Sharp Convergence Rates for Darcy's Law

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Abstract

This paper is concerned with Darcy's law for an incompressible viscous fluid flowing in a porous medium. We establish the sharp $O(\sqrt{\varepsilon})$ convergence rate in a periodically perforated and bounded domain in $\mathbb{R}^d$ for $d \geq 2$, where ε represents the size of solid obstacles. This is achieved by constructing two boundary layer correctors to control the boundary layers created by the incompressibility condition and the discrepancy of boundary values between the solution and the leading term in its asymptotic expansion. One of the correctors deals with the tangential boundary data, while the other handles the normal boundary data.

Keywords: Darcy's Law; Convergence Rate; Stokes Equations.

MR (2020) Subject Classification: 35Q35; 35B27; 76D07.

1 Introduction

This paper is concerned with Darcy's law for an incompressible viscous fluid in a porous medium. More precisely, we consider the Dirichlet problem for the steady Stokes equations,

$$\begin{cases} -\varepsilon^2 \mu \Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \operatorname{div}(u_\varepsilon) = 0 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{1.1}$$

where $\mu > 0$ is the viscosity constant, $0 < \varepsilon < 1$, and Ω_ε is a periodically perforated and bounded domain in $\mathbb{R}^d$, $d \geq 2$. In (1.1) we have normalized the velocity vector by a factor ε^2, where ε is the period. To describe the porous domain Ω_ε, we let $Y = [0,1]^d$ be a closed unit cube and Y_s (solid part) an open subset of Y with Lipschitz boundary. Throughout the paper we shall assume that $\operatorname{dist}(\partial Y, \partial Y_s) > 0$ and that $Y_f = Y \setminus \overline{Y_s}$ (the fluid part) is connected. Let Ω be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary. For $0 < \varepsilon < 1$, define

$$\Omega_\varepsilon = \Omega \setminus \bigcup_k \varepsilon \left(\overline{Y_s} + z_k \right), \tag{1.2}$$

where $z_k \in \mathbb{Z}^d$ and the union is taken over those k's for which $\varepsilon(Y + z_k) \subset \Omega$.

For $f \in L^2(\Omega; \mathbb{R}^d)$, let $(u_\varepsilon, p_\varepsilon) \in H_0^1(\Omega_\varepsilon; \mathbb{R}^d) \times L^2(\Omega_\varepsilon)$ be the weak solution of (1.1) with $\int_{\Omega_\varepsilon} p_\varepsilon \, dx = 0$. We extend u_ε to the whole domain Ω by zero and still denote the extension by u_ε. Let P_ε be the extension of p_ε to Ω, defined by (2.6). It has been known since late 1970's that as $\varepsilon \to 0$, $u_\varepsilon \to u_0$ weakly in $L^2(\Omega; \mathbb{R}^d)$ and $P_\varepsilon \to p_0$ strongly in $L^2(\Omega)$, where (u_0, p_0) is given by a Darcy law,

$$\begin{cases} u_0 = \mu^{-1} K (f - \nabla p_0) & \text{in } \Omega, \\ \operatorname{div}(u_0) = 0 & \text{in } \Omega, \\ u_0 \cdot n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

*Supported in part by NSF grant DMS-1856235 and by Simons Fellowship.

with $\int_\Omega p_0\, dx = 0$. In (1.3) the permeability matrix $K = (K_j^i)$ is a $d \times d$ positive definite and symmetric matrix defined by (2.3), and n denotes the outward unit normal to $\partial\Omega$. Furthermore, it was observed in [3] by G. Allaire that as $\varepsilon \to 0$,

$$u_\varepsilon - \mu^{-1} W(x/\varepsilon)(f - \nabla p_0) \to 0 \quad \text{strongly in } L^2(\Omega; \mathbb{R}^d), \tag{1.4}$$

where $W(y) = (W_j^i(y))$ is an 1-periodic $d \times d$ matrix defined by the cell problem (2.2) and $⨍_Y W(y)\, dy = K$. For an excellent exposition on Darcy's law and closely related topics, we refer the reader to [4] by G. Allaire and A. Mikelić.

The purpose of this paper is to study the convergence rates for $u_\varepsilon - \mu^{-1} W(x/\varepsilon)(f - \nabla p_0)$ and $P_\varepsilon - p_0$ in $L^2(\Omega)$. The following is the main result of the paper. The $O(\sqrt{\varepsilon})$ rate in (1.5) is sharp.

Theorem 1.1. *Let Ω be a bounded $C^{2,\alpha}$ domain in $\mathbb{R}^d$, $d \geq 2$ for some $\alpha > 0$. Also assume that Y_s is an open subset of $Y = [0,1]^d$ with $C^{1,\alpha}$ boundary. Let $(u_\varepsilon, p_\varepsilon) \in H_0^1(\Omega_\varepsilon; \mathbb{R}^d) \times L^2(\Omega_\varepsilon)$ be a weak solution of* (1.1)*, where $f \in C^{1,1/2}(\overline{\Omega}; \mathbb{R}^d)$ and $\int_{\Omega_\varepsilon} p_\varepsilon\, dx = 0$. Then*

$$\begin{aligned} &\|u_\varepsilon - \mu^{-1} W(x/\varepsilon)(f - \nabla p_0)\|_{L^2(\Omega)} + \|P_\varepsilon - p_0\|_{L^2(\Omega)} \\ &\qquad + \|\varepsilon \nabla u_\varepsilon - \mu^{-1} \nabla W(x/\varepsilon)(f - \nabla p_0)\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon}\, \|f\|_{C^{1,1/2}(\Omega)}, \end{aligned} \tag{1.5}$$

where C depends only on d, μ, Ω, and Y_s.

The first rigorous proof of Darcy's law by homogenization was given by L. Tartar in an appendix of [23], using an energy method. We refer the reader to [4] for references on earlier work on the formal derivation of Darcy's law, using two-scale asymptotic expansions. In [2, 3], the strong convergence of $(u_\varepsilon, P_\varepsilon)$ in $L^2(\Omega)$ was established by the method of two-scale convergence. Also see related work in [15, 20, 8, 17, 18, 19, 16].

Regarding the rate of convergence for $(u_\varepsilon, P_\varepsilon)$ in $L^2(\Omega)$, to the best of the author's knowledge, the only previous result for a bounded domain with the Dirichlet condition was obtained by E. Marušić-Paloka and A. Mikelić in [17], where a rate $O(\varepsilon^{1/6})$ was established for the case $d = 2$. See [8] for an earlier result for a unbounded domain $\Omega = (0, L) \times \mathbb{R}_+$. We remark that for Laplace's equation and systems of linear elasticity, quantitative error estimates have been established in [14, 11, 22, 10, 9]. As pointed out in [17], the simple cut-off argument, which seems to work well for standard elliptic equations and systems, does not yield any convergence rate for the Stokes equations because of the incompressibility condition. In [17], using a stream function from [24], a boundary layer corrector was constructed in the case $d = 2$ to control the boundary layer near $\partial\Omega$ created by the incompressibility condition. We mention that [17] also treated the case of nonlinear stationary Navier-Stokes equations.

We now describe our approach to the problem of convergence rates and error estimates, which is based on energy estimates. Let

$$u(x, x/\varepsilon) = \mu^{-1} W(x/\varepsilon)\big(f(x) - \nabla p_0(x)\big). \tag{1.6}$$

To address the discrepancy of boundary values between u_ε and $u(x, x/\varepsilon)$ as well as the incompressibility condition, we introduce two boundary layer correctors (Ψ_t, q_t) and (Ψ_n, q_n). Let $\partial\Omega_\varepsilon = \partial\Omega \cup \Gamma_\varepsilon$. The tangential boundary layer corrector (Ψ_t, q_t) is a weak solution of

$$\begin{cases} -\varepsilon^2 \mu \Delta \Psi_t + \nabla q_t = 0 & \text{in } \Omega_\varepsilon, \\ \operatorname{div}(\Psi_t) = 0 & \text{in } \Omega_\varepsilon, \end{cases} \tag{1.7}$$

with boundary data $\Psi_t = 0$ on Γ_ε, and

$$\Psi_t = -u(x, x/\varepsilon) + [u(x, x/\varepsilon) \cdot n]n \quad \text{on } \partial\Omega. \tag{1.8}$$

Note that $\Psi_t \cdot n = 0$ on $\partial\Omega$. By the divergence theorem and the Cauchy inequality, this gives,

$$\|\nabla \Psi_t\|^2_{L^2(\Omega_\varepsilon)} \le \|\nabla \Psi_t\|_{L^2(\partial\Omega)} \|\Psi_t\|_{L^2(\partial\Omega)}. \tag{1.9}$$

We use a localized Rellich estimate in a Lipschitz domain to show that

$$\|\nabla \Psi_t\|_{L^2(\partial\Omega)} \le C \left\{ \|\nabla_{\tan} \Psi_t\|_{L^2(\partial\Omega)} + \varepsilon^{-1/2} \|\nabla \Psi_t\|_{L^2(\Omega_\varepsilon)} \right\}, \tag{1.10}$$

where $\nabla_{\tan}\Psi_t$ denotes the tangential gradient of Ψ_t on the boundary $\partial\Omega$. The desired $O(\sqrt{\varepsilon})$ bound for $\varepsilon\|\nabla\Psi_t\|_{L^2(\Omega_\varepsilon)}$ follows from (1.9) and (1.10). See Section 4 for details.

The normal boundary layer corrector (Ψ_n, q_n) is defined as the solution of the Stokes equations (1.7) in Ω_ε, with the boundary conditions $\Psi_n = 0$ on Γ_ε, and

$$\Psi_n = -\big[u(x, x/\varepsilon) \cdot n - \gamma\big] n \quad \text{ on } \partial\Omega, \tag{1.11}$$

where

$$\gamma = \fint_{\partial\Omega} u(x, x/\varepsilon) \cdot n \, d\sigma.$$

Thanks to (1.3), we may write

$$u(x, x/\varepsilon) \cdot n = \mu^{-1} n_i \big[W^i_j(x/\varepsilon) - K^i_j\big] \Big(f_j - \frac{\partial p_0}{\partial x_j} \Big) \quad \text{ on } \partial\Omega \tag{1.12}$$

(the repeated indices are summed from 1 to d). Furthermore, there exists a 1-periodic tensor $(\phi^i_{\ell j})$ such that

$$\phi^i_{\ell j} = -\phi^\ell_{ij} \quad \text{ and } \quad W^i_j(y) - K^i_j = \frac{\partial}{\partial y_\ell} \phi^i_{\ell j}(y). \tag{1.13}$$

It follows from (1.12) and (1.13) that

$$u(x, x/\varepsilon) \cdot n = \varepsilon (2\mu)^{-1} \Big(n_i \frac{\partial}{\partial x_\ell} - n_\ell \frac{\partial}{\partial x_i} \Big) \Big(\phi^i_{\ell j}(x/\varepsilon) \Big) \cdot \Big(f_j - \frac{\partial p_0}{\partial x_j} \Big) \quad \text{ on } \partial\Omega. \tag{1.14}$$

Since $n_i \frac{\partial}{\partial x_\ell} - n_\ell \frac{\partial}{\partial x_i}$ is a tangential derivative, the formula (1.14) allows us to use an integration by parts on $\partial\Omega$ (see (5.21)), which generates the needed decay factor ε. In order to carry out this argument, we use an energy estimate to reduce the problem to the L^2 estimate for the Stokes equations in Ω, whose solutions are then represented by integrals on $\partial\Omega$, using the Poisson kernels. See Section 5 for details.

We point out that the $C^{1,\alpha}$ condition on Y_s in Theorem 1.1 is used to ensure the boundedness of ∇W, while the $C^{2,\alpha}$ condition on Ω is used for the C^2 estimates for the Stokes equations in Ω. The $C^{1,1/2}$ condition on f seems to be more or less optimal for the methods used. An $O(\sqrt{\varepsilon})$ estimate with less regularity on f would be an interesting and challenging problem.

The paper is organized as follows. In Section 2 we introduce some notations and collect several known results that will be used in later sections. In Section 3 we establish an energy estimate for the Stokes equations in Ω_ε. The tangential boundary layer corrector (Ψ_t, q_t) is constructed in Section 4, while the normal boundary layer corrector (Ψ_n, q_n) and its estimates are given in Section 5. The proof of Theorem 1.1 is contained in Section 6, where an interior corrector is constructed. In fact, a more general case is treated in Section 6, where we assume $u_\varepsilon = b \in H^1(\partial\Omega; \mathbb{R}^d)$ on $\partial\Omega$. See Theorem 6.1. Due to the discrepancy of u_ε and $u(x, x/\varepsilon)$ on $\partial\Omega$, the $O(\sqrt{\varepsilon})$ rate in Theorem 1.1 is sharp. See Remark 6.6

Throughout the paper, the repeated indices are summed from 1 to d. We will use C and c to denote positive constants that depend at most on d, μ, Ω, and Y_s. Since the value of μ is not relevant in this study, we will assume $\mu = 1$ in the rest of the paper for simplicity.

Acknowledgement. The author thanks Jinping Zhuge for several valuable comments. The author is also grateful for the valuable suggestions and corrections made by the anonymous referees.

2 Preliminaries

Let $Y = [0,1]^d$ and Y_s (solid part) be an open subset of Y with Lipschitz boundary. We assume that $\text{dist}(\partial Y, \partial Y_s) > 0$ and that (the fluid part) $Y_f = Y \setminus \overline{Y_s}$ is connected. Let

$$\omega = \bigcup_{z \in \mathbb{Z}^d} (Y_f + z) \tag{2.1}$$

be the periodic repetition of Y_f. It is easy to see that the unbounded domain ω is connected, 1-periodic, and that $\partial\omega$ is locally Lipschitz.

For $1 \le j \le d$, let $(W_j(y), \pi_j(y)) = (W_j^1(y), \dots, W_j^d(y), \pi_j(y)) \in H^1_{\text{loc}}(\omega; \mathbb{R}^d) \times L^2_{\text{loc}}(\omega)$ be the 1-periodic solution of

$$\begin{cases} -\Delta W_j + \nabla \pi_j = e_j & \text{in } \omega, \\ \text{div}(W_j) = 0 & \text{in } \omega, \\ W_j = 0 & \text{on } \partial\omega, \end{cases} \tag{2.2}$$

with $\int_{Y_f} \pi_j \, dy = 0$, where $e_j = (0, \dots, 1, \dots, 0)$ with 1 in the j^{th} place. We extend W_j to $\mathbb{R}^d$ by zero and define

$$K_j^i = \int_Y W_j^i(y) \, dy. \tag{2.3}$$

Using

$$K_j^i = \int_Y \nabla W_j^\ell \cdot \nabla W_i^\ell \, dy,$$

it is not hard to show that the $d \times d$ constant matrix (K_j^i) is symmetric and positive definite.

Thanks to the assumption $\text{dist}(\partial Y, \partial Y_s) > 0$, we have $\partial\Omega_\varepsilon = \partial\Omega \cup \Gamma_\varepsilon$ and $\text{dist}(\partial\Omega, \Gamma_\varepsilon) \ge c\varepsilon$, where

$$\Gamma_\varepsilon = \Omega \cap \partial\Omega_\varepsilon \subset \partial(\varepsilon\omega). \tag{2.4}$$

For $f \in L^2(\Omega; \mathbb{R}^d)$, let $(u_\varepsilon, p_\varepsilon)$ be a weak solution in $H^1_0(\Omega_\varepsilon; \mathbb{R}^d) \times L^2(\Omega_\varepsilon)$ of the Dirichlet problem,

$$\begin{cases} -\varepsilon^2 \Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \text{div}(u_\varepsilon) = 0 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{2.5}$$

with $\int_{\Omega_\varepsilon} p_\varepsilon \, dx = 0$. We extend u_ε to Ω by zero and still denote the extension by u_ε. Let P_ε be the extension of p_ε, defined by

$$P_\varepsilon(x) = \begin{cases} p_\varepsilon(x) & \text{if } x \in \Omega_\varepsilon, \\ \displaystyle\fint_{\varepsilon(Y_f + z_k)} p_\varepsilon & \text{if } x \in \varepsilon(Y_s + z_k) \text{ and } \varepsilon(Y + z_k) \subset \Omega \text{ for some } z_k \in \mathbb{Z}^d \end{cases} \tag{2.6}$$

(see [15]).

Theorem 2.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \ge 2$. Let $p_0 \in H^1(\Omega)$ be the weak solution of the Neumann problem,*

$$\begin{cases} \dfrac{\partial}{\partial x_i} K_j^i \Big(f_j - \dfrac{\partial p_0}{\partial x_j} \Big) = 0 & \textit{in } \Omega, \\ n_i K_j^i \Big(f_j - \dfrac{\partial p_0}{\partial x_j} \Big) = 0 & \textit{on } \partial\Omega, \end{cases} \tag{2.7}$$

with $\int_\Omega p_0\,dx = 0$*, where* $n = (n_1, \ldots, n_d)$ *denotes the outward unit normal to* $\partial\Omega$*. Then, as* $\varepsilon \to 0$*,*

$$\begin{cases} u_\varepsilon - W_j(x/\varepsilon)\Big(f_j - \dfrac{\partial p_0}{\partial x_j}\Big) \to 0 & \text{in } L^2(\Omega; \mathbb{R}^d), \\ \qquad\qquad\qquad\qquad P_\varepsilon - p_0 \to 0 & \text{in } L^2(\Omega). \end{cases} \tag{2.8}$$

As indicated in Introduction, a proof of Theorem 2.1, using the method of two-scale convergence, may be found in [2, 3]. We do not use the theorem in this paper. However, we will need several other known results stated below.

Lemma 2.2. *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^d$*,* $d \geq 2$*. Assume that* $\Gamma_\varepsilon \neq \emptyset$*. Let* $u \in H^1(\Omega_\varepsilon)$ *with* $u = 0$ *on* Γ_ε*. Then*

$$\|u\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \|\nabla u\|_{L^2(\Omega_\varepsilon)}. \tag{2.9}$$

Proof. The case $u \in H^1_0(\Omega_\varepsilon)$ is more or less well known. See e.g. [5]. The proof for the case $u \in H^1(\Omega_\varepsilon)$ with $u_\varepsilon = 0$ on Γ_ε is the same. We sketch a proof here for the reader's convenience. Suppose $\varepsilon(Y + z_k) \subset \Omega$ for some $z_k \in \mathbb{Z}^d$. Since $u = 0$ on Γ_ε, it follows by Poincaré's inequality that

$$\int_{\varepsilon(Y_f + z_k)} |u|^2\,dx \leq C\varepsilon^2 \int_{\varepsilon(Y_f + z_k)} |\nabla u|^2\,dx. \tag{2.10}$$

Similarly,

$$\int_{B(x_0, C\varepsilon)\cap\Omega_\varepsilon} |u|^2\,dx \leq C\varepsilon^2 \int_{B(x_0, C\varepsilon)\cap\Omega_\varepsilon} |\nabla u|^2\,dx, \tag{2.11}$$

if $x_0 \in \partial\Omega$ and $\varepsilon(Y + z) \subset B(x_0, C\varepsilon) \cap \Omega$ for some $z \in \mathbb{Z}^d$. The estimate (2.9) follows from (2.10) -(2.11) by a covering argument. □

Lemma 2.3. *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^d$*,* $d \geq 2$*. There exists a bounded linear operator*

$$R_\varepsilon : H^1(\Omega; \mathbb{R}^d) \to H^1(\Omega_\varepsilon, \mathbb{R}^d), \tag{2.12}$$

such that

$$\begin{cases} R_\varepsilon(u) = 0 \quad \text{on } \Gamma_\varepsilon \quad \text{and} \quad R_\varepsilon(u) = u \quad \text{on } \partial\Omega, \\ R_\varepsilon(u) \in H^1_0(\Omega_\varepsilon; \mathbb{R}^d) \quad \text{if } u \in H^1_0(\Omega; \mathbb{R}^d), \\ R_\varepsilon(u) = u \quad \text{in } \Omega_\varepsilon \quad \text{if} \quad u = 0 \text{ on } \Gamma_\varepsilon, \\ \operatorname{div}(R_\varepsilon(u)) = 0 \quad \text{in } \Omega_\varepsilon \quad \text{if} \quad \operatorname{div}(u) = 0 \text{ in } \Omega, \end{cases} \tag{2.13}$$

and

$$\varepsilon\,\|\nabla R_\varepsilon(u)\|_{L^2(\Omega_\varepsilon)} + \|R_\varepsilon(u)\|_{L^2(\Omega_\varepsilon)} \leq C\Big\{\varepsilon\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\Big\}, \tag{2.14}$$

where C *depends only on* Ω *and* Y_s*. Moreover,*

$$\|\operatorname{div}(R_\varepsilon(u))\|_{L^2(\Omega_\varepsilon)} \leq C\,\|\operatorname{div}(u)\|_{L^2(\Omega)}. \tag{2.15}$$

Proof. The proof is the similar to that of a lemma due to Tartar (in an appendix of [23], also see Lemma 1.7 in [4]). Let $u \in H^1(\Omega; \mathbb{R}^d)$. For each $\varepsilon(Y + z) \subset \Omega$, where $z \in \mathbb{Z}^d$, we define $R_\varepsilon(u)$ on $\varepsilon(Y_f + z)$ by the Dirichlet problem,

$$\begin{cases} -\varepsilon^2 \Delta R_\varepsilon(u) + \nabla q = -\varepsilon^2 \Delta u & \text{in } \varepsilon(Y_f + z), \\ \operatorname{div}(R_\varepsilon(u)) = \operatorname{div}(u) + \dfrac{1}{|\varepsilon(Y_f + z)|} \displaystyle\int_{\varepsilon(Y_s + z)} \operatorname{div}(u)\,dx & \text{in } \varepsilon(Y_f + z), \\ R_\varepsilon(u) = 0 & \text{on } \partial(\varepsilon(Y_s + z)), \\ R_\varepsilon(u) = u & \text{on } \partial(\varepsilon(Y + z)). \end{cases} \tag{2.16}$$

If $x \in \Omega_\varepsilon$ and $x \notin \varepsilon(Y_f + z)$ for any $\varepsilon(Y + z) \subset \Omega$, we let $R_\varepsilon(u)(x) = u(x)$. It is not hard to show that $R_\varepsilon(u) \in H^1(\Omega_\varepsilon; \mathbb{R}^d)$ satisfies the conditions in (2.13)-(2.15). $\square$

Lemma 2.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$. Suppose that $g \in L^2(\Omega_\varepsilon)$ and $\int_{\Omega_\varepsilon} g\, dx = 0$. Then there exists $v_\varepsilon \in H_0^1(\Omega_\varepsilon; \mathbb{R}^d)$ such that* $\operatorname{div}(v_\varepsilon) = g$ *in Ω_ε and*

$$\varepsilon \|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\|g\|_{L^2(\Omega_\varepsilon)}, \tag{2.17}$$

where C depends only on Ω and Y_s.

Proof. See e.g. [5, pp.146-148]. $\square$

We end this section with some observations on the rescaled solutions $(W_j(x/\varepsilon), \varepsilon\pi_j(x/\varepsilon))$ in Ω_ε. It follows from (2.2) by rescaling that

$$\begin{cases} -\varepsilon^2 \Delta\big\{W_j(x/\varepsilon)\big\} + \nabla\big\{\varepsilon\pi_j(x/\varepsilon)\big\} = e_j & \text{in } \varepsilon\omega, \\ \operatorname{div}(W_j(x/\varepsilon)) = 0 & \text{in } \varepsilon\omega, \\ W_j(x/\varepsilon) = 0 & \text{on } \partial(\varepsilon\omega). \end{cases} \tag{2.18}$$

We extend both W_j and π_j to $\mathbb{R}^d$ by zero. Clearly,

$$\begin{cases} \operatorname{div}(W_j(x/\varepsilon)) = 0 & \text{in } \mathbb{R}^d, \\ W_j(x/\varepsilon) = 0 & \text{on } \Gamma_\varepsilon = \Omega \cap \partial\Omega_\varepsilon. \end{cases} \tag{2.19}$$

Note that in the construction of Ω_ε, the holes near $\partial\Omega$ are not removed. As a result, the first equation in (2.18) needs to be modified for Ω_ε. In fact, a computation using integration by parts shows that

$$-\varepsilon^2 \Delta\big\{W_j(x/\varepsilon)\big\} + \nabla\big\{\varepsilon\pi_j(x/\varepsilon)\big\} = e_j + \sigma_{\varepsilon,j} \quad \text{in } \Omega_\varepsilon, \tag{2.20}$$

where $\sigma_{\varepsilon,j} \in H^{-1}(\Omega_\varepsilon; \mathbb{R}^d)$ is given by

$$\begin{aligned} &\langle \sigma_{\varepsilon,j}, \psi\rangle_{H^{-1}(\Omega_\varepsilon)\times H_0^1(\Omega_\varepsilon)} \\ &= \sum_k \left\{ -\int_{\Omega\cap\varepsilon(Y_s+z_k)} e_j \cdot \psi\, dx - \varepsilon \int_{\overline{\Omega}\cap\partial(\varepsilon(Y_s+z_k))} (\nabla W_j(x/\varepsilon)n - \pi_j(x/\varepsilon)n) \cdot \psi d\sigma \right\}. \end{aligned} \tag{2.21}$$

The sum in (2.21) is taken over those k's for which $z_k \in \mathbb{Z}^d$ and $\varepsilon(Y + z_k) \cap \partial\Omega \neq \emptyset$, and n denotes the outward unit normal. Under the assumption that ∂Y_s is $C^{1,\alpha}$, it is known that $|\nabla W_j|$ and π_j are bounded in $\mathbb{R}^d$. It follows that if $g \in H^1_{loc}(\mathbb{R}^d)$ and $\psi \in H_0^1(\Omega_\varepsilon; \mathbb{R}^d)$, then

$$|\langle \sigma_{\varepsilon,j}, g\psi\rangle_{H^{-1}(\Omega_\varepsilon)\times H_0^1(\Omega_\varepsilon)}| \leq C \sum_k \left\{ \int_{\varepsilon(Y+z_k)} |g\psi|\, dx + \varepsilon \int_{\partial(\varepsilon(Y_s+z_k))} |g\psi|\, d\sigma \right\}$$

(ψ is extended to $\mathbb{R}^d$ by zero). Using the inequality

$$\int_{\partial(\varepsilon(Y_s+z_k))} |u|^2\, d\sigma \leq C\varepsilon \int_{\varepsilon(Y+z_k)} |\nabla u|^2\, dx + C\varepsilon^{-1} \int_{\varepsilon(Y+z_k)} |u|^2\, dx,$$

(2.9) and the Cauchy inequality, one may prove that

$$|\langle \sigma_{\varepsilon,j}, g\psi\rangle_{H^{-1}(\Omega_\varepsilon)\times H_0^1(\Omega_\varepsilon)}| \leq C\varepsilon \big\{\varepsilon\|\nabla g\|_{L^2(\Sigma_{c\varepsilon})} + \|g\|_{L^2(\Sigma_{c\varepsilon})}\big\} \|\nabla\psi\|_{L^2(\Omega_\varepsilon)} \tag{2.22}$$

for any $\psi \in H_0^1(\Omega_\varepsilon; \mathbb{R}^d)$, where $\Sigma_{c\varepsilon} = \{x \in \mathbb{R}^d : \operatorname{dist}(x, \partial\Omega) < c\varepsilon\}$.

3 Energy estimates

In this section we establish the energy estimates for the Dirichlet problem,

$$\begin{cases} -\varepsilon^2 \Delta u_\varepsilon + \nabla p_\varepsilon = f + \varepsilon \operatorname{div}(F) & \text{in } \Omega_\varepsilon, \\ \operatorname{div}(u_\varepsilon) = g & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = h & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where (g, h) satisfies the compatibility condition,

$$\int_\Omega g\, dx = \int_{\partial\Omega} h \cdot n\, d\sigma. \tag{3.2}$$

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$ with Lipschitz boundary. Let $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon; \mathbb{R}^d) \times L^2(\Omega_\varepsilon)$ be a weak solution of (3.1) with $\int_{\Omega_\varepsilon} p_\varepsilon\, dx = 0$. Then*

$$\begin{aligned} &\varepsilon \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ &\leq C \left\{ \|f\|_{L^2(\Omega_\varepsilon)} + \|F\|_{L^2(\Omega_\varepsilon)} + \|g\|_{L^2(\Omega_\varepsilon)} + \|h\|_{L^2(\partial\Omega)} + \varepsilon \|h\|_{H^{1/2}(\partial\Omega)} \right\}, \end{aligned} \tag{3.3}$$

for any $0 < \varepsilon < 1$, where C depends only on Ω and Y_s.

Proof. We divide the proof into several steps.
Step 1. By Lemma 2.4, there exists $v_\varepsilon \in H_0^1(\Omega_\varepsilon; \mathbb{R}^d)$ such that $\operatorname{div}(v_\varepsilon) = p_\varepsilon$ in Ω_ε and

$$\varepsilon \|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)}. \tag{3.4}$$

By using v_ε as a test function we see that

$$\begin{aligned} \|p_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} &\leq \varepsilon^2 \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|f\|_{L^2(\Omega_\varepsilon)} \|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \varepsilon \|F\|_{L^2(\Omega_\varepsilon)} \|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ &\leq C \|p_\varepsilon\|_{L^2(\Omega)} \left\{ \varepsilon \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|f\|_{L^2(\Omega_\varepsilon)} + \|F\|_{L^2(\Omega_\varepsilon)} \right\}, \end{aligned}$$

where we have used (3.4) for the last inequality. This gives

$$\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C \left\{ \varepsilon \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|f\|_{L^2(\Omega_\varepsilon)} + \|F\|_{L^2(\Omega_\varepsilon)} \right\}. \tag{3.5}$$

Step 2. We consider the case $h = 0$ on $\partial\Omega$. This allows us to use the test function $u_\varepsilon \in H_0^1(\Omega_\varepsilon; \mathbb{R}^d)$ to obtain

$$\varepsilon^2 \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|g\|_{L^2(\Omega_\varepsilon)} + \|f\|_{L^2(\Omega_\varepsilon)} \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \varepsilon \|F\|_{L^2(\Omega_\varepsilon)} \|\nabla u_\varepsilon\|_{L^2(\Omega)},$$

where we have also used the Cauchy inequality. It follows from the inequality $\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}$ as well as the Cauchy inequality that

$$\varepsilon^2 \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq C \left\{ \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|g\|_{L^2(\Omega_\varepsilon)} + \|F\|^2_{L^2(\Omega_\varepsilon)} + \|f\|^2_{L^2(\Omega_\varepsilon)} \right\}. \tag{3.6}$$

This, together with (3.5), yields (3.3) by the Cauchy inequality.

Step 3. In the general case, we let $(H, q) \in H^1(\Omega; \mathbb{R}^d) \times L^2(\Omega)$ be a weak solution of

$$-\Delta H + \nabla q = 0 \quad \text{and} \quad \operatorname{div}(H) = \gamma \quad \text{in } \Omega,$$

with boundary data $H = h$ on $\partial\Omega$, where

$$\gamma = \frac{1}{|\Omega|} \int_{\partial\Omega} h \cdot n \, d\sigma.$$

Let $w_\varepsilon = R_\varepsilon(H)$, where R_ε is the operator given by Lemma 2.3. Note that $w_\varepsilon = 0$ on Γ_ε, $w_\varepsilon = h$ on $\partial\Omega$,

$$\varepsilon \|\nabla w_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|w_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C \left\{ \varepsilon \|\nabla H\|_{L^2(\Omega)} + \|H\|_{L^2(\Omega)} \right\}, \tag{3.7}$$

and

$$\|\mathrm{div}(w_\varepsilon)\|_{L^2(\Omega_\varepsilon)} \le C|\gamma|. \tag{3.8}$$

Thus, $u_\varepsilon - w_\varepsilon \in H_0^1(\Omega_\varepsilon; \mathbb{R}^d)$, and

$$-\varepsilon^2 \Delta(u_\varepsilon - w_\varepsilon) + \nabla p_\varepsilon = f + \varepsilon \, \mathrm{div}(F) + \varepsilon^2 \Delta w_\varepsilon \quad \text{ in } \Omega_\varepsilon.$$

Hence, by Step 2, we obtain

$$\begin{aligned} &\varepsilon \|\nabla(u_\varepsilon - w_\varepsilon)\|_{L^2(\Omega_\varepsilon)} + \|u_\varepsilon - w_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ &\le C \big\{ \|f\|_{L^2(\Omega_\varepsilon)} + \|F\|_{L^2(\Omega_\varepsilon)} + \varepsilon \|\nabla w_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|g\|_{L^2(\Omega_\varepsilon)} + |\gamma| \big\}, \end{aligned}$$

where we have used (3.8). It follows from (3.7) that

$$\begin{aligned} &\varepsilon \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ &\le C \big\{ \|f\|_{L^2(\Omega_\varepsilon)} + \|F\|_{L^2(\Omega_\varepsilon)} + \|g\|_{L^2(\Omega_\varepsilon)} + |\gamma| + \varepsilon \|\nabla H\|_{L^2(\Omega)} + \|H\|_{L^2(\Omega)} \big\}. \end{aligned} \tag{3.9}$$

Step 4. To estimate $\|\nabla H\|_{L^2(\Omega)}$ and $\|H\|_{L^2(\Omega)}$, we let

$$\widetilde{H} = H - \gamma d^{-1}(x - x_0),$$

where $x_0 \in \Omega$ is fixed. Note that

$$-\Delta \widetilde{H} + \nabla q = 0 \quad \text{ and } \quad \mathrm{div}(\widetilde{H}) = 0 \quad \text{ in } \Omega.$$

By the energy estimates,

$$\|\nabla \widetilde{H}\|_{L^2(\Omega)} \le C \|\widetilde{H}\|_{H^{1/2}(\partial\Omega)} \le C \|h\|_{H^{1/2}(\partial\Omega)},$$

and by the nontangential-maximal-function estimates for the Stokes equations in [6],

$$\|\widetilde{H}\|_{L^2(\Omega)} \le C \|\widetilde{H}\|_{L^2(\partial\Omega)} \le C \|h\|_{L^2(\partial\Omega)}.$$

It follows that

$$\varepsilon \|\nabla H\|_{L^2(\Omega)} + \|H\|_{L^2(\Omega)} \le C \big\{ \varepsilon \|h\|_{H^{1/2}(\partial\Omega)} + \|h\|_{L^2(\partial\Omega)} \big\}.$$

This, together with (3.9), completes the proof. □

Remark 3.2. If we replace the right-hand side $f + \varepsilon \, \mathrm{div}(F)$ of the first equation in (3.1) by some $\sigma_\varepsilon \in H^{-1}(\Omega_\varepsilon; \mathbb{R}^d)$ that satisfies the condition

$$|\langle \sigma_\varepsilon, \psi \rangle_{H^{-1}(\Omega_\varepsilon) \times H_0^1(\Omega_\varepsilon)}| \le \varepsilon N \|\nabla \psi\|_{L^2(\Omega_\varepsilon)}$$

for any $\psi \in H_0^1(\Omega_\varepsilon; \mathbb{R}^d)$ and some $N = N(\sigma_\varepsilon) > 0$, then the same argument as in the proof of Theorem 3.1 gives

$$\varepsilon \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C \Big\{ N + \|g\|_{L^2(\Omega_\varepsilon)} + \|h\|_{L^2(\partial\Omega)} + \varepsilon \|h\|_{H^{1/2}(\partial\Omega)} \Big\} \tag{3.10}$$

for any $0 < \varepsilon < 1$, where C depends only on Ω and Y_s. Note that if $\sigma_\varepsilon = f + \varepsilon \, \mathrm{div}(F)$, then $N = C \left\{ \|f\|_{L^2(\Omega_\varepsilon)} + \|F\|_{L^2(\Omega_\varepsilon)} \right\}$.

4 Correctors for tangential boundary data

Consider the Dirichlet problem,

$$\begin{cases} -\varepsilon^2 \Delta u_\varepsilon + \nabla p_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ \operatorname{div}(u_\varepsilon) = 0 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = h & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

with boundary data h satisfying the condition

$$h \cdot n = 0 \quad \text{on } \partial\Omega. \tag{4.2}$$

The goal of this section is to prove the following.

Theorem 4.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$. Let $(u_\varepsilon, p_\varepsilon)$ be a weak solution in $H^1(\Omega_\varepsilon; \mathbb{R}^d) \times L^2(\Omega_\varepsilon)$ of (4.1) with $\int_{\Omega_\varepsilon} p_\varepsilon \, dx = 0$, where $h \in H^1(\partial\Omega; \mathbb{R}^d)$ satisfies (4.2). Then*

$$\varepsilon \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon} \left\{ \|h\|_{L^2(\partial\Omega)} + \varepsilon \|\nabla_{\tan} h\|_{L^2(\partial\Omega)} \right\}, \tag{4.3}$$

where $\nabla_{\tan} h$ denotes the tangential gradient of h on $\partial\Omega$.

Let

$$\begin{aligned} D_r &= \left\{ (x', x_d) \in \mathbb{R}^d : \ |x'| < r \quad \text{and} \quad \psi(x') < x_d < 100d(M+1)r \right\}, \\ I_r &= \left\{ (x', \psi(x')) \in \mathbb{R}^d : \ |x'| < r \right\}, \end{aligned} \tag{4.4}$$

where $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$ is a Lipschitz function such that $\psi(0) = 0$ and $\|\nabla \psi\|_\infty \leq M$.

Lemma 4.2. *Let (v, q) be a weak solution in $H^1(D_r; \mathbb{R}^d) \times L^2(D_r)$ of the Dirichlet problem,*

$$\begin{cases} -\Delta v + \nabla q = 0 & \text{in } D_r, \\ \operatorname{div}(v) = 0 & \text{in } D_r, \\ v = g & \text{on } \partial D_r, \end{cases} \tag{4.5}$$

where $0 < r < \infty$ and $g \in H^1(\partial D_r; \mathbb{R}^d)$ satisfies the condition $\int_{\partial D_r} g \cdot n \, d\sigma = 0$. Then

$$\int_{\partial D_r} |\nabla v|^2 \, d\sigma \leq C \int_{\partial D_r} |\nabla_{\tan} v|^2 \, d\sigma, \tag{4.6}$$

where C depends only on d and M.

Proof. By dilation we may assume $r = 1$, in which case the Rellich estimate (4.6) was proved in [6, Theorem 4.15]. □

Lemma 4.3. *Let (v, q) be a weak solution of (4.5) with $r = 2$. Then*

$$\int_{I_1} |\nabla v|^2 \, d\sigma \leq C \int_{I_2} |\nabla_{\tan} g|^2 \, d\sigma + C \int_{D_2} |\nabla v|^2 \, dx, \tag{4.7}$$

where C depends only on d and M.

Proof. It follows from (4.6) that for $1 < r < 2$,

$$\int_{I_1} |\nabla v|^2 \, d\sigma \le C \int_{\partial D_r} |\nabla v|^2 \, d\sigma \le C \int_{\partial D_r} |\nabla_{\tan} v|^2 \, d\sigma.$$

Hence,

$$\int_{I_1} |\nabla v|^2 \, d\sigma \le C \int_{I_2} |\nabla_{\tan} g|^2 \, d\sigma + C \int_{D_2 \cap \partial D_r} |\nabla v|^2 \, d\sigma.$$

By integrating the inequality above in r over the interval $(1, 2)$, we obtain (4.7). $\square$

Proof of Theorem 4.1. We start with the observation,

$$\varepsilon^2 \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx = \varepsilon^2 \int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n} \cdot u_\varepsilon \, d\sigma,$$

where we have used (4.1) and (4.2). It follows by the Cauchy inequality that

$$\varepsilon^2 \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx \le \varepsilon^2 \|\nabla u_\varepsilon\|_{L^2(\partial\Omega)} \|h\|_{L^2(\partial\Omega)}. \tag{4.8}$$

We will show that

$$\int_{\partial\Omega} |\nabla u_\varepsilon|^2 \, d\sigma \le C \int_{\partial\Omega} |\nabla_{\tan} h|^2 \, d\sigma + \frac{C}{\varepsilon} \int_{\Sigma_{c\varepsilon}} |\nabla u_\varepsilon|^2 \, dx, \tag{4.9}$$

where $\Sigma_{c\varepsilon} = \{x \in \Omega : \ \operatorname{dist}(x, \partial\Omega) < c\varepsilon\} \subset \Omega_\varepsilon$. Assume (4.9) for a moment. Then

$$\|\nabla u_\varepsilon\|_{L^2(\partial\Omega)} \le C \|\nabla_{\tan} h\|_{L^2(\partial\Omega)} + C\varepsilon^{-1/2} \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}.$$

This, together with (4.8) and the Cauchy inequality, gives

$$\begin{aligned}
\varepsilon^2 \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} &\le C\varepsilon^2 \|h\|_{L^2(\partial\Omega)} \|\nabla_{\tan} h\|_{L^2(\partial\Omega)} + C\varepsilon^{3/2} \|h\|_{L^2(\partial\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon^2 \|h\|_{L^2(\partial\Omega)} \|\nabla_{\tan} h\|_{L^2(\partial\Omega)} + C\varepsilon \|h\|^2_{L^2(\partial\Omega)} + (1/2)\varepsilon^2 \|\nabla u_\varepsilon\|^2_{L^2(\Omega_e)},
\end{aligned}$$

which yields the estimate for $\varepsilon\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}$ in (4.3). The estimate for $\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}$ follows by (2.9), while the bound for $\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)}$ follows from (3.5).

It remains to prove (4.9). To this end, we shall prove in a few lines below that

$$\int_{B(x_0, c_0\varepsilon)\cap\partial\Omega} |\nabla u_\varepsilon|^2 \, d\sigma \le C \int_{B(x_0, c_1\varepsilon)\cap\partial\Omega} |\nabla_{\tan} h|^2 \, d\sigma + \frac{C}{\varepsilon} \int_{B(x_0, c_1\varepsilon)\cap\Omega} |\nabla u_\varepsilon|^2 \, dx \tag{4.10}$$

for any $x_0 \in \partial\Omega$, where $0 < c_0 < c_1$ are sufficiently small. The desired estimate (4.9) follows from (4.10) by covering $\partial\Omega$ with a finite number of balls $\{B(x_k, c_0\varepsilon)\}$ centered on $\partial\Omega$.

Finally, we note that if $v(x) = u_\varepsilon(\varepsilon x)$ and $q(x) = \varepsilon^{-1} p_\varepsilon(\varepsilon x)$, then $-\Delta v + \nabla q = 0$ and $\operatorname{div}(v) = 0$. As a result, the estimate (4.10) follows from (4.7) by a translation and rotation of the coordinate system. We point out that since the constant C in (4.7) depends only on d and M, the constant C in (4.10) depends only on d and the Lipschitz character of Ω. In particular, C does not depend on ε. $\square$

As a corollary of Theorem 4.1, we are able to construct a tangential boundary layer corrector.

Theorem 4.4. *Let Ω be bounded domain with $C^{2,\alpha}$ boundary for some $\alpha > 0$. Also assume that ∂Y_s is $C^{1,\alpha}$. Let (Ψ_t, q_t) be a weak solution in $H^1(\Omega_\varepsilon; \mathbb{R}^d) \times L^2(\Omega_\varepsilon)$ of the Dirichlet problem (4.1) with $\int_{\Omega_\varepsilon} q_t \, dx = 0$, where the boundary data h is given by*

$$h = b - W_j(x/\varepsilon)\Big(f_j - \frac{\partial p_0}{\partial x_j}\Big) + \Big[- b \cdot n + n_i W_j^i(x/\varepsilon)\Big(f_j - \frac{\partial p_0}{\partial x_j}\Big)\Big] n, \tag{4.11}$$

$f \in C^{1,1/2}(\overline{\Omega}; \mathbb{R}^d)$, $b \in H^1(\partial\Omega; \mathbb{R}^d)$ satisfies $\int_{\partial\Omega} b \cdot n \, d\sigma = 0$, and p_0 is the solution of the Neumann problem,

$$\begin{cases} K_j^i \dfrac{\partial}{\partial x_i}\Big(f_j - \dfrac{\partial p_0}{\partial x_j}\Big) = 0 & \text{in } \Omega, \\ n_i K_j^i\Big(f_j - \dfrac{\partial p_0}{\partial x_j}\Big) = b \cdot n & \text{on } \partial\Omega, \end{cases} \tag{4.12}$$

with $\int_\Omega p_0 \, dx = 0$. Then

$$\begin{aligned} &\varepsilon \|\nabla \Psi_t\|_{L^2(\Omega_\varepsilon)} + \|\Psi_t\|_{L^2(\Omega_\varepsilon)} + \|q_t\|_{L^2(\Omega_\varepsilon)} \\ &\le C\sqrt{\varepsilon}\Big\{ \|f - \nabla p_0\|_{L^2(\partial\Omega)} + \|b\|_{L^2(\partial\Omega)} + \varepsilon \|\nabla_{\tan}(f - \nabla p_0)\|_{L^2(\partial\Omega)} + \varepsilon \|\nabla_{\tan} b\|_{L^2(\partial\Omega)} \Big\} \end{aligned} \tag{4.13}$$

for any $0 < \varepsilon < 1$.

Proof. Note that $h \cdot n = 0$ on $\partial\Omega$. Also, under the assumption that ∂Y_s is $C^{1,\alpha}$, we have $W_j = W_j(y) \in C^1(\overline{\omega})$. It follows that

$$\|h\|_{L^2(\partial\Omega)} \le C\big\{ \|f - \nabla p_0\|_{L^2(\partial\Omega)} + \|b\|_{L^2(\partial\Omega)} \big\},$$

and

$$\begin{aligned} &\|\nabla_{\tan} h\|_{L^2(\partial\Omega)} \\ &\le C\Big\{ \varepsilon^{-1}\|f - \nabla p_0\|_{L^2(\partial\Omega)} + \|\nabla_{\tan}(f - \nabla p_0)\|_{L^2(\partial\Omega)} + \|\nabla_{\tan} b\|_{L^2(\partial\Omega)} + \|b\|_{L^2(\partial\Omega)} \Big\}. \end{aligned}$$

As a result, the estimate (4.13) follows readily from (4.3). $\square$

5 Correctors for normal boundary data

In this section we consider the Dirichlet problem (4.1), where the boundary data h is given by

$$h = \big\{ n_i [W_j^i(x/\varepsilon) - K_j^i] g_j - \gamma \big\} n, \tag{5.1}$$

where $g = (g_1, g_2, \dots, g_d) \in H^1(\partial\Omega; \mathbb{R}^d)$, and $\gamma \in \mathbb{R}$ is chosen so that $\int_{\partial\Omega} h \cdot n \, d\sigma = 0$, i.e.,

$$\gamma = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} n_i [W_j^i(x/\varepsilon) - K_j^i] g_j \, d\sigma. \tag{5.2}$$

The goal of this section is to prove the following.

Theorem 5.1. *Let Ω be a bounded $C^{2,\alpha}$ domain in $\mathbb{R}^d$ for some $\alpha > 0$. Also assume that ∂Y_s is $C^{1,\alpha}$. Let $(u_\varepsilon, p_\varepsilon)$ be a weak solution in $H^1(\Omega_\varepsilon; \mathbb{R}^d) \times L^2(\Omega_\varepsilon)$ of (4.1) with $\int_{\Omega_\varepsilon} p_\varepsilon \, dx = 0$, where h is given by (5.1). Then*

$$\varepsilon \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C\sqrt{\varepsilon}\ \big\{ \|g\|_{L^2(\partial\Omega)} + \sqrt{\varepsilon}\, \|\nabla_{\tan} g\|_{L^2(\partial\Omega)} \big\}, \tag{5.3}$$

for any $0 < \varepsilon < 1$.

We will prove a series of lemmas before we give the proof of Theorem 5.1. We begin with an estimate for $\|h\|_{H^{1/2}(\partial\Omega)}$.

Lemma 5.2. *Let h be given by* (5.1). *Then*

$$\|h\|_{H^{1/2}(\partial\Omega)} \leq C\varepsilon^{-1/2}\big\{\|g\|_{L^2(\partial\Omega)} + \varepsilon\|\nabla_{\tan} g\|_{L^2(\partial\Omega)}\big\} \tag{5.4}$$

for any $0 < \varepsilon < 1$.

Proof. Note that

$$\begin{aligned}\|h\|_{H^{1/2}(\partial\Omega)} &\leq C\|h\|^{1/2}_{L^2(\partial\Omega)}\|h\|^{1/2}_{H^1(\partial\Omega)}\\ &\leq C\left\{\varepsilon^{-1/2}\|h\|_{L^2(\partial\Omega)} + \varepsilon^{1/2}\|\nabla_{\tan} h\|_{L^2(\partial\Omega)}\right\},\end{aligned}$$

where we have used the Cauchy inequality. It is easy to see that $\|h\|_{L^2(\partial\Omega)} \leq C\|g\|_{L^2(\partial\Omega)}$, and

$$\|\nabla_{\tan} h\|_{L^2(\partial\Omega)} \leq C\left\{\varepsilon^{-1}\|g\|_{L^2(\partial\Omega)} + \|\nabla_{\tan} g\|_{L^2(\partial\Omega)}\right\},$$

where we have used the fact $W_j = W_j(y) \in C^1(\overline{\omega}; \mathbb{R}^d)$. It follows that

$$\|h\|_{H^{1/2}(\partial\Omega)} \leq C\varepsilon^{-1/2}\left\{\|g\|_{L^2(\partial\Omega)} + \varepsilon\|\nabla_{\tan} g\|_{L^2(\partial\Omega)}\right\}$$

for any $0 < \varepsilon < 1$. □

Lemma 5.3. *There exist 1-periodic functions $\phi^\ell_{ij} \in H^1_{loc}(\mathbb{R}^d)$, where $1 \leq i, j, \ell \leq d$, such that*

$$\frac{\partial \phi^\ell_{ij}}{\partial y_i} = W^\ell_j(y) - K^\ell_j \quad \text{and} \quad \phi^\ell_{ij} = -\phi^i_{\ell j}, \tag{5.5}$$

where the index i is summed from 1 *to* d.

Proof. The proof is the same as Lemma 3.1 in [13]. Since

$$\int_Y \left(W^\ell_j(y) - K^\ell_j\right) dy = 0,$$

one may solve the periodic boundary value problem,

$$\begin{cases}\Delta f^\ell_j = W^\ell_j - K^\ell_j & \text{in } Y,\\ f^\ell_j \text{ is 1-periodic.}\end{cases}$$

Let

$$\phi^\ell_{ij} = \frac{\partial f^\ell_j}{\partial y_i} - \frac{\partial f^i_j}{\partial y_\ell}.$$

Then $\phi^\ell_{ij} = -\phi^i_{\ell j}$. Using

$$\frac{\partial}{\partial y_\ell} W^\ell_j = 0,$$

we obtain the first equation in (5.5). □

Remark 5.4. Using (5.5), for $1 \le j \le d$, we may write

$$
\begin{aligned}
n_\ell [W_j^\ell(x/\varepsilon) - K_j^\ell] &= \varepsilon n_\ell \frac{\partial}{\partial x_i} \Big\{ \phi_{ij}^\ell(x/\varepsilon) \Big\} \\
&= \frac{\varepsilon}{2} \Big\{ n_\ell \frac{\partial}{\partial x_i} - n_i \frac{\partial}{\partial x_\ell} \Big\} \Big\{ \phi_{ij}^\ell(x/\varepsilon) \Big\},
\end{aligned}
\tag{5.6}
$$

where the skew-symmetric property is used for the last step. It follows from an integration by parts on $\partial\Omega$ that

$$
\int_{\partial\Omega} n_\ell [W_j^\ell(x/\varepsilon) - K_j^\ell] \psi \, d\sigma(x) = -\frac{\varepsilon}{2} \int_{\partial\Omega} \phi_{ij}^\ell(x/\varepsilon) \left\{ n_\ell \frac{\partial}{\partial x_i} - n_i \frac{\partial}{\partial x_\ell} \right\} \psi \, d\sigma(x). \tag{5.7}
$$

This, in particular, implies that

$$
|\gamma| \le C\varepsilon \|\nabla_{\tan} g\|_{L^2(\partial\Omega)}, \tag{5.8}
$$

where γ is given by (5.2), assuming that (ϕ_{ij}^ℓ) are bounded.

Let

$$
\Sigma_\rho = \big\{ x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) < \rho \big\}. \tag{5.9}
$$

Lemma 5.5. *Let T be the operator defined by*

$$
T(f)(x) = \int_{\partial\Omega} \frac{f(y)}{|x-y|^d} \, d\sigma(y).
$$

Then

$$
\int_{\Omega \setminus \Sigma_\varepsilon} |T(f)|^2 \, dx \le C\varepsilon^{-1} \int_{\partial\Omega} |f|^2 \, d\sigma \tag{5.10}
$$

for any $0 < \varepsilon < 1$, and

$$
\int_\Omega [\mathrm{dist}(x, \partial\Omega)]^\delta |T(f)|^2 \, dx \le C_\delta \int_{\partial\Omega} |f|^2 \, d\sigma \tag{5.11}
$$

for any $\delta > 1$.

Proof. It is not hard to see that

$$
\int_{\partial\Omega} \frac{d\sigma(y)}{|x-y|^d} \le \frac{C}{\mathrm{dist}(x, \partial\Omega)}
$$

for any $x \in \Omega$. By the Cauchy inequality,

$$
|T(f)(x)|^2 \le \frac{C}{\mathrm{dist}(x, \partial\Omega)} \int_{\partial\Omega} \frac{|f(y)|^2}{|x-y|^d} \, d\sigma(y).
$$

The estimate (5.10) follows by integrating the inequality above and using Fubini's Theorem. A similar argument gives 5.11. □

Lemma 5.6. *Let (H, q) be a weak solution in $H^1(\Omega; \mathbb{R}^d) \times L^2(\Omega)$ of the Dirichlet problem,*

$$
\begin{cases}
-\Delta H + \nabla q = 0 & \text{in } \Omega, \\
\mathrm{div}(H) = 0 & \text{in } \Omega, \\
H = h & \text{on } \partial\Omega,
\end{cases}
\tag{5.12}
$$

where h is given by (5.1)*. Then*

$$
\varepsilon \|\nabla H\|_{L^2(\Omega)} + \|H\|_{L^2(\Omega)} \le C\sqrt{\varepsilon} \Big\{ \|g\|_{L^2(\partial\Omega)} + \sqrt{\varepsilon} \, \|\nabla_{\tan} g\|_{L^2(\partial\Omega)} \Big\}, \tag{5.13}
$$

for any $0 < \varepsilon < 1$.

Proof. We first point out that by the standard energy estimates for the Stokes equations,

$$\|\nabla H\|_{L^2(\Omega)} \leq C\, \|h\|_{H^{1/2}(\partial\Omega)}.$$

In view of (5.4), this gives

$$\varepsilon\, \|\nabla H\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon}\, \left\{ \|g\|_{L^2(\partial\Omega)} + \varepsilon\, \|\nabla_{\tan} g\|_{L^2(\partial\Omega)} \right\}.$$

Next, we use the nontangential-maximal-function estimate,

$$\|(H)^*\|_{L^2(\partial\Omega)} \leq C\, \|h\|_{L^2(\partial\Omega)}, \tag{5.14}$$

to bound H on Σ_ε. The estimate (5.14) was proved in [6] for a Lipschitz domain Ω, where the nontangential maximal function $(H)^*$ is defined by

$$(H)^*(x) = \sup \big\{ |H(y)| : \ y \in \Omega \text{ and } |y - x| < C_0 \operatorname{dist}(y, \partial\Omega) \big\} \tag{5.15}$$

for $x \in \partial\Omega$. It follows that

$$\begin{aligned} \|H\|_{L^2(\Sigma_\varepsilon)} &\leq C\varepsilon^{1/2}\, \|(H)^*\|_{L^2(\partial\Omega)} \leq C\varepsilon^{1/2}\, \|h\|_{L^2(\partial\Omega)} \\ &\leq C\varepsilon^{1/2}\, \|g\|_{L^2(\partial\Omega)}. \end{aligned} \tag{5.16}$$

It remains to bound H on $\Omega \setminus \Sigma_\varepsilon$. To this end, we let $(G(x,y), \Pi(x,y))$ denote the matrix of Green functions for the Stokes equation (5.12) in Ω. That is, for each fixed $x \in \Omega$, $G(x,y) = (G^{ij}(x,y)) \in H^2_{\text{loc}}(\Omega \setminus \{x\}; \mathbb{R}^{d\times d})$ and $\Pi(x,y) = (\Pi^i(x,y)) \in L^2_{\text{loc}}(\Omega \setminus \{x\}; \mathbb{R}^d)$ satisfy

$$\begin{cases} -\Delta_y G^{ij}(x,y) + \dfrac{\partial}{\partial y_j} \Pi^i(x,y) = \delta_x \delta_{ij} & \text{in } \Omega \setminus \{x\}, \\ \dfrac{\partial}{\partial y_j} (G^{ij}(x,y)) = 0 & \text{in } \Omega \setminus \{x\}, \\ G^{ij}(x,y) = 0 & \text{for } y \in \partial\Omega, \end{cases}$$

in the sense of distribution. We also require that

$$\Pi(x,\cdot) \in L^1(\Omega; \mathbb{R}^d) \quad \text{ and } \quad \int_\Omega \Pi(x,y)\, dy = 0.$$

Under the assumption that Ω is a bounded $C^{2,\alpha}$ domain for some $\alpha > 0$, solutions of the Stokes equations (5.12) satisfy the $C^{1,1}$ estimate for H and $C^{0,1}$ estimate for q, up to the boundary. It follows that

$$\begin{aligned} |\nabla_x G(x,y)| + |\nabla_y G(x,y)| &\leq C|x-y|^{1-d}, \\ |\nabla_y G(x,y)| &\leq C \operatorname{dist}(x, \partial\Omega) |x-y|^{-d}, \\ |\nabla_x^2 G(x,y)| + |\nabla_y^2 G(x,y)| + |\nabla_x \nabla_y G(x,y)| &\leq C|x-y|^{-d}, \end{aligned} \tag{5.17}$$

and that

$$\begin{aligned} |\Pi(x,y)| &\leq C|x-y|^{1-d}, \\ |\Pi(x,y) - \Pi(x,z)| &\leq \operatorname{dist}(x, \partial\Omega) \{ |x-y|^{-d} + |x-z|^{-d} \}, \\ |\nabla_y \Pi(x,y)| &\leq C|x-y|^{-d}, \end{aligned} \tag{5.18}$$

for any $x, y \in \Omega$ and $x \neq y$, $x \neq z$. See e.g. [7, 21]. This allows us to represent the solution $H(x)$ by

$$H^i(x) = -\int_{\partial\Omega} \Big\{ n_k(y) \frac{\partial}{\partial y_k} G^{ij}(x,y) - \big[\Pi^i(x,y) - \Pi^i(x,z)\big] n_j(y) \Big\} h^j(y)\, d\sigma(y) \tag{5.19}$$

for any $x \in \Omega$, where $z \in \Omega$ and $z \neq x$ (due to the compatibility condition for h, the choice of z is arbitrary). Using (5.6), we may write $h = h^{(1)} + h^{(2)}$, where

$$\begin{aligned} h^{(1),k} &= \frac{\varepsilon}{2} \Big(n_\ell \frac{\partial}{\partial x_i} - n_i \frac{\partial}{\partial x_\ell} \Big) \Big\{ \phi^\ell_{ij}(x/\varepsilon) g_j n_k \Big\}, \\ h^{(2),k} &= -\frac{\varepsilon}{2} \phi^\ell_{ij}(x/\varepsilon) \Big(n_\ell \frac{\partial}{\partial x_i} - n_i \frac{\partial}{\partial x_\ell} \Big) \Big(g_j n_k \Big) - \gamma n_k, \end{aligned} \tag{5.20}$$

for $1 \leq k \leq d$. Let $H^{(1)}(x)$, $H^{(2)}(x)$ be given by (5.19), with h being replaced by $h^{(1)}$, $h^{(2)}$, respectively. Observe that by the divergence theorem,

$$\int_{\partial\Omega} \Big(n_\ell \frac{\partial}{\partial x_i} - n_i \frac{\partial}{\partial x_\ell} \Big) v \cdot w \, d\sigma = -\int_{\partial\Omega} v \cdot \Big(n_\ell \frac{\partial}{\partial x_i} - n_i \frac{\partial}{\partial x_\ell} \Big) w \, d\sigma \tag{5.21}$$

for $1 \leq i, \ell \leq d$. It follows that

$$\begin{aligned} & |H^{(1)}(x)| \\ & \leq C\varepsilon \int_{\partial\Omega} \Big\{ |\nabla_y G(x,y)| + |\nabla_y^2 G(x,y)| + |\nabla_y \Pi(x,y)| + |\Pi(x,y) - \Pi(x,z)| \Big\} |g(y)|\, d\sigma(y) \\ & \leq C\varepsilon \int_{\partial\Omega} \frac{|g(y)|}{|x-y|^d}\, d\sigma(y), \end{aligned}$$

where we have used the estimates in (5.17) and (5.18). In view of Lemma 5.5, we obtain

$$\|H^{(1)}\|_{L^2(\Omega\setminus\Sigma_\varepsilon)} \leq C\varepsilon^{1/2} \|g\|_{L^2(\partial\Omega)}. \tag{5.22}$$

Finally, note that

$$\begin{aligned} |H^{(2)}(x)| \leq\ & C\varepsilon \|\nabla_{\tan} g\|_{L^2(\partial\Omega)} \\ & + C\varepsilon \int_{\partial\Omega} \Big\{ |\nabla_y G(x,y)| + |\Pi(x,y) - \Pi(x,z)| \Big\} \left(|g(y)| + |\nabla_{\tan} g(y)| \right) d\sigma(y), \end{aligned}$$

where we have used (5.8). Using estimates in (5.17) and (5.18), we may deduce from (5.11) that

$$\|H^{(2)}\|_{L^2(\Omega)} \leq C\varepsilon \big\{ \|\nabla_{\tan} g\|_{L^2(\partial\Omega)} + \|g\|_{L^2(\partial\Omega)} \big\},$$

which completes the proof. □

We are now ready to give the proof of Theorem 5.1

Proof of Theorem 5.1. Let $(u_\varepsilon, p_\varepsilon)$ be a weak solution in $H^1(\Omega_\varepsilon; \mathbb{R}^d) \times L^2(\Omega_\varepsilon)$ of (4.1) with $\int_{\Omega_\varepsilon} p_\varepsilon \, dx = 0$, where h is given by (5.1). Let (H, q) be a solution of (5.12) with boundary data h. It follows from (3.9) that

$$\begin{aligned} \varepsilon \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} & \leq C \Big\{ \varepsilon \|\nabla H\|_{L^2(\Omega)} + \|H\|_{L^2(\Omega)} \Big\} \\ & \leq C\sqrt{\varepsilon} \Big\{ \|g\|_{L^2(\partial\Omega)} + \sqrt{\varepsilon} \|\nabla_{\tan} g\|_{L^2(\partial\Omega)} \Big\}, \end{aligned}$$

where we have used (5.13) for the last inequality. □

As a corollary of Theorem 5.1, we construct a normal boundary layer corrector.

Theorem 5.7. *Let Ω be a bounded $C^{2,\alpha}$ domain for some $\alpha > 0$. Also assume that ∂Y_s is $C^{1,\alpha}$. Let (Ψ_n, q_n) be a weak solution of (4.1) with $\int_{\Omega_\varepsilon} q_n \, dx = 0$, where the boundary data h is given by*

$$h = \Big\{ - n_i W_j^i(x/\varepsilon)\Big(f_j - \frac{\partial p_0}{\partial x_j}\Big) + b \cdot n - \gamma \Big\} n, \tag{5.23}$$

p_0 is defined by (4.12), b is the same as in Theorem 4.4, and $\gamma \in \mathbb{R}$ is such that $\int_{\partial\Omega} h \cdot n \, d\sigma = 0$. Then

$$\begin{aligned} &\varepsilon \|\nabla \Psi_n\|_{L^2(\Omega_\varepsilon)} + \|\Psi_n\|_{L^2(\Omega_\varepsilon)} + \|q_n\|_{L^2(\Omega_\varepsilon)} \\ &\qquad \le C\sqrt{\varepsilon} \Big\{ \|f - \nabla p_0\|_{L^2(\partial\Omega)} + \sqrt{\varepsilon} \|\nabla_{\tan}(f - \nabla p_0)\|_{L^2(\partial\Omega)} \Big\} \end{aligned} \tag{5.24}$$

for any $0 < \varepsilon < 1$. Moreover,

$$|\gamma| \le C\varepsilon \|\nabla_{\tan}(f - \nabla p_0)\|_{L^2(\partial\Omega)}. \tag{5.25}$$

Proof. Note that by the boundary condition in (4.12),

$$h = \Big\{ - n_\ell \Big[W_j^\ell(x/\varepsilon) - K_j^\ell \Big] \Big(f_j - \frac{\partial p_0}{\partial x_j} \Big) - \gamma \Big\} n \quad \text{ on } \partial\Omega.$$

As a result, the estimate (5.24) follows readily from Theorem 5.1 with $g = -(f - \nabla p_0)$. ☐

6 Convergence rates

In this section we prove the following theorem, which contains Theorem 1.1.

Theorem 6.1. *Let Ω be a bounded $C^{2,\alpha}$ domain in $\mathbb{R}^d$, $d \ge 2$ for some $\alpha > 0$. Also assume that ∂Y_s is $C^{1,\alpha}$. Let $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon; \mathbb{R}^d) \times L^2(\Omega_\varepsilon; \mathbb{R}^d)$ be a weak solution of the Dirichlet problem,*

$$\begin{cases} -\varepsilon^2 \Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \operatorname{div}(u_\varepsilon) = 0 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = b & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

where $f \in C^{1,1/2}(\overline{\Omega}; \mathbb{R}^d)$ and $b \in H^1(\partial\Omega; \mathbb{R}^d)$ satisfies the compatibility condition $\int_{\partial\Omega} b \cdot n \, d\sigma = 0$. Assume that $\int_{\Omega_\varepsilon} p_\varepsilon \, dx = 0$. Then for $0 < \varepsilon < 1$,

$$\begin{aligned} &\|u_\varepsilon - W(x/\varepsilon)(f - \nabla p_0)\|_{L^2(\Omega)} + \|P_\varepsilon - p_0\|_{L^2(\Omega)} \\ &\quad + \|\varepsilon \nabla u_\varepsilon - \nabla W(x/\varepsilon)(f - \nabla p_0)\|_{L^2(\Omega)} \\ &\qquad \le C\sqrt{\varepsilon} \Big\{ \|f\|_{C^{1,1/2}(\Omega)} + \|b \cdot n\|_{H^1(\partial\Omega)} + \|b\|_{L^2(\partial\Omega)} + \varepsilon \|\nabla_{\tan} b\|_{L^2(\partial\Omega)} \Big\}, \end{aligned} \tag{6.2}$$

where p_0 is defined by (4.12), P_ε is given by (2.6), and C depends only on Ω and Y_s.

We begin by introducing a corrector for the divergence operator. For $1 \le i, k \le d$, let $(\chi_{ik}^1(y), \dots, \chi_{ik}^d(y), \pi_{2,ik}(y)) \in H^1_{\text{loc}}(\omega; \mathbb{R}^d) \times L^2_{\text{loc}}(\omega)$ be an 1-periodic solution of

$$\begin{cases} -\Delta \chi_{ik}^j + \dfrac{\partial}{\partial y_j} \pi_{2,ik} = 0 & \text{in } \omega, \\ \dfrac{\partial}{\partial y_j} \chi_{ik}^j = -W_k^i + |Y \setminus Y_s|^{-1} K_k^i & \text{in } \omega, \\ \chi_{ik}^j = 0 & \text{on } \partial\omega. \end{cases} \tag{6.3}$$

Since the compatibility condition,

$$\int_{Y\setminus Y_s} \big\{ -W_k^i + |Y\setminus Y_s|^{-1} K_k^i \big\}\, dy = 0,$$

is satisfied, the 1-periodic solutions of (6.3) exist. Moreover, under the assumption that ∂Y_s is $C^{1,\alpha}$, the functions $\nabla \chi_{ik}^j$ and $\pi_{2,ik}$ are bounded. As usual, we extend χ_{ik}^j from ω to $\mathbb{R}^d$ by zero. Fix a function $\varphi \in C_0^\infty(B(0,1/8))$ with the properties that $\varphi \geq 0$ and $\int_{\mathbb{R}^d} \varphi\, dx = 1$. Let

$$S_\varepsilon(\psi)(x) = \psi * \varphi_\varepsilon(x) = \int_{\mathbb{R}^d} \psi(y)\varphi_\varepsilon(x-y)\, dy, \tag{6.4}$$

where $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(x/\varepsilon)$. Define $\Phi_\varepsilon(x) = (\Phi_\varepsilon^1(x), \Phi_\varepsilon^2(x), \ldots, \Phi_\varepsilon^d(x))$, where

$$\Phi_\varepsilon^j(x) = \varepsilon\eta_\varepsilon(x)\chi_{k\ell}^j(x/\varepsilon)\frac{\partial}{\partial x_\ell} S_\varepsilon\Big(f_k - \frac{\partial p_0}{\partial x_k}\Big), \tag{6.5}$$

p_0 is a solution of the Neumann problem (4.12), and η_ε is a cut-off function in $C_0^1(\Omega)$ such that $0 \leq \eta_\varepsilon \leq 1$, $\eta_\varepsilon = 1$ in $\Omega \setminus \Sigma_{3d\varepsilon}$, $\eta_\varepsilon = 0$ in $\Sigma_{2d\varepsilon}$, and $|\nabla \eta_\varepsilon| \leq C\varepsilon^{-1}$. The use of the ε-smoothing operator S_ε in (6.5) allows us to trade excessive powers of ε for lowering derivatives of $f - \nabla p_0$.

The following lemma will be useful to us.

Lemma 6.2. *Let S_ε be defined by* (6.4). *Then*

$$\|\psi - \eta_\varepsilon S_\varepsilon(\psi)\|_{L^2(\Omega)} \leq C\|\psi\|_{L^2(\Sigma_{3d\varepsilon})} + C\varepsilon\|\nabla\psi\|_{L^2(\Omega\setminus\Sigma_{d\varepsilon})} \tag{6.6}$$

for $0 < \varepsilon < 1$.

Proof. Note that

$$\|\psi - \eta_\varepsilon S_\varepsilon(\psi)\|_{L^2(\Omega)} \leq \|(1-\eta_\varepsilon)\psi\|_{L^2(\Omega)} + \|\eta_\varepsilon(\psi - S_\varepsilon(\psi)\|_{L^2(\Omega)}.$$

Clearly, the first term in the right-side hand is bounded by $\|\psi\|_{L^2(\Sigma_{3d\varepsilon})}$. To bound the second term, we use

$$\psi(x) - S_\varepsilon(\psi)(x) = \int_{\mathbb{R}^d} \varphi_\varepsilon(y)[\psi(x-y) - \psi(x)]\, dy$$

and

$$\psi(x-y) - \psi(x) = \int_0^1 (-y)\cdot\nabla\psi(x-ty)\, dt.$$

It follows that

$$\begin{aligned}
\|\eta_\varepsilon(\psi - S_\varepsilon(\psi)\|_{L^2(\Omega)} &\leq \int_{\mathbb{R}^d} \varphi_\varepsilon(y)\|\psi(\cdot - y) - \psi(\cdot)\|_{L^2(\Omega\setminus\Sigma_{2d\varepsilon})}\, dy \\
&\leq C\int_{\mathbb{R}^d} \varphi_\varepsilon(y)|y|\, dy\, \|\nabla\psi\|_{L^2(\Omega\setminus\Sigma_{d\varepsilon})} \\
&\leq C\varepsilon\|\nabla\psi\|_{L^2(\Omega\setminus\Sigma_{d\varepsilon})},
\end{aligned}$$

where we have used Minkowski's inequality. □

Note that for $x \in \Omega_\varepsilon$,

$$
\begin{aligned}
\operatorname{div}(\Phi_\varepsilon) &= \operatorname{div}(\chi_{k\ell})(x/\varepsilon)\Big[\eta_\varepsilon \frac{\partial}{\partial x_\ell} S_\varepsilon\Big(f_k - \frac{\partial p_0}{\partial x_k}\Big)\Big] + \varepsilon \chi_{k\ell}^j(x/\varepsilon)\frac{\partial}{\partial x_j}\Big[\eta_\varepsilon \frac{\partial}{\partial x_\ell} S_\varepsilon\Big(f_k - \frac{\partial p_0}{\partial x_k}\Big)\Big]\\
&= -\Big[W_k^\ell(x/\varepsilon) - \fint_{Y\setminus Y_s} W_k^\ell\Big]\Big[\eta_\varepsilon \frac{\partial}{\partial x_\ell} S_\varepsilon\Big(f_k - \frac{\partial p_0}{\partial x_k}\Big)\Big]\\
&\qquad + \varepsilon \chi_{k\ell}^j(x/\varepsilon)\frac{\partial}{\partial x_j}\Big[\eta_\varepsilon \frac{\partial}{\partial x_\ell} S_\varepsilon\Big(f_k - \frac{\partial p_0}{\partial x_k}\Big)\Big].
\end{aligned}
$$

Since

$$
\begin{aligned}
\operatorname{div}\Big(W(x/\varepsilon)(f - \nabla p_0)\Big) &= W_k^\ell(x/\varepsilon)\frac{\partial}{\partial x_\ell}\Big(f_k - \frac{\partial p_0}{\partial x_k}\Big)\\
&= \Big[W_k^\ell(x/\varepsilon) - \fint_{Y\setminus Y_s} W_k^\ell\Big]\frac{\partial}{\partial x_\ell}\Big(f_k - \frac{\partial p_0}{\partial x_k}\Big),
\end{aligned}
$$

where we have used the equation in (4.12), it follows that

$$
\begin{aligned}
&\|\operatorname{div}\Big(\Phi_\varepsilon + W(x/\varepsilon)(f - \nabla p_0)\Big)\|_{L^2(\Omega_{d\varepsilon})}\\
&\quad \le C\|\nabla(f - \nabla p_0)\|_{L^2(\Sigma_{3d\varepsilon})} + C\|\nabla[(f - \nabla p_0) - S_\varepsilon(f - \nabla p_0)]\|_{L^2(\Omega\setminus\Sigma_{2d\varepsilon})}\\
&\qquad + C\varepsilon\|\nabla^2 S_\varepsilon(f - \nabla p_0)\|_{L^2(\Omega\setminus\Sigma_{2d\varepsilon})}.
\end{aligned}
\tag{6.7}
$$

Let $(u_\varepsilon, p_\varepsilon)$ be a weak solution of (6.1) with $\int_{\Omega_\varepsilon} p_\varepsilon\, dx = 0$. Let

$$
v_\varepsilon = u_\varepsilon - \Big\{W(x/\varepsilon)(f - \nabla p_0) + \Phi_\varepsilon + \Psi_t + \Psi_n\Big\},
\tag{6.8}
$$

where Φ_ε is defined by (6.5), and Ψ_t, Ψ_n are given by Theorems 4.4 and 5.7, respectively. Using (2.20), a direct computation shows that

$$
\begin{aligned}
&-\varepsilon^2\Delta\Big\{W(x/\varepsilon)(f - \nabla p_0)\Big\} + \nabla\Big\{p_0 + \varepsilon\pi(x/\varepsilon)(f - \nabla p_0)\Big\}\\
&= f - \varepsilon^2\nabla\big(W(x/\varepsilon)\nabla(f - \nabla p_0)\big) - \varepsilon(\nabla W)(x/\varepsilon)\cdot\nabla(f - \nabla p_0) + \varepsilon\pi(x/\varepsilon)\nabla(f - \nabla p_0)\\
&\quad + \sigma_\varepsilon(f - \nabla p_0)
\end{aligned}
$$

in Ω_ε, where σ_ε is given by (2.21). It follows that

$$
\begin{aligned}
&-\varepsilon^2\Delta v_\varepsilon + \nabla\{p_\varepsilon - p_0 - p_t - p_n - \varepsilon\pi(x/\varepsilon)(f - \nabla p_0)\}\\
&= \varepsilon^2\Delta\Phi_\varepsilon + \varepsilon^2\nabla\big(W(x/\varepsilon)\nabla(f - \nabla p_0)\big) - \sigma_\varepsilon(f - \nabla p_0)\\
&\qquad + \varepsilon(\nabla W)(x/\varepsilon)\cdot\nabla(f - \nabla p_0) - \varepsilon\pi(x/\varepsilon)\nabla(f - \nabla p_0)
\end{aligned}
$$

in Ω_ε. Also, observe that

$$
\operatorname{div}(v_\varepsilon) = -\operatorname{div}\Big(\Phi_\varepsilon + W(x/\varepsilon)(f - \nabla p_0)\Big) \quad \text{in } \Omega_\varepsilon,
$$

$v_\varepsilon = 0$ on Γ_ε, and that

$$
v_\varepsilon = \gamma n \quad \text{on } \partial\Omega,
$$

where γ is a constant satisfying (5.25). Hence, by Theorem 3.1 as well as Remark 3.2 and the estimate (2.22) for σ_ε,

$$
\begin{aligned}
&\varepsilon\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\
&\le C\Big\{\varepsilon\|\nabla \Phi_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|\nabla(f-\nabla p_0)\|_{L^2(\Omega)} + \|f-\nabla p_0\|_{L^2(\Sigma_{c\varepsilon})} + \|\mathrm{div}(v_\varepsilon)\|_{L^2(\Omega_\varepsilon)} + |\gamma|\Big\} \\
&\le C\Big\{\varepsilon\|\nabla(f-\nabla p_0)\|_{L^2(\Omega)} + \|\nabla(f-\nabla p_0)\|_{L^2(\Sigma_{3d\varepsilon})} + \|f-\nabla p_0\|_{L^2(\Sigma_{c\varepsilon})} \\
&\qquad + C\|\nabla[(f-\nabla p_0) - S_\varepsilon(f-\nabla p_0)]\|_{L^2(\Omega\setminus\Sigma_{2d\varepsilon})} \\
&\qquad + \varepsilon\|\nabla^2 S_\varepsilon(f-\nabla p_0)\|_{L^2(\Omega\setminus\Sigma_{2d\varepsilon})} + \varepsilon\|\nabla_{\tan}(f-\nabla p_0)\|_{L^2(\partial\Omega)}\Big\},
\end{aligned}
\tag{6.9}
$$

where we have used (6.7) and (5.25). Let

$$
q_\varepsilon = p_\varepsilon - p_0 - q_t - q_n - \varepsilon\pi(x/\varepsilon)(f-\nabla p_0).
$$

Note that Theorem 3.1 also gives

$$
\begin{aligned}
\|q_\varepsilon - \fint_{\Omega_\varepsilon} q_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C\Big\{&\varepsilon\|\nabla(f-\nabla p_0)\|_{L^2(\Omega)} + \|\nabla(f-\nabla p_0)\|_{L^2(\Sigma_{3d\varepsilon})} \\
&+ \|f-\nabla p_0\|_{L^2(\Sigma_{c\varepsilon})} \\
&+ C\|\nabla[(f-\nabla p_0) - S_\varepsilon(f-\nabla p_0)]\|_{L^2(\Omega\setminus\Sigma_{2d\varepsilon})} \\
&+ \varepsilon\|\nabla^2 S_\varepsilon(f-\nabla p_0)\|_{L^2(\Omega\setminus\Sigma_{2d\varepsilon})} + \varepsilon\|\nabla_{\tan}(f-\nabla p_0)\|_{L^2(\partial\Omega)}\Big\}.
\end{aligned}
\tag{6.10}
$$

Lemma 6.3. *Let $(u_\varepsilon, p_\varepsilon)$ be a weak solution of (6.1) with $\int_{\Omega_\varepsilon} p_\varepsilon\, dx = 0$. Then*

$$
\begin{aligned}
&\varepsilon\|\nabla\big(u_\varepsilon - W(x/\varepsilon)(f-\nabla p_0)\big)\|_{L^2(\Omega_\varepsilon)} + \|p_\varepsilon - p_0\|_{L^2(\Omega_\varepsilon)} \\
&\le C\varepsilon^{1/2}\Big\{\varepsilon^{1/2}\|\nabla(f-\nabla p_0)\|_{L^2(\Omega)} + \varepsilon^{-1/2}\|\nabla(f-\nabla p_0)\|_{L^2(\Sigma_{3d\varepsilon})} + \varepsilon^{-1/2}\|f-\nabla p_0\|_{L^2(\Sigma_{c\varepsilon})} \\
&\qquad + \varepsilon^{-1/2}\|\nabla[(f-\nabla p_0) - S_\varepsilon(f-\nabla p_0)]\|_{L^2(\Omega\setminus\Sigma_{2d\varepsilon})} \\
&\qquad + \varepsilon^{1/2}\|\nabla^2 S_\varepsilon(f-\nabla p_0)\|_{L^2(\Omega\setminus\Sigma_{2d\varepsilon})} + \|f-\nabla p_0\|_{L^2(\partial\Omega)} \\
&\qquad + \|b\|_{L^2(\partial\Omega)} + \varepsilon\|\nabla_{\tan} b\|_{L^2(\partial\Omega)} + \sqrt{\varepsilon}\|\nabla_{\tan}(f-\nabla p_0)\|_{L^2(\partial\Omega)}\Big\}
\end{aligned}
\tag{6.11}
$$

for $0<\varepsilon<1$.

Proof. The estimate (6.11) follows readily from (6.9), (6.10), (4.13), and (5.24). □

To bound the right-hand side of (6.11), we let $p_0 = p_0^{(1)} + p_0^{(2)}$, where $p_0^{(1)}$ and $p_0^{(2)}$ are solutions of the Neumann problems,

$$
\begin{cases}
K_j^i \dfrac{\partial}{\partial x_i}\Big(f_j - \dfrac{\partial p_0^{(1)}}{\partial x_j}\Big) = 0 & \text{in } \Omega, \\
n_i K_j^i \Big(f_j - \dfrac{\partial p_0^{(1)}}{\partial x_j}\Big) = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{6.12}
$$

and

$$
\begin{cases}
K_j^i \dfrac{\partial^2 p_0^{(2)}}{\partial x_i \partial x_j} = 0 & \text{in } \Omega, \\
n_i K_j^i \dfrac{\partial p_0^{(2)}}{\partial x_j} = -b\cdot n & \text{on } \partial\Omega,
\end{cases}
\tag{6.13}
$$

respectively, with $\int_\Omega p_0^{(1)}\, dx = \int_\Omega p_0^{(2)}\, dx = 0$.

Lemma 6.4. *Let $p_0^{(1)}$ be a solution of* (6.12) *for some $f \in C^{1,1/2}(\overline{\Omega}, \mathbb{R}^d)$. Then*

$$
\begin{aligned}
\|\nabla p_0^{(1)}\|_{L^\infty(\Omega)} + \|\nabla^2 p_0^{(1)}\|_{L^\infty(\Omega)} &\le C\|f\|_{C^{1,1/2}(\overline{\Omega})}, \\
\|\nabla^2 S_\varepsilon(f)\|_{L^\infty(\Omega\setminus\Sigma_\varepsilon)} + \|\nabla^2 S_\varepsilon(\nabla p_0^{(1)})\|_{L^\infty(\Omega\setminus\Sigma_\varepsilon)} &\le C\varepsilon^{-1/2}\|f\|_{C^{1,1/2}(\overline{\Omega})},
\end{aligned}
\tag{6.14}
$$

and

$$
\|\nabla f - S_\varepsilon(\nabla f)\|_{L^\infty(\Omega\setminus\Sigma_\varepsilon)} + \|\nabla^2 p_0^{(1)} - S_\varepsilon(\nabla^2 p_0^{(1)})\|_{L^\infty(\Omega\setminus\Sigma_\varepsilon)} \le C\varepsilon^{1/2}\|f\|_{C^{1,1/2}(\overline{\Omega})}. \tag{6.15}
$$

Proof. Since Ω is a bounded $C^{2,\alpha}$ domain, the first inequality in (6.14) follows from the classical C^2 estimates, up to the boundary, for second-order elliptic equations with constant coefficients. Next, note that for $x \in \Omega \setminus \Sigma_\varepsilon$,

$$
\begin{aligned}
\frac{\partial}{\partial x_i} S_\varepsilon(\nabla f)(x) &= \varepsilon^{-1-d} \int_{\mathbb{R}^d} \frac{\partial \varphi}{\partial y_i}(y/\varepsilon) \nabla f(x-y)\, dy \\
&= \varepsilon^{-1-d} \int_{\mathbb{R}^d} \frac{\partial \varphi}{\partial y_i}(y/\varepsilon) \big[\nabla f(x-y) - \nabla f(x)\big]\, dy.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|\nabla^2 S_\varepsilon(f)\|_{L^\infty(\Omega\setminus\Sigma_\varepsilon)} &\le C\varepsilon^{-1-d} \int_{B(0,\varepsilon/4)} |\nabla\varphi(y/\varepsilon)||y|^{1/2}\, dy\, \|f\|_{C^{1,1/2}(\overline{\Omega})} \\
&\le C\varepsilon^{-1/2}\|f\|_{C^{1,1/2}(\overline{\Omega})}.
\end{aligned}
$$

By the interior $C^{2,1/2}$ estimates,

$$
|\nabla^2 p_0^{(1)}(x-y) - \nabla^2 p_0^{(1)}(y)| \le C|y|^{1/2}\Big\{\|f\|_{C^{1,1/2}(\overline{\Omega})} + \|p_0^{(1)}\|_{C^2(\overline{\Omega})}\Big\}
$$

for any $x \in \Omega \setminus \Sigma_\varepsilon$ and $|y| \le \frac{1}{4}\varepsilon$. As in the case of $\nabla^2 S_\varepsilon(f)$, this implies that

$$
|\nabla^2 S_\varepsilon(\nabla p_0^{(1)})(x)| \le C\varepsilon^{-1/2}\|f\|_{C^{1,1/2}(\overline{\Omega})}
$$

for any $x \in \Omega \setminus \Sigma_\varepsilon$. Finally, to see (6.15), we write

$$
S_\varepsilon(\nabla f)(x) - \nabla f(x) = \int_{\mathbb{R}^d} \varphi_\varepsilon(x-y)[\nabla f(x-y) - \nabla f(x)]\, dy
$$

and proceed as in the previous estimates. ∎

Lemma 6.5. *Let $p_0^{(2)}$ be a solution of* (6.13). *Then*

$$
\|\nabla p_0^{(2)}\|_{L^2(\Omega)} + \|\nabla p_0^{(2)}\|_{L^2(\partial\Omega)} + \varepsilon^{-1/2}\|\nabla p_0\|_{L^2(\Sigma_\varepsilon)} \le C\|b\cdot n\|_{L^2(\partial\Omega)}, \tag{6.16}
$$

$$
\|\nabla^2 p_0^{(2)}\|_{L^2(\Omega)} + \|\nabla^2 p_0^{(2)}\|_{L^2(\partial\Omega)} + \varepsilon^{-1/2}\|\nabla^2 p_0^{(2)}\|_{L^2(\Sigma_\varepsilon)} \le C\|b\cdot n\|_{H^1(\partial\Omega)}, \tag{6.17}
$$

$$
\varepsilon^{-1/2}\|\nabla^2 p_0^{(2)} - S_\varepsilon(\nabla^2 p_0^{(2)})\|_{L^2(\Omega\setminus\Sigma_{2\varepsilon})} \le C\|b\cdot n\|_{H^1(\partial\Omega)}, \tag{6.18}
$$

$$
\varepsilon^{1/2}\|\nabla^2 S_\varepsilon(\nabla p_0^{(2)})\|_{L^2(\Omega\setminus\Sigma_{2\varepsilon})} \le C\|b\cdot n\|_{H^1(\partial\Omega)}, \tag{6.19}
$$

for $0 < \varepsilon < 1$.

Proof. The estimates (6.16)-(6.19) follow from the nontangential-maximal-function and square-function estimates for the Neumann problems,

$$\|(\nabla p_0^{(2)})^*\|_{L^2(\partial\Omega)} + \left(\int_\Omega \operatorname{dist}(x,\partial\Omega)|\nabla^2 p_0^{(2)}(x)|^2\,dx\right)^{1/2} \le C\|b\cdot n\|_{L^2(\partial\Omega)}, \tag{6.20}$$

$$\|(\nabla^2 p_0^{(2)})^*\|_{L^2(\partial\Omega)} + \left(\int_\Omega \operatorname{dist}(x,\partial\Omega)|\nabla^3 p_0^{(2)}(x)|^2\,dx\right)^{1/2} \le C\|b\cdot n\|_{H^1(\partial\Omega)}, \tag{6.21}$$

where $(u)^*$ denotes the nontangential maximal function of u, defined by (5.15). We remark that the estimate (6.20) hols if Ω is a bounded Lipschitz domain [12], while (6.21) holds for $C^{2,\alpha}$ domains.

We only give the proof of (6.18); the others follow readily from (6.20)-(6.21). Choose $\widetilde{\eta}_\varepsilon \in C_0^1(\Omega)$ such that $\widetilde{\eta}_\varepsilon = 1$ in $\Omega \setminus \Sigma_{2\varepsilon}$, $\widetilde{\eta}_\varepsilon = 0$ in Σ_ε, and $|\nabla\widetilde{\eta}_\varepsilon| \le C\varepsilon^{-1}$. Then the left-hand side of (6.18) is bounded by

$$\varepsilon^{-1/2}\|\nabla^2 p_0^{(2)} - \widetilde{\eta}_\varepsilon S_\varepsilon(\nabla^2 p_0^{(2)})\|_{L^2(\Omega)}. \tag{6.22}$$

Using the same argument as in the proof of (6.6), we may show that (6.22) is bounded by

$$C\varepsilon^{-1/2}\|\nabla^2 p_0^{(2)}\|_{L^2(\Sigma_{3\varepsilon})} + C\varepsilon^{1/2}\|\nabla^3 p_0^{(2)}\|_{L^2(\Omega\setminus\Sigma_\varepsilon)} \le C\|b\cdot n\|_{H^1(\partial\Omega)},$$

where we have used (6.21) for the last step. □

We are now in a position to give the proof of Theorem 6.1.

Proof of Theorem 6.1. Using Lemmas 6.4 and 6.5, it is not hard to see that the right-hand side of (6.11) is bounded by

$$C\sqrt{\varepsilon}\big\{\|f\|_{C^{1,1/2}(\Omega)} + \|b\cdot n\|_{H^1(\partial\Omega)} + \|b\|_{L^2(\partial\Omega)} + \varepsilon\|\nabla_{\tan} b\|_{L^2(\partial\Omega)}\big\}.$$

As a result, we have proved that

$$\begin{aligned}&\varepsilon\|\nabla\big(u_\varepsilon - W(x/\varepsilon)(f-\nabla p_0)\big)\|_{L^2(\Omega_\varepsilon)} + \|p_\varepsilon - p_0\|_{L^2(\Omega_\varepsilon)}\\ &\qquad \le C\sqrt{\varepsilon}\left\{\|f\|_{C^{1,1/2}(\Omega)} + \|b\cdot n\|_{H^1(\partial\Omega)} + \|b\|_{L^2(\partial\Omega)} + \varepsilon\|\nabla_{\tan} b\|_{L^2(\partial\Omega)}\right\}.\end{aligned} \tag{6.23}$$

In view of Lemma 2.2, it remains to show that

$$\|P_\varepsilon - p_0\|_{L^2(\Omega)} \le C\sqrt{\varepsilon}\left\{\|f\|_{C^{1,1/2}(\Omega)} + \|b\cdot n\|_{H^1(\partial\Omega)} + \|b\|_{L^2(\partial\Omega)} + \varepsilon\|\nabla_{\tan} b\|_{L^2(\partial\Omega)}\right\}, \tag{6.24}$$

where P_ε is an extension of p_ε to Ω, defined by (2.6). To this end, we define

$$p_0^\varepsilon = \begin{cases} p_0 & \text{if } x\in\Omega_\varepsilon,\\ \displaystyle ⨍_{\varepsilon(Y_f+z_k)} p_0 & \text{if } x\in\varepsilon(Y_s+z_k) \text{ and } \varepsilon(Y+z_k)\subset\Omega \text{ for some } z_k\in\mathbb{Z}^d,\end{cases} \tag{6.25}$$

i.e., we extend $p_0|_{\Omega_\varepsilon}$ to Ω in the same manner as we do p_ε from Ω_ε to Ω. Then,

$$\begin{aligned}\|P_\varepsilon - p_0\|_{L^2(\Omega)} &\le \|P_\varepsilon - p_0^\varepsilon\|_{L^2(\Omega)} + \|p_0^\varepsilon - p_0\|_{L^2(\Omega)}\\ &= \|p_\varepsilon - p_0\|_{L^2(\Omega_\varepsilon)} + \|P_\varepsilon - p_0^\varepsilon\|_{L^2(\Omega\setminus\Omega_\varepsilon)} + \|p_0^\varepsilon - p_0\|_{L^2(\Omega\setminus\Omega_\varepsilon)}.\end{aligned}$$

Note that

$$\|P_\varepsilon - p_0^\varepsilon\|_{L^2(\Omega\setminus\Omega_\varepsilon)} \le C\|p_\varepsilon - p_0\|_{L^2(\Omega_\varepsilon)}.$$

Using Poincaré's inequality on each cell $\varepsilon(Y_f + z_k)$, we may show that

$$\|p_0^\varepsilon - p_0\|_{L^2(\Omega\setminus\Omega_\varepsilon)} \le C\varepsilon\|\nabla p_0\|_{L^2(\Omega)} \le C\varepsilon\left\{\|f\|_{L^2(\Omega)} + \|b\|_{L^2(\partial\Omega)}\right\}.$$

As a result, we have proved that

$$\|P_\varepsilon - p_0\|_{L^2(\Omega)} \le C\|p_\varepsilon - p_0\|_{L^2(\Omega_\varepsilon)} + C\varepsilon\left\{\|f\|_{L^2(\Omega)} + \|b\|_{L^2(\partial\Omega)}\right\}.$$

This completes the proof. □

Remark 6.6. Let $u(x, x/\varepsilon)$ be given by (1.6). Due to the discrepancy of u_ε and $u(x, x/\varepsilon)$ on $\partial\Omega$, the $O(\sqrt{\varepsilon})$ rate in Theorem 1.1 is sharp. Indeed, by applying the following trace inequality to the function $v = v_\varepsilon = u_\varepsilon - u(x, x/\varepsilon)$,

$$\|v\|_{L^2(\partial\Omega)} \le C\left\{\varepsilon^{-1/2}\|v\|_{L^2(\Sigma_{c\varepsilon})} + \|v\|^{1/2}_{L^2(\Sigma_{c\varepsilon})}\|\nabla v\|^{1/2}_{L^2(\Sigma_{c\varepsilon})}\right\}, \tag{6.26}$$

we obtain

$$\begin{aligned}\sqrt{\varepsilon}\|v_\varepsilon\|_{L^2(\partial\Omega)} &\le C\big\{\|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)}\big\}\\ &\le C\varepsilon\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)},\end{aligned}$$

where we have used the Cauchy inequality for the first inequality and (2.9) for the second. It follows that the error estimate

$$\varepsilon\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)} = o(\sqrt{\varepsilon}) \quad \text{ as } \varepsilon \to 0,$$

cannot hold in general. In fact, if Ω is smooth and uniformly convex, then

$$\begin{aligned}\lim_{\varepsilon\to 0} \fint_{\partial\Omega} |v_\varepsilon|^2\, d\sigma &= \lim_{\varepsilon\to 0}\fint_{\partial\Omega} |W(x/\varepsilon)(f - \nabla p_0)|^2\, d\sigma\\ &= \fint_{\partial\Omega} |K(f - \nabla p_0)|^2\, d\sigma.\end{aligned} \tag{6.27}$$

See the proof of Lemma 3.2 in [1]. Also, note that by Theorem 1.1,

$$\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon^{-1/2}\|f\|_{C^{1,1/2}(\Omega)}.$$

This, together with (6.26), yields

$$\|v_\varepsilon\|_{L^2(\partial\Omega)} \le C\left\{\varepsilon^{-1/2}\|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \big(\varepsilon^{-1/2}\|v_\varepsilon\|_{2(\Omega_\varepsilon)}\big)^{1/2}\|f\|^{1/2}_{C^{1,1/2}(\Omega)}\right\}.$$

As a result, it is not possible to have

$$\|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} = o(\sqrt{\varepsilon}) \quad \text{ as } \varepsilon \to 0,$$

unless $f = \nabla p_0$ in Ω, in which case, $v_\varepsilon \equiv 0$ in Ω_ε.

Finally, to see (6.26), choose a function $\beta \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ such that $\beta \cdot n \ge c_0 > 0$ on $\partial\Omega$, $\text{supp}(\beta) \subset \{x \in \mathbb{R}^d : \text{dist}(x, \partial\Omega) \le c\varepsilon\}$, and $|\nabla\beta| \le C\varepsilon^{-1}$. It follows by the divergence theorem that

$$\begin{aligned}c_0\int_{\partial\Omega} |v|^2\, d\sigma &\le \int_{\partial\Omega} |v|^2\, \beta\cdot n\, d\sigma \le \int_\Omega |v|^2 \text{div}(\beta)\, dx + 2\int_\Omega |v||\nabla v||\beta|\, dx\\ &\le C\varepsilon^{-1}\int_{\Sigma_{c\varepsilon}} |v|^2\, dx + C\|v\|_{L^2(\Sigma_{c\varepsilon})}\|\nabla v\|_{L^2(\Sigma_{c\varepsilon})},\end{aligned}$$

where we have used the Cauchy inequality for the last step.

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