k}{k!} \int_{\mathbb{R}} |J_i(z+y_1-y_2) - J_i(z)| dz \\
&\leq L_i^+ e^{-(\underline{d}+\underline{\alpha})t} (e^{\bar{d}t} - 1) h_i(y_1 - y_2),
\end{aligned}$$

where $h_i(x) = \int_{\mathbb{R}} |J_i(z+x) - J_i(z)| dz$, $\forall x \in \mathbb{R}$. Since $\lim_{x \rightarrow 0} h_i(x) = 0$, it follows that the family of functions $\{\mathcal{T}_i^{(2)}(t)[\psi_i](x) : 0 \leq \psi \leq L^+\}$ ($i = 1, 2$) is equicontinuous in $x \in \mathbb{R}$.

Thus, for any given sequence $\{(\phi_i)_n := \mathcal{T}_i^{(2)}(t)[(\psi_i)_n]\}_{n \geq 1} \subset \mathcal{T}_i^{(2)}(t)[\mathcal{Y}_{L^+}]$, there exist $n_m \rightarrow \infty$ and $\varphi_i^* \in C(\mathbb{R}, \mathbb{R})$ such that $\lim_{m \rightarrow \infty} (\phi_i)_{n_m}(x) = \varphi_i^*(x)$ uniformly for x in any compact subset of $\mathbb{R}$. Note that $(\psi_i)_{n_m} \in \mathcal{Y}_{L^+}$, it follows from (2.10) that $0 \leq (\phi_i)_{n_m}(x) \leq L_i^+$, and hence, $0 \leq \varphi_i^*(x) \leq L_i^+$, $\forall x \in \mathbb{R}$. Obviously, $\lim_{x \rightarrow \pm\infty} L_i^+ e^{-\theta|x|} = 0$. Hence, for any $\varepsilon_1 > 0$, there exists $K_1 > 0$ such that

$$|(\phi_i)_{n_m}(x) - \varphi_i^*(x)| e^{-\theta|x|} \leq 2L_i^+ e^{-\theta|x|} < \varepsilon_1, \quad \forall |x| \geq K_1, \quad i = 1, 2.$$

Since $\lim_{m \rightarrow +\infty} (\phi_i)_{n_m}(x) = \varphi_i^*(x)$ uniformly for $x \in [-K_1, K_1]$, there exists an integer m^* such that

$$|(\phi_i)_{n_m}(x) - \varphi_i^*(x)| e^{-\theta|x|} < \varepsilon_1, \quad \forall x \in [-K_1, K_1], \quad m \geq m^*, \quad i = 1, 2.$$

Thus, we conclude that $\lim_{m \rightarrow +\infty} \|\phi_{n_m} - \varphi^*\|_{\theta} = 0$, which implies that $\mathcal{T}^{(2)}(t)[\mathcal{Y}_{L^+}]$ is precompact in X_{θ} . Hence,

$$\alpha(\mathcal{T}(t)[\mathcal{Y}_{L^+}]) \leq \alpha(\mathcal{T}^{(1)}(t)[\mathcal{Y}_{L^+}]) + \alpha(\mathcal{T}^{(2)}(t)[\mathcal{Y}_{L^+}]) \leq e^{(\theta c - \underline{d} - \underline{\alpha})t} \alpha(\mathcal{Y}_{L^+})$$

for any $t > 0$. The proof is completed. $\square$

2.3. Construction of a nonlinear operator

In this subsection, we first convert (2.2) into an integral system, then we construct a critical nonlinear operator $\mathcal{H} : \Gamma_T \rightarrow \Gamma_T$ whose fixed points generate periodic traveling waves of (1.1).

Obviously, we can rewrite (2.2) as the following integral system

$$U_i(t, \xi) = \mathcal{T}_i(t)[U_i^0](\xi) + \int_0^t \mathcal{T}_i(t-s)[\mathcal{N}_i(V)(s, \cdot)](\xi) ds, \quad i = 1, 2, \quad (2.11)$$

where $t \in [0, T]$, $\xi \in \mathbb{R}$. Then we have the following result.

Lemma 2.10. Assume that $V = (V_1, V_2) \in \Gamma_T$ and $U^0 = (U_1^0, U_2^0) \in X_{\theta}$ with $\underline{U}(0, \cdot) \leq U^0(\cdot) \leq \overline{U}(0, \cdot)$. Then the solution $U(t, \xi) = (U_1(t, \xi), U_2(t, \xi))$ of (2.11) satisfies

$$\underline{U}(t, \xi) \leq U(t, \xi) \leq \overline{U}(t, \xi) \quad \text{for } (t, \xi) \in [0, T] \times \mathbb{R}. \quad (2.12)$$

Proof. The proof of this lemma can be divided into the following steps.

Step 1. We show that

$$\underline{U}_1(t, \xi) \leq \mathcal{T}_1(t)[\underline{U}_1(0, \cdot)](\xi) + \int_0^t \mathcal{T}_1(t-s)[\mathcal{N}_1(\underline{U}_1, \overline{U}_2)(s, \cdot)](\xi) ds, \quad (2.13)$$

$$\underline{U}_2(t, \xi) \leq \mathcal{T}_2(t)[\underline{U}_2(0, \cdot)](\xi) + \int_0^t \mathcal{T}_2(t-s)[\mathcal{N}_2(\underline{U}_1, \underline{U}_2)(s, \cdot)](\xi) ds, \quad (2.14)$$

$$\overline{U}_2(t, \xi) \geq \mathcal{T}_2(t)[\overline{U}_2(0, \cdot)](\xi) + \int_0^t \mathcal{T}_2(t-s)[\mathcal{N}_2(\overline{U}_1, \overline{U}_2)(s, \cdot)](\xi) ds. \quad (2.15)$$

We only prove (2.13), since (2.14) and (2.15) can be proved similarly. Let $\underline{V}_1(t, \xi) := \underline{U}_1(t, \xi + ct)$ and $\overline{V}_2(t, \xi) := \overline{U}_2(t, \xi + ct)$ for any $(t, \xi) \in [0, T] \times \mathbb{R}$. To prove (2.13), it suffices to show that

$$\underline{V}_1(t, \xi) \leq G_1(t)[\underline{V}_1(0, \cdot)](\xi) + \int_0^t G_1(t-s)[\mathcal{N}_1(\underline{V}_1, \overline{V}_2)(s, \cdot)](\xi) ds. \quad (2.16)$$

For any $t \in [0, T]$, by Lemma 2.6, we have

$$(\underline{V}_1)_t \leq d_1[J_1 * \underline{V}_1 - \underline{V}_1] - \alpha_1 \underline{V}_1 + \mathcal{N}_1(\underline{V}_1, \overline{V}_2), \quad \forall \xi \neq \frac{1}{\beta} \ln \frac{1}{R_1} - ct, \quad t \in [0, T].$$

Recall that

$$G_1(t-s)[\underline{V}_1(s, \cdot)](\xi) = e^{-(\alpha_1+d_1)(t-s)} \sum_{k=0}^{\infty} \frac{(d_1(t-s))^k}{k!} a_k(\underline{V}_1)(s, \cdot)(\xi).$$

Define

$$E_1(t, \xi) := -(\underline{V}_1)_t + d_1(J_1 * \underline{V}_1 - \underline{V}_1) - \alpha_1 \underline{V}_1 + \mathcal{N}_1(\underline{V}_1, \overline{V}_2)(t, \xi), \quad \forall (t, \xi) \in [0, T] \times \mathbb{R}.$$

By a direct computation, it then follows that

$$\begin{aligned} & \frac{\partial}{\partial s} G_1(t-s)[\underline{V}_1(s, \cdot)](\xi) \\ &= (\alpha_1 + d_1) e^{-(\alpha_1+d_1)(t-s)} \sum_{k=0}^{\infty} \frac{(d_1(t-s))^k}{k!} a_k(\underline{V}_1)(s, \cdot)(\xi) \\ & \quad + e^{-(\alpha_1+d_1)(t-s)} \left[\frac{\partial}{\partial s} \underline{V}_1(s, \xi) + \frac{\partial}{\partial s} \left(\sum_{k=1}^{\infty} \frac{(d_1(t-s))^k}{k!} a_k(\underline{V}_1)(s, \cdot) \right) (\xi) \right] \\ &= (\alpha_1 + d_1) G_1(t-s)[\underline{V}_1(s, \cdot)](\xi) + e^{-(\alpha_1+d_1)(t-s)} \frac{\partial}{\partial s} \underline{V}_1(s, \xi) + e^{-(\alpha_1+d_1)(t-s)} \\ & \quad \times \left[-d_1 \sum_{k=1}^{\infty} \frac{(d_1(t-s))^{k-1}}{(k-1)!} a_k(\underline{V}_1)(s, \cdot)(\xi) + \sum_{k=1}^{\infty} \frac{(d_1(t-s))^k}{k!} \frac{\partial}{\partial s} a_k(\underline{V}_1)(s, \cdot)(\xi) \right] \\ &= (\alpha_1 + d_1) G_1(t-s)[\underline{V}_1(s, \cdot)](\xi) + G_1(t-s) \left[\frac{\partial}{\partial s} \underline{V}_1(s, \cdot) \right] (\xi) \\ & \quad - d_1 e^{-(\alpha_1+d_1)(t-s)} \sum_{k=1}^{\infty} \frac{(d_1(t-s))^{k-1}}{(k-1)!} [J_1 * a_{k-1}(\underline{V}_1)(s, \cdot)](\xi) \\ &= (\alpha_1 + d_1) G_1(t-s)[\underline{V}_1(s, \cdot)](\xi) + G_1(t-s) \left[-E_1(s, \cdot) + d_1(J_1 * \underline{V}_1 - \underline{V}_1)(s, \cdot) \right. \\ & \quad \left. - \alpha_1 \underline{V}_1(s, \cdot) + \mathcal{N}_1(\underline{V}_1, \overline{V}_2)(s, \cdot) \right] (\xi) - d_1 J_1 * G_1(t-s)[\underline{V}_1(s, \cdot)](\xi) \\ &= (\alpha_1 + d_1) G_1(t-s)[\underline{V}_1(s, \cdot)](\xi) - d_1 J_1 * G_1(t-s)[\underline{V}_1(s, \cdot)](\xi) \\ & \quad + d_1 G_1(t-s) [(J_1 * \underline{V}_1 - \underline{V}_1)(s, \cdot)](\xi) - \alpha_1 G_1(t-s)[\underline{V}_1(s, \cdot)](\xi) \\ & \quad + G_1(t-s) [\mathcal{N}_1(\underline{V}_1, \overline{V}_2)(s, \cdot) - E_1(s, \cdot)](\xi) \\ &= G_1(t-s) [\mathcal{N}_1(\underline{V}_1, \overline{V}_2)(s, \cdot) - E_1(s, \cdot)](\xi), \quad 0 \leq s \leq t \leq T. \end{aligned}$$

Integrating both sides of the above equality from 0 to t , we have

$$\underline{\mathcal{V}}_1(t, \xi) - G_1(t)[\underline{\mathcal{V}}_1(0, \cdot)](\xi) = \int_0^t G_1(t-s)[\mathcal{N}_1(\underline{\mathcal{V}}_1, \overline{\mathcal{V}}_2)(s, \cdot) - E_1(s, \cdot)](\xi) ds. \quad (2.17)$$

Further, since $E_1(t, \xi) \geq 0$, $\forall \xi \neq \frac{1}{\beta} \ln \frac{1}{R_1} - ct$, it is easy to verify that

$$\int_0^t G_1(t-s)[E_1(s, \cdot)](\xi) ds \geq 0 \text{ for } t \in [0, T], \forall \xi \in \mathbb{R}.$$

Thus, (2.16) follows from (2.17), and hence (2.13) holds.

Step 2. We show that (2.12) holds. Since $V = (V_1, V_2) \in \Gamma_T$, we have $\underline{U}(t, \xi) \leq V(t, \xi) \leq \overline{U}(t, \xi)$ for $(t, \xi) \in [0, T] \times \mathbb{R}$. By $\overline{U}_1(t, \xi) = p(t)$, it then follows that

$$\overline{U}_1(t, \xi) = \mathcal{T}_1(t)[\overline{U}_1^0](\xi) + \int_0^t \mathcal{T}_1(t-s)[\overline{U}_1(s, \cdot)(\alpha_1 + r_1(s) - b_1(s)\overline{U}_1(s, \cdot))](\xi) ds.$$

In view of $U_1^0 \leq \overline{U}_1^0$, $V_1 \leq \overline{U}_1$ and the positivity of $\mathcal{T}_1(\cdot)$, we have

$$\begin{aligned} & U_1(t, \xi) - \overline{U}_1(t, \xi) \\ &= \mathcal{T}_1(t)[U_1^0](\xi) + \int_0^t \mathcal{T}_1(t-s)[V_1(s, \cdot)(\alpha_1 + r_1(s) - b_1(s)V_1(s, \cdot) - a_1(s)V_2(s, \cdot))](\xi) ds \\ &\quad - \mathcal{T}_1(t)[\overline{U}_1^0](\xi) - \int_0^t \mathcal{T}_1(t-s)[\overline{U}_1(s, \cdot)(\alpha_1 + r_1(s) - b_1(s)\overline{U}_1(s, \cdot))](\xi) ds \\ &\leq \int_0^t \mathcal{T}_1(t-s)[(\alpha_1 + r_1(s) - b_1(s)(V_1(s, \cdot) + \overline{U}_1(s, \cdot)))(V_1(s, \cdot) - \overline{U}_1(s, \cdot))](\xi) ds \\ &\leq \int_0^t \mathcal{T}_1(t-s)[(\alpha_1 + r_{1m} - 2b_{1MPM})(V_1(s, \cdot) - \overline{U}_1(s, \cdot))](\xi) ds, \quad \forall t \in (0, T]. \end{aligned}$$

By $\alpha_1 = 2b_{1MPM} + q_M a_{1M} > 2b_{1MPM} - r_{1m}$, we have $U_1(t, \xi) \leq \overline{U}_1(t, \xi)$ for any $t \in [0, T]$, $\xi \in \mathbb{R}$.

Now, we show that $U_2(t, \xi) \leq \overline{U}_2(t, \xi)$, $\forall (t, \xi) \in [0, T] \times \mathbb{R}$. Noting that $\alpha_2 = r_{2M} + 2q_M b_{2M}$, it follows from (2.15) that

$$\begin{aligned} & \overline{U}_2(t, \xi) - U_2(t, \xi) \\ &\geq \mathcal{T}_2(t)[\overline{U}_2^0](\xi) + \int_0^t \mathcal{T}_2(t-s)[\alpha_2 \overline{U}_2(s, \cdot) \\ &\quad + \overline{U}_2(s, \cdot)(-r_2(s) - b_2(s)\overline{U}_2(s, \cdot) + a_2(s)\overline{U}_1(s, \cdot))](\xi) ds \\ &\quad - \mathcal{T}_2(t)[U_2^0](\xi) - \int_0^t \mathcal{T}_2(t-s)[\alpha_2 V_2(s, \cdot) \end{aligned}$$

$$\begin{aligned}
& + V_2(s, \cdot)(-r_2(s) - b_2(s)V_2(s, \cdot) + a_2(s)V_1(s, \cdot))](\xi)ds \\
& \geq \int_0^t \mathcal{T}_2(t-s)[(\alpha_2 - r_2(s) - b_2(s)(\overline{U}_2(s, \cdot) + V_2(s, \cdot)))(\overline{U}_2(s, \cdot) - V_2(s, \cdot))](\xi)ds \\
& \geq \int_0^t \mathcal{T}_2(t-s)[(\alpha_2 - r_{2M} - 2b_{2M}q_M)(\overline{U}_2(s, \cdot) - V_2(s, \cdot))](\xi)ds = 0.
\end{aligned}$$

Thus, $U_2(t, \xi) \leq \overline{U}_2(t, \xi)$, $\forall (t, \xi) \in [0, T] \times \mathbb{R}$.

Moreover, by (2.13) and $\alpha_1 = 2b_{1M}p_M + a_{1M}q_M$, one has

$$\begin{aligned}
& U_1(t, \xi) - \underline{U}_1(t, \xi) \\
& \geq \mathcal{T}_1(t)[U_1^0 - \underline{U}_1^0](\xi) + \int_0^t \mathcal{T}_1(t-s)[\alpha_1(V_1(s, \cdot) - \underline{U}_1(s, \cdot)) + V_1(s, \cdot)(r_1(s) \\
& \quad - b_1(s)V_1(s, \cdot) - a_1(s)V_2(s, \cdot)) - \underline{U}_1(s, \cdot)(r_1(s) - b_1(s)\underline{U}_1(s, \cdot) - a_1(s)\overline{U}_2(s, \cdot))](\xi)ds \\
& \geq \int_0^t \mathcal{T}_1(t-s)[(\alpha_1 + r_1(s) - b_1(s)(V_1(s, \cdot) + \underline{U}_1(s, \cdot)) - a_1(s)\overline{U}_2(s, \cdot)) \\
& \quad \times (V_1(s, \cdot) - \underline{U}_1(s, \cdot))](\xi)ds \\
& \geq \int_0^t \mathcal{T}_1(t-s)[(\alpha_1 - q_M a_{1M} - 2b_{1M}p_M)(V_1(s, \cdot) - \underline{U}_1(s, \cdot))](\xi)ds = 0,
\end{aligned}$$

which implies that $U_1(t, \xi) \geq \underline{U}_1(t, \xi)$, $\forall (t, \xi) \in [0, T] \times \mathbb{R}$.

Similarly, using (2.14) and $\alpha_2 = r_{2M} + 2q_M b_{2M}$, one can show that $U_2(t, \xi) \geq \underline{U}_2(t, \xi)$ for all $(t, \xi) \in [0, T] \times \mathbb{R}$. This completes the proof. $\square$

Given $V = (V_1, V_2) \in \Gamma_T$ and $U^0 = (U_1^0, U_2^0) \in X_\theta$ with $\underline{U}(0, \cdot) \leq U^0(\cdot) \leq \overline{U}(0, \cdot)$, it follows from (2.11) that

$$U_i(T, \xi) = \mathcal{T}_i(T)[U_i^0](\xi) + \int_0^T \mathcal{T}_i(T-s)[\mathcal{N}_i(V)(s, \cdot)](\xi)ds, \quad i = 1, 2. \quad (2.18)$$

From Lemma 2.8, $\|\mathcal{T}(T)\|_\theta < 1$ for any $\theta \in (0, \theta_0]$. Thus, if $(U_1(T, \cdot), U_2(T, \cdot)) = (U_1^0(\cdot), U_2^0(\cdot))$, there holds

$$U_i^0 = (I_i - \mathcal{T}_i(T))^{-1} \int_0^T \mathcal{T}_i(T-s)[\mathcal{N}_i(V)(s, \cdot)]ds = \sum_{k=0}^{\infty} (\mathcal{T}_i(T))^k \int_0^T \mathcal{T}_i(T-s)[\mathcal{N}_i(V)(s, \cdot)]ds.$$

For any given $V = (V_1, V_2) \in \Gamma_T$, let $U^*(t, \xi) = (U_1^*(t, \xi), U_2^*(t, \xi))$ be the solution of the following equation:

$$U_i^*(t, \xi) = \mathcal{T}_i(t) \sum_{k=0}^{\infty} (\mathcal{T}_i(T))^k \int_0^T \mathcal{T}_i(T-s)[\mathcal{N}_i(V)(s, \cdot)](\xi)ds + \int_0^t \mathcal{T}_i(t-s)[\mathcal{N}_i(V)(s, \cdot)](\xi)ds$$

for $(t, \xi) \in [0, T] \times \mathbb{R}$. Clearly, $U^*(T, \cdot) = U^*(0, \cdot)$. Moreover, it follows from Lemma 2.10 that $U^* \in \Gamma_T$.

We now define a nonlinear operator $\mathcal{H} := (\mathcal{H}_1, \mathcal{H}_2) : \Gamma_T \rightarrow \Gamma_T$ by $\mathcal{H}(V) = U^*$; that is,

$$\begin{aligned} \mathcal{H}_i(V)(t, \xi) &= \mathcal{T}_i(t) \sum_{k=0}^{\infty} (\mathcal{T}_i(T))^k \int_0^T \mathcal{T}_i(T-s) [\mathcal{N}_i(V)(s, \cdot)](\xi) ds \\ &\quad + \int_0^t \mathcal{T}_i(t-s) [\mathcal{N}_i(V)(s, \cdot)](\xi) ds, \quad (t, \xi) \in [0, T] \times \mathbb{R}, \quad i = 1, 2. \end{aligned}$$

Before proving that $\mathcal{H}$ is α -condensing, we state an additional assumption:

(H₃) $\underline{d} > \max\{\bar{C}_1, \bar{C}_2\}$, where $\bar{C}_1 := r_{1M} + 4b_{1M}p_M + 2a_{1M}q_M + a_{1M}p_M$ and $\bar{C}_2 := 2r_{2M} + a_{2M}(p_M + q_M) + 4q_Mb_{2M}$.

We remark that (H₃) holds for enough large d_1 and d_2 , since $\bar{C}_1$ and $\bar{C}_2$ are independent of d_1 and d_2 .

Lemma 2.11. Assume that (H₁)-(H₃) hold. Then $\mathcal{H} : \Gamma_T \rightarrow \Gamma_T$ is α -condensing.

Proof. For any $\bar{U} := (\bar{U}_1, \bar{U}_2), \bar{V} := (\bar{V}_1, \bar{V}_2) \in \Gamma_T$, by some calculations, we have

$$\begin{aligned} &|\mathcal{N}_1(\bar{U})(t, \xi) - \mathcal{N}_1(\bar{V})(t, \xi)|e^{-\theta|\xi|} \\ &= |\alpha_1(\bar{U}_1(t, \xi) - \bar{V}_1(t, \xi)) + r_1(t)(\bar{U}_1(t, \xi) - \bar{V}_1(t, \xi)) - b_1(t)(\bar{U}_1^2(t, \xi) - \bar{V}_1^2(t, \xi)) \\ &\quad - a_1(t)(\bar{U}_1(t, \xi)\bar{U}_2(t, \xi) - \bar{V}_1(t, \xi)\bar{V}_2(t, \xi))|e^{-\theta|\xi|} \\ &\leq \alpha_1|\bar{U}_1(t, \xi) - \bar{V}_1(t, \xi)|e^{-\theta|\xi|} + r_1(t)|\bar{U}_1(t, \xi) - \bar{V}_1(t, \xi)|e^{-\theta|\xi|} \\ &\quad + b_1(t)|(\bar{U}_1(t, \xi) - \bar{V}_1(t, \xi))(\bar{U}_1(t, \xi) + \bar{V}_1(t, \xi))|e^{-\theta|\xi|} \\ &\quad + a_1(t)|\bar{U}_1(t, \xi)(\bar{U}_2(t, \xi) - \bar{V}_2(t, \xi)) + \bar{V}_2(t, \xi)(\bar{U}_1(t, \xi) - \bar{V}_1(t, \xi))|e^{-\theta|\xi|} \\ &\leq \alpha_1\|\bar{U} - \bar{V}\|_{\theta}^T + r_{1M}\|\bar{U} - \bar{V}\|_{\theta}^T + 2b_{1M}p_M\|\bar{U} - \bar{V}\|_{\theta}^T + a_{1M}(p_M + q_M)\|\bar{U} - \bar{V}\|_{\theta}^T \\ &= \bar{C}_1\|\bar{U} - \bar{V}\|_{\theta}^T \end{aligned}$$

and

$$\begin{aligned} &|\mathcal{N}_2(\bar{U})(t, \xi) - \mathcal{N}_2(\bar{V})(t, \xi)|e^{-\theta|\xi|} \\ &= |\alpha_2(\bar{U}_2(t, \xi) - \bar{V}_2(t, \xi)) + a_2(t)(\bar{U}_1(t, \xi)\bar{U}_2(t, \xi) - \bar{V}_1(t, \xi)\bar{V}_2(t, \xi)) \\ &\quad - r_2(t)(\bar{U}_2(t, \xi) - \bar{V}_2(t, \xi)) - b_2(t)(\bar{U}_2^2(t, \xi) - \bar{V}_2^2(t, \xi))|e^{-\theta|\xi|} \\ &\leq \alpha_2|\bar{U}_2(t, \xi) - \bar{V}_2(t, \xi)|e^{-\theta|\xi|} + a_2(t)|\bar{U}_1(t, \xi)\bar{U}_2(t, \xi) - \bar{V}_1(t, \xi)\bar{V}_2(t, \xi)|e^{-\theta|\xi|} \\ &\quad + r_2(t)|\bar{U}_2(t, \xi) - \bar{V}_2(t, \xi)|e^{-\theta|\xi|} + b_2(t)|\bar{U}_2^2(t, \xi) - \bar{V}_2^2(t, \xi)|e^{-\theta|\xi|} \leq \bar{C}_2\|\bar{U} - \bar{V}\|_{\theta}^T. \end{aligned}$$

Thus, $\|\mathcal{N}(\bar{U}) - \mathcal{N}(\bar{V})\|_{\theta}^T \leq \bar{C}\|\bar{U} - \bar{V}\|_{\theta}^T$, where $\bar{C} := \max\{\bar{C}_1, \bar{C}_2\}$. It then follows that $\alpha(\mathcal{N}(B)) \leq \bar{C}\alpha(B)$ for any bounded and closed set $B \subseteq \Gamma_T$. By Lemmas 2.9 and 2.4, for any nonempty bounded closed set $B \subseteq \Gamma_T$, we have

$$\alpha\left(\sum_{k=0}^{\infty} (\mathcal{T}_i(T))^k \int_0^T \mathcal{T}_i(T-s) [\mathcal{N}_i(B)(s, \cdot)] ds\right)$$

$$\begin{aligned}
&\leq \sum_{k=0}^{\infty} (e^{(\theta c - \underline{d} - \underline{\alpha})T})^k \alpha \left(\int_0^T \mathcal{T}_i(T-s) [\mathcal{N}_i(B)(s, \cdot)] ds \right) \\
&\leq \frac{1}{1 - e^{(\theta c - \underline{d} - \underline{\alpha})T}} \int_0^T \alpha(\mathcal{T}_i(T-s) [\mathcal{N}_i(B)(s, \cdot)]) ds \\
&\leq \frac{1}{1 - e^{(\theta c - \underline{d} - \underline{\alpha})T}} \int_0^T e^{(\theta c - \underline{d} - \underline{\alpha})(T-s)} \alpha(\mathcal{N}_i(B)(s, \cdot)) ds \\
&\leq \frac{1}{1 - e^{(\theta c - \underline{d} - \underline{\alpha})T}} \int_0^T e^{(\theta c - \underline{d} - \underline{\alpha})(T-s)} ds \alpha(\mathcal{N}_i(B)) \\
&\leq \frac{e^{(\theta c - \underline{d} - \underline{\alpha})T}}{(1 - e^{(\theta c - \underline{d} - \underline{\alpha})T})(\underline{d} + \underline{\alpha} - \theta c)} [e^{-(\theta c - \underline{d} - \underline{\alpha})T} - 1] \alpha(\mathcal{N}(B)) \\
&\leq \frac{\bar{C}}{\underline{d} + \underline{\alpha} - \theta c} \alpha(B).
\end{aligned}$$

Hence, for each $t \in [0, T]$, we have

$$\begin{aligned}
\alpha(\mathcal{H}_i(B)(t, \cdot)) &\leq \alpha \left(\mathcal{T}_i(t) \sum_{k=0}^{\infty} (\mathcal{T}_i(T))^k \int_0^T \mathcal{T}_i(T-s) [\mathcal{N}_i(B)(s, \cdot)] ds \right) \\
&\quad + \alpha \left(\int_0^t \mathcal{T}_i(t-s) [\mathcal{N}_i(B)(s, \cdot)] ds \right) \\
&\leq e^{(\theta c - \underline{d} - \underline{\alpha})t} \alpha \left(\sum_{k=0}^{\infty} (\mathcal{T}_i(T))^k \int_0^T \mathcal{T}_i(T-s) [\mathcal{N}_i(B)(s, \cdot)] ds \right) \\
&\quad + \int_0^t e^{(\theta c - \underline{d} - \underline{\alpha})(t-s)} \alpha(\mathcal{N}_i(B)(s, \cdot)) ds \\
&\leq e^{(\theta c - \underline{d} - \underline{\alpha})t} \frac{\bar{C}}{\underline{d} + \underline{\alpha} - \theta c} \alpha(B) + \frac{1 - e^{(\theta c - \underline{d} - \underline{\alpha})t}}{\underline{d} + \underline{\alpha} - \theta c} \bar{C} \alpha(B) \\
&= \frac{\bar{C}}{\underline{d} + \underline{\alpha} - \theta c} \alpha(B).
\end{aligned}$$

Similar to Lemma 3.8, we obtain that $(U_i)_t$ ($i = 1, 2$) is uniformly bounded. Therefore, $\mathcal{H}(B)(t)$ is bounded and equicontinuous. By Lemma 2.4 (i), we get $\alpha(\mathcal{H}(B)) = \sup_{t \in [0, T]} \alpha(\mathcal{H}(B)(t, \cdot))$. Thus, $\alpha(\mathcal{H}(B)) \leq \frac{\bar{C}}{\underline{d} - \theta c + \underline{\alpha}} \alpha(B)$. Since $\underline{d} > \bar{C}$, we obtain

$$\lim_{\theta \rightarrow 0^+} \frac{\bar{C}}{\underline{d} - \theta c + \underline{\alpha}} = \frac{\bar{C}}{\underline{d} + \underline{\alpha}} < 1.$$

Hence, there exists some sufficiently small $\theta > 0$ such that $\frac{\bar{C}}{\underline{d} + \underline{\alpha}} < 1$.

Consequently, for any nonempty bounded closed set $B \subseteq \Gamma_T$ with $\alpha(B) > 0$, there holds $\alpha(\mathcal{H}(B)) < \alpha(B)$; i.e., $\mathcal{H} : \Gamma_T \rightarrow \Gamma_T$ is α -condensing. This completes the proof. $\square$

2.4. Existence of periodic traveling waves

In this subsection, we show that a fixed point of the nonlinear operator $\mathcal{H}$ generates a periodic traveling wave of (1.1). In fact, we have the following result.

Theorem 2.12. *Assume that (H_1) – (H_3) hold. Then for each $c > c_*$ system (1.1) has a periodic traveling wave $U(t, \xi) = (U_1(t, \xi), U_2(t, \xi))$ satisfying $0 < U_1(t, \xi) \leq p(t)$ and $0 < U_2(t, \xi) \leq q(t)$ for $(t, \xi) \in \mathbb{R}^2$, and*

$$\lim_{\xi \rightarrow -\infty} U_1(t, \xi) = p(t), \quad \lim_{\xi \rightarrow -\infty} U_2(t, \xi) = 0 \quad \text{uniformly in } t \in \mathbb{R}. \quad (2.19)$$

If, in addition, $\overline{r_1 - a_1 q} > 0$, then $\liminf_{\xi \rightarrow +\infty} U_1(t, \xi) > 0$ uniformly in $t \in \mathbb{R}$.

Proof. By Lemma 2.11, the map $\mathcal{H} : \Gamma_T \rightarrow \Gamma_T$ is α -condensing. We can verify that $\mathcal{H} : \Gamma_T \rightarrow \Gamma_T$ is continuous with respect to $\|\cdot\|_\theta^T$. Note that Γ_T is bounded in $C([0, T] \times \mathbb{R}, \mathbb{R})$ with respect to the norm $\|\cdot\|_\theta^T$. From Lemma 2.10, we see that $\mathcal{H}^n(\Gamma_T) \subseteq \Gamma_T$ for any $n \geq 1$. It then follows that $\mathcal{H}$ is compact dissipative. By the asymptotic fixed point theorem (Lemma 2.3), $\mathcal{H}$ has a fixed point $\tilde{W} = (\tilde{W}_1, \tilde{W}_2) \in \Gamma_T$. Clearly, $\tilde{W}(0, \cdot) = \tilde{W}(T, \cdot)$ and $\underline{U} \leq \tilde{W} \leq \overline{U}$ on $[0, T] \times \mathbb{R}$. Moreover,

$$\tilde{W}_i(t, \xi) = \mathcal{T}_i(t)[\tilde{W}_i(0, \cdot)](\xi) + \int_0^t \mathcal{T}_i(t-s)[\mathcal{N}_i(\tilde{W})(s, \cdot)](\xi) ds, \quad i = 1, 2.$$

Define $U(t, \xi) = (U_1(t, \xi), U_2(t, \xi)) := \tilde{W}(t - m_0 T, \xi)$ for all $(t, \xi) \in \mathbb{R}^2$, where $m_0 \in \mathbb{Z}$ satisfies $m_0 T \leq t < (m_0 + 1)T$. It follows that $U(t + T, \xi) = U(t, \xi)$ and

$$\underline{U}(t, \xi) \leq U(t, \xi) \leq \overline{U}(t, \xi) \quad \text{for all } (t, \xi) \in \mathbb{R}^2. \quad (2.20)$$

Noting that $U(0, \xi) = \tilde{W}(0, \xi)$, we get

$$U_i(t, \xi) = \mathcal{T}_i(t)[U_i(0, \cdot)](\xi) + \int_0^t \mathcal{T}_i(t-s)[\mathcal{N}_i(U)(s, \cdot)](\xi) ds, \quad i = 1, 2.$$

Thus, $U(t, \xi)$ is a periodic traveling wave of (1.1).

It is clear that $0 \leq U_1(t, \xi) \leq p(t)$ and $0 \leq U_2(t, \xi) \leq q(t)$ for $(t, \xi) \in \mathbb{R}^2$. Now, we prove that $U_i(t, \xi) > 0$ for $(t, \xi) \in \mathbb{R}^2$, $i = 1, 2$. Assume, by contradiction, that $U_1(t_0, \xi_0) = 0$ for some $(t_0, \xi_0) \in \mathbb{R}^2$. Then from the first equation of (2.1), we have $\int_{\mathbb{R}} J_1(y) U_1(t_0, \xi_0 - y) dy = 0$. By (H_1) , there exists $y_0 > 0$ such that $J_1(y_0) \neq 0$. By the continuity of J_1 , there exists $a_0 > 0$ such that $J_1(y) \neq 0$, $\forall y \in [y_0 - a_0, y_0 + a_0]$. Since $J_1(y) = J_1(-y)$, it follows that

$$\int_{\mathbb{R}} J_1(y) U_1(t_0, \xi_0 - y) dy = \int_{\mathbb{R}} J_1(y) U_1(t_0, \xi_0 + y) dy = 0.$$

Thus $U_1(t_0, \xi_0 \pm y) = 0$, $\forall y \in [y_0 - a_0, y_0 + a_0]$, and hence,

$$U_1(t_0, y) = 0, \quad \forall y \in \xi_0 + [-y_0 - a_0, -y_0 + a_0] \cup [y_0 - a_0, y_0 + a_0].$$

Let $s_0 = \xi_0 + y_0 + a_0$ and observe that $U_1(t_0, s_0) = 0$. Thus, we can argue as above to obtain

$$U_1(t_0, y) = 0, \quad \forall y \in s_0 + [-y_0 - a_0, -y_0 + a_0] \cup [y_0 - a_0, y_0 + a_0].$$

In particular, $U_1(t_0, y) = 0, \quad \forall y \in \xi_0 + [0, 2a_0]$. Repeating the argument with $s_0 = \xi_0 + y_0 - a_0$, we have $U_1(t_0, y) = 0, \quad \forall y \in \xi_0 + [-2a_0, 0]$. It then follows that $U_1(t_0, y) = 0, \quad \forall y \in \xi_0 + [-2a_0, 2a_0]$. By induction, we see that $U_1(t_0, \xi) \equiv 0, \quad \forall \xi \in \mathbb{R}$, which contradicts the fact that $\lim_{\xi \rightarrow -\infty} U_1(t_0, \xi) = p(t_0) > 0$. Thus, $U_1(t, \xi) > 0$ for $(t, \xi) \in \mathbb{R}^2$. Similarly, we can get $U_2(t, \xi) > 0$ for $(t, \xi) \in \mathbb{R}^2$.

By (2.20) and the definitions of $\underline{U}(t, \xi)$ and $\overline{U}(t, \xi)$, (2.19) follows. It remains to show that $\liminf_{\xi \rightarrow +\infty} U_1(t, \xi) > 0$ uniformly in $t \in \mathbb{R}$. Since $U_2(t, \xi) \leq q(t)$ for $(t, \xi) \in \mathbb{R}^2$, it is clear that $u_1(t, x) := U_1(t, x + ct)$ satisfies

$$(u_1)_t \geq d_1(J_1 * u_1 - u_1) + u_1(r_1(t) - a_1(t)q(t) - b_1(t)u_1), \quad u_1(0, x) = U_1(0, x). \quad (2.21)$$

Let $v(t, x)$ be the unique solution of the following initial value problem:

$$v_t = d_1(J_1 * v - v) + v(r_1(t) - a_1(t)q(t) - b_1(t)v), \quad v(0, x) = U_1(0, x). \quad (2.22)$$

Since $\overline{r_1 - a_1 q} > 0$, it follows from the comparison theorem and the result on spreading speed for scalar time-periodic and nonlocal dispersal equations (cf. [24]) that

$$\liminf_{t \rightarrow \infty} [u_1(t, 0) - v_*(t)] \geq \liminf_{t \rightarrow \infty} [v(t, 0) - v_*(t)] = 0,$$

where $v_*(t)$ is the unique and positive T -periodic solution of the equation: $v'(t) = v(t)(r_1(t) - a_1(t)q(t) - b_1(t)v(t))$. Consequently,

$$\liminf_{n \rightarrow \infty} U_1(t, c(t + nT)) = \liminf_{n \rightarrow \infty} u_1(t + nT, 0) \geq \underline{\omega} := \frac{1}{2} \min_{t \in [0, T]} v_*(t) \text{ uniformly in } t \in \mathbb{R},$$

which implies that $\liminf_{\xi \rightarrow +\infty} U_1(t, \xi) \geq \underline{\omega} > 0$ uniformly in $t \in \mathbb{R}$. The proof is completed. $\square$

Remark 2.13. (i) Note that in Theorem 2.12, the condition that $J_i(\cdot)$ ($i = 1, 2$) has compact support can be replaced by $\mathcal{J}_i^\nu := \int_{\mathbb{R}} J_i(x) e^{\nu|x|} dx < +\infty$ for any $\nu > 0$.

(ii) For this moment, due to the occurrence of time-periodicity and nonlocal dispersal in the predator-prey system, we cannot obtain any information on the asymptotic behavior of the second component $U_2(t, \xi)$ of $(U_1(t, \xi), U_2(t, \xi))$ at $+\infty$. We leave this for our future research.

2.5. Nonexistence of periodic traveling waves

Noting that $\chi_2 := \text{supp } J_2 > 0$, $\int_{\mathbb{R}} J_2(x) dx = 1$ and $J_2 \in C(\mathbb{R})$, we have $\int_{-\chi_2/2}^{\chi_2/2} J_2(y) dy < 1$. To prove the nonexistence of periodic traveling waves with speed $c \in (0, c_*)$, we impose the following additional assumption

$$(H_4) \quad d_2 \left(1 - \int_{-\chi_2/2}^{\chi_2/2} J_2(y) dy \right) > \rho := \frac{1}{T} \int_0^T (a_2(t)p(t) - r_2(t)) dt.$$

Roughly speaking, assumption (H_4) reflects that the dispersal kernel J_2 is not too concentrated. In other words, since $J_2(x - y)$ describes the movement of predators from location y to location x , (H_4) assumes that predators do not disperse in large group. In fact we will prove the nonexistence result by applying the method of contradiction. The condition (H_4) is used to ensure the existence of a positive solution of

an auxiliary problem (2.23). Using this result, one can construct a lower solution for a related equation of system (1.1) and then easily obtain a contradiction.

Theorem 2.14. *Let (H₁)-(H₂) and (H₄) hold. Then for any $c \in (0, c_*)$, system (1.1) admits no positive periodic traveling waves satisfying (2.19).*

Proof. Suppose, by contradiction, that there exists such a periodic traveling wave satisfying (2.19) for some $0 < c < c_*$. By (H₂), it follows that $\rho > 0$. Then there exists $\varepsilon_0 \in (0, 1)$ such that

$$\rho_\varepsilon := \frac{1}{T} \int_0^T [a_2(t)(p(t) - \varepsilon) - r_2(t) - b_2(t)\varepsilon] dt > 0 \text{ for any } \varepsilon \in (0, \varepsilon_0].$$

It is easy to see that $c_* = \inf_{\lambda > 0} \frac{d_2(\int_{\mathbb{R}} J_2(y)e^{-\lambda y} dy - 1) + \rho}{\lambda}$. Thus, by (H₄), we can fix $\varepsilon \in (0, \varepsilon_0)$ such that

$$0 < c < c_*^\varepsilon := \inf_{\lambda > 0} \frac{d_2(\int_{\mathbb{R}} J_2(y)e^{-\lambda y} dy - 1) + \rho_\varepsilon}{\lambda} \text{ and } d_2\left(1 - \int_{-\chi_2/2}^{\chi_2/2} J_2(y) dy\right) \geq \rho_\varepsilon.$$

Since $\lim_{\xi \rightarrow -\infty} U_1(t, \xi) = p(t)$ and $\lim_{\xi \rightarrow -\infty} U_2(t, \xi) = 0$ uniformly in $t \in \mathbb{R}$, we can choose $M_\varepsilon > 0$ such that $p(t) - \varepsilon \leq U_1(t, \xi) \leq p(t) + \varepsilon$ and $0 < U_2(t, \xi) \leq \varepsilon$ uniformly in $t \in \mathbb{R}$, $\forall \xi < -M_\varepsilon$. Define

$$Q_\varepsilon(t) = \exp \left\{ \int_0^t [a_2(s)(p(s) - \varepsilon) - r_2(s) - b_2(s)\varepsilon] ds - \rho_\varepsilon t \right\}.$$

Clearly,

$$\frac{dQ_\varepsilon(t)}{dt} = [a_2(t)(p(t) - \varepsilon) - r_2(t) - b_2(t)\varepsilon]Q_\varepsilon(t) - \rho_\varepsilon Q_\varepsilon(t).$$

Fix a $c_0 \in (c, c_*^\varepsilon)$ and choose $\xi_1, \xi_2 \in \mathbb{R}$ with $\xi_1 = \xi_2 - \frac{\chi_2^2}{2}$ and $\xi_2 < -M_\varepsilon$. Consider the following auxiliary problem:

$$\begin{cases} c_0 h'(x) = d_2 \left(\int_{\xi_1}^{\xi_2} J_2(x-y) h(y) dy - h(x) \right) + \rho_\varepsilon h(x), & x \in (\xi_1, \xi_2], \\ h(\xi_1) = 1. \end{cases} \quad (2.23)$$

We prove the following claim:

Claim: *The initial value problem (2.23) has a solution $h(x) \in C([\xi_1, \xi_2], [0, 1])$.*

Indeed, problem (2.23) is equivalent to the integral equation

$$h(x) = e^{-\frac{d_2}{c_0}(x-\xi_1)} + \int_{\xi_1}^x e^{-\frac{d_2}{c_0}(x-s)} \left[\frac{d_2}{c_0} \int_{\xi_1}^{\xi_2} J_2(s-y) h(y) dy + \frac{\rho_\varepsilon}{c_0} h(s) \right] ds, \quad x \in [\xi_1, \xi_2].$$

Define $\tilde{\Gamma} := \{\phi \in C([\xi_1, \xi_2], [0, 1]) : \phi(\xi_1) = 1\}$, and an operator $\tilde{T} : \tilde{\Gamma} \rightarrow C[\xi_1, \xi_2]$ by

$$\tilde{T}[\phi](x) = e^{-\frac{d_2}{c_0}(x-\xi_1)} + \int_{\xi_1}^x e^{-\frac{d_2}{c_0}(x-s)} \left[\frac{d_2}{c_0} \int_{\xi_1}^{\xi_2} J_2(s-y) \phi(y) dy + \frac{\rho_\varepsilon}{c_0} \phi(s) \right] ds.$$

Note that for $x \in [\xi_1, \xi_2]$ and $s \in [\xi_1, x]$,

$$\int_{\xi_1}^{\xi_2} J_2(s-y)dy = \int_{s-\xi_2}^{s-\xi_1} J_2(y)dy \leq \int_{\xi_1-\xi_2}^{\xi_2-\xi_1} J_2(y)dy = \int_{-\chi_2/2}^{\chi_2/2} J_2(y)dy.$$

For any $\phi \in \tilde{\Gamma}$, we have

$$\begin{aligned} \tilde{T}[\phi](x) &\leq e^{-\frac{d_2}{c_0}(x-\xi_1)} + \frac{d_2}{c_0} \int_{\xi_1}^x e^{-\frac{d_2}{c_0}(x-s)} \left[\int_{\xi_1}^{\xi_2} J_2(s-y)dy + 1 - \int_{-\chi_2/2}^{\chi_2/2} J_2(y)dy \right] ds \\ &\leq e^{-\frac{d_2}{c_0}(x-\xi_1)} + \frac{d_2}{c_0} \int_{\xi_1}^x e^{-\frac{d_2}{c_0}(x-s)} ds \\ &\leq e^{-\frac{d_2}{c_0}(x-\xi_1)} + 1 - e^{-\frac{d_2}{c_0}(x-\xi_1)} = 1, \quad \forall x \in [\xi_1, \xi_2]. \end{aligned}$$

It is clear that $\tilde{T}[\phi](\cdot) \geq 0$. Hence, $\tilde{T}(\tilde{\Gamma}) \subset \tilde{\Gamma}$.

For any $x \in [\xi_1, \xi_2]$ and $\phi, \bar{\phi} \in \tilde{\Gamma}$, we have

$$\begin{aligned} |\tilde{T}[\phi](x) - \tilde{T}[\bar{\phi}](x)| &= \left| \int_{\xi_1}^x e^{-\frac{d_2}{c_0}(x-s)} \left[\frac{d_2}{c_0} \int_{\xi_1}^{\xi_2} J_2(s-y)\phi(y)dy + \frac{\rho_\varepsilon}{c_0}\phi(s) \right] ds \right. \\ &\quad \left. - \int_{\xi_1}^x e^{-\frac{d_2}{c_0}(x-s)} \left[\frac{d_2}{c_0} \int_{\xi_1}^{\xi_2} J_2(s-y)\bar{\phi}(y)dy + \frac{\rho_\varepsilon}{c_0}\bar{\phi}(s) \right] ds \right| \\ &\leq \left(\frac{d_2}{c_0} + \frac{\rho_\varepsilon}{c_0} \right) \int_{\xi_1}^x e^{-\frac{d_2}{c_0}(x-s)} ds \|\phi - \bar{\phi}\| \leq \left(1 + \frac{\rho_\varepsilon}{d_2} \right) \|\phi - \bar{\phi}\|, \end{aligned}$$

which implies that $\tilde{T}$ is continuous. Further, for any $\phi \in \tilde{\Gamma}$, we have that

$$\begin{aligned} \left| \frac{d}{dx} \tilde{T}[\phi](x) \right| &= \left| -\frac{d_2}{c_0} e^{-\frac{d_2}{c_0}(x-\xi_1)} + \frac{d_2}{c_0} \int_{\xi_1}^{\xi_2} J_2(x-y)\phi(y)dy + \frac{\rho_\varepsilon}{c_0}\phi(x) \right. \\ &\quad \left. - \frac{d_2}{c_0} \int_{\xi_1}^x e^{-\frac{d_2}{c_0}(x-s)} \left[\frac{d_2}{c_0} \int_{\xi_1}^{\xi_2} J_2(s-y)\phi(y)dy + \frac{\rho_\varepsilon}{c_0}\phi(s) \right] ds \right| \\ &\leq 2\frac{d_2}{c_0} + \frac{\rho_\varepsilon}{c_0} + \frac{d_2}{c_0} \left[\frac{d_2}{c_0} + \frac{\rho_\varepsilon}{c_0} \right] \int_{\xi_1}^x e^{-\frac{d_2}{c_0}(x-s)} ds \leq \frac{3d_2}{c_0} + \frac{2\rho_\varepsilon}{c_0}, \end{aligned}$$

which implies that $\tilde{T}(\tilde{\Gamma})$ is compact. By the Schauder's fixed point theorem, we conclude that $\tilde{T}$ has a fixed point $\tilde{h}$ in $\tilde{\Gamma}$. Hence, (2.23) has a solution $\tilde{h}(x) \in C([\xi_1, \xi_2], [0, 1])$. This proves the claim.

Define $\tilde{w}(t, x) = \tilde{h}(x)Q_\varepsilon(t)$ for $t \in \mathbb{R}$ and $x \in [\xi_1, \xi_2]$. Then, one can easily see that $\tilde{w}(t+T, x) = \tilde{w}(t, x)$ for $x \in [\xi_1, \xi_2]$ and

$$(\tilde{w})_t = d_2 \left(\int_{\xi_1}^{\xi_2} J_2(x-y) \tilde{w}(t, y) dy - \tilde{w} \right) - c_0(\tilde{w})_x + [a_2(t)(p(t) - \varepsilon) - r_2(t) - b_2(t)\varepsilon] \tilde{w}$$

for $t > 0$, $x \in (\xi_1, \xi_2]$. Since $U_2(0, \xi) > 0$ for $\xi \in (\xi_1, \xi_2)$, there exists an $\varepsilon_2 > 0$ such that $U_2(0, x) \geq \varepsilon_2 \tilde{w}(0, x)$, $x \in [\xi_1, \xi_2]$ and $U_2(t, \xi_i + (c - c_0)t) \geq \varepsilon_2 \tilde{w}(t, \xi_i)$ for $t \in [0, T]$, $i = 1, 2$.

Consider the functions $U_1(t, x + (c - c_0)t)$ and $U_2(t, x + (c - c_0)t)$, $t > 0$, $x \in [\xi_1, \xi_2]$. Denote $\tilde{U}_2(t, x) := U_2(t, x + (c - c_0)t)$. Since $(U_1(t, \xi), U_2(t, \xi))$ is a solution of system (2.1), we have

$$(\tilde{U}_2)_t = d_2(J_2 * \tilde{U}_2 - \tilde{U}_2) - c_0(\tilde{U}_2)_x + [a_2(t)U_1(t, x + (c - c_0)t) - r_2(t) - b_2(t)\tilde{U}_2]\tilde{U}_2. \quad (2.24)$$

In view of $c - c_0 < 0$ and $\xi_1 < \xi_2 < -M_\varepsilon$, we have $x + (c - c_0)t \leq -M_\varepsilon$, $t \geq 0$, $x \in [\xi_1, \xi_2]$. Noticing that $U_1(t, \xi) \geq p(t) - \varepsilon$ and $0 < U_2(t, \xi) \leq \varepsilon$, $\forall \xi < -M_\varepsilon$ uniformly in $t \in \mathbb{R}$, thus by (2.24) we can conclude that $\tilde{U}_2$ satisfies

$$(\tilde{U}_2)_t \geq d_2 \left(\int_{\xi_1}^{\xi_2} J_2(x-y) \tilde{U}_2(t, y) dy - \tilde{U}_2 \right) - c_0(\tilde{U}_2)_x + [a_2(t)(p(t) - \varepsilon) - r_2(t) - b_2(t)\varepsilon]\tilde{U}_2$$

for $t > 0$, $x \in (\xi_1, \xi_2]$. Let $\hat{U}_2(t, x) := \tilde{U}_2(t, x) - \varepsilon_2 \tilde{w}(t, x)$ for all $t \geq 0$, $x \in [\xi_1, \xi_2]$. Then, we can obtain that

$$\begin{cases} (\hat{U}_2)_t \geq d_2 \left(\int_{\xi_1}^{\xi_2} J_2(x-y) \hat{U}_2(t, y) dy - \hat{U}_2 \right) - c_0(\hat{U}_2)_x + [a_2(t)(p(t) - \varepsilon) - r_2(t) - b_2(t)\varepsilon]\hat{U}_2, \\ \hat{U}_2(0, x) \geq 0, \quad x \in [\xi_1, \xi_2], \quad \hat{U}_2(t, \xi_i) \geq 0, \quad t \geq 0, \quad i = 1, 2. \end{cases}$$

In view of the maximum principle for parabolic equations, we can conclude that $\hat{U}_2(t, x) \geq 0$ for all $t > 0$ and $x \in [\xi_1, \xi_2]$, which implies that $U_2(t, x + (c - c_0)t) \geq \varepsilon_2 \tilde{w}(t, x)$ for all $t > 0$ and $x \in [\xi_1, \xi_2]$. Since $c - c_0 < 0$ and $\tilde{h}(\tilde{x}_1) > 0$ for some $\tilde{x}_1 \in [\xi_1, \xi_2]$ with $0 \leq \tilde{x}_1 - \xi_1 \ll 1$, which contradicts $U_2(t, \tilde{x}_1 + (c - c_0)t) \rightarrow 0$ as $t \rightarrow +\infty$. This completes the proof. $\square$

3. Spreading properties

In this section, we investigate the spreading properties of solutions to system (1.1) with nonnegative compact support initial values. Let $X = BUC(\mathbb{R}, \mathbb{R}^2)$ be the space of all $\mathbb{R}^2$ -valued bounded and uniformly continuous functions on $\mathbb{R}$. We equip X with the compact open topology; i.e., a sequence $\{\varphi_n\}$ converges to φ in X if and only if $\varphi_n(x)$ converges to $\varphi(x)$ in X uniformly for x in any bounded subset of $\mathbb{R}$. Moreover, we define a norm $\|\cdot\|_X$ by

$$\|\phi\|_X = \sum_{k=1}^{\infty} \frac{\max_{|x| \leq k} |\phi(x)|}{2^k}, \quad \forall \phi \in X.$$

It follows that $(X, \|\cdot\|_X)$ is a normed space. Take $X^+ = \{u \in X : u \geq 0\}$ and

$$\tilde{X} := \{u_0 = (u_{10}, u_{20}) \in X : 0 \leq u_{10} \leq p(0), \quad 0 \leq u_{20} \leq q(0)\}.$$

Then, the topology induced by $\|\cdot\|_X$ on $\tilde{X}$ is equivalent to the compact open topology on $\tilde{X}$.

3.1. Preliminaries and main results

Set

$$c_{u_1} := \inf_{\lambda > 0} \frac{d_1(\int_{\mathbb{R}} J_1(y)e^{\lambda y} dy - 1) + \bar{r}_1}{\lambda} \quad \text{and} \quad c_{u_2} := \inf_{\lambda > 0} \frac{d_2(\int_{\mathbb{R}} J_2(y)e^{\lambda y} dy - 1) + \overline{a_2 p - r_2}}{\lambda},$$

which are the spreading speeds of

$$(u_1)_t = d_1(J_1 * u_1 - u_1) + u_1(r_1(t) - b_1(t)u_1), \quad (3.1)$$

and

$$(u_2)_t = d_2(J_2 * u_2 - u_2) + u_2(-r_2(t) - b_2(t)u_2 + a_2(t)p(t)), \quad (3.2)$$

respectively. Note that c_{u_1} denotes the spreading speed of the prey in the absence of predators and c_{u_2} represents the spreading speed of the predators with abundant prey. It is clear that $c_{u_2} = c_*$ which is defined in Section 2.

From Jin and Zhao [24, Theorem 3.5], (3.1) admits the following spreading properties.

Lemma 3.1. *Assume that $\bar{r}_1 > 0$, and let $u_1(t, x; \phi)$ be the solution of (3.1) with initial function $\phi \in X$ satisfying $0 \leq \phi \leq p(0)$. Then c_{u_1} is the spreading speed for solutions of (3.1) with compactly supported initial data; i.e.,*

(i) *For any $c > c_{u_1}$, if $\phi(x) = 0$ for x outside a bounded interval, then*

$$\lim_{t \rightarrow +\infty, |x| \geq ct} u_1(t, x; \phi) = 0;$$

(ii) *For any $c \in (0, c_{u_1})$, if $\phi \not\equiv 0$, then $\lim_{t \rightarrow +\infty, |x| \leq ct} (u_1(t, x; \phi) - p(t)) = 0$.*

Recall that $f_M := \max_{t \in [0, T]} f(t)$ and $f_m := \min_{t \in [0, T]} f(t)$ for a given function $f \in C[0, T]$. To obtain our main results, we also need the following technique assumptions:

(H₅) $r_{1m} \geq 1$.

(H₆) $d_1 > r_{1M} + \frac{1}{2}(a_{2M} + a_{1M})M_*$ and $d_2 > (\frac{1}{2}a_{1M} + \frac{3}{2}a_{2M})M_* - r_{2m}$, where $M_* := \max\{p_M, q_M\}$.

We note that condition (H₆) means that the diffusion coefficients d_1, d_2 are not too small, which indicates that, to spread successfully, both the prey and predators need to disperse with reasonable rates. Such a condition is used to discuss the smoothness of solutions of (1.1). In fact, in the proofs of two critical Lemmas 3.8 and 3.10, we need to show that some solution sequence $\{u_n(t, x)\}$ of the initial value problem of (1.1) has a convergent subsequence. Notice that the solutions of the nonlocal dispersal system have lower regularity with respect to x . Using (H₆) and the specific form of the coupling of system (1.1), we can obtain *a priori* estimate which yields that $u_n(t, x)$ and $(u_n)_t(t, x)$ have global Lipschitz constants with respect to x , which are independent of n (see e.g., (3.15)).

To establish the lower estimates of the spreading speed, we need to consider an eigenvalue problem of a time-periodic nonlocal operator. Let $R \gg \max\{\chi_1, \chi_2\} > 0$ be a given constant. Denote $\Omega_R = (-R, R)$ and

$$\mathcal{L}_{\varpi_i, \Omega}^{d_i, J_i}[\phi] := -\phi_t(t, x) + c\phi_x(t, x) + d_i \int_{\Omega} J_i(x - y)\phi(t, y)dy + (\varpi_i(t) - d_i)\phi(t, x)$$

for $(t, x) \in \mathbb{R} \times \Omega$, where $i = 1, 2$, $\Omega = \Omega_R$ or $\Omega = \mathbb{R}$, $\varpi_1(t) = r_1(t)$, and $\varpi_2(t) = a_2(t)p(t) - r_2(t)$.

Given $i \in \{1, 2\}$. We consider the following spectral problem: to find a positive and T -periodic function $\phi_i \in C^1(\mathbb{R} \times \Omega_R) \cap C(\mathbb{R} \times \bar{\Omega}_R)$ and a number $\lambda \in \mathbb{R}$ such that

$$\begin{cases} \mathcal{L}_{\varpi_i, \Omega}^{d_i, J_i}[\phi_i] + \lambda \phi_i = 0, & t \in \mathbb{R}, x \in \Omega, \\ \phi_i(t, x) = \phi_i(t + T, x), & t \in \mathbb{R}, x \in \bar{\Omega}. \end{cases} \quad (3.3)$$

Motivated by the results of Berestycki et al. [6] and Coville and Hamel [10], we introduce the following quantity which is called the *generalized principal eigenvalue* of (3.3):

$$\lambda_c(\mathcal{L}_{\varpi_i, \Omega}^{d_i, J_i}) := \sup \{ \lambda \in \mathbb{R} \mid \exists \phi_i \in C^1(\mathbb{R} \times \Omega) \cap C(\mathbb{R} \times \bar{\Omega}), \phi_i > 0 \text{ and} \\ \mathcal{L}_{\varpi_i, \Omega}^{d_i, J_i}[\phi_i] + \lambda \phi_i \leq 0 \text{ in } \mathbb{R} \times \Omega, \text{ and } \phi_i(t, x) = \phi_i(t + T, x) \text{ for } t \in \mathbb{R}, x \in \bar{\Omega} \}.$$

Coville and Hamel [10] studied the spectral problem for an autonomous version of (3.3). Indeed, when $\varpi_i(t) \equiv \tilde{\omega}_i$, where $\tilde{\omega}_i$ are constants, they showed that the generalized principal eigenvalue $\lambda_c(\mathcal{L}_{\tilde{\omega}_i, \Omega}^{d_i, J_i})$ of (3.3) is always achieved; i.e., there is a positive and T -periodic function $\phi_i \in C^1(\Omega) \cap C(\bar{\Omega})$ such that (3.3) holds with $\lambda = \lambda_c(\mathcal{L}_{\tilde{\omega}_i, \Omega}^{d_i, J_i})$. The function ϕ_i is called the *generalized principal eigenfunction associated with* $\lambda_c(\mathcal{L}_{\tilde{\omega}_i, \Omega}^{d_i, J_i})$. Moreover, the map $\tilde{\omega}_i - d_i \mapsto \lambda_c(\mathcal{L}_{\tilde{\omega}_i, \Omega}^{d_i, J_i})$ is Lipschitz continuous and $\lim_{R \rightarrow \infty} \lambda_c(\mathcal{L}_{\tilde{\omega}_i, \Omega_R}^{d_i, J_i}) = \lambda_c(\mathcal{L}_{\tilde{\omega}_i, \mathbb{R}}^{d_i, J_i})$. We mention that there are also some results on the principal spectral theory of (3.3) in the case where $c = 0$ and Ω is a bounded domain. We refer to Berestycki et al. [7], Rawal and Shen [34], Su et al. [38], Sun et al. [39] and the references therein.

We make the following assumption on the time-periodic eigenvalue problem (3.3):

(H₇) The generalized principal eigenvalue $\lambda_c(\mathcal{L}_{\varpi_i, \Omega}^{d_i, J_i})$ of (3.3) is achieved. The map $\varpi_i - d_i \mapsto \lambda_c(\mathcal{L}_{\varpi_i, \Omega}^{d_i, J_i})$ is continuous and $\lim_{R \rightarrow \infty} \lambda_c(\mathcal{L}_{\varpi_i, \Omega_R}^{d_i, J_i}) = \lambda_c(\mathcal{L}_{\varpi_i, \mathbb{R}}^{d_i, J_i})$.

The spreading properties of solutions to system (1.1) with compactly supported initial values are given in the following two theorems.

Theorem 3.2. Let (H₁)-(H₂) and (H₅)-(H₆) hold, and $u_0 \in \tilde{X}$ be nontrivially compactly supported. If $c_{u_1} > c_{u_2}$, then the solution $u = (u_1, u_2)$ of (1.1) with initial data $u_0(\cdot)$ satisfies the following properties:

- (i) For any $c > c_{u_1}$, $\lim_{t \rightarrow +\infty} \sup_{|x| \geq ct} u_1(t, x) = 0$;
- (ii) For each $c > c_{u_2}$, $\lim_{t \rightarrow +\infty} \sup_{|x| \geq ct} u_2(t, x) = 0$, and for all $c_{u_2} < c_1 < c_2 < c_{u_1}$,

$$\lim_{t \rightarrow +\infty} \sup_{c_1 t \leq |x| \leq c_2 t} |p(t) - u_1(t, x)| = 0; \quad (3.4)$$

- (iii) If, in addition, (H₇) holds, $\lambda_c(\mathcal{L}_{r_1, \mathbb{R}}^{d_1, J_1}) < 0$ and $\lambda_c(\mathcal{L}_{a_2 p - r_2, \mathbb{R}}^{d_2, J_2}) < 0$, then there exists $\varrho > 0$ such that for any $x \in \mathbb{R}$, $\liminf_{t \rightarrow +\infty} (u_1(t, x + ct), u_2(t, x + ct)) \geq (\varrho, \varrho)$.

Theorem 3.3. Let (H₁)-(H₂) and (H₅)-(H₆) hold, and $u_0 \in \tilde{X}$ be nontrivial compactly supported. If $c_{u_2} \geq c_{u_1}$, then the solution $u = (u_1, u_2)$ of (1.1) with initial data $u_0(\cdot)$ satisfies the following properties:

- (i) For any $c > c_{u_1}$, $\lim_{t \rightarrow +\infty} \sup_{|x| \geq ct} (u_1(t, x) + u_2(t, x)) = 0$;
- (ii) If, in addition, (H₇) holds, $\lambda_c(\mathcal{L}_{r_1, \mathbb{R}}^{d_1, J_1}) < 0$ and $\lambda_c(\mathcal{L}_{a_2 p - r_2, \mathbb{R}}^{d_2, J_2}) < 0$, then there exists $\varrho > 0$ such that for any $x \in \mathbb{R}$, $\liminf_{t \rightarrow +\infty} (u_1(t, x + ct), u_2(t, x + ct)) \geq (\varrho, \varrho)$.

Remark 3.4. It should be noted that the conditions:

$$(H_7) \text{ holds, } \lambda_c(\mathcal{L}_{r_1, \mathbb{R}}^{d_1, J_1}) < 0 \quad \text{and} \quad \lambda_c(\mathcal{L}_{a_2 p - r_2, \mathbb{R}}^{d_2, J_2}) < 0$$

play an important role in establishing lower estimates of the spreading speed of (1.1). As mentioned above, Coville and Hamel [10] showed that (H_7) holds for autonomous problem of (3.3). We conjecture that (H_7) also holds for the time-periodic eigenvalue problem (3.3) and leave it for future research.

3.2. Proofs of main results

In this subsection, we always assume that (H_1) – (H_2) and (H_5) – (H_6) hold and prove Theorems 3.2 and 3.3. We first prove two preliminary lemmas on the uniformly boundedness of solutions to system (1.1) and on the spatial dynamics of an auxiliary system.

Lemma 3.5. *System (1.1) admits a unique solution $u = (u_1, u_2)$ with initial value $u_0(\cdot) = (u_{10}(\cdot), u_{20}(\cdot)) \in \tilde{X}$. Moreover, $0 \leq u_1(t, x), u_2(t, x) \leq M_* = \max\{p_M, q_M\}$, $\forall t \geq 0, x \in \mathbb{R}$.*

Proof. The existence and uniqueness of solutions to system (1.1) are standard. Here, we only show that the solution $u(t, x)$ is uniformly bounded. It is easy to see that $[0, \infty)^2$ is an invariant domain for system (1.1). Thus, $u(t, x) \geq 0$. From the first equation of (1.1), we have

$$\begin{cases} (u_1)_t \leq d_1(J_1 * u_1 - u_1) + u_1(r_1(t) - b_1(t)u_1), \\ u_{10} \leq p(0). \end{cases} \quad (3.5)$$

By the comparison principle, one can see that $0 \leq u_1(x, t) \leq p(t)$. Thus, by the second equation of system (1.1), we get

$$(u_2)_t \leq d_2(J_2 * u_2 - u_2) + u_2(-r_2(t) - b_2(t)u_2 + a_2(t)p(t)).$$

Since $0 \leq u_{20} \leq q(0)$, by the comparison principle, there holds $0 \leq u_2(x, t) \leq q(t)$. Thus, $0 \leq u_1(t, x), u_2(t, x) \leq M_* = \max\{p_M, q_M\}$, $\forall t \geq 0, x \in \mathbb{R}$. This completes the proof. $\square$

In the following we consider the time-periodic nonlocal dispersal equation

$$(u_1)_t = d_1(J_1 * u_1 - u_1) + u_1(r_1(t) - b_1(t)u_1 - a_1(t)\varepsilon), \quad (3.6)$$

where $\varepsilon > 0$ is small enough such that $\overline{r_1 - a_1\varepsilon} := \frac{1}{T} \int_0^T [r_1(t) - a_1(t)\varepsilon] dt > 0$.

Lemma 3.6. *Let*

$$c_\varepsilon = \inf_{\lambda > 0} \frac{d_1(\int_{\mathbb{R}} J_1(y)e^{\lambda y} dy - 1) + \overline{r_1 - a_1\varepsilon}}{\lambda}.$$

Then the following statements hold:

- (i) c_ε is the spreading speed for solutions of (3.6) with compact support.
- (ii) c_ε and $p_\varepsilon(t)$ are nonincreasing in ε . Moreover, $\lim_{\varepsilon \rightarrow 0} c_\varepsilon = c_{u_1}$ and $\lim_{\varepsilon \rightarrow 0} p_\varepsilon(t) = p(t)$, where $p_\varepsilon(t)$ is the unique positive T -periodic solution of

$$u_1'(t) = u_1(t)(r_1(t) - b_1(t)u_1(t) - a_1(t)\varepsilon). \quad (3.7)$$

Proof. The statement (i) follows from Lemma 3.1. Next, we prove the statement (ii). Clearly, by the definitions of c_ε and c_{u_1} , we know that c_ε is nonincreasing in ε and $\lim_{\varepsilon \rightarrow 0} c_\varepsilon = c_{u_1}$. Moreover, similar to (1.3), the positive T -periodic solution $p_\varepsilon(t)$ of (3.7) has the following form:

$$p_\varepsilon(t) = \frac{p_\varepsilon^0 e^{\int_0^t (r_1(s) - a_1(s)\varepsilon) ds}}{1 + p_\varepsilon^0 \int_0^t e^{\int_0^s (r_1(\tau) - a_1(\tau)\varepsilon) d\tau} b_1(s) ds}, \quad p_\varepsilon^0 = \frac{e^{\int_0^T (r_1(s) - a_1(s)\varepsilon) ds} - 1}{\int_0^T e^{\int_0^s (r_1(\tau) - a_1(\tau)\varepsilon) d\tau} b_1(s) ds} > 0.$$

Thus, $\lim_{\varepsilon \rightarrow 0} p_\varepsilon(t) = p(t)$ and $p_\varepsilon(t)$ is nonincreasing in ε . The proof is completed. $\square$

3.2.1. Proofs of Theorems 3.2 (i)-(ii) and 3.3 (i)

In this subsection, we prove the statements (i)-(ii) of Theorem 3.2 and statement (i) of Theorem 3.3. We first prove the following critical lemma.

Lemma 3.7. *If $c > c_{u_1}$, then $\lim_{t \rightarrow +\infty} \sup_{|x| \geq ct} (u_1(t, x) + u_2(t, x)) = 0$, and if $c > c_{u_2}$, then $\lim_{t \rightarrow +\infty} \sup_{|x| \geq ct} u_2(t, x) = 0$.*

Proof. We first show that $\lim_{t \rightarrow +\infty} \sup_{|x| \geq ct} u_2(t, x) = 0$ for $c > c_{u_2}$. Define

$$\tilde{u}_2(t, x) := B_1 \varphi(t) e^{-\varsigma(x - c_{u_2} t)},$$

where $B_1 > 0$, $\varphi(t) = e^{\int_0^t (a_2(s)p(s) - r_2(s) - \overline{a_2 p - r_2}) ds}$ and $\varsigma > 0$ is the unique positive root of the equation:

$$\Delta_1(\lambda) := d_2 \left(\int_{\mathbb{R}} J_2(y) e^{\lambda y} dy - 1 \right) - c_{u_2} \lambda + \overline{a_2 p - r_2} = 0.$$

Since $0 \leq u_1 \leq p(t)$, we can show that

$$\begin{aligned} & (\tilde{u}_2)_t - d_2 [J_2 * \tilde{u}_2 - \tilde{u}_2] - \tilde{u}_2 [a_2(t)u_1 - r_2(t) - b_2(t)\tilde{u}_2] \\ & \geq (\tilde{u}_2)_t - d_2 [J_2 * \tilde{u}_2 - \tilde{u}_2] - \tilde{u}_2 [a_2(t)p(t) - r_2(t)] \\ & = c_{u_2} \varsigma \tilde{u}_2 + \tilde{u}_2 [a_2(t)p(t) - r_2(t) - \overline{a_2 p - r_2}] \\ & \quad - d_2 \left[\int_{\mathbb{R}} J_2(y) e^{\varsigma y} dy - 1 \right] \tilde{u}_2 - \tilde{u}_2 [a_2(t)p(t) - r_2(t)] = 0. \end{aligned}$$

Thus $\tilde{u}_2(t, x)$ is a super-solution of the u_2 -equation of (1.1). Since $u_{20}(x)$ is compactly supported and bounded, we can take B_1 large enough such that $u_{20}(x) \leq \tilde{u}_2(0, x)$. By the comparison principle, we have $u_2(t, x) \leq \tilde{u}_2(t, x)$ for any $(t, x) \in \mathbb{R}^2$. Thus, $\lim_{t \rightarrow +\infty} \sup_{x \geq ct} u_2(t, x) = 0$ for $c > c_{u_2}$. By comparing the function $u_2(t, x)$ with $\tilde{u}_2(t, x) := B_2 \varphi(t) e^{\varsigma(x + c_{u_2} t)}$, where $B_2 > 0$, similarly we have $\lim_{t \rightarrow +\infty} \sup_{x \leq -ct} u_2(t, x) = 0$ for $c > c_{u_2}$.

Next, we prove $\lim_{t \rightarrow +\infty} \sup_{|x| \geq ct} u_1(t, x) = 0$ for $c > c_{u_1}$. Let $\bar{u}_1$ be the solution of

$$(\bar{u}_1)_t = d_1 (J_1 * \bar{u}_1 - \bar{u}_1) + \bar{u}_1 (r_1(t) - b_1(t)\bar{u}_1)$$

associated with the initial value $\bar{u}_1(0, x) = u_{10}(x)$. Since $u_2 \geq 0$, $\bar{u}_1$ is a super-solution for the u_1 -equation in (1.1). By the comparison principle, we have $u_1(t, x) \leq \bar{u}_1(t, x)$, $(t, x) \in \mathbb{R}^2$. From Lemma 3.1, we get

$$\lim_{t \rightarrow +\infty} \sup_{|x| \geq ct} u_1(t, x) \leq \lim_{t \rightarrow +\infty} \sup_{|x| \geq ct} \bar{u}_1(t, x) = 0, \quad \forall c > c_{u_1}.$$

By the nonnegativity of u_1 , we have $\lim_{t \rightarrow +\infty} \sup_{|x| \geq ct} u_1(t, x) = 0$ for $c > c_{u_1}$.

Finally, we show that $\lim_{t \rightarrow +\infty} \sup_{|x| \geq ct} u_2(t, x) = 0$ for $c > c_{u_1}$. Define a function

$$\Delta_2(\lambda, \epsilon) := d_2 \left(\int_{\mathbb{R}} J_2(y) e^{\lambda y} dy - 1 \right) - c_{u_1} \lambda + \epsilon = 0, \quad \epsilon, \lambda \geq 0.$$

It is clear that $\Delta_2(\lambda, 0) = 0$ has two roots $\lambda_1^0 = 0$ and $\lambda_2^0 > 0$. Note that $\frac{\partial}{\partial \epsilon} \Delta_2(\lambda, \epsilon) > 0$ and $\frac{\partial^2}{\partial \lambda^2} \Delta_2(\lambda, \epsilon) > 0$ for $\lambda \geq 0$. By the continuity of $\Delta_2(\lambda, \epsilon)$, one can see that for sufficiently small $\epsilon \in (0, \overline{a_2 p - r_2})$ with $(a_{2M} - 1)\epsilon - r_{2m} < 0$, the equation $\Delta_2(\lambda, \epsilon) = 0$ has two positive roots $\lambda_1^\epsilon < \lambda_2^\epsilon$ with $\lim_{\epsilon \rightarrow 0} \lambda_1^\epsilon = \lambda_1^0 = 0$. We can further choose $\epsilon > 0$ small enough such that $\lambda_1^\epsilon < \varsigma$. Let $\bar{\lambda}_2$ be the smaller positive root of the following equation

$$\Delta_3(\lambda, \epsilon) := d_2 \left(\int_{\mathbb{R}} J_2(y) e^{\lambda y} dy - 1 \right) - c\lambda + \epsilon = 0.$$

Since $c > c_{u_1}$, it follows that $\Delta_3(\lambda, \epsilon) \leq \Delta_2(\lambda, \epsilon)$ for $\lambda, \epsilon \geq 0$. Thus $\varsigma > \lambda_1^\epsilon \geq \bar{\lambda}_2$. Moreover, for any $c > c_{u_1}$, there exists $\tilde{t} > 0$ such that for any $t \geq \tilde{t}$, $\sup_{x \geq ct} u_1(t, x) \leq \epsilon$. Define

$$\bar{u}_2(t, x) := A\bar{\phi}(t)e^{-\bar{\lambda}_2(x-ct)},$$

where $A > 0$ is large enough, $\bar{\phi}(t) = e^{\int_0^t [(a_2(s)-1)\epsilon - r_{2m}(s)] ds}$. Clearly, $\bar{\phi}(t) \leq e^{[(a_{2M}-1)\epsilon - r_{2m}]t} \leq 1$. Direct computations show that, for $t > \tilde{t}$ and $x \geq ct$,

$$\begin{aligned} & (\bar{u}_2)_t - d_2[J_2 * \bar{u}_2 - \bar{u}_2] - \bar{u}_2[a_2(t)u_1 - r_2(t) - b_2(t)\bar{u}_2] \\ & \geq (\bar{u}_2)_t - d_2[J_2 * \bar{u}_2 - \bar{u}_2] - \bar{u}_2[a_2(t)\epsilon - r_2(t)] \\ & = \bar{u}_2 \left[(a_2(t) - 1)\epsilon - r_2(t) + c\bar{\lambda}_2 - d_2 \left(\int_{\mathbb{R}} J_2(y) e^{\bar{\lambda}_2 y} dy - 1 \right) - a_2(t)\epsilon + r_2(t) \right] = 0. \end{aligned}$$

Hence, we get that $\bar{u}_2(t, x)$ is a super-solution of the u_2 -equation of (1.1) for any $t \geq \tilde{t}$ and $x \geq ct$. Since $\varsigma > \bar{\lambda}_2$, we can choose A large enough such that

$$u_2(\tilde{t}, x) \leq \tilde{u}_2(\tilde{t}, x) \leq \bar{u}_2(\tilde{t}, x), \quad \forall x > c\tilde{t}.$$

Since u_2 is bounded, we can further take A large enough such that

$$u_2(t, x) \leq \bar{u}_2(t, x) \text{ for any } ct - \chi \leq x \leq ct, \quad t \geq \tilde{t},$$

where $\chi = \max\{\chi_1, \chi_2\}$. Therefore, by the comparison theorem (cf. Zhang et al. [45, Lemma 4.7]), we get $u_2(t, x) \leq \bar{u}_2(t, x)$ for any $x \geq ct$, $t \geq \tilde{t}$, which yields that

$$\lim_{t \rightarrow +\infty} \sup_{x \geq c^0 t} u_2(t, x) = 0, \quad \forall c^0 > c.$$

Choosing c arbitrarily close to c_{u_1} , we have $\lim_{t \rightarrow +\infty} \sup_{x \geq ct} u_2(t, x) = 0$. Similarly, we can show that $\lim_{t \rightarrow +\infty} \sup_{x \leq -ct} u_2(t, x) = 0$ for $c > c_{u_1}$. This completes the proof. $\square$

The following lemma plays an important role in proving (3.4); i.e., the second part of Theorem 3.2 (ii).

Lemma 3.8. For any $c \in (c_{u_2}, c_{u_1})$, we have

$$\lim_{t \rightarrow \infty} [u_1(t, x + ct) - p(t)] = 0 \text{ uniformly on every compact subset of } \mathbb{R}. \quad (3.8)$$

Proof. By the proof of Lemma 3.7, for any given $\varepsilon > 0$, there exists $x_\varepsilon > 0$ such that

$$u_2(t, x) \leq \varepsilon \text{ for any } (t, x) \text{ such that } |x| \geq x_\varepsilon + c_{u_2}t. \quad (3.9)$$

The rest of the proof is divided into three steps.

Step 1. Choose $c' \in (c, c_{u_1})$ and claim that

$$\exists a > 0, x_1 \in \mathbb{R}, \eta_1 > 0, \text{ s.t. } \liminf_{t \rightarrow +\infty} \inf_{x \in (-a, a)} u_1\left(\frac{ct}{c'}, x + ct + x_1\right) \geq \eta_1. \quad (3.10)$$

Take

$$J_{1\sigma}(y) = \frac{1}{\sigma} J_1\left(\frac{y}{\sigma}\right) e^{\frac{y}{\sigma c_{u_1}}}, \quad 0 < \sigma < 1. \quad (3.11)$$

Given any $a > 0$, consider the following eigenvalue problem:

$$\begin{cases} d_1\left(\int_{\mathbb{R}} J_{1\sigma}(y) \psi_{2a}(x - y) dy - \psi_{2a}(x)\right) = -\lambda_{2a} \psi_{2a}(x), x \in (-2a, 2a), \\ \psi_{2a}(x) = 0, x \in \mathbb{R} \setminus (-2a, 2a), \\ \|\psi_{2a}\|_\infty = 1. \end{cases} \quad (3.12)$$

From Garcia and Rossi [19, Theorems 2.1 and 1.4], the above eigenvalue problem has a principal eigenvalue λ_{2a} with a principal eigenfunction $\psi_{2a} \in C[-2a, 2a]$. Thanks to the strong maximum principle, the non-negative eigenfunction ψ_{2a} is strictly positive in $[-2a, 2a]$ (cf. Chasseigne et al. [9, Remark 3.1]). Moreover, since $\int_{\mathbb{R}} J_{1\sigma}(y) dy \geq 1$, we have $\lambda_{2a} \leq 0$ for sufficiently large a .

Take $N_0 := \max_{z \in [-2a, 2a]} |\frac{\psi'_{2a}(z)}{\psi_{2a}(z)}|$. We can take $\varepsilon > 0$ such that $\frac{c'}{c_{u_1}} < 1 - a_{1M}\varepsilon$. Let $x_1 = x_\varepsilon + 2a/\sigma$. Choose $a, \sigma, \eta > 0$ such that $\lambda_{2a} \leq 0$ and

$$\frac{c'}{c_{u_1}} + c'\sigma N_0 - \left(1 - b_{1M}\eta e^{\frac{2a/\sigma - x_1}{c_{u_1}}} - a_{1M}\varepsilon\right) \leq 0. \quad (3.13)$$

Define

$$\underline{u}_1(t, x) := \eta e^{-\frac{1}{c_{u_1}}(x - c't)} \psi_{2a}(\sigma(x - c't - x_1)).$$

Next, we show that $u_1 \geq \underline{u}_1$.

For $-2a < \sigma(x - c't - x_1) < 2a$, we have $|x| \geq x_1 + c't - 2a/\sigma \geq x_\varepsilon + c_{u_2}t$. From (3.9), we have $u_2(t, x) \leq \varepsilon$. By (H₅), it then follows from (3.13) that

$$\begin{aligned} H[\underline{u}_1](t, x) &:= (\underline{u}_1)_t - d_1(J_1 * \underline{u}_1 - \underline{u}_1) - \underline{u}_1(r_1(t) - b_1(t)\underline{u}_1 - a_1(t)u_2) \\ &\leq (\underline{u}_1)_t - d_1(J_1 * \underline{u}_1 - \underline{u}_1) - \underline{u}_1(r_1(t) - b_1(t)\underline{u}_1 - a_1(t)\varepsilon) \\ &= \frac{c'}{c_{u_1}} \eta e^{-\frac{x - c't}{c_{u_1}}} \psi_{2a}(\sigma(x - c't - x_1)) - c'\sigma \eta e^{-\frac{x - c't}{c_{u_1}}} \psi'_{2a}(\sigma(x - c't - x_1)) \\ &\quad - \eta d_1 e^{-\frac{x - c't}{c_{u_1}}} \left[\int_{\mathbb{R}} J_1(y) e^{\frac{y}{c_{u_1}}} \psi_{2a}(\sigma(x - c't - y - x_1)) dy - \psi_{2a}(\sigma(x - x_1 - c't)) \right] \end{aligned}$$

$$\begin{aligned}
& -\underline{u}_1(r_1(t) - b_1(t)\underline{u}_1 - a_1(t)\varepsilon) \\
& = \frac{c'}{c_{u_1}}\underline{u}_1 - c'\sigma\eta e^{-\frac{x-c't}{c_{u_1}}} \psi'_{2a}(\sigma(x - c't - x_1)) - \underline{u}_1(r_1(t) - b_1(t)\underline{u}_1 - a_1(t)\varepsilon) \\
& \quad - \eta d_1 e^{-\frac{x-c't}{c_{u_1}}} \left[\int_{\mathbb{R}} J_{1\sigma}(y) \psi_{2a}(\sigma(x - c't - x_1) - y) dy - \psi_{2a}(\sigma(x - c't - x_1)) \right] \\
& = \underline{u}_1 \left[\frac{c'}{c_{u_1}} - c'\sigma \frac{\psi'_{2a}(\sigma(x - c't - x_1))}{\psi_{2a}(\sigma(x - c't - x_1))} + \lambda_{2a} - r_1(t) + b_1(t)\underline{u}_1 + a_1(t)\varepsilon \right] \\
& \leq \underline{u}_1 \left[\frac{c'}{c_{u_1}} + c'\sigma N_0 - 1 + b_{1M}\eta e^{\frac{2a/\sigma - x_1}{c_{u_1}}} + a_{1M}\varepsilon \right] \leq 0
\end{aligned}$$

for $-2a < \sigma(x - c't - x_1) < 2a$. Hence, we have $H[\underline{u}_1](t, x) \leq 0$ for all $x \in \mathbb{R}$ and $t > 0$.

Recalling that u_1 is positive for any positive time and $\psi_{2a}(\cdot)$ has compact support, we can assume that $u_1(1, x) \geq \underline{u}_1(1, x)$ for all $x \in \mathbb{R}$. Hence, by the comparison principle, we get $u_1(t, x) \geq \underline{u}_1(t, x)$ for $x \in \mathbb{R}$ and $t \geq 1$, which implies that

$$u_1\left(\frac{ct}{c'}, x\right) > \underline{u}_1\left(\frac{ct}{c'}, x\right) = \eta e^{-\frac{1}{c_{u_1}}(x-ct)} \psi_{2a}(\sigma(x - ct - x_1)) \text{ for } t \geq \frac{c'}{c}, x \in \mathbb{R}.$$

Note that $\sigma \in (0, 1)$. Therefore, for $t \geq \frac{c'}{c}$ and $x \in (-a, a)$, we have

$$u_1\left(\frac{ct}{c'}, x + ct + x_1\right) \geq \eta e^{-\frac{1}{c_{u_1}}(x+x_1)} \psi_{2a}(\sigma x) \geq \eta e^{-\frac{1}{c_{u_1}}(a+x_1)} \min_{z \in [-a, a]} \psi_{2a}(z) =: \eta_1 > 0,$$

and hence, (3.10) holds.

Step 2. We now claim that there exist $a > 0$, $\eta_2 > 0$ and $x_2 \in \mathbb{R}$ such that

$$\liminf_{t \rightarrow +\infty} \inf_{t' \in [\frac{\varepsilon t}{c'}, t], x \in (-\frac{a}{2}, \frac{a}{2})} u_1(t', x + ct + x_2) \geq \eta_2. \quad (3.14)$$

Take $a > 0$ and $\varepsilon > 0$ such that $r_{1m} > a_{1M}\varepsilon + \lambda_a$. Then choose $\eta' \in (0, \eta_1)$ such that $b_{1M}\eta' < r_{1m} - a_{1M}\varepsilon - \lambda_a$. Fix $t > 0$ and define

$$\underline{u}_1(t', x) := \eta' \psi_a(x - ct - x_1), \quad t' \in \left[\frac{ct}{c'}, t\right],$$

where (λ_a, ψ_a) satisfies (3.12) with $J_{1\sigma}(\cdot) = J_1(\cdot)$ and $2a$ replaced by a .

For $-a < x - ct - x_1 < a$, $|x| \geq x_1 + ct - a \geq x_\varepsilon + c_{u_2}t$. Then $u_2 \leq \varepsilon$, and hence

$$\begin{aligned}
& (\underline{u}_1)_{t'}(t', x) - d_1(J_1 * \underline{u}_1 - \underline{u}_1)(t', x) - \underline{u}_1(t', x)[r_1(t') - b_1(t')\underline{u}_1(t', x) - a_1(t')u_2(t', x)] \\
& \leq (\underline{u}_1)_{t'}(t', x) - d_1(J_1 * \underline{u}_1 - \underline{u}_1)(t', x) - \underline{u}_1(t', x)[r_1(t') - b_1(t')\underline{u}_1(t', x) - a_1(t')\varepsilon] \\
& = -d_1 \left[\int_{\mathbb{R}} J_1(y) \eta' \psi_a(x - y - x_1 - ct) dy - \eta' \psi_a(x - x_1 - ct) \right] \\
& \quad - \underline{u}_1(t', x)[r_1(t') - b_1(t')\underline{u}_1(t', x) - a_1(t')\varepsilon] \\
& = \underline{u}_1(t', x)[\lambda_a - r_1(t') + b_1(t')\underline{u}_1(t', x) + a_1(t')\varepsilon] \\
& \leq \underline{u}_1(t', x)[\lambda_a - r_{1m} + b_{1M}\eta' + a_{1M}\varepsilon] \leq 0.
\end{aligned}$$

Therefore, we have $H[\underline{u}_1](t', x) \leq 0$ for all $x \in \mathbb{R}$ and $t' \in \left[\frac{ct}{c'}, t\right]$.

It follows from (3.10) that for $t \gg 1$ and $x \in (-a, a)$,

$$u_1\left(\frac{ct}{c'}, x + ct + x_1\right) \geq \eta_1 \geq \eta' \psi_a(x).$$

Since $\psi_a(x) = 0$ for $|x| \geq a$, we have $u_1\left(\frac{ct}{c'}, x + ct + x_1\right) \geq \eta' \psi_a(x)$ for $x \in \mathbb{R}$ and $t \gg 1$. Noting that $\eta' \psi_a(x) = \underline{u}_1\left(\frac{ct}{c'}, x + ct + x_1\right)$, we have $u_1\left(\frac{ct}{c'}, x\right) \geq \underline{u}_1\left(\frac{ct}{c'}, x\right)$ for $x \in \mathbb{R}$ and $t \gg 1$. Hence, by the comparison principle, we have $u_1(t', x) \geq \underline{u}_1(t', x)$ for $x \in \mathbb{R}$, $t' \in [\frac{ct}{c'}, t]$ and $t \gg 1$. It thus follows that (3.14) holds with $\eta_2 := \eta' \inf_{x \in (-\frac{a}{2}, \frac{a}{2})} \psi_a(x)$.

Step 3. We now prove (3.8). Let $\{t_n\}_{n \in \mathbb{Z}} = \{nT\}_{n \in \mathbb{Z}}$. Define

$$u_{1n}(t, x) := u_1(t_n + t, x + ct_n), \quad u_{2n}(t, x) := u_2(t_n + t, x + ct_n)$$

for $(t, x) \in [-t_n, +\infty) \times \mathbb{R}$. By the periodicity of $a_i(t)$, $b_i(t)$ and $r_i(t)$, we can see that $u_n(t, x) = (u_{1n}(t, x), u_{2n}(t, x))$ satisfies

$$\begin{cases} (u_{1n})_t = d_1[J_1 * u_{1n} - u_{1n}] + u_{1n}[r_1(t) - b_1(t)u_{1n} - a_1(t)u_{2n}], \\ (u_{2n})_t = d_2[J_2 * u_{2n} - u_{2n}] + u_{2n}[-r_2(t) - b_2(t)u_{2n} + a_2(t)u_{1n}], \\ u_{10,n}(-t_n, x) := u_1(0, x + ct_n), \quad u_{20,n}(-t_n, x) := u_2(0, x + ct_n) \end{cases}$$

for $(t, x) \in [-t_n, +\infty) \times \mathbb{R}$.

By Lemma 3.5, $0 \leq u_{1n}(t, x), u_{2n}(t, x) \leq M_*$ for $(t, x) \in [-t_n, +\infty) \times \mathbb{R}$. Next, we prove the following *a priori* estimate on $u_n(t, x)$:

$$|(u_{in})_t|, |(u_{in})_{tt}| \leq C \text{ and } |u_{in}(t, x + \gamma) - u_{in}(t, x)|, |(u_{in})_t(t, x + \gamma) - (u_{in})_t(t, x)| \leq C\gamma, \quad (3.15)$$

for $i = 1, 2$, $t \geq -t_n$, $x \in \mathbb{R}$ and some positive number C and any $\gamma > 0$.

In fact, it is clear that

$$\begin{aligned} |(u_{1n})_t| &\leq M_*(2d_1 + r_{1M} + M_*b_{1M} + a_{1M}M_*) =: C_1, \\ |(u_{2n})_t| &\leq M_*(2d_2 + r_{2M} + a_{2M}M_* + b_{2M}M_*) =: C_2, \end{aligned}$$

and

$$\begin{aligned} |(u_{1n})_{tt}| &\leq d_1|J_1 * (u_{1n})_t| + d_1|(u_{1n})_t| + |(u_{1n})_t|[r_1(t) + b_1(t)|u_{1n}| + a_1(t)|u_{2n}|] \\ &\quad + |u_{1n}|[|r'_1(t)| + b_1(t)|(u_{1n})_t| + |b'_1(t)||u_{1n}| + a_1(t)|(u_{2n})_t| + |a'_1(t)||u_{2n}|] \\ &\leq 2d_1C_1 + C_1(b_{1M}M_* + r_{1M} + a_{1M}M_*) \\ &\quad + M_*\left[\max_{t \in [0, T]}(|r'_1(t)| + |a'_1(t)|M_* + |b'_1(t)|M_*) + C_2a_{1M} + C_1b_{1M}\right] =: C_3, \\ |(u_{2n})_{tt}| &\leq d_2|J_2 * (u_{2n})_t| + d_2|(u_{2n})_t| + |(u_{2n})_t|[a_2(t)|u_{1n}| + r_2(t) + b_2(t)|u_{2n}|] \\ &\quad + |u_{2n}|[|a'_2(t)||u_{1n}| + a_2(t)|(u_{1n})_t| + |r'_2(t)| + b_2(t)|(u_{2n})_t| + |b'_2(t)||u_{2n}|] \\ &\leq 2d_2C_2 + C_2(M_*a_{2M} + r_{2M} + b_{2M}M_*) \\ &\quad + M_*\left[\max_{t \in [0, T]}(|a'_2(t)|M_* + |r'_2(t)| + |b'_2(t)|M_*) + C_1a_{2M} + C_2b_{2M}\right] =: C_4. \end{aligned}$$

Since $J'_i \in L^1$ by (H_1) , there exist $L_i > 0$, $i = 1, 2$, for any $\gamma > 0$, such that

$$\begin{aligned} \int_{\mathbb{R}} |J_i(x + \gamma - y) - J_i(x - y)| dy &= \gamma \int_{\mathbb{R}} \left| \int_0^1 J'_i(x + \theta_1 \gamma - y) d\theta_1 \right| dy \\ &\leq \gamma \int_0^1 \int_{\mathbb{R}} |J'_i(x + \theta_1 \gamma - y)| dy d\theta_1 \leq L_i \gamma. \end{aligned}$$

Let $\check{U}_{in}(t, x) = u_{in}(t, x + \gamma) - u_{in}(t, x)$, $i = 1, 2$. Then it follows that

$$\begin{aligned} (\check{U}_{1n})_t(t, x) &= d_1 \int_{\mathbb{R}} (J_1(x + \gamma - y) - J_1(x - y)) u_{1n}(t, y) dy - d_1 \check{U}_{1n}(t, x) \\ &\quad + r_1(t) \check{U}_{1n}(t, x) - b_1(t) \check{U}_{1n}(t, x) (u_{1n}(t, x + \gamma) + u_{1n}(t, x)) \\ &\quad - a_1(t) u_{1n}(t, x + \gamma) \check{U}_{2n}(t, x) - a_1(t) \check{U}_{1n}(t, x) u_{2n}(t, x) \end{aligned}$$

and

$$\begin{aligned} (\check{U}_{2n})_t(t, x) &= d_2 \int_{\mathbb{R}} (J_2(x + \gamma - y) - J_2(x - y)) u_{2n}(t, y) dy - d_2 \check{U}_{2n}(t, x) \\ &\quad + a_2(t) [u_{1n}(t, x + \gamma) \check{U}_{2n}(t, x) + \check{U}_{1n}(t, x) u_{2n}(t, x)] \\ &\quad - r_2(t) \check{U}_{2n}(t, x) - b_2(t) \check{U}_{2n}(t, x) (u_{2n}(t, x + \gamma) + u_{2n}(t, x)). \end{aligned}$$

Hence,

$$\begin{aligned} (\check{U}_{1n}^2)_t(t, x) &= 2\check{U}_{1n}(t, x) \left[d_1 \int_{\mathbb{R}} (J_1(x + \gamma - y) - J_1(x - y)) u_{1n}(t, y) dy \right] - 2d_1 \check{U}_{1n}^2(t, x) \\ &\quad + 2r_1(t) \check{U}_{1n}^2(t, x) - 2b_1(t) \check{U}_{1n}^2(t, x) (u_{1n}(t, x + \gamma) + u_{1n}(t, x)) \\ &\quad - 2a_1(t) u_{1n}(t, x + \gamma) \check{U}_{1n}(t, x) \check{U}_{2n}(t, x) - 2a_1(t) u_{2n}(t, x) \check{U}_{1n}^2(t, x) \end{aligned}$$

and

$$\begin{aligned} (\check{U}_{2n}^2)_t(t, x) &= 2\check{U}_{2n}(t, x) \left[d_2 \int_{\mathbb{R}} (J_2(x + \gamma - y) - J_2(x - y)) u_{2n}(t, y) dy \right] - 2d_2 \check{U}_{2n}^2(t, x) \\ &\quad + 2a_2(t) u_{1n}(t, x + \gamma) \check{U}_{2n}^2(t, x) + 2a_2(t) u_{2n}(t, x) \check{U}_{1n}(t, x) \check{U}_{2n}(t, x) \\ &\quad - 2r_2(t) \check{U}_{2n}^2(t, x) - 2b_2(t) \check{U}_{2n}^2(t, x) (u_{2n}(t, x + \gamma) + u_{2n}(t, x)). \end{aligned}$$

By the above two equalities, we have

$$\begin{aligned} &(\check{U}_{1n}^2)_t(t, x) + (\check{U}_{2n}^2)_t(t, x) \\ &\leq 2\check{U}_{1n}(t, x) \left[d_1 \int_{\mathbb{R}} (J_1(x + \gamma - y) - J_1(x - y)) u_{1n}(t, y) dy \right] - 2d_1 \check{U}_{1n}^2(t, x) \\ &\quad + 2r_1(t) \check{U}_{1n}^2(t, x) + a_1(t) u_{1n}(t, x + \gamma) (\check{U}_{1n}^2(t, x) + \check{U}_{2n}^2(t, x)) - 2r_2(t) \check{U}_{2n}^2(t, x) \\ &\quad + 2\check{U}_{2n}(t, x) \left[d_2 \int_{\mathbb{R}} (J_2(x + \gamma - y) - J_2(x - y)) u_{2n}(t, y) dy \right] - 2d_2 \check{U}_{2n}^2(t, x) \\ &\quad + 2a_2(t) u_{1n}(t, x + \gamma) \check{U}_{2n}^2(t, x) + a_2(t) u_{2n}(t, x) (\check{U}_{1n}^2(t, x) + \check{U}_{2n}^2(t, x)) \end{aligned}$$

$$\begin{aligned}
&\leq 4d_1L_1\gamma M_*^2 + 4d_2L_2\gamma M_*^2 \\
&\quad + 2\left[r_1(t) - d_1 + \frac{1}{2}a_2(t)u_{2n}(t, x) + \frac{1}{2}a_1(t)u_{1n}(t, x + \gamma)\right]\check{U}_{1n}^2(t, x) \\
&\quad + 2\left[\frac{1}{2}a_2(t)u_{2n}(t, x) - d_2 + \left(a_2(t) + \frac{1}{2}a_1(t)\right)u_{1n}(t, x + \gamma) - r_2(t)\right]\check{U}_{2n}^2(t, x) \\
&\leq 4d_1L_1\gamma M_*^2 + 2\left(r_{1M} + \frac{1}{2}a_{2M}M_* + \frac{1}{2}a_{1M}M_* - d_1\right)\check{U}_{1n}^2(t, x) \\
&\quad + 4d_2L_2\gamma M_*^2 + 2\left(\frac{3}{2}a_{2M}M_* + \frac{1}{2}a_{1M}M_* - r_{2m} - d_2\right)\check{U}_{2n}^2(t, x) \\
&= \left(4d_1L_1M_*^2 + 4d_2L_2M_*^2\right)\gamma - k_1\check{U}_{1n}^2(t, x) - k_2\check{U}_{2n}^2(t, x),
\end{aligned}$$

where

$$k_1 := 2\left(d_1 - r_{1M} - \frac{1}{2}a_{2M}M_* - \frac{1}{2}a_{1M}M_*\right), \quad k_2 := 2\left(d_2 - \frac{3}{2}a_{2M}M_* - \frac{1}{2}a_{1M}M_* + r_{2m}\right).$$

From (H₆), we see that $\underline{k} = \min\{k_1, k_2\} > 0$. Then

$$\begin{aligned}
(\check{U}_{1n}^2(t, x) + \check{U}_{2n}^2(t, x))_t &\leq \left(4d_1L_1M_*^2 + 4d_2L_2M_*^2\right)\gamma - \underline{k}[\check{U}_{1n}^2(t, x) + \check{U}_{2n}^2(t, x)] \\
&:= C_0\gamma - \underline{k}[\check{U}_{1n}^2(t, x) + \check{U}_{2n}^2(t, x)],
\end{aligned}$$

where $C_0 = 4d_1L_1M_*^2 + 4d_2L_2M_*^2$. Since $\check{U}_{1n}^2 + \check{U}_{2n}^2$ is bounded, it is easy to verify that $\check{U}_{1n}^2(t, x) + \check{U}_{2n}^2(t, x) \leq C_0\gamma/\underline{k}$. That is,

$$|u_{1n}(t, x + \gamma) - u_{1n}(t, x)|^2 + |u_{2n}(t, x + \gamma) - u_{2n}(t, x)|^2 \leq C_0\gamma/\underline{k}.$$

Thus, for any $\gamma > 0$, we have

$$|u_{1n}(t, x + \gamma) - u_{1n}(t, x)| \leq C_5\gamma, \quad |u_{2n}(t, x + \gamma) - u_{2n}(t, x)| \leq C_6\gamma, \quad \forall t > -t_n, \quad x \in \mathbb{R}.$$

Moreover, for any $t > -t_n$, $x \in \mathbb{R}$, we get

$$\begin{aligned}
&|(u_{1n})_t(t, x + \gamma) - (u_{1n})_t(t, x)| \\
&= |d_1[J_1 * u_{1n}(t, x + \gamma) - u_{1n}(t, x + \gamma)] - d_1[J_1 * u_{1n}(t, x) - u_{1n}(t, x)] \\
&\quad + u_{1n}(t, x + \gamma)[r_1(t) - b_1(t)u_{1n}(t, x + \gamma) - a_1(t)u_{2n}(t, x + \gamma)] \\
&\quad - u_{1n}(t, x)[r_1(t) - b_1(t)u_{1n}(t, x) - a_1(t)u_{2n}(t, x)]| \\
&\leq [2d_1 + r_{1M} + 2M_*b_{1M} + a_{1M}M_*]C_5\gamma + a_{1M}M_*C_6\gamma =: C_7\gamma,
\end{aligned}$$

and similarly,

$$|(u_{2n})_t(t, x + \gamma) - (u_{2n})_t(t, x)| \leq [2d_2 + r_{2M} + a_{2M}M_* + 2b_{2M}M_*]C_6\gamma + a_{2M}M_*C_5\gamma =: C_8\gamma.$$

Taking $C = \max\{C_1, \dots, C_8\}$, the results about *a priori* estimates on $u_n(t, x)$ hold.

Therefore, by the Arzela-Ascoli theorem, there exists a subsequence of $\{t_n\}$, still denoted by $\{t_n\}$, such that $(u_{1n}(t, x), u_{2n}(t, x))$ converges to $(u_{1\infty}(t, x), u_{2\infty}(t, x))$ locally uniformly in $(t, x) \in \mathbb{R}^2$ as $n \rightarrow \infty$. From (3.9), we see that $u_{1\infty}(t, x)$ satisfies

$$(u_{1\infty})_t - d_1[J_1 * u_{1\infty} - u_{1\infty}] - u_{1\infty}[r_1(t) - b_1(t)u_{1\infty} - a_1(t)\varepsilon] \geq 0.$$

By (3.14), we obtain

$$\liminf_{t_n \rightarrow +\infty} \inf_{t' \in [\frac{ct_n}{c}, t_n], x \in (-\frac{a}{2}, \frac{a}{2})} u_1(t', x + x_2 + ct_n) \geq \eta_2. \quad (3.16)$$

Note that $t + t_n \in [\frac{ct_n}{c}, t_n]$ for $t \leq 0$ with $|t| \ll 1$. Since $u_{1n}(t, x + x_2) = u_1(t + t_n, x + ct_n + x_2)$, it follows from (3.16) that $\inf_{x \in (-\frac{a}{2}, \frac{a}{2})} u_{1\infty}(t, x + x_2) \geq \eta_2$ for any $t \leq 0$ with $|t| \ll 1$. Let $u_{1\varepsilon}$ be the solution of

$$(u_{1\varepsilon})_t(t, x) = d_1(J_1 * u_{1\varepsilon} - u_{1\varepsilon})(t, x) + u_{1\varepsilon}(t, x)[r_1(t) - b_1(t)u_{1\varepsilon}(t, x) - a_1(t)\varepsilon]$$

with initial data $u_{1\varepsilon}(0, x) = \hat{g}(x)$, where $\hat{g}(x) \in C(\mathbb{R}, [0, \eta_2])$ is defined by

$$\hat{g}(x) = \begin{cases} \eta_2, & x \in \left(-\frac{a}{4} + x_2, \frac{a}{4} + x_2\right), \\ \text{nondecreasing}, & x \in \left(-\frac{a}{2} + x_2, -\frac{a}{4} + x_2\right), \\ \text{nonincreasing}, & x \in \left(\frac{a}{4} + x_2, \frac{a}{2} + x_2\right), \\ 0, & x \in \mathbb{R} \setminus \left(-\frac{a}{2} + x_2, \frac{a}{2} + x_2\right). \end{cases}$$

By the comparison principle, we have $u_{1\infty}(t, x) \geq u_{1\varepsilon}(t, x)$ for any $(t, x) \in \mathbb{R}^2$. Moreover, by Lemma 3.6 (i), we know that $\lim_{t \rightarrow \infty} [u_{1\varepsilon}(t, x) - p_\varepsilon(t)] = 0$ converges locally uniformly in $x \in \mathbb{R}$. Since $\lim_{\varepsilon \rightarrow 0} p_\varepsilon(t) = p(t)$, we may assume that $p_\varepsilon(t) > p(t) - \varepsilon$. Thus, $\lim_{t \rightarrow \infty} [u_{1\infty}(t, x) - p_\varepsilon(t)] \geq 0$ locally uniformly in $x \in \mathbb{R}$. By the definition of $u_{1\infty}$, we then obtain that

$$\liminf_{t \rightarrow +\infty} [u_1(t, x + ct) - p(t)] \geq \liminf_{t \rightarrow +\infty} [u_1(t, x + ct) - p_\varepsilon(t) - \varepsilon] > -\varepsilon$$

locally uniformly in x . By the arbitrariness of ε , $\lim_{t \rightarrow \infty} [u_1(t, x + ct) - p(t)] = 0$ locally uniformly on every compact subset of $\mathbb{R}$. The proof is completed. $\square$

Proofs of Theorems 3.2 (i)-(ii) and 3.3 (i). From Lemma 3.7, we see that the statement (i) of Theorems 3.2-3.3 and the first part of statement (ii) of Theorem 3.2 hold. Thus, we only need to prove the second part of Theorem 3.3 (ii); i.e., (3.4). If it is not true, then we can assume that there exist two sequences $\{t_n\}$ and $\{x_n\}$ satisfying $c_1 t_n < x_n < c_2 t_n$ and $t_n \rightarrow +\infty$, as $n \rightarrow +\infty$ such that

$$\limsup_{n \rightarrow +\infty} [u_1(t_n, x_n) - p(t_n)] < 0.$$

Let $c_n = \frac{x_n}{t_n}$. Then $c_n \in (c_1, c_2) \subset (c_{u_2}, c_{u_1})$. Thus, there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $\lim_{j \rightarrow +\infty} c_{n_j} = c \in [c_1, c_2]$. By Lemma 3.8, it then follows that

$$u_1(t_{n_j}, x_{n_j}) - p(t_{n_j}) = u_1(t_{n_j}, c_{n_j} t_{n_j}) - p(t_{n_j}) \rightarrow 0 \text{ as } j \rightarrow +\infty,$$

which contradicts $\limsup_{n \rightarrow +\infty} [u_1(t_n, x_n) - p(t_n)] < 0$. This completes the proof of Theorems 3.2 (i)-(ii) and 3.3 (i). $\square$

3.2.2. Proofs of Theorems 3.2 (iii) and 3.3 (ii)

In this subsection, we prove the statement (iii) of Theorem 3.2 and statement (ii) of Theorem 3.3 by using the persistence theory in dynamical systems.

Definition 3.9. Let (Z_0, ρ_0) be a metric space with the metric ρ_0 .

- (i) $\{\Psi_t\}_{t \geq 0}$ is a T -periodic semiflow on (Z_0, ρ_0) provided that $\{\Psi_t\}_{t \geq 0}$ satisfies: (1) $\Psi_0(v) = v$, $\forall v \in Z_0$;
- (2) $\Psi_t(\Psi_T(v)) = \Psi_{t+T}(v)$, $\forall t \geq 0$, $v \in Z_0$; (3) $\Psi_t(v)$ is continuous in (t, v) on $[0, +\infty) \times Z_0$;

(ii) Ψ_T is called the Poincaré map associated with the periodic semiflow $\{\Psi_t\}_{t \geq 0}$.

Let $w(t, x) = u(t, x + ct; u_0)$. Clearly, $w(0, \cdot) = u_0(\cdot)$, and $w(t, x) =: w(t, x; u_0)$ satisfies

$$\begin{cases} (w_1)_t = d_1(J_1 * w_1 - w_1) + c(w_1)_x + w_1(r_1(t) - b_1(t)w_1 - a_1(t)w_2), \\ (w_2)_t = d_2(J_2 * w_2 - w_2) + c(w_2)_x + w_2(-r_2(t) - b_2(t)w_2 + a_2(t)w_1). \end{cases} \quad (3.17)$$

We can see that to prove Theorems 3.2 (iii) and 3.3 (ii), it suffices to prove the persistence of $w(t, x)$. Define a family of operators $\{\Phi_t\}_{t \geq 0}$ on X^+ by

$$\Phi_t(u_0)(x) = w(t, x; u_0) \text{ for } t \geq 0, x \in \mathbb{R} \text{ and } u_0 \in X^+.$$

One can easily show that $\{\Phi_t\}_{t \geq 0}$ is a T -periodic semiflow on X^+ .

For any $y \in \mathbb{R}$, let T_y be the translation operator on X defined by $T_y(\varphi)(x) = \varphi(x - y)$, $\forall x \in \mathbb{R}$, $\varphi \in X$. It then follows that $\tilde{\Phi}_t := T_{ct} \circ \Phi_t$, $\forall t \geq 0$, is also a semiflow on X^+ , and for any $u_0 \in X^+$, $\tilde{w}(t, z) := \tilde{w}(t, z; u_0) = \tilde{\Phi}_t(u_0)(z)$ satisfies

$$\begin{cases} (\tilde{w}_1)_t = d_1(J_1 * \tilde{w}_1 - \tilde{w}_1) + \tilde{w}_1(r_1(t) - b_1(t)\tilde{w}_1 - a_1(t)\tilde{w}_2), \\ (\tilde{w}_2)_t = d_2(J_2 * \tilde{w}_2 - \tilde{w}_2) + \tilde{w}_2(-r_2(t) - b_2(t)\tilde{w}_2 + a_2(t)\tilde{w}_1). \end{cases}$$

Clearly, $w(t, x) = \tilde{w}(t, x + ct) := \tilde{w}(t, z)$. We then have the following result.

Lemma 3.10. Φ_T is α -contracting in the sense that $\lim_{n \rightarrow +\infty} \alpha(\Phi_T^n(B)) = 0$ for any bounded set $B \subseteq \tilde{X}$. Further, Φ_T has a global attractor in $\tilde{X}$.

Proof. Let B be a given bounded subset in $\tilde{X}$. Motivated by Hsu et al. [23, Lemma 4.1], we first prove that Φ_t is asymptotically compact on B in the sense that for any sequences $\{u_{0,n}\} \in B$ and $\{t_n\} \rightarrow +\infty$, there exist subsequences $\{u_{0,n_j}\}$ and $\{t_{n_j}\} \rightarrow +\infty$ such that $\Phi_{t_{n_j}}(u_{0,n_j})$ converges with respect to the compact open topology as $j \rightarrow +\infty$. Let

$$w_n(t, x) = (w_{1n}(t, x), w_{2n}(t, x)) = \Phi_t(u_{0,n})(x), \quad \forall u_{0,n} \in \tilde{X}, t \geq 0, x \in \mathbb{R}.$$

By Lemma 3.5, we know that the family of functions $\{\Phi_{t_n}(u_{0,n})(x)\}_{n \geq 1}$ is uniformly bounded on $\mathbb{R}$ for all $n \geq 1$. In view of the Arzela-Ascoli theorem, it suffices to show that $\{\Phi_{t_n}(u_{0,n})(x)\}_{n \geq 1}$ is equicontinuous in $x \in \mathbb{R}$ for all $n \geq 1$. Since $w(t_n, x) = \tilde{w}(t_n, x + ct_n)$, it is equivalent to show that $\{\tilde{\Phi}_{t_n}(u_{0,n})(z)\}_{n \geq 1}$ is equicontinuous in $z \in \mathbb{R}$ for all $n \geq 1$.

Define $(\bar{w}_{1n}(t, z), \bar{w}_{2n}(t, z)) := (\tilde{w}_{1n}(t + t_n, z), \tilde{w}_{2n}(t + t_n, z))$ for all $t \geq -t_n$ and $z \in \mathbb{R}$. Clearly, $(\bar{w}_{1n}(0, z), \bar{w}_{2n}(0, z)) = (\tilde{w}_{1n}(t_n, z), \tilde{w}_{2n}(t_n, z)) = \tilde{\Phi}_{t_n}(u_{0,n})(z)$, $\forall n \geq 1, z \in \mathbb{R}$. Since $J'_i \in L^1$ by (H₁), there exist $L_i > 0$, $i = 1, 2$, such that

$$\int_{\mathbb{R}} |J_i(z_1 - y) - J_i(z_2 - y)| dy \leq L_i |z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{R}.$$

Note that $\bar{w}_{1n}(t, z), \bar{w}_{2n}(t, z) \leq M_*$, $\forall n \geq 1, t \geq 0, z \in \mathbb{R}$. Thus, for any $\varepsilon > 0$, if $|z_1 - z_2| < \bar{\delta}_i := \frac{\varepsilon}{L_i} > 0$, then

$$\left| \int_{\mathbb{R}} (J_i(z_1 - y) - J_i(z_2 - y)) \bar{w}_{in}(t, y) dy \right| < M_* \varepsilon, \quad i = 1, 2.$$

For any $z_1, z_2 \in \mathbb{R}$, let $\bar{V}_{1n}(t) := \bar{w}_{1n}(t, z_1) - \bar{w}_{1n}(t, z_2)$, $\bar{V}_{2n}(t) := \bar{w}_{2n}(t, z_1) - \bar{w}_{2n}(t, z_2)$. Then it follows that

$$\begin{aligned}
& (\bar{V}_{1n}^2(t) + \bar{V}_{2n}^2(t))_t \\
&= 2\bar{V}_{1n}(t) \left[d_1 \int_{\mathbb{R}} (J_1(z_1 - y) - J_1(z_2 - y)) \bar{w}_{1n}(t, y) dy \right] - 2d_1 \bar{V}_{1n}^2(t) \\
&\quad + 2r_1(t + t_n) \bar{V}_{1n}^2(t) - 2b_1(t + t_n) \bar{V}_{1n}^2(t) (\bar{w}_{1n}(t, z_1) + \bar{w}_{1n}(t, z_2)) \\
&\quad - 2a_1(t + t_n) \bar{w}_{2n}(t, z_1) \bar{V}_{1n}^2(t) - 2a_1(t + t_n) \bar{w}_{1n}(t, z_2) \bar{V}_{1n}(t) \bar{V}_{2n}(t) \\
&\quad + 2\bar{V}_{2n}(t) \left[d_2 \int_{\mathbb{R}} (J_2(z_1 - y) - J_2(z_2 - y)) \bar{w}_{2n}(t, y) dy \right] - 2d_2 \bar{V}_{2n}^2(t) \\
&\quad + 2a_2(t + t_n) [\bar{w}_{1n}(t, z_1) \bar{V}_{2n}^2(t) + \bar{V}_{1n}(t) \bar{V}_{2n}(t) \bar{w}_{2n}(t, z_2)] \\
&\quad - 2r_2(t + t_n) \bar{V}_{2n}^2(t) - 2b_2(t + t_n) \bar{V}_{2n}^2(t) (\bar{w}_{2n}(t, z_1) + \bar{w}_{2n}(t, z_2)) \\
&\leq 2\bar{V}_{1n}(t) \left[d_1 \int_{\mathbb{R}} (J_1(z_1 - y) - J_1(z_2 - y)) \bar{w}_{1n}(t, y) dy \right] - 2d_1 \bar{V}_{1n}^2(t) \\
&\quad + 2r_1(t + t_n) \bar{V}_{1n}^2(t) + a_1(t + t_n) \bar{w}_{1n}(t, z_2) (\bar{V}_{1n}^2(t) + \bar{V}_{2n}^2(t)) - 2r_2(t + t_n) \bar{V}_{2n}^2(t) \\
&\quad + 2\bar{V}_{2n}(t) \left[d_2 \int_{\mathbb{R}} (J_2(z_1 - y) - J_2(z_2 - y)) \bar{w}_{2n}(t, y) dy \right] - 2d_2 \bar{V}_{2n}^2(t) \\
&\quad + 2a_2(t + t_n) \bar{w}_{1n}(t, z_1) \bar{V}_{2n}^2(t) + a_2(t + t_n) \bar{w}_{2n}(t, z_2) (\bar{V}_{1n}^2(t) + \bar{V}_{2n}^2(t)) \\
&\leq 4d_1 M_*^2 \varepsilon + 4d_2 M_*^2 \varepsilon - 2 \left[d_1 - r_{1M} - \frac{1}{2} a_{1M} M_* - \frac{1}{2} a_{2M} M_* \right] \bar{V}_{1n}^2(t) \\
&\quad - 2 \left[d_2 + r_{2m} - \frac{1}{2} a_{1M} M_* - \frac{3}{2} a_{2M} M_* \right] \bar{V}_{2n}^2(t) \\
&\leq \underline{C} \varepsilon - \underline{k} (\bar{V}_{1n}^2(t) + \bar{V}_{2n}^2(t)),
\end{aligned}$$

where $\underline{C} = 4d_1 M_*^2 + 4d_2 M_*^2$ and $\underline{k} = \min\{k_1, k_2\}$. By the variation of constants formula and the comparison argument, we have

$$\bar{V}_{1n}^2(t) + \bar{V}_{2n}^2(t) \leq e^{-\underline{k}(t-s)} (\bar{V}_{1n}^2(s) + \bar{V}_{2n}^2(s)) + \underline{C} \varepsilon \int_s^t e^{-\underline{k}(t-\vartheta)} d\vartheta.$$

Letting $t = 0$ and $s = -t_n$ in the above inequality, we further obtain

$$\bar{V}_{1n}^2(0) + \bar{V}_{2n}^2(0) \leq e^{-\underline{k}t_n} (\bar{V}_{1n}^2(-t_n) + \bar{V}_{2n}^2(-t_n)) + \underline{C} \varepsilon / \underline{k}.$$

It then follows that

$$\sum_{i=1}^2 |\tilde{w}_{in}(t_n, z_1) - \tilde{w}_{in}(t_n, z_2)|^2 \leq \sum_{i=1}^2 |\tilde{w}_{in}(0, z_1) - \tilde{w}_{in}(0, z_2)|^2 + \underline{C} \varepsilon / \underline{k}.$$

In view of $u_{0,n} \in B \subseteq \tilde{X}$, thus $\tilde{w}_{1n}(0, z_1)$ and $\tilde{w}_{2n}(0, z_2)$ are uniformly continuous for $z_1, z_2 \in \mathbb{R}$. Hence, there exists $\bar{\delta}_3 > 0$ such that $|\tilde{w}_{in}(0, z_1) - \tilde{w}_{in}(0, z_2)| \leq \varepsilon^{\frac{1}{2}}$, $i = 1, 2$, provided that $|z_1 - z_2| \leq \bar{\delta}_3$. Thus, for any $z_1, z_2 \in \mathbb{R}$ with $|z_1 - z_2| \leq \bar{\delta} := \min\{\bar{\delta}_1, \bar{\delta}_2, \bar{\delta}_3\}$,

$$|\tilde{w}_{in}(t_n, z_1) - \tilde{w}_{in}(t_n, z_2)| \leq (2 + \underline{C} \varepsilon / \underline{k}) \varepsilon, \quad i = 1, 2.$$

Consequently, $\tilde{\Phi}_T^n$ is asymptotically compact on B , and so is Φ_T^n .

Now we consider the omega limit set of B for the Poincaré map Φ_T ; i.e.,

$$\omega(B) := \{u_0 \in \tilde{X} : \lim_{j \rightarrow +\infty} \Phi_T^{n_j}(u_{0,j}) = u_0 \text{ for some sequences } u_{0,j} \in B \text{ and } n_j \rightarrow +\infty\}.$$

Since Φ_T^n is asymptotically compact on B , it then follows that $\omega(B)$ is a nonempty, compact, and invariant set for Φ_T in $\tilde{X}$, and $\omega(B)$ attracts B (see, e.g., the proof of Sell and You [36, Lemma 23.1 (2)] for continuous-time semiflows). In view of Zhao [53, Lemma 1.1.2 (b)], one has

$$\alpha(\Phi_T^n(B)) \leq \alpha(\omega(B)) + \delta(\Phi_T^n(B), \omega(B)) = \delta(\Phi_T^n(B), \omega(B)) \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Then Φ_T is α -contracting, and thus, Φ_T is asymptotically smooth. Note that Φ_T is uniformly bounded, it follows from Zhao [53, Theorem 1.1.3(b)] (see also Magal and Zhao [28, Lemma 2.1 (b)]), that Φ_T has a global attractor in $\tilde{X}$. This completes the proof. $\square$

We are now ready to give the proofs of Theorems 3.2 (iii) and 3.3 (ii).

Proofs of Theorems 3.2 (iii) and 3.3 (ii). The proofs of these statements can be divided into the following steps.

Step 1. Let

$$P := \tilde{X}, \quad P_0 := \{u_0 \in P : u_{10} \neq 0 \text{ and } u_{20} \neq 0\}, \text{ and } \partial P_0 := P \setminus P_0.$$

By the strong maximum principle of parabolic equations and Lemma 3.5, it is easy to see that for any initial data $u_0 \in P_0$, the solution $w(t, x)$ of (3.17) satisfies $0 < w_1(t, x) \leq M_*$, $0 < w_2(t, x) \leq M_*$ for all $t > 0$ and $x \in \mathbb{R}$. It follows that $\Phi_T^n(P_0) = w(nT, \cdot; P_0) \subset P_0$, $\forall n \in \mathbb{N}$. Let

$$M_\partial := \{u_0 \in \partial P_0 : \Phi_T^n(u_0) \in \partial P_0, \forall n \in \mathbb{N}\}$$

and $\omega(u_0)$ be the omega limit set of the orbit $\gamma^+(u_0) := \{\Phi_T^n(u_0), \forall n \in \mathbb{N}\}$. Now we prove the following claim.

Claim. $\bigcup_{u_0 \in M_\partial} \omega(u_0) \subseteq \{(0, 0), (p(0), 0)\}$.

Indeed, for any given $u_0 \in M_\partial$, we have $\Phi_T^n(u_0) \in \partial P_0$, $\forall n \in \mathbb{N}$, which implies that $w_1(nT, \cdot; u_0) \equiv 0$ or $w_2(nT, \cdot; u_0) \equiv 0$. Moreover, for all $t \geq 0$, we have $w_1(t, \cdot; u_0) \equiv 0$ or $w_2(t, \cdot; u_0) \equiv 0$. If $w_1(t, \cdot; u_0) \equiv 0$ for all $t \geq 0$, then $w_2(t, \cdot; u_0)$ satisfies

$$\begin{cases} (w_2)_t = d_2(J_2 * w_2 - w_2) + c(w_2)_x + w_2(-r_2(t) - b_2(t)w_2), & t > 0, x \in \mathbb{R}, \\ w_2(0, x) = u_{20}(x) \geq 0, & x \in \mathbb{R}. \end{cases}$$

Let $z(x, t)$ be the unique solution of the following initial value problem:

$$\begin{cases} z_t = d_2(J_2 * z - z) + cz_x - r_2(t)z, & t > 0, x \in \mathbb{R}, \\ z(0, x) = u_{20}(x) \geq 0, & x \in \mathbb{R}. \end{cases}$$

By similar discussions in section 2.2, we have

$$z(t, x) = e^{-d_2 t} e^{-\int_0^t r_2(\tau) d\tau} \sum_{k=0}^{\infty} \frac{(d_2 t)^k}{k!} a_k(u_{20})(x + ct).$$

Noting that $a_k(u_{20})(\cdot) \leq M_*$, $\forall k$, we obtain

$$0 \leq z(t, x) \leq M_* e^{-\int_0^t r_2(\tau) d\tau} \leq M_* e^{-r_2 m t} \rightarrow 0 \text{ as } t \rightarrow \infty.$$

It then follows from $0 \leq w_2(t, \cdot) \leq z(t, \cdot)$ that $\lim_{t \rightarrow +\infty} w_2(t, \cdot) = 0$ uniformly for x in any bounded subset of $\mathbb{R}$. If $w_1(\tau_0, \cdot; u_0) \not\equiv 0$ for some $\tau_0 > 0$, then by the strong maximum principle, we can get $w_1(t, \cdot; u_0) > 0$ for all $t > \tau_0$. This implies that $w_2(t, \cdot; u_0) \equiv 0$ for all $t > \tau_0$. Thus, $w_1(t, \cdot; u_0)$ satisfies

$$\begin{cases} (w_1)_t = d_1(J_1 * w_1 - w_1) + c(w_1)_x + w_1(r_1(t) - b_1(t)w_1), & t > \tau_0, x \in \mathbb{R}, \\ w_1(0, x) = u_{10}(x) \geq 0, & x \in \mathbb{R}. \end{cases}$$

It follows that either $\lim_{t \rightarrow \infty} [w_1(t, \cdot) - p(t)] = 0$ or $\lim_{t \rightarrow \infty} w_1(t, \cdot) = 0$ uniformly for x in any bounded subset of $\mathbb{R}$. Hence, the claim holds.

Step 2. We are going to prove that $(0, 0)$ is a uniform weak repeller for P_0 in the sense that there exists a $\delta_1 > 0$ such that

$$\limsup_{n \rightarrow +\infty} \|\Phi_T^n(u_0) - (0, 0)\|_X \geq \delta_1 \text{ for all } u_0 \in P_0.$$

Given any $\epsilon > 0$, by the continuous dependence of $\Phi_t(u_0)$ on the initial value with respect to compact open topology, there exists $\delta_1 > 0$ such that for all $u_0 \in P_0$ with $\|u_0\|_X < \delta_1$, there holds $\|\Phi_t(u_0)\|_X < \epsilon$ for any $t \in [0, T]$.

Suppose, by contradiction, there exists $u_0 \in P_0$ such that $\limsup_{n \rightarrow +\infty} \|\Phi_T^n(u_0)\|_X < \delta_1$. Then there exists $n_0 \geq 1$ such that $\|\Phi_T^n(u_0)\|_X < \delta_1$, $\forall n \geq n_0$. For any $t \geq n_0 T$, let $t = nT + t'$ with $n \geq n_0$ and $t' \in [0, T]$, we have

$$\|\Phi_t(u_0)\|_X = \|\Phi_{t'}(\Phi_T^n(u_0))\|_X < \epsilon.$$

In particular,

$$0 < w_i(t, x; u_0) < \epsilon, \quad t \geq n_0 T, \quad x \in \mathbb{R}, \quad i = 1, 2. \quad (3.18)$$

Take $r_0 = b_{1M} + a_{1M}$. Consequently, it follows that the equation for $w_1(t, x; u_0)$ satisfies

$$\begin{aligned} (w_1)_t &\geq d_1(J_1 * w_1 - w_1) + c(w_1)_x + w_1(r_1(t) - b_1(t)\epsilon - a_1(t)\epsilon) \\ &\geq d_1(J_1 * w_1 - w_1) + c(w_1)_x + w_1[r_1(t) - (b_{1M} + a_{1M})\epsilon] \\ &= d_1(J_1 * w_1 - w_1) + c(w_1)_x + w_1(r_1(t) - r_0\epsilon), \quad t \geq n_0 T, \quad x \in \mathbb{R}. \end{aligned}$$

It views of $u_0 \in P_0$, by the strong maximum principle, we have $w_1(n_0 T, \cdot) > 0$. Since $\lim_{R \rightarrow \infty} \lambda_c(\mathcal{L}_{r_1, \Omega_R}^{d_1, J_1}) = \lambda_c(\mathcal{L}_{r_1, \mathbb{R}}^{d_1, J_1}) < 0$, we can choose $R \gg \max\{\chi_1, \chi_2\} > 0$ such that $\lambda_c(\mathcal{L}_{r_1, \Omega_R}^{d_1, J_1}) < 0$. By (H₇), we can choose $\epsilon > 0$ small enough such that $\lambda_c(\epsilon) := \lambda_c(\mathcal{L}_{r_1 - r_0\epsilon, \Omega_R}^{d_1, J_1}) < 0$. Choose $\tilde{\alpha}_1 > 0$ such that $w_1(n_0 T, x) \geq \tilde{\alpha}_1 e^{-\lambda_c(\epsilon)n_0 T} \tilde{\phi}_1(n_0 T, x)$, $x \in \bar{\Omega}_R$, where $\tilde{\phi}_1(t, \cdot) > 0$ is the generalized principal eigenfunction corresponding to the generalized principal eigenvalue $\lambda_c(\epsilon)$ of eigenvalue problem

$$\begin{cases} \mathcal{L}_{r_1 - r_0\epsilon, \Omega_R}^{d_1, J_1}[\tilde{\phi}_1](t, x) + \lambda_c(\epsilon)\tilde{\phi}_1(t, x) = 0, & t \in \mathbb{R}, x \in \Omega_R, \\ \tilde{\phi}_1(t, x) = \tilde{\phi}_1(t + T, x) > 0, & t \in \mathbb{R}, x \in \bar{\Omega}_R. \end{cases}$$

Define the function $\underline{w}_1(t, x)$ as follows

$$\underline{w}_1(t, x) = \tilde{\alpha}_1 e^{-\lambda_c(\epsilon)t} \tilde{\phi}_1(t, x), \quad t \geq n_0 T, \quad x \in \bar{\Omega}_R.$$

Then $\underline{w}_1(t, x)$ satisfies

$$\begin{cases} (\underline{w}_1)_t = d_1(\int_{\Omega_R} J_1(x-y)\underline{w}_1(t, y)dy - \underline{w}_1(t, x)) \\ \quad + c(\underline{w}_1)_x + \underline{w}_1(r_1(t) - r_0\epsilon), \quad t \geq n_0 T, \quad x \in \Omega_R, \\ \underline{w}_1(n_0 T, x) = \tilde{\alpha}_1 e^{-\lambda_c(\epsilon)n_0 T} \tilde{\phi}_1(n_0 T, x) \leq w_1(n_0 T, x), \quad x \in \bar{\Omega}_R. \end{cases}$$

It then follows from the comparison principle that

$$w_1(t, x) \geq \underline{w}_1(t, x) = \tilde{\alpha}_1 e^{-\lambda_c(\epsilon)t} \tilde{\phi}_1(t, x), \quad \forall t \geq n_0 T, \quad x \in \bar{\Omega}_R.$$

Since $\tilde{\phi}_1(t, x)$ is a positive T -periodic function in t , we have $\lim_{t \rightarrow +\infty} w_1(t, x; u_0) = +\infty$, $\forall x \in \bar{\Omega}_R$. This is a contradiction to (3.18). Hence, we conclude that $(0, 0)$ is a uniform weak repeller and $\{(0, 0)\}$ is an isolated invariant set in P_0 .

Step 3. We are going to prove that $(p(0), 0)$ is a uniformly weak repeller in the sense that there exists a $\delta_2 > 0$ such that

$$\limsup_{n \rightarrow +\infty} \|\Phi_T^n(u_0) - (p(0), 0)\|_X \geq \delta_2 \text{ for all } u_0 \in P_0.$$

Given any $\epsilon > 0$. By the continuous dependence of $\Phi_t(u_0)$ on the initial data with respect to compact open topology, there exists $\delta_2 > 0$ such that for any $u_0 \in P_0$ with $\|u_0 - (p(0), 0)\|_X < \delta_2$, we have $\|\Phi_t(u_0) - \Phi_t(p(0), 0)\|_X < \epsilon$, $\forall t \in [0, T]$.

Suppose, by contradiction, there exists $u_0 \in P_0$ such that $\limsup_{n \rightarrow +\infty} \|\Phi_T^n(u_0) - (p(0), 0)\|_X < \delta_2$. Then there exists $n_0 \geq 1$ such that $\|\Phi_T^n(u_0) - (p(0), 0)\|_X < \delta_2$, $\forall n \geq n_0$. For any $t \geq n_0 T$, let $t = nT + t''$ with $n \geq n_0$ and $t'' \in [0, T]$, we have

$$\|\Phi_t(u_0) - \Phi_t(p(0), 0)\|_X = \|\Phi_{t''}(\Phi_T^n(u_0)) - \Phi_{t''}(p(0), 0)\|_X < \epsilon.$$

Then

$$w_1(t, x; u_0) > p(t) - \epsilon, \quad 0 < w_2(t, x; u_0) < \epsilon, \quad \forall t \geq n_0 T, \quad x \in \mathbb{R}. \quad (3.19)$$

Take $\tilde{r}_2 = b_{2M} + a_{2M}$. Therefore, it follows that the equation for $w_2(t, x; u_0)$ satisfies

$$\begin{aligned} (w_2)_t &\geq d_2(J_2 * w_2 - w_2) + c(w_2)_x + w_2[a_2(t)p(t) - r_2(t) - (b_{2M} + a_{2M})\epsilon] \\ &= d_2(J_2 * w_2 - w_2) + c(w_2)_x + w_2[a_2(t)p(t) - r_2(t) - \tilde{r}_2\epsilon], \quad t \geq n_0 T, \quad x \in \mathbb{R}. \end{aligned}$$

It views of $u_0 \in P_0$, by the maximum principle, we have $w_2(n_0 T, \cdot) > 0$. Since

$$\lim_{R \rightarrow \infty} \lambda_c(\mathcal{L}_{a_2 p - r_2, \Omega_R}^{d_2, J_2}) = \lambda_c(\mathcal{L}_{a_2 p - r_2, \mathbb{R}}^{d_2, J_2}) < 0,$$

we can choose $R \gg \max\{\chi_1, \chi_2\} > 0$ such that $\lambda_c(\mathcal{L}_{a_2 p - r_2, \Omega_R}^{d_2, J_2}) < 0$. Using (H₇), there exists $\epsilon > 0$ such that $\tilde{\lambda}_c(\epsilon) := \lambda_c(\mathcal{L}_{a_2 p - r_2 - \tilde{r}_2 \epsilon, \Omega_R}^{d_2, J_2}) < 0$. Thus, we can take $\tilde{\alpha}_2 > 0$ such that $w_2(n_0 T, x) \geq \tilde{\alpha}_2 e^{-\tilde{\lambda}_c(\epsilon)n_0 T} \tilde{\phi}_2(n_0 T, x)$, $x \in \bar{\Omega}_R$, where $\tilde{\phi}_2(t, \cdot) > 0$ is the generalized principal eigenfunction corresponding to the generalized principal eigenvalue $\tilde{\lambda}_c(\epsilon)$ of the eigenvalue problem

$$\begin{cases} \mathcal{L}_{a_2 p - r_2 - \tilde{r}_2 \epsilon, \Omega_R}^{d_2, J_2}[\tilde{\phi}_2](t, x) + \tilde{\lambda}_c(\epsilon)\tilde{\phi}_2(t, x) = 0, & t \in \mathbb{R}, x \in \Omega_R, \\ \tilde{\phi}_2(t, x) = \tilde{\phi}_2(t+T, x) > 0, & t \in \mathbb{R}, x \in \bar{\Omega}_R. \end{cases}$$

Let $\underline{w}_2(t, x)$ be a function defined as

$$\underline{w}_2(t, x) = \tilde{\alpha}_2 e^{-\tilde{\lambda}_c(\epsilon)t} \tilde{\phi}_2(t, x), \quad t \geq n_0 T, \quad x \in \bar{\Omega}_R.$$

Then $\underline{w}_2(t, x)$ satisfies

$$\begin{cases} (\underline{w}_2)_t = d_2(\int_{\Omega_R} J_2(x-y)\underline{w}_2(t, y)dy - \underline{w}_2(t, x)) + c(\underline{w}_2)_x \\ \quad + \underline{w}_2(a_2(t)p(t) - r_2(t) - \tilde{r}_2\epsilon), \quad t \geq n_0 T, \quad x \in \Omega_R, \\ \underline{w}_2(n_0 T, x) = \tilde{\alpha}_2 e^{-\tilde{\lambda}_c(\epsilon)n_0 T} \tilde{\phi}_2(n_0 T, x) \leq w_2(n_0 T, x), \quad x \in \bar{\Omega}_R. \end{cases}$$

It follows from the comparison principle that

$$w_2(t, x) \geq \underline{w}_2(t, x) = \tilde{\alpha}_2 e^{-\tilde{\lambda}_c(\epsilon)t} \tilde{\phi}_2(t, x), \quad \forall t \geq n_0 T, \quad x \in \bar{\Omega}_R.$$

Since $\tilde{\phi}_2(t, x)$ is a positive T -periodic function in t , we have $\lim_{t \rightarrow +\infty} w_2(t, x; u_0) = +\infty, \forall x \in \bar{\Omega}_R$. This contradicts (3.19). Hence, we conclude that $(p(0), 0)$ is a uniform weak repeller and $\{(p(0), 0)\}$ is an isolated invariant set in P_0 .

Step 4. Completion of the proof. The above results yield that

$$W^s(\{(0, 0)\}) \cap P_0 = \emptyset, \quad W^s(\{(p(0), 0)\}) \cap P_0 = \emptyset,$$

where $W^s(\{(0, 0)\})$ and $W^s(\{(p(0), 0)\})$ are the stable set of $(0, 0)$ and $(p(0), 0)$, respectively. Moreover, it is easy to verify that there are no subsets of $\{(0, 0), (p(0), 0)\}$ forming a cycle in ∂P_0 . Further, since Φ_T admits a global attractor on P , it then follows from Zhao [53, Theorem 1.3.1 and Remark 1.3.1] that $\Phi_T : P \rightarrow P$ is uniformly persistent with respect to $(P_0, \partial P_0)$; that is, there exists a $\hat{\delta} > 0$ such that

$$\liminf_{n \rightarrow +\infty} d(\Phi_T^n(u_0), \partial P_0) \geq \hat{\delta}, \quad \forall u_0 \in P_0.$$

By Zhao [53, Theorem 1.3.6], we obtain that $\Phi_T : P_0 \rightarrow P_0$ has a global attractor A_0 .

Next, we define a continuous function $\tilde{p} : P \rightarrow [0, +\infty)$ by

$$\tilde{p}(u_0) := \min \left\{ \inf_{x \in \mathbb{R}} u_{10}(x), \inf_{x \in \mathbb{R}} u_{20}(x) \right\}, \quad \forall u_0 \in P.$$

Since $A_0 = \Phi_T(A_0)$ (i.e. A_0 is invariant for Φ_T) $\subset \text{int}(P)$, we have $\psi_1(\cdot) > 0, \psi_2(\cdot) > 0$ for all $(\psi_1, \psi_2) \in A_0$. By $\Phi_t(P_0) \subset P_0, \forall t \geq 0$, we have $\cup_{t \in [0, T]} \Phi_t(A_0) \subset P_0$. Note that $\Phi_T(A_0) = A_0, \Phi_t$ is a T -periodic semiflow and $\lim_{t \rightarrow \infty} d(\Phi_t(u_0), \Phi_t(A_0)) = 0, \forall u_0 \in P_0$ (see the proof of Zhao [53, Theorem 3.1.1]), we get $\lim_{t \rightarrow \infty} d(\Phi_t(u_0), \cup_{t \in [0, T]} \Phi_t(A_0)) = 0$. By the continuity of $\Phi_t(u_0)$ for $(t, u_0) \in [0, +\infty) \times P$ and the compactness of $[0, T] \times A_0$, it follows that $\cup_{t \in [0, T]} \Phi_t(A_0)$ is a compact subset of P_0 . Thus, $\min_{u_0 \in \cup_{t \in [0, T]} \Phi_t(A_0)} \tilde{p}(u_0) > 0$. Hence, there exists $\varepsilon_* > 0$ such that for any $u_0 \in P_0$,

$$\liminf_{t \rightarrow +\infty} \min \{w_1(t, x), w_2(t, x)\} = \liminf_{t \rightarrow +\infty} \tilde{p}(\Phi_t(u_0)) \geq \varepsilon_*.$$

Further, there exists $\varrho \in (0, \varepsilon_*)$ such that $\liminf_{t \rightarrow +\infty} w_i(t, x) \geq \varrho$; i.e., $\liminf_{t \rightarrow +\infty} u_i(t, x+ct) \geq \varrho, i = 1, 2$. This completes the proofs of the statement (iii) of Theorem 3.2 and statement (ii) of Theorem 3.3. $\square$

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