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The Global Sections of Chiral de Rham Complexes on Compact Ricci-flat Kähler Manifolds II

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Abstract: We give a complete description of the vertex algebra of global sections of the chiral de Rham complex of an arbitrary compact Ricci-flat Kähler manifold.

1. Introduction

The chiral de Rham complex Ω_X^{ch} is a sheaf of vertex superalgebras that exists on any smooth manifold X in either the algebraic, complex analytic or smooth settings. It is bigraded by degree and conformal weight, and contains the ordinary de Rham sheaf as the weight zero component. The de Rham differential extends to a square-zero differential on the entire structure which preserves conformal weight and raises the degree by one. This sheaf was introduced by Malikov, Schechtman and Vaintrob in [13], and has attracted significant attention in both the physics and mathematics literature. The space of global sections $\Gamma(X, \Omega_X^{\mathrm{ch}})$ is always a vertex superalgebra, and it is known to have extra structure when X is endowed with geometric structures. For example, if X has a Riemannian metric, it has an $\mathcal{N} = 1$ superconformal structure, and when X is Kähler or hyperkähler this is enhanced to an $\mathcal{N} = 2$ structure and $\mathcal{N} = 4$ structure, respectively [1]. In these cases, certain covariantly closed differential forms on X also give rise to fields in $\Gamma(X, \Omega_X^{\mathrm{ch}})$. For example, when X is Calabi-Yau it was shown in [3] that $\Gamma(X, \Omega_X^{\mathrm{ch}})$ contains a subalgebra generated by 8 fields that was introduced by Odake [15].

Until recently, a complete description of $\Gamma(X, \Omega_X^{\mathrm{ch}})$ was not known in any examples other than an affine space or a torus. In [2], for any congruence subgroup $G \subset SL(2, \mathbb{R})$, Dai constructed a basis of the G-invariant global sections of the chiral de Rham complex on the upper half plane, which are holomorphic at the cusps. The vertex operations are determined by a modification of the Rankin-Cohen brackets of modular forms. In [18], the second author showed that for a compact Ricci-flat Kähler manifold with holonomy

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group $SU(d)$ or $Sp(\frac{d}{2})$, $\Gamma(X, \Omega_X^{\mathrm{ch}})$ is isomorphic to a certain subalgebra of the $bc\beta\gamma$-system of rank $d = \dim X$ which is invariant under the action of an infinite-dimensional Lie algebra of Cartan type. An explicit description of this invariant space was conjectured in [18] and this conjecture was proven in the case $d = 2$ using results on the invariant theory of arc spaces developed in [11]. This allowed a complete description of $\Gamma(X, \Omega_X^{\mathrm{ch}})$ for all K3 surfaces; it is isomorphic to the simple (small) $\mathcal{N} = 4$ superconformal algebra with central charge $c = 6$ [16, 17].

Very recently, in a series of papers [7–9] we have proven the arc space analogues of the first and second fundamental theorems of invariant theory for the general linear, special linear, and symplectic groups. This was achieved by providing a standard monomial basis for these invariant spaces that extends the standard monomial basis in the classical setting. These results provide the needed ingredients to complete the description of $\Gamma(X, \Omega_X^{\mathrm{ch}})$ for a general compact Ricci-flat Kähler manifold. Unfortunately, this approach does not generalize to Kähler manifolds which are not Ricci-flat, since there is no method to describe the global sections of tensor powers of the tangent and cotangent bundles. In general, $\Gamma(X, \Omega_X^{\mathrm{ch}})$ need not be isomorphic to a subalgebra of a free field algebra which is invariant under a Lie algebra of Cartan type.

The plan of the paper is following. In Sect. 2, we introduce the $\beta\gamma - bc$ system. In Sect. 3, we introduce the Lie algebras of Cartan type and their actions on the $\beta\gamma - bc$ systems. In Sect. 4, we calculate the subspaces of invariant elements in $\beta\gamma - bc$ systems under the action of special series and Hamiltonian series of Lie algebras of Cartan type, by reducing this to the invariant theory of arc spaces. Finally, in Sect. 5, we calculate the space of global sections of the chiral de Rham complexes on compact Ricci-flat Kähler manifolds.

2. $\beta\gamma - bc$ System

2.1. Vertex algebras. In this paper, we will follow the formalism of vertex algebras developed in [6]. A vertex algebra is the data $(\mathcal{A}, Y, L_{-1}, 1)$. In this notation,

(1) $\mathcal{A}$ is a $\mathbb{Z}_2$-graded vector space over $\mathbb{C}$. The $\mathbb{Z}_2$-grading is called parity, and $|a|$ denotes the parity of a homogeneous element $a \in \mathcal{A}$.
(2) Y is an even linear map

$$Y : \mathcal{A} \to \mathrm{End}(\mathcal{A})[[z, z^{-1}]], \qquad Y(a) = a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}.$$

Here z is a formal variable and $a(z)$ is called the field corresponding to a.
(3) $1 \in \mathcal{A}$ is called the vacuum vector.
(4) L_{-1} is an even endomorphism of $\mathcal{A}$.

They satisfy the following axioms:

- *Vacuum axiom.* $L_{-1}1 = 0$; $1(z) = Id$; for $a \in \mathcal{A}, n \geq 0, a_{(n)}1 = 0$ and $a_{(-1)}1 = a$;
- *Translation invariance axiom.* For $a \in \mathcal{A}$, $[L_{-1}, Y(a)] = \partial a(z)$;
- *Locality axiom.* Let z, w be formal variables. For homogeneous $a, b \in \mathcal{A}$, $(z - w)^k [a(z), b(w)] = 0$ for some $k \geq 0$, where $[a(z), b(w)] = a(z)b(w) - (-1)^{|a||b|} b(w)a(z)$.

For $a, b \in \mathcal{A}$, $n \in \mathbb{Z}_{\geq 0}$, $a_{(n)}b$ is their n^{th} product and their operator product expansion (OPE) is

$$a(z)b(w) \sim \sum_{n\geq 0} (a_{(n)}b)(w)(z-w)^{-n-1}.$$

The Wick product of $a(z)$ and $b(z)$ is $: a(z)b(z) : = (a_{(-1)}b)(z)$. The other negative products are given by

$$: \partial^n a(z)b(z) : = n!(a_{(-n-1)}b)(z).$$

For $a_1, \dots, a_k \in \mathcal{A}$, their iterated Wick product is defined to be

$$: a_1(z) \cdots a_k(z) : = : a_1(z)b(z) :, \qquad b(z) = : a_2(z) \cdots a_k(z) : .$$

We often omit the formal variable z when no confusion can arise.

We say that $\mathcal{A}$ is generated by a subset $\{\alpha^i \mid i \in I\}$ if $\mathcal{A}$ is spanned by all words in the letters α^i, and all products, for $i \in I$ and $n \in \mathbb{Z}$. We say that $\mathcal{A}$ is strongly generated by $\{\alpha^i \mid i \in I\}$ if $\mathcal{A}$ is spanned by words in the letters α^i, and all products for $n < 0$. Equivalently, $\mathcal{A}$ is spanned by the monomials

$$\{: \partial^{k_1}\alpha^{i_1} \cdots \partial^{k_m}\alpha^{i_m} : \mid i_1, \dots, i_m \in I,\ k_1, \dots, k_m \geq 0\}.$$

For $a, b \in \mathcal{A}$, the following identities will be frequently used.

$$: ab :_{(n)} = \sum_{k<0} a_{(k)}b_{(n-k-1)} + (-1)^{|a||b|} \sum_{k \geq 0} b_{(n-k-1)}a_{(k)}, \tag{2.1}$$

$$a_{(n)}b = \sum_{k\in\mathbb{Z}} (-1)^{k+1}(-1)^{|a||b|}(b_{(k)}a)_{(n-k-1)}1. \tag{2.2}$$

2.2. $\beta\gamma - bc$ system. Let V be a d-dimensional complex vector space. The $\beta\gamma$-system $\mathcal{S}(V)$ and bc-system $\mathcal{E}(V)$ were introduced in [4]. The $\beta\gamma$-system $\mathcal{S}(V)$ is strongly generated by even elements $\beta^{x'}(z)$, $x' \in V$ and $\gamma^x(z)$, $x \in V^*$. The nontrivial OPEs among these generators are

$$\beta^{x'}(z)\gamma^x(w) \sim \langle x, x' \rangle (z-w)^{-1}.$$

The bc-system $\mathcal{E}(V)$ is strongly generated by odd elements $b^{x'}(z), x' \in V$ and $c^x(z), x \in V^*$. The nontrivial OPEs among these generators are

$$b^{x'}(z)c^x(w) \sim \langle x, x' \rangle (z-w)^{-1}.$$

Here for $P = \beta, \gamma, b$ or c, we assume $a_1 P^{x_1} + a_2 P^{x_2} = P^{a_1x_1+a_2x_2}$.

Let

$$\mathcal{W}(V) := \mathcal{S}(V) \otimes \mathcal{E}(V).$$

Let $\alpha^x = \partial\gamma^x$. Then $\beta^{x'}$ and α^x satisfy

$$\beta^{x'}(z)\alpha^x(w) \sim \langle x, x' \rangle (z-w)^{-2}.$$

Let $\mathcal{S}_+(V)$ be the subalgebra of $\mathcal{S}(V)$ generated by $\beta^{x'}$ and α^x, so that $\mathcal{S}_+(V)$ is a system of $2d$ free bosons. Let

$$\mathcal{W}_+(V) := \mathcal{S}_+(V) \otimes \mathcal{E}(V).$$

If V' is a vector space and $\psi : V \to V'$ is a linear isomorphism, let $\psi^* : V'^* \to V^*$ be the induced map on dual spaces. Then ψ induces an isomorphism of vertex algebras

$$\begin{aligned} &\mathcal{W}(\psi) : \mathcal{W}(V) \to \mathcal{W}(V'), \\ &\beta^{x'} \mapsto \beta^{\psi(x')}, \qquad b^{x'} \mapsto b^{\psi(x')}, \qquad \gamma^x \mapsto \gamma^{(\psi^*)^{-1}(x)}, \qquad c^x \mapsto c^{(\psi^*)^{-1}(x)}. \end{aligned} \tag{2.3}$$

Note that $\mathcal{W}(\psi)$ restricts to an isomorphism $\mathcal{W}_+(V) \cong \mathcal{W}_+(V')$.

Fix $x'_1, \dots, x'_d$, a basis of V and let $x_1, \dots, x_d$ be the dual basis of V^*. Let S_0 be the set of $\beta^{x'_i}_{(n)}, \alpha^{x_i}_{(n)}, b^{x'_i}_{(n)}, c^{x_i}_{(n)}$, $1 \le i \le d$, $n < 0$. These operators are supercommutative. Let $SW(V) = \mathbb{C}[S_0]$ be the algebra generated by these operators. There is a canonical isomorphism of $SW(V) \otimes_{\mathbb{C}} \mathbb{C}[\gamma^{x_1}_{(-1)}]$ modules,

$$\tilde{\pi} : SW(V) \otimes_{\mathbb{C}} \mathbb{C}[\gamma^{x_1}_{(-1)}, \dots, \gamma^{x_d}_{(-1)}] \to \mathcal{W}(V), \qquad a \otimes f \mapsto af1.$$

In particular, $\mathcal{W}(V)$ is a free $\mathbb{C}[\gamma^{x_1}_{(-1)}, \dots, \gamma^{x_d}_{(-1)}]$-module. Restricting $\tilde{\pi}$ to $SW(V) \otimes \{1\}$, we get an isomorphism of $SW(V)$ modules,

$$\pi : SW(V) \to \mathcal{W}_+(V), \quad a \mapsto a1. \tag{2.4}$$

2.3. Subalgebras of $\mathcal{W}_+(V)$. Let

$$\begin{aligned} &Q(z) = \sum_{i=1}^d : \beta^{x'_i}(z) c^{x_i}(z) :, \qquad L(z) = \sum_{i=1}^d (: \beta^{x'_i}(z) \partial \gamma^{x_i}(z) : - : b^{x'_i}(z) \partial c^{x_i}(z) :), \\ &J(z) = -\sum_{i=1}^d : b^{x'_i}(z) c^{x_i}(z) :, \qquad G(z) = \sum_{i=1}^d : b^{x'_i}(z) \partial \gamma^{x_i}(z) :, \end{aligned} \tag{2.5}$$

Note that L is a Virasoro field in $\mathcal{W}(V)$ of central charge zero, and $b^{x'_i}, c^{x_i}, \beta^{x'_i}, \gamma^{x_i}$ are primary of weights $1, 0, 1, 0$ with respect to L. Also, J generates a Heisenberg algebra and the zero mode $J_{(0)}$ induces an additional $\mathbb{Z}$-grading called the degree; note that $b^{x'_i}$, c^{x_i}, $\beta^{x'_i}$, γ^{x_i} have degrees $-1, 1, 0, 0$. Finally, we recall that L can be replaced with the Virasoro field $T = L - \frac{1}{2}\partial J$. This has central charge $c = 3d$, and $b^{x'_i}, c^{x_i}, \beta^{x'_i}, \gamma^{x_i}$ are primary of weights $\frac{1}{2}, \frac{1}{2}, 1, 0$ with respect to T. The subalgebra of $\mathcal{W}_+(V)$ generated by Q, T, J, G (equivalently, Q, L, J, G) is isomorphic to the $\mathcal{N} = 2$ superconformal algebra with central charge $c = 3d$.

Next, let

$$\begin{aligned} &D(z) = : b^{x'_1}(z) b^{x'_2}(z) \cdots b^{x'_d}(z) :, \qquad E(z) = : c^{x_1}(z) c^{x_2}(z) \cdots c^{x_d}(z) :, \\ &B(z) = Q(z)_{(0)} D(z), \qquad C(z) = G(z)_{(0)} E(z). \end{aligned} \tag{2.6}$$

If $d = 2l$ is even, let

$$\begin{aligned} &D'(z) = \sum_{i=1}^l : b^{x'_{2i-1}}(z) b^{x'_{2i}}(z) :, \qquad E'(z) = \sum_{i=1}^l : c^{x_{2i-1}}(z) c^{x_{2i}}(z) :, \\ &B'(z) = Q(z)_{(0)} D'(z), \qquad C'(z) = G(z)_{(0)} E'(z). \end{aligned} \tag{2.7}$$

Definition 2.1. Let $\mathcal{A}_0(V)$ be the vertex algebra generated by the fields (2.5) and (2.6). Let $\mathcal{A}_1(V)$ be the vertex algebra generated by the fields (2.5) and (2.7).

The algebra $\mathcal{A}_0(V)$ was introduced by Odake in [15] and was studied extensively in the case $d = 3$. It is easy to verify that the fields (2.5) and (2.6) strongly generate $\mathcal{A}_0(V)$. Similarly, $\mathcal{A}_1(V)$ is strongly generated by the fields (2.5) and (2.7), and is isomorphic to the simple small $\mathcal{N} = 4$ superconformal vertex algebra with central charge $c = 3d$. In [18], we have shown that $\mathcal{W}_+(V)$ is a unitary representation of $\mathcal{A}_0(V)$ and $\mathcal{A}_1(V)$.

3. Lie Algebras of Cartan Type and their Action on $\beta\gamma - bc$ System

3.1. Lie algebras of Cartan type. The space of algebraic vector fields on V is a graded Lie algebra

$$\mathrm{Vect}(V) = \oplus_{n\geq -1}\mathrm{Vect}_n(V), \quad \mathrm{Vect}_n(V) = \mathrm{Sym}^{n+1}(V^*)\otimes V.$$

If $x_1, \dots, x_d$ is a basis of V^*, then any element $v \in \mathrm{Vect}_n(V)$ can be written as $v = \sum_{i=1}^d P_i \frac{\partial}{\partial x_i}$, where P_i is a homogeneous polynomial of degree $n+1$. For $\sum_{i=1}^d P_i \frac{\partial}{\partial x_i} \in \mathrm{Vect}_n(V, \omega_0)$ and $\sum_{j=1}^d P'_j \frac{\partial}{\partial x_j} \in \mathrm{Vect}_m(V)$,

$$\left[\sum_{i=1}^d P_i \frac{\partial}{\partial x_i}, \sum_{j=1}^d P'_j \frac{\partial}{\partial x_j}\right] = \sum_{i,j}(P_i \frac{\partial P'_j}{\partial x_i}\frac{\partial}{\partial x_j} - P'_j \frac{\partial P_i}{\partial x_j}\frac{\partial}{\partial x_i}) \in \mathrm{Vect}_{n+m}(V).$$

This Lie algebra is called the *general series*. For a k-form $\omega \in \wedge^k V^*$, let

$$\mathrm{Vect}_n(V,\omega) = \{v \in \mathrm{Vect}_n(V) | L_v\omega = 0\},$$
$$\mathrm{Vect}(V,\omega) = \bigoplus_{n\geq -1} \mathrm{Vect}_n(V,\omega).$$

Here L_v is the Lie derivative of v. Note that $\mathrm{Vect}(V,\omega)$ is a graded Lie subalgebra of $\mathrm{Vect}(V)$. We now consider $\mathrm{Vect}(V,\omega)$ for some particular choices of ω.

(1) If $\omega_0 = dx_1 \wedge \cdots \wedge dx_d$,

$$\mathrm{Vect}_n(V,\omega_0) = \left\{\sum_{i=1}^d P_i \frac{\partial}{\partial x_i} \in \mathrm{Vect}_n(V) | \sum \frac{\partial}{\partial x_i} P_i = 0\right\}.$$

The Lie algebra $\mathrm{Vect}(V,\omega_0)$ is called the *special series*. $\mathrm{Vect}_0(V,\omega_0)$ is a Lie algebra isomorphic to $\mathfrak{sl}_d(\mathbb{C})$.

(2) If $d = 2l$ is even and $\omega_1 = \sum_{i=1}^l dx_{2i-1} \wedge dx_{2i}$. The Lie algebra $\mathrm{Vect}(V,\omega_1)$ is called the *Hamiltonian series*, and $\mathrm{Vect}_0(V,\omega_1)$ is a Lie algebra isomorphic to $\mathfrak{sp}_d(\mathbb{C})$.

(3) If $d = 2l+1$ and $\omega = dx_{2l+1} + \sum_{i=1}^l (x_{l+i}dx_i - x_i dx_{l+i})$. The Lie algebra

$$\{v \in \mathrm{Vect}(V) | L_v\omega = P\omega,\ P \in \mathrm{Sym}^*(V^*)\}$$

is called the *contact series*.

The general series, special series, Hamiltonian series and contact series are called the Lie algebras of Cartan type and constitute an important class of simple infinite dimensional Lie algebras. In this paper, we consider the special series and Hamiltonian series.

3.2. The actions of Lie algebras of Cartan type on $\beta\gamma - bc$ systems. $\mathrm{Vect}(V)$ has a canonical action on $\mathcal{W}(V)$ according to the Part III of [14]. Let $\mathcal{L} : \mathrm{Vect}(V) \to \mathrm{Der}(\mathcal{W}(V))$ be the map given by

$$\mathcal{L}\left(\sum_i P_i(x_1, \ldots, x_d)\frac{\partial}{\partial x_i}\right) = \sum_i (Q_{(0)} : P_i(\gamma^{x_1}, \ldots, \gamma^{x_d})b^{x_i'} :)_{(0)}. \tag{3.1}$$

Clearly $\mathcal{L}$ is a homomorphism of Lie algebras.

4. Vect(V, ω_i)-Invariants

For $R \subset \mathcal{W}(V)$, let

$$R^{\mathrm{Vect}(V,\omega_i)} = \{a \in R \mid \mathcal{L}(g)a = 0, \text{ for any } g \in \mathrm{Vect}(V, \omega_i)\}$$

be the space of $\mathrm{Vect}(V, \omega_i)$-invariants. In [18], the second author has shown that

Lemma 4.1. $\mathcal{A}_0(V) \subset \mathcal{W}(V)^{Vect(V,\omega_0)}$ *and* $\mathcal{A}_1(V) \subset \mathcal{W}(V)^{Vect(V,\omega_1)}$.

Theorem 4.2. *If* $d = \dim V = 2$, $\mathcal{W}(V)^{Vect(V,\omega_0)} = \mathcal{A}_0(V)$.

It was conjectured in [18] that for all d, $\mathcal{W}(V)^{\mathrm{Vect}(V,\omega_0)} = \mathcal{A}_0(V)$ and $\mathcal{W}(V)^{\mathrm{Vect}(V,\omega_1)} = \mathcal{A}_1(V)$. In this section, we will prove this conjecture.

4.1. $Vect_0(V, \omega_i)[t]$-invariants. Let $\mathfrak{g}_0 = \mathrm{Vect}_0(V, \omega_i)$. Let $\mathfrak{g}_0[t] = \oplus_{n\geq 0}\mathfrak{g}_0 t^n$ be the Lie algebra given by

$$[g_i t^i, g_j t^j] = [g_i, g_j]t^{i+j}, \quad \text{for} \quad g_i, g_j \in \mathfrak{g}_0.$$

The action of $\mathfrak{g}_0$ on V induces an action of $\mathfrak{g}_0[t]$ on $SW(V)$, which is given by

$$\begin{aligned}
gt^n \beta^{x_i'}_{(-k)} &= \beta^{gx_i'}_{(-k+n)}, \ n < k, & gt^n \beta^{gx_i'}_{(-k)} &= 0, \ n \geq k,\\
gt^n b^{x_i'}_{(-k)} &= b^{gx_i'}_{(-k+n)}, \ n < k, & gt^n b^{gx_i'}_{(-k)} &= 0, \ n \geq k,\\
gt^n c^{x_i}_{(-k)} &= c^{gx_i}_{(-k+n)}, \ n < k, & gt^n c^{gx_i}_{(-k)} &= 0, \ n \geq k,\\
gt^n \gamma^{x_i}_{(-k)} &= \gamma^{gx_i}_{(-k+n)}, \ n < k-1, & gt^n \gamma^{gx_i}_{(-k)} &= 0, \ n \geq k-1.
\end{aligned}$$

Note that $SW(V)$ is a ring with a derivation ∂, given by $\partial P_{(-k)} = kP_{(-k-1)}$, for $P = \beta^{x_i'}, b^{x_i'}, c^{x_i}$ and α^{x_i}. For $R \subset SW(V)$, let $R^{\mathfrak{g}_0[t]}$ denote the subspace of $\mathfrak{g}_0[t]$-invariants in R.

As preparation for the next lemma, we recall the following results from [8,9,12]. Given an algebraic group G over $\mathbb{C}$ and a finite-dimensional G-module V, the arc space $J_\infty(G)$ is an algebraic group which acts on the arc space $J_\infty(V)$. The quotient morphism $V \to V/\!\!/G$ induces a morphism $J_\infty(V) \to J_\infty(V/\!\!/G)$, so we have a morphism

$$J_\infty(V)/\!\!/J_\infty(G) \to J_\infty(V/\!\!/G). \tag{4.1}$$

In particular, we have a ring homomorphism

$$\mathbb{C}[J_\infty(V/\!\!/G)] \to \mathbb{C}[J_\infty(V)]^{J_\infty(G)}. \tag{4.2}$$

If $V /\!\!/ G$ is smooth or a complete intersection, and $\mathbb{C}[V]$ has no nontrivial one-dimensional G-invariant subspaces, it was shown in [11] that (4.2) is an isomorphism, although in general it is neither injective nor surjective. If (4.2) is surjective, it follows that $\mathbb{C}[J_\infty(V)]^{J_\infty(G)}$ is generated as a differential algebra by the subalgebra $\mathbb{C}[V]^G$.

More explicitly, let $V_j \cong V$ for $j \geq 0$, and fix a basis $\{x_{1,j}, \dots, x_{n,j}\}$ for V_j. Let $S = \mathbb{C}[\bigoplus_{j\geq 0} V_j]$. The map $\mathbb{C}[J_\infty(V)] = \mathbb{C}[x_1^{(j)}, \dots, x_n^{(j)} | \ j \geq 0] \to S$ sending $x_i^{(j)} \mapsto x_{i,j}$ is an isomorphism of differential algebras, where the differential ∂ on S is given by $\partial(x_{i,j}) = (j+1)x_{i,j+1}$. In particular, the subalgebra $S_0 = \mathbb{C}[V_0]$ generates S as a differential algebra.

For $j \geq 0$, let $\tilde{V}_j \cong V$ and let $L = \bigwedge \bigoplus_{j\geq 0} \tilde{V}_j$. Fix a basis $\{y_{1,j}, \dots, y_{n,j}\}$ for $\tilde{V}_j^*$ and extend the differential on S to an even differential ∂ on $S \otimes L$, defined on generators by $\partial(y_{i,j}) = (j+1)y_{i,j+1}$. There is an action of $J_\infty(G)$ on $S \otimes L$, and we may consider the invariant ring $(S \otimes L)^{J_\infty(G)}$. Let $L_0 = \bigwedge(\tilde{V}_0) \subset L$, and let $\langle (S_0 \otimes L_0)^G \rangle$ be the differential algebra generated by $(S_0 \otimes L_0)^G$, which lies in $(S \otimes L)^{J_\infty(G)}$.

Since G acts on the direct sum $V^{\oplus k}$ of k copies of V, we have a map

$$\mathbb{C}[J_\infty(V^{\oplus k} /\!\!/ G)] \to \mathbb{C}[J_\infty(V^{\oplus k})]^{J_\infty(G)}. \tag{4.3}$$

Theorem 4.3 ([12, Thm. 7.1]). Suppose that (4.3) is an isomorphism for all $k \geq 1$. Then $(S \otimes L)^{J_\infty(G)} = \langle (S_0 \otimes L_0)^G \rangle$.

In fact, under the above hypothesis, all differential algebraic relations in $(S \otimes L)^{J_\infty(G)}$ are consequences of relations among the generators of $(S_0 \otimes L_0)^G$, and their derivatives [10, Thm. 3.1 (2)]), but this stronger fact will not be needed in this paper. By [8, Cor. 1.5], the hypothesis of Theorem 4.3 is satisfied in the case $G = Sp_{2d}$ and $V = \mathbb{C}^{2d}$.

In the case $G = SL_d$ and $V = \mathbb{C}^d \oplus (\mathbb{C}^d)^*$, this hypothesis is not satisfied since (4.3) is surjective for all k but fails to be injective when $k \geq d+3$; see [9, Thm. 1.2]. However, the surjectivity of (4.3) for all k in this case is enough for our purposes, due to the following:

Theorem 4.4 ([10, Thm. 3.1 (1)]). Suppose that (4.3) is surjective for all $k \geq 1$. Then $(S \otimes L)^{G_\infty} = \langle (S_0 \otimes L_0)^G \rangle$.

This applies to the case of $G = SL_d$ and $V = \mathbb{C}^d \oplus (\mathbb{C}^d)^*$. Note that if (4.3) fails to be injective for some k, it need not be the case that all differential algebraic relations in $(S \otimes L)^{G_\infty}$ are consequences of relations in $(S_0 \otimes L_0)^G$ and their derivatives, but this does not affect our results.

Lemma 4.5. *Recall the isomorphism $\pi : SW(V) \to \mathcal{W}_+(V)$ given by (2.4).*

(1) If $\mathfrak{g}_0 = Vect_0(V, \omega_0)$, as a ring with a derivation ∂, $SW(V)^{\mathfrak{g}_0[t]}$ is generated by

$$\pi^{-1}(Q(z)), \pi^{-1}(L(z)), \pi^{-1}(G(z)), \pi^{-1}(J(z)), \pi^{-1}(E(z)),$$
$$\pi^{-1}(B(z)), \pi^{-1}(C(z)), \pi^{-1}(D(z)).$$

(2) If $\mathfrak{g}_0 = Vect_0(V, \omega_1)$, as a ring with a derivation ∂, $SW(V)^{\mathfrak{g}_0[t]}$ is generated by

$$\pi^{-1}(Q(z)), \pi^{-1}(L(z)), \pi^{-1}(G(z)), \pi^{-1}(J(z)), \pi^{-1}(E'(z)),$$
$$\pi^{-1}(B'(z)), \pi^{-1}(C'(z)), \pi^{-1}(D'(z)).$$

Proof. For the first statement, $\mathfrak{g}_0 = \mathfrak{sl}_d$, and we have an isomorphism of $\mathfrak{sl}_d[t]$-modules

$$SW(V) \cong \mathbb{C}[J_\infty(V \oplus V^*)] \otimes \bigwedge \left(\oplus_{j\geq 0} \left(V_j \oplus V_j^*\right)\right) = S \otimes L,$$

where $V = \mathbb{C}^d$. Under the linear isomorphism (2.4), the above fields correspond to the generators of the subalgebra $(S_0 \otimes L_0)^{SL_d}$, which by Theorem 4.4 generate $(S \otimes L)^{J_\infty(SL_d)} = (S \otimes L)^{\mathfrak{sl}_d[t]}$ as a differential algebra.

The second statement is proven in the same way using Theorem 4.3, since $\mathfrak{g}_0 = \mathfrak{sp}_{2d}$ and we have an isomorphism of $\mathfrak{sp}_{2d}[t]$-modules

$$SW(V) \cong \mathbb{C}[J_\infty(V)] \otimes \bigwedge \left(\oplus_{j\geq 0} V_j\right) = S \otimes L,$$

where $V = \mathbb{C}^{2d}$. Then the above fields correspond to the generators of $(S_0 \otimes L_0)^{Sp_{2d}}$, and hence generate $SW(V)^{\mathfrak{sp}_{2d}[t]}$ as a differential algebra. $\square$

4.2. Vect(V, ω_i)-invariants. Let $SW_n(V)$ be the linear subspace of $SW(V)$ which is spanned by the monomials of $\gamma_{(i-1)}, \beta_{(i)}, b_{(i)}, c_{(i)}, i < 0$ with the property that the number of c in the monomial plus double of the number of γ in the monomial is n. We then have the grading

$$SW(V) = \oplus_{n\geq 0} SW_n(V).$$

Since the action of $\mathfrak{g}_0[t]$ on $SW(V)$ preserves $SW_n(V)$, $SW(V)^{\mathfrak{g}_0[t]} = \oplus_{n\geq 0} SW_n(V)^{\mathfrak{g}_0[t]}$.

Lemma 4.6. *Let $a \in \mathcal{W}(V)^{Vect(V,\omega_i)}$ be homogeneous with respect to conformal weight. Then*

(1) $a \in \mathcal{W}_+(V)$. In particular, we may write $a = \pi(a_k + a_{k-1} + \cdots)$ where π is given by (2.4), and $a_n \in SW_n(V)$.
(2) The leading term a_k is $Vect_0(V, \omega_i)[t]$-invariant.

Proof. It is easy to see that $\mathrm{Vect}_{-1}(V, \omega_i) = \mathrm{Vect}_{-1}(V)$. So for any $a \in \mathcal{W}(V)^{\mathrm{Vect}(V,\omega_i)}$, $\mathcal{L}(\frac{\partial}{\partial x_j})a = \beta^{x'_j}_{(0)} a = 0$ for any $1 \leq j \leq d$. Therefore $\gamma^{x_i}_{(-1)}$ does not appear in a, so that $a \in \mathcal{W}_+(V)$. Since a has fixed conformal weight, it is apparent that it has the form $a = \pi(a_k + a_{k-1} + \cdots)$ with $a_n \in SW_n(V)$. This proves (1).

Next, let $\mathfrak{g}_j = \mathrm{Vect}_j(V, \omega_i)$. It is easy to see that π is $\mathfrak{g}_0$-equivariant. So a_k is $\mathfrak{g}_0$-invariant. Let $v_1 = x_1^2 \frac{\partial}{\partial x_2} \in \mathfrak{g}_1$ and $g_1 = x_1 \frac{\partial}{\partial x_2} \in \mathfrak{g}_0$. Let

$$K_1 = \sum_{l\geq 1} \gamma^{x_1}_{(-l-1)} g_1 t^l.$$

We have

$$0 = \mathcal{L}(v_1)a = (2 : (: \gamma^{x_1} c^{x_1} :) b^{x'_2} :_{(0)} + : (: \gamma^{x_1} \gamma^{x_1} :) \beta^{x'_2} :_{(0)}) \pi(a_k + a_{k-1} + \cdots).$$

Consider the homogeneous component $SW_{k+2}(V)$:

$$0 = (\mathcal{L}(v_1)a)_{k+2} = 2 \sum_{l=1}^{\infty} \gamma^{x_1}_{(-l-1)} g_1 t^l a_k = 2K_1 a_k.$$

Similarly, let $v_0 = x_1^2 \frac{\partial}{\partial x_1} - 2x_1 x_2 \frac{\partial}{\partial x_2} \in \mathfrak{g}_1$ and $g_0 = x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} \in \mathfrak{g}_0$. Let

$$K_0 = \sum_{l \geq 1} \gamma^{x_1}_{(-l-1)} g_0 t^l - \sum_{l \geq 1} \gamma^{x_2}_{(-l-1)} g_1 t^l.$$

We have $K_0 a_k = 0$.

Inductively, let $K_n = [K_0, K_{n-1}]$, for $n \geq 2$, then $K_n a_k = 0$.

$$[K_0, \gamma^{x_1}_{(-l)}] = \sum_{s=2}^{l-2} \gamma^{x_1}_{(-s)} \gamma^{x_1}_{(-l+s)}.$$

$$[K_0, g_1 t^j] = 2 \sum_{s \geq 1} \gamma^{x_1}_{(-s-1)} g_1 t^{j+s}.$$

So inductively, we obtain

$$K_n = \sum_{l_i \geq 1} c_{l_1,\ldots,l_n} (\prod_{i=1}^{n} \gamma^{x_1}_{-l_i-1}) g_1 t^{l_1+\cdots+l_n}.$$

Here $c_{l_1,\ldots,l_n}$ are positive numbers. When l is large enough, $g_1 t^l a_k = 0$. Let L be the largest number such that $g_1 t^L a_k \neq 0$. If $L \geq 1$ then

$$0 = K_L a_k = c_{1,\ldots,1} (\gamma^{x_1}_{-2}))^L g_1 t^L a_k \neq 0.$$

So $g_1 t^L a_k = 0$.. Since $\mathfrak{g}_0$ is a simple Lie algebra, $\mathfrak{g}_0[t]$ is generated by $\mathfrak{g}_0$ and $g_1 t$. So a_k is $\mathfrak{g}_0[t]$-invariant. □

Theorem 4.7. $\mathcal{W}(V)^{Vect(V,\omega_0)} = \mathcal{A}_0(V)$; $\mathcal{W}(V)^{Vect(V,\omega_1)} = \mathcal{A}_1(V)$.

Proof. For the first equation, let $\mathfrak{g}_0 = \mathrm{Vect}_0(V, \omega_0)$. By Lemma 4.5, any $a_k \in SW_k(V)^{\mathfrak{g}_0[t]}$ can be represented as a polynomial in $\partial^l \pi^{-1}(Q(z))$, $\partial^l \pi^{-1}(L(z))$, $\partial^l \pi^{-1}(G(z))$,$\partial^l \pi^{-1}(J(z))$, $\partial^l \pi^{-1}(E(z))$, $\partial^l \pi^{-1}(B(z))$, $\partial^l \pi^{-1}(C(z))$, $\partial^l \pi^{-1}(D(z))$. Let b be the corresponding normally ordered polynomial in $\partial^l L(z)$, $\partial^l G(z)$, $\partial^l J(z)$, $\partial^l E(z)$, $\partial^l B(z)$, $\partial^l C(z)$, $\partial^l D(z)$. We have $\pi^{-1}(b) = b_k + b_{k-1} + \cdots$, $b_n \in SW_n(V)$ with $b_k = a_k$.

If $a \in \mathcal{W}(V)^{\mathrm{Vect}(V,\omega_0)}$, $\pi^{-1}(a) = a_k + a_{k-1} + \cdots$. By Lemma 4.6, $a_k \in SW_k(V)^{\mathfrak{g}_0[t]}$. So there is a $b \in \mathcal{A}_0(V)$ with $\pi^{-1}(b) = b_k + b_{k-1} + \cdots$ and $a_k = b_k$. Thus $\pi^{-1}(a-b) = (a_{k-1} - b_{k-1}) + \cdots$ and $a - b$ is $\mathrm{Vect}(V, \omega_0)$-invariant. By induction on k, we conclude $a \in \mathcal{A}_0(V)$. So $\mathcal{W}(V)^{\mathrm{Vect}(V,\omega_0)} = \mathcal{A}_0(V)$.

The proof for the second equation is similar. □

5. Chiral de Rham Complex

Let $\mathcal{W} = \mathcal{W}(\mathbb{C}^d)$ and $x'_1, \ldots, x'_d$ be a standard basis of $\mathbb{C}^d$. Then $\mathcal{W}$ has strong generators $\beta^i = \beta^{x'_i}$, $b^i = b^{x'_i}$, $\gamma^i = \gamma^{x_i}$ and $c^i = c^{x_i}$, and is a free $\mathbb{C}[\gamma^1_{(-1)}, \ldots, \gamma^d_{(-1)}]$ module. If X is a complex manifold and $(U, \gamma^1, \ldots, \gamma^d)$ is a complex coordinate system of X, $\mathcal{O}(U)$ is a $\mathbb{C}[\gamma^1_{(-1)}, \ldots, \gamma^d_{(-1)}]$-module by identifying the action of $\gamma^i_{(-1)}$ with the product of γ^i. The chiral de Rham complex Ω^{ch}_X is a sheaf of vertex algebras on X whose algebra of sections $\Omega^{\mathrm{ch}}_X(U)$ is given by

$$\Omega^{\mathrm{ch}}_X(U) = \mathcal{W} \otimes_{\mathbb{C}[\gamma^1_{(-1)},\ldots,\gamma^d_{(-1)}]} \mathcal{O}(U).$$

In particular, $\Omega_X^{\mathrm{ch}}(U)$ is the vertex algebra with strong generators $\beta^i(z)$, $b^i(z)$, $c^i(z)$ and $f(z)$, $f \in \mathcal{O}(U)$. The nontrivial OPEs among these generators are

$$\beta^i(z)f(w) \sim \frac{\partial f}{\partial \gamma^i}(w)(z-w)^{-1}, \quad b^i(z)c^j(w) \sim \delta^i_j(z-w)^{-1},$$

as well as the normally ordered product relations

$$: f(z)g(z) := fg(z), \text{ for } f, g \in \mathcal{O}(U).$$

Let $\tilde{\gamma}^1, \dots, \tilde{\gamma}^d$ be another set of coordinates on U, with

$$\tilde{\gamma}^i = f^i(\gamma^1, \dots, \gamma^d), \quad \gamma^i = g^i(\tilde{\gamma}^1, \dots, \tilde{\gamma}^d).$$

We have the following coordinate change equations:

$$\begin{aligned}
\partial\tilde{\gamma}^i(z) &= \sum_{j=1}^{d} : \frac{\partial f^i}{\partial \gamma^j}(z)\partial\gamma^j(z) :, \\
\tilde{b}^i(z) &= \sum_{j=1}^{d} : \frac{\partial g^j}{\partial \tilde{\gamma}^i}(f(\gamma))(z)b^j(z) :, \\
\tilde{c}^i(z) &= \sum_{j=1}^{d} : \frac{\partial f^i}{\partial \gamma^j}(z)c^j(z) :, \\
\tilde{\beta}^i(z) &= \sum_{j=1}^{d} : \frac{\partial g^j}{\partial \tilde{\gamma}^i}(f(\gamma))(z)\beta^j(z) : + \sum_{k=1}^{d} :(: \frac{\partial}{\partial \gamma^k}(\frac{\partial g^j}{\partial \tilde{\gamma}^i}(f(\gamma)))(z)c^k(z) :)b^j(z) : .
\end{aligned} \tag{5.1}$$

5.1. Global sections. There are four sections $Q(z)$, $L(z)$, $J(z)$ and $G(z)$ from (2.5) in $\Omega_X^{\mathrm{ch}}(U)$. For a general complex manifold X, $L(z)$ and $G(z)$ are globally defined and have the same form in any local coordinate system. The fields $Q(z)$ and $J(z)$ are globally defined if and only if the first Chern class $c_1(TX)$ vanishes, but their zero modes $Q_{(0)}$ and $J_{(0)}$, are always globally defined [13]. The operators $L_{(1)}$ and $J_{(0)}$ give Ω_X^{ch} a $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$-grading by conformal weight k and degree l, respectively.

$$\Omega_X^{\mathrm{ch}} = \bigoplus_{k,l} \Omega_X^{\mathrm{ch}}[k,l].$$

Note that the zero mode $Q_{(0)}$ of $Q(z)$ is the chiral de Rham differential, and it preserves conformal weight and raises the degree by one.

If X is a Calabi-Yau manifold with a nowhere vanishing holomorphic d-form w_0, let $(U, \gamma_1, \dots, \gamma_d)$ be a coordinate system on X such that locally,

$$w_0|_U = d\gamma^1 \cdots d\gamma^d.$$

The eight sections $Q(z)$, $L(z)$, $J(z)$, $G(z)$, $B(z)$, $C(z)$, $D(z)$ and $E(z)$ from (2.5) and (2.6) in $\Omega_X^{\mathrm{ch}}(U)$ are globally defined on X [3].

If X is a hyperkähler manifold with holomorphic symplectic form w_1, let $(U, \gamma_1, \ldots, \gamma_d)$ be a coordinate system on X such that locally,

$$w_1|_U = \sum_{i=1}^{\frac{d}{2}} d\gamma^{2i-1} \wedge d\gamma^{2i}.$$

Then the eight sections $Q(z)$, $L(z)$, $J(z)$, $G(z)$, $B'(z)$, $C'(z)$, $D'(z)$ and $E'(z)$ from (2.5) and (2.7) in $\Omega_X^{\mathrm{ch}}(U)$ are globally defined on X [1].

Definition 5.1. If X is a Calabi-Yau manifold with a nowhere vanishing holomorphic d-form, let $\mathcal{A}_0(X)$ be the vertex algebra which is strongly generated by the eight global sections given by $Q(z)$, $L(z)$, $J(z)$, $G(z)$, $B(z)$, $C(z)$, $D(z)$ and $E(z)$ on X.

If X is a hyperkähler manifold, let $\mathcal{A}_1(X)$ be the vertex algebra which is strongly generated by the eight global sections given by $Q(z)$, $L(z)$, $J(z)$, $G(z)$, $B'(z)$, $C'(z)$, $D'(z)$ and $E'(z)$ on X.

The following theorem was proven in [18].

Theorem 5.2. *If X is a d-dimensional compact Kähler manifold with holonomy group $G = SU(d)$ and w_0 is a nowhere vanishing holomorphic d-form, then*

$$\Gamma(X, \Omega_X^{ch}) \cong \mathcal{W}_+(T_xX)^{Vect(T_xX, w_0|_x)}.$$

If X is a d-dimensional compact Kähler manifold with holonomy group $G = Sp(\frac{d}{2})$ and w_1 is a holomorphic symplectic form, then the space of global section of Ω_X^{ch}

$$\Gamma(X, \Omega_X^{ch}) \cong \mathcal{W}_+(T_xX)^{Vect(T_xX, w_1|_x)}.$$

Thus we have

Theorem 5.3. *If X is a d-dimensional compact Kähler manifold with holonomy group $G = SU(d)$, then*

$$\mathcal{A}_0(X) = \Gamma(X, \Omega_X^{ch}) \cong \mathcal{A}_0(\mathbb{C}^d).$$

If X is a d-dimensional compact Kähler manifold with holonomy group $G = Sp(\frac{d}{2})$, then the eight global sections given by $Q(z)$, $L(z)$, $J(z)$, $G(z)$, $B'(z)$, $C'(z)$, $D'(z)$ and $E'(z)$ strongly generate

$$\mathcal{A}_1(X) = \Gamma(X, \Omega_X^{ch}) \cong \mathcal{A}_1(\mathbb{C}^d).$$

Proof. If X is a d-dimensional compact Kähler manifold with holonomy group $G = SU(d)$, there must be a nowhere vanishing holomorphic d-form w_0. By Theorem 5.2, $\Gamma(X, \Omega_X^{\mathrm{ch}}) \cong \mathcal{W}_+(T_xX)^{\mathrm{Vect}(T_xX, w_0|_x)}$. By Theorem 4.7, $\mathcal{W}_+(T_xX)^{\mathrm{Vect}(T_xX, w_0|_x)}$ is isomorphic to $\mathcal{A}_0(\mathbb{C}^d)$. So $\Gamma(X, \Omega_X^{\mathrm{ch}}) \cong \mathcal{A}_0(\mathbb{C}^d)$. The isomorphism maps the global sections given by $Q(z)$, $L(z)$, $J(z)$, $G(z)$, $B(z)$, $C(z)$, $D(z)$ and $E(z)$ to $Q(z)$, $L(z)$, $J(z)$, $G(z)$, $B(z)$, $C(z)$, $D(z)$ and $E(z)$ themselves. So the eight global sections given by $Q(z)$, $L(z)$, $J(z)$, $G(z)$, $B(z)$, $C(z)$, $D(z)$ and $E(z)$ strongly generate $\Gamma(X, \Omega_X^{\mathrm{ch}}) \cong \mathcal{A}_0(\mathbb{C}^d)$.

Similarly, if X is a d-dimensional compact Kähler manifold with holonomy group $G = Sp(\frac{d}{s})$, there must be a holomorphic symplectic form w_1. Then by Theorem 5.2, $\Gamma(X, \Omega_X^{\mathrm{ch}}) \cong \mathcal{W}_+(T_xX)^{\mathrm{Vect}(T_xX, w_0|_x)}$. By Theorem 4.7, $\mathcal{W}_+(T_xX)^{\mathrm{Vect}(T_xX, w_0|_x)}$ is isomorphic to $\mathcal{A}_0(\mathbb{C}^d)$. So $\Gamma(X, \Omega_X^{\mathrm{ch}}) \cong \mathcal{A}_0(\mathbb{C}^d)$. The isomorphism maps the global sections given by $Q(z)$, $L(z)$, $J(z)$, $G(z)$, $B'(z)$, $C'(z)$, $D'(z)$ and $E'(z)$ to $Q(z)$, $L(z)$, $J(z)$, $G(z)$, $B'(z)$, $C'(z)$, $D'(z)$ and $E'(z)$ themselves. So the eight global sections given by $Q(z)$, $L(z)$, $J(z)$, $G(z)$, $B'(z)$, $C'(z)$, $D'(z)$ and $E'(z)$ strongly generate $\Gamma(X, \Omega_X^{\mathrm{ch}}) \cong \mathcal{A}_0(\mathbb{C}^d)$. □

5.2. Covering maps. Let X and Y be compact complex manifolds and let $p : Y \to X$ be a covering map. By the definition of chiral de Rham complex, the inverse image sheaf $p^{-1}\Omega_X^{\mathrm{ch}} = \Omega_Y^{\mathrm{ch}}$. A global section of Ω_X^{ch} pulls back to a global section of Ω_Y^{ch}. Let

$$p^* : \Gamma(X, \Omega_X^{\mathrm{ch}}) \to \Gamma(Y, \Omega_Y^{\mathrm{ch}})$$

be the pullback map. If p is an isomorphism, then p^* is clearly an isomorphism.

If p is a finite normal covering map, let $G(Y, p)$ be its covering transformation group. For any $g \in G(y, p)$, the action of g on Y, $\rho(g) : Y \to Y$ induces an automorphism $\rho(g)^* : \Gamma(Y, \Omega_Y^{\mathrm{ch}}) \to \Gamma(Y, \Omega_Y^{\mathrm{ch}})$. Let $\Gamma(Y, \Omega_Y^{\mathrm{ch}})^{G(Y,p)}$ be the invariant subalgebra under the induced action of $G(Y, p)$.

Proposition 5.4. *p^* induces an isomorphism of vertex algebras $\Gamma(X, \Omega_X^{ch}) \to \Gamma(Y, \Omega_Y^{ch})^{G(Y,p)}$.*

Proof. Obviously, p^* is an injective morphism of vertex algebras. For any $g \in G(Y, p)$, $p \circ \rho(g) = p$. So $\rho(g)^* \circ p^* = p^*$. For any section $a \in \Gamma(X, \Omega_X^{\mathrm{ch}})$, $\rho(g)^*(p^*(a)) = p^*(a)$, so $p^*(a)$ is $G(Y, p)$-invariant.

On the other hand, assume p is an n-sheet covering map. There is an open cover $\{U_\alpha\}$ of X such that each $p^{-1}(U_\alpha)$ is the disjoint union of open sets $V_{\alpha,i}$ in Y, and $p|_{V_{\alpha,i}} : V_{\alpha,i} \to U_\alpha$ is an isomorphism. Let $\tilde{a} \in \Gamma(Y, \Omega_Y^{\mathrm{ch}})^{G(Y,p)}$, and define

$$a_\alpha = \frac{1}{n}\sum_{i=1}^{n}((p|_{V_{\alpha,i}})^{-1})^*(\tilde{a}|_{V_{\alpha,i}}).$$

For another open set U_β in the open cover, it is easy to see $a_\alpha|_{U_\alpha \cap U_\beta} = a_\beta|_{U_\alpha \cap U_\beta}$, so there is an $a \in \Gamma(X, \Omega_X^{\mathrm{ch}})$ with $a|_{U_\alpha} = a_\alpha$. It is easy to see that $p^*(a) = \tilde{a}$, since $\tilde{a}$ is $G(Y, p)$-invariant. □

5.3. Global sections: general case. For a compact Ricci-flat Kähler manifold, we have the following properties (Proposition 6.22, 6.23 in [5]).

Proposition 5.5. *Let X be a compact Ricci-flat Kähler manifold. Then X admits a finite cover isomorphic to the product Kähler manifold $T^{2l} \times X_1 \times X_2 \cdots \times X_k$, where T^{2l} is a flat Kähler torus and X_j is a compact, simply connected, irreducible, Ricci-flat Kähler manifold for $j = 1, \ldots, k$.*

Proposition 5.6. *Let X be a compact, simply-connected, irreducible, Ricci-flat Kähler manifold of dimension d. Then either $d \geq 2$ and its holonomy group is $SU(d)$, or $d \geq 4$ is even and its holonomy group is $Sp(\frac{d}{2})$. Conversely, if X is a compact Kähler manifold and its holonomy group is $SU(d)$ or $Sp(\frac{d}{2})$, then X is Ricci-flat and irreducible and X has finite fundamental group.*

Lemma 5.7. *Let G be a group and let $N \cong \mathbb{Z}^k$ be a subgroup with finite index in G. Then there is a subgroup M of N such that the index of M in N is finite, and M is a normal subgroup of G.*

Proof. Let $N_1, \dots, N_l$ be all of the conjugate subgroups of N in G, and let $M = \cap N_i$, so that M is a normal subgroup of G. Since the index of N_i in G is finite, for any $g \in G$, there is an integer $m_i > 0$ such that $g^{m_i} \in N_i$. Let $m = m_1 \cdots m_l$. Then $g^m \in N_i$ for all $1 \le i \le l$, so that $g^m \in M$. If $g_1, \dots, g_k$ are generators of N, there exist positive integers $m^1, \dots, m^k$ such that $g_i^{m^i} \in M$. Since N is a free abelian group, the index of M in N is no more than $m^1 \cdots m^k$. □

Proposition 5.8. *The finite covering map in Proposition 5.5 can be chosen to be a normal covering map.*

Proof. Let $Y = T^{2l} \times X_1 \times X_2 \cdots \times X_k$ be the finite cover in Proposition 5.5, and let $p : Y \to X$ be the covering map. It induces an injective morphism of fundamental groups $p_* : \pi_1(Y, y) \to \pi_1(X, x)$ for $x = p(y)$. Since each X_i is simply connected, $\pi_1(Y) \cong \pi_1(T^{2l}) \cong \mathbb{Z}^{2l}$. Since p is a finite covering map, the index of $p_*(\pi_1(Y, y))$ in $\pi_1(X, x)$ is finite. By Lemma 5.7, there is a finite index subgroup M of $\pi_1(Y, y)$, such that $p_*(M)$ is a normal subgroup of $\pi_1(X, x)$. We have a covering map $p_1 : Y \to Y$ (given by the covering map $T^{2l} \to T^{2l}$) with $p_{1*}(\pi_1(Y, y_1)) = M \subset \pi_1(Y, y)$ for some $y_1 \in p_1^{-1}(y)$. Then the covering map $p \circ p_1 : Y \to X$ is a finite normal covering map since $p_* \circ p_{1*}(\pi_1(Y, y_1)) = p_*(M)$ is a normal subgroup of $\pi_1(X, x)$ with finite index. □

Theorem 5.9. *Let X be a compact Ricci-flat Kähler manifold. Let $Y = T^{2l} \times X_1 \times X_2 \cdots \times X_k$ be the finite cover of X in Proposition 5.5. Let $p : Y \to X$ be the finite normal covering map in Proposition 5.8, and let $G(Y, p)$ be the covering transformation group. Then*

$$\Gamma(X, \Omega_X^{ch}) \cong (\Gamma(T^{2l}, \Omega_{T^{2l}}^{ch}) \bigotimes (\otimes_{i=1}^n \mathcal{A}_0(X_i)) \bigotimes (\otimes_{i=n+1}^k \mathcal{A}_1(X_i)))^{G(Y,p)}$$

through p^*.

Proof. Assume the dimension of X_i is d_i. By Proposition 5.6, we can assume the holonomy group of X_i is $SU(d_i)$ for $1 \le i \le n$ and the holonomy group of X_j is $Sp(\frac{d_j}{2})$ for $n < j \le k$. By Theorem 5.3,

$$\Gamma(Y, \Omega_Y^{\mathrm{ch}}) = \Gamma(T^{2l}, \Omega_{T^{2l}}^{\mathrm{ch}}) \bigotimes (\otimes_{i=1}^n \mathcal{A}_0(X_i)) \bigotimes (\otimes_{i=n+1}^k \mathcal{A}_1(X_i))$$

By Proposition 5.4, $\Gamma(Y, \Omega_Y^{\mathrm{ch}}) \cong \Gamma(Y, \Omega_Y^{\mathrm{ch}})^{G(Y,p)}$ through p^*. So

$$\Gamma(X, \Omega_X^{\mathrm{ch}}) \cong (\Gamma(T^{2l}, \Omega_{T^{2l}}^{\mathrm{ch}}) \bigotimes (\otimes_{i=1}^n \mathcal{A}_0(X_i)) \bigotimes (\otimes_{i=n+1}^k \mathcal{A}_1(X_i)))^{G(Y,p)}$$

through p^*. □

Since $\Gamma(T^{2l}, \Omega_{T^{2l}}^{\mathrm{ch}}) \cong \mathcal{W}_+(\mathbb{C}^{2l})$, the above theorem gives the space of global sections of chiral de Rham complex on compact Ricci-flat Kähler manifolds explicitly.

Declarations

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References

1. Ben-Zvi, D., Heluani, R., Szczesny, M.: Supersymmetry of the chiral de Rham complex. Compos. Math. **144**, 503–521 (2008)
2. Dai, X.: Chiral de Rham complex on the upper half plane and modular forms, arXiv: 2011.07696
3. Ekstrand, J., Heluani, R., Källén, J., Zabzine, M.: Chiral de Rham complex on Riemannian manifolds and special holonomy. Comm. Math. Phys. **318**(3), 575–613 (2013)
4. Friedan, D., Martinec, E., Shenker, S.: Conformal invariance, supersymmetry and string theory. Nucl. Phys. B **271**, 93–165 (1986)
5. Joyce, D.D.: Compact Manifolds with Special Holonomy. Oxford University Press, Oxford (2000)
6. Kac, V.: Vertex Algebras for Beginners. University Lecture Series 10. Providence, RI: American
7. Linshaw, A., Song, B.: Standard monomials and invariant theory for arc spaces I: general linear group, arXiv:2108.06864
8. Linshaw, A., Song, B.: Standard monomials and invariant theory for arc spaces II: symplectic group, arXiv:2108.08989
9. Linshaw, A., Song, B.: Standard monomials and invariant theory for arc spaces III: special linear group, arXiv:2108.08991
10. Linshaw, A., Song, B.: Cosets of affine vertex algebras via arc spaces, arXiv:2109.09050
11. Linshaw, A., Schwarz, G., Song, B.: Jet schemes and invariant theory. Ann. I. Fourier **65**(6), 2571–2599 (2015)
12. Linshaw, A., Schwarz, G., Song, B.: Arc spaces and the vertex algebra commutant problem. Adv. Math. **277**, 338–364 (2015)
13. Malikov, F., Schechtman, V., Vaintrob, A.: Chiral de Rham complex. Commun. Math. Phys. **204**, 439–473 (1999)
14. Malikov, F., Schechtman, V.: Chiral de Rham complex. II, Differential topology, infinite-dimensional Lie algebras, and applications, pp. 149–188 (1999)
15. Odake, S.: Extension of $N = 2$ superconformal algebra and Calabi-Yau compactification. Modern Phys. Lett. A **4**(6), 557–568 (1989)
16. Song, B.: The global sections of the chiral de Rham complex on a Kummer surface. Int. Math. Res. Not. IMRN **14**, 4271–4296 (2016)
17. Song, B.: Vector bundles induced from jet schemes. Trans. Am. Math. Soc. **374**(4), 2661–2685 (2021)
18. Song, B.: The global sections of chiral de Rham complexes on compact Ricci-flat Kähler manifolds. Commun. Math. Phys. **382**, 351–379 (2021)

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