Sufficient Conditions for Optimality in Finite-Horizon Two-Player Zero-Sum Hybrid Games

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***Abstract*— A finite-horizon two-player zero-sum game under dynamic constraints given in terms of hybrid dynamical systems is formulated in this paper. Sufficient conditions that consist of a hybrid version of the Hamilton-Jacobi-Isaacs equations are provided to guarantee that a pure strategy is a saddle-point equilibrium for the game. It is shown that when the players select the optimal strategy, the value function can be evaluated without needing computing solutions. Using this framework, a finite-horizon optimal control problem under an adversarial action with decision-making agents exhibiting hybrid dynamics is addressed.**

I. Introduction

Games involving multiple players with different interests emerge in multi-agent systems in noncooperative settings, e.g., [1], [2]. Generally speaking, a game is an optimization problem with multiple players, constraints that enforce the "rules" of the game, and payoff functions to be optimized through the selection of decision variables. Constraints on the actions taken by the players formulated as dynamic relationships (i.e., involving time) lead to *dynamic games*. Differential games pertain to the case when these constraints are given in terms of differential equations; see, e.g., [3] and the references therein. Interestingly, the combination of physics, computing, and networks lead to dynamic constraints that exhibit both continuous and discrete behavior. In particular, intermittent information availability, resets of variables such as expiring timers, and other nonsmooth and instantaneous changes of the state lead to dynamic constraints that can be conveniently captured using hybrid system models.

Unfortunately, tools for the design of algorithms for games with such hybrid dynamic constraints, which we refer to as *hybrid games* [4], are not as developed as those for differential games. Additional constraints arise when the optimization problem is to be solved in a finite horizon, which is typically studied using backward induction tools [5]. Nevertheless, setting a priori a specific combination of the amount of continuous evolution and discrete evolution allowed to a hybrid system significantly restricts the set in which the optimization problem is solved. Motivated by the lack of tools for the design of algorithms for finite-horizon hybrid games, following [4], we formulate a framework for the study of two-player zero-sum games with generic hybrid dynamic constraints as in [6]. Specifically, in this paper, we formulate a finite horizon optimization problem in which the cost functional includes a stage cost that penalizes the continuous evolution (or flow) and the discrete evolution (or jumps) of the variables, as well as their final value, via a terminal cost. The dynamic constraint in the game is hybrid and given in terms of hybrid equations [7], as

$$\mathcal{H}\begin{cases} \dot{x} = F(x, u_{C1}, u_{C2}) & (x, u_{C1}, u_{C2}) \in C \\ x^+ = G(x, u_{D1}, u_{D2}) & (x, u_{D1}, u_{D2}) \in D \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $(u_{C1}, u_{D1}) \in \mathbb{R}^{m_{C1}} \times \mathbb{R}^{m_{D1}}$ is the input chosen by player P_1, and $(u_{C2}, u_{D2}) \in \mathbb{R}^{m_{C2}} \times \mathbb{R}^{m_{D2}}$ is the input chosen by player P_2. The *flow map* $F : \mathbb{R}^n \times \mathbb{R}^{m_{C1} \times m_{C2}} \to \mathbb{R}^n$ captures the continuous evolution of the system on the *flow set* C. The *jump map* $G : \mathbb{R}^n \times \mathbb{R}^{m_{D1} \times m_{D2}} \to \mathbb{R}^n$ describes the discrete evolution of the system on the *jump set* D. In this framework, the data of the hybrid system $\mathcal{H}$ is given by (C, F, D, G).

The conditions on the finite-horizon optimization problem formulated in this paper are similar to their counterparts in the differential/dynamic game theory literature. Nevertheless, in contrast to [4] and conventional finite-horizon game theory, the notion of terminal time herein allows for state trajectories with terminal times belonging to a set $\mathcal{T}$, called the terminal set. To account for hybrid time domains, which are introduced in Section II, a hybrid time domain-like geometry is assumed for $\mathcal{T}$ as in [8].

For the broad class of systems modeled as in (1), when solutions for a given input are unique, we consider a cost functional $\mathcal{J} : \mathbb{R}^n \times \mathbb{R}^{m_{C1} \times m_{C2}} \times \mathbb{R}^{m_{D1} \times m_{D2}}$ associated to the solution to $\mathcal{H}$ from $\xi \in \mathbb{R}^n$ and study the problem

$$\underset{(u_{C1}, u_{D1})}{\text{minimize}} \ \underset{(u_{C2}, u_{D2})}{\text{maximize}} \ \mathcal{J}(\xi, u_{C1}, u_{C2}, u_{D1}, u_{D2}) \tag{2}$$

over the set of input actions with terminal time in $\mathcal{T}$, as a zero-sum two-player finite-horizon hybrid game. The main contributions of this paper are summarized as follows.

- We present a framework for studying finite-horizon two-player zero-sum games with generic hybrid dynamic constraints.
- We present sufficient conditions based on Hamilton-Jacobi-Isaacs-like equations to attain a finite-horizon saddle-point equilibrium and evaluate the game value function without computation of solutions.

The remainder of this paper is organized as follows. Preliminary concepts are introduced in Section II. In Section III, we formulate and analyze two-player zero-sum hybrid games with finite-horizon, from a rigorous problem formulation

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to saddle-equilibria design in Theorem 3.6. A numerical example and an application are presented in Section IV, displaying the versatility of the approach. Section V provides conclusions, closing remarks, and future work. Due to space constraints, proofs and other details are not included and will be published elsewhere.

Notation. Given two vectors x, y, we use the equivalent notation $(x, y) = [x^\top y^\top]^\top$. The symbol $\mathbb{N}$ denotes the set of natural numbers including zero. The symbol $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative reals.

II. Preliminaries

A. Hybrid Systems with Inputs

The concept of hybrid time domain, in which solutions to the dynamical system $\mathcal{H}$ are fully described, is presented.

Definition 2.1: (Hybrid time domain) *A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if, for each $(T, J) \in E$, the set $E \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain, i.e., it can be written in the form $\bigcup_{j=0}^{J}([t_j, t_{j+1}] \times \{j\})$ for some finite nondecreasing sequence of times $\{t_j\}_{j=0}^{J+1}$ with $t_{J+1} = T$. Each element $(t, j) \in E$ denotes the elapsed hybrid time, which indicates that t seconds of flow time and j jumps have occurred.*

A hybrid signal is a function defined on a hybrid time domain. Given a hybrid signal ϕ and $j \in \mathbb{N}$, we define $I_\phi^j = \{t : (t, j) \in \operatorname{dom} \phi\}$.

Definition 2.2: (Hybrid arc) *A hybrid signal $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$ is called a hybrid arc if, for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t, j)$ is locally absolutely continuous on the interval I_ϕ^j. A hybrid arc ϕ is compact if $\operatorname{dom} \phi$ is compact.*

In this article, the same symbols are used to denote an input action and its values. The context clarifies the meaning of u as follows: "the function u," "the signal u," or "the hybrid signal u" that appears in "the solution pair (ϕ, u)" refer to the input action, whereas "u" refers to the input value as a point in $\mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ in any other case. The reader can replace "the function u" with "u_ϕ," the input action yielding the system to a response described by the hybrid arc ϕ.

Definition 2.3: (Hybrid Input) *A hybrid signal u is a hybrid input if, for each $j \in \mathbb{N}$, the function $t \mapsto u(t, j)$ is Lebesgue measurable and locally essentially bounded on the interval I_u^j.*

Let $\mathcal{X}$ be the set of hybrid arcs $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$ and $\mathcal{U} = \mathcal{U}_C \times \mathcal{U}_D$ the set of hybrid inputs $u = (u_C, u_D) : \operatorname{dom} u \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$, where $u_C = (u_{C1}, u_{C2})$, $m_{C1} + m_{C2} = m_C$, $u_D = (u_{D1}, u_{D2})$, and $m_{D1} + m_{D2} = m_D$.

We say that a solution pair (ϕ, u) to $\mathcal{H}$ (see [6, Definition 2.6]) is maximal if it cannot be extended, and we say it is complete when $\operatorname{dom} \phi$ is unbounded. We denote by $\hat{\mathcal{S}}_\mathcal{H}(M)$ the set of solution pairs (ϕ, u) to (1) such that $\phi(0, 0) \in M$. The set $\mathcal{S}_\mathcal{H}(M) \subset \hat{\mathcal{S}}_\mathcal{H}(M)$ denotes all maximal solution pairs. Given $\xi \in \mathbb{R}^n$, we denote by $\mathcal{U}_\mathcal{H}(\xi)$ the set of input actions u that render maximal solutions to $\mathcal{H}$ from ξ.

We define the projections of C and D onto $\mathbb{R}^n$, respectively as $\Pi(C) := \{\xi \in \mathbb{R}^n : \exists u_C \in \mathbb{R}^{m_C} : (\xi, u_C) \in C\}$, $\Pi(D) := \{\xi \in \mathbb{R}^n : \exists u_D \in \mathbb{R}^{m_D} : (\xi, u_D) \in D\}$ and the set-valued maps $\Pi_u(x, C) := \{u_C \in \mathbb{R}^{m_C} : (x, u_C) \in C\}$, $\Pi_u(x, D) := \{u_D \in \mathbb{R}^{m_D} : (x, u_D) \in D\}$ denoting the set of input values available for a given state. Likewise, we define $\operatorname{dom}_t \phi := \{t \in \mathbb{R}_{\geq 0} : \exists j \in \mathbb{N}_{\geq 0} \ : \ (t, j) \in \operatorname{dom} \phi\}$, $\sup_t \operatorname{dom} \phi := \sup\{t \in \mathbb{R}_{\geq 0} : \exists j \in \mathbb{N}_{\geq 0} \ : \ (t, j) \in \operatorname{dom} \phi\}$, $\sup_j \operatorname{dom} \phi := \sup\{j \in \mathbb{N}_{\geq 0} : \exists t \in \mathbb{R}_{\geq 0} \ : \ (t, j) \in \operatorname{dom} \phi\}$, and $\sup \operatorname{dom} \phi = (\sup_t \operatorname{dom} \phi, \sup_j \operatorname{dom} \phi)$.

III. Finite-horizon Two-player Zero-sum Hybrid Games

A. Motivation

Consider a system denoted $\mathcal{H}$, with state $x \in \mathbb{R}$, input $u_C := (u_{C1}, u_{C2}) \in \mathbb{R}^2$, and dynamics

$$\begin{aligned} \dot{x} &= F(x, u_C) := ax + Bu_C & x \in [0, \delta] \\ x^+ &= G(x) := \sigma & x = \mu \end{aligned} \tag{3}$$

where $a < 0, B = [b_1 \ b_2]$, with $b_1, b_2 \in \mathbb{R}$, and $\delta \geq \mu > \sigma > 0$. Here, u_{C1} is designed by player P_1, which aims to minimize a cost functional $\mathcal{J}$, while player P_2 seeks to maximize it by choosing u_{C2}. The terminal set $\mathcal{T}$ describes the hybrid time domain of the set of solutions over which the optimization problem is solved and is defined as

$$\mathcal{T} := \{(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N} : \max\{T/\delta_p, J\} = \tau_p\} \tag{4}$$

where $\tau_p \in \mathbb{N} \setminus \{0\}$ defines the number of jumps and $\delta_p > 0$ determines the ordinary time t allowed by $\mathcal{T}$. Applying classical continuous-time or discrete-time game theoretical tools to solve this optimization problem might lead to suboptimal input actions due to solutions to $\mathcal{H}$ potentially exhibiting both continuous and discrete behavior. Indeed, solutions starting from δ can either jump or flow at μ[1]. In Figure 1, the response ϕ_h to the hybrid system $\mathcal{H}$ from $\xi = \delta = 2$ for a certain input action displays this behavior. For the case in which the cost functional $\mathcal{J}$ penalizes both the continuous evolution and the discrete evolution, the associated cost to ϕ_h, denoted J_h, is calculated using the hybrid methods developed in this paper. In contrast, the costs J_c and J_d are computed using continuous-time methods and discrete-time methods, respectively. As the plot shows, existing tools are incapable of properly evaluating the cost of solutions to hybrid systems. [2]

We are interested in designing feedback laws, potentially time-dependent, to solve finite-horizon two-player hybrid games. This motivates the need for a hybrid zero-sum game formulation for scenarios with finite horizon and results providing sufficient conditions to certify optimality in a min-max sense of feedback laws for hybrid systems. In addition,

[1]The domain of u_C determines whether jump or flow occurs from μ.

[2]Code at https://github.com/HybridSystemsLab/HybridGames-FiniteHorizon

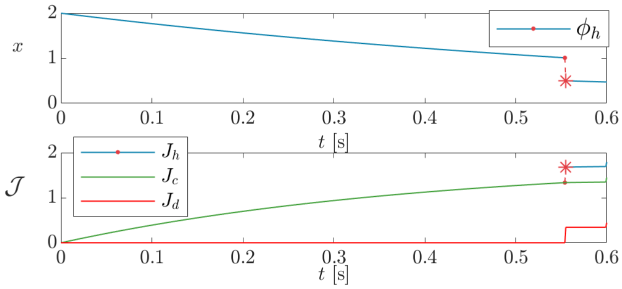

Fig. 1. A solution to (3) (blue) and its cost with time horizon of 2 jumps or 0.6 seconds. The cost computed with continuous-time methods is displayed in green, and with discrete-time methods is displayed in red. The parameters used are $a = -1, b_1 = b_2 = 1, \delta = \xi = 2, \mu = 1, \sigma = 0.5, Q_C = 1, R_{C1} = 1.304$, and $R_{C2} = -4$.

we are interested in solutions that guarantee optimality without the need of computing solutions.

B. Formulation

Following the formulation in [3], for each $i \in \{1, 2\}$, consider the i-th player P_i with dynamics described by $\mathcal{H}_i$ as in (1) with data (C_i, F_i, D_i, G_i), state $x_i \in \mathbb{R}^{n_i}$, and input $u_i = (u_{Ci}, u_{Di}) \in \mathbb{R}^{m_{Ci}} \times \mathbb{R}^{m_{Di}}$, where $C_i \subset \mathbb{R}^n \times \mathbb{R}^{m_C}$, $F_i : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}^{n_i}$, $D_i \subset \mathbb{R}^n \times \mathbb{R}^{m_D}$ and $G_i : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}^{n_i}$, with $n_1 + n_2 = n$. We denote by $\mathcal{U}_i = \mathcal{U}_{Ci} \times \mathcal{U}_{Di}$ the set of hybrid inputs for $\mathcal{H}_i$; see Definition 2.3.

Definition 3.1: (Elements of a two-player zero-sum hybrid game) *A two-player zero-sum hybrid game is composed by*

1) *The state $x = (x_1, x_2) \in \mathbb{R}^n$, where, for each $i \in \{1, 2\}$, $x_i \in \mathbb{R}^{n_i}$ is the state of player P_i.*
2) *The set of joint input actions $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$ with elements $u = (u_1, u_2)$, where, for each $i \in \{1, 2\}$, u_i is a hybrid input. For each $i \in \{1, 2\}$, P_i selects u_i independently from u_{3-i}, thus allowing the joint input action u to have components u_i that each player chooses independently.*
3) *The dynamics of the game, described as in* (1) *and denoted by $\mathcal{H}$, with data*
$$\begin{aligned} C &:= C_1 \cap C_2 \\ F(x, u_C) &:= (F_1(x, u_C), F_2(x, u_C)) \\ D &:= D_1 \cup D_2 \\ G(x, u_D) &:= \{\hat{G}_i(x, u_D) : (x, u_D) \in D_i, i \in \{1, 2\}\} \end{aligned}$$
where $\hat{G}_1(x, u_D) = (G_1(x, u_D), I_{n_2})$, and $\hat{G}_2(x, u_D) = (I_{n_1}, G_2(x, u_D))$.
4) *For each $i \in \{1, 2\}$, a strategy space K_i of P_i defined as a collection of mappings $\gamma_i : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^{m_{Ci}} \times \mathbb{R}^{m_{Di}}$. The strategy space of the game K is the collection of mappings with elements $\gamma = (\gamma_1, \gamma_2)$, where $\gamma_i \in K_i$ for each $i \in \{1, 2\}$. Each $\gamma_i \in K_i$ is said to be a permissible pure*[3] *strategy for P_i.*
5) *A scalar-valued functional $(\xi, u) \mapsto \mathcal{J}_i(\xi, u)$ defined for each $i \in \{1, 2\}$, and called the cost associated to P_i. We refer to a single cost functional $\mathcal{J} = \mathcal{J}_1 = -\mathcal{J}_2$ as the cost associated to the solution to $\mathcal{H}$ from ξ, and its structure is defined for each type of game.*

When the mathematical description of a game is in Kuhn's extensive form, the evolution of the game, the decision making process, the sharing of information between the players, and their outcomes are described in the formulation. This allows to guarantee the game admits a solution. For the formulation in Definition 3.1 to be in Kuhn's extensive form, additional assumptions are required such that each strategy has a unique cost correspondence. For a given initial condition, a given strategy potentially leads to nonunique solutions to $\mathcal{H}$, each of which may have a different cost[4].

Given the formulation of the elements of a zero-sum hybrid game in Definition 3.1, its solution is defined as follows.

Definition 3.2: (Saddle-point equilibrium) *Consider a two-player zero-sum game, with dynamics $\mathcal{H}$ as in (1) with $\mathcal{J}_1 = \mathcal{J}, \mathcal{J}_2 = -\mathcal{J}$, for a given cost functional $\mathcal{J} : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$. We say that a strategy $\gamma = (\gamma_1, \gamma_2) \in K$ is a saddle-point equilibrium if for each $\xi \in \Pi(\overline{C} \cup D)$, every hybrid input $u^* = (u_1^*, u_2^*)$ rendering a maximal response ϕ^* to $\mathcal{H}$ from ξ, with components defined as $\operatorname{dom} \phi^* \ni (t, j) \mapsto u_i^*(t, j) = \gamma_i(t, j, \phi^*(t, j))$ for each $i \in \{1, 2\}$, satisfies*

$$\mathcal{J}(\xi, (u_1^*, u_2)) \leq \mathcal{J}(\xi, u^*) \leq \mathcal{J}(\xi, (u_1, u_2^*)) \qquad (5)$$

for all hybrid inputs u_1 such that there exists ϕ such that $(\phi, (u_1, u_2^*)) \in \mathcal{S}_{\mathcal{H}}(\xi)$, and for all hybrid inputs u_2 such that there exists ϕ such that $(\phi, (u_1^*, u_2)) \in \mathcal{S}_{\mathcal{H}}(\xi)$.

Definition 3.2 is a generalization of the classical pure strategy Nash equilibrium [3, (6.3)] to the case where the players exhibit hybrid dynamics and opposite optimization goals. In words, we refer to the strategy $\gamma^* = (\gamma_1^*, \gamma_2^*)$ as a saddle-point when a player P_i cannot improve the cost $\mathcal{J}_i$ by playing any strategy different from γ_i^* when the player P_{3-i} is playing the strategy of the saddle-point, γ_{3-i}^*. Notice that the saddle-point, as a solution to the zero-sum two-player game, is a strategy in K, though the concept of a solution to a hybrid system $\mathcal{H}$ is a hybrid arc.

Next, we formulate a finite-horizon optimization problem to solve the two-player zero-sum hybrid game and provide sufficient conditions to characterize the pure strategy saddle-point equilibrium. Consider a two-player zero-sum hybrid game with dynamics $\mathcal{H}$ as in (1) for given (C, F, D, G). The cost evaluation tools employed in approaches based on dynamic programming require uniqueness of solutions to $\mathcal{H}$ for a given input action u from an initial condition ξ; this justifies the following assumption.

[3]This is in contrast to when K_i is defined as a probability distribution, in which case $\gamma_i \in K_i$ is referred to as a mixed strategy.

[4]Notice that a given strategy γ can lead to multiple input actions due to a nonempty $C \cap D$.

Assumption 3.3: *The flow map F and the flow set C are such that solutions to*

$$\dot{x} = F(x, u_C) \qquad (x, u_C) \in C$$

are unique for each input u_C. The jump map G is single-valued.

Under Assumption 3.3, solutions to $\mathcal{H}$ are unique[5] for each $u \in \mathcal{U}$.

Given a solution (ϕ, u) to $\mathcal{H}$, $(T, J) \in \operatorname{dom}(\phi, u)$ is referred to the terminal time of (ϕ, u) if $t \leq T$ and $j \leq J$ for all $(t, j) \in \operatorname{dom}(\phi, u)$. Given $\mathcal{T} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$, let us denote by $\hat{\mathcal{S}}_{\mathcal{H}}^{\mathcal{T}}(M) \subset \hat{\mathcal{S}}_{\mathcal{H}}(M)$ the set of compact solutions to $\mathcal{H}$ from M, with terminal time in $\mathcal{T}$, i.e., if $(\phi, u) \in \hat{\mathcal{S}}_{\mathcal{H}}^{\mathcal{T}}(M)$ and $\max \operatorname{dom}(\phi, u) = (T, J)$, then $(T, J) \in \mathcal{T}$. We denote by $\mathcal{U}_{\mathcal{H}}^{\mathcal{T}}(M)$ the set of input actions u such that compact solutions to $\mathcal{H}$ from M for u have terminal time in $\mathcal{T}$.

Given $\xi \in \Pi(C \cup D)$, a joint input action $u = (u_C, u_D) \in \mathcal{U}$, the stage cost for flows $L_C : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$, the stage cost for jumps $L_D : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, and the terminal cost $q : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}$, we define the cost associated to the solution (ϕ, u) to $\mathcal{H}$ from ξ with terminal time $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, under Assumption 3.3, as

$$\begin{aligned}\mathcal{J}(\xi, u) := \sum_{j=0}^{J} \int_{t_j}^{t_{j+1}} & L_C(t, j, \phi(t, j), u_C(t, j)) dt \\ & + \sum_{j=0}^{J-1} L_D(t, j, \phi(t_{j+1}, j), u_D(t_{j+1}, j)) \\ & \qquad + q(T, J, \phi(T, J))\end{aligned} \tag{6}$$

where $t_{J+1} = T$ and $\{t_j\}_{j=0}^{\sup_j \operatorname{dom} \phi}$ is a nondecreasing sequence associated to the definition of the hybrid time domain of ϕ; see Definition 2.2. For this scenario, the terminal set is defined as in (4). Let us also denote the set of points contained by the window described by $\mathcal{T}$ and the coordinate axes as

$$\mathcal{T}_{\leq \tau_p} := \{(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N} : \max\{t/\delta_p, j\} \leq \tau_p\} \tag{7}$$

Using the formulation above, the two-player zero-sum game consists of solving the following problem.

Problem ($\star$): Given $\xi \in \mathbb{R}^n$, $\mathcal{T} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$, under Assumption 3.3, solve

$$\underset{u = (u_1, u_2) \in \mathcal{U}_{\mathcal{H}}^{\mathcal{T}}(\xi)}{\underset{u_1}{\text{minimize}} \ \underset{u_2}{\text{maximize}}} \ \mathcal{J}(\xi, u) \tag{8}$$

where $\mathcal{U}_{\mathcal{H}}^{\mathcal{T}}$ is the set of joint input actions yielding solutions with terminal time in $\mathcal{T}$.

Remark 3.4: (Finite-horizon saddle-point equilibrium and min-max control) *A solution to Problem ($\star$), when it exists, can be expressed in terms of the pure strategy saddle-point equilibrium γ for the two-player zero-sum finite-horizon game. Each $u^* = (u_1^*, u_2^*)$ rendering a response ϕ^* such that $(\phi^*, u^*) \in \mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)$, defined as $\operatorname{dom} \phi^* \ni (t, j) \mapsto u_i^*(t, j) = \gamma_i(t, j, \phi^*(t, j))$ for each $i \in \{1, 2\}$, satisfies*

$$u^* = \underset{u = (u_1, u_2) \in \mathcal{U}_{\mathcal{H}}^{\mathcal{T}}(\xi)}{\arg \min_{u_1} \max_{u_2}} \ \mathcal{J}(\xi, u) = \underset{u = (u_1, u_2) \in \mathcal{U}_{\mathcal{H}}^{\mathcal{T}}(\xi)}{\arg \max_{u_2} \min_{u_1}} \ \mathcal{J}(\xi, u)$$

and it is referred to as a min-max control at ξ.

Definition 3.5: (Value function) *Given $\xi \in \Pi(\overline{C} \cup D)$, and parameters $\delta_p > 0$ and $\tau_p \in \mathbb{N} \setminus \{0\}$ defining the set $\mathcal{T} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ as in (4), under Assumption 3.3, the value function at ξ is given by*

$$\mathcal{J}_{\mathcal{T}}^*(\xi) := \underset{u = (u_1, u_2) \in \hat{\mathcal{S}}_{\mathcal{H}}^{\mathcal{T}}(\xi)}{\min_{u_1} \max_{u_2}} \ \mathcal{J}(\xi, u) = \underset{u = (u_1, u_2) \in \hat{\mathcal{S}}_{\mathcal{H}}^{\mathcal{T}}(\xi)}{\max_{u_2} \min_{u_1}} \ \mathcal{J}(\xi, u) \tag{9}$$

If there does not exist (ϕ, u) from ξ such that $\operatorname{dom} \phi$ ever enters $\mathcal{T}$, i.e. if $\hat{\mathcal{S}}_{\mathcal{H}}^{\mathcal{T}}(\xi)$ is empty, then $\mathcal{J}_{\mathcal{T}}^*(\xi) := \infty$.

C. Design of Saddle-Point Equilibrium for Finite-Horizon Hybrid Games

The following theorem provides sufficient conditions to characterize the value function $\mathcal{J}_{\mathcal{T}}^*$ and the feedback law that attains it. It addresses the solution to Problem ($\star$) for each $\xi \in \Pi(\overline{C} \cup D)$, showing that the optimizer is the saddle-point equilibrium.

Theorem 3.6: (Hamilton-Jacobi-Isaacs (HJI) equations for Problem ($\star$)) *Given a two-player zero-sum hybrid game with dynamics $\mathcal{H}$ as in (1) described by (C, F, D, G) satisfying Assumption 3.3, stage costs $L_C : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$ and $L_D : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, terminal cost $q : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}$, and parameters $\delta_p > 0$ and $\tau_p \in \mathbb{N} \setminus \{0\}$ defining the sets $\mathcal{T}, \mathcal{T}_{\leq \tau_p} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ as in (4) and (7), respectively, suppose the following hold:*

1) *There exists a function $V : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on a neighborhood of $\mathcal{T}_{\leq \tau_p} \times \Pi(C)$ satisfying the Hamilton-Jacobi-Isaacs hybrid PDEs given as*

$$\begin{aligned} 0 &= \underset{u_C = (u_{C1}, u_{C2}) \in \Pi_u(x, C)}{\min_{u_{C1}} \max_{u_{C2}}} \Big\{ L_C(t, j, x, u_C) + \frac{\partial V}{\partial t}(t, j, x) \\ &\qquad\qquad + \frac{\partial V}{\partial x}(t, j, x) F(x, u_C) \Big\} \\ &= \underset{u_C = (u_{C1}, u_{C2}) \in \Pi_u(x, C)}{\max_{u_{C2}} \min_{u_{C1}}} \Big\{ L_C(t, j, x, u_C) + \frac{\partial V}{\partial t}(t, j, x) \\ &\qquad\qquad + \frac{\partial V}{\partial x}(t, j, x) F(x, u_C) \Big\} \\ &\qquad\qquad \forall ((t, j), x) \in \mathcal{T}_{\leq \tau_p} \times \Pi(C) \end{aligned} \tag{10}$$

$$\begin{aligned} V(t, j, x) &= \underset{u_D = (u_{D1}, u_{D2}) \in \Pi_u(x, D)}{\min_{u_{D1}} \max_{u_{D2}}} \{ L_D(t, j, x, u_D) \\ &\qquad\qquad + V(t, j + 1, G(x, u_D)) \} \\ &= \underset{u_D = (u_{D1}, u_{D2}) \in \Pi_u(x, D)}{\max_{u_{D2}} \min_{u_{D1}}} \{ L_D(t, j, x, u_D) \\ &\qquad\qquad + V(t, j + 1, G(x, u_D)) \} \\ &\qquad\qquad \forall ((t, j), x) \in \mathcal{T}_{\leq \tau_p} \times \Pi(D) \end{aligned} \tag{11}$$

[5] Under Assumption 3.3, the domain of the input u specifies whether at $C \cap D$ there is a jump or flow.

2) For each $\xi \in \Pi(\overline{C} \cup D)$, each $(\phi, u) \in \mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)$ satisfies

$$V(t, j, \phi(t, j)) = q(t, j, \phi(t, j)) \qquad \forall (t, j) \in \operatorname{dom} \phi \cap \mathcal{T} \quad (12)$$

Then

$$\mathcal{J}_{\mathcal{T}}^{*}(\xi) = V(0, 0, \xi) \qquad \forall \xi \in \Pi(\overline{C} \cup D), \qquad (13)$$

and any feedback law $\gamma := (\gamma_C, \gamma_D) : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ with values

$$\gamma_C(t, j, x) \in \underset{u_C = (u_{C1}, u_{C2}) \in \Pi_u(x, C)}{\arg \min_{u_{C1}} \max_{u_{C2}}} \left\{ L_C(t, j, x, u_C) + \frac{\partial V}{\partial x}(t, j, x) F(x, u_C) \right\} \qquad \forall ((t, j), x) \in \mathcal{T}_{\leq \tau_p} \times \Pi(C)$$

and

$$\gamma_D(t, j, x) \in \underset{u_D = (u_{D1}, u_{D2}) \in \Pi_u(x, D)}{\arg \min_{u_{D1}} \max_{u_{D2}}} \{ L_D(t, j, x, u_D) + V(t, j+1, G(x, u_D)) \} \qquad \forall ((t, j), x) \in \mathcal{T}_{\leq \tau_p} \times \Pi(D)$$

is a pure strategy saddle-point equilibrium for the two-player zero-sum finite-horizon hybrid game with $\mathcal{J}_1 = \mathcal{J}$, $\mathcal{J}_2 = -\mathcal{J}$.

IV. Examples

We characterize the pure strategy saddle-point equilibrium and the value function for the example introduced in Section III.A.

Example 4.1: (Hybrid game with nonunique solutions for a given strategy) For the system in (3), consider the parameters δ_p and τ_p defining the sets $\mathcal{T}, \mathcal{T}_{\leq \tau_p} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ as in (4) and (7), respectively, and the cost functions $L_C(t, j, x, u_C) := x^2 Q_C + u_C^\top R_C u_C$, $L_D(t, j, x) := p(t)(x^2 - \sigma^2)$, and terminal cost $q(t, j, x) := p(t) x^2$, defining $\mathcal{J}$ as in (6), with $R_C := \begin{bmatrix} R_{C1} & 0 \\ 0 & R_{C2} \end{bmatrix}$, $Q_C, R_{C1}, -R_{C2} > 0$ and $p(t) > 0$ for all $t \geq 0$. The function p is such that

$$\frac{\mathrm{d}p}{\mathrm{d}t}(t) = -Q_C - 2p(t)a + p(t)^2 (b_1^2 R_{C1}^{-1} + b_2^2 R_{C2}^{-1}) \qquad (14)$$

for all $t \in [0, \delta_p \tau_p]$. The function $V(t, j, x) := p(t) x^2$ is such that

$$\underset{u_C = (u_{C1}, u_{C2}) \in \mathbb{R}^2}{\min_{u_{C1}} \max_{u_{C2}}} \left\{ L_C(t, j, x, u_C) + \tfrac{\partial V}{\partial x}(t, j, x) F(x, u_C) + \tfrac{\partial V}{\partial t}(t, j, x) \right\} = 0 \qquad (15)$$

holds for all (t, j, x) such that $x \in [0, \delta]$ and $(t, j) \in \mathcal{T}_{\leq \tau_p}$. The min-max in (15) is attained by $\gamma_C(t, j, x) = (-R_{C1}^{-1} b_1 p(t) x, -R_{C2}^{-1} b_2 p(t) x)$. In particular, thanks to (14), we have

$$L_C(t, j, x, \gamma_C(t, j, x)) + \tfrac{\partial V}{\partial x}(t, j, x) F(x, \gamma_C(t, j, x)) + \tfrac{\partial V}{\partial t}(t, j, x) = 0$$

Then, $V(t, j, x) = p(t) x^2$ is a solution to (10). In addition, the function V is such that

$$\underset{(u_{D1}, u_{D2}) \in \mathbb{R}^2}{\min_{u_{D1}} \max_{u_{D2}}} \{ L_D(t, j, x) + V(t, j+1, G(x)) \} = p(t) x^2 \qquad (16)$$

at $x = \mu$, which makes $V(t, j, x) = p(t) x^2$ a solution to (11). Given that V is continuously differentiable on $\mathbb{R} \times \mathbb{R}_{\geq 0}$ thanks to p being continuously differentiable on $\mathbb{R}_{\geq 0}$, and that (10) and (11) hold thanks to (15) and (16), from Theorem 3.6 we have that the value function is $\mathcal{J}_{\mathcal{T}}^{*}(\xi) := p(0)\xi^2$ for any $\xi \in [0, \delta] \cup \{\mu\}$.

Notice that solutions can potentially flow or jump at $x = \mu$. The set of all maximal responses from $\xi = \delta$ is denoted $\mathcal{R}_\kappa(\xi) = \{\phi_c, \phi_h\}$, where the continuous response ϕ_c is such that $\operatorname{dom} \phi_c = \mathbb{R}_{\geq 0} \times \{0\}$, and is given by $\phi_c(t, 0) = (\delta \exp((a - R_{C1}^{-1} b_1 p(t) - R_{C2}^{-1} b_2 p(t))t), t, 0)$ for all $t \in [0, \delta_p \tau_p]$. In simple words, ϕ_c flows from δ to 0 in $\delta_p \tau_p$ seconds. If $\tau_p > 1$, a jump can occur in the given time horizon, and the maximal response ϕ_h has domain $\operatorname{dom} \phi_h = ([0, t^h] \times \{0\}) \cup ([t^h, \delta_p \tau_p) \times \{1\})$, where $t^h \leq \delta_p \tau_p$, and is given by $\phi_h(t, 0) = (\delta \exp((a - R_{C1}^{-1} b_1 p(t) - R_{C2}^{-1} b_2 p(t))t), t, 0)$, $\phi_h(t, 1) = (\sigma \exp((a - R_{C1}^{-1} b_1 p(t) - R_{C2}^{-1} b_2 p(t))(t - t^h)), t, 1)$. In simple words, ϕ_h flows from δ to μ in t^h seconds, then it jumps to σ and flows to 0 for $\delta_p \tau_p - t^h$ seconds. Figure 2 illustrates this behavior. By denoting the corresponding input signals as $u_c = \gamma(t, j, \phi_c)$ and $u_h = \gamma(t, j, \phi_h)$, we show in the bottom plot of Figure 2 that the cost of the solutions (ϕ_c, u_c) and (ϕ_h, u_h) equal $p(0)\delta^2$. This corresponds to the saddle-point equilibrium in

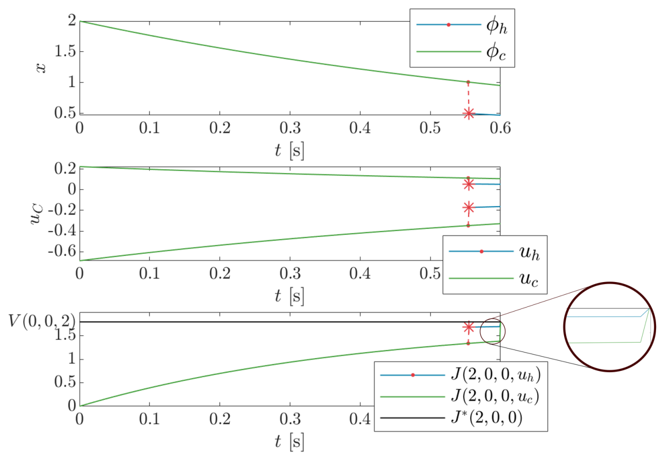

Fig. 2. Nonnunique solutions for a given strategy with time horizon defined by $\delta_p = 0.3$ and $\tau_p = 2$, attaining min-max optimal cost for $a = -1, b_1 = b_2 = 1, \delta = \xi = 2, \mu = 1, \sigma = 0.5, Q_C = 1, R_{C1} = 1.304, R_{C2} = -4$, and $p(0) = 0.4481$. The continuous solution is shown in green, and the hybrid solution is shown in blue.

Definition 3.2 with every maximal solution rendered by γ from $(t, j, \xi) = (0, 0, 2)$ attaining the optimal cost. □

Example 4.2: (Bouncing ball) Inspired by the problem in [9], consider a simplified model of a juggling system as in [10], with state $x \in \mathbb{R}^2$, input $u_D := (u_{D1}, u_{D2}) \in \mathbb{R}^2$, and dynamics

$$\begin{aligned} \dot{x} &= F(x) := (x_2, -1) && x \in \mathbb{R}_{\geq 0} \times \mathbb{R} \\ x^+ &= G(x, u_D) := (0, -\lambda x_2 + u_{D1} + u_{D2}) && (x, u_D) \in \{0\} \times \mathbb{R}_{\leq 0} \times \mathbb{R}^2 \end{aligned} \qquad (17)$$

where u_{D1} is the control input, u_{D2} is the action of an attacker, and $\lambda \in (0, 1)$ is the coefficient of restitution of the ball. The scenario in which u_{D1} is designed to minimize a cost functional $\mathcal{J}$ under the presence of the worst-case attack u_{D2} is formulated as a two-player zero-sum finite-horizon hybrid game. With the aim of pursuing minimum energy and distance to the origin at jumps, consider the cost functions $L_C(x, u_C) := 0$, $L_D(x, u_D) := x_2^2 Q_D + u_D^\top R_D u_D$, and terminal cost $q(x) := \frac{1}{2}x_2^2 + x_1$ defining $\mathcal{J}$ as in (6), with $R_D := \left[\begin{smallmatrix} R_{D1} & 0 \\ 0 & R_{D2} \end{smallmatrix}\right]$ and Q_D, R_{D1}, $-R_{D2} > 0$. Here, u_{D1} is designed by player P_1, which aims to minimize $\mathcal{J}$, while player P_2 seeks to maximize it by choosing u_{D2}. The function $V(x) := x_1 + \frac{1}{2}x_2^2$ is such that $\frac{\partial V}{\partial x}(x)F(x) = 0$ for all $x \in \mathbb{R}_{\geq 0} \times \mathbb{R}$, making V a solution to (10). In addition, the function V is such that

$$\min_{u_{D1}} \max_{u_{D2}} \underset{u_D=(u_{D1},u_{D2})\in\mathbb{R}^2}{} \{L_D(x, u_D) + V(G(x, u_D))\} = \tfrac{1}{2}x_2^2 \tag{18}$$

for all $(x, u_D) \in \{0\} \times \mathbb{R}_{\leq 0} \times \mathbb{R}^2$, and attained by $\gamma_D(x) = (\gamma_{D1}(x), \gamma_{D2}(x))$ with $\gamma_{D1}(x) = \frac{R_{D2}\lambda}{R_{D1}+R_{D2}+2R_{D1}R_{D2}} x_2$ and $\gamma_{D2}(z) = \frac{R_{D1}\lambda}{R_{D1}+R_{D2}+2R_{D1}R_{D2}} x_2$ when

$$Q_D = \frac{-2R_{D1}R_{D2}\lambda^2 + R_{D1} + R_{D2} + 2R_{D1}R_{D2}}{2R_{D1} + 2R_{D2} + 4R_{D1}R_{D2}}, \tag{19}$$

which makes V a solution to (11). Thus, given that V is continuously differentiable on $\mathbb{R}^2$, and that (10) and (11) hold thanks to (18) and (19), from Theorem 3.6, the value function is $\mathcal{J}_\mathcal{T}^*(\xi_1, \xi_2) := \frac{\xi_2^2}{2} + \xi_1$. Figure 3 displays this behavior.

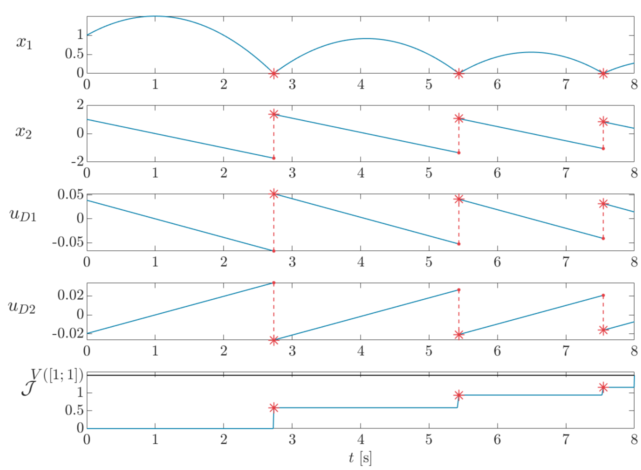

Fig. 3. Bouncing ball solutions attaining minimum cost under worst-case u_2, with $\tau_p = 100, \delta_p = 2/25, \lambda = 0.8, R_{D1} = 10, R_{D2} = -20$, and $Q_D = 0.189$.

In Figure 4, we let the players select feedback laws close with the Nash equilibrium and calculate the cost associated to the new laws. The variation of the cost along the changes in the feedback laws makes evident the saddle-point geometry. □

V. Conclusion and Future Work

In this paper, we employed a game theoretical approach to formulating games in which a control action is designed by a player P_1 to accomplish an optimization objective within a finite hybrid horizon while another player, P_2, has

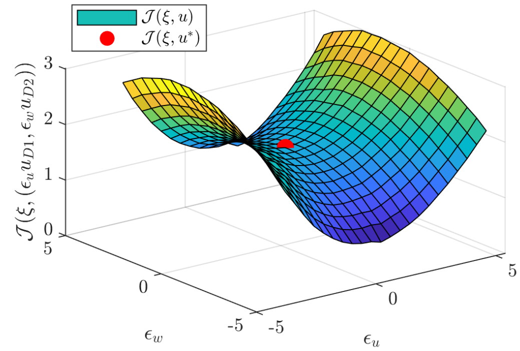

Fig. 4. Saddle point behavior in the cost of solutions to bouncing ball from $\xi = (1, 1)$ when the feedback gains vary around the optimal value. The cost is evaluated on solutions $(\phi, u) \in \mathcal{S}_\mathcal{H}^\mathcal{T}(\xi)$ with feedback law variations specified by ϵ_u and ϵ_u in $u = (\epsilon_u \gamma_1(t, j, \phi), \epsilon_w \gamma_2(t, j, \phi))$.

the goal to maximize it under hybrid dynamic constraints. A general formulation of hybrid games was proposed and used as the basis to state sufficient conditions in terms of Hamilton-Jacobi-Isaacs hybrid PDEs to attain the solution of the game, namely, the saddle-point equilibrium. The main result ensures that by playing the equilibrium action, P_1 minimizes the cost under the equilibrium action by player P_2. Future work includes studying necessary conditions to characterize the value function and finding bounds in the cost based on approximate versions of the optimal conditions.

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