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SIGNED AREA ENUMERATION FOR LATTICE WALKS

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Abstract. We give a summary of recent progress on the signed area enumeration of closed walks on planar lattices. Several connections are made with quantum mechanics and statistical mechanics. Explicit combinatorial formulae are proposed which rely on sums labelled by the multicompositions of the length of the walks.

Keywords: lattice walks, signed area enumeration, Hofstadter model, exclusion statistics.

1. Introduction

The seminal problem of the signed area enumeration of walks on planar lattices of various kinds has been around for a long time. It is well known that this purely combinatorial problem can be equivalently reformulated in the realm of Hofstadter-like quantum mechanics models (note that, in physics, the "signed area" is often called the "algebraic area"). Recently, in [16], this problem has been given a boost in the form of an explicit enumeration formula which in turn could be reinterpreted [5, 14, 15] in terms of statistical mechanics models with exclusion statistics, again a purely quantum concept. It is a striking fact that an enumeration quest regarding classical random walks should be in the end so intimately connected to quantum physics.

In this note we give a summary of this recent progress starting with the original signed area enumeration problem for closed walks on a square lattice and then enlarging the perspective to other kind of lattices and walks via the statistical mechanics reinterpretation. The first question we address is: Among the $\binom{N}{N/2}^2$ closed N-step walks that one can draw on a square lattice starting from and returning to a given point (note that N is then necessarily even), how many of them enclose a given signed area A?

The signed area enclosed by a directed walk is weighted by its *winding number*: If the walk moves around a region in a counterclockwise (positive) direction, its area counts as positive, otherwise negative; if the walk winds around more than once, the area is counted with multiplicity. These regions inside the walk are called winding sectors.

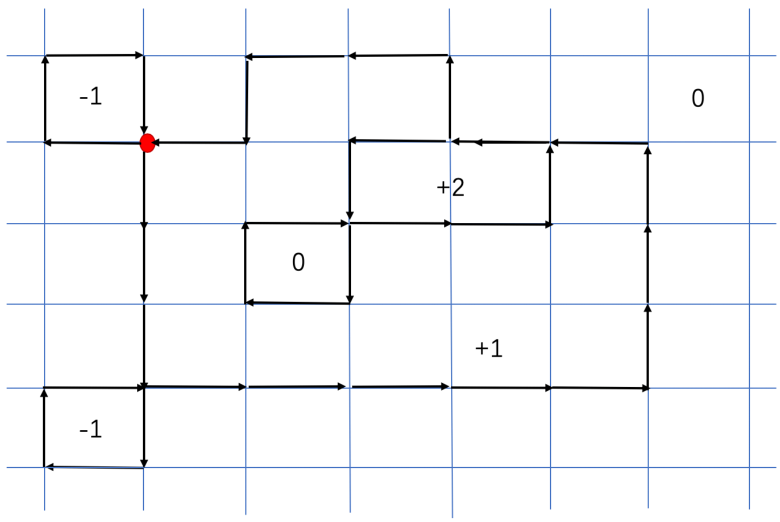

Figure 1. A closed walk of length $N = 36$ starting from and returning to the same bullet red point with its various winding sectors $m = +2, +1, 0, -1, -1$ (containing respectively $2, 14, 1, 1, 1$ unit lattice cells). Note the double arrow on the horizontal link (above the $+2$ sector) which indicates that the walk has moved twice on this link, here in the same left direction.

In Figure 1, the 0-winding sector inside the walk arises from a superposition of a $+1$ and a -1 winding. Summing the areas of each sector, with the corresponding multiplicative weigth, gives the signed area $A = (-1) \times 2 + (+0) \times 1 + (+1) \times 14 + (+2) \times 2 = 16$.

More formally, if $\gamma : [0, 1] \to \mathbb{R}^2$ is a closed path that begins and ends at the origin, the signed area of this path is

$$A = \int_{\mathbb{R}^2} \eta(\gamma, \mathbf{x}) d\mathbf{x} = \sum_{m=-\infty}^{\infty} m S_m$$

where $\eta(\gamma, \mathbf{x})$ is the winding number of γ around the point $\mathbf{x} \in \mathbb{R}^2$, and where S_m denotes the classical area of the m-winding sectors inside the path (i.e. the number of unit lattice cells it encloses with winding number m, where m can be positive or negative).

Winding sectors for continuous Brownian curves as well as for discrete lattice walks have been the subject of study for a long time. In this respect, we note in the last few years some advances in [2] where[1] an explicit formula for the expected area $\langle S_m \rangle$ of the m-winding sectors inside square lattice walks is proposed, to the exception of the 0-winding sector, for the simple reason that the latter is difficult to distinguish from the outside (i.e., 0-winding again) sector, which is of infinite size. Taking the continuous limit allows to recover the results previously obtained in [4] for Brownian curves. One notes that for Brownian curves the expected area $\langle S_0 \rangle$ of the 0-winding sectors is also known by other means thanks to the SLE machinery [7]. However, it remains an open problem for discrete lattice walks.

[1]Note that Timothy Budd gave a talk on his article [2] at the conference Lattice Paths, Combinatorics and Interactions, at CIRM, in 2021. A video of his talk is available on the website of this conference.

Counting the number of closed walks of length N on the square lattice enclosing a signed area A can be achieved in a most straightforward way by introducing two lattice hopping operators (symbols) u and v respectively in the right and up directions, as well as operators u^{-1} and v^{-1} corresponding to hops in the left and down direction. A directed walk on the square lattice starting at the origin is then represented by the ordered product of the hopping operators corresponding to its individual steps. By convention we order the operators from right to left as we trace the steps of the walk. Clearly the set of all walks of length N on the lattice is reproduced by the 4^N terms in the expansion of

$$\left(u + u^{-1} + v + v^{-1}\right)^N \tag{1.1}$$

into monomials of products of symbols, each with N factors. The operator $u+u^{-1}+v+v^{-1}$ can be considered as the generator of walks.

We are interested in closed walks, and in counting their multiplicity according to their signed area. To this end, we endow the above operators with the relations

$$v\ u = \mathrm{Q}\ u\ v\ ,\ \ uu^{-1} = u^{-1}u = vv^{-1} = v^{-1}v = 1 \tag{1.2}$$

with Q a central element (that is, Q commutes with all operators). This allows us to reduce all terms in (1.1) into monomials of the form $v^n u^m$, (m, n) being the lattice coordinates of the end of the walk, with coefficients powers of Q. In particular, closed walks correspond to the monomial $v^0 u^0$ in (1.1).

The non-commutativity relation $vu = \mathrm{Q}\ uv$ has the effect of flipping a right-up two-step segment into an up-right one, producing a factor of Q, and similarly for the other relations implied by (1.2) such as, e.g., $v\,u^{-1} = \mathrm{Q}^{-1}u^{-1}v$. In each case, the coefficient is Q to the power of the signed area of the unit lattice cells left behind by the exchange. Repeated application of these relations reduces each walk to a right-angular one $v^n u^m$ with an overall coefficient Q to the power of the total signed area of the original walk closed by joining its end to the origin with a vertical and a horizontal straight walk (in that order). In particular, closed walks will correspond to monomials Q^A, with A the signed area of the walk. Therefore, the $v^0 u^0$ part in

$$[v\text{^}0\ u\text{^}0]\quad \left(u + u^{-1} + v + v^{-1}\right)^N = \sum_{A=-N/2+1}^{N/2-1} C_N(A)\ \mathrm{Q}^A \tag{1.3}$$

provides the numbers $C_N(A)$ which count the closed walks of length N enclosing a signed area A. For example, one easily checks that $[v^0u^0]\left(u + u^{-1} + v + v^{-1}\right)^4 = 28 + 4\mathrm{Q} + 4\mathrm{Q}^{-1}$, indicating that among the $\binom{4}{2}^2 = 36$ closed walks making 4 steps $C_4(0) = 28$ enclose a signed area $A = 0$ and $C_4(1) = C_4(-1) = 4$ enclose a signed area $A = \pm 1$. The non-commutativity relation written in the form

$$v^{-1}u^{-1}v\ u = \mathrm{Q}$$

expresses the fact the the elementary walk circling one lattice cell in the positive direction has signed area 1.

2. The Hofstadter model

In any irreducible representation of the operator relations (1.2) the central element Q will be represented by a number. Restricting to unitary representations, for which $u^\dagger = u^{-1}$, $v^\dagger = v^{-1}$, Q will necessarily be a complex number of norm unity, i.e., a phase. This provides a mapping between the u, v representation for walks and quantum mechanics, interpreting u and v as unitary operators acting on a quantum Hilbert space, and the Hermitian operator $u + u^{-1} + v + v^{-1}$ as the Hamiltonian of a quantum system.

In fact, such a quantum system exists and corresponds to a well-known model in physics. Interpreting u and v as operators that generate hops of a quantum particle by one link on the square lattice, the non-commutativity relation (1.2) indicates that translations of the particle in the horizontal and vertical directions do not commute. This can be interpreted as that the particle is charged and coupled to a homogeneous magnetic field B perpendicular to the lattice. Writing $\mathrm{Q} = \mathrm{e}^{\mathrm{i}2\pi\Phi/\Phi_o}$, Φ is then the flux of B through the unit lattice cell and $\Phi_o = h/c$ is the flux quantum (h is the Planck constant and c the particle's charge). The Hermitian operator

$$H = u + u^{-1} + v + v^{-1}$$

then becomes a Hamiltonian modelling a quantum particle hopping on a square lattice and coupled to a perpendicular magnetic field. This model is known as the Hofstadter model [8].

To make the physics connection completely explicit, we note that in quantum mechanics the hopping operators u and v are written as

$$u = \mathrm{e}^{\mathrm{i}(p_x - cA_x)/\hbar} \quad \text{and} \quad v = \mathrm{e}^{\mathrm{i}(p_y - cA_y)/\hbar},$$

where $A_x = -By$ and $A_y = 0$ are the two components of the vector potential of the magnetic field in the Landau gauge and $p_x = -\mathrm{i}\hbar\partial_x$ and $p_y = -\mathrm{i}\hbar\partial_y$ those of the momentum operator (where we use the standard notation $\hbar = h/(2\pi)$). The relation (1.2) thus follows from the Heisenberg commutators $[x, p_x] = [y, p_y] = \mathrm{i}\hbar$ and the Baker–Campbell–Hausdorff formula. One then introduces the quantum state $\Psi_{m,n}$ representing the probability amplitude of the particle being at lattice site (m, n), on which hopping operators act as

$$u\Psi_{m,n} = \mathrm{e}^{\mathrm{i}cnB/\hbar}\Psi_{m+1,n} \ , \quad v\Psi_{m,n} = \Psi_{m,n+1}.$$

Note that the magnetic flux per unit lattice cell is $\Phi = B$, and thus $cB/\hbar = 2\pi\Phi/(h/c) = 2\pi\Phi/\Phi_o$, so the factor appearing in the action of u on $\psi_{m,n}$ is Q^n. Using translation invariance in the horizontal direction we can further choose $\Psi_{m,n}$ to be an eigenstate of p_x, that is, $\Psi_{m,n} = \mathrm{e}^{imk_x}\Phi_n$. The action of u, v on Φ_n becomes

$$u\Phi_n = \mathrm{e}^{\mathrm{i}k_x}\mathrm{Q}^n\Phi_n \ , \ v\Phi_n = \Phi_{n+1}.$$

For the Hofstader model, the Schrödinger equation $H\Psi = E\Psi$ (which determines the eigenvalue E of the spectrum) can be rewritten as

$$(u + u^{-1} + v + v^{-1})\Psi_{m,n} = E\Psi_{m,n} \Rightarrow \Phi_{n+1} + \Phi_{n-1} + (\mathrm{Q}^n\mathrm{e}^{\mathrm{i}k_x} + \mathrm{Q}^{-n}\mathrm{e}^{-\mathrm{i}k_x})\Phi_n = E\Phi_n.$$

Going a step further, a simplification arises when the flux is *rational* (i.e. when one has $\mathrm{Q} = \mathrm{e}^{\mathrm{i}2\pi p/q}$ with p, q two coprime integers): It induces a q-periodicity of the Schrödinger equation in the vertical direction. Then, as Bloch's theorem states that solutions to the Schrödinger equation in a periodic potential take the form of a plane wave modulated by a periodic function $\tilde{\Phi}_n$, we can write

$$\Phi_n = \mathrm{e}^{\mathrm{i}nk_y}\tilde{\Phi}_n \ , \ \tilde{\Phi}_{n+q} = \tilde{\Phi}_n. \tag{2.1}$$

Indeed, for this rational flux $Q^q = 1$ and u^q, v^q become Casimirs (a physicist's term for central elements), and the choice of Bloch states (2.1) can be interpreted mathematically as choosing an irreducible representation of the u, v algebra. Acting on such states, u^q and v^q become

$$u^q = \mathrm{e}^{\mathrm{i}qk_x} \ , \ v^q = \mathrm{e}^{\mathrm{i}qk_y}.$$

One ends up with u and v, acting on $\tilde{\Phi}_n$, becoming the $q \times q$ matrices

$$u = \mathrm{e}^{\mathrm{i}k_x} \begin{pmatrix} Q & 0 & 0 & \cdots & 0 & 0 \\ 0 & Q^2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & Q^3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Q^{q-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad \text{and} \quad v = \mathrm{e}^{\mathrm{i}k_y} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

involving two real quantities k_x and k_y. Finding the energy spectrum (which depends on k_x and k_y) reduces to computing the eigenvalues $E_1, E_2, \ldots, E_q$ of the $q \times q$ Hamiltonian matrix

$$H_q = \begin{pmatrix} Q\mathrm{e}^{\mathrm{i}k_x} + Q^{-1}\mathrm{e}^{-\mathrm{i}k_x} & \mathrm{e}^{\mathrm{i}k_y} & 0 & \cdots & 0 & \mathrm{e}^{-\mathrm{i}k_y} \\ \mathrm{e}^{-\mathrm{i}k_y} & Q^2\mathrm{e}^{\mathrm{i}k_x} + Q^{-2}\mathrm{e}^{-\mathrm{i}k_x} & \mathrm{e}^{\mathrm{i}k_y} & \cdots & 0 & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}k_y} & () & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & () & \mathrm{e}^{\mathrm{i}k_y} \\ \mathrm{e}^{\mathrm{i}k_y} & 0 & 0 & \cdots & \mathrm{e}^{-\mathrm{i}k_y} & Q^q\mathrm{e}^{\mathrm{i}k_x} + Q^{-q}\mathrm{e}^{-\mathrm{i}k_x} \end{pmatrix}.$$

All the machinery of quantum mechanics is now at our disposal. Selecting as in (1.3) the v^0u^0 monomial of $\left(u + u^{-1} + v + v^{-1}\right)^N$ translates in the quantum world to computing the trace of H_q^N. The quantum trace is defined as

$$\mathbf{Tr}\, H_q^N := \frac{1}{q}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{dk_x}{2\pi}\frac{dk_y}{2\pi}\, \mathrm{tr}\, H_q^N = \frac{1}{q}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{dk_x}{2\pi}\frac{dk_y}{2\pi} \sum_{i=1}^{q} E_i^N, \tag{2.2}$$

that is, one sums over the q eigenvalues E_i of H_q (yielding the standard matrix trace $\mathrm{tr}\, H_q^N$) and integrates over k_x and k_y while enforcing a continuous normalization in k_x, k_y and we rescale by a factor $1/q$ (thus, if one considers for example the $q \times q$ identity matrix I_q, one has $\mathrm{tr}\, I_q = q$, while $\mathbf{Tr}\, I_q = 1$). Under this definition of the trace, $\mathbf{Tr}\, u^m v^n = \delta_{m,0}\delta_{n,0}$ (integration over k_x, k_y eliminates the traces of terms involving u^{qm} and v^{qn}). We thus get a first striking result (note that this was also obtained by another approach in [1]):

(a bit over the top)

Theorem 2.1. *The signed area enumeration of closed paths is given by*

$$\sum_A C_N(A)\, Q^A = \boldsymbol{Tr}\, H_q^N. \tag{2.3}$$

3. The signed area enumeration

It is known [3] that the determinant of the matrix $I_q - zH_q$ is

$$\det(I_q - zH_q) = \sum_{n=0}^{\lfloor q/2 \rfloor} (-1)^n Z(n) z^{2n} - 2\Big(\cos(qk_x) + \cos(qk_y)\Big) z^q,$$

where the $Z(n)$'s are independent of k_x and k_y and $Z(0) = 1$. Kreft [12] was able to rewrite them in a closed form as trigonometric multiple nested sums

$$Z(n) = \sum_{k_1=1}^{q-2n+2} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} s_{k_1+2n-2} s_{k_2+2n-4} \cdots s_{k_{n-1}+2} s_{k_n}, \tag{3.1}$$

where

$$s_k = 4\sin^2(\pi kp/q). \tag{3.2}$$

We will call s_k the spectral function of the model. This will be the starting point for the signed area enumeration. We give here a summary of the procedure, more details can be found in [14, 16].

First introduce the coefficients $b(n)$ via

$$\log\left(\sum_{n=0}^{\lfloor q/2 \rfloor} Z(n) z^n\right) = \sum_{n=1}^{\infty} b(n) z^n. \tag{3.3}$$

The $b(n)$ are related to the desired traces. Start by noting that

$$\mathbf{Tr}\ H_q^{2n} = \frac{1}{q} \mathrm{tr}\, H_q^{2n} \text{ for } n < q.$$

Indeed, $\mathrm{tr}\, u^n v^m = q\ \delta_{n,0}\delta_{m,0}$ for $m, n < q$ and so the values of the Casimirs k_x, k_y do not appear, making the integration over them in (2.2) trivial. Then the identity

$$\log\ \det(I_q - zH_q) = \mathrm{tr}\ \log(I_q - zH_q) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}\, H_q^n$$

implies that the quantum trace (2.2) is proportional to $b(n)$ for $n < q$

$$\mathbf{Tr}\, H_q^{2n} = 2n(-1)^{n+1} \frac{1}{q} b(n).$$

By keeping q as a free parameter and extending it to arbitrarily big values, the quantum trace can be calculated for all n.

Note that the term of order z^n in $\det(I_q - zH_q)$ is $-(z^n/n)\mathrm{tr}\, H_q^n$ plus terms involving products of traces, $\mathrm{tr}\, H_q^{n_1}\, \mathrm{tr}\, H_q^{n_2} \cdots$ with $n_1 + n_2 + \cdots = n$. As each trace $\mathrm{tr}\, H_q^n$ contributes an overall factor q, the $b(n)$ can also be obtained as the order q term in $Z(n)$, ignoring terms of higher order $q^2, \ldots, q^n$. Since each sum in (3.1) contributes a factor of q, only single sums will appear in $b(n)$. Explicitly, the $b(n)$ can be calculated to be

$$b(n) = (-1)^{n+1} \sum_{\substack{l_1, l_2, \ldots, l_j \\ \text{composition of } n}} c(l_1, l_2, \ldots, l_j) \sum_{k=1}^{q-j+1} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1}. \tag{3.4}$$

Their structure is encoded in the coefficients $c(l_1, l_2, \ldots, l_j)$ labelled by the compositions $l_1, l_2, \ldots, l_j$ of n (defined as the ordered partitions of n: There are 2^{n-1} compositions of n, for example $3 = 3, 2+1, 1+2, 1+1+1$), with[2]

$$c(l_1, l_2, \ldots, l_j) = \frac{\binom{l_1+l_2}{l_1}}{l_1+l_2} \; l_2 \frac{\binom{l_2+l_3}{l_2}}{l_2+l_3} \; \cdots \; l_{j-1} \frac{\binom{l_{j-1}+l_j}{l_{j-1}}}{l_{j-1}+l_j}.$$

As stated, only single trigonometric sums appear in (3.4). Putting everything together,

$$\textbf{Tr}\; H_q^N = \sum_{\substack{l_1, l_2, \ldots, l_j \\ \text{composition of } N/2}} c(l_1, l_2, \ldots, l_j)\, \frac{N}{q} \sum_{k=1}^{q-j+1} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1}.$$

We need to state N is even

The trigonometric single sums can be computed, keeping q as a free parameter, as mentioned before. Finally, one can extract from (2.3) the desired number of closed walks of given area; this gives the following theorem.

Theorem 3.1. *The number of closed walks of length N enclosing a given signed area A is*

$$C_N(A) = \; N \times \sum_{\substack{l_1, l_2, \ldots, l_j \\ \textit{composition of } N/2}} \frac{\binom{l_1+l_2}{l_1}}{l_1+l_2} \; l_2 \frac{\binom{l_2+l_3}{l_2}}{l_2+l_3} \; \cdots \; l_{j-1} \frac{\binom{l_{j-1}+l_j}{l_{j-1}}}{l_{j-1}+l_j} \; \times$$

$$\sum_{k_3=0}^{2l_3} \sum_{k_4=0}^{2l_4} \cdots \sum_{k_j=0}^{2l_j} \prod_{i=3}^{j} \binom{2l_i}{k_i} \binom{2l_1}{l_1 + A + \sum_{i=3}^{j}(i-2)(k_i - l_i)} \binom{2l_2}{l_2 - A - \sum_{i=3}^{j}(i-1)(k_i - l_i)}.$$

This formula grows quickly in complexity since the number of compositions on which one has to sum increases like $2^{N/2}$ with the number of steps. Still it has the benefit of being explicit and, as far as we are aware of, the first one available. If one also takes into account the inner sums in Theorem 3.1 the number of terms for large N grows like 2.14^N due to the k_i individual summations from 0 to $2l_i$ for each given composition $l_1, l_2, \ldots, l_j$ of n (see[3] the entry `A060801` in the On-line Encyclopedia of Integer Sequences). This is much less than the direct enumeration of closed walks which is of order $4^N/(\pi N/2)$.

In the limit of the elementary lattice size $a \to 0$ and the walk length $N \to \infty$ with the scaling $Na^2 = 2t$, walks go over to Brownian motion curves and we recover the continuum limit of a particle moving on the plane in a constant magnetic field. To implement this limit, we rescale the lattice cell area to a^2, which amounts to setting $A \to A/a^2$ in $C_N(A)$. We conjecture that

$$\frac{N\, C_N(A/a^2)}{\binom{N}{N/2}^2} \to \frac{\pi}{\cosh^2(\pi A/t)},$$

consistently with the law for the distribution of the signed area enclosed by Brownian curves after a time t (obtained by Paul Lévy in 1950; see [10, 13]). It can also be obtained directly in the continuum limit by considering the partition function of a quantum particle in a magnetic field with a Landau level energy spectrum. For small values of A the convergence has been checked numerically for N up to 140 to improve with increasing N.

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[2] The expression for $c(l_1, l_2, \ldots, l_j)$ is closely related to the enumeration of Dyck paths (up to a factor l_1), see [11] p. 516. We have further elaborated on this relation in [6].

[3] We thank one of the referees for drawing our attention to this point.

4. Exclusion statistics

The quantities $Z(n)$ and $b(n)$ introduced previously admit a statistical mechanical interpretation. Let us write the spectral function s_k in (3.2) as $s_k = \mathrm{e}^{-\beta\epsilon_k}$ (β is the inverse temperature) and interpret is as the Boltzmann factor for a quantum 1-body spectrum ϵ_k labelled by an integer k. The structure of $Z(n)$ in (3.1) then precisely corresponds to an n-body partition function for a gas of particles with 1-body spectrum ϵ_k and exclusion statistics $g = 2$: The $+2$ shifts in the spectral function arguments ensure that no two particles can occupy adjacent quantum states. Exclusion statistics is, again, a purely quantum concept which describes the statistical mechanical properties of identical particles. Ordinary particles are either bosons ($g = 0$), which can occupy the same quantum state, or fermions ($g = 1$), which cannot occupy the same quantum state. We see that square lattice walks map to systems with statistics beyond Fermi exclusion, in which particles can occupy neither the same state nor adjacent states. In a sense, each particle excludes two quantum states, thus $g = 2$. In general, for g-exclusion particles the n-body partition function (3.1) would become

$$Z(n) = \sum_{k_1=1}^{q-gn+g} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} s_{k_1+gn-g} s_{k_2+gn-2g} \cdots s_{k_{n-1}+g} s_{k_n} \tag{4.1}$$

with a shift g instead of 2 in the arguments of the spectral function. In line with (3.3), (3.4) the associated n-th cluster coefficient can be shown to take the form

$$b(n) = (-1)^{n+1} \sum_{\substack{l_1, l_2, \ldots, l_j \\ g-\text{composition of } n}} c_g(l_1, l_2, \ldots, l_j) \sum_{k=1}^{q-j+1} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1}, \tag{4.2}$$

where

$$c_g(l_1, l_2, \ldots, l_j) = \frac{(l_1 + \cdots + l_{g-1} - 1)!}{l_1! \cdots l_{g-1}!} \prod_{i=1}^{j-g+1} \binom{l_i + \cdots + l_{i+g-1} - 1}{l_{i+g-1}}.$$

In (4.2) one sums over all g-compositions of the integer n, obtained by inserting at will inside the usual compositions (i.e., the 2-compositions) no more than $g - 2$ zeroes in succession. For example, for $n = 3$ and $g = 3$ one has 9 such 3-compositions:

$3, 2+1, 1+2, 1+1+1, 2+0+1, 1+0+2, 1+0+1+1, 1+1+0+1, 1+0+1+0+1.$

For general g there are g^{n-1} such g-compositions of the integer n (see [9] for an analysis of these extended compositions, also called multicompositions).

One has reached the conclusion that the signed area enumeration for walks on the square lattice is described by a quantum gas of particles with statistical exclusion $g = 2$. To relate this explicitly to properties of the Hofstadter Hamiltonian itself, let us perform on the hopping lattice operators u and v the modular transformation

$$u \to -u\, v, \;\; v \to v,$$

which leave their own commutation relation invariant to get the new Hamiltonian

$$H = -u\, v - v^{-1}u^{-1} + v + v^{-1} \tag{4.3}$$

still describing the same walks but on a deformed lattice.

We now consider the following matrix associated to the Hamiltonian (4.3)

$$I_q - zH_q = \begin{pmatrix} 1 & -(1-\mathrm{Q})z & 0 & \cdots & 0 & 0 \\ -(1-\frac{1}{\mathrm{Q}})z & 1 & -(1-\mathrm{Q}^2)z & \cdots & 0 & 0 \\ 0 & -(1-\frac{1}{\mathrm{Q}^2})z & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -(1-\mathrm{Q}^{q-1})z \\ 0 & 0 & 0 & \cdots & -(1-\frac{1}{\mathrm{Q}^{q-1}})z & 1 \end{pmatrix}, \tag{4.4}$$

where we set $k_x = k_y = 0$ for simplicity (since, as we explained in Section 3, k_x and k_y do not appear in the counting formula). The Hofstadter spectral function (3.2) becomes

Shall we add the explanation that the sign in u -> -uv is needed to make corners vanish?

$$s_k = (1-\mathrm{Q}^k)(1-\frac{1}{\mathrm{Q}^k}).$$

The matrix (4.4) is a particular case of the more general class of matrices having the following shape[4]

$$I_q - zH_q = \begin{pmatrix} 1 & -f(1)z & 0 & \cdots & 0 & -g(q)z \\ -g(1)z & 1 & -f(2)z & \cdots & 0 & 0 \\ 0 & -g(2)z & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -f(q-1)z \\ -f(q)z & 0 & 0 & \cdots & -g(q-1)z & 1 \end{pmatrix} \tag{4.5}$$

and associated spectral functions

$$s_k = f(k)g(k),$$

which become the building blocks of the $Z(n)$'s in (3.1) (up to spurious "umklapp" terms, a name deriving from momentum periodicity effects on lattice quantum models, which disappear if $f(q)$ and $g(q)$ both vanish).

For statistics $g = 3$, the matrix (4.5) generalizes in a natural way to

$$I_q - zH_q = \begin{pmatrix} 1 & -f(1)z & 0 & 0 & \cdots & 0 & -g(q-1)z & 0 \\ 0 & 1 & -f(2)z & 0 & \cdots & 0 & 0 & -g(q)z \\ -g(1)z & 0 & 1 & -f(3)z & \cdots & 0 & 0 & 0 \\ 0 & -g(2)z & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -f(q-2)z & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -f(q-1)z \\ -f(q)z & 0 & 0 & 0 & \cdots & -g(q-2)z & 0 & 1 \end{pmatrix}, \tag{4.6}$$

that is, with an extra vanishing paradiagonal below the unity main diagonal, which is the manifestation of the stronger $g = 3$ exclusion.

[4]We hope that the reader will easily distinguish between too similar (but unrelated!) notations: the *function* $g(k)$ for the entries of the matrix $I_q - zH_q$, and the *integer* g (the parameter of the exclusion model, a standard notation in the literature).

The spectral function corresponding to this matrix is

$$s_k = g(k)f(k)f(k+1)$$

and the determinant $\det(I_q - zH_q)$ assumes the form (4.1) with $g = 3$. For g-exclusion the generalization of (4.6) amounts to a Hamiltonian of the form

$$H = F(u)v + v^{1-g}G(u), \tag{4.7}$$

and $I - zH$ is then a matrix with $g - 2$ vanishing paradiagonals below the main diagonal (here q is left arbitrary but is understood to be larger than g). The spectral parameters of this matrix are

$$F(\mathrm{Q}^k) = f(k), \quad G(\mathrm{Q}^k) = g(k),$$

and the spectral function is

$$s_k = g(k)f(k)f(k+1)\dots f(k+g-2).$$

Clearly the Hofstadter Hamiltonian (4.3), which rewrites as $H = (1-u)v + v^{1-2}(1-u^{-1})$, is a particular case of (4.7) with $g = 2$ and $F(u) = 1 - u$, $G(u) = 1 - u^{-1}$.

5. Chiral walks on the triangular lattice

Let us illustrate this mechanism in the case of $g = 3$ exclusion with the specific example of chiral walks on a triangular lattice. The three hopping operators U, V and $W = \mathrm{Q}U^{-1}V^{-1}$ described in Figure 2 are such that $VU = \mathrm{Q}^2UV$.

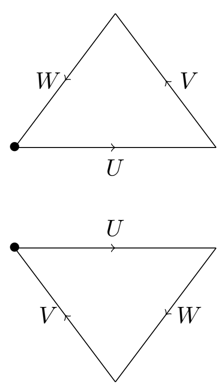

Figure 2. The three hopping operators U, V and W on the triangular lattice.

The triangular lattice Hamiltonian is

$$H = U + V + W.$$

It generates walks composed of triangles either pointing up and winding in the positive direction, or pointing down and winding in the negative direction. In this sense, the walks are chiral. The factor Q in the definition of W and the factor Q^2 in the commutation of U, V are chosen so that up-pointing (positive) triangles are assigned area $+1$. Figure 3 depicts some examples of chiral walks on the triangular lattice.

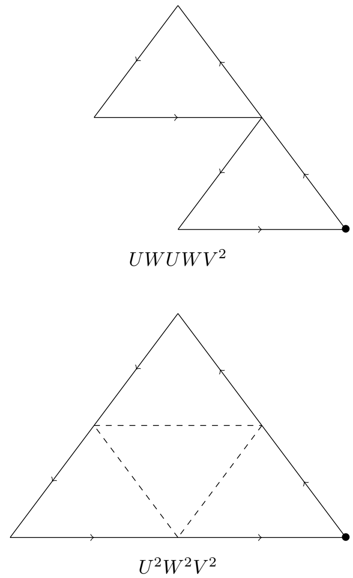

FIGURE 3. Examples of closed chiral walks on the triangular lattice.

To bring H to the exclusion form (4.7) one chooses the representation $U = -\mathrm{i}\, u\, v$ and $V = \mathrm{i}\, u^{-1}\, v$, with u and v as before, in which case H rewrites as

$$H = \mathrm{i}(-u + u^{-1})v + v^{-2}.$$

In this form, H is indeed a Hamiltonian of the type (4.7) for $g = 3$ exclusion, with $F(u) = \mathrm{i}(-u + u^{-1})$, $G(u) = 1$, spectral parameters

$$f(k) = -\mathrm{i}(\mathrm{Q}^k - \frac{1}{\mathrm{Q}^k}), \quad g(k) = 1,$$

spectral function

$$s_k = g(k)f(k)f(k+1) = 4\sin(2\pi pk/q)\sin\Big(2\pi p(k+1)/q\Big), \tag{5.1}$$

and matrix

$$I_q - zH_q = \begin{pmatrix} 1 & \mathrm{i}(\mathrm{Q}-\frac{1}{\mathrm{Q}})z & 0 & 0 & \cdots & 0 & -z & 0 \\ 0 & 1 & \mathrm{i}(\mathrm{Q}^2-\frac{1}{\mathrm{Q}^2})z & 0 & \cdots & 0 & 0 & -z \\ -z & 0 & 1 & \mathrm{i}(\mathrm{Q}^3-\frac{1}{\mathrm{Q}^3})z & \cdots & 0 & 0 & 0 \\ 0 & -z & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \mathrm{i}(\mathrm{Q}^{q-2}-\frac{1}{\mathrm{Q}^{q-2}})z & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \mathrm{i}(\mathrm{Q}^{q-1}-\frac{1}{\mathrm{Q}^{q-1}})z \\ 0 & 0 & 0 & 0 & \cdots & -z & 0 & 1 \end{pmatrix},$$

which is of the type (4.6) with a vanishing bottom-left entry. The non-Hermiticity of the triangular Hamiltonian, and thus of $I_q - zH_q$, is a consequence of the chiral nature of the walks.

The triangular signed area enumeration follows [14], yielding an expression similar to Theorem 3.1 with the trigonometric single sums appearing in (4.2) involving the triangular spectral function (5.1) and the sum done over all 3-compositions of the length of the triangular walks.

6. Conclusion

In conclusion, we have shown how tools from quantum and statistical physics allow for an explicit enumeration of closed walks of fixed length and signed area on planar lattices.

The enumeration formulae rely on an explicit sum over compositions, and their number of terms grows quickly with the length of the walk (although much less quickly than a brute-force counting formula). It would be certainly rewarding to rewrite it as a sum with a smaller number of terms. The use of symmetry on the lattice or alternative ways to write the generator of walks (Hamiltonian) may offer promise towards this goal. We leave this issue as well as other questions of interest to the lattice walk combinatorics community.

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