Critical Sets of Elliptic Equations with Rapidly Oscillating Coefficients in Two Dimensions

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Abstract

In this paper we continue the study of critical sets of solutions u_ε of second-order elliptic equations in divergence form with rapidly oscillating and periodic coefficients. In [18], by controling the "turning" of approximate tangent planes, we show that the $(d-2)$-dimensional Hausdorff measures of the critical sets are bounded uniformly with respect to the period ε, provided that doubling indices for solutions are bounded. In this paper we use a different approach, based on the reduction of the doubling indices of u_ε, to study the two-dimensional case. The proof relies on the fact that the critical set of a homogeneous harmonic polynomial of degree two or higher in dimension two contains only one point.

Keywords: Critical Set; Homogenization; Doubling Index.

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1 Introduction

In this paper we continue the study of critical points of solutions of elliptic equations in homogenization. More precisely, we consider a family of second-order elliptic operators in divergence form,

$$\mathcal{L}_\varepsilon = -\text{div}(A(x/\varepsilon)\nabla), \tag{1.1}$$

where $0 < \varepsilon \le 1$ and $A(y) = (a_{ij}(y))$ is a $d \times d$ matrix-valued function in $\mathbb{R}^d$. Throughout the paper, unless indicated otherwise, we shall assume that

- (ellipticity) there exists some $\lambda \in (0, 1]$ such that

$$\lambda|\xi|^2 \le \langle A(y)\xi, \xi\rangle \quad \text{and} \quad |\langle A(y)\xi, \zeta\rangle| \le \lambda^{-1}|\xi||\zeta| \quad \text{for any } y, \xi, \zeta \in \mathbb{R}^d; \tag{1.2}$$

- (periodicity) A is periodic with respect to some lattice Γ of $\mathbb{R}^d$,

$$A(y + z) = A(y) \quad \text{for any } y \in \mathbb{R}^d \text{ and } z \in \Gamma; \tag{1.3}$$

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- (smoothness) there exists some $M > 0$ such that

$$|A(x) - A(y)| \le M|x-y| \quad \text{for any } x, y \in \mathbb{R}^d. \tag{1.4}$$

We will use the notation,

$$E_r = \left\{ x \in \mathbb{R}^d : \ 2\langle (\widehat{A} + (\widehat{A})^T)^{-1} x, x\rangle < r^2 \right\} \tag{1.5}$$

for $r > 0$, where $\widehat{A}$ denotes the homogenized matrix for A. If $\widehat{A} + (\widehat{A})^T = 2I$, then $E_r = B(0, r)$.

Let $\chi(y) = (\chi_j(y))$ denote the first-order corrector for $\mathcal{L}_\varepsilon$. We will also assume that the periodic matrix $I + \nabla\chi$ is nonsingular and that

$$\det(I + \nabla\chi) \ge \mu \tag{1.6}$$

for some $\mu > 0$. Let u_ε be a non-constant weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in E_2 and

$$\mathcal{C}(u_\varepsilon) = \left\{ x : \ |\nabla u_\varepsilon(x)| = 0 \right\}, \tag{1.7}$$

the critical set of u_ε. Suppose that $u_\varepsilon(0) = 0$ and

$$\fint_{E_2} u_\varepsilon^2 \le 4^N \fint_{E_1} u_\varepsilon^2 \tag{1.8}$$

for some $N \ge 1$. Under the conditions (1.2), (1.3), (1.4) and (1.6), it is proved in [18] by the present authors that

$$|\left\{ x : \text{dist}(x, \mathcal{C}(u_\varepsilon) \cap E_{1/2}) < r \right\}| \le C(N) r^2 \tag{1.9}$$

for $0 < r < 1$, and consequently,

$$\mathcal{H}^{d-2}\left\{ x \in E_{1/2} : \ |\nabla u_\varepsilon(x)| = 0 \right\} \le C(N), \tag{1.10}$$

where $C(N)$ depends at most on d, λ, Γ, M, μ, and N. This is the first result on geometric measure estimates, that are uniform in $\varepsilon > 0$, for critical sets of solutions of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$. We mention that in [17], the following uniform bound of the nodal sets,

$$\mathcal{H}^{d-1}\left\{ x \in E_{1/2} : \ u_\varepsilon(x) = 0 \right\} \le C(N), \tag{1.11}$$

was established by the present authors, under the conditions (1.2), (1.3) and (1.4). Classical results in the study of nodal, singular, and critical sets for solutions and eigenfunctions of elliptic operators may be found in [10, 15, 16, 11, 13, 12, 14]. We refer the reader to [9, 21, 5, 19, 20] and their references for more recent work in this area. Since the bounding constants $C(N)$ depend on the smoothness of coefficients, the quantitative results for $\mathcal{L}_1$ in the references mentioned above do not extend to the operator $\mathcal{L}_\varepsilon$.

The proof of (1.9) in [18] is based on an estimate of "turning" for the projection of a non-constant solution u_ε onto the subspace of spherical harmonic order ℓ, when the doubling index for u_ε on a sphere $\partial B(0, r)$ is trapped between $\ell - \delta$ and $\ell + \delta$, for r between 1 and a minimal radius $r^* \geq C_0\varepsilon$. In this paper we provide a different and much simpler proof for the two-dimensional case. Our approach is based on the reduction of the doubling index and relies on the fact that the critical set of a homogeneous harmonic polynomial of degree 2 or higher in dimension two contains only one point. We note that the condition (1.6) holds in the case $d = 2$ if A is periodic and Hölder continuous.

The following is the main result of the paper.

Theorem 1.1. *Let $d = 2$. Assume that $A = A(y)$ satisfies the conditions* (1.2), (1.3) *and* (1.4)*. Let $u_\varepsilon \in H^1(E_2)$ be a non-constant weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $E_2 \subset \mathbb{R}^2$. Suppose that $u_\varepsilon(0) = 0$ and* (1.8) *holds for some $N \geq 1$. Then*

$$\#(E_{1/2} \cap \mathcal{C}(u_\varepsilon)) \leq C(N), \tag{1.12}$$

where $C(N)$ depends at most on λ, Γ, M, and N.

Throughout the paper we will use C and c to denote constants that may depend on d, λ in (1.2), Γ in (1.3), M in (1.4), and μ in (1.6). If a constant also depends on other parameters, such as the doubling index of a solution, it will be stated explicitly.

2 Homogenization

Let $d \geq 2$ and $A = A(y)$ be a $d \times d$ matrix satisfying (1.2) and (1.3). The first-order corrector $\chi = \chi(y) = (\chi_j(y))$ is defined by the cell problem,

$$\begin{cases} \mathcal{L}_1(\chi_j) = -\dfrac{\partial}{\partial y_i}(a_{ij}) \quad \text{in } Y, \\ \displaystyle\fint_Y \chi_j = 0 \quad \text{and} \quad \chi_j \text{ is Y-periodic,} \end{cases} \tag{2.1}$$

for $1 \leq j \leq d$ (the index i is summed from 1 to d), where Y is the fundamental domain for the lattice Γ. The homogenized operator $\mathcal{L}_0$ is given by

$$\mathcal{L}_0 = -\text{div}(\widehat{A}\nabla), \tag{2.2}$$

where, for $\xi \in \mathbb{R}^d$,

$$\langle \widehat{A}\xi, \xi \rangle = \fint_Y \langle A\nabla v_\xi, \nabla v_\xi \rangle \tag{2.3}$$

and $v_\xi(y) = \langle \xi, y + \chi(y) \rangle$. It follows from (2.1) that $\mathcal{L}_1(y_j + \chi_j) = 0$ in $\mathbb{R}^d$. Thus, by De Giorgi - Nash estimates, χ_j is Hölder continuous. Furthermore, if A is Hölder continuous, i.e., there exist $\alpha \in (0, 1]$ and $M_\alpha > 0$ such that

$$|A(x) - A(y)| \leq M_\alpha |x - y|^\alpha \quad \text{for any } x, y \in \mathbb{R}^d, \tag{2.4}$$

so is $\nabla \chi_j$.

Theorem 2.1. *Let $d = 2$. Suppose A satisfies* (1.2), (1.3) *and* (2.4)*. Then*

$$\det(I + \nabla\chi)(x) \geq \mu \tag{2.5}$$

for any $x \in \mathbb{R}^2$, where $\mu > 0$ depends only on λ, Γ and (α, M_α).

Proof. This theorem was more or less proved in [2, 3], although it is not stated explicitly. Also see related work in [1, 6, 4] and the references therein. We give an outline of the proof here.

Step 1. Let $u_i = x_i + \chi_i(x)$ for $i = 1, 2$, and $U = (u_1, u_2)$. Use the continuity and boundedness of χ_i to show that $U : \mathbb{R}^2 \to \mathbb{R}^2$ is onto.

Step 2. Show that $U : \mathbb{R}^2 \to \mathbb{R}^2$ is one-to-one and U^{-1} is continuous. As a result, $U : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism. The smoothness condition (2.4) is not needed. See [2] for details.

Step 3. Let $\xi \in \mathbb{R}^2$ with $|\xi| = 1$. Consider the function $u_\xi = \langle U, \xi \rangle$. Note that $\operatorname{div}(A(x)\nabla u_\xi) = 0$ in $\mathbb{R}^2$. To prove $\det(I + \nabla\chi) > 0$, it suffices to show that

$$|\nabla u_\xi(x)| > 0$$

for any $x \in \mathbb{R}^2$. To this end, fix $y_0 \in \mathbb{R}^2$ and $r_0 > 0$. Let $\Omega = U^{-1}(B(y_0, r_0))$. Note that $\operatorname{div}(A\nabla u_\xi) = 0$ in Ω, and $g = u|_{\partial\Omega}$ is unimodal. This implies $|\nabla u_\xi(x_0)| > 0$, where $U(x_0) = y_0$. See [3] for details.

Step 4. Use $\det(I+\nabla\chi) > 0$ and a compactness argument to show that $\det(I+\nabla\chi) \geq \mu$, where $\mu > 0$ depends only on λ, Γ and (α, M_α). □

Remark 2.2. The estimate (2.5) fails if $d \geq 3$. See [7, 8] for a counter-example.

By a change of variables we may assume that

$$\widehat{A} + (\widehat{A})^T = 2I. \tag{2.6}$$

See Remark 2.3 in [18]. This ensures that solutions of the homogenized equation $\mathcal{L}_0(u_0) = 0$ are harmonic. The following compactness theorem will be used in the next section.

Theorem 2.3. *Let u_j be a solution of $\operatorname{div}(A^j(x/\varepsilon_j)\nabla u_j) = 0$ in $B(0, r_0)$, where $\varepsilon_j \to 0$ and A^j satisfies (1.2), (1.3), (2.4) and (2.6). Suppose that $\{u_j\}$ is bounded in $L^2(B(0, r_0))$. Then there exists a subsequence, still denoted by $\{u_j\}$ and a harmonic function u_0 in $B(0, r_0)$, such that $u_j \to u_0$ weakly in $L^2(B(0, r_0))$ and weakly in $H^1(B(0, r))$ for any $0 < r < r_0$. Moreover,*

$$\|u_j - u_0\|_{L^\infty(B(0,r))} \to 0, \tag{2.7}$$

$$\|\nabla u_j - (I + \nabla\chi^j(x/\varepsilon_j))\nabla u_0\|_{L^\infty(B(0,r))} \to 0, \tag{2.8}$$

for any $0 < r < r_0$, where χ^j denotes the first-order correctors for the matrix A^j.

Proof. See Theorem 2.7 and Remark 2.8 in [18]. □

3 Doubling indices and critical sets

Let $d \geq 2$. As in [18], we introduce a doubling index for a continuous function u on a ball $B(x_0, r)$, defined by

$$N^*(u, x_0, r) = \log_4 \frac{\fint_{\partial B(x_0,r)} (u - u(x_0))^2}{\fint_{\partial B(x_0,r/2)} (u - u(x_0))^2}, \tag{3.1}$$

assuming $\|u - u(x_0)\|_{L^2(\partial B(x_0,t))} \neq 0$ for $0 < t \leq r$. Define

$$\mathcal{M}(\lambda, \Gamma, M) = \Big\{ A = A(y) : \ A \text{ satisfies (1.2), (1.3), (1.4), and (2.6)} \Big\}. \tag{3.2}$$

Theorem 3.1. *Let $L \geq 2$ and $\delta_0 \in (0, 1/2]$. Assume that $A \in \mathcal{M}(\lambda, \Gamma, M)$. There exists $\varepsilon_0 = \varepsilon_0(L, \delta_0) > 0$ such that if $0 < \varepsilon < \varepsilon_0 r$ and $u_\varepsilon \in H^1(B(x_0, r))$ is a non-constant solution of $\text{div}(A(x/\varepsilon)\nabla u_\varepsilon) = 0$ in $B(x_0, r)$ for some $r > 0$ and $x_0 \in \mathbb{R}^d$, with the properties that,*

$$N^*(u_\varepsilon, x_0, r) \leq L + 1 \quad \text{and} \quad N^*(u_\varepsilon, x_0, r/2) \leq \ell + \delta_0, \tag{3.3}$$

where $\ell \in \mathbb{N}$ and $1 \leq \ell \leq L$, then

$$N^*(u_\varepsilon, x_0, r/4) \leq \ell + \delta_0. \tag{3.4}$$

If, in addition, $2^J \varepsilon < \varepsilon_0 r$ for some integer $J \geq 0$, then

$$N^*(u_\varepsilon, x_0, r/2^j) \leq \ell + \delta_0 \quad \text{for } j = 2, \ldots, J + 2. \tag{3.5}$$

Proof. This is proved in [18, Theorem 3.1]. □

Theorem 3.2. *Let $L \geq 2$ and $\delta_1 \in (0, 1/2]$. Assume that $A \in \mathcal{M}(\lambda, \Gamma, M)$. There exists $\varepsilon_1 = \varepsilon_1(L, \delta_1) > 0$ such that if $0 < \varepsilon < \varepsilon_1 r$, $u_\varepsilon \in H^1(B(x_0, r))$, $\text{div}(A(x/\varepsilon)\nabla u_\varepsilon) = 0$ in $B(x_0, r)$ for some $x_0 \in \mathbb{R}^d$ and $r > 0$,*

$$N^*(u_\varepsilon, x_0, r) \leq L + 1 \quad \text{and} \quad N^*(u_\varepsilon, x_0, r/2) \leq \ell - \delta_1, \tag{3.6}$$

where $\ell \in \mathbb{N}$ and $1 \leq \ell \leq L$, then

$$N^*(u_\varepsilon, x_0, \delta_1 r/(8\ell)) \leq \ell - 1 + \delta_1. \tag{3.7}$$

Proof. This is proved in [18, Theorem 3.4]. □

Define

$$\mathcal{A}(\lambda, \Gamma, M, \mu) = \Big\{ A = A(y) : A \text{ satisfies (1.2), (1.3), (1.4), (1.6) and (2.6)} \Big\}. \tag{3.8}$$

Theorem 3.3. *Let $L \geq 2$ and $A \in \mathcal{A}(\lambda, \Gamma, M, \mu)$. There exists $\varepsilon_0 = \varepsilon_0(L) > 0$ such that if $u_\varepsilon \in H^1(B(0, 1))$ is a non-constant solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $B(0, 1)$ for some $0 < \varepsilon < \varepsilon_0$, $N^*(u_\varepsilon, 0, 1) \leq L$, and*

$$N^*(u_\varepsilon, 0, 1/2) \leq 3/2, \tag{3.9}$$

then $|\nabla u_\varepsilon(0)| \neq 0$.

Proof. This is proved in [18, Theorem 3.5]. □

Fix $L \geq 1$, $\varepsilon > 0$, $r > 0$, and $x_0 \in \mathbb{R}^d$. Define

$$\begin{aligned} \mathcal{F}(L, \varepsilon, r, x_0) = \Big\{ & u \in H^1(B(x_0, 2r)) : \quad u \text{ is not constant, } u(x_0) = 0, \\ & \text{div}(A(x/\varepsilon)\nabla u) = 0 \text{ in } B(x_0, 2r) \text{ for some } A \in \mathcal{A}(\lambda, \Gamma, M, \mu), \\ & \text{and } N^*(u, x, r) \leq 2L, \ N^*(u, x, r/2) \leq L \ \text{ for all } x \in B(x_0, r/2) \Big\}, \end{aligned} \tag{3.10}$$

and

$$
\mathcal{E}(L,\varepsilon,r)= \sup\left\{ \frac{\mathcal{H}^{d-2}(\mathcal{C}(u)\cap B(x_0,r/4))}{r^{d-2}}:\ u\in\mathcal{F}(L,\varepsilon,r,x_0)\quad\text{for some } x_0\in\mathbb{R}^d\right\}, \tag{3.11}
$$

where $\mathcal{C}(u)$ denotes the critical set of u,

$$
\mathcal{C}(u)=\big\{x:\ |\nabla u(x)|=0\big\}.
$$

Since $u\in\mathcal{F}(L,\varepsilon,r,x_0)$ implies $u(\cdot+x_0)\in\mathcal{F}(L,\varepsilon,r,0)$, it follows that

$$
\mathcal{E}(L,\varepsilon,r)=\sup\left\{ \frac{\mathcal{H}^{d-2}(\mathcal{C}(u)\cap B(0,r/4))}{r^{d-2}}:\quad u\in\mathcal{F}(L,\varepsilon,r,0)\right\}. \tag{3.12}
$$

By a simple covering argument, it is not hard to see that if $0<r_1\le r_2/2$, then

$$
\mathcal{E}(L,\varepsilon,r_2)\le C\left(\frac{r_2}{r_1}\right)^2\mathcal{E}(L,\varepsilon,r_1), \tag{3.13}
$$

where C depends only on d.

Lemma 3.4. *For any $\theta>0$,*

$$
\mathcal{E}(L,\varepsilon,r)=\mathcal{E}(L,\theta^{-1}\varepsilon,\theta^{-1}r). \tag{3.14}
$$

Proof. This follows from the observation that if $u\in\mathcal{F}(L,\varepsilon,r,0)$ and $v(x)=u(\theta x)$, then $v\in\mathcal{F}(L,\theta^{-1}\varepsilon,\theta^{-1}r,0)$ and

$$
\mathcal{H}^{d-2}(\mathcal{C}(u)\cap B(0,r/4))=\theta^{d-2}\mathcal{H}^{d-2}(\mathcal{C}(v)\cap B(0,\theta^{-1}r/4)).
$$

□

Theorem 3.5. *If $0<r\le\varepsilon_0^{-1}\varepsilon$ for some $\varepsilon_0>0$, then*

$$
\mathcal{E}(L,\varepsilon,r)\le C(L,\varepsilon_0), \tag{3.15}
$$

where $C(L,\varepsilon_0)$ depends on ε_0 and L.

Proof. Note that by (3.14),

$$
\mathcal{E}(L,\varepsilon,r)=\mathcal{E}(L,r^{-1}\varepsilon,1).
$$

Since $r^{-1}\varepsilon\ge\varepsilon_0$ and A satisfies (1.2) and (1.4), the estimate (3.15) follows readily from [21] for the operator $\mathcal{L}_1$ (see [11] for the case $d=2$ and [12] for the case of smooth coefficients). Indeed, the coefficient matrix $\widetilde{A}(x)=A(x/(r^{-1}\varepsilon))$ satisfies the Lipschitz condition (1.4) with $M\varepsilon_0^{-1}$ in the place of M. Moreover, the conditions $N^*(u,x,1)\le 2L$ and $N^*(u,x,1/2)\le L$ for $x\in B(0,1/2)$ implies that

$$
\int_{B(0,1)}|\nabla u|^2\le C\int_{B(0,1)}u^2\le C\int_{\partial B(0,1)}u^2,
$$

where C depends on L. The periodicity condition is not needed. □

Theorem 3.6. *Fix $L \geq 2$ and $\delta_0 \in (0, 1/2]$. There exists $\varepsilon_0 > 0$, depending on L and δ_0, such that if $0 < \varepsilon < \varepsilon_0 r$, $\ell \in \mathbb{N}$ and $2 \leq \ell \leq L$, then*

$$\mathcal{E}(\ell - \delta_0, \varepsilon, r) \leq C_0 \mathcal{E}(\ell - 1 + \delta_0, \varepsilon, c_0 r), \tag{3.16}$$

where $c_0 = \delta_0/(8\ell)$ and C_0 depends on L and δ_0.

Proof. In view of Lemma 3.4 we may assume $r = 1$. By the definition of $\mathcal{E}(\ell - 1 + \delta_0, \varepsilon, c_0)$, it suffices to show that if $u \in \mathcal{F}(\ell - \delta_0, \varepsilon, 1, 0)$, then

$$N^*(u, x_0, c_0) \leq \ell - 1 + \delta_0 \quad \text{and} \quad N^*(u, x_0, c_0/2) \leq \ell - 1 + \delta_0, \tag{3.17}$$

for any $x_0 \in B(0, 1/2)$, provided that $0 < \varepsilon < \varepsilon_0$. By covering $B(0, 1/4)$ with a finite number of balls $\{B(y_j, c_0/4) : j = 1, 2, \ldots, k_0\}$, where $k_0 \leq C(d)/(c_0)^d$ and $y_j \in B(0, 1/4)$, this would imply that $u \in \mathcal{F}(\ell - 1 + \delta_0, \varepsilon, c_0, y_j)$ for $1 \leq j \leq k_0$. As a result,

$$\begin{aligned} \mathcal{H}^{d-2}(\mathcal{C}(u) \cap B(0, 1/4)) &\leq \sum_j \mathcal{H}^{d-2}(\mathcal{C}(u) \cap B(y_j, c_0/4)) \\ &\leq k_0 c_0^{d-2} \mathcal{E}(\ell - 1 + \delta_0, \varepsilon, c_0), \end{aligned}$$

from which the estimate (3.16) with $r = 1$ follows.

To see (3.17), we note that $N^*(u, x_0, 1) \leq 2(\ell - \delta_0)$ and $N^*(u, x_0, 1/2) \leq \ell - \delta_0$ for any $x_0 \in B(0, 1/2)$. By Theorem 3.2 we have $N^*(u, x_0, c_0) \leq \ell - 1 + \delta_0$, where $c_0 = \delta_0/(8\ell)$. Observe that if ε is sufficiently small, we may use Theorem 3.1 to obtain $N^*(u, x_0, 2c_0) \leq C(L)$. Applying Theorem 3.1 again gives $N^*(u, x_0, c_0/2) \leq \ell - 1 + \delta_0$. □

Theorem 3.7. *There exists $\varepsilon_0 > 0$ such that*

$$\mathcal{E}(3/2, \varepsilon, r) = 0 \tag{3.18}$$

for any $0 < \varepsilon < \varepsilon_0 r$.

Proof. By Lemma 3.4 we may assume $r = 1$. Let $u \in \mathcal{F}(3/2, \varepsilon, 1, 0)$. Then $N^*(u, x_0, 1) \leq 3$ and $N^*(u, x_0, 1/2) \leq 3/2$ for any $x_0 \in B(0, 1/2)$. By Theorem 3.3 we obtain $|\nabla u(x_0)| \neq 0$, if $0 < \varepsilon < \varepsilon_0$. Thus $\mathcal{C}(u) \cap B(0, 1/4) = \emptyset$ and consequently, $\mathcal{E}(3/2, \varepsilon, 1) = 0$. □

4 Proof of Theorem 1.1

Throughout this section we assume $d = 2$ and A satisfies (1.2), (1.3) and (1.4). Note that by Theorem 2.1, the matrix A satisfies the invertibility condition (1.6).

Lemma 4.1. *Let $d = 2$ and fix $L \geq 2$. There exist $\varepsilon_0, \delta_0 \in (0, 1/4)$, depending on L, such that if $0 < \varepsilon < \varepsilon_0 r$, $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $B(x_0, r)$, u_ε is not constant,*

$$N^*(u_\varepsilon, x_0, r) \leq 2(\ell + \delta_0), \quad N^*(u_\varepsilon, x_0, r/2) \leq \ell + \delta_0, \tag{4.1}$$

and u_ε has a critical point in $B(x_0, 3r/4) \setminus B(x_0, r/128)$, where $\ell \in \mathbb{N}$ and $2 \leq \ell \leq L$, then

$$N^*(u_\varepsilon, x_0, r/4) \leq \ell - \delta_0. \tag{4.2}$$

Proof. By translation and dilation it suffices to consider the case $x_0 = 0$ and $r = 1$. To prove (4.2), we argue by contradiction. Suppose there exist sequences $\{\varepsilon_j\} \subset \mathbb{R}_+$ and $\{u_j\} \subset H^1(B(0,1))$ such that $\varepsilon_j \to 0$, $\text{div}(A^j(x/\varepsilon_j)\nabla u_j) = 0$ in $B(0,1)$ for some A^j satisfying (1.2), (1.3) and (1.4), u_j is not constant, $\nabla u_j(y_j) = 0$ for some $y_j \in B(0,3/4) \setminus B(0,1/128)$,

$$N^*(u_j,0,1) \le 2(\ell + (1/j)), \quad N^*(u_j,0,1/2) \le \ell + (1/j),$$

and that

$$N^*(u_j,0,1/4) > \ell - (1/j).$$

We may assume that $u_j(0) = 0$ and

$$\fint_{\partial B(0,1/2)} u_j^2 = 1.$$

Since $N^*(u_j,0,1) \le 2\ell + 2$, this implies that $\{u_j\}$ is bounded in $L^2(\partial B(0,1))$. It follows that $\{u_j\}$ is bounded in $L^2(B(0,1))$. Thus, in view of Theorem 2.3, by passing to a subsequence, we may assume that $u_j \to u_0$ weakly in $L^2(B(0,1))$ and strongly in $L^2(B(0,r))$ for any $0 < r < 1$, where u_0 is harmonic in $B(0,1)$. Moreover,

$$\|u_j - u_0\|_{L^\infty(B(0,3/4))} \to 0, \tag{4.3}$$

and

$$\|\nabla u_j - (I + \nabla\chi^j(x/\varepsilon_j))\nabla u_0\|_{L^\infty(B(0,3/4))} \to 0, \tag{4.4}$$

where χ^j denotes the first-order correctors for the matrix A^j.

Next, by letting $j \to \infty$, we obtain $u_0(0) = 0$ and

$$1 = \fint_{\partial B(0,1/2)} u_0^2.$$

Hence, u_0 is not constant. Moreover,

$$N^*(u_0,0,1/2) \le \ell \quad \text{and} \quad N^*(u_0,0,1/4) \ge \ell.$$

By the monotonicity of $N^*(u_0,0,r)$ for harmonic functions, we obtain

$$N^*(u_0,0,1/2) = N^*(u_0,0,1/4) = \ell.$$

It follows that u_0 is a homogeneous harmonic polynomial of degree ℓ. Since $d = 2$, this implies that $|\nabla u_0(x)| \ne 0$ for any $x \ne 0$. However, since $|\nabla u_j(y_j)| = 0$ and

$$\det(I + \nabla\chi^j(y_j/\varepsilon_j)) \ge \mu > 0,$$

in view of (4.4), we conclude that $|\nabla u_0(y_j)| \to 0$ as $j \to \infty$. Since $1 \ge |y_j| \ge (1/128)$, we obtain a contradiction. □

Lemma 4.2. *Fix $L \ge 2$. There exist $\varepsilon_0, \delta_0, \theta \in (0,1/4)$, depending on L, such that if $\varepsilon_0^{-1}\varepsilon \le r \le 1$, $\ell \in \mathbb{N}$ and $2 \le \ell \le L$,*

$$\mathcal{E}(\ell + \delta_0, \varepsilon, r) \le \max\left\{\mathcal{E}(\ell + \delta_0, \varepsilon, r/2), C_0\mathcal{E}(\ell - 1 + \delta_0, \varepsilon, \theta r)\right\}, \tag{4.5}$$

where C_0 depends on L.

Proof. By Lemma 3.4 we may assume $r = 1$. Let $u \in \mathcal{F}(\ell + \delta_0, \varepsilon, 1, 0)$, where $\delta_0 \in (0, 1/4)$ is given by Lemma 4.1. Consider the cover

$$\{B(x, 1/40) : x \in \mathcal{C}(u) \cap B(0, 1/4)\}.$$

Let $\{B(y_j, 1/40) : j = 1, 2, \ldots, k_0\}$ be a Vitali subcover; i.e., $y_j \in \mathcal{C}(u) \cap B(0, 1/4)$,

$$\mathcal{C}(u) \cap B(0, 1/4) \subset \bigcup_{j=1}^{k_0} B(y_j, 1/8),$$

and $B(y_i, 1/40) \cap B(y_j, 1/40) = \emptyset$ for $i \neq j$. We have two cases: $k_0 = 1$ and $k_0 \geq 2$. Note that $1 \leq k_0 \leq C_0$ for some absolute constant C_0.

If $k_0 = 1$, then

$$\mathcal{C}(u) \cap B(0, 1/4) \subset \mathcal{C}(u) \cap B(y_1, 1/8).$$

Since $u \in \mathcal{F}(\ell + \delta_0, \varepsilon, 1, 0)$ and $B(y_1, 1/4) \subset B(0, 1/2)$, we have $u \in \mathcal{F}(\ell + \delta_0, \varepsilon, y_1, 1/2)$. It follows that

$$\#(\mathcal{C}(u) \cap B(0, 1/4)) \leq \mathcal{E}(\ell + \delta_0, \varepsilon, 1/2). \tag{4.6}$$

Suppose $k_0 \geq 2$. Then $u \in \mathcal{F}(\ell - \delta_0, \varepsilon, y_j, 1/2)$ for $1 \leq j \leq k_0$. Indeed, let $x \in B(y_j, 1/4)$. If $|x - y_j| \geq (1/128)$, then $y_j \in B(x, 1/2) \setminus B(x, 1/128)$. On the other hand, if $|x - y_j| < (1/128)$ and $i \neq j$, then

$$\begin{aligned} |y_i - x| &\geq |y_i - y_j| - |y_j - x| \\ &\geq (1/20) - (1/128) \geq (1/128), \end{aligned}$$

and

$$\begin{aligned} |y_i - x| &\leq |y_i - y_j| + |y_j - x| \\ &< (1/2) + (1/128) < (3/4). \end{aligned}$$

Hence, $y_i \in B(x, 3/4) \setminus B(x, 1/128)$ for $i \neq j$. In both cases, by Lemma 4.1, we obtain

$$N^*(u, x, 1/2) \leq \ell + \delta_0 \leq 2(\ell - \delta_0) \quad \text{and} \quad N^*(u, x, 1/4) \leq \ell - \delta_0,$$

for any $x \in B(y_j, 1/4)$, provided that $0 < \varepsilon < \varepsilon_0$. As a result, $u \in \mathcal{F}(\ell - \delta_0, \varepsilon, y_j, 1/2)$ for $1 \leq j \leq k_0$. It follows that

$$\begin{aligned} \#(\mathcal{C}(u) \cap B(0, 1/4)) &\leq \sum_{j=1}^{k_0} \#(\mathcal{C}(u) \cap B(y_j, 1/8)) \\ &\leq k_0 \mathcal{E}(\ell - \delta_0, \varepsilon, 1/2) \\ &\leq C_0 \mathcal{E}(\ell - 1 + \delta_0, \varepsilon, c_0/2), \end{aligned} \tag{4.7}$$

where $c_0 = \delta_0/(8\ell)$ and we have used Theorem 3.6 for the last inequality. By (3.13) we may replace $c_0/2$ in (4.7) by $\theta = \delta_0/(32L)$. $\square$

We are now in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. It suffices to consider the case $0 < \varepsilon < \varepsilon_0$, where $\varepsilon_0 = \varepsilon_0(N) > 0$ is sufficiently small. The case $\varepsilon \ge \varepsilon_0$ is covered by [11, 21].

Let $u_\varepsilon \in H^1(E_2)$ be a non-constant solution of $\text{div}(A(x/\varepsilon)\nabla u_\varepsilon) = 0$ in E_2, where A satisfies the conditions (1.2), (1.3) and (1.4). Suppose that $u_\varepsilon(0) = 0$ and the doubling condition (1.8) holds for some $N \ge 1$. Since $d = 2$, by Theorem 2.1, the invertibility condition (1.6) is satisfied. By a change of variables we may assume $\widehat{A} + (\widehat{A})^T = 2I$. As a result, $E_r = B(0,r)$ and u_ε satisfies the condition

$$\fint_{B(0,2)} u_\varepsilon^2 \le 4^N \fint_{B(0,1)} u_\varepsilon^2. \tag{4.8}$$

By the doubling inequality for u_ε in [17, Theorem 1.2], this gives

$$\fint_{B(x,r)} u_\varepsilon^2 \le C(N) \fint_{B(x,r/2)} u_\varepsilon^2, \tag{4.9}$$

for any $x \in B(0,3/4)$ and $0 < r < 1$. Hence, for $x \in B(0,1/2)$ and $1/2 \le r \le 1$,

$$\begin{aligned}\fint_{\partial B(x,r)} u_\varepsilon^2 &\le C \fint_{B(x,5r/4)} u_\varepsilon^2 \\ &\le C \fint_{B(x,r/2)} u_\varepsilon^2 \le C \fint_{\partial B(x,r/2)} u_\varepsilon^2,\end{aligned}$$

where C depends on N. Consequently, $N^*(u_\varepsilon, x, 1) \le L$ and $N^*(u_\varepsilon, x, 1/2) \le L$ for any $x \in B(0,1/2)$, where L depends on N. This shows that $u_\varepsilon \in \mathcal{F}(L, \varepsilon, 1, 0)$ for some integer $L \ge 2$.

Let $\varepsilon_0, \delta_0, \theta_0 \in (0, 1/4)$ be given by Lemma 4.2. We assume that ε_0 is so small that Theorem 3.7 holds. We will show that for any $\ell \in \mathbb{N}$ and $1 \le \ell \le L$,

$$\mathcal{E}(\ell + \delta_0, \varepsilon, r) \le C, \tag{4.10}$$

where $0 < r \le 1$ and C depends on L. This yields

$$\#(\mathcal{C}(u_\varepsilon) \cap B(0,1/4)) \le C(N). \tag{4.11}$$

By a simple covering argument we replace $B(0,1/4)$ in (4.11) by $B(0,1/2)$.

To prove the estimate (4.10), we use an induction argument on ℓ. To this end, we first note that (4.10) holds for $\ell = 1$. Indeed, if $0 < \varepsilon < \varepsilon_0 r$,

$$\mathcal{E}(1 + \delta_0, \varepsilon, r) \le \mathcal{E}(3/2, \varepsilon, r) = 0,$$

by Theorem 3.7. If $\varepsilon \ge \varepsilon_0 r$, we may use Theorem 3.5 to obtain

$$\mathcal{E}(1 + \delta_0, \varepsilon, r) \le C(L, \varepsilon_0).$$

Next, suppose (4.10) holds for some $\ell < L$. If $0 < \varepsilon < \varepsilon_0 r$, we use Lemma 4.2 to obtain

$$\mathcal{E}(\ell + 1 + \delta_0, \varepsilon, r) \le \max\big\{\mathcal{E}(\ell + 1 + \delta_0, \varepsilon, r/2), C\big\}, \tag{4.12}$$

where C depends on L. By Theorem 3.5, the estimate above also holds for $\varepsilon \geq \varepsilon_0 r$. By an induction argument on j, this implies that

$$\mathcal{E}(\ell+1+\delta_0, \varepsilon, r) \leq \max\left\{\mathcal{E}(\ell+1+\delta_0, \varepsilon, 2^{-j}r), C\right\} \tag{4.13}$$

for any $j \geq 1$. Finally, we choose j so large that $2^{-j}r \leq \varepsilon_0^{-1}\varepsilon$. By Theorem 3.5 we obtain

$$\mathcal{E}(\ell+1+\delta_0, \varepsilon, r) \leq C,$$

which completes the proof of (4.10).

□

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