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Exploiting delay differential equations solved by Eta functions as suitable mathematical tools for the investigation of thickness controlling in rolling mill

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ABSTRACT

This work is dedicated to introducing the properties and application of Eta functions. We derive the properties of the Eta function, such as the generating function, integral representation, and the Laplace transform. Also, some properties of the Eta-based functions are introduced. To show the advantages of the Eta-based functions in the computational method, we develop a new numerical method to solve the state-dependent and time-dependent neutral delay differential equation based on the Eta-based function. We introduce the operational matrix of derivative for the Eta-base functions to develop the new numerical method. This method uses the operational matrix of derivative and collocation method to convert the delay differential equation to a system of nonlinear algebraic equations. We derive the technique's error bound and establish the method's accuracy by solving some examples, which are state-dependent and time-dependent delay differential equations. In the end, we study the model of the metal forming process by rolling the mill using the new numerical method to show the advantages of using the Eta-based function for solving a more practical problem.

1. Introduction

Eta functions have been introduced by Ixaru [1] for solving the Schrödinger equation. These functions are a powerful tools for deriving the approximation of functions with trigonometric or hyperbolic variation which has oscillatory character (see [2–9] and references therein). The new set of based functions, the Eta-based function, has been introduced using the Eta functions. An essential property of the Eta-based functions is that they tend to the polynomial when the involved frequencies tend to zero. Thus, the Eta-based functions are suitable for attaining a good approximation of high oscillatory functions and polynomials. This property brings the excellent opportunity to approximate the solution of the dynamical systems when we do not know the behavior of the exact solution. Recently, Mashayekhi et al. [10] have used the Eta-based functions to find the least-squares approximation of a function. The paper results show the advantages of using the Eta-based function compared to sets of orthogonal or nonorthogonal functions for finding the least-squares approximation of a high oscillatory function.

Delay differential equations (DDEs) have been used for modeling many phenomena in electrical engineering [11], drilling system [12, 13], ecology [14], biology [15] and robotics [16]. It is well known that it is challenging to solve a delay system analytically. Many researchers have devoted considerable attention to finding numerical methods for solving delay differential equations, especially by Runge–Kutta and spectral methods. Of the numerous papers we mention Runge–Kutta methods [17,18], Legendre spectral method [19,20], Jacobi spectral method [21], hat spectral method [22], Galerkin method [23], modified homotopy perturbation method [24] and Chebyshev cardinal functions [25]. Also, the analysis of the existence of the solution and its properties for the neutral delay differential equations have been studied in [26,27].

To the best of our knowledge, prior analysis of delay differential equations has addressed some, but not all, of the following issues

- Solving the multi-pantograph delay differential equation.
- Solving the state-dependent and time-dependent delay differential equation.
- Developing the method with less CPU time than existing approaches to solving DDEs.
- Developing an accurate method to solve DDEs in a large domain.
- Studying the behavior of DDEs when the exact solution is unknown.

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This paper will introduce some new properties of the Eta-based function to address these issues by solving a general form of DDEs. To solve this, we consider a general form of state-dependent and time-dependent neutral delay equations, and we use Eta-based functions' properties to introduce the derivative's operational matrix. The method first expands the solution of DDES as an Eta-based function with unknown coefficients. The operational matrix of the derivative is then utilized to reduce the solution of DDES to the solution of algebraic equations. We show the accuracy of the method by solving some numerical examples. The results of the numerical examples show the Eta-based functions are much better than the Legendre polynomials when the exact solution of the DDEs is trigonometric functions. Also, in this case, trigonometric and Eta-based functions have the same accuracy. The Eta-based functions are much better than the Legendre polynomials and trigonometric functions when the exact solution of DDEs is an exponential function or has one of the following forms

$$\Delta(t) = \delta_1(t)\sin(\omega t) + \delta_2(t)\cos(\omega t) \text{ or } \Delta(t) = \delta_1(t)\sinh(\omega t) + \delta_2(t)\cosh(\omega t),$$

where ω is a constant and $\delta_1(t)$, $\delta_2(t)$ are polynomials. The results of the numerical models show the ability of the present method to solve all kinds of DDEs with higher accuracy and less CPU time compared to existing approaches. Also, these results demonstrate the power of the technique to solve DDEs in a large domain. It will be seen that the present method can be applied to the model of the metal forming process by the rolling mill. Since the exact solution of the rolling mill thickness control system is unknown, using the Eta-based functions allowed us to consider all possible situations for the exact solution, including trigonometric functions, hyperbolic functions, or polynomials. The details of the work presented in this paper are listed as follows:

Section 2 presents some applicable definitions of generalized hypergeometric functions, Bessel functions, Eta-based functions, and the best approximation. We use these definitions to introduce some properties of Eta functions, including the generating function and integral representation of the Eta function. We also present the Laplace transform of the Eta function based on the Mittag–Leffler function, and we derive the connection between the Eta-based and Bessel functions. We also introduce the operational matrix of the derivative and dual operational matrix of Eta-based functions. These matrices will be used in Section 3 to develop a numerical method for solving DDEs based on the Eta-based functions. Section 3 introduces a new direct computational method for solving the state-dependent and time-dependent neutral delay equation using the Eta-based functions [10]. We use the operational matrix of derivative for the Eta-base functions and collocation method to reduce the delay differential equations to a set of nonlinear equations. Then we solve these nonlinear equations. We derive the error bounds of the numerical method in Section 4. In this section, we show the rate of convergence in the present process depends on the number of Eta-based functions, and the error bounds tend to be zero by increasing the number of Eta-based functions. Numerical examples are presented in Section 5 to demonstrate the efficiency and accuracy of the proposed method. These numerical examples include the state-dependent and time-dependent delay differential equations. In Section 6, to show the application of the Eta-based function for a more realistic example, we use the new numerical method to study the model of the metal forming process by rolling mill [28], which is a state-dependent delay differential equation. In metalworking, moving is a metal forming process in which metal stock is passed through one or more rolls to reduce the thickness and make the thickness uniform. A rolling mill is equipment used for the rolling process of metal, and it can complete the entire process. Roll stands holding pairs of rolls are grouped into rolling mills that can quickly process metal, typically steel, into products such as structural steel, bar stock, and rails. In this paper, using the new numerical method, we study the behavior of the metal forming process by the rolling mill. This study shows the rate of change of the metal thickness depends on desired metal thicknesses after passing the rolling mill, the initial thickness of the metal, and the thickness sensor. In the end, a conclusion is drawn in Section 7.

2. Eta functions and Eta-based functions

In this section, first, we provide some preliminaries helpful in designing our numerical method and error analysis. Then, we introduce some new properties of Eta and Eta-based functions.

2.1. Generalized hypergeometric functions

For real parameters $p_1, \ldots, p_\alpha$ and $q_1, \ldots, q_\beta$ ($q_j \neq 0, -1, -2, \ldots$ $j = 1, \ldots, \beta$), we define the generalized hypergeometric function ${}_\alpha F_\beta(p_1, \ldots, p_\alpha; q_1, \ldots, q_\beta; Y)$ according to Eq. (1) [29]

$${}_\alpha F_\beta(p_1, \ldots, p_\alpha; q_1, \ldots, q_\beta; Y) = \sum_{k=0}^{\infty} \frac{(p_1)_k \ldots (p_\alpha)_k}{(q_1)_k \ldots (q_\beta)_k} \frac{Y^k}{k!}, \tag{1}$$

where $(p)_k$ is the Pochhammer symbol defined, in terms of the Gamma function $\Gamma(.)$, by

$$(p)_0 = 1, \quad (p)_k = p(p+1)(p+2)\ldots(p+k-1) = \Gamma(p+k)/\Gamma(p), \quad k \in \mathbb{N}. \tag{2}$$

If $\alpha \leq \beta$, the series is absolutely convergent for all values of Y, if $\alpha = \beta + 1$, the series is convergent for $|Y| < 1$ and for $|Y| = 1$ the series is conditionally convergent. If $\alpha > \beta + 1$, the series is divergent.

2.2. Bessel functions

The Bessel function of the first kind of real order μ has the series expansion as stated in Eq. (3) [29]:

$$J_\mu(Y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\mu+1)} \left(\frac{Y}{2}\right)^{2k+\mu}. \tag{3}$$

The infinite series in Eq. (3) will converge for all values of Y. The modified Bessel functions of the first kind are defined by Eq. (4) [29] as

$$I_\mu(Y) = i^{-\mu} J_\mu(iY), \tag{4}$$

where $i = \sqrt{-1}$ is the complex unit. It is easy to show the modified Bessel functions of the first kind are a real function of Y.

2.3. Eta functions

Eta functions, denoted by $\eta_n(Y)$, $n > 0$ and $Y \neq 0$, are defined in terms of the recurrence relation (5) [1,2]:

$$\eta_n(Y) = \frac{\eta_{n-2}(Y) - (2n-1)\eta_{n-1}(Y)}{Y}, \qquad n = 1, 2, 3, \ldots \tag{5}$$

where

$$\eta_{-1}(Y) = \begin{cases} \cos(|Y|^{\frac{1}{2}}) & Y \leq 0, \\ \cosh(Y^{\frac{1}{2}}) & Y > 0, \end{cases} \qquad \eta_0(Y) = \begin{cases} \frac{\sin(|Y|^{\frac{1}{2}})}{|Y|^{\frac{1}{2}}} & Y < 0, \\ 1 & Y = 0, \\ \frac{\sinh(Y^{\frac{1}{2}})}{Y^{\frac{1}{2}}} & Y > 0. \end{cases} \tag{6}$$

These functions have the following values at $Y = 0$:

$$\eta_n(0) = \frac{1}{(2n+1)!!}, \qquad n = 1, 2, \ldots \tag{7}$$

where !! is a double factorial [1]. Eta functions have some well-known properties such as:

- **Series expansion:**

$$\begin{aligned} \eta_n(Y) &= 2^n \sum_{k=0}^{\infty} \frac{(k+n)!}{(2k+2n+1)!} \frac{Y^k}{k!} \\ &= 2^{-(n+1)} \sqrt{\pi} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+n+\frac{3}{2})} \frac{(\frac{Y}{4})^k}{k!}, \quad n = 0, 1, \ldots \end{aligned} \tag{8}$$

where for the second equality, we used the Legendre duplication formula according to Eq. (9)

$$\Gamma(2k) = \frac{2^{2k-1}}{\sqrt{\pi}}\Gamma(k)\Gamma(k+\frac{1}{2}), \qquad Re\ k > 0. \tag{9}$$

- **Differentiation properties:**

$$\eta_n'(Y) = \frac{1}{2}\eta_{n+1}(Y), \quad n = -1, 0, 1, 2, \dots \tag{10}$$

- **Generating differential equation:** $\eta_n(Y)$, $(n = 0, 1, 2, \dots)$ is the suitably normalized regular solution of differential equation (11)

$$Yz'' + \frac{1}{2}(2n+3)z' - \frac{1}{4}z = 0. \tag{11}$$

We will derive three more Eta-functions properties in the following theorems, including generating function, integral representation, and Laplace transform.

Theorem 2.1 (*Generating Function*)**.** *The generating function of the Eta functions is obtained according to Eq. (12) as*

$$\sqrt{\frac{\pi}{2}}e^{\frac{t}{2}+\frac{Y}{2t}} \times \frac{1}{\sqrt{t}}Erf\sqrt{\frac{t}{2}} = \sum_{n=0}^{\infty}\eta_n(Y)t^n, \tag{12}$$

where Erf, is an Error function (also called probability integral) as stated in Eq. (13)

$$Erf\ t = \frac{2}{\sqrt{\pi}}\int_0^t e^{-s^2}ds = \frac{2}{\sqrt{\pi}}e^{-t^2}\sum_{\gamma=0}^{\infty}\frac{2^\gamma t^{2\gamma+1}}{(2\gamma+1)!!}. \tag{13}$$

Proof. Using the Taylor series of $e^{\frac{Y}{2t}}$ and Eq. (13) we have

$$\begin{aligned}
&\sqrt{\frac{\pi}{2}}e^{\frac{t}{2}+\frac{Y}{2t}} \times \frac{1}{\sqrt{t}}Erf\sqrt{\frac{t}{2}} = \left(e^{\frac{Y}{2t}}\right) \times \left(\sqrt{\frac{\pi}{2}}e^{\frac{t}{2}} \times \frac{1}{\sqrt{t}}Erf\sqrt{\frac{t}{2}}\right)\\
&= \left(\sum_{\nu=0}^{\infty}\frac{\left(\frac{Y}{2t}\right)^\nu}{\nu!}\right)\left(\sqrt{\frac{\pi}{2}}e^{\frac{t}{2}} \times \frac{1}{\sqrt{t}} \times \frac{2}{\sqrt{\pi}}e^{-\frac{t}{2}}\sum_{\gamma=0}^{\infty}\frac{2^\gamma\left(\frac{t}{2}\right)^{\frac{2\gamma+1}{2}}}{(2\gamma+1)!!}\right)\\
&= \sum_{\nu=0}^{\infty}\frac{\left(\frac{Y}{2t}\right)^\nu}{\nu!}\ \sum_{\gamma=0}^{\infty}\frac{(t)^\gamma}{(2\gamma+1)!!}\\
&= \sum_{\nu=0}^{\infty}\frac{\left(\frac{Y}{2t}\right)^\nu}{\nu!}\ \sum_{\gamma=0}^{\infty}\frac{\gamma!(2t)^\gamma}{(2\gamma+1)!} = \sum_{\nu,\gamma=0}^{\infty}\frac{2^{\gamma-\nu}\gamma!}{(2\gamma+1)!\nu!}Y^\nu t^{\gamma-\nu},
\end{aligned} \tag{14}$$

now we want to pick out the coefficient of t^n in this expansion. For a fixed value of γ the coefficient of t^n is obtained by taking $\gamma - \nu = n$, *i.e.*, $\gamma = \nu + n$. Thus, for this special value of γ in Eq. (14), the coefficient of t^n can be obtained from the following relation

$$\frac{2^n\gamma!}{(2\gamma+1)!(\gamma-n)!}Y^{\gamma-n} = \text{the coefficient of } t^n. \tag{15}$$

The total coefficient of t^n in Eq. (14) is obtained by summing over all allowed values of γ. Since $\nu = \gamma - n$ and $\nu \geq 0$, we should have $\gamma \geq n$ so using Eq. (15), the total coefficient of t^n will be as

$$\sum_{\gamma=n}^{\infty}\frac{2^n\gamma!}{(2\gamma+1)!(\gamma-n)!}Y^{\gamma-n} = \sum_{k=0}^{\infty}\frac{2^n(n+k)!}{(2n+2k+1)!}\frac{Y^k}{k!} = \eta_n(Y), \tag{16}$$

where we have set $k = \gamma - n$.

Theorem 2.2 (*Integral Representation*)**.** *Eta functions of order n can be represented by the integral Eq. (17) as*

$$\eta_n(Y) = \frac{2^{-(n+1)}\sqrt{\pi}}{2\pi i}\int_{-\infty}^{0^+}t^{-(n+\frac{3}{2})}e^{t+\frac{Y}{4t}}dt. \tag{17}$$

Proof. We have

$$\begin{aligned}
\int_{-\infty}^{0^+}t^{-(n+\frac{3}{2})}e^t \times e^{\frac{Y}{4t}}dt &= \int_{-\infty}^{0^+}t^{-(n+\frac{3}{2})}e^t\sum_{k=0}^{\infty}\frac{\left(\frac{Y}{4t}\right)^k}{k!}dt\\
&= \int_{-\infty}^{0^+}t^{-(n+\frac{3}{2})}e^t\sum_{k=0}^{\infty}(4t)^{-k}\frac{Y^k}{k!}dt\\
&= \sum_{k=0}^{\infty}\frac{\left(\frac{Y}{4}\right)^k}{k!}\int_{-\infty}^{0^+}t^{-(k+n+\frac{3}{2})}e^t dt,
\end{aligned} \tag{18}$$

we also recall the Hankel's representation conforming to Eq. (19) [30],

$$\int_{-\infty}^{0^+}t^{-(k+n+\frac{3}{2})}e^t dt = \frac{2\pi i}{\Gamma(k+n+\frac{3}{2})}, \tag{19}$$

substituting Eq. (19) into (18) leads to

$$\int_{-\infty}^{0^+}t^{-(n+\frac{3}{2})}e^t \times e^{\frac{Y}{4t}}dt = \sum_{k=0}^{\infty}\frac{2\pi i}{\Gamma(k+n+\frac{3}{2})}\frac{\left(\frac{Y}{4}\right)^k}{k!}, \tag{20}$$

consequently, from Eqs. (8) and (20), we obtain

$$\begin{aligned}
&\frac{2^{-(n+1)}\sqrt{\pi}}{2\pi i}\int_{-\infty}^{0^+}t^{-(n+\frac{3}{2})}e^{t+\frac{Y}{4t}}dt\\
&= 2^{-(n+1)}\sqrt{\pi}\sum_{k=0}^{\infty}\frac{1}{\Gamma(k+n+\frac{3}{2})}\frac{\left(\frac{Y}{4}\right)^k}{k!} = \eta_n(Y).
\end{aligned} \tag{21}$$

Theorem 2.3 (*Laplace Transform*)**.** *The Laplace transform of Eta functions is expressed following Eq. (22)*

$$\mathcal{L}\{\eta_n(Y);s\} = \frac{2^{-(n+1)}\sqrt{\pi}}{s}E_{1,n+\frac{3}{2}}\left(\frac{1}{4s}\right). \tag{22}$$

Proof. Using Eq. (8) we have

$$\begin{aligned}
\mathcal{L}\{\eta_n(Y);s\} &= \int_0^\infty e^{-sY}\eta_n(Y)dY\\
&= \int_0^\infty e^{-sY} \times 2^{-(n+1)}\sqrt{\pi}\sum_{k=0}^{\infty}\frac{(\frac{Y}{4})^k}{k!\Gamma(k+n+\frac{3}{2})}dY\\
&= 2^{-(n+1)}\sqrt{\pi}\sum_{k=0}^{\infty}\frac{1}{4^k k!\Gamma(k+n+\frac{3}{2})}\int_0^\infty e^{-sY}Y^k dY\\
&= \frac{2^{-(n+1)}\sqrt{\pi}}{s}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4s}\right)^k}{\Gamma(k+n+\frac{3}{2})} = \frac{2^{-(n+1)}\sqrt{\pi}}{s}E_{1,n+\frac{3}{2}}\left(\frac{1}{4s}\right),
\end{aligned} \tag{23}$$

where

$$E_{\alpha,\beta}(Y) = \sum_{k=0}^{\infty}\frac{Y^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0,\ \beta \in \mathbb{C}, \tag{24}$$

is the generalized Mittag–Leffler function [29].

2.4. Eta-based functions

Eta-based functions are defined according to Eq. (25) as

$$\varphi_n(t) = t^{n-1}\eta_{\lfloor\frac{n}{2}\rfloor-1}(Y(t)), \ n = 1, 2, \dots \tag{25}$$

where $\lfloor\frac{n}{2}\rfloor$ is the integer part of $\frac{n}{2}$, and $Y(t) = -\xi^2t^2$ in the trigonometric case and $Y(t) = \xi^2t^2$ in the hyperbolic case. These functions have the following properties

$$\varphi_{n+1}(t) = t\varphi_n(t), \quad \text{for even number } n \geq 2. \tag{26}$$

$$\varphi_{n+2}(t) = \frac{t\varphi_{n-1}(t) - (n-1)\varphi_n(t)}{\mp\xi^2}, \quad \text{for even number } n \geq 2, \quad \xi \neq 0, \tag{27}$$

where the upper/lower sign is for oscillatory/hyperbolic case.

An essential property of the Eta-based functions is that they tend to the classical power function (or polynomial) when $\xi = 0$ [2]. In the following theorem, the relation between Eta-based functions and the Bessel functions is presented.

Theorem 2.4. *The Eta-based functions $\varphi_n(t)$ can be defined by the Bessel functions as stated in Eq.* (28)

$$\varphi_n(t) = \begin{cases} \sqrt{\frac{\pi}{2}}\xi^{\frac{1}{2}-\lfloor\frac{n}{2}\rfloor} t^{n-\lfloor\frac{n}{2}\rfloor-\frac{1}{2}} J_{\lfloor\frac{n}{2}\rfloor-\frac{1}{2}}(\xi t), & Y = -\xi^2 t^2, \\ \sqrt{\frac{\pi}{2}}\xi^{\frac{1}{2}-\lfloor\frac{n}{2}\rfloor} t^{n-\lfloor\frac{n}{2}\rfloor-\frac{1}{2}} I_{\lfloor\frac{n}{2}\rfloor-\frac{1}{2}}(\xi t), & Y = \xi^2 t^2. \end{cases} \tag{28}$$

Proof. From Eqs. (8), (25) and using the Legendre duplication formula (9), we have

$$\varphi_n(t) = 2^{\lfloor\frac{n}{2}\rfloor-1} t^{n-1} \sum_{k=0}^{\infty} \frac{\Gamma(k+\lfloor\frac{n}{2}\rfloor)}{\Gamma(2k+2\lfloor\frac{n}{2}\rfloor)} \frac{Y^k}{k!}$$
$$= 2^{\lfloor\frac{n}{2}\rfloor-1} t^{n-1} \sum_{k=0}^{\infty} \frac{\sqrt{\pi}2^{1-2k-2\lfloor\frac{n}{2}\rfloor}}{\Gamma(k+\lfloor\frac{n}{2}\rfloor+\frac{1}{2})} \frac{Y^k}{k!}$$

$$= \begin{cases} \sqrt{\frac{\pi}{2}} t^{n-\lfloor\frac{n}{2}\rfloor-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-\xi^2)^k}{\Gamma(k+\lfloor\frac{n}{2}\rfloor+\frac{1}{2})} \frac{(\frac{t}{2})^{2k+\lfloor\frac{n}{2}\rfloor-\frac{1}{2}}}{k!} \\ \quad = \sqrt{\frac{\pi}{2}}\xi^{\frac{1}{2}-\lfloor\frac{n}{2}\rfloor} t^{n-\lfloor\frac{n}{2}\rfloor-\frac{1}{2}} J_{\lfloor\frac{n}{2}\rfloor-\frac{1}{2}}(\xi t), \quad Y = -\xi^2 t^2, \\ \sqrt{\frac{\pi}{2}} t^{n-\lfloor\frac{n}{2}\rfloor-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(\xi^2)^k}{\Gamma(k+\lfloor\frac{n}{2}\rfloor+\frac{1}{2})} \frac{(\frac{t}{2})^{2k+\lfloor\frac{n}{2}\rfloor-\frac{1}{2}}}{k!} \\ \quad = \sqrt{\frac{\pi}{2}}\xi^{\frac{1}{2}-\lfloor\frac{n}{2}\rfloor} t^{n-\lfloor\frac{n}{2}\rfloor-\frac{1}{2}} I_{\lfloor\frac{n}{2}\rfloor-\frac{1}{2}}(\xi t), \quad Y = \xi^2 t^2. \end{cases} \tag{29}$$

In the following theorem, we derive the product of two Eta-based functions.

Theorem 2.5. *For $\xi \neq 0$, the product of two Eta-based functions $\varphi_n(t)\varphi_m(t)$ can be obtained as reported by Eq.* (30)

$$\begin{aligned} \varphi_n(t)\varphi_m(t) &= \pi 2^{-\left(\lfloor\frac{n}{2}\rfloor+\lfloor\frac{m}{2}\rfloor\right)} t^{n+m-2} \sum_{k=0}^{\infty} \frac{1}{(k+\lfloor\frac{n}{2}\rfloor-\frac{1}{2})!(k+\lfloor\frac{m}{2}\rfloor-\frac{1}{2})!} \\ &\quad \times \binom{2k+\lfloor\frac{n}{2}\rfloor+\lfloor\frac{m}{2}\rfloor-1}{k} \left(\frac{\mp\xi^2 t^2}{4}\right)^k \\ &= \pi 2^{-\left(\lfloor\frac{n}{2}\rfloor+\lfloor\frac{m}{2}\rfloor\right)} t^{n+m-2} \\ &\quad \times \frac{{}_2F_3\left(\frac{1}{2}\lfloor\frac{n}{2}\rfloor+\frac{1}{2}\lfloor\frac{m}{2}\rfloor, \frac{1}{2}+\frac{1}{2}\lfloor\frac{n}{2}\rfloor+\frac{1}{2}\lfloor\frac{m}{2}\rfloor; \frac{1}{2}+\lfloor\frac{m}{2}\rfloor, \frac{1}{2}+\lfloor\frac{n}{2}\rfloor, \lfloor\frac{n}{2}\rfloor+\lfloor\frac{m}{2}\rfloor; \mp\xi^2 t^2\right)}{\Gamma(\lfloor\frac{n}{2}\rfloor+\frac{1}{2})\Gamma(\lfloor\frac{m}{2}\rfloor+\frac{1}{2})}. \end{aligned} \tag{30}$$

Proof. Using the definition of Eta-based functions and Cauchy's rule for multiplication of power series, we have

$$\varphi_n(t)\varphi_m(t) = \pi \sum_{k=0}^{\infty} \frac{2^{-2k-\lfloor\frac{n}{2}\rfloor}(\mp\xi^2)^k}{\Gamma(k+\lfloor\frac{n}{2}\rfloor+\frac{1}{2})} \frac{t^{2k+n-1}}{k!} \sum_{k=0}^{\infty} \frac{2^{-2k-\lfloor\frac{m}{2}\rfloor}(\mp\xi^2)^k}{\Gamma(k+\lfloor\frac{m}{2}\rfloor+\frac{1}{2})} \frac{t^{2k+m-1}}{k!}$$

$$\begin{aligned} &= \pi 2^{-\left(\lfloor\frac{n}{2}\rfloor+\lfloor\frac{m}{2}\rfloor\right)} t^{n+m-2} \\ &\quad \times \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{1}{(j+\lfloor\frac{n}{2}\rfloor-\frac{1}{2})!\, j!\, (k-j+\lfloor\frac{m}{2}\rfloor-\frac{1}{2})!\, (k-j)!} \left(\frac{\mp\xi^2 t^2}{4}\right)^k \\ &= \pi 2^{-\left(\lfloor\frac{n}{2}\rfloor+\lfloor\frac{m}{2}\rfloor\right)} t^{n+m-2} \\ &\quad \times \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{1}{(k+\left\lfloor\frac{n}{2}\right\rfloor-\frac{1}{2})!(k+\lfloor\frac{m}{2}\rfloor-\frac{1}{2})!} \binom{k+\lfloor\frac{n}{2}\rfloor-\frac{1}{2}}{k-j} \binom{k+\lfloor\frac{m}{2}\rfloor-\frac{1}{2}}{j} \left(\frac{\mp\xi^2 t^2}{4}\right)^k \end{aligned}$$

$$\begin{aligned} &= \pi 2^{-\left(\lfloor\frac{n}{2}\rfloor+\lfloor\frac{m}{2}\rfloor\right)} t^{n+m-2} \sum_{k=0}^{\infty} \frac{1}{(k+\lfloor\frac{n}{2}\rfloor-\frac{1}{2})!(k+\lfloor\frac{m}{2}\rfloor-\frac{1}{2})!} \\ &\quad \times \binom{2k+\lfloor\frac{n}{2}\rfloor+\lfloor\frac{m}{2}\rfloor-1}{k} \left(\frac{\mp\xi^2 t^2}{4}\right)^k. \end{aligned} \tag{31}$$

Using the hypergeometric series, Eq. (1), and Legendre duplication formula, Eq. (9), and Eq. (31) we have

$$\begin{aligned} \varphi_n(t)\varphi_m(t) &= \pi 2^{-\left(\lfloor\frac{n}{2}\rfloor+\lfloor\frac{m}{2}\rfloor\right)} t^{n+m-2} \sum_{k=0}^{\infty} \frac{1}{(k+\lfloor\frac{n}{2}\rfloor-\frac{1}{2})!(k+\lfloor\frac{m}{2}\rfloor-\frac{1}{2})!} \\ &\quad \times \binom{2k+\lfloor\frac{n}{2}\rfloor+\lfloor\frac{m}{2}\rfloor-1}{k} \left(\frac{\mp\xi^2 t^2}{4}\right)^k \\ &= \pi 2^{-\left(\lfloor\frac{n}{2}\rfloor+\lfloor\frac{m}{2}\rfloor\right)} t^{n+m-2} \\ &\quad \times \sum_{k=0}^{\infty} \frac{\Gamma(2k+\lfloor\frac{n}{2}\rfloor+\lfloor\frac{m}{2}\rfloor)}{\Gamma(k+\lfloor\frac{n}{2}\rfloor+\frac{1}{2})\Gamma(k+\lfloor\frac{m}{2}\rfloor+\frac{1}{2})\Gamma(k+\lfloor\frac{n}{2}\rfloor+\lfloor\frac{m}{2}\rfloor)} \frac{\left(\frac{\mp\xi^2 t^2}{4}\right)^k}{k!} \\ &= \pi 2^{-\left(\lfloor\frac{n}{2}\rfloor+\lfloor\frac{m}{2}\rfloor\right)} t^{n+m-2} \\ &\quad \times \frac{{}_2F_3\left(\frac{1}{2}\lfloor\frac{n}{2}\rfloor+\frac{1}{2}\lfloor\frac{m}{2}\rfloor, \frac{1}{2}+\frac{1}{2}\lfloor\frac{n}{2}\rfloor+\frac{1}{2}\lfloor\frac{m}{2}\rfloor; \frac{1}{2}+\lfloor\frac{m}{2}\rfloor, \frac{1}{2}+\lfloor\frac{n}{2}\rfloor, \lfloor\frac{n}{2}\rfloor+\lfloor\frac{m}{2}\rfloor; \mp\xi^2 t^2\right)}{\Gamma(\lfloor\frac{n}{2}\rfloor+\frac{1}{2})\Gamma(\lfloor\frac{m}{2}\rfloor+\frac{1}{2})}, \end{aligned} \tag{32}$$

this completes the proof.

2.5. Best approximation and operational matrices

Suppose $f(t) \in L^2[0,1]$ and

$$f_N^H(t) = H^T A = a_1 h_1(t) + a_2 h_2(t) + \cdots + a_N h_N(t), \tag{33}$$

is the best approximation to f out of H where

$$H(t) = [h_1(t), h_2(t), \ldots, h_N(t)]^T, \qquad A = [a_1, a_2, \ldots, a_N]^T, \tag{34}$$

are the base functions and coefficients vector. We have two next theorems if we choose the Eta-based functions as basis functions in Eq. (33).

Theorem 2.6 (*Operational Matrix of Derivative*)**.** *The derivative of the $H(t) = [\varphi_1(t), \varphi_2(t), \ldots, \varphi_N(t)]^T$ where $\varphi_i(t)$ defined in Eq.* (25) *satisfies the following relation*

$$H'(t) = D(t)H(t), \tag{35}$$

where $D(t) = [d_{ij}]_{N\times N}$ is the operational matrix of derivative.

Proof. We present the proof for two cases, $\xi = 0$ and $\xi \neq 0$. First we consider $\xi \neq 0$. From Eqs. (10) and (25) we have

$$\varphi'_n(t) = (n-1)t^{n-2}\eta_{\lfloor\frac{n}{2}\rfloor-1}(Y(t)) \mp \xi^2 t^n \eta_{\lfloor\frac{n}{2}\rfloor}(Y(t)), \tag{36}$$

using Eqs. (26), (27) and Eq. (36), we have

$$\varphi'_n(t) = \begin{cases} \mp\xi^2\varphi_{n+1} & n = 1, \\ \varphi_{n-1}(t) & n \text{ is even}, \\ t\varphi_{n-2}(t) + \varphi_{n-1}(t) & n \text{ is odd}. \end{cases} \tag{37}$$

Using Eq. (37) we have

$$D(t) = \begin{cases} D_1(t) & \text{If } N \text{ is even}, \\ D_2(t) & \text{If } N \text{ is odd}, \end{cases} \tag{38}$$

where

$$D_1(t) = \begin{bmatrix} 0 & \mp\xi^2 & & & & & & \\ 1 & 0 & 0 & & & & & \\ t & 1 & 0 & 0 & & & & \\ & 0 & 1 & 0 & 0 & & & \\ & & t & 1 & 0 & 0 & & \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & t & 1 & 0 & 0 \\ & & & & & 0 & 1 & 0 \end{bmatrix},$$

$$D_2(t) = \begin{bmatrix} 0 & \mp\xi^2 & & & & & & \\ 1 & 0 & 0 & & & & & \\ t & 1 & 0 & 0 & & & & \\ & 0 & 1 & 0 & 0 & & & \\ & & t & 1 & 0 & 0 & & \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & 0 & 1 & 0 & 0 \\ & & & & & t & 1 & 0 \end{bmatrix}.$$

Now let to consider $\xi = 0$. Using Eqs. (7) and (25), the derivative of $\varphi_n(t)$ is obtained conforming to Eq. (39)

$$\varphi_n'(t) = \begin{cases} 0 & n = 1, \\ \varphi_{n-1}(t) & n \text{ is even}, \\ (n-1)\varphi_{n-1}(t) & n \text{ is odd}. \end{cases} \tag{39}$$

Using Eq. (39) we have

$$D(t) = \begin{cases} D_3 & \text{If } N \text{ is even}, \\ D_4 & \text{If } N \text{ is odd}, \end{cases} \tag{40}$$

where

$$D_3 = \begin{bmatrix} 0 & 0 & & & & & & & \\ 1 & 0 & 0 & & & & & & \\ & 2 & 0 & 0 & & & & & \\ & & 1 & 0 & 0 & & & & \\ & & & 4 & 0 & 0 & & & \\ & & & & \ddots & \ddots & \ddots & & \\ & & & & & 1 & 0 & 0 & \\ & & & & & & N-2 & 0 & 0 \\ & & & & & & & 1 & 0 \end{bmatrix},$$

$$D_4 = \begin{bmatrix} 0 & 0 & & & & & & & \\ 1 & 0 & 0 & & & & & & \\ & 2 & 0 & 0 & & & & & \\ & & 1 & 0 & 0 & & & & \\ & & & 4 & 0 & 0 & & & \\ & & & & \ddots & \ddots & \ddots & & \\ & & & & & N-3 & 0 & 0 & \\ & & & & & & 1 & 0 & 0 \\ & & & & & & & N-1 & 0 \end{bmatrix}.$$

As you can see, in this case, $D(t)$ does not depend on t.

Theorem 2.7 (*Dual Operational Matrix*). *The dual operational matrix of the* $H(t) = [\varphi_1(t), \varphi_2(t), \dots, \varphi_N(t)]^T$ *can be obtained according to Eq. (41) as*

$$\int_0^1 H(t)H^T(t)dt = Q_H, \tag{41}$$

where Q_H *is the* $N \times N$ *dual operational matrix and*

$$Q_H = \begin{bmatrix} \phi(1,1) & \phi(1,2) & \cdots & \phi(1,N) \\ \phi(2,1) & \phi(2,2) & \cdots & \phi(2,N) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(N,1) & \phi(N,2) & \cdots & \phi(N,N) \end{bmatrix}, \tag{42}$$

in which

$$\phi(n,m) = \begin{cases} \dfrac{\pi 2^{-\left(\lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor\right)}}{\Gamma(\lfloor \frac{n}{2} \rfloor + \frac{1}{2})\Gamma(\lfloor \frac{m}{2} \rfloor + \frac{1}{2})} \times \displaystyle\sum_{k=0}^{\infty} \frac{(\mp\xi^2)^k (\frac{1}{2}\lfloor \frac{n}{2} \rfloor + \frac{1}{2}\lfloor \frac{m}{2} \rfloor)_k (\frac{1}{2} + \frac{1}{2}\lfloor \frac{n}{2} \rfloor + \frac{1}{2}\lfloor \frac{m}{2} \rfloor)_k (\frac{n+m-1}{2})_k}{(n+m-1)k!(\frac{1}{2} + \lfloor \frac{m}{2} \rfloor)_k (\frac{1}{2} + \lfloor \frac{n}{2} \rfloor)_k (\lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor)_k (\frac{n+m+1}{2})_k} & \text{If } \xi \neq 0, \\ \dfrac{1}{(n+m-1)!(2\lfloor \frac{m}{2} \rfloor - 1)!!(2\lfloor \frac{n}{2} \rfloor - 1)!!} & \text{If } \xi = 0. \end{cases} \tag{43}$$

Proof. Since

$$\int_0^1 H(t)H^T(t)dt = \begin{bmatrix} \int_0^1 \varphi_1(t)\varphi_1(t)dt & \int_0^1 \varphi_1(t)\varphi_2(t)dt & \cdots & \int_0^1 \varphi_1(t)\varphi_N(t)dt \\ \int_0^1 \varphi_2(t)\varphi_1(t)dt & \int_0^1 \varphi_2(t)\varphi_2(t)dt & \cdots & \int_0^1 \varphi_2(t)\varphi_N(t)dt \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^1 \varphi_N(t)\varphi_1(t)dt & \int_0^1 \varphi_N(t)\varphi_2(t)dt & \cdots & \int_0^1 \varphi_N(t)\varphi_N(t)dt \end{bmatrix},$$

we have

$$\phi(i,j) = \int_0^1 \varphi_i(t)\varphi_j(t)dt, \tag{44}$$

by integrating of the product of two Eta-based functions given in Eq. (30) on $[0,1]$, the result is obtained directly for $\xi \neq 0$. For $\xi = 0$, the result can be obtained by using Eqs. (7) and (25).

3. State-dependent and time-dependent neutral delay equation

In this section, we use the Eta-based function to develop the new numerical method for the state-dependent and time-dependent neutral delay equation as stated in Eq. (45)

$$\begin{cases} x'(t) = g(t, x(t), x(t - \Theta_1(t, x(t))), x'(t - \Theta_2(t, x(t)))), \\ x(0) = x_0, \qquad 0 \le t \le 1. \end{cases} \tag{45}$$

In Eq. (45),

$$x(t) = [x_1(t), x_2(t), \dots, x_\rho(t)]^T \in \mathbb{R}^\rho, \tag{46}$$

is a real-valued ρ-vector function and

$$g(t) = [g_1(t), g_2(t), \dots, g_\rho(t)]^T, \tag{47}$$

is assumed to be a sufficiently smooth real-valued ρ-vector function. Also, Θ_1, Θ_2 are assumed to be continuous functions for all $t \in [0,1]$.

3.1. Numerical method

This section is devoted to presenting a new numerical method for solving the problem given in Eq. (45). Using Eq. (33) the best approximation of $x_i(t)$, $i = 1, 2, \dots, \rho$ is

$$x_i(t) = H^T(t)A_i, \tag{48}$$

and

$$x(t) = \hat{H}(t)\hat{A}, \tag{49}$$

where $\hat{A}$ is a $\rho N \times 1$ vector given by

$$\hat{A} = [A_1, A_2, \dots, A_\rho]^T, \tag{50}$$

and

$$\hat{H}(t) = I_\rho \otimes H^T(t), \tag{51}$$

in which I_ρ is the ρ dimensional identity matrix, $\hat{H}(t)$ is $\rho \times \rho N$ matrix as well, and $\otimes$ denotes Kronecker product [31].

Using Eq. (48) and Theorem 2.6, we have

$$x_i'(t) = H^T(t)D^T(t)A_i. \tag{52}$$

Using Eqs. (46) and (48) we get

$$x'(t) = \hat{H}(t)\hat{D}(t)\hat{A}, \tag{53}$$

where $\hat{D}(t)$ is $\rho N \times \rho N$ matrix as

$$\hat{D}(t) = I_\rho \otimes D^T(t).$$

Substituting Eqs. (49) and (53) into (45), we have

$$\hat{H}(t)\hat{D}(t)\hat{A} = g(t, \hat{H}(t)\hat{A}, \hat{H}(t-\Theta_1(t,\hat{H}(t)\hat{A}))\hat{A}, \hat{H}(t-\Theta_2(t,\hat{H}(t)\hat{A}))\hat{D}(t-\Theta_2(t,\hat{H}(t)\hat{A}))\hat{A}). \tag{54}$$

Next we collocate Eq. (54) at the Chebyshev nodes (see [32]) in [0, 1]

$$t_j = \frac{1}{2}\cos\frac{\pi(2j+1)}{2(N+1)} + \frac{1}{2}, \quad j = 0,1,\ldots,N-1, \tag{55}$$

to obtain a system of ρN nonlinear equations as

$$W = \hat{H}(t_j)\hat{D}(t_j)\hat{A} - g(t_j, \hat{H}(t_j)\hat{A}, \hat{H}(t_j-\Theta_1(t_j,\hat{H}(t_j)\hat{A}))\hat{A}, \hat{H}(t_j - \Theta_2(t_j,\hat{H}(t_j)\hat{A})) \times \hat{D}(t_j-\Theta_2(t_j,\hat{H}(t_j)\hat{A}))\hat{A}) = 0. \tag{56}$$

Similar to Eq. (49), corresponding matrix form for the initial condition $x(0) = x_0$ is according to Eq. (57)

$$V = \hat{H}(0)\hat{A} - x_0 = 0. \tag{57}$$

Replacing V instead of the ρ last row of W, we have a set of ρN nonlinear equations which can be solved for the elements of $\hat{A}$ using the well Newton's iterative method. Finally, we calculate $x(t)$ given in Eq. (49).

4. Error estimate

This section aims to estimate the error norm for the numerical method presented in Section 3.1. For ease of exposition but without any loss of generality, we describe convergence analysis for $\rho = 1$ and $x_1 = x$. At first, we suppose that $H^\mu(0,1)$ with $\mu \geq 0$ is a Sobolev space equipped with the norm confirming to Eq. (58)

$$\| x \|_{H^\mu(0,1)} = \left(\sum_{j=0}^{\mu}\int_0^1 |x^{(j)}(t)|^2 w(t)dt\right)^{\frac{1}{2}} = \left(\sum_{j=0}^{\mu}\| x^{(j)} \|^2_{L^2(0,1)}\right)^{\frac{1}{2}}. \tag{58}$$

To continue the error discussion, the following Theorem from [33] is recalled.

Theorem 4.1. *Assume that x be a member of Sobolev space $H^\mu(0,1)$ with $\mu \geq 0$, and $P_n(2t-1)$ be the well-known shifted Legendre polynomials defined on the interval* [0, 1]. *Let*

$$\sum_{n=0}^{N} \mathbf{a}_n P_n(2t-1) \in \Pi_N, \tag{59}$$

denotes the best approximation of x using the set of shifted Legendre polynomials, where Π_N is the space of all polynomials of degree less than or equal to N. Then we have

$$\left\| x - \sum_{n=0}^{N}\mathbf{a}_n P_n(2t-1)\right\|_{L^2(0,1)} \leq cN^{-\mu}|x|_{H^{\mu;N}(0,1)}, \tag{60}$$

where c is a constant positive independent of N and x and

$$|x|_{H^{\mu;N}(0,1)} = \left(\sum_{i=\min\{\mu,N+1\}}^{\mu}\left\|x^{(i)}\right\|_2^2\right)^{\frac{1}{2}}. \tag{61}$$

Theorem 4.2. *Suppose that x be a member of Sobolev space $H^\mu(0,1)$ with $\mu \geq 0$, and φ_n be the Eta-based functions defined on the interval* [0, 1]. *Assume that $x_N(t) = \sum_{n=1}^N a_n\varphi_n(t)$ denotes the approximation of x using the set of Eta-based functions. Then we have*

$$\left\| x - \sum_{n=1}^{N} a_n\varphi_n(t)\right\|_{L^2(0,1)} \leq cN^{-\mu}|x|_{H^{\mu;N}(0,1)} + \sum_{n=1}^{N}\sqrt{\pi}2^{-\lfloor\frac{n}{2}\rfloor}\varepsilon|a_n|. \tag{62}$$

Proof. Using Eqs. (8) and (25) we have

$$\begin{aligned}
\|x(t)-x_N(t)\|_{L^2(0,1)} &= \left\|x - \sum_{n=1}^N a_n\varphi_n(t)\right\|_{L^2(0,1)} \\
&= \left\|x(t) - \sum_{n=1}^N a_n\sqrt{\pi}2^{-\lfloor\frac{n}{2}\rfloor}\sum_{k=0}^{\infty}\frac{(\mp\xi^2)^k}{\Gamma\left(k+\lfloor\frac{n}{2}\rfloor+\frac{1}{2}\right)}\frac{t^{2k+n-1}}{4^k k!}\right\|_{L^2(0,1)} \\
&= \left\|x(t) - \sum_{n=1}^N a_n\sqrt{\pi}2^{-\lfloor\frac{n}{2}\rfloor}\left(\sum_{k=0}^{M}\frac{(\mp\xi^2)^k}{\Gamma\left(k+\lfloor\frac{n}{2}\rfloor+\frac{1}{2}\right)}\frac{t^{2k+n-1}}{4^k k!} + \sum_{k=M+1}^{\infty}\frac{(\mp\xi^2)^k}{\Gamma\left(k+\lfloor\frac{n}{2}\rfloor+\frac{1}{2}\right)}\frac{t^{2k+n-1}}{4^k k!}\right)\right\|_{L^2(0,1)} \\
&\leq \left\|x(t) - \sum_{n=1}^N a_n\sqrt{\pi}2^{-\lfloor\frac{n}{2}\rfloor}\sum_{k=0}^{M}\frac{(\mp\xi^2)^k}{\Gamma\left(k+\lfloor\frac{n}{2}\rfloor+\frac{1}{2}\right)}\frac{t^{2k+n-1}}{4^k k!}\right\|_{L^2(0,1)} \\
&+ \left\|\sum_{n=1}^N a_n\sqrt{\pi}2^{-\lfloor\frac{n}{2}\rfloor}\sum_{k=M+1}^{\infty}\frac{(\mp\xi^2)^k}{\Gamma\left(k+\lfloor\frac{n}{2}\rfloor+\frac{1}{2}\right)}\frac{t^{2k+n-1}}{4^k k!}\right\|_{L^2(0,1)}.
\end{aligned} \tag{63}$$

Since the best approximation of a given function $x \in H^\mu(0,1)$ is unique (For more details about the uniqueness of the best approximation, please see [34] page 334), using Theorem 4.1, we have

$$\begin{aligned}
\left\|x(t) - \sum_{n=1}^N\sum_{k=0}^M \frac{\sqrt{\pi}a_n(\pm\xi^2)^k}{2^{\lfloor\frac{n}{2}\rfloor}\Gamma\left(k+\lfloor\frac{n}{2}\rfloor+\frac{1}{2}\right)}\frac{t^{2k+n-1}}{k!}\right\|_{L^2(0,1)}
&= \left\|x(t) - \sum_{n=0}^{N-1} a_n P_n(2t-1)\right\|_{L^2(0,1)} \\
&\leq cN^{-\mu}|x|_{H^{\mu;N}(0,1)}.
\end{aligned} \tag{64}$$

Also, we know that the series $\sum_{k=0}^{\infty}\frac{(\mp\xi^2)^k}{\Gamma\left(k+\lfloor\frac{n}{2}\rfloor+\frac{1}{2}\right)}\frac{t^{2k+n-1}}{4^k k!}$ is convergent so we get

$$\sum_{k=M+1}^{\infty}\frac{(\mp\xi^2)^k}{\Gamma\left(k+\lfloor\frac{n}{2}\rfloor+\frac{1}{2}\right)}\frac{t^{2k+n-1}}{4^k k!} < \varepsilon, \tag{65}$$

using Eq. (65) for all values of $|t| < 1$ we have

$$\begin{aligned}
&\left\|\sum_{n=1}^N a_n\sqrt{\pi}2^{-\lfloor\frac{n}{2}\rfloor}\sum_{k=M+1}^{\infty}\frac{(\mp\xi^2)^k}{\Gamma\left(k+\lfloor\frac{n}{2}\rfloor+\frac{1}{2}\right)}\frac{t^{2k+n-1}}{4^k k!}\right\|_{L^2(0,1)} \leq \\
&\sum_{n=1}^N\left(\sqrt{\pi}2^{-\lfloor\frac{n}{2}\rfloor}|a_n|\left\|\sum_{k=M+1}^{\infty}\frac{(\mp\xi^2)^k}{\Gamma\left(k+\lfloor\frac{n}{2}\rfloor+\frac{1}{2}\right)}\frac{t^{2k+n-1}}{4^k k!}\right\|_{L^2(0,1)}\right) \leq \\
&\sum_{n=1}^N\sqrt{\pi}2^{-\lfloor\frac{n}{2}\rfloor}\varepsilon|a_n|.
\end{aligned} \tag{66}$$

Using Eqs. (63), (64) and (66), we obtain the error estimate $x - x_N$ and this completes the proof.

In the following Theorem, we obtain the error bound of the numerical method presented in Section 3.1.

Theorem 4.3. *Let $x \in H^\mu(0,1)$ be the exact solution of Eq. (45) and $\bar{x}_N = H^T\bar{A} = \sum_{n=1}^N \bar{a}_n\varphi_n(t)$ be the approximate solution of this equation obtained by the method proposed in Section 3.1. Then, we have*

$$\begin{aligned}
\|x-\bar{x}_N\|_{L^2(0,1)} &\leq cN^{-\mu}|x|_{H^{\mu;N}(0,1)} + \sum_{n=1}^N\sqrt{\pi}2^{-\lfloor\frac{n}{2}\rfloor}\varepsilon|a_n| \\
&+\|A-\bar{A}\|_{L^2(0,1)}\left(\sum_{n=1}^N\frac{\pi 2^{-2\lfloor\frac{n}{2}\rfloor}}{(-1+2n)\{\Gamma(\lfloor\frac{n}{2}\rfloor+\frac{1}{2})\}^2}\,{}_2F_3\right. \\
&\times\left.\left(n-\tfrac{1}{2},\lfloor\tfrac{n}{2}\rfloor;n+\tfrac{1}{2},2\lfloor\tfrac{n}{2}\rfloor,\lfloor\tfrac{n}{2}\rfloor+\tfrac{1}{2};\mp\xi^2\right)\right)^{\frac{1}{2}}.
\end{aligned} \tag{67}$$

Proof. Let assume $x_N(t) = H^T A = \sum_{n=1}^{N} a_n\varphi_n(t)$, we have

$$\|x - \bar{x}_N\|_{L^2(0,1)} \le \|x - x_N\|_{L^2(0,1)} + \|x_N - \bar{x}_N\|_{L^2(0,1)}. \quad (68)$$

Using Theorem 4.2, we have the upper bound for $\|x - x_N\|_{L^2(0,1)}$ so we need to introduce an upper bound for $\|x_N - \bar{x}_N\|_{L^2(0,1)}$. To this aim, using the Schwarz's inequality, we find

$$\begin{aligned}\|x_N - \bar{x}_N\|^2_{L^2(0,1)} &= \left\|\sum_{n=1}^{N} a_n\varphi_n(t) - \sum_{n=1}^{N}\bar{a}_n\varphi_n(t)\right\|^2_{L^2(0,1)}\\ &= \left\|\sum_{n=1}^{N}(a_n - \bar{a}_n)\varphi_n(t)\right\|^2_{L^2(0,1)}\\ &= \int_0^1 \left|\sum_{n=1}^{N}(a_n - \bar{a}_n)\varphi_n(t)\right|^2 dt\\ &\le \left(\sum_{n=1}^{N}|a_n - \bar{a}_n|^2\right)\int_0^1 \sum_{n=1}^{N}|\varphi_n(t)|^2 dt.\end{aligned} \quad (69)$$

Using Theorem 2.5 for $n = m$ and Eq. (69) we have

$$\begin{aligned}\|x_N - \bar{x}_N\|^2_{L^2(0,1)} &\le \left(\sum_{n=1}^{N}|a_n - \bar{a}_n|^2\right)\\ &\times \int_0^1 \sum_{n=1}^{N} \frac{\pi 2^{-2\lfloor\frac{n}{2}\rfloor} t^{2n-2}}{\{\Gamma(\lfloor\frac{n}{2}\rfloor + \frac{1}{2})\}^2}\, {}_1F_2\left(\lfloor\tfrac{n}{2}\rfloor; 2\lfloor\tfrac{n}{2}\rfloor, \lfloor\tfrac{n}{2}\rfloor + \tfrac{1}{2}; \mp\xi^2 t^2\right) dt\\ &= \left(\sum_{n=1}^{N}|a_n - \bar{a}_n|^2\right)\\ &\times \sum_{n=1}^{N} \frac{\pi 2^{-2\lfloor\frac{n}{2}\rfloor}}{(-1+2n)\{\Gamma(\lfloor\frac{n}{2}\rfloor + \frac{1}{2})\}^2}\, {}_2F_3\left(n - \tfrac{1}{2}, \lfloor\tfrac{n}{2}\rfloor; n + \tfrac{1}{2}, 2\lfloor\tfrac{n}{2}\rfloor, \lfloor\tfrac{n}{2}\rfloor + \tfrac{1}{2}; \mp\xi^2\right),\end{aligned} \quad (70)$$

which completes the proof.

Remark. The result of Theorem 4.3 shows the rate of convergence in the present method depends on the number of Eta-based functions, so by increasing the number of Eta-based functions, the error bounds tend to zero. For more details, please see example 5.3.

5. Numerical example

In this section, we assess the new numerical method presented in Section 3.1 to derive the numerical solution of Eq. (45) for different cases. We consider different formats of the delay term, including a zero delay term, a pantograph delay where the delay term is represented as $x(qt)$, and a time-dependent delay where the delay term is expressed as $x(\tau(t))$, and a state-dependent delay where a delay term is introduced as $x(t - \Theta(t, x(t)))$. To show the advantages of the Eta-based function, we consider three cases for the set of base functions $H(t)$ in Eq. (33). In each example, we present the absolute error for each case to compare the results.

Case 1: We choose Eta-based functions as a base. In this case

$$H(t) = [\varphi_1(t), \varphi_2(t), \ldots, \varphi_N(t)]^T \quad (71)$$

is defined on $t \in [0, 1]$ where $\varphi_i(t)$ has been introduced in Eq. (25).
Case 2: We choose Legendre polynomials as a base. Legendre polynomials, $P_n(t)$, is defined on the interval $(-1, 1)$ using the following recursive formula

$$P_n(t) = 2P_{n-1}(t) - P_{n-2}(t),\ n = 2, 3, \ldots, \quad (72)$$

where $P_0(t) = 1$ and $P_1(t) = t$. These polynomials are orthogonal with respect to the weight $w(t) = 1$ on the interval $[-1, 1]$. In this case

$$H(t) = [P_0(2t - 1), P_1(2t - 1), \ldots, P_{N-1}(2t - 1)]^T \quad (73)$$

is defined for $t \in [0, 1]$.
Case 3: We choose $H(t) = [\psi_0(t), \psi_1(t), \ldots, \psi_{N-1}(t)]^T$ as a base where

$$\psi_i(t) = \begin{cases} cos(i \times t), & \text{if i is even}\\ sin(i \times t), & \text{if i is odd}\end{cases}$$

and $t \in [0, 1]$. In some specific cases we could consider only $sin(i \times t)$ or $cos(i \times t)$ as the base.

Table 1
Absolute error for $x_1(t)$ in example 5.1.

t	Eta-based functions N = 2	Legendre polynomials N = 5	Trigonometric functions N = 5
0.2	0.0	1.9×10^{-4}	7.9×10^{-2}
0.4	0.0	1.9×10^{-5}	7.9×10^{-2}
0.6	0.0	1.7×10^{-4}	8.3×10^{-2}
0.8	4.4×10^{-16}	2.9×10^{-4}	9.0×10^{-2}

Table 2
Absolute error for $x_2(t)$ in example 5.1.

t	Eta-based functions N = 2	Legendre polynomials N = 5	Trigonometric functions N = 5
0.2	2.2×10^{-16}	1.0×10^{-3}	1.3×10^{-2}
0.4	2.2×10^{-16}	7.5×10^{-4}	4.2×10^{-2}
0.6	4.4×10^{-16}	5.7×10^{-4}	6.4×10^{-2}
0.8	8.8×10^{-16}	4.5×10^{-4}	8.2×10^{-2}

Table 3
Absolute error for example 5.2.

t	Eta-based functions N = 4	Legendre polynomials N = 4	Trigonometric functions N = 4
0.2	3.60×10^{-16}	2.05×10^{-2}	5.05×10^{-2}
0.4	5.27×10^{-16}	2.50×10^{-2}	6.16×10^{-2}
0.6	6.10×10^{-16}	2.67×10^{-2}	6.47×10^{-2}
0.8	6.66×10^{-16}	3.17×10^{-2}	7.71×10^{-2}

We implemented our method and performed our numerical simulations with Mathematica 12.

5.1. Two-dimensional linear systems with a zero delay term

In this example, we consider a zero delay term to show the effectiveness of the Eta-based functions for solving the ordinary differential equations. We consider Eq. (45) with $\rho = 2$, $g_1(t) = x_1(t) + x_2(t)$ and $g_2(t) = -2x_1(t) - x_2(t)$. We have a two dimensional differential system with the exact solution $x_1(t) = 3sin(t) + 2cos(t)$ and $x_2(t) = -5sin(t) + cos(t)$. Tables 1 and 2 show the absolute error for this case.

In this example, reaching the absolute error of order $\mathcal{O}(10^{-16})$, the CPU time taken in Legendre polynomials was almost 271 times greater than that in Eta-based functions, and this accuracy was not achieved when we used trigonometric functions.

5.2. Pantograph delay differential equation

In this example, we consider Eq. (45) where $\rho = 1$ and $g(t) = x(\frac{t}{2}) + \frac{e^t(t+1)}{2} + \frac{e^{-t}(t-1)}{2} - \frac{t}{2}\sinh\frac{t}{2}$. In this case, we have a delay differential equation of pantograph type with an exact solution $x(t) = tsinh(t)$. In this case, for reaching the absolute error of order $\mathcal{O}(10^{-16})$, the CPU time taken in Legendre polynomials was almost 42 times greater than that in Eta-based functions, and this accuracy was not achieved when we used trigonometric functions. The absolute error is presented in Table 3. In this table, we choose four first terms of the base for all three different choices of base functions.

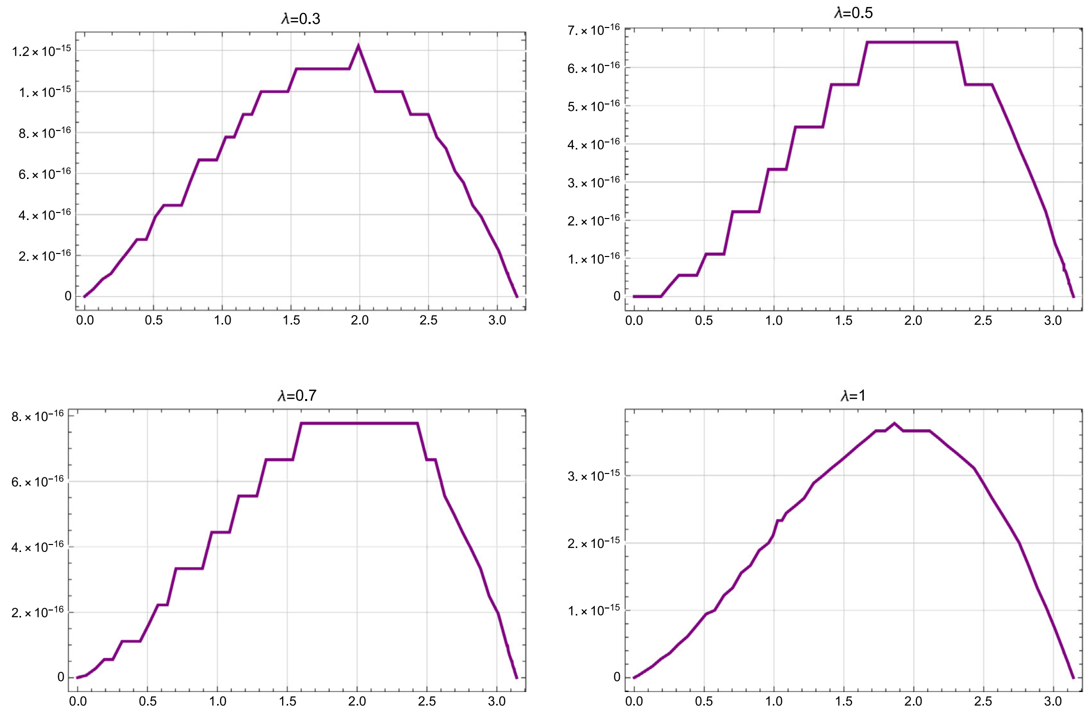

Fig. 1. Error function $|x_{exact} - x_{approximate}|$ for example 5.5.

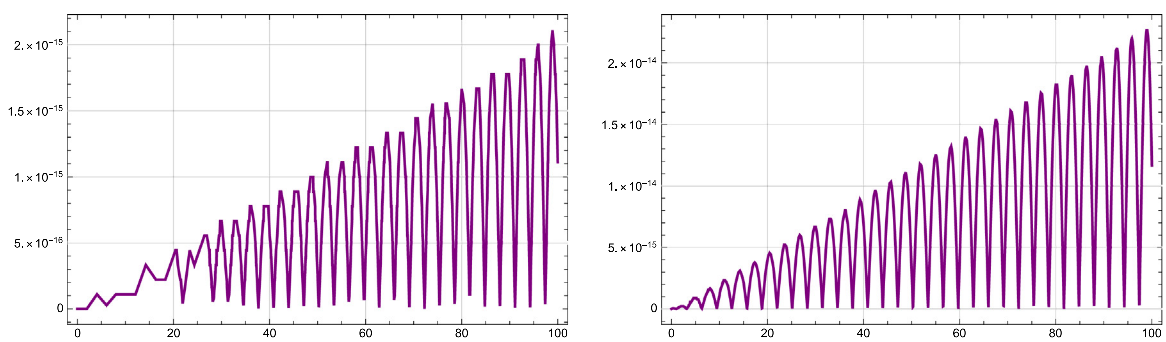

Fig. 2. Error function for $x_1(t)$ (left) and $x_2(t)$ (right) for $t \in [0, 100]$ and $N = 3$ in example 5.6.

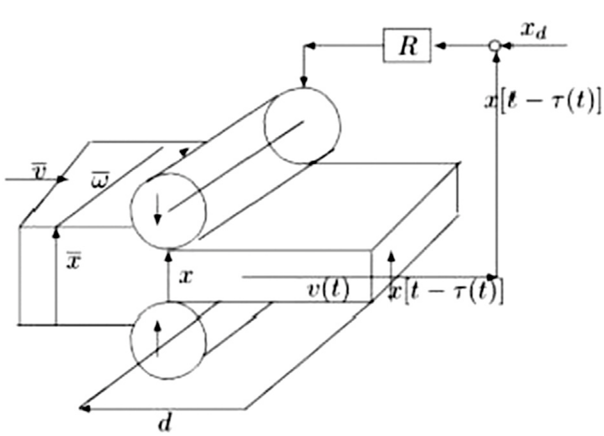

Fig. 3. Schematic diagram of rolling mill.

Table 4
Absolute error for example 5.3.

t	Eta-based functions			Trigonometric functions		
	N = 3	N = 7	N = 11	N = 3	N = 7	N = 11
0.2	1.2×10^{-2}	1.2×10^{-5}	1.6×10^{-10}	5.2×10^{-3}	6.9×10^{-4}	4.2×10^{-5}
0.4	1.5×10^{-2}	9.5×10^{-6}	1.2×10^{-10}	7.5×10^{-3}	5.3×10^{-4}	3.0×10^{-5}
0.6	1.4×10^{-2}	7.1×10^{-6}	9.1×10^{-11}	5.8×10^{-3}	4.0×10^{-4}	2.2×10^{-5}
0.8	1.0×10^{-2}	4.9×10^{-6}	6.3×10^{-11}	2.3×10^{-3}	2.6×10^{-4}	1.6×10^{-6}

Table 5
CPU time used corresponding to Eta-based functions for solving example 5.3.

Absolute error	$\mathcal{O}(10^{-2})$	$\mathcal{O}(10^{-3})$	$\mathcal{O}(10^{-5})$	$\mathcal{O}(10^{-10})$
CPU time	$(N = 3)$ 0.001	$(N = 5)$ 0.016	$(N = 7)$ 0.031	$(N = 11)$ 0.157

5.3. Multi pantograph delay differential equation

In this example, we consider the multi-pantograph delay differential equation. We assume, in Eq. (45), $\rho = 1$ and $g(t) = -x(t) - e^{-\frac{t}{2}} \sin(\frac{t}{2}) x(\frac{t}{2}) - 2e^{-\frac{3t}{4}} \cos(\frac{t}{2}) \sin(\frac{t}{4}) x(\frac{t}{4})$. The exact solution is $x(t) = e^{-t} cos(t)$. The absolute error is presented in Table 4. Also, CPU time used (in seconds) for different values of N is given in Table 5.

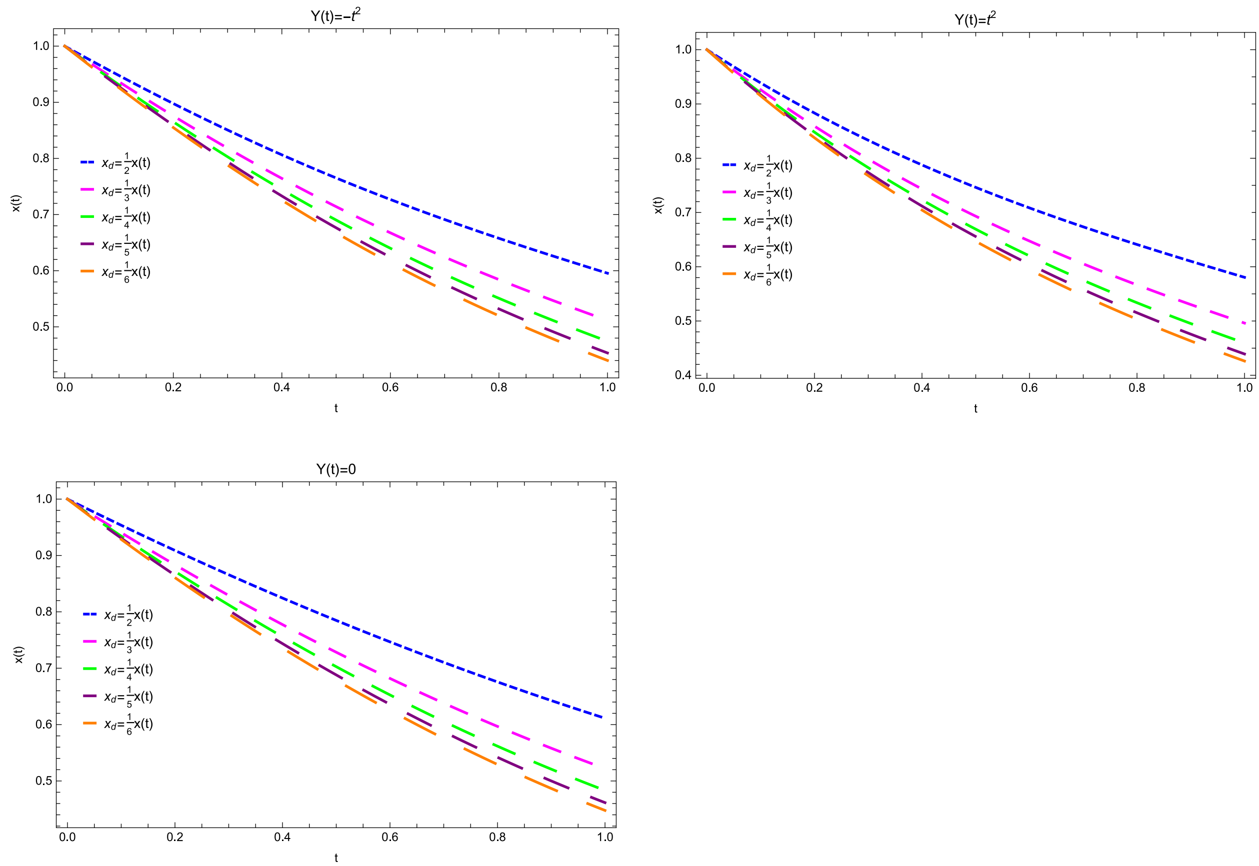

Fig. 4. Comparative plots of the metal thickness ($x(t)$) for Eq. (78) with $N = 3$, $K = 1$ and initial conditions $x(0) = 1$, $\tau(0) = 0$ with five different cases for ζ.

Table 6
Absolute error for example 5.4.

t	Eta-based functions $N = 3$	Legendre polynomials $N = 3$	Trigonometric functions $N = 3$
0.2	1.3877×10^{-16}	4.0251×10^{-2}	3.9415×10^{-2}
0.4	1.6653×10^{-16}	4.9112×10^{-2}	6.3251×10^{-2}
0.6	2.2204×10^{-16}	3.4987×10^{-2}	6.1952×10^{-2}
0.8	2.2204×10^{-16}	6.9415×10^{-3}	3.7040×10^{-2}

Table 7
Absolute error for example 5.5 at the point $t = \pi$ with different choices of λ.

λ	Methods	Errors	CPU time
0.3	Eta-based functions with N = 3	1.0479×10^{-26}	0.15
	Numerical method [19] with N = 2, M = 9	0.11×10^{-10}	2538.69
0.7	Eta-based functions with N = 3	2.21161×10^{-29}	0.33
	Numerical method [19] with N = 2, M = 9	0.21×10^{-11}	3418.12
1.0	Eta-based functions with N = 3	7.25971×10^{-25}	0.41
	Numerical method [19] with N = 4, M = 5	0.46×10^{-4}	3332.34

5.4. Time-dependent neutral delay differential equation

To examine the effectiveness of the proposed method for time-dependent neutral delay differential equations, we consider Eq. (45) with $\rho = 1$ and $g(t) = -x(\Theta(t)) + x'(\Theta(t)) + cosh(t) - \frac{1}{t+1}$. Also, we assume $\Theta(t) = ln(t+1)$. The exact solution is chosen as $x(t) = sinh(t)$. Table 6 shows the absolute error for this case. In this example, reaching the absolute error of order $\mathcal{O}(10^{-16})$, the CPU time taken in Legendre polynomials was almost 66 times greater than that in Eta-based functions, and the CPU time taken in trigonometric functions was nearly 22 times greater than that in Eta-based functions.

Table 8
Absolute error for $x_1(t)$ in example 5.6.

t	Eta-based functions N = 3	Legendre polynomials N = 3	Trigonometric functions N = 3
0.2	0.0	4.63391×10^{-2}	2.77556×10^{-17}
0.4	0.0	8.01172×10^{-2}	5.55112×10^{-17}
0.6	0.0	1.08939×10^{-1}	1.11022×10^{-16}
0.8	0.0	1.39789×10^{-1}	1.11022×10^{-16}

Table 9
Absolute error for $x_2(t)$ in example 5.6.

t	Eta-based functions N = 3	Legendre polynomials N = 3	Trigonometric functions N = 3
0.2	5.55112×10^{-17}	2.94881×10^{-2}	2.77556×10^{-17}
0.4	5.55112×10^{-17}	4.50456×10^{-2}	5.55112×10^{-17}
0.6	0.0	5.42770×10^{-2}	0.0
0.8	0.0	6.41681×10^{-2}	0.0

5.5. State-dependent neutral delay differential equation

In this example, we consider a state-dependent delay differential equation with delay term $\Theta(t) = tx^2(t)$. In this case we assume $\rho = 1$ and

$$g(t) = cos(t)(1 + x(\Theta(t))) + \lambda x(t)x'(\Theta(t)) + (1-\lambda)sin(t)cos(tsin^2(t)) - sin(t + sin^2(t)).$$

We have a neutral delay differential equation with a state-dependent delay with an exact solution $x(t) = sin(t)$. This example was considered in [19] by using the hybrid functions approximation method. Using Eta-based functions as a base, we have compared the present scenario with the numerical method presented in [19]. The error bound and the CPU time (in seconds) have been shown in Table 7.

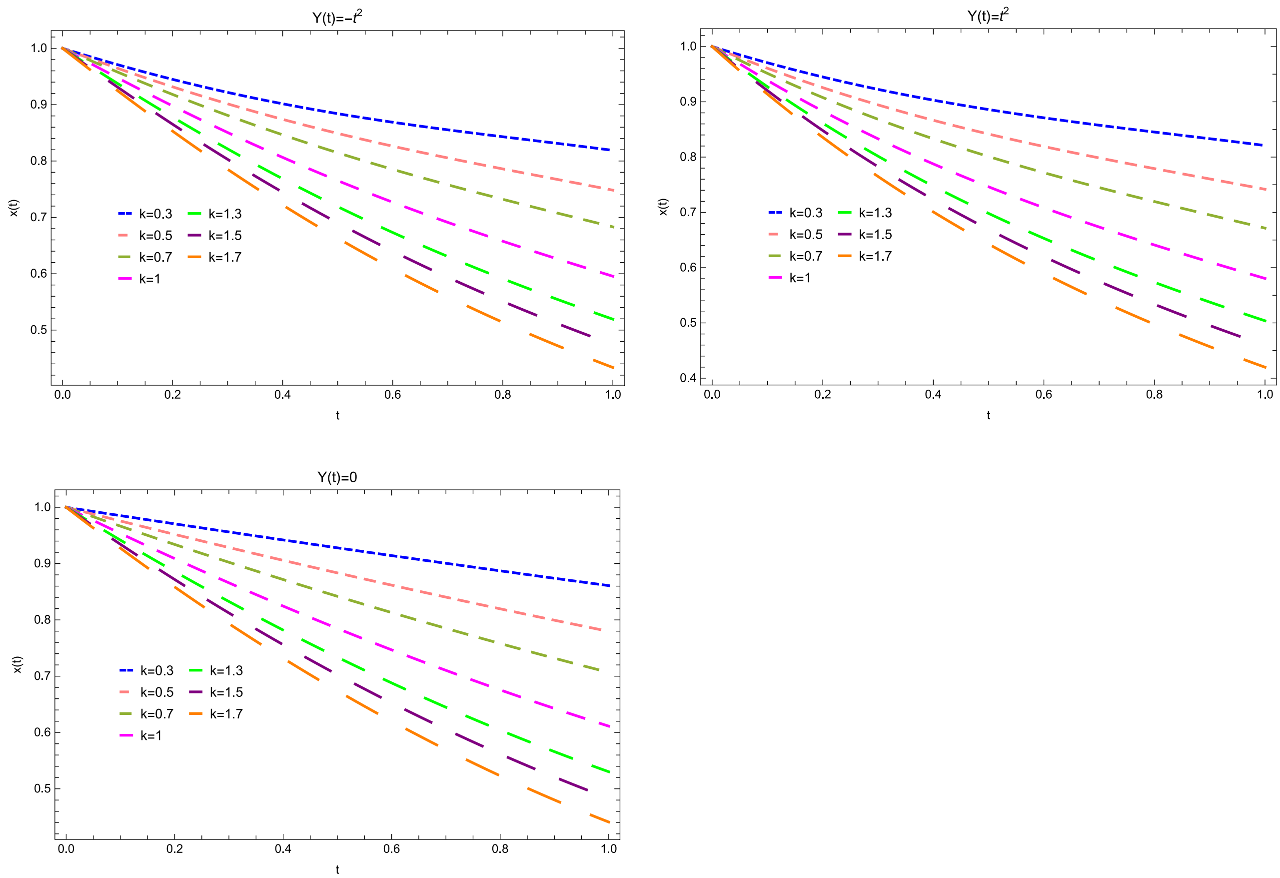

Fig. 5. Comparative plots of the metal thickness ($x(t)$) for Eq. (78) with $N = 3$, $\zeta = 0.5$ and initial conditions $x(0) = 1$, $\tau(0) = 0$ with seven different cases for K.

Table 10
CPU time used corresponding to Eta-based functions, Legendre polynomials, and Trigonometric functions for example 5.6.

Base function	Eta-based functions	Legendre polynomials	Trigonometric functions
CPU time to reach the absolute error of order $\mathcal{O}(10^{-17})$	$(N = 3)$ 0.016	$(N = 14)$ 1825.6	$(N = 3)$ 0.016

Our new numerical method is computationally much better than the proposed method in [19] especially when we ask for higher accuracy and less computational time-consuming in state-dependent delay differential equations. Absolute errors for different values of λ with $N = 3$ are also shown in Fig. 1.

5.6. System of state-dependent delay differential equations

Finally, we apply the new method to derive the numerical solution of a two-dimensional system of state-dependent delay differential equations. In Eq. (45) assume that $\rho = 2$, $g_1(t) = x_1(x_2(t)) + cos(t) - sin(sint)$ and $g_2(t) = x_1(t) - x_2(t) + cost$, where $x_1(0) = x_2(0) = 0$. An exact solution of this system is $x_1(t) = x_2(t) = sin(t)$. Tables 8 and 9 show the absolute error for $t \in [0, 1]$. In these tables, we choose three first terms of the base for all three different choices of base functions. The CPU times (in seconds) to reach the absolute error of order $\mathcal{O}(10^{-17})$ are given in Table 10. Also absolute errors for $t \in [0, 100]$ are plotted in Fig. 2.

Remark. These numerical examples show the Eta-based functions are much better than the Legendre polynomials when the exact solution of the DDEs is trigonometric functions. Also, in this case, trigonometric and Eta-based functions have the same accuracy. The Eta-based functions are much better than the Legendre polynomials and trigonometric functions when the exact solution of DDEs is an exponential function or has one of the following forms

$$\Delta(t) = \delta_1(t)\sin(\omega t) + \delta_2(t)\cos(\omega t) \text{ or } \Delta(t) = \delta_1(t)\sinh(\omega t) + \delta_2(t)\cosh(\omega t),$$

where ω is a constant and $\delta_1(t)$, $\delta_2(t)$ are polynomials. These results are consistent with the reported results in [10]. This section's results convince us to use Eta-based functions to study the behavior of the rolling mill thickness control system in the next section. Since the exact solution of the rolling mill thickness control system is unknown, using the Eta-based functions allowed us to consider all possible situations for the exact solution, including trigonometric functions, hyperbolic functions, or polynomials.

6. Rolling mill thickness control system with state-dependent delay

In this section, we study the behavior of the rolling mill thickness control system (for more details, please see [28]) using the method presented in Section 3.1 to show the application of the Eta-based function for a more practical problem. Consider a billet of metal of thickness $\bar{x}$ and width $\bar{\omega}$ enters a simplified model of the rolling mill with a velocity $\bar{v}$ and leaves it with a velocity $v(t)$ having now the thickness $x(t)$ and width $\omega(t)$ (Fig. 3). The sheet thickness is measured at a certain constant d from the rolls and then is used for control by the controller R with the set value x_d.

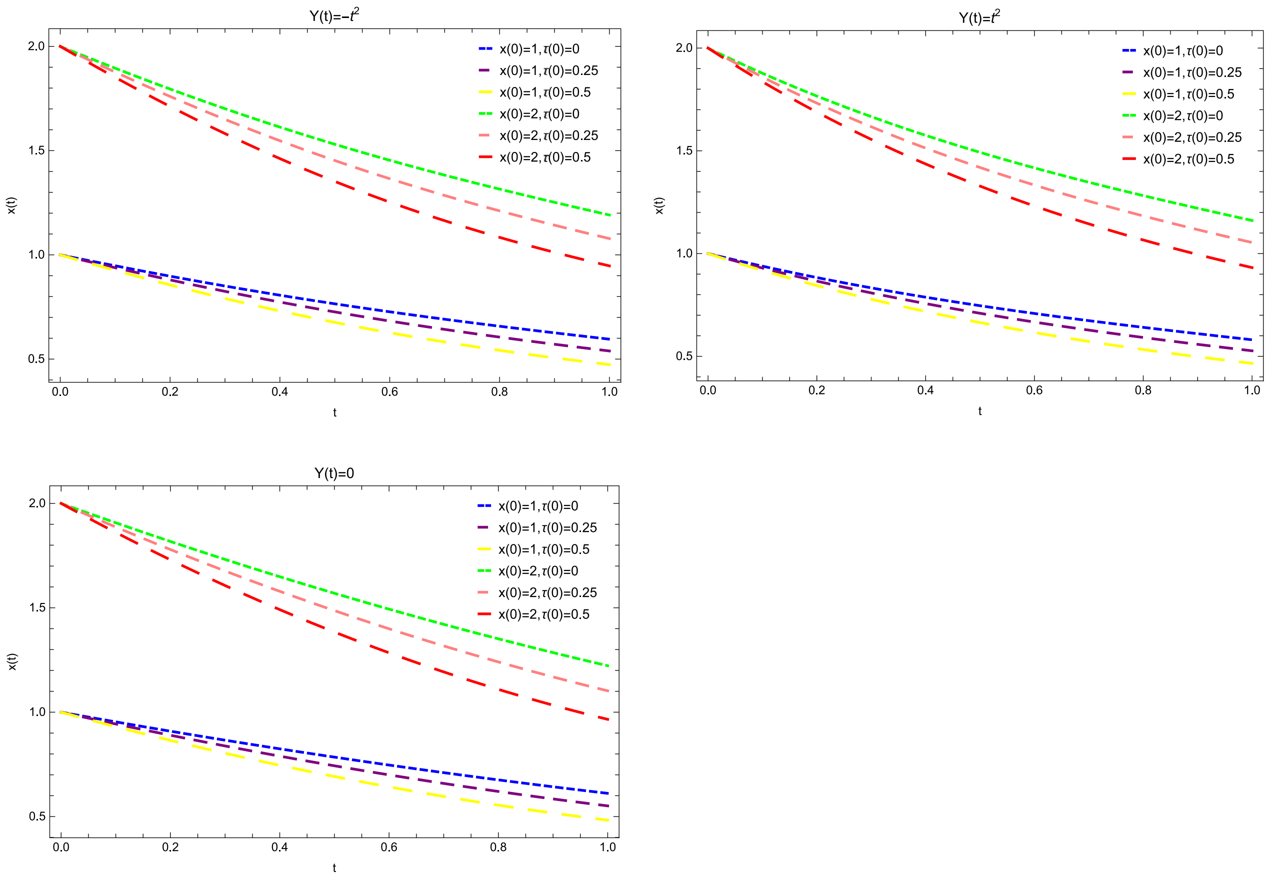

Fig. 6. Behavior of the metal thickness ($x(t)$) for Eq. (78) with $N = 3$, $\zeta = 0.5$, $K = 1$ and six different cases for initial conditions.

Assuming the density of metal is constant, and by using the law of conservation of mass, the equation expressing that the volume of metal stays constant can be written as

$$\bar{x}.\ \bar{\omega}.\ \bar{v} = x(t).\omega(t).v(t).$$

The width $\omega(t)$ is usually kept constant in relation to width $\bar{\omega}$, so we have

$$x(t).v(t) = const. \tag{74}$$

Since the velocity $v(t)$ is time-dependent, then delay τ Between the rolls' passage and the thickness sensor is a time-dependent function.

The distance traveled by the metal is obtained from the following equation

$$d = \int_{t-\tau(t)}^{t} v(s)ds. \tag{75}$$

Substituting Eq. (74) in Eq. (75) and differentiating lead to

$$\frac{1}{x(t)} - \frac{1-\tau'(t)}{x(t-\tau(t))} = 0. \tag{76}$$

Suppose that the regulator is formulated as

$$x'(t) = -Kx(t-\tau(t)) + Kx_d(t), \tag{77}$$

then, by using Eqs. (76) and (77) the following system is acquired

$$\begin{cases} x'(t) = -Kx(t-\tau(t)) + Kx_d(t), \\ \tau'(t) = 1 - \frac{x(t-\tau(t))}{x(t)}, \end{cases} \tag{78}$$

which corresponds to a system of state-dependent delay differential equations.

Suppose that the thickness of metal after passing the distance d of rolling mill according to the equation $x_d(t) = \zeta x(t)$, $0 < \zeta < 1$. Now, we apply the presented method in Section 3.1 to study the behavior of Eq. (78) in three different cases.

At first, we consider different desired metal thicknesses after passing the rolling mill (changing the values of ζ) while the parameter $K = 1$ and $x(0) = 1$ are fixed. Fig. 4 shows this case's thickness behavior ($x(t)$). Next, we assume the desired thickness of the metal plate after passing the rolling mill is fixed ($x_d(t) = 0.5x(t)$) and change the parameter K. Also, in this case, we assume the value of the initial thickness of the metal is fixed ($x(0) = 1$). Fig. 5 shows this case's thickness behavior ($x(t)$). Finally, we change the value of the initial thickness of the metal $x(0)$ and the initial delay amount between the rolling passage and the thickness sensor $\tau(0)$ while we assume $K = 1$ $\zeta = 0.5$. Fig. 6 shows this case's thickness behavior ($x(t)$).

Since we do not have an exact solution for Eq. (78), we have assumed all possible values $Y(t) = t^2$, $Y(t) = -t^2$ and $Y(t) = 0$ of frequencies, which introduce hyperbolic, trigonometric and polynomial functions.

The results show that metal thickness ($x(t)$) decreases over time despite different frequency choices. Also, these results demonstrate the rate of change of $x(t)$ depends on desired metal thicknesses after passing the rolling mill (ζ), the initial thickness of the metal ($x(0)$), thickness sensor ($\tau(0)$) and parameter K.

Also, Fig. 7 shows the state-dependent delay decreases over time by changing the initial thickness of the metal ($x(0)$) and thickness sensor ($\tau(0)$).

7. Conclusion

We have derived some new properties of the Eta and Eta-based functions. We have developed a new numerical method for solving the state-dependent and time-dependent neutral delay differential equation by introducing the operational matrix of derivative for the Eta-base

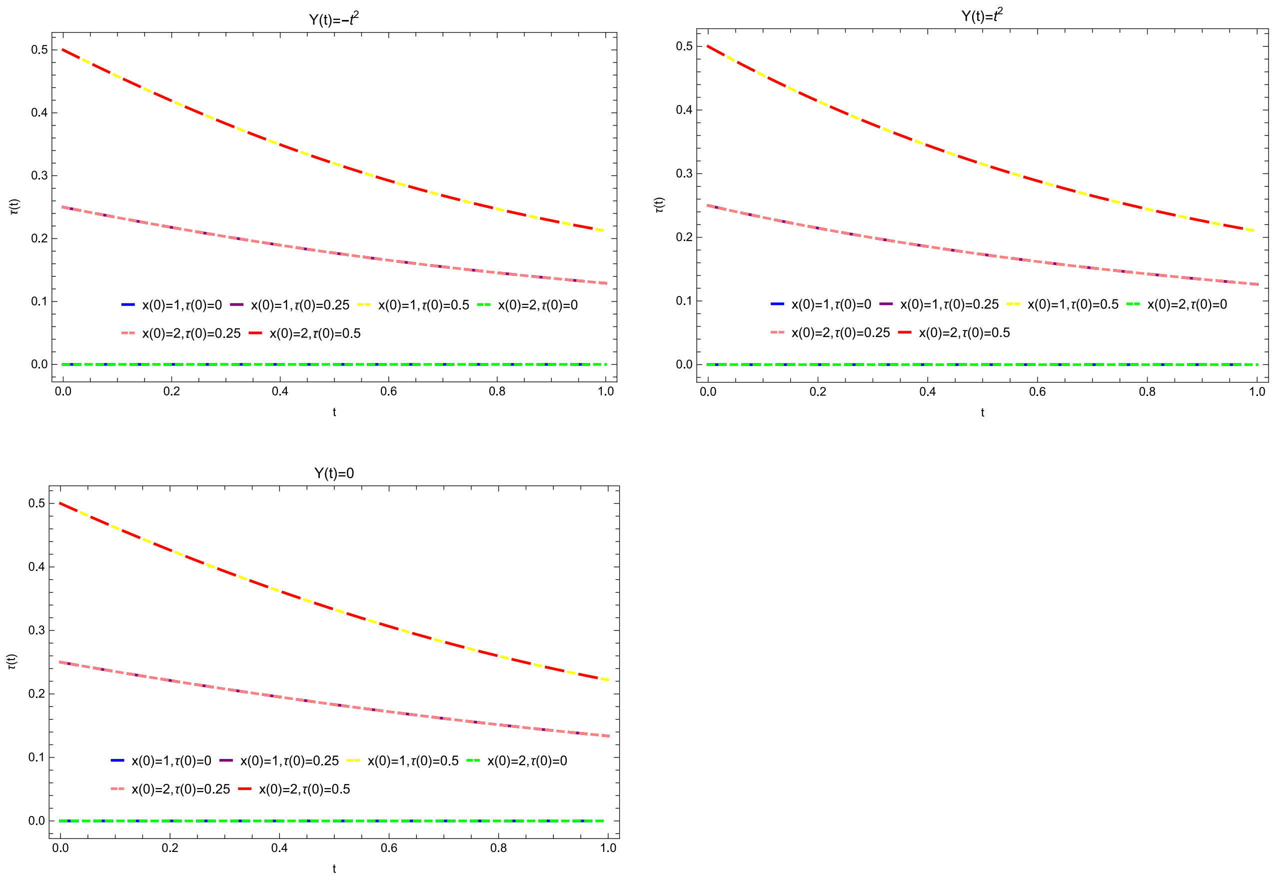

Fig. 7. Behavior of the state-dependent delay for Eq. (78) with $N = 3$, $\zeta = 0.5$, $K = 1$ and six different cases for initial conditions.

functions. The results of the numerical examples show the advantages of using the Eta-based functions for solving the state-dependent and time-dependent neutral delay differential equation. When we do not know the behavior of the exact solution, using the Eta-based functions allowed us to consider all possible situations for the precise solution, including a trigonometric function, hyperbolic functions, or polynomials. In the end, we used the new numerical method to study the behavior of the rolling mill thickness control system. This study shows the rate of change of the metal thickness depends on desired metal thicknesses, the initial thickness of the metal, and the initial thickness sensor.

CRediT authorship contribution statement

S. Sedaghat: Conceptualization, Developing the theory, Writing – original draft, Writing – review & editing. **S. Mashayekhi:** Conceptualization, Designing the study, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

[1] Ixaru Liviu Gr. Numerical methods for differential equations and applications. 1984.
[2] Ixaru L Gr. Operations on oscillatory functions. Comput Phys Comm 1997;105(1):1–19.
[3] Ixaru L Gr. Numerical computation of the coefficients in exponential fitting. Numer Algorithms 2021;87(3):1097–106.
[4] Ixaru L Gr. Exponential and trigonometrical fittings: user-friendly expressions for the coefficients. Numer Algorithms 2019;82(3):1085–96.
[5] Ixaru Liviu Gr, Berghe Guido Vanden. Exponential fitting, Vol. 568. Springer Science & Business Media; 2004.
[6] Cardone Angelamaria, Ixaru Liviu Gr, Paternoster Beatrice. Exponential fitting direct quadrature methods for Volterra integral equations. Numer Algorithms 2010;55(4):467–80.
[7] Coleman JP, Ixaru L Gr. Truncation errors in exponential fitting for oscillatory problems. SIAM J Numer Anal 2006;44(4):1441–65.
[8] Ixaru L Gr. LILIX—a package for the solution of the coupled channel Schrödinger equation. Comput Phys Comm 2002;147(3):834–52.
[9] Conte Dajana, Esposito Elena, Paternoster Beatrice, Ixaru L Gr. Some new uses of the ηm (Z) functions. Comput Phys Comm 2010;181(1):128–37.
[10] Mashayekhi Somayeh, Ixaru L Gr. The least-squares fit of highly oscillatory functions using Eta-based functions. J Comput Appl Math 2020;376:112839.
[11] Hernández-Díez José-Enrique, Méndez-Barrios César-Fernando, Niculescu Silviu-Iulian, Bárcenas-Bárcenas Ernesto. A current sensorless delay–based control scheme for MPPT–boost converters in photovoltaic systems. IEEE Access 2020;8:174449–62.
[12] Aarsnes Ulf Jakob F, van de Wouw Nathan. Dynamics of a distributed drill string system: Characteristic parameters and stability maps. J Sound Vib 2018;417:376–412.
[13] Aarsnes Ulf Jakob F, van de Wouw Nathan. Axial and torsional self-excited vibrations of a distributed drill-string. J Sound Vib 2019;444:127–51.
[14] Morozov Andrew, Abbott Karen, Cuddington Kim, Francis Tessa, Gellner Gabriel, Hastings Alan, Lai Ying-Cheng, Petrovskii Sergei, Scranton Katherine, Zeeman Mary Lou. Long transients in ecology: Theory and applications. Phys Life Rev 2020;32:1–40.
[15] Estrada-Rodriguez Gissell, Gimperlein Heiko, Painter Kevin J, Stocek Jakub. Space-time fractional diffusion in cell movement models with delay. Math Models Methods Appl Sci 2019;29(01):65–88.

[16] Taylor-King Jake P, Franz Benjamin, Yates Christian A, Erban Radek. Mathematical modelling of turning delays in swarm robotics. IMA J Appl Math 2015;80(5):1454–74.
[17] Enright Wayne H, Hayashi Hiroshi. A delay differential equation solver based on a continuous Runge–Kutta method with defect control. Numer Algorithms 1997;16(3):349–64.
[18] Bellen Alfredo. Preservation of superconvergence in the numerical integration of delay differential equations with proportional delay. IMA J Numer Anal 2002;22(4):529–36.
[19] Sedaghat Salameh, Nemati Somayeh, Ordokhani Yadollah. Application of the hybrid functions to solve neutral delay functional differential equations. Int J Comput Math 2017;94(3):503–14.
[20] Nemati S, Lima PM, Sedaghat S. Legendre wavelet collocation method combined with the Gauss–Jacobi quadrature for solving fractional delay-type integro-differential equations. Appl Numer Math 2020;149:99–112.
[21] Mashayekhi S, Sedaghat S. Fractional model of stem cell population dynamics. Chaos Solitons Fractals 2021;146:110919.
[22] Nemati S, Lima P, Sedaghat S. An effective numerical method for solving fractional pantograph differential equations using modification of hat functions. Appl Numer Math 2018;131:174–89.
[23] Huang Qiumei, Xu Xiuxiu, Brunner Hermann. Continuous Galerkin methods on quasi-geometric meshes for delay differential equations of pantograph type. Discrete Contin Dyn Syst 2016;36(10):5423.
[24] Abolhasani M, Ghaneai H, Heydari M. Modified homotopy perturbation method for solving delay differential equations. Appl Sci Rep 2010;16(2):89–92.
[25] Heydari M, Loghmani GB, Hosseini SM. Operational matrices of Chebyshev cardinal functions and their application for solving delay differential equations arising in electrodynamics with error estimation. Appl Math Model 2013;37(14–15):7789–809.
[26] Bellen Alfredo, Guglielmi Nicola. Solving neutral delay differential equations with state-dependent delays. J Comput Appl Math 2009;229(2):350–62.
[27] Jackiewicz Zdzislaw. Existence and uniqueness of solutions of neutral delay-differential equations with state dependent delays. Funkcial Ekvac 1987;30(1):9–17.
[28] Górecki Henryk. Optimization and control of dynamic systems: foundations, main developments, examples and challenges, Vol. 107. Springer; 2017.
[29] Kilbas Anatoliĭ. Theory and applications of fractional differential equations.
[30] Bateman Harry. Higher transcendental functions [Volumes I-Iii], Vol. 1. McGraw-Hill Book Company; 1953.
[31] Slyusar VI. A family of face products of matrices and its properties. Cybernetics and Systems 1999;35(3):379–84.
[32] Mason John C, Handscomb David C. Chebyshev polynomials. Chapman and Hall/CRC; 2002.
[33] Canuto Claudio, Hussaini M Yousuff, Quarteroni Alfio, Zang Thomas A. Spectral methods: fundamentals in single domains. Springer Science & Business Media; 2007.
[34] Kreyszig Erwin. Introductory functional analysis with applications, Vol. 17. John Wiley & Sons; 1991.