A Mixed Integer Approach to Solve Hybrid Model Predictive Control Problems

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Abstract— **This paper presents an algorithm for solving the optimization problem associated with hybrid model predictive control for a class of discretized hybrid control systems. The proposed approach consists of reformulating the optimal control problem as a mixed integer quadratic problem (MIQP), which can be solved using well-established algorithms in the literature. Specifically, the given discretized hybrid control system is transformed into a mixed logical dynamical (MLD) system that, for the class of discretized hybrid control systems considered, gives rise to an MIQP. The MLD model is obtained through an intermediate step that transforms the discretized hybrid control system into a discrete-time control system.**

I. Introduction

Model predictive control (MPC) is a powerful feedback control technique as it assures asymptotic stability with optimality and constraint satisfaction [1]. Being an optimization-based technique, MPC tends to be computationally expensive and heavily depends on the performance of the optimization scheme employed. It is well documented that MPC may require substantial amount of time to compute due to the time required for the optimization scheme to terminate [2], [3].

As the optimal control problem (OCP) that is to be solved at each control recomputation event is, in general, a nonlinear programming problem, there are many approaches and algorithms available in the literature. For instance, the OCP associated with MPC can be formulated to predict the state and perform optimization sequentially or simultaneously. The OCP itself can be solved using myriad of methods, such as sequential quadratic programming techniques, penalty methods, Lagrangian-based approaches, interior point methods, among others. The survey [4] outlines these approaches and solution methods. These techniques are applicable to important classes of MPC problems for continuous-time and discrete-time systems. On the other hand, techniques for the solution of MPC problems for hybrid systems are much less developed. When the hybrid system is modeled as a piecewise affine (PWA) system or as a mixed logical dynamical (MLD) system, and the cost functions are quadratic, the OCP can be formulated as a mixed-integer quadratic problem (MIQP) [5], [6]. This approach is quite powerful as it enables the use of MIQP solvers available in the literature.

Motivated by the success of formulating the OCP as an MIQP for hybrid systems modeled as PWA or MLD systems, we propose an MIQP approach for the solution to the hybrid MPC problem formulated in [7], [8]. In these articles, hybrid systems are modeled by *hybrid equations*, which are given in terms of constrained differential and difference equations. A general theory of robust asymptotic stability and hybrid control design for such class of systems is available in [9] and [10], respectively, where the versatility and generality of the framework is displayed in several applications. Although key theoretical aspects of hybrid MPC are addressed in [7], [8], the numerical solution of the associated (hybrid) OCP is not investigated therein.

As a first step towards efficient methods for the solution to such problems, we consider the discretized version of hybrid equations considered in [11] (see also [12]) and the hybrid MPC problem therein, and propose an efficient method to compute solutions to the associated OCP. To accommodate the binary variables involved, we employ the so-called McCormick Relaxation to reformulate the hybrid MPC problem as an MIQP. For the class of discretized hybrid equations considered—specifically, those with linear flow and jump maps, and flow and jump sets given in terms of inequalities involving a function of the state and input—and following the ideas in [13], we derive a (discrete-time) MLD system model of the hybrid equation and formulate the OCP associated with hybrid MPC as an MIQP. Our approach consists of transforming the discretized hybrid equation into a discrete-time system with binary variables, followed by its reformulation as an MLD system. To mathematically formalize these transformations, we establish equivalences between the trajectories (or solutions) to each system. With such relationships in place, we relate the OCP solution associated with the equivalent MLD system—which can be obtained using MIQP solvers— to the solution to the OCP associated with the discretized hybrid equation. Consequently, our results provide an MIQP solution to hybrid MPC for the class of systems considered. In addition, we present an algorithm that implements our approach and we illustrate it in a controlled bouncing ball system.

The outline of this paper is as follows. Section II presents the definitions of discretized hybrid control systems, mixed logical dynamical systems, and their solutions. The MPC problem for discretized hybrid dynamical system is formulated in Section III. In Section IV, we detail the mixed integer formulation of discretized hybrid model predictive control, followed by numerical simulations. Concluding remarks are provided in Section V.

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II. Preliminaries

A. Notation

We denote by $\mathbb{R}$ the set of real numbers, $\mathbb{R}_{\geq 0}$ its nonnegative subset, and by $\mathbb{N}$ the set of nonnegative integers. Boolean "or," "and," and "not" are denoted by $\vee$, $\wedge$, and $\sim$, respectively. The standard projection onto $\mathbb{R}^n$ is defined by the function $\Pi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$, such that $\Pi(x, y) = x$. The n-dimensional identity matrix is denoted by I_n.

B. Hybrid Control Systems

In this paper, we consider an affine discretized hybrid control system given by

$$\mathcal{H}_d : \begin{cases} x^+ = f(x, u) := A_f x + B_f u + c_f & (x, u) \in C \\ x^+ = g(x, u) := A_g x + B_g u + c_g & (x, u) \in D, \end{cases} \tag{1}$$

where $(x, u) \in C \cup D \cup g(D) =: \mathcal{X} \subset \mathbb{R}^n \times \mathbb{R}^m$ are the state and the input of the system, respectively. The set C is called the flow set and D is called the jump set. The affine functions $f : C \to \mathbb{R}^n$ and $g : D \to \mathbb{R}^n$ are the flow and jump maps, respectively.

Definition 1: A set $E \subset \mathbb{N} \times \mathbb{N}$ is called a discrete hybrid time domain if, for each $(K, J) \in E$, there exists a nondecreasing sequence $\{k_j\}_{j=0}^{J+1}$ such that $k_0 = 0$, $k_{j+1} \in \mathbb{N}$ for each $j \in \{1, 2, ..., J\}$, and

$$E \cap (\{0, 1, ..., K\} \times \{0, 1, ..., J\}) = \bigcup_{j=0}^{J} \bigcup_{k=K_j}^{K_{j+1}} (k, j).$$

•

The state and input are presented by discrete hybrid time $(k, j) \in \mathbb{N} \times \mathbb{N}$, where k and j index the evolution during discretized flow and jumps, respectively.

Definition 2 (Solution pair [11]): A pair $(x : \operatorname{dom} x \to \mathbb{R}^n,\ u : \operatorname{dom} u \to \mathbb{R}^m)$ is a solution pair to $\mathcal{H}_d$ if the following conditions hold:

(S1) $\operatorname{dom} x = \operatorname{dom} u$ is a discrete hybrid time domain.

(S2) $(x(0, 0), u(0, 0)) \in \mathcal{X}$.

(S3) For each $(k, j) \in \operatorname{dom} x$ such that $(k + 1, j) \in \operatorname{dom} x$,
$$x(k+1, j) = f(x(k, j), u(k, j)) \quad (x(k, j), u(k, j)) \in C.$$

(S4) For each $(k, j) \in \operatorname{dom} x$ such that $(k, j + 1) \in \operatorname{dom} x$,
$$x(k, j+1) = g(x(k, j), u(k, j)) \quad (x(k, j), u(k, j)) \in D.$$

•

Throughout, $\widehat{\mathcal{S}}_{\mathcal{H}_d}(x_0)$ denotes the set to solution pairs (x, u) to $\mathcal{H}_d$ such that $x(0, 0) = x_0$. The pair $(L, J) \in \operatorname{dom}(x, u)$ is called the terminal time of the solution pair (x, u) if $k \leq K$ and $j \leq J$ for all $(k, j) \in \operatorname{dom}(x, u)$.

C. Mixed Logical Dynamical Systems

A general MLD model is given by [13]

$$\hat{x}^+ = A\hat{x} + B_1\hat{u} + B_2\hat{\delta} + B_3 z + B_4 \tag{2}$$

$$\text{subject to } E_2\hat{\delta} + E_3 z \leq E_1\hat{u} + E_4\hat{x} + E_5, \tag{3}$$

where $\hat{x} \in \mathbb{R}^n$ is the state and $\hat{u} \in \mathbb{R}^m$ is the input of the system. The auxiliary continuous and binary variables are represented by $z \in \mathbb{R}^{n_d}$ and $\hat{\delta} \in \{0, 1\}^{m_d}$, respectively. All of these variables have binary and continuous values. The matrices A, $\{B_i\}_{i=1}^4$, and $\{E_i\}_{i=1}^5$ have appropriate dimensions.

The MLD model in (2)-(3) can be expressed as

$$\mathcal{H}_{MLD} : \begin{cases} \hat{x}^+ = \Phi(z, \hat{\delta}, \hat{x}, \hat{u}) \\ \Psi(z, \hat{\delta}, \hat{x}, \hat{u}) \leq 0, \end{cases} \tag{4}$$

where

$$\begin{aligned} \Phi(z, \hat{\delta}, \hat{x}, \hat{u}) &:= A\hat{x} + B_1\hat{u} + B_2\hat{\delta} + B_3 z + B_4, \\ \Psi(z, \hat{\delta}, \hat{x}, \hat{u}) &:= E_2\hat{\delta} + E_3 z - E_1\hat{u} - E_4\hat{x} - E_5. \end{aligned} \tag{5}$$

A solution to $\mathcal{H}_{MLD}$ is defined as follows.

Definition 3: A function

$$\mathcal{M} \ni \ell \mapsto \big(z(\ell), \hat{\delta}(\ell), \hat{x}(\ell), \hat{u}(\ell)\big)$$

is a solution to $\mathcal{H}_{MLD}$ if it satisfies

$$\begin{aligned} &\hat{x}(\ell + 1) = \Phi(z(\ell), \hat{\delta}(\ell), \hat{x}(\ell), \hat{u}(\ell)) && \forall \ell : \ell, \ell + 1 \in \mathcal{M}, \\ &\Psi(z(\ell), \hat{\delta}(\ell), \hat{x}(\ell), \hat{u}(\ell)) \leq 0 && \forall \ell \in \mathcal{M} \end{aligned} \tag{6}$$

where $\mathcal{M}$ is of the form $\{0, 1, \ldots, K\}$, with K finite, or equal to $\mathbb{N}$. •

III. Hybrid Model Predictive Control for Discretized Hybrid Control Systems

This section formulates a Model Predictive Control (MPC) problem for discretized hybrid dynamical system given by $\mathcal{H}_d$ in (1). Based on the framework given in [14], we first introduce some details related to MPC for discretized hybrid systems.

A. Prediction Horizon

A fixed end-time optimal control problem can be appropriately used in continuous/discrete-time MPC that optimal controls are updated periodically, and each computed control input has the same terminal time. Due to the nature of (discrete) hybrid time domains, using a fixed end-time optimal control problem is restrictive [14]. For (discretized) hybrid dynamical systems, the solutions might flow or jump, so the prediction horizon must accommodate solutions having different discrete hybrid time domains. To address these issues, as in [14], we define the prediction horizon $\mathcal{T} \subset \mathbb{N} \times \mathbb{N}$ as

$$\mathcal{T} := \{(k, j) \in \mathbb{N} \times \mathbb{N} : \max\{k, j\} = \tau_p\} \tag{7}$$

where τ_p is a given positive integer. Thus, for some $\tau_p \in \{1, 2, \ldots\}$, the terminal time (T, J) of every feasible solution pair satisfies $\max\{T, J\} = \tau_p$.

B. Cost Functional

Given a solution pair (x, u) to $\mathcal{H}_d$ with compact domain and terminal time (L, J), let $\{K_j\}_{j=0}^{J+1}$ be a nondecreasing sequence such that

$$\operatorname{dom}(x, u) = \bigcup_{j=0}^{J} \bigcup_{k=K_j}^{K_{j+1}} (k, j)$$

and $K_{J+1} = L$, and $X \subset \Pi(\mathcal{X})$ be the terminal constraint set. If $x(K, J) \in X$, then the cost of the pair (x, u) is given by

$$\mathcal{J}(x,u) := \Big(\sum_{j=0}^{J} \sum_{k=K_j}^{K_{j+1}-1} L_C(x(k,j), u(k,j)) \Big) + \Big(\sum_{j=0}^{J-1} L_D(x(k,j), u(k,j)) \Big) + V(x(L,J)). \tag{8}$$

In (8), L_C is called the *flow cost* and is defined on the flow set C, L_D is called the *jump cost* and is defined on the jump set D, and V is called the *terminal cost* and is defined on the terminal constraint set X.

C. Hybrid Optimal Control Problem

Given the terminal constraint set X and the prediction horizon $\mathcal{T}$, the minimization of the cost functional $\mathcal{J}$ is performed over solution pairs of $\mathcal{H}_d$ with initial condition x_0.

Problem 1: Given an initial condition $x_0 \in \mathbb{R}^n$,

$$\begin{aligned} \text{minimize} \quad & \mathcal{J}(x,u) \\ \text{subject to} \quad & (x,u) \in \widehat{\mathcal{S}}_{\mathcal{H}_d}(x_0) \\ & x(L,J) \in X \\ & (L,J) \in \mathcal{T}, \end{aligned} \tag{9}$$

where the constraints $x(L, J) \in X$ and $(L, J) \in \mathcal{T}$ dictate that solutions pairs have terminal conditions in X and terminal times in $\mathcal{T}$, respectively. •

If a solution pair (x, u) satisfies the constraints in (9) with $x(0, 0) = x_0$, then we call it a *feasible solution*. A feasible solution is called the *optimal solution* if it minimizes $\mathcal{J}$.

In the next section, we show that the model $\mathcal{H}_d$ and Problem 1 can be reformulated as a mixed integer quadratic problem (MIQP) and solve it with an MIQP solver.

IV. A Mixed Integer Formulation of Discretized Hybrid Model Predictive Control

We formulate a version of Problem 1 that can be solved using mixed integer tools. To this end, we proceed as follows:

Step 1) The discretized hybrid control system $\mathcal{H}_d$ is converted into a discrete-time control system, denoted $\tilde{\mathcal{H}}_d$;

Step 2) The new discrete-time control system $\tilde{\mathcal{H}}_d$ is converted into an MLD system, denoted $\mathcal{H}_{MLD}$;

Step 3) Problem 1 is formulated for $\mathcal{H}_{MLD}$ and solved using mixed integer tools.

The conversion in Step 1 is an intermediate step leading to a model that can be recast as an MLD system. This conversion is technical and will be published elsewhere. For this reformulation to be possible, we impose the following structure on the flow set and the jump set of $\mathcal{H}_d$.

Assumption 1: The flow set C is given as

$$C = C_1 \cup C_2 \tag{10}$$

and the jump set D is given as

$$D = D_1 \cap D_2 \tag{11}$$

where, for each $i \in \{1, 2\}$,

$$C_i = \{(x,u) \in \mathcal{X} : h_i(x,u) - \sigma_i \le 0\}, \tag{12}$$
$$D_i = \{(x,u) \in \mathcal{X} : h_i(x,u) + \sigma_i \ge 0\}, \tag{13}$$

$h_i : \mathcal{X} \to \mathbb{R}$ is defined as

$$h_i(x,u) = h_{i1}^\top x + h_{i2}^\top u$$

with h_{i1} and h_{i2} vectors of appropriate dimension and $\sigma_i \ge 0$ is a constant. •

We exploit the MLD system structure enabled by Assumption 1 to formulate an MIQP version of Problem 1. For this purpose, we impose the following assumption on the flow cost, jump cost, and terminal cost in the cost functional $\mathcal{J}$ in (8).

Assumption 2: The flow cost L_C, the jump cost L_D, and the terminal cost V are given by

$$\begin{aligned} & L_C(x,u) = x^\top Q_c x + u^\top R_c u, \\ & L_D(x,u) = x^\top Q_d x + u^\top R_d u, \quad V(x) = x^\top P x \end{aligned} \tag{14}$$

for each $(x, u) \in \mathcal{X}$, where $P \succeq 0$, $Q_c \succeq 0$, $R_c \succ 0$, $Q_d \succeq 0$, and $R_d \succ 0$. •

A. Recasting $\mathcal{H}_d$ as an MLD system

Using McCormick Relaxation (also known as binary decomposition) from [15] and [16], we formulate the following lemma that allows for the discrete-time system $\mathcal{H}_d$ in (1) to be transformed into an MLD system $\mathcal{H}_{MLD}$ as in (4).

Lemma 1: Consider a compact set $\Lambda \subset \mathbb{R}^n$ and a continuous function $p : \Lambda \to \mathbb{R}$. Define

$$M := \max_{x \in \Lambda} p(x), \quad m := \min_{x \in \Lambda} p(x). \tag{15}$$

Given functions $\delta : \Lambda \to \{0, 1\}$ and $z : \Lambda \to \mathbb{R}$,

$$z(x) = \delta(x) p(x) \qquad \forall x \in \Lambda \tag{16}$$

holds, if and only if, for each $x \in \Lambda$, the following hold:

$$z(x) \le M\delta(x), \tag{17a}$$
$$z(x) \ge m\delta(x), \tag{17b}$$
$$z(x) \le p(x) - m(1 - \delta(x)), \tag{17c}$$
$$z(x) \ge p(x) - M(1 - \delta(x)). \tag{17d}$$

□

Now, we are ready to formulate an MLD system associated with the discretized hybrid system $\mathcal{H}_d$.

For each $i \in \{1, 2\}$, we define set-valued maps

$$\mathcal{U}_i : C_i \cup D_i \rightrightarrows \{0, 1\}$$

as follows:

$$\mathcal{U}_i(x,u) := \begin{cases} 1 & \text{if} \quad (x,u) \in C_i \setminus D_i \\ 0 & \text{if} \quad (x,u) \in D_i \setminus C_i \\ \{0,1\} & \text{if} \quad (x,u) \in C_i \cap D_i. \end{cases} \tag{18}$$

Theorem 1: Suppose the discretized hybrid dynamical system $\mathcal{H}_d$ in (1) with data $(C, A_f, B_f, c_f, D, A_g, B_g, c_g)$ satisfies Assumption 1 and $\mathcal{X}$ is a compact set. Let A, B_i, E_j for all $i \in \{1, 2, \ldots 4\}$ and $j \in \{1, 2, \ldots, 5\}$ in (5) take the following values:

$$A := A_g,\ B_1 := B_g,\ B_2 := \begin{bmatrix} c_f - c_g & c_f - c_g \end{bmatrix},$$

$$B_3 := \begin{bmatrix} A_f - A_g & B_f - B_g & c_f - c_g \end{bmatrix},\ B_4 := c_g,$$

$$E_1 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ h_{12} \\ -h_{12} \\ h_{22} \\ -h_{22} \\ 0 \\ I_m \\ 0 \\ -I_m \end{bmatrix},\ E_2 := \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \\ -M_1 & -M_1 \\ -M_2 & -M_2 \\ m_1 & m_1 \\ m_2 & m_2 \\ m_{31} + \sigma_1 & 0 \\ M_{31} - \sigma_1 & 0 \\ 0 & m_{32} + \sigma_2 \\ 0 & M_{32} - \sigma_2 \\ -m_1 & -m_1 \\ -m_2 & -m_2 \\ M_1 & M_1 \\ M_2 & M_2 \end{bmatrix},$$

$$E_3 := \begin{bmatrix} 0 & 0 & -I_n \\ 0 & 0 & I_n \\ 0 & 0 & I_n \\ 0 & 0 & -I_n \\ I_n & 0 & -M_1 \\ 0 & I_n & -M_2 \\ -I_n & 0 & m_1 \\ 0 & -I_n & m_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I_n & 0 & -m_1 \\ 0 & I_n & -m_2 \\ -I_n & & M_1 \\ 0 & I_n & M_2 \end{bmatrix}, E_4 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ h_{11} \\ -h_{11} \\ h_{21} \\ -h_{21} \\ I_n \\ 0 \\ -I_n \\ 0 \end{bmatrix}, E_5 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sigma_1 \\ M_{31} \\ \sigma_2 \\ M_{32} \\ -m_1 \\ -m_2 \\ M_1 \\ M_2 \end{bmatrix}, \tag{19}$$

where

$$\begin{aligned} &M_1 := \max\{x : (x, u) \in \mathcal{X}\}, \quad m_1 := \min\{x : (x, u) \in \mathcal{X}\}, \\ &M_2 := \max\{\tilde{u} : (x, u) \in \mathcal{X}\}, \quad m_2 := \min\{u : (x, u) \in \mathcal{X}\}, \\ &M_{31} := \max\{h_1(x, u) : (x, u) \in \mathcal{X}\}, \\ &m_{31} := \min\{h_1(x, u) : (x, u) \in \mathcal{X}\}, \\ &M_{32} := \max\{h_2(x, u) : (x, u) \in \mathcal{X}\}, \\ &m_{32} := \min\{h_2(x, u) : (x, u) \in \mathcal{X}\}, \end{aligned} \tag{20}$$

where $\sigma_1, \sigma_2, h_{11}, h_{12}, h_{21}, h_{22}, h_1, h_2$ are given parameters and functions that come from (10) and (11). Let $\mathcal{H}_d$ be defined as in (1). Then, for each solution $(k, j) \mapsto (x(k, j), u(k, j))$ to $\mathcal{H}_d$, the function $\ell \mapsto \big(z(\ell), \hat{\delta}_1(\ell), \hat{\delta}_2(\ell), \hat{x}(\ell), \hat{u}(\ell)\big)$ is defined as

$$\begin{cases} \hat{\delta}_1(\ell) \in \mathcal{U}_{f1}(x(k, j), u(k, j)), & \text{(21a)} \\ \hat{\delta}_2(\ell) \in \mathcal{U}_{f2}(x(k, j), u(k, j)), & \text{(21b)} \\ \hat{u}(\ell) := u(k, j), & \text{(21c)} \\ \hat{x}(\ell) := x(k, j), & \text{(21d)} \\ z(\ell) = \begin{bmatrix} z_1(\ell) \\ z_2(\ell) \\ z_3(\ell) \end{bmatrix} := & \\ \quad \begin{bmatrix} (\hat{\delta}_1(\ell) + \hat{\delta}_2(\ell) - \hat{\delta}_1(\ell)\hat{\delta}_2(\ell))x(k, j) \\ (\hat{\delta}_1(\ell) + \hat{\delta}_2(\ell) - \hat{\delta}_1(\ell)\hat{\delta}_2(\ell))u(k, j) \\ \hat{\delta}_1(\ell)\hat{\delta}_2(\ell) \end{bmatrix}, & \text{(21e)} \end{cases}$$

for each $\ell = k + j$ with $(k, j) \in \operatorname{dom}(x, u)$ is a solution to $\mathcal{H}_{MLD}$ in (4) with

$$\Psi(z, \hat{\delta}_1, \hat{\delta}_2, \hat{x}, \hat{u}) = \begin{pmatrix} -z_3 \\ z_3 - \hat{\delta}_1 \\ z_3 - \hat{\delta}_2 \\ -z_3 + \hat{\delta}_1 + \hat{\delta}_2 - 1 \\ z_1 - M_1(\hat{\delta}_1 + \hat{\delta}_2 + z_3) \\ z_2 - M_2(\hat{\delta}_1 + \hat{\delta}_2 + z_3) \\ m_1(\hat{\delta}_1 + \hat{\delta}_2 + z_3) - z_1 \\ m_2(\hat{\delta}_1 + \hat{\delta}_2 + z_3) - z_2 \\ (m_{31} + \sigma_1)\hat{\delta}_1 - h_1(\hat{x}, \hat{u}) - \sigma_1 \\ (M_{31} - \sigma_1)\hat{\delta}_1 + h_1(\hat{x}, \hat{u}) - M_{31} \\ (m_{32} + \sigma_2)\hat{\delta}_2 - h_2(\hat{x}, \hat{u}) - \sigma_2 \\ (M_{32} - \sigma_2)\hat{\delta}_2 + h_2(\hat{x}, \hat{u}) - M_{32} \\ z_1 - \hat{x} + m_1(1 - (\hat{\delta}_1 + \hat{\delta}_2 + z_3)) \\ z_2 - \hat{u} + m_2(1 - (\hat{\delta}_1 + \hat{\delta}_2 + z_3)) \\ \hat{x} - z_1 - M_1(1 - (\hat{\delta}_1 + \hat{\delta}_2 + z_3)) \\ \hat{u} - z_2 - M_2(1 - (\hat{\delta}_1 + \hat{\delta}_2 + z_3)) \end{pmatrix}. \tag{22}$$

and

$$\begin{aligned} \Phi(z, \hat{\delta}_1, \hat{\delta}_2, \hat{x}, \hat{u}) = {} & (A_f - A_g)z_1 + (B_f - B_g)z_2 \\ & + (c_f - c_g)(\hat{\delta}_1 + \hat{\delta}_2 + z_3) + A_g\hat{x} + B_g\hat{u} + c_g \end{aligned} \tag{23}$$

defined for each $z = (z_1, z_2, z_3) \in \mathbb{R}^{n+m} \times \{0, 1\}$ and each $\hat{\delta} = (\hat{\delta}_1, \hat{\delta}_2) \in \{0, 1\}^2$. Furthermore, for each solution

$$\ell \mapsto \big(z(\ell), \hat{\delta}_1(\ell), \hat{\delta}_2(\ell), \hat{x}(\ell), \hat{u}(\ell)\big)$$

to $\mathcal{H}_{MLD}$, the function $(k, j) \mapsto (x(k, j), u(k, j))$ defined as

$$\begin{cases} x(k, j) := \hat{x}(\ell), & \text{(24a)} \\ u(k, j) := \hat{u}(\ell), & \text{(24b)} \end{cases}$$

for each $k = \sum_{i=1}^{\ell}(\hat{\delta}_1(i) + \hat{\delta}_2(i) - \hat{\delta}_1(i)\hat{\delta}_2(i))$ and $j = \ell - k$ with $\ell \in \operatorname{dom}(z, \hat{\delta}_1, \hat{\delta}_2, \hat{x}, \hat{u})$, is a solution to $\mathcal{H}_d$. $\square$

B. MIQP version of the Hybrid Optimal Control Problem

Now, we use Theorem 1, to convert the hybrid optimal control problem in Problem 1 to an MIQP problem. To enforce the prediction horizon constraint, we add two auxiliary variables $\hat{r}_c$ and $\hat{r}_d$ to the proposed $\mathcal{H}_{MLD}$ system in (4). By including $\hat{r}_c$ and $\hat{r}_d$ we can keep track of flows and the number of jumps elapsed. To this end, we rewrite the MLD system with new variables as follows:

$$\mathcal{H}_{\mathcal{MLD}}:\begin{cases}\hat{\zeta}^+ = \begin{bmatrix}\hat{x}^+\\ \hat{r}_c^+\\ \hat{r}_d^+\end{bmatrix} = \begin{bmatrix}\Phi(z,\hat{\delta}_1,\hat{\delta}_2,\hat{x},\hat{u})\\ \rho(\hat{\delta}_1,\hat{\delta}_2)+r_c\\ 1-\rho(\hat{\delta}_1,\hat{\delta}_2)+r_d\end{bmatrix}\\ \\ \Psi(z,\hat{\delta}_1,\hat{\delta}_2,\hat{x},\hat{u}) \preceq 0,\end{cases} \tag{25}$$

where $\hat{\zeta} := (\hat{x}, \hat{r}_c, \hat{r}_d)$,

$$\rho(\hat{\delta}_1,\hat{\delta}_2) = \hat{\delta}_1 + \hat{\delta}_2 - \hat{\delta}_1\hat{\delta}_2$$

and $\hat{\delta}_1, \hat{\delta}_2, z$, and $\hat{x}$ are given in (21), $\Psi(z,\hat{\delta}_1,\hat{\delta}_2,\hat{x},\hat{u})$ and $\Phi(z,\hat{\delta}_1,\hat{\delta}_2,\hat{x},\hat{u})$ are given in (22) and (23), respectively.

Now, considering (21e), $z_1(\ell) = \rho(\hat{\delta}_1(\ell),\hat{\delta}_2(\ell))\hat{x}(\ell)$ and $z_2(\ell) = \rho(\hat{\delta}_1(\ell),\hat{\delta}_2(\ell))\hat{u}(\ell)$, $\mathcal{J}$ in (8) is written as

$$\begin{aligned}\widehat{\mathcal{J}}(z,\hat{\zeta},\hat{u}) = \sum_{\ell=0}^{N-1}\Bigg(&\Big(z_1(\ell)^\top Q_c\hat{x}(\ell) + z_2(\ell)^\top R_c\hat{u}(\ell)\Big)\\ &+\Big(\hat{x}(\ell)^\top Q_d\hat{x}(\ell) + \hat{u}(\ell)^\top R_d\hat{u}(\ell)\Big)\\ &-\Big(z_1(\ell)^\top Q_d\hat{x}(\ell) + z_2(\ell)^\top R_d\hat{u}(\ell)\Big)\Bigg) + \hat{x}(N)^\top P\hat{x}(N)\end{aligned} \tag{26}$$

with $\mathcal{H}_{\mathcal{MLD}}$ defined in (25). The corresponding MIQP problem to Problem 1 to be solved is as follows.

Problem 2: Given an initial condition $z_0 = (z_0, \hat{\zeta}_0) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \{0\} \times \{0\}$

$$\begin{aligned}&\text{minimize} && \widehat{\mathcal{J}}(z,\hat{\zeta},\hat{u})\\ &\text{subject to} && (z,\hat{\delta}_1,\hat{\delta}_2,\hat{\zeta},\hat{u}) \in \widehat{\mathcal{S}}_{\mathcal{H}_{\mathcal{MLD}}}(z_0)\\ &&& \hat{x}(N) \in X\\ &&& (r_c(N), r_d(N)) \in \mathcal{T},\end{aligned} \tag{27}$$

where[1] $N \in \{\tau_p, \tau_p+1, \dots, 2\tau_p\}$ is the terminal time of $(z,\hat{\delta}_1,\hat{\delta}_2,\hat{\zeta},\hat{u})$, and $\widehat{\mathcal{S}}_{\mathcal{H}_{\mathcal{MLD}}}(z_0)$ is the set of solution pairs of $\mathcal{H}_{\mathcal{MLD}}$ from z_0. •

Using (23) and (2), a solution to $\mathcal{H}_{\mathcal{MLD}}$ is given by

$$\begin{aligned}\hat{x}(\ell) = \sum_{i=0}^{\ell-1} A^i\big(&B_1\hat{u}(\ell-1-i)\\ &+B_2\rho(\hat{\delta}_1(\ell-1-i),\hat{\delta}_2(\ell-1-i))\\ &+B_3 z(\ell-1-i)+B_4\big) + A^\ell\hat{x}_0,\end{aligned} \tag{28}$$

for each $\ell = k + j$ with $(k,j) \in \mathrm{dom}(x,u)$, where A and $\{B_i\}_{i=1}^4$ are given in (19). Substituting (28) into (26) and (22), and defining the vectors

$$\mathcal{Z}(\ell) = \big[z(\ell), \hat{r}_c(\ell), \hat{r}_d(\ell), \hat{u}(\ell), \rho(\hat{\delta}_1(\ell),\hat{\delta}_2(\ell))\big]^\top \tag{29}$$

$$\mathcal{V} = \begin{bmatrix}\mathcal{Z}(0)\\ \mathcal{Z}(1)\\ \vdots\\ \mathcal{Z}(N-1)\end{bmatrix}, \tag{30}$$

Problem 2 is rewritten as follows.

Problem 3: Given $\hat{x}_0$ and an initial condition $\mathcal{Z}(0) = \mathcal{Z}_0 \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \{0\} \times \{0\} \times \mathbb{R}^m \times \{0,1\}$ and $\mathcal{V}$ defined in (30)

$$\begin{aligned}&\text{minimize} && \mathcal{V}^\top S_1\mathcal{V} + 2(S_2 + \hat{x}_0^\top S_3)\mathcal{V}\\ &\text{subject to} && F_1\mathcal{V} \preceq F_2 + F_3\hat{x}_0,\end{aligned} \tag{31}$$

with $\{S_i, F_i\}_{i=1}^3$ appropriately defined (see footnote 2). •

Theorem 2: Suppose a solution to Problem 3 is given by $\mathcal{V}$ and $\mathcal{Z}(\ell)$ for all $\ell \in \{0,1,\dots,N-1\}$ is given in (29). Then,

$$((x(0,0),u(0,0)),\dots,(x(L,J),u(L,J)))$$

is a solution to Problem 1, where $L + J = N - 1$ and

$$\begin{cases}x(k,j) := \displaystyle\sum_{i=0}^{\ell-1} A^i\big(B_1\hat{u}(\ell-1-i)\\ \qquad\qquad + B_2\rho(\hat{\delta}_1(\ell-1-i),\hat{\delta}_2(\ell-1-i))\\ \qquad\qquad + B_3 z(\ell-1-i)+B_4\big) + A^\ell\hat{x}_0,\\ u(k,j) := \hat{u}(\ell),\end{cases} \tag{32}$$

for each $k = \sum_{i=1}^{\ell}(\hat{\delta}_1(i) + \hat{\delta}_2(i) - \hat{\delta}_1(i)\hat{\delta}_2(i))$ and $j = \ell - k$ with $\ell \in \mathrm{dom}\Big(z,\hat{\delta}_1,\hat{\delta}_2,\hat{\zeta},\hat{u}\Big)$. □

C. Implementation of Hybrid MPC using MIQP solvers

Using Problem 3, an algorithm for solving a hybrid MPC problem with an MIQP solver is given as follows.

Algorithm 1: Implementation of Hybrid MPC by using MIQP Problem 3

Set $i = 0$, $\ell_0 = 0$, $\hat{x}(0) = \hat{x}_0$, and $\mathcal{Z}(0) = \mathcal{Z}_0$;
while *true* **do**
  Solve Problem 3 to obtain the optimal solution $\mathcal{V}^*$;
  while $\max\{\hat{r}_c(\ell-\ell_i), \hat{r}_d(\ell-\ell_i)\} \le \tau_c$ **do**
    generate trajectory $\hat{x}$ using (32);
  end
  set $i = i+1$, $\ell_i = \ell$, $\hat{\zeta}_0 = (\hat{x}(\ell_i), 0, 0)$
end

where $\tau_c \le \tau_p$ is a positive integer number and used to parametrize the control horizon. The control horizon regulates the optimization times and it has the same structure as the prediction horizon $\mathcal{T}$ defined in (7).

[1]We allow N to take up to the value $2\tau_p$ to account for the maximum number of discrete steps, which could happen due to solutions exhibiting both flow and jumps.

Example 1: (Discretized Controlled Bouncing Ball) Consider a ball bouncing vertically on a horizontal surface. In [17, p. 27] the bouncing ball is modeled as a point mass with height x_1 and vertical velocity x_2. The motion of the ball evolves according to the following discretized hybrid control system:

$$x_1^+ = x_1 + T_s x_2 - T_s^2 \delta, \quad x_2^+ = x_2 - T_s \delta \quad \text{when} \quad x_1 \geq 0 \tag{33}$$

$$x_1^+ = x_1 - T_s x_2, \quad x_2^+ = -\lambda x_2 + u \quad \text{when} \quad x_1 = 0 \quad x_2 \leq 0, \tag{34}$$

where $\delta = 9.8$, $\lambda \in [0,1]$, and T_s are the gravitational constant, the coefficient of restitution, and the sample time, respectively. The data of the discretized hybrid dynamical system in (1) for the state and input in a compact set is as follows: $A_f = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}, A_g = \begin{bmatrix} 1 & -T_s \\ 0 & -\lambda \end{bmatrix}, c_f = \begin{bmatrix} -T_s^2\delta \\ -T_s\delta \end{bmatrix}, c_g = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B_f = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B_g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathcal{X} = \{(x,u) | x \in [x_{1_{\min}}, x_{1_{\max}}] \times [x_{2_{\min}}, x_{2_{\max}}], u \in [u_{\min}, u_{\max}]\} \cap L_V(c), C = \{(x,u) \in \mathcal{X} : x_1 \geq 0\}$, and $D = \{(x,u) \in \mathcal{X} : x_1 = 0, x_2 \leq 0\}$, where $L_V(c)$ is the sublevel set of function $V(x_1,x_2) = \frac{1}{2}x_2^2 + \delta x_1$ for a constant c. We need to restrict C and D to a compact set that is forward invariant. To this end, the set $\mathcal{X}$ is defined to be the sublevel set of the function $V(x_1,x_2)$ which is a Lyapunov function for the system [18]. To represent the flow and jump sets, in (1) and (1), in the form given in Assumption 1, we choose the functions $h_1(x_1,x_2) = -x_1$, $h_2(x_1,x_2) = -x_2$ and $\sigma_i = 0$ for each $i \in \{1,2\}$ and $x_{1_{\min}} = 0$, $x_{1_{\max}} = 10$, $x_{2_{\min}} = -10$, $x_{2_{\max}} = 10$, $u_{\min} = -0.01$, and $u_{\max} = 0.01$. The control objective is to minimize the cost functional (8) with $Q_c = 0.2I_2$, $R_c = 0.01$, $Q_d = 0.2I_2$, $R_d = 0.01$, and $P = 0.1I_2$. Also, the prediction and control horizon are given with $\tau_p = 2$ and $\tau_c = 1$, respectively. As shown in Fig 1 , when $(x,u) \in C \setminus D$, then $\rho(\hat{\delta}_1, \hat{\delta}_2) = 1$ and the solution flows according to $f(x,u) = A_f x + B_f u$. When $(x,u) \in D \setminus C$, then $\rho(\hat{\delta}_1, \hat{\delta}_2) = 0$ and the solution jumps according to $g(x,u) = A_g x + B_g u$. Finally, if $(x,u) \in C \cap D$, then $\rho(\hat{\delta}_1, \hat{\delta}_2) \in \{0,1\}$ and the solution will either jump or flow, and also the control input has adhered to the intended restriction as given in $\mathcal{X}$. [2] $\square$

V. Conclusion

In this paper, a new mixed-integer model predictive control approach for discretized hybrid systems is presented. To solve the formulated MPC problem for the discretized hybrid dynamical system, boolean algebra is employed to formulate a mixed integer quadratic program for the transformed MLD system. The proposed approach consists of converting the discretized hybrid system into a nonlinear discrete-time system, and transforming the converted nonlinear discrete-time system into an MLD system using McCormick Relaxation.

[2]Files for this simulation can be found at the following hyperlink: https://github.com/HybridSystemsLab/HybridMPCMLDBouncingBall.git. Definitions of the matrices $\{S_i, F_i\}_{i=1}^3$ in Theorem 2 are given in `costfunction.m`, `F1.m`, and `b.m` at this hyperlink.

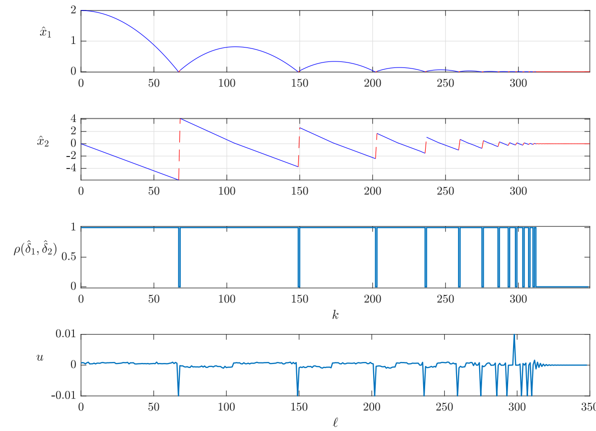

Fig. 1: Simulation result for the controlled bouncing ball in Example 1.

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