Meta-Diagrams for 2-Parameter Persistence

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─── **Abstract** ───

We first introduce the notion of meta-rank for a 2-parameter persistence module, an invariant that captures the information behind images of morphisms between 1D slices of the module. We then define the meta-diagram of a 2-parameter persistence module to be the Möbius inversion of the meta-rank, resulting in a function that takes values from signed 1-parameter persistence modules. We show that the meta-rank and meta-diagram contain information equivalent to the rank invariant and the signed barcode. This equivalence leads to computational benefits, as we introduce an algorithm for computing the meta-rank and meta-diagram of a 2-parameter module M indexed by a bifiltration of n simplices in $O(n^3)$ time. This implies an improvement upon the existing algorithm for computing the signed barcode, which has $O(n^4)$ time complexity. This also allows us to improve the existing upper bound on the number of rectangles in the rank decomposition of M from $O(n^4)$ to $O(n^3)$. In addition, we define notions of erosion distance between meta-ranks and between meta-diagrams, and show that under these distances, meta-ranks and meta-diagrams are stable with respect to the interleaving distance. Lastly, the meta-diagram can be visualized in an intuitive fashion as a persistence diagram of diagrams, which generalizes the well-understood persistence diagram in the 1-parameter setting.

1 Introduction

In the case of a 1-parameter persistence module, the persistence diagram (or barcode) captures its complete information up to isomorphism via a collection of intervals. The persistence diagram is represented as a multi-set of points in the plane, whose coordinates are the birth and death times of intervals, each of which encodes the lifetime of a topological feature. This compact representation of a persistence module enables its interpretability and facilitates
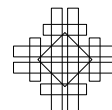
its visualization. When moving to the multiparameter setting, the situation becomes much more complex as a multiparameter persistence module may contain indecomposable pieces that are not entirely determined by intervals or do not admit a finite discrete description [9].

Such an increased complexity has led to the study of other invariants for multiparameter persistence modules. The first invariant is the *rank invariant* [9], which captures the information from the images of internal linear maps in a persistence module across all dimensions. Patel noticed that the persistence diagram in the 1-parameter setting is equivalent to the *Möbius inversion* [25] of the rank function [24]. He then defined the generalized persistence diagram as the Möbius inversion of a function defined on a subset of intervals of $\mathbb{R}$, denoted Dgm, with values in some abelian group.

The idea of Möbius inversion has been extended in many directions. Kim and Mémoli defined generalized persistence diagrams for modules on posets [12, 17]. Patel and McCleary extended Patel's generalized persistence diagrams to work for persistence modules indexed over finite lattices [22]. Botnan et al. [6] implicitly studied the Möbius inversion of the rank function for 2-parameter modules, leading to a notion of a diagram with domain all rectangles in $\mathbb{Z}^2$. Asashiba et al. used Möbius inversion on a finite 2D grid to define interval-decomposable approximations [1]. Morozov and Patel [23] defined a generalized persistence diagram in the 2-parameter setting via Möbius inversion of the birth-death function and provided an algorithm for its computation. Their algorithm has some similarity with ours: it utilizes the vineyards algorithm [13] to study a 2-parameter persistence module, by slicing it over 1D paths.

Our work also involves the idea of slicing a 2-parameter module. This idea of slicing appears in the fibered barcode [10, 20], which is equivalent to the rank function. To obtain insight into the structure of a 2-parameter persistence module M, Lesnick and Wright [20] explored a set of 1-parameter modules obtained via restricting M onto all possible lines of non-negative slope. Buchet and Escolar [8] showed that any 1-parameter persistence module with finite support could be found as a restriction of some indecomposable 2-parameter persistence module with finite support. Furthermore, Dey et al. [15] showed that certain zigzag (sub)modules of a 2-parameter module can be used to compute the generalized rank invariant, whose Möbius inversion is the generalized persistence diagram defined by Kim and Mémoli. Our work considers the images between slices of a 2-parameter module, which is related to the work by Bauer and Schmal [3].

In [7], Botnan et al. introduced the notion of *rank decomposition*, which is equivalent to the generalized persistence diagram formed by Möbius inversion of the rank function, under some additional conditions. Botnan et al. further demonstrated that the process of converting a module to a rank decomposition is stable with respect to the matching distance [18]. Additionally, they introduced a visualization of this rank decomposition via a *signed barcode*, which highlights the diagonals of rectangles appearing in the rank decomposition, along with their multiplicity. They visualized the value of the signed barcode with a 2-parameter persistence module generated by clustering a point cloud with a scale and a density parameter.

Unlike the previous results that perform Möbius inversion over a higher-dimensional poset such as $\mathbb{Z}^2$, our work involves Möbius inversion over a finite subcollection of intervals of $\mathbb{R}$, which leads to a simpler inversion formula. In this work, we introduce the notion of *meta-rank* for a 2-parameter persistence module, which is a map from Dgm to isomorphism classes of persistence modules. Instead of looking at images of linear maps between vector spaces (as with the usual rank invariant), the meta-rank considers images of the maps between 1-parameter persistence modules formed by slicing a 2-parameter persistence module

$$M_x^a \xrightarrow{\phi_x(a \leq b)} M_x^b \xrightarrow{\phi_x(b \leq c)} M_x^c$$
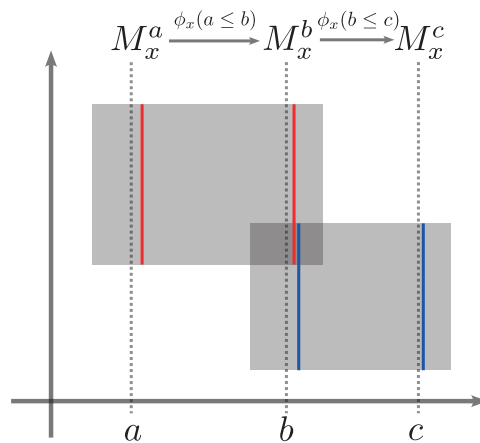
Figure 1 Slicing a 2-parameter module M along vertical lines yields 1-parameter modules, such as M_x^a, M_x^b, and M_x^c. There are morphisms between these 1-parameter modules induced by the internal morphisms of M, and the meta-rank captures the information about these morphisms. For example, if M is defined as the direct sum of the two interval modules given by the two shaded rectangles, then the meta-rank of M on $[a, b)$ is the image of $\phi_x(a \leq b)$, which has a barcode consisting of the red interval. The meta-rank of M on $[b, c)$ has a barcode consisting of the blue interval, and the meta-rank of M on $[a, c)$ is 0, as $\phi_x(a \leq c) = \phi_x(b \leq c) \circ \phi_x(a \leq b) = 0$.
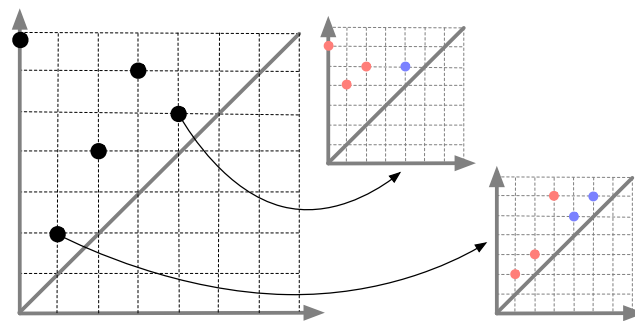
Figure 2 A meta-diagram viewed as a persistence diagram of signed diagrams (red and blue mean positive and negative signs respectively).

along vertical and horizontal lines, see Figure 1. We then define the meta-diagram as the Möbius inversion of the meta-rank, giving a map from Dgm to isomorphism classes of signed persistence modules. This contrasts Botnan et al.'s approach [7] of using Möbius inversion in 2D, as our Möbius inversion formula over Dgm is simpler and consists of fewer terms.

Contributions. The meta-rank and meta-diagram turn out to contain information equivalent to the rank invariant (Proposition 12) and signed barcode (Proposition 27) respectively. Therefore, both meta-rank and meta-diagram can be regarded as these known invariants seen from a different perspective. However, this different viewpoint brings forth several advantages as listed below that make the meta-rank and meta-diagram stand out on their own right:

1. The meta-rank and meta-diagram of a 2-parameter persistence module M induced by a bifiltration of a simplicial complex with n simplices can be computed in $O(n^3)$ time.
2. This immediately implies an improvement of the $O(n^4)$ algorithm of Botnan et al. [7] for computing the signed barcodes.

3. The $O(n^3)$ time algorithm for computing meta-rank and meta-diagram also implicitly improves the bound on the number of signed bars in the rank decomposition of M to $O(n^3)$ from the current known bound of $O(n^4)$. This addresses an open question whether the size of the signed barcode is bounded tightly by the number of rectangles or not.

4. The meta-diagram can be viewed as a persistence diagram of signed diagrams as illustrated in Figure 2. Such an intuitive visualization generalizes the classic persistence diagram – a known technique in topological data analysis (TDA) – to summarize persistent homology.

5. The meta-diagram also generalizes the concept of a sliced barcode well-known in TDA [20]. It ensembles sliced bars on a set of lines, but not forgetting the maps between slices induced by the module M being sliced.

For omitted proofs and further details, see the full version [11].

2   Preliminaries

We regard a poset $(P, \leq)$ as a category, with objects the elements $p \in P$, and a unique morphism $p \to q$ if and only if $p \leq q$; this is referred to as the *poset category* for $(P, \leq)$. When it is clear from the context, we will denote the poset category by P.

Fix a field **k**, and assume all vector spaces have coefficients in **k** throughout this paper. Let **vec** denote the category of finite-dimensional vector spaces with linear maps between them. A *persistence module*, or *module* for short, is a functor $M : P \to \mathbf{vec}$. For any $p \in P$, we denote the vector space $M_p := M(p)$, and for any $p \leq q \in P$, we denote the linear map $\varphi_M(p \leq q) := M(p \leq q)$. When M is apparent, we drop the subscript from φ_M. We call P the *indexing poset* for M. We focus on the cases when the indexing poset is $\mathbb{R}$ or $\mathbb{R}^2$, equipped with the usual order and product order, respectively. Definitions and statements we make follow analogously when the indexing poset is $\mathbb{Z}$ or $\mathbb{Z}^2$, which we will cover briefly in Section 5. If the indexing poset for M is $P \subseteq \mathbb{R}$, then M is a *1-parameter (or 1D) persistence module*. If the indexing poset for M is $P \subseteq \mathbb{R}^2$, with P not totally-ordered, then M is a *2-parameter (or 2D) persistence module*, or a *bimodule* for short.

Following [21], we require that persistence modules be *constructible*:

▶ **Definition 1.** *A module $M : \mathbb{R} \to \mathbf{vec}$ is* constructible *if there exists a finite set $S := \{s_1 < \ldots < s_n\} \subset \mathbb{R}$ such that:*
- *For $a < s_1$, $M(a) = 0$;*
- *For $s_i \leq a \leq b < s_{i+1}$, $\varphi_M(a \leq b)$ is an isomorphism;*
- *For $s_n \leq a \leq b$, $\varphi_M(a \leq b)$ is an isomorphism.*
Similarly, a bimodule $M : \mathbb{R}^2 \to \mathbf{vec}$ is constructible *if there exists a finite set $S := \{s_1 < \ldots < s_n\} \subset \mathbb{R}$ such that:*
- *If $x < s_1$ or $y < s_1$, then $M((x, y)) = 0$,*
- *For $s_i \leq x_1 \leq x_2 < s_{i+1}$ and $s_j \leq y_1 \leq y_2 < s_{j+1}$, $\varphi_M((x_1, y_1) \leq (x_2, y_2))$ is an isomorphism,*
- *If $x_1 \geq s_n$ or $y_1 \geq s_n$ and $(x_1, y_1) \leq (x_2, y_2)$, then $\varphi_M((x_1, y_1) \leq (x_2, y_2))$ is an isomorphism.*
In either case, such a module is S-constructible.

If a module is S-constructible, unless otherwise stated, assume $S = \{s_1 < \ldots < s_n\}$. If M is S-constructible, then M is S'-constructible for any $S' \supseteq S$. For the rest of the paper, we assume any given persistence module is constructible.

Of particular importance in the study of 1- and 2-parameter persistence modules are the notions of interval modules and interval decomposable modules. We state the definitions:

▶ **Definition 2.** *For a poset $(P, \leq)$, an* interval *of P is a non-empty subset $I \subset P$ such that:*
- *(convexity) If $p, r \in I$ and $q \in P$ with $p \leq q \leq r$, then $q \in I$.*
- *(connectivity) For any $p, q \in I$, there is a sequence $p = r_0, r_1, \ldots, r_n = q$ of elements of I, where for all $0 \leq i \leq n - 1$, either $r_i \geq r_{i+1}$ or $r_i \leq r_{i+1}$.*

We denote the collection of all intervals of P as $\mathbf{Int}(P)$.

For $I \in \mathbf{Int}(P)$, the *interval module* $\mathbf{k}^I$ is the persistence module indexed over P, with:

$$\mathbf{k}_p^I = \begin{cases} \mathbf{k} & \text{if } p \in I \\ 0 & \text{otherwise} \end{cases}, \qquad \varphi_{\mathbf{k}^I}(p \leq q) = \begin{cases} \mathrm{id}_{\mathbf{k}} & \text{if } p \leq q \in I \\ 0 & \text{otherwise} \end{cases}.$$

Given any $M, N : P \to \mathbf{vec}$, the direct sum $M \oplus N$ is defined point-wise at each $p \in P$. We say a nontrivial $M : P \to \mathbf{vec}$ is *decomposable* if M is isomorphic to $N_1 \oplus N_2$ for some non-trivial $N_1, N_2 : P \to \mathbf{vec}$, which we denote by $M \cong N_1 \oplus N_2$. Otherwise, M is *indecomposable*. Interval modules are indecomposable [5].

A persistence module $M : P \to \mathbf{vec}$ is *interval decomposable* if it is isomorphic to a direct sum of interval modules. That is, if there is a multiset of intervals $\mathrm{barc}(M)$, such that:

$$M \cong \bigoplus_{I \in \mathrm{barc}(M)} \mathbf{k}^I$$

If this multiset exists, we call it the *barcode* of M. If it exists, $\mathrm{barc}(M)$ is well-defined as a result of the Azumaya-Krull-Remak-Schmidt theorem [2]. Thus, in the case where M is interval decomposable, $\mathrm{barc}(M)$ is a complete descriptor of the isomorphism type of M.

Of particular importance in this work are *right-open rectangles*, which are intervals $R \subset \mathbb{R}^2$ of the form $R = [a_1, b_1) \times [a_2, b_2)$. If M can be decomposed as a direct sum of interval modules $\mathbf{k}^R$ with R a right-open rectangle, then we say M is *rectangle decomposable*.

1-parameter persistence modules are particularly nice, as they are always interval decomposable [14]. As a result, the barcode is a complete invariant for 1-parameter persistence modules. On the other hand, bimodules do not necessarily decompose in this way. In fact, there is no complete discrete descriptor for bimodules [9].

A number of invariants have been proposed to study bimodules. One of the first and the most notable invariant is the *rank invariant* [9] recalled in Definition 3.

▶ **Definition 3** ([9]). *For P a poset, define $\mathbf{D}(P) := \{(a, b) \in P \times P \,|\, a \leq b\}$. For $M : P \to \mathbf{vec}$, the* rank invariant *of M, $\mathrm{rank}_M : \mathbf{D}(P) \to \mathbb{Z}_{\geq 0}$, is defined point-wisely as:*

$$\mathrm{rank}_M(a, b) := \mathrm{rank}(\varphi_M(a \leq b))$$

For a bimodule, the rank invariant is inherently a 4D object, making it difficult to visualize directly. RIVET [20] visualizes the rank invariant indirectly through the fibered barcode. In [7], Botnan et al. defined the *signed barcode* based on the notion of a *rank decomposition*:

▶ **Definition 4** ([7]). *Let $M : \mathbb{R}^n \to \mathbf{vec}$ be a persistence module with rank function rank_M. Suppose $\mathscr{R}, \mathscr{S}$ are multisets of intervals from $\mathbb{R}^n$. Define $\mathbf{k}_{\mathscr{R}} := \oplus_{I \in \mathscr{R}} \mathbf{k}^R$, and similarly $\mathbf{k}_{\mathscr{S}}$. Then $(\mathscr{R}, \mathscr{S})$ is a* rank decomposition *for rank_M if as integral functions:*

$$\mathrm{rank}_M = \mathrm{rank}_{\mathscr{R}} - \mathrm{rank}_{\mathscr{S}}.$$

If $\mathscr{R}, \mathscr{S}$ consist of right-open rectangles, then the pair is a rank decomposition by rectangles. We have:

▶ **Theorem 5** ([7], Theorem 3.3). *Every finitely presented $M : \mathbb{R}^2 \to \mathbf{vec}$ admits a unique minimal rank decomposition by rectangles.*

Here minimality comes in the sense that $\mathscr{R} \cap \mathscr{S} = \emptyset$. The signed barcode then visualizes the rank function in $\mathbb{R}^2$ by showing the diagonals of the rectangles in $\mathscr{R}$ and $\mathscr{S}$.
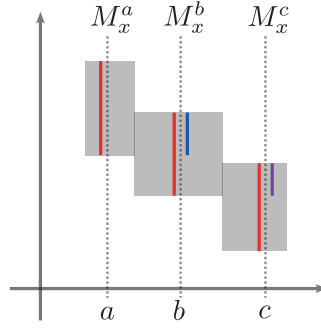
Figure 3 An illustration of M and its barcode for some values of $\mathbf{mrk}_M$ in Example 9.

3   Meta-Rank

In this section, we introduce the *meta-rank*. While the rank invariant captures the information of images between pairs of vector spaces in a persistence module, the meta-rank captures the information of images between two 1-parameter persistence modules obtained via slicing a bimodule. We begin with some preliminary definitions:

▶ **Definition 6.** *Let $M : \mathbb{R}^2 \to \mathbf{vec}$ be a bimodule. For $s \in \mathbb{R}$, define the vertical slice $M_x^s : \mathbb{R} \to \mathbf{vec}$ point-wise as $M_x^s(a) := M(s,a)$, and with morphisms from a to b as $\varphi_x^s(a \leq b) := \varphi((s,a) \leq (s,b))$. Analogously, define the horizontal slice $M_y^s : \mathbb{R} \to \mathbf{vec}$ by setting $M_y^s(a) := M(a,s)$ and $\varphi_y^s(a \leq b) := \varphi((a,s) \leq (b,s))$ for all $a \leq b \in \mathbb{R}$.*

Define a morphism of 1-parameter persistence modules $\phi_x(s \leq t) : M_x^s \to M_x^t$ for $s \leq t \in \mathbb{R}$ by $\phi_x(s \leq t)(a) := \varphi((s,a) \leq (t,a))$. Analogously, define $\phi_y(s \leq t) : M_y^s \to M_y^t$ for $s \leq t \in \mathbb{R}$ by $\phi_y(s \leq t)(a) := \varphi((a,s) \leq (a,t))$.

Denote by $\mathbf{Pvec}$ the isomorphism classes of persistence modules over $\mathbb{R}$. Each element of $\mathbf{Pvec}$ can be uniquely represented by its barcode, which is what we do in practice. We recall the definition of Dgm from [24], which serves as the domain for the meta-rank:

▶ **Definition 7** ([24]). *Define* Dgm *as the poset of all half-open intervals $[p,q) \subset \mathbb{R}$ for $p < q$, and all half-infinite intervals $[p, \infty) \subset \mathbb{R}$. The poset relation is inclusion.*

▶ **Definition 8.** *Suppose $M : \mathbb{R}^2 \to \mathbf{vec}$ is S-constructible. Define the* horizontal meta-rank $\mathbf{mrk}_{M,x} : \mathrm{Dgm} \to \mathbf{Pvec}$ *as follows:*
- *For $I = [s, s_i)$ with $s_i \in S$, $\mathbf{mrk}_{M,x}(I) := [\mathrm{im}(\phi_x(s \leq s_i - \delta))]$, for some $\delta > 0$ such that $s_i - \delta \geq s$ and $s_i - \delta \geq s_{i-1}$.*
- *For $I = [s, \infty)$, $\mathbf{mrk}_{M,x}(I) := [\mathrm{im}(\phi_x(s \leq s_n))]$.*
- *For all other $I = [s, t)$, $\mathbf{mrk}_{M,x}(I) := [\mathrm{im}(\phi_x(s \leq t))]$.*

Analogously, define the vertical meta-rank, $\mathbf{mrk}_{M,y} : \mathrm{Dgm} \to \mathbf{Pvec}$ *by replacing each instance of x above with y.*

The results in this paper are stated in terms of the horizontal meta-rank, but hold analogously for the vertical meta-rank. To simplify notation, we henceforth denote $\mathbf{mrk}_{M,x}$ as $\mathbf{mrk}_M$. When there is no confusion, we drop the subscript from $\mathbf{mrk}_M$.

▶ **Example 9.** As illustrated in Figure 3, let I be the connected gray interval and define the bimodule $M := \mathbf{k}^I$. The barcodes for the 1-parameter modules M_x^a, M_x^b, and M_x^c are shown in red next to their corresponding vertical slices. The barcode for $\mathbf{mrk}_M([a,b))$ consists of the blue interval, which is the overlap of the bars in M_x^a and M_x^b, $\mathrm{barc}(M_x^a) \cap \mathrm{barc}(M_x^b)$.
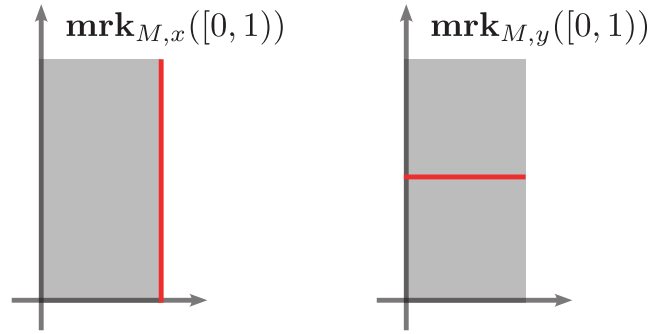
Figure 4 An illustration of M, depicting $\mathbf{mrk}_{M,x}([0,1)) \neq \mathbf{mrk}_{M,y}([0,1))$.

Similarly, $\mathbf{mrk}_M([b,c))$ has a barcode consisting of the purple interval, which is the overlap of the bars in M_x^b and M_x^c. As the bars in the barcodes for M_x^a and M_x^c have no overlap, $\mathrm{im}(\phi_x(a \leq c)) = 0$, therefore $\mathbf{mrk}_M([a,c)) = 0$.

▶ **Remark 10.** In general, $\mathbf{mrk}_x \neq \mathbf{mrk}_y$. For example, consider the right-open rectangle R with the lower-left corner the origin, and the upper right corner $(1,2)$, as in Figure 4. Let $M := \mathbf{k}^R$. As illustrated, $\mathbf{mrk}_{M,x}([0,1)) = [0,2) \neq [0,1) = \mathbf{mrk}_{M,y}([0,1))$.

The following Proposition 11 allows us to compute the meta-rank of a bimodule via the meta-ranks of its indecomposable summands:

▶ **Proposition 11.** *For* $M, N : \mathbb{R}^2 \to \mathbf{vec}$, *we have:*

$$\mathbf{mrk}_M \oplus \mathbf{mrk}_N = \mathbf{mrk}_{M \oplus N}$$

where $\mathbf{mrk}_M \oplus \mathbf{mrk}_N : \mathrm{Dgm} \to \mathbf{Pvec}$ *is defined as:*

$$(\mathbf{mrk}_M \oplus \mathbf{mrk}_N)([s,t)) := [\mathbf{mrk}_M([s,t)) \oplus \mathbf{mrk}_N([s,t))].$$

For a finite $S \subseteq \mathbb{R}$, let $\overline{S} := S \cup \{\infty\}$. Define $\overline{S}_> : \mathbb{R} \cup \{\infty\} \to \overline{S}$ as $\overline{S}_>(t) := \min\{s \in \overline{S} \mid s > t\}$. For $M \in \mathbf{Pvec}$, $[b,d) \in \mathrm{Dgm}$, let $\#[b,d) \in M$ denote the multiplicity of $[b,d) \in \mathrm{barc}(M)$. The rank invariant and the meta-rank contain equivalent information:

▶ **Proposition 12.** *For* $M : \mathbb{R}^2 \to \mathbf{vec}$, *one can compute* rank_M *from* $\mathbf{mrk}_M$ *and one can compute* $\mathbf{mrk}_M$ *from* rank_M. *In particular, given* $(s,y) \leq (t,y') \in \mathbb{R}^2$,

$$\mathrm{rank}_M((s,y),(t,y')) = \#[b_i,d_i) \in \mathbf{mrk}_M([s,\overline{S}_>(t))) \text{ s.t. } b_i \leq y \leq y' < d_i.$$

That is, the rank is the number of intervals in $\mathrm{barc}(\mathbf{mrk}_M([s,\overline{S}_>(t))))$ *containing* $[y,y']$.

The reason for needing $\overline{S}_>(t)$ for the right endpoint is that if $t \in S$, $\mathbf{mrk}_M([s,t))$ does not capture the information of the image of $\phi_x(s \leq t)$, only the image of $\phi_x(s \leq t - \delta)$.

Finally, we discuss the stability of the meta-rank. The meta-rank is stable with respect to a notion of erosion distance, based on that of Patel [24]. We introduce truncated barcode:

▶ **Definition 13.** *For* $\epsilon \geq 0$, *and* $I = [s,t) \in \mathrm{Dgm}$, *define* $I[\epsilon :] := [s+\epsilon, t)$. *For* $M : \mathbb{R} \to \mathbf{vec}$ *define:* $\mathrm{barc}_\epsilon(M) := \{I[\epsilon :] \mid I \in \mathrm{barc}(M)\}$. *If* $I = [s,t) \in \mathrm{barc}(M)$ *has* $t - s \leq \epsilon$, *then* I *has no corresponding interval in* $\mathrm{barc}_\epsilon(M)$.

▶ **Definition 14.** *For* $M, N : \mathbb{R} \to \mathbf{vec}$, *we say* $M \preceq_\epsilon N$ *if there exists an injective function on barcodes* $\iota : \mathrm{barc}_\epsilon(M) \hookrightarrow \mathrm{barc}(N)$ *such that for all* $J \in \mathrm{barc}_\epsilon(M)$, $J \subseteq \iota(J)$.

For $\epsilon \geq 0$, $M \in \mathbf{Pvec}$, let M^ϵ refer to the ϵ-shift of M [19], with $M^\epsilon(a) := M(a + \epsilon)$ and $\varphi_{M^\epsilon}(a \leq b) := \varphi_M(a + \epsilon \leq b + \epsilon)$. For $I = [s, t) \in \mathrm{Dgm}$ and $a, b \in \mathbb{R}$, let $I_a^b := [s + a, t + b)$, with the convention $\infty + b := \infty$ for any $b \in \mathbb{R}$. We now define the erosion distance:

▶ **Definition 15.** *Let* $M, N : \mathbb{R}^2 \to \mathbf{vec}$. *Define the erosion distance as follows:*

$$\mathrm{d_E}(\mathbf{mrk}_M, \mathbf{mrk}_N) := \inf\{\epsilon > 0 \,|\, \forall I \in \mathrm{Dgm}, \mathbf{mrk}_M(I_{-\epsilon}^\epsilon)^\epsilon \preceq_{2\epsilon} \mathbf{mrk}_N(I) \,\mathrm{and}$$
$$\mathbf{mrk}_N(I_{-\epsilon}^\epsilon)^\epsilon \preceq_{2\epsilon} \mathbf{mrk}_M(I)\}$$

if the set we are infimizing over is empty, we set $\mathrm{d_E}(\mathbf{mrk}_M, \mathbf{mrk}_N) := \infty$.

▶ **Proposition 16.** $\mathrm{d_E}$ *as defined in Definition 15 is an extended pseudometric on the collection of meta-ranks of constructible bimodules* $M : \mathbb{R}^2 \to \mathbf{vec}$.

We compare bimodules M and N using the multiparameter interleaving distance [19]. The ϵ-shift and the truncation of the barcode in Definition 13 are necessary for stability, due to the interleaving distance being based on diagonal shifts of bimodules, whereas the meta-rank is based on horizontal maps instead of diagonal ones. We have the following:

▶ **Theorem 17.** *For constructible* $M, N : \mathbb{R}^2 \to \mathbf{vec}$, *we have:*

$$\mathrm{d_E}(\mathbf{mrk}_M, \mathbf{mrk}_N) \leq \mathrm{d_I}(M, N)$$

4 Meta-Diagram

We use the Möbius inversion formula from Patel [24] on the meta-rank function to get a *meta-diagram*. This formula involves negative signs, so we need a notion of signed persistence modules. Our ideas are inspired by the work of Betthauser et al. [4], where we consider breaking a function into positive and negative parts.

▶ **Definition 18.** *A* signed 1-parameter persistence module *is an ordered pair* (M, N), *where* $M, N : \mathbb{Z} \to \mathbf{vec}$ *are 1-parameter persistence modules.* M *is the* positively signed *module, and* N *is the* negatively signed *module.*

▶ **Definition 19.** *View* $\mathbf{Pvec}$ *as a commutative monoid with operation* $\oplus$ *given by* $[M] \oplus [N] :=$ $[M \oplus N]$, *and identity element* $[0]$. *Define* $\mathbf{SPvec}$ *to be the Grothendieck group of* $\mathbf{Pvec}$.

Each element of $\mathbf{SPvec}$ is an isomorphism class of ordered pairs $[([M^+], [M^-])]$. From the completeness of barcodes for 1-parameter persistence modules, we assume without loss of generality that each element M^+, M^- is given by $* := \oplus_{I \in \mathrm{barc}(*)} \mathbf{k}^I$ and drop the internal equivalence class notation to write an element of $\mathbf{SPvec}$ as $[(M^+, M^-)]$. Proposition 20 allows us to make a canonical choice of representative for each element of $\mathbf{SPvec}$:

▶ **Proposition 20.** *Let* $A \in \mathbf{SPvec}$. *Then there is a unique representative* $A = [(M^+, M^-)]$ *with* $\mathrm{barc}(M^+) \cap \mathrm{barc}(M^-) = \emptyset$.

As a result of Proposition 20, when convenient, we represent an element of $\mathbf{SPvec}$ uniquely by the sum of barcodes of this special representative, as in the following example:

▶ **Example 21.** Consider $[(N^+, N^-)] \in \mathbf{SPvec}$ where $\mathrm{barc}(N^+) = \{[0, 4], [1, 3], [2, 4]\}$ and $\mathrm{barc}(N^-) = \{[1, 3], [3, 4]\}$. By Proposition 20, $[(N^+, N^-)]$ is uniquely represented by $[(M^+, M^-)]$ with $\mathrm{barc}(M^+) = \{[0, 4], [2, 4]\}$ and $\mathrm{barc}(M^-) = \{[3, 4]\}$. In practice, we will denote this element of $\mathbf{SPvec}$ as $[0, 4] + [2, 4] - [3, 4] \in \mathbf{SPvec}$. If $M, N \in \mathbf{Pvec}$, denote by $M + N$ the element $[(M \oplus N, 0)] \in \mathbf{SPvec}$, and denote by $M - N$ the element $[(M, N)] \in \mathbf{SPvec}$. For an illustration, see Figure 5.
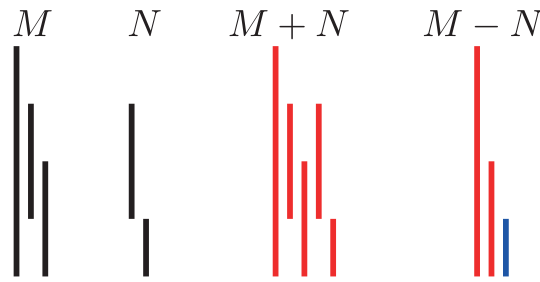
Figure 5 Illustration of the barcodes for $M, N \in \mathbf{Pvec}$ and $M + N, M - N \in \mathbf{SPvec}$. For $M + N$ and $M - N$, a red interval is positively signed and a blue interval is negatively signed.

With this notion of signed persistence module in hand, we now use a modified version of the Möbius inversion formula from [24] to define a meta-diagram:

▶ **Definition 22.** *Let* $M : \mathbb{R}^2 \to \mathbf{vec}$ *be S-constructible. Define the* horizontal meta-diagram *to be the function* $\mathbf{mdgm}_M : \mathrm{Dgm} \to \mathbf{SPvec}$ *via the Möbius inversion formula:*

$$\mathbf{mdgm}_{M,x}([s_i, s_j)) := \mathbf{mrk}_{M,x}([s_i, s_j)) - \mathbf{mrk}_{M,x}([s_i, s_{j+1}))$$
$$+ \mathbf{mrk}_{M,x}([s_{i-1}, s_{j+1})) - \mathbf{mrk}_{M,x}([s_{i-1}, s_j))$$
$$\mathbf{mdgm}_{M,x}([s_i, \infty)) := \mathbf{mrk}_{M,x}([s_i, \infty)) - \mathbf{mrk}_{M,x}([s_{i-1}, \infty))$$

where s_0 *is any value* $s_0 < s_1$ *and* s_{n+1} *is any value* $s_{n+1} > s_n$*. For any other* $[s, t) \in \mathrm{Dgm}$*, set* $\mathbf{mdgm}_{M,x}([s, t)) := 0$*. Define the* vertical meta-diagram *by replacing each instance of x above with y.*

We henceforth let $\mathbf{mdgm}$ refer to the horizontal meta-diagram of M, dropping the subscript when there is no confusion. The following Möbius inversion formula describes the relation between the meta-rank and meta-diagram. It is the direct analogue of [24, Theorem 4.1].

▶ **Proposition 23.** *For* $[s, t) \in \mathrm{Dgm}$*, we have:*

$$\mathbf{mrk}([s, t)) = \sum_{\substack{I \in \mathrm{Dgm} \\ I \supseteq [s, t)}} \mathbf{mdgm}(I)$$

▶ **Proposition 24.** *For* $M, N : \mathbb{R}^2 \to \mathbf{vec}$*, we have:*

$$\mathbf{mdgm}_M \oplus \mathbf{mdgm}_N = \mathbf{mdgm}_{M \oplus N},$$

where $\mathbf{mdgm}_M \oplus \mathbf{mdgm}_N : \mathrm{Dgm} \to \mathbf{SPvec}$ *is defined by*

$$(\mathbf{mdgm}_M \oplus \mathbf{mdgm}_N)([s, t)) := [\mathbf{mdgm}_M([s, t))^+ \oplus \mathbf{mdgm}_N([s, t))^+,$$
$$\mathbf{mdgm}_M([s, t))^- \oplus \mathbf{mdgm}_N([s, t))^-].$$

Proposition 24 allows us to compute meta-diagrams straightforwardly if we have an indecomposable decomposition of a module. In particular, by Proposition 25, meta-diagrams are simply computable for rectangle decomposable modules.

▶ **Proposition 25.** *Suppose* $M = \mathbf{k}^R$ *is an* $\mathbb{R}^2$*-indexed interval module supported on the right-open rectangle R, with lower-left corner* (s, t) *and upper-right corner* (s', t')*. We have:*

$$\mathbf{mdgm}_M([a, b)) = \begin{cases} [t, t') & \text{if } a = s \text{ and } b = s'; \\ 0 & \text{otherwise.} \end{cases}$$

▶ **Corollary 26.** *Let $M = \oplus_{R \in \mathrm{barc}(M)} \mathbf{k}^R$ be rectangle decomposable. Then the interval $[t, t']$ appears in $\mathbf{mdgm}([s, s'))$ with multiplicity n if and only if the right-open rectangle with lower-left corner (s, t) and upper right corner (s', t') appears in $\mathrm{barc}(M)$ with multiplicity n.*

4.1 Equivalence With Rank Decomposition via Rectangles

For $M : \mathbb{R}^2 \to \mathbf{vec}$, the rank decomposition by rectangles contains the same information as the rank invariant, which by Proposition 12 contains the same information as the meta-rank. We now show one can directly go from the meta-diagram to the rank decomposition:

▶ **Proposition 27.** *Let $M : \mathbb{R}^2 \to \mathbf{vec}$ be constructible. Define as follows:*

$$\mathscr{R} := \bigcup_{I \in \mathrm{Dgm}} \left(\bigcup_{[a,b] \in \mathbf{mdgm}_M(I)} I \times [a, b) \right),$$

$$\mathscr{S} := \bigcup_{I \in \mathrm{Dgm}} \left(\bigcup_{-[a,b] \in \mathbf{mdgm}_M(I)} I \times [a, b) \right),$$

where all unions are the multiset union. Then $(\mathscr{R}, \mathscr{S})$ is a rank decomposition for M.

Proof. It suffices to show that for all $w_1 := (x_1, y_1) \leq w_2 := (x_2, y_2) \in \mathbb{R}^2$, $\mathrm{rank}_M(w_1, w_2) = \mathrm{rank}_{\mathbf{k}_{\mathscr{R}}}(w_1, w_2) - \mathrm{rank}_{\mathbf{k}_{\mathscr{S}}}(w_1, w_2)$. Suppose $w_1 \leq w_2 \in \mathbb{R}^2$ as above. By Proposition 12,

$$\mathrm{rank}_M(w_1, w_2) = \#[b_i, d_i) \in \mathbf{mrk}_M([x_1, x_2']) \quad \text{s.t.} \quad b_i \leq y_1 \leq y_2 < d_i,$$

where for notational simplicity, $x_2' := \overline{S}_{>}(x_2)$.

Now fix $[b, d)$ such that $b \leq y_1 \leq y_2 < d$. By Proposition 23, we have:

$$\#[b, d) \in \mathbf{mrk}_M([x_1, x_2']) = \#[b, d) \in \sum_{\substack{I \in \mathrm{Dgm} \\ I \supseteq [x_1, x_2']}} \mathbf{mdgm}_M(I)$$

$$= \left(\#[b, d) \in \sum_{\substack{I \in \mathrm{Dgm} \\ I \supseteq [x_1, x_2']}} \mathbf{mdgm}_M^+(I) \right) - \left(\#[b, d) \in \sum_{\substack{I \in \mathrm{Dgm} \\ I \supseteq [x_1, x_2']}} \mathbf{mdgm}_M^-(I) \right)$$

By Proposition 25 and Corollary 26, the term $\#[b, d) \in \sum_{\substack{I \in \mathrm{Dgm} \\ I \supseteq [x_1, x_2']}} \mathbf{mdgm}^+(I)$ is the number of times $I \times [b, d)$ appears in $\mathscr{R}$ across all $I \supseteq [x_1, x_2']$, and the term $\#[b, d) \in \sum_{\substack{I \in \mathrm{Dgm} \\ I \supseteq [x_1, x_2']}} \mathbf{mdgm}^-(I)$ is the number of times $I \times [b, d)$ appears in $\mathscr{S}$ across all $I \supseteq [x_1, x_2']$.

Thus, we see that $\mathrm{rank}_M(w_1, w_2)$ is equal to the number of rectangles in $\mathscr{R}$ containing w_1 and w_2 minus the number of rectangles in $\mathscr{S}$ containing w_1 and w_2. From the definition of rectangle module and the fact that rank commutes with direct sums, the first term is $\mathrm{rank}(\mathbf{k}_{\mathscr{R}})(w_1, w_2)$ and the second term is $\mathrm{rank}(\mathbf{k}_{\mathscr{S}})(w_1, w_2)$, and so we get:

$$\mathrm{rank}_M(w_1, w_2) = \mathrm{rank}_{\mathbf{k}_{\mathscr{R}}}(w_1, w_2) - \mathrm{rank}_{\mathbf{k}_{\mathscr{S}}}(w_1, w_2) \qquad \blacktriangleleft$$

4.2 Stability of Meta-Diagrams

We now show a stability result for meta-diagrams. We need to modify the notion of erosion distance to do so, as meta-diagrams have negatively signed parts. We proceed by adding the positive part of one meta-diagram to the negative part of the other. This idea stems from Betthauser et al.'s work [4], and was also used in the stability of rank decompositions in [7].

▶ **Definition 28.** *For $M, N : \mathbb{R}^2 \to \mathbf{vec}$, define $\mathrm{PN}(M, N) : \mathrm{Dgm} \to \mathbf{vec}$ as*

$$\mathrm{PN}(M, N)([s, t)) := \mathbf{mdgm}_M^+([s, t)) + \mathbf{mdgm}_N^-([s, t))$$

$\mathrm{PN}(M, N)$ is a non-negatively signed 1-parameter persistence module for all $[s, t) \in \mathrm{Dgm}$, allowing us to make use of the previous notion of $\preceq_\epsilon$ (Definition 14) to define an erosion distance for meta-diagrams. Unlike meta-ranks which have a continuous support, a meta-diagram is only supported on $(\overline{S})^2$ for some finite $S \subset \mathbb{R}$. As a result, we first modify the notion of erosion distance to fit the discrete setting.

Define maps $\overline{S}_\geq, \overline{S}_\leq : \mathbb{R} \cup \{\infty\} \to \overline{S}$ by $\overline{S}_\geq(x) := \min\{s \in \overline{S} \mid x \geq s\}$ and $\overline{S}_\leq(x) := \max\{s \in \overline{S} \mid x \leq s\}$, or some value less than s_1 if this set is empty. We say S is *evenly-spaced* if there exists $c \in \mathbb{R}$ such that $s_{i+1} - s_i = c$ for all $1 \leq i \leq n - 1$. In the following, fix an evenly-spaced finite $S \subset \mathbb{R}$.

▶ **Definition 29.** *For S-constructible $M, N : \mathbb{R}^2 \to \mathbf{vec}$, define the erosion distance:*

$$\begin{aligned}
\mathrm{d}_{\mathrm{E}}^S(\mathbf{mdgm}_M, \mathbf{mdgm}_N) := \inf\{&\epsilon \geq 0 \mid \forall s \leq t \in \overline{S}, \\
&\mathrm{PN}(M, N)([\overline{S}_\leq(s - \epsilon), \overline{S}_\geq(s + \epsilon))^\epsilon \preceq_{2\epsilon} \mathrm{PN}(N, M)([s, t)) \,\text{and} \\
&\mathrm{PN}(N, M)([\overline{S}_\leq(s - \epsilon), \overline{S}_\geq(s + \epsilon))^\epsilon \preceq_{2\epsilon} \mathrm{PN}(M, N)([s, t))\}
\end{aligned}$$

We have the following stability result for meta-diagrams,

▶ **Theorem 30.** *For S-constructible $M, N : \mathbb{R}^2 \to \mathbf{vec}$, with S evenly-spaced, we have*

$$\mathrm{d}_{\mathrm{E}}^S(\mathbf{mdgm}_M, \mathbf{mdgm}_N) \leq \mathrm{d}_{\mathrm{I}}(M, N).$$

For a stability result when S is not evenly-spaced, see the full version [11].

5 Algorithms

In this section, we provide algorithms for computing meta-ranks and meta-diagrams. The input to these algorithms is a simplex-wise bifiltration:

▶ **Definition 31.** *Let $n \in \mathbb{Z}$, and $[n]$ denote the poset $\{1, \ldots, n\}$ with the usual order. Let K be a simplicial complex, and $\mathrm{sub}(K)$ denote all subsets of K which are themselves simplicial complexes. A filtration is a function $F : [n] \to \mathrm{sub}(K)$ such that for $a \leq b$, $F(a) \subseteq F(b)$. We say a filtration is simplex-wise if for all $1 \leq a \leq n - 1$, either $F(a + 1) = F(a)$ or $F(a + 1) = F(a) \cup \{\sigma\}$ for some $\sigma \in K \setminus F(a)$. In the latter case, we denote this with $F(a) \xrightarrow{\sigma} F(a + 1)$. We say $\sigma \in \mathrm{sub}(K)$ arrives at a if $\sigma \in F(a)$ and $\sigma \notin F(a - 1)$.*

Define $P_n := [n] \times [n]$ equipped with the product order. A bifiltration is a function $F : P_n \to \mathrm{sub}(K)$. We say a bifiltration is simplex-wise if for all $(a, b) \in P_n$, for $(x, y) = (a+1, b)$ or $(a, b + 1)$, if $(x, y) \in P_n$, then either $F((x, y)) = F((a, b))$, or $F((a, b)) \xrightarrow{\sigma} F((x, y))$ for some $\sigma \notin F((a, b))$.

Applying homology to a bifiltration yields a bimodule defined on P_n. Our theoretical background in previous sections focused on the case of bimodules defined over $\mathbb{R}^2$. The same ideas and major results follow similarly for a module defined over P_n. We quickly highlight the differences in definitions when working with modules defined on P_n. The following definitions are re-phrasings of the horizontal meta-rank and horizontal meta-diagram for modules indexed over P_n, but as before, the statements are directly analogous in the vertical setting. Let $\mathbf{Int}([n])$ refer to all intervals of $[n]$, which consists of $\{[a, b] \mid a \le b,\, a, b \in [n]\}$.

▶ **Definition 32.** *For $M : P_n \to \mathbf{vec}$, define the meta-rank, $\mathbf{mrk}_M : \mathbf{Int}([n]) \to \mathbf{Pvec}$ by*

$$\mathbf{mrk}_M([s, t]) := [\mathrm{im}(\phi_x(s \le t))]$$

▶ **Definition 33.** *For $M : P_n \to \mathbf{vec}$, define the meta-diagram, $\mathbf{mdgm}_M : \mathbf{Int}([n]) \to \mathbf{SPvec}$ as follows: if $1 < s \le t < n$, define:*

$$\begin{aligned}
\mathbf{mdgm}_M([s, t]) :=\; & \mathbf{mrk}_M([s, t]) - \mathbf{mrk}_M([s, t + 1]) \\
& + \mathbf{mrk}_M([s - 1, t + 1]) - \mathbf{mrk}_M([s - 1, t]), \\
\mathbf{mdgm}_M([s, n]) :=\; & \mathbf{mrk}_M([s, n]) - \mathbf{mrk}_M([s - 1, n]), \\
\mathbf{mdgm}_M([1, t]) :=\; & \mathbf{mrk}_M([1, t]) - \mathbf{mrk}_M([1, t + 1]), \ \text{and} \\
\mathbf{mdgm}_M([1, n]) :=\; & \mathbf{mrk}_M([1, n]).
\end{aligned}$$

5.1   Overview of the Algorithm

Henceforth, assume $F : P_n \to \mathrm{sub}(K)$ is a simplex-wise bifiltration, and M is the result of applying homology to F. Our algorithm to compute the meta-rank relies on the vineyards algorithm from [13]. The algorithm starts with F as the input. Define γ_1 to be the path in P going from $(1, 1) \to (1, n) \to (n, n)$, i.e. the path along the top-left boundary of P. We compute the $D = RU$ decomposition for the interval decomposition of the persistence module given by the 1-parameter filtration found by slicing F over γ_1, which we denote F_{γ_1}. This decomposition gives us all the persistence intervals and persistence pairs (σ_i, σ_j) and unpaired simplices corresponding to each interval, the former corresponding to a finite interval and the latter an infinite interval. To simplify notation, for every unpaired simplex corresponding to an infinite interval, we pair it with an implicit simplex arriving in an extended F at $(n + 1, n + 1)$. We store the persistence intervals in an ordered list, which we denote `intervals`. All intervals in `intervals` restricted to $[1, n]$ constitute together the 1-parameter persistence module M_x^1, which is precisely $\mathbf{mrk}_M([1, 1])$. We then store $\mathbf{mrk}_M([1, 1])$ as a list, with the same ordering as `intervals`, leaving an empty placeholder whenever an interval does not intersect $[1, n]$.

We sweep γ through P, over one square at a time, going down through the first column, until we reach γ_2, the path $(1, 1) \to (2, 1) \to (2, n) \to (n, n)$. From there, we repeat the process column-by-column until we reach γ_n, the path $(1, 1) \to (n, 1) \to (n, n)$; see Figure 6.

After each change of a single vertex in our intermediary paths γ stemming from swapping the upper-left boundary of a single square to the lower-right one, the resulting filtration F_γ either remains the same, or changes in one of the ways illustrated in Figure 7.

After passing through each square, we update each interval in `intervals` in-place. If F_γ remains the same, then there is no change to `intervals`. If F_γ changes by altering the arrival time of a single simplex, then the pairings do not change, and the interval corresponding to the shifted simplex either extends by one or shrinks by one. If a transposition occurs, see Figure 7 (left), then we use the transposition update process from the vineyards algorithm.
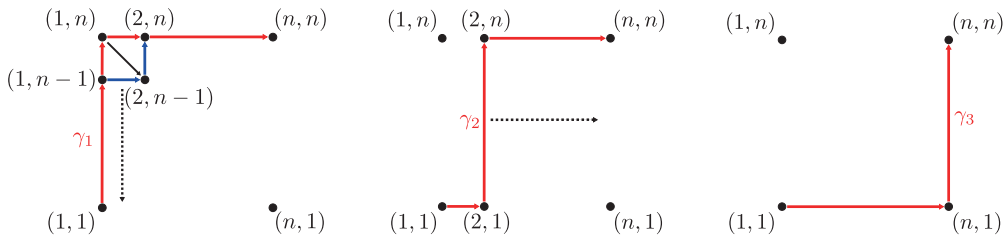
Figure 6 We start with γ_1 on the left, and then push γ_1 through the square to track along the lower-right corner of the square (in blue). We repeat this process, descending down each square in the first column until we reach γ_2 (middle). Then we repeat this process column-by-column until we've reached γ_n (right).
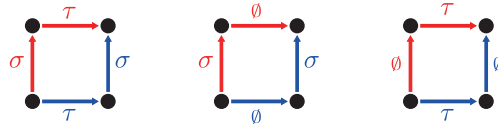
Figure 7 Three possible ways in which F_γ can change via being pushed through a one-by-one square. In our algorithm, γ always starts along the red path, then shifts to the blue path.

If we start at γ_1, then when we reach F_{γ_2}, we can restrict each interval in `intervals` to $[2, n+1]$ and shift it back down one, and this corresponds to $\mathbf{mrk}_M([2,2])$, which we store using the same rules as we did with $\mathbf{mrk}_M([1,1])$.

Since we are storing all intervals in meta-ranks in this ordered fashion, we can take any interval in $\mathbf{mrk}_M([2,2])$, and see where it came from in $\mathbf{mrk}_M([1,1])$, which would be the interval stored at the same index in both lists. By taking the intersection, we get the corresponding interval which we put into this location in the list $\mathbf{mrk}_M([1,2])$. We repeat the process of modifying γ one vertex at a time to get the paths γ_i from $(1,1) \to (i,1) \to (i,n) \to (n,n)$ as above, updating `intervals` and getting $\mathbf{mrk}_M([i,i])$ by taking appropriate intersections and shifts. Since every list of intervals we store maintains this ordering, we can take any interval in $\mathbf{mrk}_M([i,i])$, and see the corresponding interval it was previously (if any) in $\mathbf{mrk}_M([k, i-1])$ for all $1 \le k \le i-1$. Then by intersecting the interval in $\mathbf{mrk}_M([i,i])$ with its corresponding interval in $\mathbf{mrk}_M([k, i-1])$, we get a new corresponding interval in $\mathbf{mrk}_M([k,i])$. We repeat this process iteratively with i going from 1 to n, which at the end computes all of $\mathbf{mrk}_M : \mathbf{Int}([n]) \to \mathbf{Pvec}$.

We now describe what can happen to the intervals as we pass over a single square in which a transposition occurs, swapping σ_i and σ_{i+1}. From the analysis in [13, Section 3], if the pairing function changes, then the intervals themselves do not change. If the pairing function remains the same, then two of the persistence intervals will change. Suppose σ_i is paired with τ_i and σ_{i+1} is paired with τ_{i+1}. There are four possibilities, see Figure 8.

We describe the algorithm in Algorithm 1. The output of Algorithm 1 will be $\mathbf{mrk}_M$, stored as a collection of lists of the barcodes $\mathbf{mrk}_M([s,t])$ for all $s \le t \in [n]$.

We now prove the correctness of Algorithm 1.

▶ **Proposition 34.** *For $i \in [n]$, $\mathbf{mrk}_M([i,i])$ is found by taking each interval in the barcode for F_{γ_i}, shifting it down by $i-1$, and then taking the intersection with $[1,n]$.*

▶ **Proposition 35.** *Let $1 < i \le n$, and suppose we know $\mathbf{mrk}_M([i,i])$ and $\mathbf{mrk}_M([k, i-1])$ for all $1 \le k \le i-1$, and that these lists of intervals are stored in the ordered fashion previously described. From this information, we can compute $\mathbf{mrk}_M([k,i])$.*
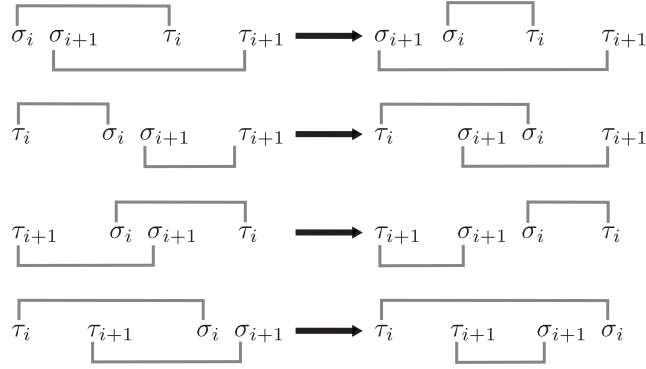
Figure 8 Four cases in which intervals change after a transposition. Observe that in each case, both intervals change, and this change is in exactly one coordinate.

Algorithm 1 METARANK(F).

Step 1. Compute $D = RU$ for F_{γ_1}, getting the ordered list `intervals` and the pairing for each interval.

Step 2. for each interval in `intervals`, intersect the interval with $[1, n]$, and store the result in the ordered list $\mathbf{mrk}([1, 1])$.

Step 3. For $i := 1$ to $n - 1$, do

    Step 3.1. For $j := n$ down to 2, do

        * update D, R, U, and `intervals` via the vineyards algorithm, as γ sweeps through the square with upper-left corner (i, j) and lower-right corner $(i + 1, j - 1)$.

    Step 3.2. For each interval in `intervals`, shift the interval down by $i - 1$, and intersect the interval with $[1, n]$, storing the result in the ordered list $\mathbf{mrk}([i, i])$.

    Step 3.3. For $k := 1$ to $i - 1$, do

        * For each interval in $\mathbf{mrk}([i, i])$, intersect with the corresponding interval in $\mathbf{mrk}([k, i - 1])$. Store this intersection in the ordered list $\mathbf{mrk}([k, i])$.

▶ **Theorem 36.** *Algorithm 1 correctly computes the meta-rank for the bimodule M induced by homology of the input bifiltration F, and runs in time $O(n^3)$. As a result, the number of rectangles in the rank decomposition for M is also $O(n^3)$.*

Proof. By Proposition 34, we can compute $\mathbf{mrk}_M([1, 1])$, and further $\mathbf{mrk}_M([i, i])$ for all $i \in [n]$. Then we can use Proposition 35 iteratively to fill in $\mathbf{mrk}_M([k, i])$ for all $1 \leq k < i \leq n$, and we are done.

For the runtime analysis, first observe that the initial $D = RU$ computation in Step 1 takes $O(n^3)$ time, and `intervals` can be computed from the decomposition in linear time. The loop in Step 2 also takes linear time, as the size of `intervals` is $O(n)$ which is fixed throughout. Step 3 consists of a for loop with $O(n)$ iterations. Step 3.1 consists of a for loop with $O(n)$ iterations, and each loop inside performs an update over a square using the vineyards approach. A single update takes $O(n)$ time in the worst case, so Step 3.1 takes $O(n^2)$ time. Step 3.2 runs in linear time for the same reason as Step 2. Step 3.3 consists of a for loop with $O(n)$ iterations, with each iteration taking $O(n)$ operations as the size of each $\mathbf{mrk}_M([k, i])$ is the same as `intervals`. Hence, Step 3.3 has total runtime $O(n^2)$. Thus, each loop in Step 3 consists of substeps that run in $O(n^2)$ time, $O(n)$ time, and $O(n^2)$ time respectively, incurring a total cost of $O(n^3)$ over $O(n)$ iterations.

To summarize, we have a step with $O(n^3)$ cost, followed by a step with $O(n)$ cost, followed by a step with $O(n^3)$ cost, so the algorithm runs in $O(n^3)$ time.

By Definition 33, we can compute $\mathbf{mdgm}_M$ from $\mathbf{mrk}_M$ in $O(n^3)$ time, implying the number of non-zero intervals in $\mathbf{mdgm}_M$ is $O(n^3)$. By Proposition 27, each non-zero interval in $\mathbf{mdgm}_M$ corresponds uniquely to a single rectangle in the rank decomposition of M, and so the number of such rectangles is likewise $O(n^3)$.   ◀

6 Discussion

We conclude with some open questions. First, we would like to extend our approach to the d-parameter setting. We expect that a proper extension would satisfy relationships with the rank invariant and rank decompositions similar to Proposition 12 and Proposition 27. Such an extension would also lead to a "recursive" formulation of the persistence diagram of diagrams illustrated in Figure 2. Next, Theorem 36 implies that the number of rectangles needed in a rank decomposition for a bimodule is bounded above by $O(n^3)$. It is not known whether this bound is tight. Lastly, there have been multiple recent works that use algorithmic ideas from 1-parameter persistence to compute invariants in the multiparameter setting [15, 16, 23]. We wish to explore in what ways these approaches can create new algorithms or improve upon existing ones for computing the invariants of multi-parameter persistence modules.

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