Gauge and scalar fields on $\mathbb{CP}^2$: A gauge-invariant analysis. II. The measure for gauge fields and a 4d WZW theory

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We consider the volume of the gauge orbit space for gauge fields on four-dimensional complex projective space. The analysis uses a parametrization of gauge fields where gauge transformations act homogeneously on the fields, facilitating a manifestly gauge-invariant analysis. The volume element contains a four-dimensional Wess-Zumino-Witten (WZW) action for a Hermitian matrix-valued field. There is also a masslike term for certain components of the gauge field. We discuss how the mass term could be related to results from lattice simulations as well as Schwinger-Dyson equations. We argue for a kinematic regime where the Yang-Mills theory can be approximated by the 4d-WZW theory. The result is suggestive of the instanton liquid picture of QCD. Further it is also indicative of the mechanism for confinement being similar to what happens in two dimensions.

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I. INTRODUCTION

In this paper, we continue our analysis of quantum fields on the manifold $\mathbb{CP}^2$ focusing on the gauge-invariant volume element relevant to the functional integration over gauge fields [1]. As is well known, in a gauge theory, the physical degrees of freedom correspond to the space of all gauge potentials ($\mathcal{A}$) modulo the set of all gauge transformations which are set to the identity at some chosen point of the spacetime manifold ($\mathcal{G}_*$). The volume element on the gauge orbit space $\mathcal{C} = \mathcal{A}/\mathcal{G}_*$ is what appears in the functional formulation of gauge theories, eliminating the redundant degrees of freedom corresponding to gauge transformations. Despite being a key foundational ingredient for the quantum description of gauge theories, there is no satisfactory expression for this volume element for four-dimensional non-Abelian gauge theories [2]. While perturbation theory is well understood by use of gauge fixing and Faddeev-Popov ghosts, or, equivalently, via the Becchi-Rouet-Stora-Tyutin procedure, an analytic approach to nonperturbative questions such as confinement remains elusive.

The motivation to consider non-Abelian gauge theories on the manifold $\mathbb{CP}^2$ is from two and three dimensions. In two dimensions, it is possible to calculate the volume element for $\mathcal{C}$ exactly in terms of a Wess-Zumino-Witten (WZW) action [3]. The same calculation placed within a Hamiltonian formulation of $(2+1)$-dimensional gauge theories has led to an analytic formula for the string tension [4,5] which agrees very well, to within about 2%, with the results from numerical simulations [6]. Another prediction regarding the Casimir effect for non-Abelian gauge theories also seems to agree, to within one percent or so, with numerical estimates [7]. The key feature which facilitated such calculations was the complex structure of the two-dimensional spaces and an associated parametrization of gauge potentials which allowed for factoring out gauge transformations and reduction to gauge-invariant degrees of freedom in a simple way. The manifold $\mathbb{CP}^2$ then emerges as the natural candidate for a similar scenario in four dimensions. This is a complex Kähler manifold with a standard choice of metric as the Fubini-Study metric which is given in local complex coordinates z^a, $\bar{z}^{\bar{a}}$, $a = 1, 2$, $\bar{a} = 1, 2$, by

$$ds^2 = \frac{dz \cdot d\bar{z}}{(1 + z \cdot \bar{z}/r^2)} - \frac{\bar{z} \cdot dz z \cdot d\bar{z}}{r^2(1 + z \cdot \bar{z}/r^2)^2} = g_{a\bar{a}} dz^a d\bar{z}^{\bar{a}}. \quad (1)$$

Here r is a scale parameter defining the volume of the space as $\pi^2 r^4/2$. As $r \to \infty$, the metric becomes flat (albeit modulo some global issues), so that one can compare results with expectations in flat space. The main advantage is that $\mathbb{CP}^2 = SU(3)/U(2)$, so that, utilizing group

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theoretic techniques, one can obtain a parametrization of gauge fields similar to what was obtained in two dimensions. This was indicated in [8] and explained in detail in [1]. In the latter paper, we calculated the leading quantum corrections, i.e., monomials of fields and derivatives of the lowest dimensions generated by loops, due to a chiral scalar field on $\mathbb{CP}^2$ coupled to gauge fields. The effective action from integrating out the scalar fields comprised of a quadratic divergence corresponding to a possible gauge-invariant mass term, standard logarithmic divergences corresponding to wave function and/or coupling constant renormalization and a finite WZW action, which is a dimensionally upgraded version of the 2d-WZW action. The natural next set of questions will be about contributions due to the gauge fields themselves, with the volume element on $\mathcal{C}$ being a key part of the one-loop results. This is the subject of the present paper.

The organization of the paper is as follows. The parametrization of the gauge fields and the factoring out of the gauge degrees of freedom are reviewed in Sec. II. The formal expression for the volume element for the gauge orbit space $\mathcal{C} = \mathcal{A}/\mathcal{G}_*$ is given in Sec. III, where we also identify the relevant Jacobian determinant to be calculated. In addition to the scalar propagator on $\mathbb{CP}^2$ (with hypercharge $Y = 0$), which was calculated in [1], we will need the propagator for an antisymmetric rank-2 tensor (with $Y = -2$). This is calculated in Sec. IV. A covariant point-splitting regularization, consistent with the isometries of $\mathbb{CP}^2$, is discussed in Sec. V. In Sec. VI, we give the key results of our calculations. We have calculated the terms of the lowest scaling dimension (≤4), which are presumably the most relevant for the long-wavelength modes of the fields. These include a four-dimensional WZW term with a finite coefficient, a mass-like term with a quadratically divergent coefficient and a set of log-divergent terms of dimension 4. The physical implications of these results are discussed in Sec. VII. There are two appendixes which give details of the calculations for the WZW term and the UV-divergent terms.

II. PARAMETRIZATION OF GAUGE FIELDS

We start by recalling the parametrization of the gauge fields introduced in [1]. The manifold $\mathbb{CP}^2$ is taken to be the group coset space $\mathbb{CP}^2 = SU(3)/U(2)$, so that it can be coordinatized in terms of a group element $g \in SU(3)$, with identification $g \sim gh$, $h \in U(2) \subset SU(3)$. Thus $U(2)$ defines the local isotropy group. As a result, vectors, tensors, etc., transform as specific nontrivial representations of $U(2)$.

Consider the group $g \in SU(3)$ defined in its fundamental representation as a 3×3 unitary matrix g of unit determinant. It may be taken to be of the form $g = \exp(it_a \varphi^a)$, where t_a form a basis for traceless Hermitian 3×3 matrices, with $\mathrm{Tr}(t_a t_b) = \frac{1}{2}\delta_{ab}$, and φ^a are the coordinates for $SU(3)$. The $SU(2) \subset U(2)$ subgroup is the standard isospin subgroup, defined by the upper left 2×2 block, and corresponding to the generators t_a, $a = 1, 2, 3$. The $U(1)$ part of $U(2)$ is defined by the hypercharge transformations, with the generator $Y = 2t_8/\sqrt{3}$. We also define right translation operators on g by $R_a g = g t_a$. In terms of the frame fields E_i^a for $SU(3)$, we may write these as differential operators:

$$g^{-1}dg = -it_a E_i^a d\varphi^i, \quad R_a = i(E^{-1})_a^i \frac{\partial}{\partial \varphi^i}, \quad R_a g = g t_a. \tag{2}$$

Translations on $\mathbb{CP}^2$ correspond to the coset directions t_a, $a = 4, 5, 6, 7$, and we can define the complex translation operators as

$$R_{\pm 1} = R_4 \pm iR_5, \qquad R_{\pm 2} = R_6 \pm iR_7. \tag{3}$$

These will be denoted by R_i and $R_{\bar{i}}$, $i, \bar{i} = 1, 2$.

Functions on $\mathbb{CP}^2$ are invariant under the $U(2)$ subgroup. So they admit a mode expansion of the form

$$F(g) = \sum_{s,A} C_A^{(s)} D_{A,w}^{(s)}(g) = \sum_{s,A} C_A^{(s)} \langle s, A|\hat{g}|s, w\rangle. \tag{4}$$

Here $\mathcal{D}_{AB}^{(s)}(g) = \langle s, A|\hat{g}|s, B\rangle$ are the finite-dimensional unitary representation matrices for $SU(3)$ and in (4) the states on the right, namely, $|s, w\rangle$ are invariant under the $U(2)$ subgroup of $SU(3)$. The action of R_a on $\mathcal{D}_{AB}^{(s)}(g)$ is given by

$$R_a \langle s,A|\hat{g}|s,B\rangle = \langle s,A|\hat{g}T_a|s,B\rangle = \langle s,A|\hat{g}|s,C\rangle (T_a)_{CB}, \tag{5}$$

where T_a are the matrix representatives of t_a in the representation designated by s. The invariance condition for functions may therefore be stated as

$$T_a|s, w\rangle = 0, \qquad a = 1, 2, 3, 8. \tag{6}$$

In more detail, we can specify a state $|s, A\rangle = |a_1 a_2 \ldots a_p, b_1 b_2 \ldots b_q\rangle$ corresponding to a finite-dimensional $SU(3)$ representation of the form $T_{b_1 b_2 \ldots b_q}^{a_1 a_2 \ldots a_p}$, $a_i, b_j = 1, 2, 3$. We will refer to this as a (p, q)-type representation. These are totally symmetric in all the upper indices a_i's and totally symmetric in all the lower indices b_j's with the trace (or any contraction between any choice of upper and lower indices) vanishing. Under the action of $g \in SU(3)$, $T_{b_1 b_2 \ldots b_q}^{a_1 a_2 \ldots a_p}$ transform as

$$T_{b_1 b_2 \ldots b_q}^{a_1 a_2 \ldots a_p} \to (g^{*a_1 a_1'} g^{*a_2 a_2'} \ldots)(g_{b_1 b_1'} g_{b_2 b_2'} \ldots) T_{b_1' b_2' \ldots b_q'}^{a_1' a_2' \ldots a_p'} \tag{7}$$

For functions, we need $\langle s, A|\hat{g}|s, w\rangle$, with states of the (p, p) type, with $|s, A\rangle$ being general and

$$|s,A\rangle = |333\ldots,333\ldots\rangle. \tag{8}$$

Vectors on $\mathbb{CP}^2$ must transform the same way as $R_{\pm i}$, so they must be doublets of $SU(2)\subset SU(3)$ and must carry hypercharge $Y=\pm 1$. Derivatives of functions of the form $R_{\pm i}f$ obviously satisfy this requirement. Another possibility is given by representations of the type $(p+3,p)$ and $(p,p+3)$, so that a general parametrization of a vector takes the form

$$A_i = -R_i f - \eta_{i\bar{i}}\epsilon^{\bar{i}\bar{j}}\sum_{s,A} C_A^{(s)}\langle s,A|\hat{g}|\bar{j}33\ldots,33\ldots\rangle,$$
$$\bar{A}_{\bar{i}} = -R_{\bar{i}}\bar{f} - \eta_{\bar{i}i}\epsilon^{ij}\sum_{s^*,A} C_A^{(s^*)}\langle s^*,A|\hat{g}|33\ldots,j33\ldots\rangle. \tag{9}$$

The particular state $|\bar{j}33\ldots,33\ldots\rangle$ [of the $(p+3,p)$ type] can be obtained, by the application of $R_{\bar{j}}$, from the $SU(2)$ invariant states, with all indices equal to 3, as

$$\eta_{i\bar{i}}\epsilon^{\bar{i}\bar{j}}|\bar{j}33\ldots,33\ldots\rangle = \eta_{i\bar{i}}\epsilon^{\bar{i}\bar{j}}R_{\bar{j}}|33\ldots,33\ldots\rangle, \tag{10}$$

where $\eta_{i\bar{i}}=\delta_{i\bar{i}}$ is the metric for $\mathbb{CP}^2$ in the tangent frame and $\epsilon^{\bar{i}\bar{j}}$ is the Levi-Civita tensor. With a similar result for the conjugate representation, the parametrization (9) can be written as

$$A_i = -R_i f - \eta_{i\bar{i}}\epsilon^{\bar{i}\bar{j}}R_{\bar{j}}\chi,$$
$$\bar{A}_{\bar{i}} = -R_{\bar{i}}\bar{f} - \eta_{\bar{i}i}\epsilon^{ij}R_j\bar{\chi}. \tag{11}$$

In a coordinate basis, rather than the tangent frames we have used above, this becomes

$$A_k = -\nabla_k f + g_{k\bar{k}}\bar{\nabla}_{\bar{m}}\chi^{\bar{k}\bar{m}}, \quad \bar{A}_{\bar{k}} = \bar{\nabla}_{\bar{k}}\bar{f} - g_{\bar{k}k}\nabla_m\bar{\chi}^{km}, \tag{12}$$

where $\chi^{\bar{k}\bar{m}}$ and $\bar{\chi}^{km}$ are antisymmetric rank-two tensors, which for dimensional reasons, are proportional to $\epsilon^{\bar{k}\bar{m}}$ and ϵ^{km} and so can be reduced to χ and $\bar{\chi}$.

The non-Abelian generalization of (12) leads to the following parametrization for the gauge potentials:

$$A_i = -\nabla_i M M^{-1} + g_{i\bar{i}}\bar{D}_{\bar{j}}\phi^{\bar{i}\bar{j}},$$
$$\bar{A}_{\bar{i}} = M^{\dagger-1}\bar{\nabla}_{\bar{i}}M^\dagger - g_{\bar{i}i}D_j\phi^{\dagger ij}. \tag{13}$$

In these expressions, M and $M^\dagger$ are complex matrices which are elements of $SL(N,\mathbb{C})$ if the gauge group is $SU(N)$; i.e., they are $N\times N$ complex matrices with determinant equal to 1. (More generally, M and $M^\dagger$ will be in the complexification of the gauge group.) Further, $\phi^{\bar{i}\bar{j}}=\epsilon^{\bar{i}\bar{j}}\phi$ and $\phi^{\dagger ij}=\epsilon^{ij}\phi^\dagger$ are also complex $N\times N$ matrices in the Lie algebra of $SL(N,\mathbb{C})$. We also use an anti-Hermitian basis for the fields, so that $\bar{A}_{\bar{i}}=-(A_i)^\dagger$. The derivatives D_j and $\bar{D}_{\bar{j}}$ in (13) are defined by

$$D_j\Phi = \nabla_j\Phi + [-\nabla_j M M^{-1},\Phi],$$
$$\bar{D}_{\bar{j}}\Phi = \bar{\nabla}_{\bar{j}}\Phi + [M^{\dagger-1}\bar{\nabla}_{\bar{j}}M^\dagger,\Phi]. \tag{14}$$

We have written these in terms of the action on a field Φ (like ϕ or $\phi^\dagger$) which transforms under the adjoint representation of the gauge group, i.e., as $\Phi\to U\Phi U^\dagger$, where $U\in SU(N)$ is the gauge transformation. The gauge transformation of the matrices M and $M^\dagger$ is given by $M\to UM$ and $M^\dagger\to M^\dagger U^\dagger$. We can then see that the potentials in (13) transform as connections. It is sufficient to use just $(-\nabla_j MM^{-1}, M^{\dagger-1}\bar{\nabla}_{\bar{j}}M^\dagger)$ in D_j and $\bar{D}_{\bar{j}}$ to ensure that $D_j\Phi$ and $\bar{D}_{\bar{j}}\Phi$ transform covariantly under gauge transformations. $(\nabla_j,\bar{\nabla}_{\bar{j}})$ are also taken to be Levi-Civita covariant, so that (13) behave correctly under gauge and coordinate transformations.

It is possible to write the parametrization (13) in terms of manifestly gauge-invariant fields by using the identities

$$\begin{aligned}\bar{D}_{\bar{j}}\phi^{\bar{i}\bar{j}} &= \bar{\nabla}_{\bar{j}}\phi^{\bar{i}\bar{j}} + [M^{\dagger-1}\bar{\nabla}_{\bar{j}}M^\dagger,\phi^{\bar{i}\bar{j}}]\\ &= M[\bar{\nabla}_{\bar{j}}(M^{-1}\phi M)^{\bar{i}\bar{j}} + [H^{-1}\bar{\nabla}_{\bar{j}}H,(M^{-1}\phi M)^{\bar{i}\bar{j}}]]M^{-1}\\ &= M(\bar{\mathcal{D}}_{\bar{j}}(M^{-1}\phi M)^{\bar{i}\bar{j}})M^{-1} = M(\bar{\mathcal{D}}_{\bar{j}}\chi^{\bar{i}\bar{j}})M^{-1},\\ D_j\phi^{\dagger ij} &= M^{\dagger-1}(\mathcal{D}_j\chi^{\dagger ij})M^\dagger.\end{aligned} \tag{15}$$

Here $\chi^{\bar{i}\bar{j}}=\epsilon^{\bar{i}\bar{j}}(M^{-1}\phi M)$, $\chi^{\dagger ij}=\epsilon^{ij}(M^\dagger\phi^\dagger M^{\dagger-1})$ and $H=M^\dagger M$. The derivatives $\bar{\mathcal{D}}_{\bar{j}}$ and $\mathcal{D}_j$ are defined using the connections $H^{-1}\bar{\nabla}_{\bar{j}}H$ and $-\nabla_j HH^{-1}$; i.e.,

$$\bar{\mathcal{D}}_{\bar{j}}\Phi = \bar{\nabla}_{\bar{j}}\Phi + [H^{-1}\bar{\nabla}_{\bar{j}}H,\Phi],$$
$$\mathcal{D}_j\Phi = \nabla_j\Phi + [-\nabla_j HH^{-1},\Phi]. \tag{16}$$

By virtue of (15), the parametrization (13) can be written as

$$A_i = -\nabla_i MM^{-1} + M(g_{i\bar{i}}\bar{\mathcal{D}}_{\bar{j}}\chi^{\bar{i}\bar{j}})M^{-1},$$
$$\bar{A}_{\bar{i}} = M^{\dagger-1}\bar{\nabla}_{\bar{i}}M^\dagger + M^{\dagger-1}(-g_{\bar{i}i}\mathcal{D}_j\chi^{\dagger ij})M^\dagger. \tag{17}$$

Another equivalent version is given by

$$\begin{aligned}A_i &= -\nabla_i MM^{-1} - Ma_iM^{-1},\\ \bar{A}_{\bar{i}} &= M^{\dagger-1}\bar{\nabla}_{\bar{i}}M^\dagger + M^{\dagger-1}\bar{a}_{\bar{i}}M^\dagger,\\ a_i &= -M^{-1}g_{i\bar{i}}\bar{D}_{\bar{j}}\phi^{\bar{i}\bar{j}}M = -g_{i\bar{i}}\bar{\mathcal{D}}_{\bar{j}}\chi^{\bar{i}\bar{j}},\\ \bar{a}_{\bar{i}} &= -M^\dagger g_{\bar{i}i}D_j\phi^{\dagger ij}M^{\dagger-1} = -g_{\bar{i}i}\mathcal{D}_j\chi^{\dagger ij} = a_i{}^\dagger.\end{aligned} \tag{18}$$

Notice that a_i and $\bar{a}_{\bar{i}}$ obey the conditions

$$g^{\bar{k}i}\bar{\mathcal{D}}_{\bar{k}}a_i = -\bar{\mathcal{D}}_{\bar{i}}\bar{\mathcal{D}}_{\bar{j}}\chi^{\bar{i}\bar{j}} = 0, \qquad g^{k\bar{i}}\mathcal{D}_k\bar{a}_{\bar{i}} = 0, \tag{19}$$

so that, effectively, they have only one independent component each.

The gauge-invariant degrees of freedom are given by $H = M^\dagger M$ and $\chi = M^{-1}\phi M$, $\chi^\dagger = M^\dagger\phi^\dagger M^{\dagger -1}$. Equivalently, they may be taken as $H = M^\dagger M$ and a_i, $\bar{a}_{\bar{i}}$, where there are the additional constraints (19). Yet another choice, also equivalent to the above mentioned ones, would be $\chi' = M^\dagger \phi M^{\dagger -1}$, $\chi'^\dagger = M^{-1}\phi^\dagger M$ and $H = M^\dagger M$. From the point of view of carrying out the functional integration, these fields are the coordinates for the gauge-orbit space $\mathcal{C}$. A polar decomposition of M as $M = U\rho$, where ρ is Hermitian, allows us to factor out the gauge degrees of freedom and define a volume measure on $\mathcal{C} = \mathcal{A}/\mathcal{G}_*$. This will be taken up in the next section.

An interesting feature which is worthy of comment is the holomorphic ambiguity or redundancy of the parametrization (13) or (18). Notice that $(M, a_i, M^\dagger, \bar{a}_{\bar{i}})$ and $(M\bar{V}(\bar{x}), \bar{V}^{-1}(\bar{x})a_i\bar{V}(\bar{x}), V(x)M^\dagger, V(x)\bar{a}_{\bar{i}}V^{-1}(x))$ lead to the same gauge potentials, where $V(x)$ is an $SL(N,\mathbb{C})$ matrix with matrix elements which are holomorphic functions and $\bar{V}(\bar{x})$ is a similar antiholomorphic matrix. Thus there is a certain ambiguity in how the matrices M and $M^\dagger$ can be chosen if $(A_i, \bar{A}_{\bar{i}})$ are given. On $\mathbb{CP}^2$, there are no globally defined holomorphic or antiholomorphic functions, except for a constant. So there are no additional degrees of freedom associated with this. However, as mentioned in [1], this feature may be useful with locally defined V and $\bar{V}$ to express fields in a nonsingular way in various coordinate patches.

We close this section with another comment on the uniqueness of the parametrization of the fields. This will be important for the metric and volume element which we consider in the next section. We have argued, based on the group theoretic counting of functional degrees of freedom, that any vector can be parametrized as in (11); hence any gauge potential on $\mathbb{CP}^2$ can be written in terms of $M, M^\dagger, \chi$, and $\chi^\dagger$. The construction of A_i and $\bar{A}_{\bar{i}}$ from the data $(M, M^\dagger, \chi, \chi^\dagger)$ is thus clear. Conversely, we can ask whether we can construct $(M, M^\dagger, \chi, \chi^\dagger)$ from $(A_i, \bar{A}_{\bar{i}})$. This can indeed be done, as explained in some detail in [1]. The fields $(M, M^\dagger, \chi, \chi^\dagger)$ will be nonlocal functions of $A_i, \bar{A}_{\bar{i}}$ and their derivatives, consistent with the fact that there is no way to factor out the gauge degrees of freedom and obtain (unconstrained) gauge-invariant degrees of freedom in a local way in terms of A_i and $\bar{A}_{\bar{i}}$.

III. THE METRIC AND VOLUME ELEMENT FOR $\mathcal{C}$

We now turn to the metric on the space of gauge potentials ($\mathcal{A}$) and the reduction of the associated volume element to the gauge orbit space $\mathcal{A}/\mathcal{G}_*$. The starting point is the standard Euclidean metric on the space of the fields A, given by

$$ds^2 = -\int d\mu g^{i\bar{i}} \mathrm{Tr}(\delta A_i \delta \bar{A}_{\bar{i}}). \tag{20}$$

Here $d\mu$ denotes the volume element for $\mathbb{CP}^2$. In terms of the parametrization of the fields (13), we find

$$\begin{aligned}\delta A_i &= -D_i\theta + g_{i\bar{i}}\epsilon^{\bar{i}\bar{j}}(\bar{D}_{\bar{j}}\delta\phi + [\bar{D}_{\bar{j}}\theta^\dagger, \phi])\\ &= -D_i\theta + [\theta^\dagger, M a_i M^{-1}] + g_{i\bar{i}}\epsilon^{\bar{i}\bar{j}}\bar{D}_{\bar{j}}\delta\phi',\\ \delta\bar{A}_{\bar{i}} &= \bar{D}_{\bar{i}}\theta^\dagger - g_{\bar{i}i}\epsilon^{ij}(D_j\delta\phi^\dagger - [D_j\theta, \phi^\dagger])\\ &= \bar{D}_{\bar{i}}\theta^\dagger + [\theta, M^{\dagger -1}\bar{a}_{\bar{i}}M^\dagger] - g_{\bar{i}i}\epsilon^{ij}D_j\delta\phi'^\dagger,\end{aligned} \tag{21}$$

where $\theta = \delta M M^{-1}$, $\theta^\dagger = M^{\dagger -1}\delta M^\dagger$, $\delta\phi' = \delta\phi + [\theta^\dagger, \phi]$, and $\delta\phi'^\dagger = \delta\phi^\dagger - [\theta, \phi^\dagger]$. We have also used the definition of a_i, $\bar{a}_{\bar{i}}$ from (18). Upon using (21) in (20) and carrying out some integrations by parts, we find

$$\begin{aligned}ds^2 = \int d\mu \mathrm{Tr}[&\theta^\dagger(-\bar{D}\cdot D)\theta + \delta\phi'(-\bar{D}\cdot D)\delta\phi'^\dagger\\ &+ \theta^\dagger g^{i\bar{i}}[M a_i M^{-1}, [M^{\dagger -1}\bar{a}_{\bar{i}}M^\dagger, \theta]]\\ &+ \theta g^{i\bar{i}}[M^{\dagger -1}\bar{a}_{\bar{i}}M^\dagger, D_i\theta] - \theta^\dagger g^{i\bar{i}}[M a_i M^{-1}, \bar{D}_{\bar{i}}\theta]\\ &+ \theta\epsilon^{\bar{i}\bar{j}}[M^{\dagger -1}\bar{a}_{\bar{i}}M^\dagger, \bar{D}_{\bar{j}}\delta\phi']\\ &- \theta^\dagger\epsilon^{ij}[M a_i M^{-1}, D_j\delta\phi'^\dagger]].\end{aligned} \tag{22}$$

We can write this as a quadratic form:

$$ds^2 = \frac{1}{2}\int d\mu\, \xi^{\dagger\alpha}\mathcal{M}_{\alpha\beta}\xi^\beta, \qquad \xi = (\theta, \theta^\dagger, \delta\phi', \delta\phi'^\dagger). \tag{23}$$

We have written this in terms of components in the Lie algebra of the gauge group $SU(N)$ by taking $\xi = \xi^\alpha(-it^\alpha)$, where t^α form a basis for the Lie algebra. It is useful to compare this with the metric

$$\begin{aligned}ds_0^2 &= \frac{1}{2}\int d\mu\, \xi^{\dagger\alpha}\xi^\alpha = \int d\mu[\theta^{\dagger\alpha}\theta^\alpha + \delta\phi'^\alpha\delta\phi'^{\dagger\alpha}]\\ &= \int d\mu[(M^{\dagger -1}\delta M^\dagger)^\alpha(\delta M M^{-1})^\alpha + H^{\alpha\beta}_{\mathrm{Adj}}\delta\chi'^\alpha\delta\chi'^{\dagger\beta}].\end{aligned} \tag{24}$$

In the second line, we used the fact that $\delta\chi' = \delta(M^\dagger\phi M^{\dagger -1}) = M^\dagger\delta\phi' M^{\dagger -1}$ and its conjugate version to rewrite the last term. Further $H^{\alpha\beta}_{\mathrm{Adj}}$ is the adjoint representation of H defined by $H^{\alpha\beta}_{\mathrm{Adj}} = -2\mathrm{Tr}(H^{-1}t^\alpha H t^\beta)$.

From the structure of (23), we see that the volume element takes the form $dV = \sqrt{\det\mathcal{M}}dV_0$, where dV_0 is the volume element corresponding to (24). The first term in (24) is the integral over $\mathbb{CP}^2$ of the Cartan-Killing metric for $SL(N,\mathbb{C})$. The corresponding volume element is thus the product over all points of the Haar measure for $SL(N,\mathbb{C})$. Using the polar decomposition $M = U\rho$, where ρ is Hermitian, and writing out the differential form of the top rank, we can see that this gives the volume element $\prod_x [dU]d\mu(H)$, where dU is the volume of $SU(N)$ and $d\mu(H)$ is the Haar measure for $SL(N,\mathbb{C})/SU(N)$. Further, since $H^{\alpha\beta}$ has unit determinant, we can write the volume element of ds_0^2 as

$$dV_0 = \prod_x [dU] d\mu(H) d\chi' d\chi'^\dagger. \tag{25}$$

Finally going back to (23), we get the corresponding volume element as

$$dV = \sqrt{\det \mathcal{M}} \prod_x [dU] d\mu(H) d\chi' d\chi'^\dagger. \tag{26}$$

We can now factor out the volume of gauge transformations, i.e., factor out $\prod_x [dU]$ to obtain the volume element for $\mathcal{C} = \mathcal{A}/\mathcal{G}_*$ as

$$d\mu[\mathcal{C}] = \sqrt{\det \mathcal{M}} \prod_x d\mu(H) d\chi' d\chi'^\dagger. \tag{27}$$

The differentials appearing in this expression are for gauge-invariant fields.

The next step is the calculation of the determinant of $\mathcal{M}$. For this, it is simpler to write the determinant as a functional integral over a set of auxiliary fields B, $\bar{B}$, C, and $\bar{C}$. Explicitly,

$$\frac{1}{\sqrt{\det \mathcal{M}}} = \int [dB d\bar{B} dC d\bar{C}] e^{-S_0 - S_1}, \tag{28}$$

where

$$S_0 = \int d\mu [\bar{C}^\alpha (-\bar{D} \cdot D)^{\alpha\beta} C^\beta + B^\alpha (-\bar{D} \cdot D)^{\alpha\beta} \bar{B}^\beta], \tag{29}$$

$$\begin{aligned} S_1 = \int d\mu [&\bar{C}^\alpha (M a M^{-1} \cdot M^{\dagger -1} \bar{a} M^\dagger)^{\alpha\beta} C^\beta \\ &+ C^\alpha (M^{\dagger -1} \bar{a} M^\dagger \cdot D)^{\alpha\beta} C^\beta + \bar{C}^\alpha (-M a M^{-1} \cdot \bar{D})^{\alpha\beta} \bar{C}^\beta \\ &+ C^\alpha (-\epsilon^{\bar{i}\bar{j}} M^{\dagger -1} \bar{a}_{\bar{i}} M^\dagger \bar{D}_{\bar{j}})^{\alpha\beta} B^\beta \\ &+ \bar{C}^\alpha (\epsilon^{ij} M a_i M^{-1} D_j)^{\alpha\beta} \bar{B}^\beta]. \end{aligned} \tag{30}$$

Equation (28) expresses the required determinant in the form of a functional integral for a standard field theory. We have taken C and $\bar{C}$ to behave like θ and $\theta^\dagger$, so they are scalar fields. B and $\bar{B}$ correspond to $\delta\phi'$ and $\delta\phi'^\dagger$, so they are fields with $Y = 2$ and -2, respectively. The terms in S_1 involve powers of a and $\bar{a}$, while the differential operators in S_0 only depend on H via the covariant derivatives. Our strategy will be to evaluate the integral in (28) as a perturbation series by expanding e^{-S_1} in powers of S_1. The integral over e^{-S_0} will define the lowest-order result. Writing $\Gamma = \log \sqrt{\det \mathcal{M}}$, we find $\Gamma = \Gamma_0 + \Delta\Gamma$, with

$$\begin{aligned} e^{\Gamma_0} &= [\det(-\bar{D} \cdot D)_{Y=0}][\det(-\bar{D} \cdot D)_{Y=-2}], \\ e^{-\Delta\Gamma} &= \langle e^{-S_1} \rangle = \left[\frac{1}{\int e^{-S_0}}\right] \int [dB d\bar{B} dC d\bar{C}] e^{-S_0 - S_1}. \end{aligned} \tag{31}$$

The evaluation of the determinant $\det(-\bar{D} \cdot D)_{Y=0}$ was considered in [1]. This was done by writing its variation as

$$\delta(\mathrm{Tr} \log(-\bar{D} \cdot D)_{Y=0}) = \int d\mu \mathrm{Tr}[\delta(M^{\dagger -1} \bar{\nabla} M^\dagger) M^{\dagger -1} \langle \hat{J} \rangle M^\dagger + \text{Hermitian conjugate}], \tag{32}$$

where

$$\langle \hat{J}(x) \rangle = -\mathcal{D}_x \mathcal{G}(x,y)|_{y \to x}, \quad \mathcal{G}(x,y) = \left(\frac{1}{-\bar{\nabla} \cdot \mathcal{D}}\right)_{x,y}. \tag{33}$$

The free "propagator" $G = (-\bar{\nabla} \cdot \nabla)^{-1}$ for scalar fields on $\mathbb{CP}^2$ was calculated exactly. The propagator $\mathcal{G}(x,y)$ could then be obtained by expansion in powers of $\nabla H H^{-1}$, since $\mathcal{D}$, as defined in (16), involves this connection. With regularization to take care of singularities as $y \to x$ in the expression for $\langle \hat{J}(x) \rangle$, we obtained the leading terms in the expansion of $\log \det(-\bar{D} \cdot D)_{Y=0}$ in terms of the monomials of H and its derivatives of increasing scaling dimension. The lowest-order term was a WZW action for H, somewhat surprisingly, with a finite coefficient. The next set of terms of dimension four were the H-dependent part of $\mathrm{Tr}(F^2)$ and $\mathrm{Tr}(g^{i\bar{i}} F_{i\bar{i}})^2$, with a logarithmically divergent coefficient. There are also terms of higher dimension which will have finite coefficients; these were not explicitly calculated in [1]. We do not display the leading terms of $\det(-\bar{D} \cdot D)_{Y=0}$ at this point; they will be given later, together with some of the other terms in (31) which will be calculated presently.

The calculation of the leading terms in $\det(-\bar{D} \cdot D)_{Y=-2}$ will proceed as in the case of $\det(-\bar{D} \cdot D)_{Y=0}$. Since we will again expand in powers of $\nabla H H^{-1}$, the key ingredient for this will be the free propagator $(-\bar{\nabla} \cdot \nabla)^{-1}$, now for $Y = -2$ fields. This, as well as the issue of regularization for such propagators, will be taken up in the next section.

Once we have the propagators, the calculation of terms resulting from S_1 will be straightforward. We will focus on the terms of scaling dimension ≤ 4, corresponding to possible quadratic and logarithmic divergences. For these terms, it will suffice to consider terms up to four powers of S_1.

IV. THE PROPAGATOR FOR $Y = -2$ FIELDS

In this section, we calculate the propagator for fields with $Y = -2$, such as the field $\bar{B}$ in (29) and (30). This field was identified as the variation of $\phi^\dagger$ in $\phi^{\dagger ij} = \epsilon^{ij} \phi^\dagger$, where $\phi^{\dagger ij}$ is a second-rank antisymmetric tensor. The Laplace operator on a second-rank antisymmetric tensor with holomorphic indices will lead to the equation for the propagator for $\bar{B}$. Rather than using the tangent frame basis, if we use the coordinate basis for $\mathbb{CP}^2$, we can write

$$\delta\phi^{\dagger ij} = \frac{\epsilon^{ij}}{\sqrt{g}} \delta\phi^\dagger = \frac{\epsilon^{ij}}{\sqrt{g}} \bar{B}. \tag{34}$$

Since $\mathbb{CP}^2$ is a Kähler manifold with potential $K = \log(1 + \bar{z} \cdot z)$, the metric and Levi-Civita connections are easily worked out as

$$
\begin{aligned}
g_{a\bar{a}} &= \frac{\eta_{a\bar{a}}}{(1+\bar{z}\cdot z)} - \frac{\eta_{a\bar{b}}\eta_{\bar{a}b}\bar{z}^{\bar{b}}z^b}{(1+\bar{z}\cdot z)^2},\\
g^{a\bar{a}} &= (1+\bar{z}\cdot z)(\eta^{a\bar{a}} + z^a\bar{z}^{\bar{a}}),\\
\Gamma^a_{bc} &= -\frac{(\delta^a_b\eta_{c\bar{c}} + \delta^a_c\eta_{b\bar{c}})\bar{z}^{\bar{c}}}{(1+\bar{z}\cdot z)}, \qquad \Gamma^{\bar{a}}_{\bar{b}\,\bar{c}} = (\Gamma^a_{bc})^*. \qquad (35)
\end{aligned}
$$

The other components of the connection vanish. Using these results, we find

$$
\begin{aligned}
&\int d\mu g_{\bar{i}m}(\bar{\nabla}_{\bar{j}}\delta\phi^{\bar{i}\,\bar{j}})(\nabla_n\delta\phi^{\dagger mn})\\
&= \int d\mu B\Big[-(1+\bar{z}\cdot z)(\bar{\partial}\cdot\partial + \bar{z}\cdot\bar{\partial}z\cdot\partial) + 3 + \frac{9}{4}\bar{z}\cdot z\\
&\quad -\frac{3}{2}(1+\bar{z}\cdot z)(z\cdot\partial - \bar{z}\cdot\bar{\partial})\Big]\bar{B}. \qquad (36)
\end{aligned}
$$

This identifies the kinetic operator on $\bar{B}$. The propagator thus obeys the equation

$$
\begin{aligned}
&\Big[-(1+\bar{z}\cdot z)(\bar{\partial}\cdot\partial + \bar{z}\cdot\bar{\partial}z\cdot\partial) + 3 + \frac{9}{4}\bar{z}\cdot z\\
&\quad -\frac{3}{2}(1+\bar{z}\cdot z)(z\cdot\partial - \bar{z}\cdot\bar{\partial})\Big]\tilde{G}(z,y) = \frac{\delta^{(4)}(z,y)}{(\det g)}, \qquad (37)
\end{aligned}
$$

where $\tilde{G}(z,y)$ is the free propagator for $Y=-2$ fields, to be distinguished from $G(z,y)$, which denotes the free propagator for scalars with $Y=0$. In the case of scalar fields, we could take the propagator G to be a function of the distance between the two points corresponding to z and y. However, in the present case, there is an additional phase factor due to the presence of the operator $z\cdot\partial - \bar{z}\cdot\bar{\partial}$ in (37). This is ultimately due to $Y=-2$ for the field, which implies that it couples to the Y component of the curvature and connection. To isolate the phase factor, we note that the $\bar{B}$ field has the mode expansion

$$
\bar{B} = \sum_{s,A}\bar{B}^{(s)}_A\langle s,A|\hat{g}|\underbrace{33\ldots}_{p};\underbrace{33\ldots}_{p+3}\rangle \qquad (38)
$$

As a result, the propagator for the $Y=-2$ fields takes the form

$$
\begin{aligned}
\tilde{G}(z,y) &= \sum_p \frac{d_p}{(p+2)(p+3)}\langle\underbrace{33\ldots}_{p};\underbrace{33\ldots}_{p+3}|\hat{g}^\dagger_y\hat{g}_z|\underbrace{33\ldots}_{p};\underbrace{33\ldots}_{p+3}\rangle\\
&\sim [(g^\dagger_y g_z)_{33}]^3\sum[[(g^\dagger_y g_z)_{33}(g^\dagger_z g_y)_{33}]^p + \cdots], \qquad (39)
\end{aligned}
$$

where $d_p = \frac{1}{2}(p+1)(p+4)(2p+5)$ is the dimension of the $(p,p+3)$ representation. In the second line of this equation, we have indicated the result in terms of products of $g^\dagger_y g_z$ and its conjugate, where g is the 3×3 matrix $g\in SU(3)$ used to define local coordinates in the coordinate patch we are using. There are also lower powers of $(g^\dagger_y g_z)_{33}(g^\dagger_z g_y)_{33}$ for each representation, as indicated by the ellipsis in the square brackets. The point is that there is a common term $[(g^\dagger_y g_z)_{33}]^3$ which leads to a phase factor. To see this more explicitly, we note that $g\in SU(3)$ is related to the local coordinates as

$$
(g_z)_{i3} = \frac{z^i}{\sqrt{1+\bar{z}\cdot z}}, \quad i=1,2, \quad (g_z)_{33} = \frac{1}{\sqrt{1+\bar{z}\cdot z}}. \qquad (40)
$$

The square of the distance, s, between two points with local coordinates z^i, y^i can be taken to be

$$
\begin{aligned}
s(z,y) = \sigma^2_{z,y} &= \frac{1}{(g^\dagger_y g_z)_{33}(g^\dagger_z g_y)_{33}} - 1\\
&= \frac{(1+\bar{z}\cdot z)(1+\bar{y}\cdot y)}{(1+\bar{y}\cdot z)(1+\bar{z}\cdot y)} - 1. \qquad (41)
\end{aligned}
$$

This is clearly translationally invariant [under left translations $g\to ug$, $u\in SU(3)$] and agrees with $s=\bar{z}\cdot z$ when $y=0$, i.e., for $g_y=1$. Thus, we see from (39) that $\tilde{G}$ is a function of s, apart from the prefactor

$$
\begin{aligned}
[(g^\dagger_y g_z)_{33}]^3 &= \left[\frac{(g^\dagger_y g_z)_{33}}{(g^\dagger_z g_y)_{33}}\right]^{\frac{3}{2}}|(g^\dagger_y g_z)_{33}|^3 = \left[\frac{(g^\dagger_y g_z)_{33}}{(g^\dagger_z g_y)_{33}}\right]^{\frac{3}{2}}\frac{1}{(1+s)^{\frac{3}{2}}}\\
&= \left[\frac{(1+\bar{y}\cdot z)}{(1+\bar{z}\cdot y)}\right]^{\frac{3}{2}}\frac{1}{(1+s)^{\frac{3}{2}}}. \qquad (42)
\end{aligned}
$$

Combining this with (39) we see that a general ansatz for the propagator can be taken as

$$
\tilde{G}(z,y) = \left[\frac{(1+\bar{y}\cdot z)}{(1+\bar{z}\cdot y)}\right]^{\frac{3}{2}}F(s). \qquad (43)
$$

Having identified the phase factor, we can now use (37) to calculate $F(s)$. For $z\neq y$, or $s\neq 0$, the δ function has no support and we can check that the homogeneous equation is solved by

$$
\begin{aligned}
F(s) &= C_1(1+s)^{\frac{3}{2}}\left[\frac{1+2s}{s(1+s)} + 2\log\left(\frac{s}{1+s}\right)\right]\\
&\quad + C_2(1+s)^{\frac{3}{2}}, \qquad (44)
\end{aligned}
$$

where C_1 and C_2 are arbitrary constants. Considering the short-distance behavior, we see that we need $C_1=\frac{1}{2}$ to reproduce the δ function on the right-hand side of (37). Further, there should be no singularity as $s\to\infty$; this identifies $C_2=0$. Combining (43) and (44), we get the propagator for $Y=-2$ fields as

$$\tilde{G}(z,y) = (1+s)^{\frac{3}{2}}\left[\frac{1}{2s}+\frac{1}{2(1+s)}+\log\left(\frac{s}{1+s}\right)\right]\left[\frac{(1+\bar{y}\cdot z)}{(1+\bar{z}\cdot y)}\right]^{\frac{3}{2}} = F(s)\left[\frac{(1+\bar{y}\cdot z)}{(1+\bar{z}\cdot y)}\right]^{\frac{3}{2}}, \tag{45}$$

where $s = \sigma_{z,y}^2$ is as given in (41). The last factor which is the phase can also be viewed as arising from the path-ordered integral of the Levi-Civita connection along a line joining y and z.

V. REGULARIZATION

The regularization of the propagator for $Y = -2$ fields will use the same point splitting which was used for scalar fields in [1]. The key idea was to define the regularized version of $G(z,y)$ as $G(z,y')$, where y' specifies a point displaced from y by a small distance of order $\sqrt{\epsilon}$. Thus ϵ will serve as the regularization parameter, with $\epsilon \to 0$ recovering the unregularized results. The shift $y \to y'$ must be done in a way consistent with the isometries and gauge symmetries. It is useful to work this out in terms of the homogeneous coordinates $Z = (Z_1, Z_2, Z_3)$, $Y = (Y_1, Y_2, Y_3)$ for the points z, y, with the identifications $Z \sim \lambda Z$, $Y \sim \lambda' Y$, and $\lambda, \lambda' \in \mathbb{C} - \{0\}$. In terms of these coordinates, the distance between the points, given in (41), can be written as

$$s = \sigma_{z,y}^2 = \frac{\bar{Z}\cdot Z \bar{Y}\cdot Y}{\bar{Z}\cdot Y \bar{Y}\cdot Z} - 1 \equiv \sigma^2(Z,Y). \tag{46}$$

Notice that this expression has invariance under the scaling $Z \to \lambda Z$, $Y \to \lambda' Y$. In a particular coordinate patch with $Z^3, Y^3 \neq 0$, we can write

$$Z = Z^3(z^1, z^2, 1) = Z^3\sqrt{1+\bar{z}\cdot z}(g_{13}, g_{23}, g_{33}),$$
$$Y = Y^3(y^1, y^2, 1) = Y^3\sqrt{1+\bar{y}\cdot y}(g'_{13}, g'_{23}, g'_{33}), \tag{47}$$

where $z^i = Z^i/Z^3$ and $y^i = Y^i/Y^3$. We then recover the expression given in (41). With s written in homogeneous coordinates as in (46), (45) gives a globally valid expression for the propagator.

For the point-splitting regularization, we shift the point Y to Y', which is chosen to be

$$Y' = Y + \alpha\left(\frac{W\bar{Y}\cdot Y}{\bar{Y}\cdot W} - Y\right). \tag{48}$$

Here α is a small complex number, $|\alpha| \sim \sqrt{\epsilon}$, and W parametrizes the shift of coordinates. The added term is chosen so as to have the same scaling behavior as Y. We then find that

$$1 + \sigma^2(Z,Y') = \frac{(1+\sigma^2(Z,Y))(1+\alpha\bar{\alpha}\sigma^2(Y,W))}{[1+\alpha(\frac{\bar{Z}\cdot W\bar{Y}\cdot Y}{\bar{Y}\cdot W\bar{Z}\cdot Y} - 1)][1+\bar{\alpha}(\frac{\bar{W}\cdot Z\bar{Y}\cdot Y}{\bar{W}\cdot Y\bar{Y}\cdot Z} - 1)]}. \tag{49}$$

In Eq. (45), we can replace all factors of s by $\sigma^2(Z,Y')$ given by (49). Under coordinate transformations, the propagator must transform as $u^\dagger(z)\tilde{G}(z,y)u(y)$, where $u^\dagger$ is the hypercharge phase transformation with $Y = -2$. (u will correspond to $Y = 2$.) To preserve this property, we add an extra phase factor joining y and y'. We can view this extra term as a path-ordered integral for the Levi-Civita connection along a line joining y and y', analogous to the Wilson line for the gauge fields (see below) that is needed to maintain gauge covariance of the regulator. Thus the UV-regularized form of the propagator is given by

$$\tilde{G}_{\mathrm{Reg}}(z,y) = F(\sigma^2(Z,Y'))\left[\frac{(1+\bar{y}'\cdot z)}{(1+\bar{z}\cdot y')}\right]^{\frac{3}{2}}\left[\frac{(1+\bar{y}\cdot y')}{(1+\bar{y}'\cdot y)}\right]^{\frac{3}{2}}, \tag{50}$$

where as in [1] we will do a suitable angular averaging over the displacement due to point splitting. (It will turn out that, for some of the calculations we do, one of the arguments of the propagator can be shifted to the origin by virtue of translational invariance. The details of how this can be done in terms of the homogeneous coordinates are given in [1]; see also Appendix A of this paper.)

Since $\sigma^2(Z,Y)$ and $\sigma^2(Z,Y')$ are covariant quantities respecting the full isometry [namely, left $SU(3)$ transformation on Z or Y] of $\mathbb{CP}^2$, the procedure we have outlined provides a covariant point-splitting regularization. However, we will need to modify this slightly to take account of covariance with respect to gauge transformations as well. We have so far considered the free propagator. In the presence of gauge fields, the propagator is $\tilde{\mathcal{G}}(z,y) = ((-\bar{D}\cdot D)^{-1}_{z,y})_{Y=-2}$. We carry out the explicit calculations by expanding this in powers of the gauge field as

$$\tilde{\mathcal{G}}(z,y) = \tilde{G}(z,y) + \int_{y_1}\tilde{G}(z,y_1)\mathbb{V}_{y_1}\tilde{G}(y_1,y) + \int_{y_1,y_2}\tilde{G}(z,y_1)\mathbb{V}_{y_1}\tilde{G}(y_1,y_2)\mathbb{V}_{y_2}\tilde{G}(y_2,y) + \cdots, \tag{51}$$

where $\mathbb{V} = \bar{A}\cdot\partial + A\cdot\bar{\partial} + (\bar{\partial}\cdot A) + \bar{A}\cdot A$. The propagator $\tilde{\mathcal{G}}(z,y)$ transforms as $\tilde{\mathcal{G}}(z,y) \to U(z)\tilde{\mathcal{G}}(z,y)U^\dagger(y)$ under the gauge transformation $M \to UM$, $M^\dagger \to M^\dagger U^\dagger$. This should be respected in calculating currents such as $\langle\hat{J}(z)\rangle = -D_z\tilde{\mathcal{G}}(z,y)]_{y\to z}$ in (32) and (33), with regularization. In other words, even though we shift y to y', the gauge transformation must maintain the action of $U^\dagger(y)$ at the second argument to be consistent with its role in the current $\langle \hat{J}(x) \rangle$. This means that a gauge-invariant point splitting is given by

$$\tilde{\mathcal{G}}_{\rm Reg}(z,y) = \tilde{\mathcal{G}}(z,y')\mathcal{P}\exp\left(-\int_y^{y'}(M^{\dagger-1}\bar{\nabla}M^\dagger - \nabla M M^{-1})\right)$$
$$= \left[\tilde{G}_{\rm Reg}(z,y) + \int_{y_1}\tilde{G}(z,y_1)\mathbb{V}(y_1)\tilde{G}_{\rm Reg}(y_1,y) + \cdots\right]$$
$$\times \mathcal{P}\exp\left(-\int_y^{y+\delta y}(M^{\dagger-1}\bar{\nabla}M^\dagger - \nabla M M^{-1})\right). \tag{52}$$

Here $\tilde{G}_{\rm Reg}$ is as in (50) and $y' = y + \delta y$, with $\delta y^a \delta \bar{y}^{\bar{a}} \to \epsilon \eta^{a\bar{a}}$ in taking the small ϵ limit in a symmetric way. (This is for the case when y can be set to zero, which will cover the calculations for which we need this factor.) The path-ordered exponential helps to convert the $U^\dagger(y')$ [due to $\tilde{\mathcal{G}}(z,y')$] back to $U^\dagger(y)$. (Such a step was already included for the isometries since we added an extra phase factor in $\tilde{G}_{\rm Reg}$ connecting y to y'.) Because it involves the integral of one-forms, it is adequate to use local coordinates y, y' in (52).

Turning to the infrared side of calculations, note that there are no infrared divergences since $\mathbb{CP}^2$ is a compact space of finite volume. However, we are only calculating the leading terms in Γ, up to terms of scaling dimensions ≤ 4. Terms of higher dimension will be ultraviolet finite. While they are irrelevant for issues of renormalization, we need to identify a kinematic regime where such terms are suppressed to be able to make any conclusions with the terms we calculate. If we use an infrared cutoff λ, then the terms of higher scaling dimension will carry inverse powers of this cutoff and will be suppressed for modes of the fields with small momenta compared to λ. This is what we do here. (This rationale for the infrared cutoff is explained in more detail in [1].) The WZW action will be special because it is a term of lower dimension *yet appears with a finite coefficient*. Just for that particular term, we do the calculations both with and without an infrared cutoff.

The details of the UV- and IR-regularized calculations will be given in Appendix B, but here we note that the IR regularization is easily incorporated by using a simple integral representation for $F(s)$ in (45) and including a lower cutoff. Explicitly, we write

$$F(s) = \frac{1}{r^2}\int_{\lambda r^2}^\infty dt e^{-ts}\left[\frac{1}{2}(1+s)^{-\frac{1}{2}} + (1+s)^{\frac{3}{2}}\left(\frac{1}{t}(e^{-t}-1) + e^{-t}\left(1+\frac{t}{2}\right)\right)\right]. \tag{53}$$

We have introduced r^2 via the scaling of coordinates. The infrared cutoff λ appears as the lower limit of the integration over t. When λ is set to zero, we clearly reproduce $F(s)$ in (45). This result, combined with (52), can be used for calculating the effective action.

VI. RESULTS

We are now ready to present the results regarding the volume element for $\mathcal{A}/\mathcal{G}_*$. As mentioned at the end of Sec. III, we will consider the expansion of Γ as a series in terms of increasing scaling dimension, focusing on those with dimension ≤ 4. These are the terms we can expect to be relevant for the long-wavelength modes of the fields; they are potentially UV-divergent terms, up to a possible logarithmic divergence. For scaling dimension 2, the possible terms correspond to a masslike term for a and $\bar{a}$ (with a coefficient of order $1/\epsilon$) and a WZW term $S_{\rm wzw}(H)$, which, somewhat surprisingly, has a finite coefficient. There are also terms of dimension 4 which arise with a coefficient of order $\log \epsilon$.

A. The WZW action

All purely H-dependent terms, such as the WZW action, come from Γ_0 as defined in (31). As mentioned above, the determinant $\det(-\bar{D}\cdot D)_{Y=0}$ for scalar fields was already found in [1]. Using the result from there

$$\mathrm{Tr}\,\log(-\bar{D}\cdot D)_{Y=0} = C_{Y=0} S_{\rm wzw}(H) + \cdots, \tag{54}$$

$$C_{Y=0} = \frac{1}{\pi r^2}\left[1 - \log 2 + \frac{3}{2}e^{-\lambda r^2} + \frac{\lambda r^2}{4}\right] + \frac{1}{\pi r^2}\left[(E_1(\lambda r^2) - E_1(2\lambda r^2)) - \frac{1}{2}e^{-\lambda r^2}(1 - e^{-\lambda r^2})\right] + \frac{1}{\pi r^2}\left[\frac{(1-e^{-\lambda r^2})^2}{4\lambda r^2} + \lambda r^2(e^{\lambda r^2} - 1)E_1(2\lambda r^2)\right]. \tag{55}$$

Here $S_{\rm wzw}(H)$ is the four-dimensional WZW action given by [9,10]

$$S_{\rm wzw}(H) = \frac{1}{2\pi}\int \frac{\pi^2}{2} d\mu g^{a\bar{a}} \mathrm{Tr}(\nabla_a H \bar{\nabla}_{\bar{a}} H^{-1}) - \frac{i}{24\pi}\int \omega \wedge \mathrm{Tr}(H^{-1}dH)^3$$
$$= \frac{\pi}{4}\int d\mu g^{a\bar{a}} \mathrm{Tr}(\nabla_a H \bar{\nabla}_{\bar{a}} H^{-1}) - \frac{i}{24\pi}\int \omega \wedge \mathrm{Tr}(H^{-1}dH)^3, \tag{56}$$

where ω is the Kähler two-form on $\mathbb{CP}^2$. This can be expressed in local coordinates as

$$\omega = i g_{a\bar{a}} dz^a d\bar{z}^{\bar{a}} \tag{57}$$

with $g_{a\bar{a}}$ given by the Fubini-Study metric (1). The last term in (56) is, as usual, over a five-manifold which has $\mathbb{CP}^2$ as the boundary. We have normalized the volume of $\mathbb{CP}^2$ to 1, so there is an extra factor of $\pi^2/2$ in (56) compared to the standard normalizations. (Also, we use a slightly different convention for the normalization of ω, compared to [8].)

It is straightforward to simplify the expression for $C_{Y=0}$ in (55) to show that in the absence of an infrared cutoff ($\lambda \to 0$) we get a finite result with no infrared divergence:

$$C_{Y=0} = \frac{5}{2\pi r^2}, \qquad \lambda \to 0. \tag{58}$$

On the other hand, for $\lambda r^2 \gg 1$,

$$C_{Y=0} = \frac{\lambda}{4\pi} + \mathcal{O}(1), \qquad \lambda r^2 \gg 1. \tag{59}$$

Turning to the $Y = -2$ part of Γ_0, i.e., $\mathrm{Tr}\log(-\bar{D}\cdot D)_{Y=-2}$, we can proceed analogously as in [1] by defining and calculating the current $\langle \hat{J}(H,1)\rangle$. As for the scalar case

$$\begin{aligned}\delta(\mathrm{Tr}\log(-\bar{D}\cdot D)_{Y=-2}) \\ = \int d\mu \mathrm{Tr}[\delta(M^{\dagger-1}\bar{\nabla}M^{\dagger})M^{\dagger-1}\langle\hat{J}\rangle M^{\dagger} + \mathrm{H.c.}], \\ \langle\hat{J}\rangle = -\mathcal{D}_x\tilde{\mathcal{G}}(x,y)|_{y\to x},\end{aligned} \tag{60}$$

where the only difference is that $\tilde{\mathcal{G}}(x,y) = (-\bar{\nabla}\cdot\mathcal{D})^{-1}_{x,y}$ is now defined as an expansion in free propagators for $Y = -2$ fields on $\mathbb{CP}^2$ (with the covariant derivatives ∇ including the appropriate Levi-Civita connections).

Expanding $\langle\hat{J}\rangle$ in powers of ∇HH^{-1}, and using the UV- and IR-regularized propagators as defined in (52) and (53), we can in principle find $\langle\hat{J}\rangle$ as we did for the scalar case. However, as the result is much more complicated for $Y = -2$ fields, instead, here we present the results in the two relevant limits, one with no infrared cutoff ($\lambda \to 0$) and the other for $\lambda r^2 \gg 1$:

$$\begin{aligned}\langle\hat{J}_a\rangle = -\frac{\pi}{2}C_{Y=-2}\nabla_a HH^{-1} + \cdots, \\ \mathrm{Tr}\log(-\bar{D}\cdot D)_{Y=-2} = C_{Y=-2}S_{\mathrm{wzw}}(H) + \cdots,\end{aligned} \tag{61}$$

$$C_{Y=-2} = \frac{1}{\pi r^2}, \qquad \lambda \to 0, \tag{62}$$

$$C_{Y=-2} = \frac{\lambda}{4\pi} + \mathcal{O}(1), \qquad \lambda r^2 \gg 1. \tag{63}$$

The details of this calculation are given in Appendix A. We may also note that the transition from the variation and the current in (60) and (61) to the integrated version $S_{\mathrm{wzw}}(H)$ relies on the four-dimensional version of the Polyakov-Wiegmann identity, which gives

$$\delta S_{\mathrm{wzw}}(H) = -\frac{\pi}{2}\int d\mu g^{a\bar{a}}\mathrm{Tr}[\bar{\nabla}_{\bar{a}}(\delta M^{\dagger}M^{\dagger-1})\nabla_a HH^{-1}]. \tag{64}$$

As for the scalar case, other than the WZW action, Γ_0 will also include $\log\epsilon$ terms (see below) and finite terms as well (if we include terms with scaling dimension > 4, involving more derivatives on H). We will not calculate the finite terms in this paper.

Combining the results for the $Y = 0$ and $Y = -2$ fields, we have

$$\begin{aligned}\Gamma_0 &= \mathrm{Tr}\log(-\bar{D}\cdot D)_{Y=0} + \mathrm{Tr}\log(-\bar{D}\cdot D)_{Y=-2} \\ &= CS_{\mathrm{wzw}}(H) + \cdots,\end{aligned} \tag{65}$$

where $C = C_{Y=0} + C_{Y=-2}$. In the case of no infrared cutoff the WZW action has a finite coefficient

$$C = \frac{7}{2\pi r^2}, \qquad \lambda \to 0. \tag{66}$$

For $\lambda r^2 \gg 1$,

$$C = \frac{\lambda}{2\pi}, \qquad \lambda r^2 \gg 1. \tag{67}$$

B. The mass term

Now we turn to terms with quadratic divergence. In Γ_0 the only term that could have $1/\epsilon$ divergence is the WZW action. However, as we have seen, for this term all divergences cancel out, giving an overall finite coefficient.

Among the next set of terms in $\Delta\Gamma$ in (31), there are a couple of possible candidates for quadratic divergence. However, the only term which survives is a mass term for a and $\bar{a}$. This is expected as the only quadratically divergent term in $\Delta\Gamma$ that is invariant under the holomorphic transformation $M^{\dagger} \to VM^{\dagger}$ is a mass term $\mathrm{Tr}[\bar{a}HaH^{-1}]$.

Looking back at (31) and expanding the exponential in S_1 we can write

$$\Delta\Gamma = \langle S_1\rangle - \frac{1}{2!}\langle S_1^2\rangle + \frac{1}{3!}\langle S_1^3\rangle \cdots, \tag{68}$$

where the sum is over connected diagrams. S_1 is defined in (30), and

$$\begin{aligned}\langle C(x)\bar{C}(y)\rangle &= \left(\frac{1}{-\bar{D}\cdot D}\right)_{Y=0} = \mathcal{G}(x,y), \\ \langle\bar{B}(x)B(y)\rangle &= \left(\frac{1}{-\bar{D}\cdot D}\right)_{Y=-2} = \tilde{\mathcal{G}}(x,y).\end{aligned} \tag{69}$$

Since each term in the expansion of $\Delta\Gamma$ is a trace over the operators MaM^{-1}, $M^{\dagger-1}\bar{a}M^{\dagger}$, D, $\bar{D}$ and $(-\bar{D}\cdot D)^{-1}$ in each such trace we can factor out $M^{\dagger}$, effectively setting $M^{\dagger} \to 1$

and $M \to H$.[1] Quadratically divergent terms will come only from the first two terms in the expansion. Performing similar calculations as in the case of the current in $\delta\Gamma_0$, we get

$$\Delta\Gamma = \left(-\frac{1}{4\epsilon} + \frac{1}{2r^2}\log\epsilon\right)\int d\mu g^{a\bar{a}}\mathrm{Tr}(\bar{a}_{\bar{a}} H a_a H^{-1}) + \mathcal{O}(\log\epsilon). \tag{70}$$

The only quadratically divergent term is thus a mass term for a and $\bar{a}$.

C. The log-divergent terms

Logarithmically divergent terms can arise from both Γ_0 and $\Delta\Gamma$. Combining all such contributions,

$$\begin{aligned}\Gamma_{\log\epsilon} &= \Gamma_{0\log\epsilon} + \Delta\Gamma_{\log\epsilon}\\ &= \frac{\log\epsilon}{12}\int \mathrm{Tr}[(g^{a\bar{a}}\bar{\nabla}_{\bar{a}}(\nabla_a H H^{-1}))^2 + (g^{a\bar{a}}\bar{a}_{\bar{a}} H a_a H^{-1})^2\\ &\quad + g^{a\bar{a}} g^{b\bar{b}}[\bar{a}_{\bar{a}}, H a_a H^{-1}]\bar{\nabla}_{\bar{b}}(\nabla_b H H^{-1})\\ &\quad - g^{a\bar{a}} g^{b\bar{b}}(\bar{\nabla}_{\bar{a}}(\nabla_b H H^{-1})[\bar{a}_{\bar{b}}, H a_a H^{-1}]\\ &\quad + \bar{\nabla}_{\bar{a}}\bar{a}_{\bar{b}}\mathcal{D}_a(H a_b H^{-1}))],\end{aligned} \tag{71}$$

where the trace is in the adjoint representation. (Once again, the details are given in Appendix B.)

VII. DISCUSSION

The key results we have obtained are the following. We introduced a parametrization of the gauge potentials [see Eq. (18) in Sec. II], which allows for the explicit factoring out of gauge transformations and consequent reduction to gauge-invariant degrees of freedom. In particular, the volume element for the gauge orbit space $\mathcal{C} = \mathcal{A}/\mathcal{G}_*$ can be written as

$$d\mu[\mathcal{C}] = e^{\Gamma}\prod_x d\mu(H) d\chi' d\chi'^{\dagger}. \tag{72}$$

Unlike the case of two dimensions, an exact calculation of Γ is not possible in four dimensions. We have calculated the first set of terms in Γ corresponding to monomials of the fields and their derivatives of scaling dimension ≤ 4. These are the terms which can be expected to be potentially ultraviolet divergent. The first of these terms is a four-dimensional WZW action $S_{\mathrm{wzw}}(H)$ for $H \in SL(N,\mathbb{C})/SU(N)$ [for an $SU(N)$ gauge theory], given in (65). *A priori*, on dimensional grounds, one may expect this term to be quadratically divergent, but, somewhat surprisingly, it arises with a finite coefficient. The sign of the coefficient is "correct" in the sense of ensuring convergence for functional integration over H. [In [1], we calculated a similar contribution due to chiral scalar fields on $\mathbb{CP}^2$. The coefficient of $S_{\mathrm{wzw}}(H)$ was such that it tends to destabilize the theory for long-wavelength modes of H. It is interesting that chiral scalars on $\mathbb{CP}^2$ have this destabilizing effect. For the volume element for $\mathcal{C}$, there is no such issue.]

The second term in Γ is a mass term for the components a_i and $\bar{a}_{\bar{i}}$ of the potentials, which are related to χ' and $\chi'^{\dagger}$ as $a_i = -g_{i\bar{i}}\epsilon^{\bar{i}\bar{j}} H^{-1}(\bar{\nabla}_{\bar{j}}\chi')H$ and $\bar{a}_{\bar{i}} = -g_{\bar{i}i}\epsilon^{ij} H(\nabla_j \chi'^{\dagger})H^{-1}$, respectively. The coefficient of this term, say, μ^2_{div}, is quadratically divergent. Since ultraviolet divergences are related to products of operators at the same point, i.e., to ultralocal geometry and hence not sensitive to global geometry, the divergence shows that this term should survive even if we take the large r limit of $\mathbb{CP}^2$. Further, since it is consistent with gauge invariance and all the isometries of $\mathbb{CP}^2$, there is no reason to reject this. This means that the functional measure has to be defined by introducing a similar counterterm (with a coefficient μ^2_{counter}) from the beginning and then renormalizing by setting $\mu^2_{\mathrm{div}} + \mu^2_{\mathrm{counter}}$ to a finite value μ^2_{Ren}. This finite renormalized value μ_{Ren} has the dimensions of mass and will serve as the mass parameter defining the theory. This is in accordance with the fact that dimensional transmutation, with the introduction of an arbitrary scale factor, is needed to define four-dimensional gauge theories. In the usual perturbative approach, this would enter via loop corrections and the running coupling constant, but in our formulation, it appears as what is needed to make the volume element for $\mathcal{C}$ (or the measure of functional integration) well defined. If loop calculations are carried out in our approach, the results will be functions of μ_{Ren}; there will be no need for additional dimensional parameters; what is conventionally considered as Λ_{QCD} will be related to μ_{Ren}.

There is also a set of terms which are of scaling dimension 4 with a logarithmically divergent coefficient; see (71). Unlike the case of flat space, these terms do not combine into $\mathrm{Tr}(F^2)$ since the reduced isometries of $\mathbb{CP}^2$ allow for additional tensor structures. Presumably, to make the volume element well defined, similar counterterms have to be introduced *a priori* and renormalization has to be carried out. Finally there is also an infinity of terms of scaling dimension > 4, which are ultraviolet finite, which we have not calculated. They are presumably less relevant to the dynamics of long-wavelength modes of the fields compared to the terms we have calculated.

Returning to the mass term (70), we note that the possibility of a soft gluon mass has been proposed already in the 1980s [11]. Lattice simulations of the gluon propagator in the Landau gauge also indicate its saturation to a finite value at low momenta, consistent with a propagator mass [12]. At least qualitatively, we need an

[1]Alternatively, in the path integral formulation of $e^{-\Delta\Gamma}$ one can make the transformations $C \to M^{\dagger}C$, $\bar{C} \to \bar{C}M^{\dagger-1}$, $B \to BM^{\dagger-1}$ and $\bar{B} \to M^{\dagger}\bar{B}$. Since $\det M^{\dagger} = 1$ this redefinition of the fields does not affect the volume element for the path integral.

analytic understanding of these lattice results. Indeed, a number of papers have analyzed the Schwinger-Dyson equations of QCD with a view to showing that the gluon self-energy is nonvanishing at zero momentum, along with attempts to extract quantitative predictions from it [13,14]. The appearance of a possible gauge-invariant mass term in our analysis provides a parallel track of viewing such analyses.

Perhaps the most striking and qualitatively new feature of our analysis is the appearance of the WZW action $S_{\rm wzw}(H)$. With the above given argument for a nonzero mass term (70), it is then possible to consider a kinematic regime of momenta $\ll \mu_{\rm Ren}$ where we can neglect the massive components a_i and $\bar{a}_{\bar{i}}$ and consider a reduced theory where

$$A_a \simeq -\nabla_a M M^{-1}, \qquad \bar{A}_{\bar{a}} \simeq M^{\dagger -1} \bar{\nabla}_{\bar{a}} M^{\dagger}. \tag{73}$$

The volume element then takes the form

$$\begin{aligned} d\mu[\mathcal{C}] &= e^{\Gamma} d\mu(H), \\ \Gamma &\simeq C S_{\rm wzw}(H) + C_1 \int {\rm Tr}(g^{\bar{a}a} \bar{\nabla}_{\bar{a}} (\nabla_a H H^{-1}))^2 + \cdots \\ &\approx C S_{\rm wzw}(H) + \cdots. \end{aligned} \tag{74}$$

[Here C_1 is the renormalized value of the coefficient of the term in (71), after the $\log \epsilon$ divergence is eliminated.] In the last line of (74) we have neglected the term quartic in the derivatives as it is less significant for long-wavelength modes compared to $S_{\rm wzw}(H)$. The theory defined by (74) should be applicable for $\lambda \ll \mu_{\rm Ren}^2$, with $\lambda r^2 \gg 1$. This theory is the four-dimensional WZW theory on $\mathbb{CP}^2$. So our conclusion is that we expect that for fields of modes of wavelength small compared to $\mu_{\rm Ren}$, the four-dimensional Yang-Mills theory can be approximated by a 4d-WZW theory for the field $H \in G^{\mathbb{C}}/G = SL(N, \mathbb{C})/SU(N)$.

The 4d-WZW theory, we may note, also has a history going back to the 1980s, appearing first in the work of Donaldson in the context of anti-self-dual instantons [9]. The same theory is obtained in the Kähler-Chern-Simons theory [10] which attempted to generalize the 2d-WZW theory to four dimensions, a paradigm similar to the WZW-CS relation in two and three dimensions [15]. As shown in [10] and elaborated in [16,17], this action also leads to a holomorphically factorized current algebra, analogous to the case in two dimensions. 4d-WZW theories have also been found in higher-dimensional quantum Hall systems [18]. They also describe the target space dynamics of (world-sheet) $\mathcal{N} = 2$ heterotic superstrings [19]. More recently, such theories have been analyzed in [20] in the context of holomorphic field theories on twistor space.

The critical points of the action $S_{\rm wzw}(H)$ are anti-self-dual instantons. They are related to holomorphic vector bundles, with M and $M^{\dagger}$ defining the holomorphic frames for the bundle. What is interesting is that there is some evidence, based on lattice simulations, that the correlation functions for gauge fields and hadrons seem to be dominated by instantons at low energies; see, for example, [21,22]. While it is difficult to see instanton dominance analytically for fields on $\mathbb{R}^4$, the present result that the theory can be approximated by the 4d-WZW theory along the lines argued above provides some analytical evidence for an instanton liquid picture.

Finally, there is another aspect of the 4d-WZW theory which is worth pointing out. In the $(2+1)$-dimensional analysis considered in [4,5] the expectation value of the Wilson loop operator (in a representation indicated as R) takes the form

$$\begin{aligned} \langle W_R(C) \rangle &= \mathcal{N} \int d\mu(H) e^{2 c_A S^{(2d)}_{\rm wzw}(H)} \\ &\quad \times \exp\left(-\frac{8\pi}{e^4 c_A} \int {\rm Tr}(\bar{\nabla}(\nabla H H^{-1}))^2 \right) \\ &\quad \times {\rm Tr}\Big[\mathcal{P} e^{\oint_C \nabla H H^{-1}} \Big] \\ &\sim e^{-\sigma_R {\rm Area}(C)}, \qquad \sigma_R = e^4 \frac{c_A c_R}{4\pi}, \end{aligned} \tag{75}$$

where $S^{(2d)}_{\rm wzw}(H)$ is the 2d-WZW action for H, c_R and c_A are the values of the quadratic Casimir operators for the representation R and for the adjoint representation, respectively. e^2 is the coupling constant of the Yang-Mills theory. Notice that, as $e^2 \to \infty$, which is the limit where the integrand in (75) defines the 2d-WZW theory for H, the expectation value of $W(C)$ vanishes for any curve C enclosing any nonzero area. If we consider evaluating $\langle W(C) \rangle$ in terms of correlators for the current $\nabla H H^{-1}$, the leading term due to the two-point function is of the form

$$\oint \oint dz dz' \langle \nabla H H^{-1}(z) \nabla H H^{-1}(z') \rangle \sim -c_R \oint \oint \frac{dz dz'}{(z - z')^2}. \tag{76}$$

The UV singularity of this integral is not regularized when $e^2 \to \infty$, and this is the genesis of the vanishing of $\langle W(C) \rangle$.

We see that a similar situation is obtained in the theory defined in the 4d-theory (74). Using the Polyakov-Wiegmann identity

$$\begin{aligned} S_{\rm wzw}(NH) &= S_{\rm wzw}(H) + S_{\rm wzw}(N) \\ &\quad - \frac{\pi}{2} \int g^{\bar{a}a} {\rm Tr}(N^{-1} \bar{\nabla}_{\bar{a}} N \nabla_a H H^{-1}), \end{aligned} \tag{77}$$

we obtain

$$\int d\mu(H)\exp\bigg(CS_{\rm wzw}(H) - C\frac{\pi}{2}\int g^{\bar{a}a}\mathrm{Tr}(N^{-1}\bar{\nabla}_{\bar{a}}N\nabla_a HH^{-1})\bigg) = e^{-CS_{\rm wzw}(N)}. \tag{78}$$

By taking small variations of N, we then find

$$\langle g^{\bar{a}a}\bar{\nabla}_{\bar{a}}(\nabla_a HH^{-1})^\alpha(x) g^{\bar{b}b}\bar{\nabla}_{\bar{b}}(\nabla_b HH^{-1})^\beta(y)\rangle = -\frac{4}{\pi C} g^{\bar{a}a}\bar{\nabla}_{\bar{a},y}\nabla_{a,y}\delta(y,x)\delta^{\alpha\beta}. \tag{79}$$

The two-point function for the currents can be obtained from this as

$$\langle(\nabla_a HH^{-1})^\alpha(x)(\nabla_b HH^{-1})^\beta(y)\rangle = \frac{4}{\pi C}\nabla_{a,x}\nabla_{b,y}G(y,x)\delta^{\alpha\beta}, \tag{80}$$

where $G(y,x)$ is the propagator for scalars on $\mathbb{CP}^2$ given in [1] as

$$G(y,x) = \frac{1}{2s} - \frac{1}{2}\log\left(\frac{s}{1+s}\right) - \frac{3}{4}, \tag{81}$$

where $s = \sigma^2_{y,x}$ is given by (41). We see that the logarithmic term of $G(y,x)$ in (80) can indeed reproduce a result similar to what was obtained in $(2+1)$ dimensions. It is not possible to make a more complete analysis at this stage, but clearly the similarity with $(2+1)$ dimensions shows that the reduced theory (74) in terms of the 4d-WZW action is worthy of further investigation. This will be left to future work.

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APPENDIX A: CALCULATING $\langle\hat{J}\rangle$ FOR THE $Y = -2$ FIELDS

In this Appendix we outline the calculations of the current $\langle\hat{J}\rangle$ in (60) for $Y = -2$ fields. The procedure is analogous to the scalar calculation in [1]. The only difference is in the propagator and the connections of the gradients as given in (35).

The current is

$$\begin{aligned}\langle\hat{J}\rangle &= -\mathcal{D}_x\tilde{\mathcal{G}}_{\rm Reg}(x,y)|_{y\to x}\\ &= \mathrm{Term\,1} + \mathrm{Term\,2} + \mathrm{Term\,3} + \cdots,\\ \mathrm{Term\,1} &= -\nabla_{xa}\tilde{G}(x,y')\left[\frac{(1+y'\cdot\bar{y})}{(1+\bar{y}'\cdot y)}\right]^{\frac{3}{2}} \times\mathcal{P}\exp\left(\int_y^{y'}\nabla HH^{-1}\right)\bigg|_{y\to x},\\ \mathrm{Term\,2} &= (\nabla_a HH^{-1})_x\tilde{G}(x,x')\left[\frac{(1+x'\cdot\bar{x})}{(1+\bar{x}'\cdot x)}\right]^{\frac{3}{2}},\\ \mathrm{Term\,3} &= \int_z \nabla_{xa}\tilde{G}(x,z)g_z^{b\bar{b}}(\nabla_b HH^{-1})_z\bar{\nabla}_{z\bar{b}}\tilde{G}(z,x') \times\left[\frac{(1+x'\cdot\bar{x})}{(1+\bar{x}'\cdot x)}\right]^{\frac{3}{2}}.\end{aligned} \tag{A1}$$

The primed coordinate is given in (48) and we perform an angular integration over w and α with the conditions that $|\alpha|^2 = \epsilon$ and $\sigma^2(x,w) = 1$.

As in the scalar calculation (in [1]), we make a coordinate transformation $w \to w'$ such that $w^a = x^a + (e_x^{-1})^a_b\frac{w'^b}{1-\bar{x}\cdot w'}$, where e^{-1} are the tangent frame fields. Under this transformation $\sigma^2(x,w) = |w'|^2$. Term 1 in (A1) becomes

$$\begin{aligned}\mathrm{Term\,1} &= -\int_\alpha\delta(|\alpha|^2-\epsilon)\int_w\delta(\sigma^2(x,w)-1) \times\left\{\nabla_{xa}\tilde{G}(x,y')\left[\frac{(1+y'\cdot\bar{y})}{(1+\bar{y}'\cdot y)}\right]^{\frac{3}{2}} \times(y'-y)^b(\nabla_b HH^{-1})_y|_{y\to x}\right\}\\ &= (\nabla_a HH^{-1})_x\frac{\epsilon}{2}\left((1+\epsilon)F'(\epsilon) - \frac{3}{2}F(\epsilon)\right),\end{aligned} \tag{A2}$$

where the extra term $-\frac{3}{2}F(\epsilon)$ (as compared to the scalar case) comes from the spin connection in ∇_a and the phase factor of $\tilde{G}(x,y')$. Including the scale r in (A2), we take $\epsilon \to \epsilon/r^2$ and $F(s)$ is the IR-regulated propagator

$$F(s) = \frac{1}{r^2}\int_{\lambda r^2}^\infty dt e^{-ts}\left[\frac{1}{2}(1+s)^{-\frac{1}{2}} + (1+s)^{\frac{3}{2}}\left(\frac{1}{t}(e^{-t}-1) + e^{-t}\left(1+\frac{t}{2}\right)\right)\right]. \tag{A3}$$

Term 1 then becomes

$$\mathrm{Term\,1} = (\nabla_a HH^{-1})_x\left(-\frac{1}{4\epsilon} - \frac{1}{8r^2}\right). \tag{A4}$$

Performing similar calculations as for Term 1 it is straightforward to find that

$$\text{Term 2} = (\nabla_a HH^{-1})_x \left[\frac{1}{2\epsilon} + \frac{1}{r^2}\log\left(\frac{\epsilon}{r^2}\right) - \frac{\lambda}{2} - \frac{1}{4r^2} + \frac{1}{2r^2}e^{-\lambda r^2}(3+\lambda r^2) + \frac{1}{r^2}(E_1(\lambda r^2) + \log(\lambda r^2) + \gamma)\right], \tag{A5}$$

where E_1 is the exponential integral

$$E_1(w) = \int_1^\infty \frac{dt}{t} e^{-wt}. \tag{A6}$$

For Term 3 we need three coordinate transformations:

(i) a transformation $w \to w'$ such that $\sigma^2(x,w) = 1$, as explained above;

(ii) a transformation $z \to z'$ such that $z^a = x^a + (e_x^{-1})^a_b \frac{z'^b}{1-\bar{x}\cdot z'}$, setting $\sigma^2(x,z) = |z'|^2$;

(iii) finally, a transformation $z' \to \tilde{z}$, which can be given in homogenous coordinates as $\tilde{Z}/\tilde{Z}_3 = \bar{\tilde{W}}_3 Z'/(\bar{\tilde{W}}'\cdot Z')$, where $\tilde{W}' = (\alpha W'_1, \alpha W'_2, W'_3) = W'_3(\alpha w'_1, \alpha w'_2, 1)$.

The first two transformations effectively eliminate x from the integrals by translating the integration variables. The last one is useful because $1 + \sigma^2(z', \alpha w') = (1+\epsilon)(1 + |\tilde{z}|^2)$ which significantly simplifies the integrals in $\tilde{z}$ and w'. These transformations are the same as in [1] (where more details are given as well) and, as was the case there, the integration measures remain unchanged. With these changes, Term 3 becomes

$$\begin{aligned}
\text{Term 3} &= (\nabla_b HH^{-1})_x \int_\alpha \delta(|\alpha|^2 - \epsilon) \int_w \delta(\sigma^2(x,w) - 1) \int d\mu(z) \nabla_{xa}\tilde{G}(x,z) g_z^{b\bar{b}} \bar{\nabla}_{z\bar{b}} \tilde{G}(z,x') \left[\frac{(1+x'\cdot\bar{x})}{(1+\bar{x}'\cdot x)}\right]^{\frac{3}{2}} \\
&= -(\nabla_b HH^{-1})_x \int d\mu(\tilde{z})(1+|\tilde{z}|^2)(e_x^{-1})^b_m \tilde{z}^m \eta_{a\bar{a}}(e_x)^{\bar{a}}_{\bar{m}} \bar{\tilde{z}}^{\bar{m}} \left(F'(|\tilde{z}|^2)(1+|\tilde{z}|^2) - \frac{3}{2}F(|\tilde{z}|^2)\right) \\
&\quad \times \left(F'(|\tilde{z}|^2(1+\epsilon)+\epsilon)(1+|\tilde{z}|^2)(1+\epsilon) + \frac{3}{2}F(|\tilde{z}|^2(1+\epsilon)+\epsilon)\right) \\
&= -(\nabla_a HH^{-1})_x \int_0^\infty ds s^2 \left(F'(s) - \frac{3}{2(1+s)}F(s)\right)\left(F'(s(1+\epsilon)+\epsilon)(1+\epsilon) + \frac{3}{2(1+s)}F(s(1+\epsilon)+\epsilon)\right) \\
&= \mathcal{I}(\nabla_a HH^{-1})_x.
\end{aligned} \tag{A7}$$

The exact calculation of $\mathcal{I}$ is more complicated than for its scalar counterpart. So instead of calculating it for arbitrary λ, we find it in the two limits, namely, in the case of no infrared cutoff, i.e., $\lambda \to 0$, and in the case of $\lambda r^2 \gg 1$. Including the scale factor r, we find

$$\begin{aligned}
\mathcal{I} &= -\frac{1}{4\epsilon} - \frac{1}{r^2}\log\left(\frac{\epsilon}{r^2}\right) - \frac{13}{8r^2}, \qquad \lambda \to 0, \\
\mathcal{I} &= -\frac{1}{4\epsilon} - \frac{1}{r^2}\log(\lambda\epsilon) + \frac{3\lambda}{8} + \mathcal{O}(1), \qquad \lambda r^2 \gg 1.
\end{aligned} \tag{A8}$$

Combining (A4), (A5), and (A8) and taking the appropriate limits,

$$\begin{aligned}
\langle \hat{J}_a \rangle &= -\frac{\pi}{2} C_{Y=-2} \nabla_a HH^{-1} + \cdots, \\
C_{Y=-2} &= \frac{1}{\pi r^2}, \qquad \lambda \to 0, \\
C_{Y=-2} &= \frac{\lambda}{4\pi} + \mathcal{O}(1), \qquad \lambda r^2 \gg 1,
\end{aligned} \tag{A9}$$

which are the results in (62) and (63).

APPENDIX B: ULTRAVIOLET-DIVERGENT TERMS

In this Appendix we go over some of the calculations leading to the mass term and log-divergent terms in (70) and (71). For the mass term we proceed as in Appendix A. As for terms that have at most a log divergence we calculate them in the flat limit ($r \to \infty$), afterward restoring r in the metric and the volume element.[2]

We take the flat limit after performing similar coordinate transformations as in Appendix A, most significantly, the transformation $z' \to \tilde{z}$ that sets $1 + \sigma^2(z', \alpha w') = (1 + |\tilde{z}|^2)(1+\epsilon)$. Thus, in the $r \to \infty$ limit, we can take the regulated propagator to be

$$\tilde{G}_{\text{Reg}}(x,y) \to \frac{1}{2(|x-y|^2 + \epsilon)}. \tag{B1}$$

[2]On dimensional grounds, for terms that are at most log divergent, r can only appear in terms that go as $\mathcal{O}(x/r)$. To preserve the symmetries of the space such terms can only appear in the metric (or its inverse or the volume element for the space). By contrast, a term that can have $1/\epsilon$ divergence can also have terms of order $(\log \epsilon)/r^2$.

Here we only included the first term in the propagator, as in the $r \to \infty$ limit it is the only term that survives for both the scalar and $Y = -2$ fields. The IR regulator does not show up in UV-divergent terms.

In Γ_0 the only term that can have $1/\epsilon$ divergence is the WZW action which happens to have a finite coefficient (see above). For log-divergent terms we expand $\langle \hat{J} \rangle$ further in $\nabla H H^{-1}$. However, since we calculate them in the limit $r \to \infty$, the result will be the same for scalar and $Y = -2$ fields. Thus we can use our result from [1], where we calculated the log-divergent term for the scalar current. Here, we simply need to double it, as there is one term coming from the scalar part of Γ_0 and one from the $Y = -2$ part:

$$\Gamma_0 = \frac{\log \epsilon}{12} \int \mathrm{Tr}(\bar{\nabla}(\nabla H H^{-1}))^2 + \text{finite}. \tag{B2}$$

For $\Delta\Gamma$ we can take $M \to H$ and $M^\dagger \to 1$, as discussed in Sec. VI B. Thus,

$$\begin{aligned} \Delta\Gamma &= \langle S_1 \rangle - \frac{1}{2!}\langle S_1^2 \rangle + \frac{1}{3!}\langle S_1^3 \rangle + \cdots \\ &= \Delta\Gamma^{(1)} + \Delta\Gamma^{(2)} + \Delta\Gamma^{(3)} + \cdots, \end{aligned} \tag{B3}$$

where

$$\begin{aligned} S_1 = \int d\mu [&\bar{C}^\alpha (H a H^{-1} \cdot \bar{a})^{\alpha\beta} C^\beta + C^\alpha (\bar{a} \cdot \mathcal{D})^{\alpha\beta} C^\beta \\ &+ \bar{C}^\alpha (-H a H^{-1} \cdot \bar{\nabla})^{\alpha\beta} \bar{C}^\beta + C^\alpha (-\epsilon^{\bar{i}\bar{j}} \bar{a}_{\bar{i}} \bar{\nabla}_{\bar{j}})^{\alpha\beta} B^\beta \\ &+ \bar{C}^\alpha (\epsilon^{ij} H a_i H^{-1} \nabla_j)^{\alpha\beta} \bar{B}^\beta], \end{aligned} \tag{B4}$$

$\langle C^\alpha \bar{C}^\beta \rangle = \mathcal{G}^{\text{scalar}} = (-\bar{\nabla} \cdot \mathcal{D})^{-1}_{\text{scalar}}$, and $\langle \bar{B}^\alpha B^\beta \rangle = \tilde{\mathcal{G}}^{Y=-2} = (-\bar{\nabla} \cdot \mathcal{D})^{-1}_{Y=-2}$. In carrying out various calculations, we will be expanding $\mathcal{G}^{\text{scalar}}$ and $\tilde{\mathcal{G}}^{Y=-2}$ in terms of the corresponding free propagators G and $\tilde{G}$, as in (51). For $Y = -2$, the UV-regularized form of the free propagator is given in (45) and (50). The scalar propagator was given in [1] and has the form

$$G(z, y) = \frac{1}{2s} - \frac{1}{2}\log\left(\frac{s}{1+s}\right) - \frac{3}{4}, \qquad s = \sigma^2_{z,y}, \tag{B5}$$

with the replacement of s by $\sigma^2(Z, Y')$ to take account of regularization.

For the mass term, we keep r finite and perform similar calculations as in Appendix A. The log-divergent terms can be obtained by calculating in the flat space limit and then upgrading the metric and volume factors to the curved space expressions. Then, expanding each $\mathcal{G}$ in $\nabla H H^{-1}$, we find the following UV-divergent terms:

$$\Delta\Gamma^{(1)} = \langle S_1 \rangle = \int \mathrm{Tr} H a H^{-1} \bar{a} \mathcal{G}^{\text{scalar}}_{\text{Reg}} = \left(\frac{1}{2\epsilon} - \frac{1}{2r^2}\log\epsilon\right) \int \mathrm{Tr} H a H^{-1} \bar{a} + \frac{\log\epsilon}{4} \int \mathrm{Tr} H a H^{-1} \bar{a}\, \bar{\nabla}(\nabla H H^{-1}), \tag{B6}$$

$$\begin{aligned} \Delta\Gamma^{(2)} &= -\frac{1}{2!}\langle S_1^2 \rangle \\ &= -\frac{1}{2}\int \mathrm{Tr}[H a H^{-1} \bar{a} \mathcal{G}^{\text{scalar}} H a H^{-1} \bar{a} \mathcal{G}^{\text{scalar}}_{\text{Reg}} - 4 \bar{a} \cdot \mathcal{D} \mathcal{G}^{\text{scalar}} H a H^{-1} \cdot \bar{\nabla} \mathcal{G}'^{\text{scalar}}_{\text{Reg}} + 2 \epsilon^{\bar{i}\bar{j}} \bar{\nabla}_{\bar{i}} (\bar{a}_{\bar{j}} \mathcal{G}^{\text{scalar}}) \epsilon^{ij} H a_i H^{-1} \mathcal{D}_j \tilde{\mathcal{G}}^{Y=-2}_{\text{Reg}}] \\ &= \left(-\frac{3}{4\epsilon} + \frac{1}{r^2}\log\epsilon\right) \int \mathrm{Tr} H a H^{-1} \bar{a} + \log\epsilon \int \mathrm{Tr}\Bigg[\frac{1}{4}(H a H^{-1} \bar{a})^2 - \frac{5}{12} H a H^{-1} \bar{a}\, \bar{\nabla}(\nabla H H^{-1}) \\ &\quad - \frac{1}{12} g^{a\bar{a}} g^{b\bar{b}} \bar{\nabla}_{\bar{a}} \bar{a}_{\bar{b}} \mathcal{D}_a (H a_b H^{-1}) - \frac{1}{12} g^{a\bar{a}} g^{b\bar{b}} \bar{\nabla}_{\bar{b}} (\nabla_a H H^{-1}) [\bar{a}_{\bar{a}}, H a_b H^{-1}]\Bigg], \end{aligned} \tag{B7}$$

$$\begin{aligned} \Delta\Gamma^{(3)} &= \frac{1}{3!}\langle S_1^3 \rangle \\ &= \frac{1}{3}\int \mathrm{Tr}[H a H^{-1} \bar{a} \mathcal{G}^{\text{scalar}} H a H^{-1} \bar{a} \mathcal{G}^{\text{scalar}} H a H^{-1} \bar{a} \mathcal{G}^{\text{scalar}}_{\text{Reg}} - 12 \bar{a} \cdot \mathcal{D} \mathcal{G}^{\text{scalar}} H a H^{-1} \bar{a} \mathcal{G}^{\text{scalar}} H a H^{-1} \cdot \bar{\nabla} \mathcal{G}'^{\text{scalar}}_{\text{Reg}} \\ &\quad + 3 \epsilon^{\bar{i}\bar{j}} \bar{\nabla}_{\bar{i}} (\bar{a}_{\bar{j}} \mathcal{G}^{\text{scalar}}) H a H^{-1} \bar{a} \mathcal{G}^{\text{scalar}} \epsilon^{ij} H a_i H^{-1} \mathcal{D}_j \tilde{\mathcal{G}}^{Y=-2}_{\text{Reg}}] \\ &= -\frac{5}{4}\log\epsilon \int \mathrm{Tr}(H a H^{-1} \bar{a})^2, \end{aligned} \tag{B8}$$

$$\begin{aligned}\Delta\Gamma^{(4)} &= -\frac{1}{4!}\langle S_1^4\rangle \\ &= -\frac{1}{4}\int \mathrm{Tr}[16\bar{a}\cdot\mathcal{D}\mathcal{G}^{\mathrm{scalar}}HaH^{-1}\cdot\bar{\nabla}\mathcal{G}'^{\mathrm{scalar}}\bar{a}\cdot\mathcal{D}\mathcal{G}^{\mathrm{scalar}}HaH^{-1}\cdot\bar{\nabla}\mathcal{G}'^{\mathrm{scalar}}_{\mathrm{Reg}} \\ &\quad - 16HaH^{-1}\cdot\bar{\nabla}\mathcal{G}'^{\mathrm{scalar}}\bar{a}\cdot\mathcal{D}\mathcal{G}^{\mathrm{scalar}}\epsilon^{ij}Ha_iH^{-1}\mathcal{D}_j\tilde{\mathcal{G}}^{\mathrm{Y}=-2}\epsilon^{\bar{i}\bar{j}}\bar{\nabla}_{\bar{i}}(\bar{a}_{\bar{j}}\mathcal{G}^{\mathrm{scalar}}_{\mathrm{Reg}}) \\ &\quad + 2\epsilon^{\bar{i}\bar{j}}\bar{\nabla}_{\bar{i}}(\bar{a}_{\bar{j}}\mathcal{G}^{\mathrm{scalar}})\epsilon^{ij}Ha_iH^{-1}\mathcal{D}_j\tilde{\mathcal{G}}^{\mathrm{Y}=-2}\epsilon^{\bar{k}\bar{l}}\bar{\nabla}_{\bar{k}}(\bar{a}_{\bar{l}}\mathcal{G}^{\mathrm{scalar}})\epsilon^{kl}Ha_kH^{-1}\mathcal{D}_l\tilde{\mathcal{G}}^{\mathrm{Y}=-2}_{\mathrm{Reg}}] \\ &= \frac{13}{12}\log\epsilon\int \mathrm{Tr}(HaH^{-1}\bar{a})^2, \end{aligned} \tag{B9}$$

where in the above $\mathcal{G}' = (-\mathcal{D}\cdot\bar{\nabla})^{-1}$ and the traces are in the adjoint representation.

Combining the four terms

$$\begin{aligned}\Delta\Gamma &= \left(-\frac{1}{4\epsilon}+\frac{1}{2r^2}\log\epsilon\right)\int \mathrm{Tr}HaH^{-1}\bar{a} + \frac{1}{12}\log\epsilon\int \mathrm{Tr}[[\bar{a},HaH^{-1}]\bar{\nabla}(\nabla HH^{-1}) + (HaH^{-1}\bar{a})^2 \\ &\quad - g^{a\bar{a}}g^{b\bar{b}}(\bar{\nabla}_{\bar{a}}\bar{a}_{\bar{b}}\mathcal{D}_a(Ha_bH^{-1}) + \bar{\nabla}_{\bar{b}}(\nabla_a HH^{-1})[\bar{a}_{\bar{a}},Ha_bH^{-1}])]. \end{aligned} \tag{B10}$$

Putting together Γ_0 and $\Delta\Gamma$ from (B2) and (B10) above

$$\begin{aligned}\Gamma &= \Gamma_0 + \Delta\Gamma \\ &= \left(-\frac{1}{4\epsilon}+\frac{1}{2r^2}\log\epsilon\right)\int \mathrm{Tr}\bar{a}HaH^{-1} + \frac{1}{12}\log\epsilon\int \mathrm{Tr}[(\bar{\nabla}(\nabla HH^{-1}))^2 + [\bar{a},HaH^{-1}]\bar{\nabla}(\nabla HH^{-1}) + (\bar{a}HaH^{-1})^2 \\ &\quad - g^{a\bar{a}}g^{b\bar{b}}(\bar{\nabla}_{\bar{b}}(\nabla_a HH^{-1})[\bar{a}_{\bar{a}},Ha_bH^{-1}] + \bar{\nabla}_{\bar{a}}\bar{a}_{\bar{b}}\mathcal{D}_a(Ha_bH^{-1}))], \end{aligned} \tag{B11}$$

which are the results for the mass term and log-divergent terms in (70) and (71).

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[1] D. Karabali, A. Maj, and V. P. Nair, preceding paper, Phys. Rev. D **106**, 085012 (2022).

[2] Some of the general issues concerning the volume element for the gauge orbit space are discussed in I. M. Singer, Phys. Scr. **24**, 817 (1980); Commun. Math. Phys. **60**, 7 (1978); P. K. Mitter and C. M. Viallet, Commun. Math. Phys. **79**, 457 (1981); Phys. Lett. **85B**, 246 (1979); M. Asorey and P. K. Mitter, Commun. Math. Phys. **80**, 43 (1981); O. Babelon and C. M. Viallet, Commun. Math. Phys. **81**, 515 (1981); Phys. Lett. **103B**, 45 (1981); P. Orland, arXiv:hep-th/9607134; Phys. Rev. D **70**, 045014 (2004).

[3] K. Gawedzki and A. Kupiainen, Phys. Lett. B **215**, 119 (1988); Nucl. Phys. **B320**, 649 (1989); M. Bos and V. P. Nair, Int. J. Mod. Phys. A **05**, 959 (1990).

[4] D. Karabali and V. P. Nair, Nucl. Phys. **B464**, 135 (1996); Phys. Lett. B **379**, 141 (1996); D. Karabali, Chanju Kim, and V. P. Nair, Nucl. Phys. **B524**, 661 (1998); Phys. Lett. B **434**, 103 (1998); D. Karabali and V. P. Nair, Phys. Rev. D **77**, 025014 (2008); D. Karabali, V. P. Nair, and A. Yelnikov, Nucl. Phys. **B824**, 387 (2010).

[5] For a short review, see V. P. Nair, Proc. Sci., QCD-TNT09 (**2011**) 030, arXiv:1201.0977.

[6] M. Teper, Phys. Rev. D **59**, 014512 (1999); B. Lucini and M. Teper, Phys. Rev. D **64**, 105019 (2001); B. Bringoltz and M. Teper, Phys. Lett. B **645**, 383 (2007); A. Athenadorou and M. Teper, J. High Energy Phys. 02 (2017) 015; B. H. Wellegehausen, A. Wipf, and C. Wozar, Phys. Rev. D **83**, 016001 (2011); J. Kiskis and R. Narayanan, J. High Energy Phys. 09 (2008) 080.

[7] For the numerical lattice-based evaluation of the Casimir energy for non-Abelian gauge theories, see M. N. Chernodub, V. A. Goy, A. V. Molochkov, and Ha Huu Nguyen, Phys. Rev. Lett. **121**, 191601 (2018). For the theoretical work on the Casimir energy and the analytic calculation of the propagator mass, see D. Karabali and V. P. Nair, Phys. Rev. D **98**, 105009 (2018).

[8] V. P. Nair, Phys. Rev. D **88**, 105027 (2013).

[9] S. K. Donaldson, Proc. London Math. Soc. **s3-50**, 1 (1985).

[10] V. P. Nair and J. Schiff, Phys. Lett. B **246**, 423 (1990); Nucl. Phys. **B371**, 329 (1992).

[11] J. M. Cornwall, Phys. Rev. D **10**, 500 (1974); **26**, 1453 (1982); C. Bernard, Nucl. Phys. **B219**, 341 (1983).

[12] I. L. Bogolubsky, E. M. Ilgenfritz, M. Muller-Preussker, and A. Sternbeck, Phys. Lett. B **676**, 69 (2009); I. L. Bogolubsky, E. M. Ilgenfritz, M. Muller-Preussker, and A. Sternbeck, arXiv:0710.1968; P. O. Bowman, U. M. Heller, D. B. Leinweber, M. B. Parappilly, A. Sternbeck, L. von Smekal, A. G. Williams, and J. Zhang, Phys. Rev. D **76**, 094505 (2007); O. Oliveira and P. J. Silva, *Proc. Sci.*, LAT2009 (2009) 226 [arXiv:0910.2897]; A. Cucchieri and T. Mendes, *Proc. Sci.*, LAT2007 (2007) 297 [arXiv:0710.0412]; A. Cucchieri and T. Mendes, Phys. Rev. Lett. **100**, 241601 (2008); A. Cucchieri and T. Mendes, Phys. Rev. D **81**, 016005 (2010).

[13] The literature on this is enormous, but a good status report with references to earlier work is J. M. Cornwall, J. Papavassiliou, and D. Binosi, *The Pinch Technique and Its Application to Nonabelian Gauge Theories* (Cambridge University Press, Cambridge, England, 2011); A complementary review is N. Vandersickel and D. Zwanziger, Phys. Rep. **520**, 175 (2012).

[14] A more recent review article which also updates the status of this approach is A. C. Aguilar, D. Binosi, and J. Papavassiliou, Front. Phys. **11**, 111203 (2016).

[15] E. Witten, Commun. Math. Phys. **121**, 351 (1989).

[16] A. Losev, Gregory Moore, Nikita Nekrasov, and Samson Shatashvili, Nucl. Phys. B, Proc. Suppl. **46**, 130 (1995); Nucl. Phys. **B484**, 196 (1997).

[17] T. Inami, H. Kanno, T. Ueno, and C.-S. Xiong, Phys. Lett. B **399**, 97 (1997); T. Inami, H. Kanno, and T. Ueno, Mod. Phys. Lett. A **12**, 2757 (1997).

[18] D. Karabali and V. P. Nair, Nucl. Phys. **B641**, 533 (2002); **B679**, 427 (2004); **B697**, 513 (2004); V. P. Nair and S. Randjbar-Daemi, Nucl. Phys. **B679**, 447 (2004); D. Karabali, V. P. Nair, and R. Randjbar-Daemi, in *From Fields to Strings: Circumnavigating Theoretical Physics*, edited by Ian Kogan Memorial Collection, M. Shifman, A. Vainshtein, and J. Wheater (World Scientific, Singapore, 2004), pp. 831–876.

[19] H. Ooguri and C. Vafa, Nucl. Phys. **B367**, 83 (1991).

[20] K. Costello, arXiv:2111.08879.

[21] This subject also has an enormous literature. For a review, see T. Schäfer and E. V. Shuryak, Rev. Mod. Phys. **70**, 323 (1998); D. Diakonov, Proc. Int. Sch. Phys. Fermi **130**, 397 (1996).

[22] Another recent review, with a useful discussion of the issues involved, is A. Athenodorou, Ph. Boucaud, F. De Soto, J. Rodríguez-Quintero, and S. Zafeiropoulos, J. High Energy Phys. 02 (2018) 140.