Entanglement Entropy & Matter-Gravity Couplings for Fuzzy Geometry

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In this talk I discuss some features of the entanglement entropy for fuzzy geometry, focusing on its dependence on the background fields and the spin connection of the emergent continuous manifold in a large N limit. Using the Landau-Hall paradigm for fuzzy geometry, this is argued to be given by a generalized Chern-Simons form, making a point of connection with the thermodynamic view of gravity. Matter-gravity couplings are also considered in the same framework; they naturally lead to certain specific nonminimal couplings involving powers of the curvature.

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1. Introduction

In this talk, I will present two results related to the use of fuzzy geometry as the underlying structure for a theory of gravity. This is based on the work published in [1] and [2]. Let me begin by recalling that in fuzzy geometry we have an N-dimensional Hilbert space of states $\mathcal{H}_N$, which maybe viewed as describing the (dynamical) degrees of freedom pertaining to space itself. Observables must be defined in terms of the algebra of matrices or linear transformations acting on $\mathcal{H}_N$; we will refer to this algebra as $\mathcal{M}_N$. Space as a continuum is obtained as an approximation in the limit $N \to \infty$. This limit, however, can be ambiguous if we do not specify additional data beyond just the abstract structure of $\mathcal{H}_N$ itself. Such data usually take the form of a Dirac operator or Laplace operator, but the key point is that the choice of such an operator leads to a specification of the metric (and other geometric) data on the emergent space and hence the procedure for making this choice (presumably implemented dynamically based on matter content) should be considered as what is meant by a theory of gravity in the fuzzy context. Rather than specifying a Laplace or Dirac operator at the level of $\mathcal{H}_N$, one can think of the possible large N limits as parametrized in terms of the gauge fields and spin connections in the final emergent continuous manifold. In other words, we can take the starting data for fuzzy geometry as $(\mathcal{H}_N, \mathcal{M}_N)$, along with a procedure for taking large N limits. This procedure can be stated in terms of a set of gauge fields and spin connections on the final emergent manifold.[1] A theory of gravity is then a prescription for choosing one specific large N limit, which we may consider as "optimal", the field equations for gravity are this optimization condition. This is the setting for the results discussed below.

The first of the two results is about the entanglement entropy (EE) for a fuzzy space, i.e., entanglement pertaining to the degrees of freedom of space itself. In the usual way, here we are considering a division of space into two regions, and looking for a (reduced) density matrix relevant for local observables in one of these regions. The result for the entropy should thus follow from a suitable reduction of $\mathcal{H}_N$. The dominant term for the EE will be, as usual, proportional to the area of the interface. One can even see that there are some features reminiscent of the type III$_1$ von Neumann algebra for the local observables [1]. But here my focus will be on the dependence of this EE on the gauge fields and spin connection in the emergent continuum. We will argue that this dependence is given by a generalized Chern-Simons (CS) form [1]. The interest in the question of how EE may be related to the gauge fields and spin connection and how it can inform the issue of gravity for fuzzy spaces are due to the following observations. As has been known for many years, the extremization of the Bekenstein-Hawking (BH) entropies for Rindler horizons of all accelerated observers in a spacetime leads to the Einstein equations of gravity in vacuum [3]. Secondly, at least in some contexts, the BH entropy can also be viewed as an entanglement entropy [4]. Finally for gravity in (2+1) dimensions (and for CS-type gravity in higher dimensions) the field equations correspond to the extremization of appropriate CS forms [5, 6]. We see that the connection between EE for fuzzy spaces and CS forms becomes relevant within this circle of known results on gravity.

Our second result is on the interaction of matter fields with the gauge fields and spin connection mentioned above. Again, to make our statement regarding this a little more precise, consider the

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[1]It may be worth emphasizing that the gauge fields we are talking about here are not the usual electroweak or chromodynamic gauge fields of the standard model. We are talking about the gauge fields relevant to the definition of the fuzzy space itself.

situation where $\mathcal{H}_N$ is the space of sections of a power of the canonical line bundle on a compact Kähler space, i.e., arising from the geometric quantization of the symplectic structure $\Omega = n\,\omega \equiv da$, where ω is the Kähler two-form. This shows clearly that a (which is an Abelian gauge field defining Ω and hence the structure of $\mathcal{H}_N$) should be one of the gauge fields of interest, relevant to defining the fuzzy space itself. And more generally we can consider $A = a + A'$ where we add some perturbation to the starting a, without changing the topological class of Ω. We could also consider perturbations of the spin connection on this Kähler space. As we will see later, we could further include additional structures corresponding to nonabelian gauge fields as well. Let $S(A)$ denote the CS form of the gauge fields and the spin connection, we will make the specific nature of this term very explicit later. The second result I shall discuss is that the coupling of matter fields is then given by $S(A + \mathbb{A})$ where the $U(1)$ field A is shifted by the Poincaré-Cartan one-form $\mathbb{A}$ for matter [2].[2] This will give the action in the first order Hamiltonian formulation. In terms of the Lagrangian, the resulting action is of the form

$$S = \int (\text{polynomial in } R, F) \times \mathcal{L}_{\text{matter}} \tag{1}$$

This implies that in addition to the minimal coupling of mater fields to gauge fields and gravity, which is to be expected, there are specific nonminimal terms which is a polynomial in the curvatures (R) and gauge field strengths (F). The matter Lagrangian will have the usual covariant derivatives of fields and so on, but the density for integration over the volume has a polynomial in terms of F, R, in addition to the usual $\sqrt{\det g}$ factor. It may be interesting to mention at this point that such couplings have been used in some attempts to explain galactic rotation curves and similar phenomena usually attributed to dark matter [7].

2. Field dependence of entanglement entropy

2.1 How does field dependence arise for the EE?

Turning to details, consider the complex projective space $\mathbb{CP}^k$. We can carry out the geometric quantization of $\Omega = n\omega = da$ in the holomorphic polarization. The corresponding sections of the line bundle are holomorphic, they are the "wave functions" which give a realization of the Hilbert space $\mathcal{H}_N$. We can also view these wave functions as defining the lowest Landau level (LLL) of a quantum Hall problem, with Ω as the background magnetic field. In other words, these wave functions are the lowest eigenstates of a Laplace operator, with covariant derivatives defined with the connection a. While the quantum Hall analogy is not necessary for our analysis, it is a useful picture giving intuition about some of the arguments to follow [8, 9]. For example, in this picture, the state describing the fuzzy space is the fully occupied LLL with one fermion for each state. If we consider the fermion field expanded as

$$\psi = \sum_s b_s \, u_s(x) + \text{higher Landau levels} \tag{2}$$

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[2]This is somewhat converse to the usual approach. Instead of starting with a matter Lagrangian and "gauging" it with, say, spin connection to obtain the coupling to gravity, here we start with $S(A)$ given in terms of the spin connection and any other gauge fields needed to define the fuzzy space and then shift A to obtain the matter coupling.

where b_s are fermion annihilation operators for the LLL and u_s are the wave functions mentioned above, fuzzy $\mathbb{CP}^k$ is the state $b_0^\dagger b_1^\dagger \cdots b_N^\dagger |0\rangle$ or, equivalently, it is described by the density matrix

$$\rho = b_0^\dagger b_1^\dagger \cdots b_N^\dagger |0\rangle \langle 0| b_N \cdots b_1 b_0 \tag{3}$$

This is a pure state and if we reduce it to a subset of b's (by tracing over the rest of them) we still obtain a pure state and hence no entropy. However, if we consider dividing $\mathbb{CP}^k$ into two regions, say, D and its complement, then we do get a nontrivial entropy upon reducing ρ to ρ_{Red} relevant for local observables in D. This entropy is given by

$$S_{\text{EE}} = - \sum_s \left[\lambda_s \log \lambda_s + (1 - \lambda_s) \log(1 - \lambda_s) \right], \qquad \lambda_s = \int_D u_s^* u_s \tag{4}$$

I will not go over the derivation of this formula since it has appeared in the literature before [10] and has been used in related work [1, 11] and in the previous talk [12] to calculate the entanglement entropy for quantum Hall droplets.

A few comments might be appropriate at this point before we proceed to discuss the field dependence of the entropy. First of all, we may note that the wave functions u_s have support everywhere on $\mathbb{CP}^k$, although they are exponentially small away from where $u_s^* u_s$ is a maximum. This allows for a "leakage" of $u_s^* u_s$ across any interface between D and its complement. This is the essence of how a nonzero S_{EE} can arise. This is similar to what happens in relativistic quantum field theory where the two-point function, for example, for a scalar field, is nonzero even for spacelike separations. This fact is, in turn, related to the nonfactorizability of the vacuum wave functional for the field, to the Reeh-Schlieder theorem and the result that the reduced density matrices for local observables are KMS (Kubo-Martin-Schwinger) states with nonzero entropy.

Secondly, notice that the wave functions carry information about the gauge field a and the spin connection (and metric) on $\mathbb{CP}^k$. In taking the large N limit of an operator $\hat{F}$ on $\mathcal{H}_N$, we construct the symbol which is a function (on $\mathbb{CP}^k$) associated to it given by

$$F = \sum_{r,s} u_r \, F_{rs} \, u_s^* \tag{5}$$

where $F_{rs} = \langle r| \hat{F} |s\rangle$ are the matrix elements of the operator $\hat{F}$. Operator products are then realized as star products of the symbols, i.e.,

$$\sum_{r,s} u_r \, \langle r| \hat{F}\hat{G} |s\rangle \, u_s^* = \sum_{r,s,k} u_r \left(F_{rk} G_{ks} \right) u_s^* = F * G \tag{6}$$

In the large N limit, the star product simplifies, becoming commutative, so that the operator algebra $\mathcal{M}_N$ tends to the commutative algebra of pointwise multiplication of the symbols. This is the procedure for obtaining the continuum description. (See [8] for explicit calculations for $\mathbb{CP}^k$.) The expressions for the symbols and the star product will, of course, depend on the gauge field a and the spin connection on the space via the dependence of the wave functions u_s on the same. Each choice of the background field gives a particular large N limit. It is in this sense that I mentioned that the large N limits are parametrized by the background gauge fields. More generally, one can consider the lowest eigenfunctions of the Laplacian with more general background fields, including

nonabelian gauge fields as well as general spin connections. So the data defining the wave functions and hence the large N limits will then be these more general background fields.[3] (As we consider different choices for these background fields, it is important to keep N the same, so that the Hilbert space $\mathcal{H}_N$ has the same abstract structure. This can be ensured by keeping all background fields in the same topological class as defined by an index theorem; this will be explained below.)

2.2 Moser's lemma and Chern-Simons term

We can now turn to the crucial question we posed earlier. How do $\lambda_s = \int_D u_s^* u_s$ depend on the background gauge fields and the spin connection? For analyzing this, we will again use $\mathbb{CP}^k$, starting with $\Omega = nda$, and the standard spin connection for the Fubini-Study metric, and making perturbations to both. Towards the calculation of the change in λ_s, we define an "occupancy matrix" P for $\mathcal{H}_N$ as

$$(P)_{ab} = \begin{cases} \delta_{ab} & a, b = 0, 1, \cdots, s-1 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

(We are still considering the state (3) with all states occupied, P is just an auxiliary quantity to help with the argument here.) As mentioned above, the large N simplification of matrices on $\mathcal{H}_N$ is facilitated by using functions which are the symbols [8]. From (5), the symbol or the function corresponding to the matrix P is $(P)_{s-1} = \sum_{a=0}^{s-1} u_a^* u_a$. Therefore we can write

$$u_s^* u_s = (P)_s - (P)_{s-1} \tag{8}$$

To obtain the background field dependence of this quantity, we look at $W \equiv \mathrm{Tr}(PA_0)_s$, in terms of which

$$\frac{\delta W}{\delta A_0} = (P)_s + *\text{-product corrections} \tag{9}$$

Here A_0 is just a dummy variable used to define W; it can be set to zero after the functional derivative is taken, although it may be viewed as the time-component of a $U(1)$ gauge field. The point is that W is easier to calculate. Once we have W, equations (9), (8) will lead us to $u_s^* u_s$, including its dependence on the gauge fields and spin connections.

First let us focus on changing the background values only for the $U(1)$ field which occurs in Ω, i.e., just for the canonical one-form. The symplectic form is thus given by $\Omega = (n\omega + dA') = \Omega_0 + dA'$. We can now relate $\mathrm{Tr}(PA_0)$ calculated with $\Omega_0 + dA'$ (i.e., calculated with the wave functions corresponding to $\Omega_0 + dA'$) to a trace calculated with Ω_0 as

$$\mathrm{Tr}(PA_0)_{\Omega_0+dA'} = \mathrm{Tr}(P\mathcal{A})_{\Omega_0} = \int (P) * \mathcal{A} = \int (P)\,\mathcal{A} + \cdots \tag{10}$$

where $\mathcal{A}$ is to be determined in terms of A' and Ω_0. The integral in (10) is over the whole manifold. The point is that Ω_0 and $\Omega_0 + dA'$ belong to the same topological class, so they can be related by a diffeomorphism, i.e., by a change of coordinates as $v \to v - w$. Here we are essentially using Moser's lemma familiar from classical mechanics. The condition we need is

$$\Omega_0 + dA' \Big]_{v-w} = \Omega_0 \Big]_v, \qquad \mathcal{A} = A_0 \Big]_{v-w} \tag{11}$$

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[3]Again, as mentioned in footnote 1, these are not the gauge fields of the standard model, these are part of the structure defining the large N limit of the fuzzy space itself.

In terms of the one-form potentials, this becomes

$$a + A'\Big]_{v-w} - a\Big]_v = df \approx 0 \tag{12}$$

Here f is any function on the manifold and $\approx$ indicates equality up to an exact form. Taking a (with $da = \Omega_0$) as the term of the zeroth order, and A' as of the first order, we can solve for w as a series in the perturbation. For the first two orders, the condition (12) reduces to [4]

$$
\begin{aligned}
& w_1^j \partial_j a_i + a_j \partial_i w_1^j - A_i' \approx 0 \\
& w_2^j \partial_j a_i + a_j \partial_i w_2^j + w_1^j \partial_j A_i' + A_j' \partial_i w_1^j - \frac{1}{2} w_1^k w_1^l \partial_k \partial_l a_i - w_1^k \partial_k a_j \partial_i w_1^j \approx 0
\end{aligned} \tag{13}
$$

By adding certain exact one-forms, (since we only have the weak equality $\approx$), these equations can be solved as

$$
\begin{aligned}
w_1^j &= -(\Omega_0^{-1})^{jk} A_k' \\
w_2^j &= -(\Omega_0^{-1})^{jk} \left[(\partial_k A_l' - \partial_l A_k') w_1^l + \frac{1}{2} w_1^m w_1^n \partial_m (\Omega_0)_{nk} + \frac{1}{2} (w_1^m \partial_k w_1^n)(\Omega_0)_{mn} \right]
\end{aligned} \tag{14}
$$

We can now calculate $\mathcal{A} = A_0\Big]_{v-w}$ as

$$
\begin{aligned}
\mathcal{A} =\ & A_0 - (\Omega_0)^{-1 ij} A_i' \partial_j A_0 + (\Omega_0)^{-1 ij} \left[\left(\frac{1}{2} A_j' \partial_k A_i' - A_j' \partial_i A_k' \right) (\Omega_0)^{-1 kl} \partial_l A_0 \right. \\
& \left. - \frac{1}{2} A_j' A_k' \partial_i \left((\Omega_0)^{-1 kl} \partial_l A_0 \right) \right] + \cdots
\end{aligned} \tag{15}
$$

We can now multiply this expression by P and integrate over the whole manifold to get

$$
\begin{aligned}
W =\ & \int A_0 \left[P + d \left(P k \Omega_0^{k-1} A' \right) + \frac{P}{2} k(k-1) \Omega_0^{k-2} dA' A' + \cdots \right) \right] \\
& + \frac{k}{2} \int P \Omega_0^{k-1} d \left[A'(\Omega_0)^{-1 ij} A_i' \partial_j A_0 \right] + \cdots
\end{aligned} \tag{16}
$$

The factors of k, $(k-1)$, etc. arise from writing the expression in terms of differential forms. Again using integration by parts and rearranging factors, we can write the last term in terms of A_0, rather than its derivative, to obtain

$$
\begin{aligned}
W =\ & \int A_0 \left[P + d \left(P k \Omega_0^{k-1} A' \right) + \frac{P}{2} k(k-1) \Omega_0^{k-2} dA' A' + \cdots \right) \right] \\
& - \frac{1}{2} \int A_0 d \left(\Omega_0^{k-1} A'(\Omega_0)^{-1 ij} \partial_i P A_j' \right) + \cdots
\end{aligned} \tag{17}
$$

Taking the functional derivative with respect to A_0 we get

$$
\begin{aligned}
\Omega_0^k P\Big]_{a+A'} =\ & \Omega_0^k P + d \left(P k \Omega_0^{k-1} A' \right) + \frac{P}{2} k(k-1) \Omega_0^{k-2} dA' A' + \cdots \right) \\
& - \frac{1}{2} d \left(\Omega_0^{k-1} A'(\Omega_0)^{-1 ij} \partial_i P A_j' \right)
\end{aligned}
$$

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[4]One can, of course, include higher orders, but these will suffice to illustrate the argument.

$$= \Omega_0^k P + d\big(PQ(a+A',a)\big) - \frac{1}{2}d\left(Q(a+A',a)(\Omega_0)^{-1ij}\partial_i P\, A'_j\right) \qquad (18)$$

where

$$Q(a+A',a) = k\int_0^1 dt (da + t\, dA')^{k-1}\, A' \qquad (19)$$

The left hand side of (18) corresponds to calculations with $a + A'$, while the P's on the right hand side are in terms of u^*u with just a, i.e., with $A' = 0$. The whole expression is linear in P, so we can easily take the difference of such terms with P_s and P_{s-1}, to obtain the field-dependent corrections to $u_s^* u_s$. This result is

$$\Omega_0^k \left[(u_s^* u_s)_{A'_i \neq 0} - (u_s^* u_s)_{A'_i = 0}\right] = d\left[(u_s^* u_s)_{A'_i=0}\, Q(a+A',a)\right]$$
$$-\frac{k}{2}d\left[\Omega_0^{k-1} A'(\Omega_0^{-1})^{ij}\partial_i(u_s^* u_s)_{A'=0}\, A'_j\right] + \cdots \qquad (20)$$

(For more details on these calculations using the idea of Moser's lemma, see [1, 13]. These calculations can also be done using the symbols and star products, see [8, 9].) We can now integrate this result over D to obtain the change in λ_s due to $a \to a + A'$ as

$$\lambda_s \Big]_{A'_i \neq 0} - \lambda_s \Big]_{A'_i = 0} = \oint_{\partial D} Q(a+A',a)\,(u_s^* u_s) - \frac{1}{2}\oint_{\partial D}\left[Q(a+A',a)\,(\Omega_0^{-1})^{ij}\partial_i(u_s^* u_s)\, A'_j\right] + \cdots \qquad (21)$$

In the integrands on the right hand side, $u_s^* u_s$ and other quantities are evaluated on ∂D.

The quantity $Q(a + A', a)$ which is given in (19) and appears in this formula is a generalized Chern-Simons form connecting two nonzero gauge potentials $A_1 = a$, $A_2 = a + A'$. If $\mathcal{P}(F)$ is an invariant polynomial (such as an index density) expressed as the symmetrized trace of a product of k F's, we define $Q(A_2, A_1)$ by

$$Q(A_2, A_1) = k\int_0^1 dt\, \mathcal{P}(A_2 - A_1, F_t, F_t, \cdots, F_t), \qquad A_t = A_1 + t(A_2 - A_1) \qquad (22)$$

This leads to $\mathcal{P}(F_2) - \mathcal{P}(F_1) = d\,Q(A_2, A_1)$. Thus up to an exact form, $Q(A_2, A_1)$ is the difference of two CS terms, but equation (22) gives a specific formula for the extra exact form. Such generalized CS forms are important in defining the anomaly with nontrivial background fields, see [14]. For our simple Abelian case, choosing $\mathcal{P}(F) = F^k$, $A_2 = a + A'$, $A_1 = a$,

$$Q(a+A',a) = k\int_0^1 dt (da + t\, dA')^{k-1}\, A' \qquad (23)$$

which agrees with (19). Upon comparison with (21), we see that the first correction to λ_s is indeed proportional to this generalized CS form. The second term in (21), which has additional Ω_0^{-1}, is subdominant at large N. Since $\delta S_{\mathrm{EE}} = -\sum_s \delta\lambda_s \log(\lambda_s/(1-\lambda_s))$, we can write

$$\delta S_{\mathrm{EE}} = \oint Q(a+A',a)\sum_s C_s\,(u_s^* u_s)_{\partial D} + \cdots, \qquad C_s = \log(\lambda_s/(1-\lambda_s)) \qquad (24)$$

Restating, our conclusion from this analysis is:

> The leading field-dependent correction to the entanglement entropy is proportional to the generalized Chern-Simons term $Q(a + A', a)$.

2.3 Higher dimensions, the Dolbeault index

This result can be generalized to include nonabelian gauge fields and arbitrary gravitational backgrounds. The wave functions for the LLL are holomorphic, they belong to the kernel of $\bar{\partial}$-operator and so the number of states is given by the Dolbeault index theorem [15]. Since the bulk part of $\text{Tr}\,(PA_0)$ is related to the number of states, it can also be calculated directly using the Dolbeault index density. The theorem states that the index is given as

$$\text{Index}(\bar{\partial}_V) = \int \text{td}(T_c K) \wedge \text{ch}(V) \tag{25}$$

where $\text{td}(T_c K)$ is the Todd class on the complex tangent space of K and $\text{ch}(V)$ is the Chern character of the vector bundle V. The vector bundle refers to the fact that we can have gauge fields in addition to the gravitational fields. The Todd class is given by

$$
\begin{aligned}
\text{td} \;&=\; \prod_i \frac{x_i}{1 - e^{-x_i}} \\
&=\; 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1 c_2 + \frac{1}{720}(-c_4 + c_1 c_3 + 3c_2^2 + 4c_1^2 c_2 - c_1^4) + \cdots
\end{aligned} \tag{26}
$$

where the first line gives the formula in terms of the splitting principle and the second gives the expansion for low dimensions in terms of the Chern classes c_i, which, for any vector bundle with curvature $\mathcal{F}$, are defined by

$$\det\left(1 + \frac{i\mathcal{F}}{2\pi}t\right) = \sum_i c_i\, t^i \tag{27}$$

The Chern character is defined by

$$\text{ch}(V) = \text{Tr}\left(e^{i\mathcal{F}/2\pi}\right) = \dim V + \text{Tr}\,\frac{i\mathcal{F}}{2\pi} + \frac{1}{2!}\text{Tr}\,\frac{i\mathcal{F} \wedge i\mathcal{F}}{(2\pi)^2} + \cdots \tag{28}$$

With these formulae, one can see that the index, for low dimensions, is contained in the expansion

$$
\begin{aligned}
\text{Index}(\bar{\partial}_V) \;&=\; \int \dim V\,\text{Tr}\left(\frac{iR}{4\pi}\right) + \text{Tr}\left(\frac{iF}{2\pi}\right) + \frac{\dim V}{12}(c_1^2 + c_2) + \frac{1}{2}\text{Tr}\left(\frac{iF}{2\pi}\right)^2 + \cdots \\
c_1 \;&=\; \text{Tr}\,\frac{iR}{2\pi}, \qquad c_2 = \frac{1}{2}\left[\left(\text{Tr}\,\frac{iR}{2\pi}\right)^2 - \text{Tr}\left(\frac{iR}{2\pi}\right)^2\right]
\end{aligned} \tag{29}
$$

Taking this index density as the invariant polynomial, we can define a corresponding generalized Chern-Simons term by the same formula as before, i.e.,

$$Q(A_2, A_1) = k\int_0^1 dt\,\mathcal{P}(A_2 - A_1, \mathcal{F}_t, \mathcal{F}_t, \cdots, \mathcal{F}_t), \qquad A_t = A_1 + t(A_2 - A_1) \tag{30}$$

where $\mathcal{F}$ can be F or R, with A referring to a gauge potential or the spin connection, respectively. Again, this definition (30) is consistent with $\mathcal{P}(\mathcal{F}_2) - \mathcal{P}(\mathcal{F}_1) = d\,Q(A_2, A_1)$. The fields A_2 and A_1 belong to the same topological class in the sense that the integrated index is the same for both, and one can continuously connect A_2 to A_1 as in $A_t = A_1 + t(A_2 - A_1)$. This is the meaning of the phrase "without changing the topological class of Ω" in the introduction. The rest of the argument for the

field-dependence of λ_s is similar to the Abelian case. (For more details, see [1].) Equation (21) still holds with $Q(a + A, a)$ replaced by the generalized CS form for the Dolbeault index given in (30). The formula for the leading correction to the entropy is the same as (24), with Q corresponding to the Dolbeault index density. So we can now restate our result more generally as:

> The leading field-dependent correction to the entanglement entropy is proportional to the generalized Chern-Simons term $Q(a + A', a)$ associated to the Dolbeault index density.

This completes my statement of the first result mentioned in the introduction.

3. The condensed matter perspective

Although this meeting is on particle physics and gravity, it may be interesting to change hats for a quick aside and view our result from a condensed matter perspective. We are considering the Hall state where all one-particle lowest Landau level (LLL) states are filled, so the density matrix (3) corresponds to the $\nu = 1$ integer quantum Hall state. The fermions in the LLL may be viewed as a droplet of a fluid, which is effectively incompressible because of the Pauli exclusion principle. For the same reason, the many-particle state is highly correlated. For simplicity, let me consider the two-dimensional case, i.e., $\mathbb{CP}^1 \sim S^2$ or its flat limit as we take the radius to be large. (The generalization to higher dimensional quantum Hall systems is straightforward.) The symplectic form Ω_0 is the magnetic field, a being the electromagnetic vector potential. The responses of the state to variations of this field (i.e., under $a \to a + A'$) and to variations of the spin connection are related to the electrical conductivity and the Hall viscosity, respectively. These are clearly quantities of physical interest. In the same spirit, one can ask about the entanglement entropy which can arise when we restrict attention to observables defined locally in some region of the droplet. As is to be expected, there will be a formally divergent constant term proportional to the phase volume [12], but the dependence on the background fields, from our result, is of the form

$$\delta S_{\text{EE}} \sim \frac{1}{2\pi} \oint_{\partial D} \left[A - a + \frac{1}{2}(\alpha - \alpha_0) \right] \sum_s C_s (u_s^* u_s)_{\partial D} \tag{31}$$

where a corresponds to the starting constant magnetic field, $A - a = A'$ gives the perturbation to it, and likewise, α_0 is the spin connection for $\mathbb{CP}^1$ with the Fubini-Study metric and $\alpha - \alpha_0$ is the perturbation to it. The EE has been argued to be useful in characterizing topological phases of matter, and so, in this context, I expect the result (31) should be of interest.

4. Coupling matter fields

We now return to the main topic and the second result mentioned in the beginning. Again, we will consider the manifold $\mathbb{CP}^k$, with possible perturbations to the gauge fields and spin connection within the same topological class as before. Consider now a matter system with the degrees of freedom described by a set of variables $\{q_\lambda\}$ and conjugate variables $\{p_\lambda\}$. The relevant physical

quantity for the quantum dynamics of these degrees of freedom is the transformation kernel, which, for an infinitesimal change ϵ of time, is

$$\langle q' | e^{-iH\epsilon} | q \rangle = \int [dp] \exp \left[ip_\lambda (q'_\lambda - q_\lambda) - H(p,q)\, \epsilon \right] \tag{32}$$

The variables q_λ may be viewed as the coefficients in the mode expansion of an $N \times N$ matrix $\hat{q}$ with matrix elements

$$\hat{q}_{ij} = \sum_\lambda q_\lambda (T_\lambda)_{ij}, \qquad i,j = 1,2,\cdots,N, \tag{33}$$

where $\{T_\lambda\}$ form an orthonormal basis for matrices acting on $\mathcal{H}_N$. (This is how the matter system is placed in the fuzzy space.) As in the case of P, we can rewrite the $\{T_\lambda\}$ in terms of functions which are symbols associated to the matrices. In the present situation it is slightly easier to use the so-called contravariant symbols which form the basis for the Berezin-Toeplitz quantization procedure [16, 17]. The contravariant symbol ϕ associated to $\hat{q}_{ij}$ is defined by [5]

$$\hat{q}_{ij} = \int d\mu \, u_i^* \, \phi \, u_j \tag{34}$$

We can now convert the set of variables $\{q_\lambda, p_\lambda\}$ to symbols and products of them to star products; for example, we can write

$$
\begin{aligned}
\sum_\lambda p_\lambda p_\lambda &= \sum_{\lambda,\lambda'} \mathrm{Tr}(p_\lambda T_\lambda)(p_{\lambda'} T_{\lambda'}) = \sum_{i,j} \int d\mu d\mu' \left[u_i^*(z)\Pi(z)u_j(z) \right] \left[u_j^*(z')\Pi(z')u_i(z') \right] \\
&\equiv \int d\mu \sum_i u_i^* (\Pi * \Pi) u_i = \int d\mu \, \rho \, \Pi * \Pi
\end{aligned}
\tag{35}
$$

The transition from the first line to the second line of this equation uses the definition of the star product for the contravariant symbols. Using this equation, and similar formulae for other terms in (32), the transformation kernel can be converted to the path integral

$$
\begin{aligned}
Z &= \mathcal{N} \int [D\Pi \, D\phi] \exp\left(i \int dt \, d\mu \, \rho \, \left[\Pi * \dot{\phi} - H(\Pi,\phi) \right] \right) \\
&= \mathcal{N} \int [D\Pi \, D\phi] \exp\left(- \int d\mu \, \rho \, \mathbb{A} \right)
\end{aligned}
\tag{36}
$$

where $\mathbb{A}$ is given by

$$\mathbb{A} = -i \left(\Pi * \dot{\phi} - H(\Pi,\phi) \right) \, dt \tag{37}$$

$\mathbb{A}$ is the Poincaré-Cartan one-form for the dynamics of the variables $\{q_\lambda, p_\lambda\}$, i.e., for matter fields. (We have also generalized (32) to a finite change of time.)

In the path integral (36) and in the definition of $\mathbb{A}$ in (37), all the products with ϕ and the conjugate variable Π involve star-products. The Hamiltonian will contain such products as well, among the fields and their conjugates. As an example, let T_α be the basis matrices obeying the

---

[5] The contravariant symbol is a classical function which, upon "quantization" according to the formula (34), leads to the quantum version, i.e., to the matrix $\hat{q}$. The symbol we used in (8) started with the matrix and obtained a classical function from it. That function is referred to as the covariant symbol. In either case, the definitions are made using the wave functions u_i.

$SU(k + 1)$ algebra in the $N \times N$ representation; they form a subset of the full basis $\{T_\lambda\}$ and commutators with T_α serve as derivatives and local rotations. Then we can consider a Hamiltonian of the form

$$H = \frac{1}{2}\mathrm{Tr}\left[\hat{\Pi}\hat{\Pi} + \beta_1 [T_\alpha, \hat{q}][T_\alpha, \hat{q}] + m_0^2 \hat{q}\hat{q}\right] + g_0 \,\mathrm{Tr}(\hat{q}^4) \tag{38}$$

where β_1, m_0, g_0 are constants. This Hamiltonian, if we evaluate the trace using the expansion (33) (and a similar one for $\hat{\Pi}$) is expressed entirely in terms of $\{q_\lambda, p_\lambda\}$. Using symbols and star products according to (34), (35), H in (38) becomes

$$H(\Pi, \phi) = \int d\mu\, \rho \left[\frac{1}{2}\left(\Pi * \Pi + \alpha_1 (\nabla_\alpha \phi) * (\nabla_\alpha \phi) + m_0^2 \phi * \phi\right) + g_0\, \phi * \phi * \phi * \phi\right] \tag{39}$$

Going back to (36), the key point to note is the presence of the factor ρ which gives the density of states. As I mentioned earlier, the number of states is given by the Dolbeault index theorem, so we can identify the density ρ with the Dolbeault index density (up to terms which are total derivatives and hence vanish upon integration). In the case of the entanglement entropy, we considered the difference of this index density for two different connections, which led to the $(2k - 1)$-form $Q(a + A', a)$, appropriate for integration over the $(2k - 1)$-dimensional interface between D and its complement. It is also possible to "integrate up" from the index density to define a $(2k + 1)$-form on $\mathbb{CP}^k \times \mathbb{R}$, $\mathbb{R}$ being the time direction. This CS $(2k + 1)$-form is designed to have the property that upon variation with respect to A_0 (the time-component of $U(1)$ field A) it will give the Dolbeault index density. Explicitly this CS form is given by [15]

$$S_{\mathrm{eff}} = \int \left[\mathrm{td}(T_c K) \wedge \sum_p (CS)_{2p+1}(A)\right]_{2k+1} + 2\pi \int \Omega_{2k+1}^{\mathrm{grav}} \tag{40}$$

where $(CS)_{2p+1}$ is the usual Chern-Simons $(2p + 1)$-form for gauge fields and $\Omega_{2k+1}^{\mathrm{grav}}$ is defined by

$$\dim V [\mathrm{td}(T_c K)]_{2k+2} = d\, \Omega_{2k+1}^{\mathrm{grav}} \tag{41}$$

S_{eff} in (40) depends on the gauge fields A, spin connection α. Because it is the integral of a differential form, S_{eff} in (40) can have, at most, one power of the time-component of the Abelian part of A, i.e. with $A_0 dt$. We can write (40) as

$$S_{\mathrm{eff}} = \int \rho\, A_0\, dt + S_{\mathrm{eff}}^{(0)} \tag{42}$$

where $S_{\mathrm{eff}}^{(0)}$ is independent of A_0. The coefficient of A_0, by construction, is the index density ρ, which is also the field-dependent generalization of $\sum_i u_i^* u_i$. Since Poincaré-Cartan one-form $\mathbb{A}$ has only the time-component, we see that we can also write

$$S_{\mathrm{eff}}(A + \mathbb{A}, \alpha) = \int \rho\, \mathbb{A} + S_{\mathrm{eff}}(A, \alpha) \tag{43}$$

Going back to (36), we see that we can write the exponent in as $S_{\mathrm{eff}}(A + \mathbb{A}, \alpha) - S_{\mathrm{eff}}(A, \alpha)$. Rather than subtracting out $S_{\mathrm{eff}}(A, \alpha)$, we will keep this in the path integral, as it does correspond to the dynamics of the background fields themselves [18]. This finally brings us to our second result:

The path integral describing the dynamics of matter fields coupled to gauge and gravitational fields defining the large N limit for fuzzy spaces is given by

$$Z = \mathcal{N} \int [D\Pi\, D\phi]\, \exp\Big(iS_{\text{eff}}(A + \mathbb{A})\Big) \qquad (44)$$

where $\mathbb{A}$ is the Poincaré-Cartan form for the matter dynamics and S_{eff} is as given in (40).

As an example, in four dimensions, the matter coupling takes the form

$$S_{\text{matter}} = \frac{1}{32\pi^2} \int (i\mathbb{A}) \left[\dim V\Big(F_{\mu\nu}F_{\alpha\beta} + \frac{1}{24}R^{ab}_{\mu\nu}R^{ab}_{\alpha\beta}\Big) + \text{Tr}(t_a t_b)\bar{F}^a_{\mu\nu}\bar{F}^b_{\alpha\beta} \right] dx^\mu \cdots dx^\beta \qquad (45)$$

where $F = (-i)\frac{1}{2}F_{\mu\nu}dx^\mu\, dx^\nu$ is the Abelian gauge field, and $\bar{F} = (-it_a)\frac{1}{2}F^a_{\mu\nu}dx^\mu\, dx^\nu$ is the nonabelian part of the background gauge fields, see footnote 1 for a clarifying remark on this. $R^{ab}_{\mu\nu}$ is the Riemann curvature tensor. Thus $\mathbb{A}$ is multiplied by a polynomial of fields and curvatures whose form is determined by the Dolbeault index. If the integration over the "momenta" Π is done in (44), one can write it in terms of a spacetime action. In this case the matter part takes the form mentioned in equation (1).

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