ARTICLE IN PRESS
JID:YAIMA AID:109188 /FLA [m1L; v1.338] P.1 (1-63)
Advances in Mathematics ••• (••••) ••••••

Contents lists available at ScienceDirect

Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim

Elliptic finite-band potentials of a non-self-adjoint Dirac operator

Gino Biondini [a], Xu-Dan Luo [b], Jeffrey Oregero [c], Alexander Tovbis [c]

[a] *Department of Mathematics, State University of New York at Buffalo, Buffalo, NY 14260, United States of America*
[b] *Key Laboratory of Mathematics Mechanization, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China*
[c] *Department of Mathematics, University of Central Florida, Orlando, FL 32816, United States of America*

A R T I C L E I N F O

Article history:
Received 28 February 2023
Accepted 16 June 2023
Available online xxxx
Communicated by C. Fefferman

Keywords:
Dirac operators
Spectral theory
Nonlinear Schrödinger equation
Non-self-adjoint problems

A B S T R A C T

We present an explicit two-parameter family of finite-band Jacobi elliptic potentials given by $q \equiv A\,\mathrm{dn}(x;m)$, where $m \in (0,1)$ and A can be taken to be positive without loss of generality, for a non-self-adjoint Dirac operator L, which connects two well-known limiting cases of the plane wave ($m = 0$) and of the sech potential ($m = 1$). We show that, if $A \in \mathbb{N}$, then the spectrum consists of $\mathbb{R}$ plus $2A$ Schwarz symmetric segments (bands) on $\mathrm{i}\mathbb{R}$. This characterization of the spectrum is obtained by relating the periodic and antiperiodic eigenvalue problems for the Dirac operator to corresponding eigenvalue problems for tridiagonal operators acting on Fourier coefficients in a weighted Hilbert space, and to appropriate connection problems for Heun's equation. Conversely, if $A \notin \mathbb{N}$, then the spectrum of L consists of infinitely many bands in $\mathbb{C}$. When $A \in \mathbb{N}$, the corresponding potentials generate finite-genus solutions for all the positive and negative flows associated with the focusing nonlinear

E-mail address: biondini@buffalo.edu (G. Biondini).

https://doi.org/10.1016/j.aim.2023.109188

Schrödinger hierarchy, including the modified Korteweg-deVries equation and the sine-Gordon equation.

Contents

1. Introduction and main results

1.1. Background

In this work we study a non-self-adjoint Dirac operator with a Jacobi elliptic potential, namely,

$$L\phi = z\phi\,, \quad z \in \mathbb{C}\,, \tag{1.1}$$

where $\phi(x;z) = (\phi_1, \phi_2)^{\mathrm{T}}$, the superscript "T" denoting matrix transpose, L is given formally by

$$L := \mathrm{i}\sigma_3(\partial_x - Q(x))\,, \qquad Q(x) = \begin{pmatrix} 0 & q(x) \\ -\overline{q(x)} & 0 \end{pmatrix}, \quad x \in \mathbb{R}\,, \tag{1.2}$$

the potential $Q(x)$ is l-periodic, $\sigma_3 := \mathrm{diag}(1, -1)$ (cf. Appendix A.1) and overline denotes the complex conjugate. In particular, let

$$q(x; A, m) = A\,\mathrm{dn}(x;m)\,, \tag{1.3}$$

where $\mathrm{dn}(x;m)$ is one of the three basic Jacobi elliptic functions (cf. [43,79]), and $m \in (0,1)$ is the elliptic parameter. Finally, A is an arbitrary constant, which one can take to be real and positive without loss of generality. (It is easy to see that $\arg A \neq 0$ leaves the spectrum invariant.) We will do so throughout this work. Recall that $\mathrm{dn}(x;m)$ has minimal period $l = 2K$ along the real x-axis, where $K := K(m)$ is the complete elliptic integral of the first kind [43,79]. Also recall that $\mathrm{dn}(x;0) \equiv 1$ and $\mathrm{dn}(x;1) \equiv \mathrm{sech}\,x$. Both of the limiting cases $m = 0$ and $m = 1$ are exactly solvable (i.e., the spectrum is known in closed form), and therefore provide convenient "bookends" for the results of this work.

There are several factors that motivate the present study. A first one is that Dirac operators arise naturally in quantum field theory [54,104], and therefore the identification of exactly solvable potentials is relevant in that context. A second one is the obvious similarity between the study of (1.1) and that of eigenvalue problems for the time-independent Schrödinger equation, namely

$$(-\Delta + V(x))\,\phi = \lambda\,\phi\,, \tag{1.4}$$

where Δ denotes the n-dimensional Laplacian operator and $\phi : \mathbb{R}^n \to \mathbb{C}$, which has been an integral component of mathematical physics since its first appearance in the 1920's (e.g., see [38,76,82]), and which received renewed interest in the late 1960's and 1970's (e.g., see [1,24,62,78,98]) thanks to the connection with infinite-dimensional integrable systems. Namely, the fact that the one-dimensional time-independent Schrödinger equation [i.e., (1.4) with $n = 1$] is the first half of the Lax pair for the *Korteweg-deVries* (KdV) equation [36,66]. As a result, the study of direct and inverse spectral problems for the

Schrödinger operator played a key role in the development of the so-called *inverse scattering transform* (IST) to solve the initial value problem for the KdV equation [36,66]. The direct and inverse scattering theory was later made more rigorous, and generalizations of the theory were also studied [6,7,20,24,27,52,53,67,73,75,77,93]. In particular, the so-called finite-gap (or finite-band) solution became a primary object of study.

Similar problems have been considered for (1.1), since it comprises the first half of the Lax pair associated to the *nonlinear Schrödinger* (NLS) equation, namely, the partial differential equation (PDE)

$$\mathrm{i}q_t + q_{xx} + 2s|q|^2 q = 0\,. \tag{1.5}$$

Here $q : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, subscripts x and t denote partial differentiation and, as usual, the sign $s = \pm 1$ denotes the focusing and defocusing cases, respectively. Similarly to the KdV equation, the NLS equation is an infinite-dimensional Hamiltonian system. Also, similarly to the KdV equation, the NLS equation is a ubiquitous physical model. In particular, (1.5) is a universal model describing the slow modulations of a weakly monochromatic dispersive wave envelope, and therefore appears in many physical contexts, such as deep water waves, nonlinear optics, plasmas, ferromagnetics and Bose-Einstein condensates (e.g., see [1,3,81]). Therefore, the study of the NLS equation is of both theoretical and applicative interest.

In 1972 [105], Zakharov and Shabat showed that (1.5) is the compatibility condition of the matrix Lax pair

$$\phi_x = (-\mathrm{i}z\,\sigma_3 + Q(x,t))\,\phi\,, \tag{1.6a}$$

$$\phi_t = (-2\mathrm{i}z^2\sigma_3 + H(x,t,z))\,\phi\,, \tag{1.6b}$$

with σ_3 as above, and

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ -s\overline{q(x,t)} & 0 \end{pmatrix}, \qquad H(x,t,z) = 2zQ - \mathrm{i}\sigma_3(Q^2 - Q_x)\,. \tag{1.7}$$

Following [105], (1.6a) [i.e., the first half of the Lax pair] is referred to as the *Zakharov-Shabat* (ZS) scattering problem. It is easy to see that (1.6a) [with $s = 1$] is equivalent to (1.1). Thus, the solution to (1.5) [with $s = 1$] comprises the scattering potential q in (1.1). Moreover, one can also show that time evolution of q according to the focusing NLS equation (1.5) [with $s = 1$] amounts to an isospectral deformation of the potential for the Dirac operator (1.2).

Scattering theory for the Zakharov-Shabat system have been studied extensively over the years. In [105] the IST for (1.5) in the focusing case with localized data, i.e., with $q(x,t=0) \in L^1(\mathbb{R})$, was formulated. Corresponding results for the defocusing case with constant boundary conditions (BCs), i.e., $|q(x,t)| \to q_o \neq 0$ as $x \to \pm\infty$, were obtained in [106]. The theory was then revisited and elucidated in [1,32,78]. When $q \in L^1(\mathbb{R})$, the isospectral data is composed of two pieces: an absolutely continuous spectrum, and a set of discrete eigenvalues. When q is periodic, however, the isospectral data is purely absolutely continuous and has a band and gap structure.

Of particular interest is the effort to find classes of potentials for which the scattering problem can be solved exactly. Satsuma and Yajima [86] considered the case of $q(x) = A\,\mathrm{sech}\,x$, with A an arbitrary positive constant, and obtained a complete representation of eigenfunctions and scattering data. Their work was later generalized by Tovbis and Venakides [95] to potentials of the type $q(x) = A\,\mathrm{sech}\,x\,\mathrm{e}^{-ia\log(\cosh x)}$, with A as above and a an arbitrary real constant. These results were then used in [57,96] to study the behavior of solutions of the focusing NLS equation in the semiclassical limit. More recently, Trillo et al. [35] obtained similar results for potentials of the type $q(x) = A\tanh x$ in the defocusing case. In all of these cases, the ZS scattering problem is reduced to connection problems for the hypergeometric equation. Finally, Klaus and Shaw [59,60] identified classes of "single-lobe" potentials for which the point spectrum is purely imaginary.

The above-mentioned works considered potentials that are either localized or tend to constant boundary conditions as $|x| \to \infty$. Spectral problems for the Schrödinger operator with a periodic potential similar to the one considered here are also a classical subject, and their study goes back to Lamé [65], and Ince [47–49], where the spectrum for a two-parameter family of potentials was studied, and necessary and sufficient conditions in order for such potentials to give rise to a spectrum with a finite number of gaps were derived (see also [4,15,28,39,72]). More recently, these results were generalized in [92] and [88], and in seminal work a characterization of all elliptic algebro-geometric solutions of the KdV and AKNS hierarchies was given by Gesztesy and Weikard in [39–41].

Finite-band potentials for the focusing and defocusing ZS scattering problems have also been studied [8,40,51,63,87]. In particular, the special case of genus-one potentials was explicitly considered in [16,56], and the stability of those solutions was recently studied in [22]. On the other hand, the identification of exactly solvable cases for periodic potentials is generally challenging, and few families of finite-band potentials for (1.2) have been studied in detail (see [40,41]).

Here we present an explicit, two-parameter family of finite-band potentials of the focusing ZS system and we characterize the resulting spectrum. We also show that (1.1) with the potential (1.3) can be reduced to certain connection problems for Heun's equation. Unlike the case of the hypergeometric equation, the connection problem for Heun's equation has not been solved in general [85]. Still, special cases can be solved exactly. For example, for certain classes of periodic potentials it turns out that Hill's equation [i.e., (1.4) with $n = 1$ and periodic potential] can be mapped to a Heun equation. Classical works [47,48,72] where the spectrum of Hill's equation for a multi-parameter family of potentials was studied, resulted in the derivation of necessary and sufficient conditions for such potentials to give rise to a spectrum with a finite number of bands and gaps. Importantly, the absence of a gap in the spectrum of the Hill operator corresponds uniquely to the coexistence of solutions, namely, the existence of two linearly independent peri-

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

odic, or antiperiodic, solutions to the given ordinary differential equation (ODE) [72]. More recently, those results were strengthened in [100,102] and [45]. The results of this work provide a direct analogue of all these results for the Dirac operator (1.2) as well as for the Hill operator with PT-symmetric potential.

1.2. Main results

We first introduce some definitions in order to state the main results of this work (see Appendix A.1 for further notations and standard definitions).

Definition 1.1 *(Lax spectrum).* The Lax spectrum of the matrix-valued differential expression L in (1.2) is the set

$$\Sigma(L) := \left\{ z \in \mathbb{C} : L\phi = z\phi,\ 0 < \|\phi\|_{L^\infty(\mathbb{R};\mathbb{C}^2)} < \infty \right\}, \tag{1.8}$$

i.e., the set of complex numbers z such that (1.1) has at least one bounded nonzero solution.

It can be proved that for q locally integrable the Lax spectrum defined above equals the spectrum of the maximal operator associated with L in $L^2(\mathbb{R};\mathbb{C}^2)$, the space of square-integrable two-component vector-valued functions, namely the set $\{z \in \mathbb{C} : z \notin \rho(L)\}$, where $\rho(L)$ is the resolvent set of L (see [84] p. 249). Moreover, it is well known that if q is l-periodic with minimal period l, then $\Sigma(L)$ is purely continuous and comprised of an at most countable collection of regular analytic arcs, referred to as *bands*, in the spectral plane [40,84]. Throughout this work we will occasionally use spectrum as a synonym for the Lax spectrum. Further properties of the Lax spectrum are discussed in Section 2. If the potential q is such that there are at most finitely many bands we say that q is a *finite-band potential* (see Definition 2.5). The class of finite-band potentials plays a key role in the IST for the NLS equation on the torus [8,37,71]. In particular, it was shown in [51] that the potential can be reconstructed from the knowledge of two key spectral data: (i) the periodic and antiperiodic eigenvalues of L (i.e., the set of values z associated with periodic or antiperiodic eigenfunctions, respectively), which correspond to endpoints of spectral bands, and (ii) the Dirichlet (or auxiliary) eigenvalues of L, defined as the set of zeros of the 1,2 entry of the monodromy matrix (see Section 2 for precise definitions of all these quantities). To specify the dependence of solutions associated with (1.3) on the parameters A, m, we will also occasionally use the notation $\Sigma(L; A, m)$ to denote the Lax spectrum.

Theorem 1.2. *Consider (1.1) with $q \equiv A\,\mathrm{dn}(x; m)$, $m \in (0, 1)$, and $A > 0$. Then the potential q is finite-band if and only if $A \in \mathbb{N}$. Moreover, if $A \in \mathbb{N}$, then:*

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

$$\Sigma(L;A,m)\subset\mathbb{R}\cup(-\mathrm{i}A,\mathrm{i}A)\,, \tag{1.9}$$

and q is a $2A$-band (i.e., a genus $2A-1$) potential of the Dirac operator (1.2).

(Of course it is well known that $\Sigma(L)$ is Schwarz symmetric and $\mathbb{R}\subset\Sigma(L)$ [71,74].) Theorem 1.2 is a consequence of the following more detailed description of the spectrum:

Theorem 1.3. *Assume the conditions of Theorem 1.2. If $A\in\mathbb{N}$ then:*

1. *For any $m\in(0,1)$, the non-real part of the Lax spectrum, $\Sigma(L;A,m)\setminus\mathbb{R}$, is a proper subset of $(-\mathrm{i}A,\mathrm{i}A)$. (For $m=0$, the Lax spectrum is $\Sigma(L;A,0)=\mathbb{R}\cup[-\mathrm{i}A,\mathrm{i}A]$.)*
2. *For any $m\in(0,1)$, there are exactly $2A$ symmetric bands of $\Sigma(L;A,m)$ along $(-\mathrm{i}A,\mathrm{i}A)$, separated by $2A-1$ open gaps. The central gap (i.e., the gap surrounding the origin) contains an eigenvalue at $z=0$, which is periodic when A is even and antiperiodic when A is odd.*
3. *For any $m\in[0,1)$, $\mathbb{R}\subset\Sigma(L;A,m)$ contains infinitely many interlaced periodic and antiperiodic eigenvalues, symmetrically located with respect to $z=0$.*
4. *Each periodic/antiperiodic eigenvalue $z\in\mathbb{R}$ has geometric multiplicity two and each periodic/antiperiodic eigenvalue $z\in(-\mathrm{i}A,\mathrm{i}A)\setminus\{0\}$ has geometric multiplicity one.*
5. *Each periodic/antiperiodic eigenvalue $z\in\mathbb{R}$ is simultaneously a Dirichlet eigenvalue. All these Dirichlet eigenvalues are immovable.*
6. *Each of the open $2A-1$ gaps on $(-\mathrm{i}A,\mathrm{i}A)$ contains exactly one movable Dirichlet eigenvalue. Thus, all of the $2A-1$ movable Dirichlet eigenvalues of the finite-band solution with genus $2A-1$ are located in the gaps of the interval $(-\mathrm{i}A,\mathrm{i}A)$.*

Recall that a movable Dirichlet eigenvalue is a Dirichlet eigenvalue whose location changes when changing the normalization of the monodromy matrix, whereas the location of immovable Dirichlet eigenvalues is independent of the normalization of the monodromy matrix. For an N-band potential there are a total of $N-1$ movable Dirichlet eigenvalues (cf. Definition 2.12 and [34,37]).

Theorem 1.4. *Assume the conditions of Theorem 1.2. If $A\notin\mathbb{N}$, then:*

1. *For any $m\in(0,1)$, each periodic or antiperiodic eigenvalue has geometric multiplicity one.*
2. *There are no periodic or antiperiodic eigenvalues on $\mathbb{R}$.*
3. *There are infinitely many spines (spectral bands emanating transversally from the real axis) at the real critical points of the Floquet discriminant (i.e., the trace of the monodromy matrix).*

Time evolution according to the NLS equation is an isospectral deformation of a potential of (1.2). Thus, by Theorem 1.2, if $A\in\mathbb{N}$, the initial condition $q(x,0)=$

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

$A\,\mathrm{dn}(x;m)$ generates a genus $2A-1$ solution of the focusing NLS equation; conversely, if $A \notin \mathbb{N}$, the corresponding solution is not finite-genus.

After various preliminaries in Section 2, the proof of Theorems 1.2 and 1.3 involves several steps:

- In Section 3 we map (1.1) into Hill's equation with a complex potential, and in Section 4 we map Hill's equation into a second-order trigonometric ODE.
- In Section 4.2 we map the trigonometric ODE into a three-term recurrence relation for the Fourier coefficients.
- In Section 4.3 we demonstrate that, when $A \in \mathbb{N}$, each periodic or antiperiodic eigenvalue of L is associated to a corresponding ascending or descending semi-infinite Fourier series.
- In Section 5 we map the trigonometric ODE into Heun's equation and relate the periodic and antiperiodic eigenvalue problems for (1.1) with potential (1.3) to a connection problem for Heun's equation.
- Moreover, in Section 5 we show that the periodic and antiperiodic eigenvalues of (1.2) with potential (1.3) correspond to the eigenvalues of certain tridiagonal operators that encode the recurrence relations for the coefficients of the Frobenius series solution of Heun's equation at the origin and at infinity.
- In Section 6 we establish that all eigenvalues of the above-mentioned tridiagonal operators are real.

The determination of the precise number of spectral bands for any $m \in (0,1)$ is proved in Section 8. Finally, Theorem 1.4 is proved in Section 7. Notation, standard definitions, several technical statements and additional results and observations are relegated to the appendices.

2. Preliminaries

We begin by briefly reviewing basic properties of the Lax spectrum. Unless stated otherwise, all statements in this section hold for operators L with arbitrary continuous l-periodic potentials.

2.1. Bloch-Floquet theory

While it is natural to pose (1.1) on the whole real x-axis, all of the requisite information for the spectral theory is contained in the period interval of the potential, namely, $I_{x_o} := [x_o, x_o + l]$, where $x = x_o$ is an arbitrary base point. Consider the *Floquet boundary conditions* (BCs):

$$\mathrm{BC}_\nu(L) := \{\phi \;:\; \phi(x_o + l; z) = \mathrm{e}^{\mathrm{i}\nu l}\phi(x_o; z)\,,\ \nu \in \mathbb{R}\}\,. \tag{2.1}$$

Definition 2.1 *(Floquet eigenvalues of the Dirac operator).* Let the operator $L : H^1(I_{x_o};\mathbb{C}^2) \to L^2(I_{x_o};\mathbb{C}^2)$ be defined by (1.2). Let $\mathrm{dom}(L) := \{\phi \in H^1(I_{x_o};\mathbb{C}^2) : \phi \in \mathrm{BC}_\nu(L)\}$. The set of Floquet eigenvalues of L is given by

$$\Sigma_\nu(L) := \{z \in \mathbb{C} : \exists \phi \not\equiv 0 \in \mathrm{dom}(L) \text{ s.t. } L\phi = z\phi\}\,. \tag{2.2}$$

In particular, $\nu = 2n\pi/l$, $n \in \mathbb{Z}$, identifies periodic eigenfunctions, while $\nu = (2n-1)\pi/l$, $n \in \mathbb{Z}$, identifies antiperiodic eigenfunctions. We will call the corresponding eigenvalues periodic and antiperiodic, respectively, and we will denote the set of periodic and antiperiodic eigenvalues by $\Sigma_\pm(L)$, respectively.

(H^1 denotes the space of square-integrable functions with square-integrable first derivative.) It is well-known that $\Sigma_\nu(L)$ is discrete and countably infinite [15,25].

Next we review the theory of linear homogeneous ODEs with periodic coefficients and important connections to the Lax spectrum. We set the base point $x_o = 0$ without loss of generality. Recall, the *Floquet solutions (or Floquet eigenfunctions)* of (1.1) are solutions such that

$$\phi(x + l; z) = \mu\, \phi(x; z)\,, \tag{2.3}$$

where $\mu := \mu(z)$ is the Floquet multiplier. Then by Floquet's Theorem (see [15,33]) all bounded (in x) Floquet solutions of (1.1) have the form $\phi(x; z) = \mathrm{e}^{\mathrm{i}\nu x}\psi(x; z)$, where $\psi(x + l; z) = \psi(x; z)$ and $\nu := \nu(z) \in \mathbb{R}$. Thus, a solution of (1.1) is bounded for all $x \in \mathbb{R}$ if and only if $|\mu| = 1$, in which case one has the relation

$$\mu = \mathrm{e}^{\mathrm{i}\nu l}\,, \tag{2.4}$$

with $\nu \in \mathbb{R}$. The quantity $\mathrm{i}\nu$ is the Floquet exponent. (With a slight abuse of terminology, we will often simply refer to ν as the Floquet exponent for brevity.) The Floquet multipliers are the eigenvalues of the *monodromy matrix* $M := M(z)$, defined by $Y(x+l; z) = Y(x; z)M(z)$, where $Y(x; z)$ is any fundamental matrix solution of (1.1). It is well-known that the monodromy matrix is entire as a function of z [71,74]. Note that $\det M(z) \equiv 1\ \forall z \in \mathbb{C}$ by Abel's formula, since (1.2) is traceless. Thus, the eigenvalues of M are given by the roots of the quadratic equation $\mu^2 - 2\Delta\,\mu + 1 = 0$, where $\Delta := \Delta(z)$ is the *Floquet discriminant*, i.e.,

$$\Delta(z) = \tfrac{1}{2}\,\mathrm{tr}\, M(z)\,. \tag{2.5}$$

Further, $\mu_\pm = \Delta \pm \sqrt{\Delta^2 - 1}$. Thus (1.1) admits bounded solutions if and only if $-1 \le \Delta \le 1$.

Remark 2.2. For $q \in C(\mathbb{R})$ one has $\Delta(z) = \cos(zl) + o(1)$ as $z \to \infty$ along the real z-axis (see [71,74]).

The above considerations yield an equivalent representation of the Lax spectrum (see [15,28,84]):

Theorem 2.3. *The Lax spectrum* $\Sigma(L)$ *is given by*

$$\Sigma(L) = \{z \in \mathbb{C} : \Delta(z) \in [-1,1]\}\,. \tag{2.6}$$

Additionally, for any fixed $\nu \in \mathbb{R}$ *the Floquet eigenvalues are given by*

$$\Sigma_\nu(L) = \{z \in \mathbb{C} : \Delta(z) = \cos(\nu l)\}\,. \tag{2.7}$$

For each $\nu \in \mathbb{R}$ *the set* $\Sigma_\nu(L)$ *is discrete and the only accumulation point occurs at infinity. Moreover,*

$$\Sigma(L) = \bigcup_{\nu \in [0,2\pi/l)} \Sigma_\nu(L)\,. \tag{2.8}$$

Remark 2.4. By (2.3), (2.4) and (2.7), the values $z \in \mathbb{C}$ for which $\Delta(z) = \pm 1$ are the periodic and antiperiodic eigenvalues $z \in \Sigma_\pm(L)$ (see Definition 2.1), respectively. The periodic and antiperiodic eigenvalues correspond to band edges of the Lax spectrum. Further, $\Sigma_\nu(L) \cap \Sigma_{\nu'}(L) = \emptyset$ for all $\nu \neq \nu' \mod 2\pi/l$.

2.2. General properties of the Lax spectrum

Owing to (2.6), the Lax spectrum (1.8) is located along the zero level curves of $\operatorname{Im}\Delta(z)$, i.e., $\Gamma := \{z \in \mathbb{C} : \operatorname{Im}\Delta(z) = 0\}$. Moreover, Γ is the union of an at most countable set of regular analytic curves Γ_n [40], each starting from infinity and ending at infinity:

$$\Gamma = \cup_{n\in\mathbb{N}} \Gamma_n\,. \tag{2.9}$$

(The precise details of the map $n \mapsto \Gamma_n$ are not important for the present purposes.) Different curves $\Gamma_i \neq \Gamma_j$ (and therefore different spectral bands) can intersect at saddle points of $\Delta(z)$. However, two distinct Γ_n can intersect at most once, as a result of the fact that each Γ_n is a level curve of the harmonic function $\operatorname{Im}\Delta(z)$. Thus the Lax spectrum $\Sigma(L)$ cannot contain any closed curves in the finite z-plane.

Definition 2.5 *(Spectral band).* A spectral band is a maximally connected regular analytic arc along Γ_n where $\Delta(z) \in [-1,1]$ holds. Each finite portion of Γ_n where $|\operatorname{Re}\Delta(z)| > 1$, delimited by a band on either side, is called a spectral gap.

Lemma 2.6. *([12]) The real* z*-axis is the only band extending to infinity;* $\Sigma(L)$ *contains no closed curves in the finite* z*-plane; and the resolvent set* $\varrho := \mathbb{C} \setminus \Sigma(L)$ *is comprised of two connected components.*

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

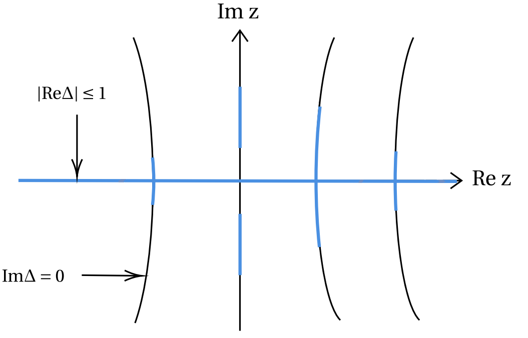

Fig. 1. Schematic diagram of the Lax spectrum for a generic potential.

With the above definition, the Lax spectrum can be decomposed into bands and gaps along each Γ_n as in a self-adjoint problem, with the crucial difference that here bands and gaps are not restricted to lie along the real z-axis (as they would be in a self-adjoint problem), but lie instead along arcs of Γ_n. Fig. 1 provides a schematic illustration of these concepts.

We call a spectral band intersecting the real or imaginary z-axis transversally a *spine* [74]. Generically, the Lax spectrum of the operator (1.2) includes infinitely many spines emanating from (infinitely many) critical points that extend to ($\pm$) infinity along the real z-axis [25,74], in which case we call q an *infinite-band potential*. Otherwise, we call q a *finite-band potential*. Specifically, if there are N bands (not including the real z-axis) we say that q is an N-*band potential*. The corresponding solutions of the focusing NLS equation are described in terms of Riemann Θ-functions determined by hyperelliptic Riemann surfaces of genus $G = N - 1$ (see [8,34,37,51,91]). For example, $q \equiv A$ is a genus-0 (i.e., a 1-band) potential of the Dirac operator (1.2), and $q \equiv \mathrm{dn}(x; m)$ is a genus-1 (i.e., a 2-band) potential.

Remark 2.7. The following sets play a key role in the analysis:

- Periodic/antiperiodic points: points $z_\pm \in \mathbb{C}$ such that $\Delta(z_\pm) = \pm 1$ (note $z_\pm \in \Sigma_\pm(L)$);
- Critical points: points $z_c \in \mathbb{C}$ such that $\partial_z \Delta(z_c) = 0$.

We denote by $\Phi(x; z)$ the fundamental matrix solution of (1.1) normalized so that $\Phi(0; z) \equiv \mathbf{1}$, where $\mathbf{1}$ is the 2×2 identity matrix. The trace and the eigenvalues of the monodromy matrix $M(z)$ are independent of the particular fundamental matrix solution chosen, and therefore so is the Floquet discriminant $\Delta(z)$ and the Lax spectrum $\Sigma(L)$. Nonetheless, it will be convenient to use $\Phi(x; z)$, so that $M(z)$ is simply given by

$$M(z) = \Phi(l; z)\,. \tag{2.10}$$

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

Remark 2.8. It is straightforward to see that, for all $z \in \mathbb{C}$, the monodromy matrix satisfies the same symmetries as the scattering matrix for the IST on the line (e.g., see [1,2,71])

$$M^{-1}(z) = \sigma_2 M^{\mathrm{T}}(z)\,\sigma_2\,, \tag{2.11a}$$

$$\overline{M(\overline{z})} = \sigma_2 M(z)\,\sigma_2\,. \tag{2.11b}$$

Moreover, it is also straightforward to verify the following additional symmetries (e.g., see [12]). If q is real, then

$$M(-\overline{z}) = \overline{M(z)}\,, \qquad z \in \mathbb{C}\,. \tag{2.12a}$$

Moreover, if q is even, then

$$M(-\overline{z}) = \sigma_1 \overline{M^{-1}(z)} \sigma_1\,, \qquad z \in \mathbb{C}\,, \tag{2.12b}$$

while if q is odd, then

$$M(-\overline{z}) = \sigma_2 \overline{M^{-1}(z)} \sigma_2\,, \qquad z \in \mathbb{C}\,, \tag{2.12c}$$

where σ_1 and σ_2 are the first and second Pauli spin matrices, respectively (see Appendix A.1).

The symmetry (2.11b) for the monodromy matrix implies that the Floquet discriminant satisfies the Schwarz symmetry

$$\overline{\Delta(\overline{z})} = \Delta(z)\,, \qquad z \in \mathbb{C}\,. \tag{2.13}$$

Moreover, if q is real or even or odd, (2.12) implies additionally that $\Delta(z)$ is an even function:

$$\Delta(-z) = \Delta(z)\,, \qquad z \in \mathbb{C}\,. \tag{2.14}$$

As a result, one has:

Lemma 2.9. *If q is real or even or odd, $\Sigma(L)$ is symmetric about the real and imaginary z-axes. Thus, the Floquet eigenvalues come in symmetric quartets $\{z, \overline{z}, -z, -\overline{z}\}$.*

For q real and even, it follows from (2.12a) and (2.12b) that

$$M(z) = \Delta(z)\,\mathbf{1} + c(z)\,\sigma_3 - \mathrm{i}s(z)\,\sigma_2\,, \qquad z \in \mathbb{C}\,. \tag{2.15}$$

Obviously, (2.15) together with the fact that $\det M(z) \equiv 1$, imply the relation

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

$$\Delta^2(z) = 1 + c^2(z) - s^2(z)\,, \qquad z \in \mathbb{C}\,. \tag{2.16}$$

Equation (2.12a) also implies that, if q is real, $M(z)$ is real when $z \in \mathrm{i}\mathbb{R}$. Moreover, for q real and even, one has:

$$\Delta(z) = \Delta(\overline{z}) = \overline{\Delta(z)}, \quad s(z) = s(\overline{z}) = \overline{s(z)}, \quad c(z) = -c(\overline{z}) = \overline{c(z)}\,, \qquad z \in \mathrm{i}\mathbb{R}\,. \tag{2.17}$$

That is, $\Delta(z)$, $s(z)$ and $c(z)$ are all real for $z \in \mathrm{i}\mathbb{R}$. For $z \in \mathbb{R}$, $\Delta(z)$ and $s(z)$ are real, whereas $c(z)$ is purely imaginary. Finally, since $M(z)$ is entire, (2.17) also implies

$$s(-z) = s(z)\,, \qquad c(-z) = -c(z)\,, \qquad z \in \mathbb{C}\,. \tag{2.18}$$

Next we show that the Lax spectrum of (1.2) with a non-constant potential is confined to an *open* strip in the spectral plane. The following Lemma is proved in Appendix A.2, and is instrumental for this work:

Lemma 2.10. *Suppose $q \in C(\mathbb{R})$ is l-periodic. (i) If q is not constant and $z \in \Sigma(L)$, then $|\operatorname{Im} z| < \|q\|_\infty$. (ii) If q is real or even or odd, and $\Sigma_\pm(L) \subset \mathbb{R} \cup \mathrm{i}\mathbb{R}$, then $\Sigma(L) \subset \mathbb{R} \cup \mathrm{i}\mathbb{R}$ and q is finite-band.*

To solve the inverse problem in the IST (namely, reconstructing the potential from the scattering data), an auxiliary set of spectral data is also needed–the Dirichlet eigenvalues [34,51]:

Definition 2.11 *(Dirichlet eigenvalues).* Let $M(z)$ be defined by (2.10). The set of Dirichlet eigenvalues (see [34]) with base point $x_0 = 0$ is defined as

$$\Sigma_{\mathrm{Dir}}(L; x_o = 0) := \{z \in \mathbb{C} : s(z) = 0\}\,. \tag{2.19}$$

In contrast to the Lax spectrum, the Dirichlet eigenvalues are not invariant with respect to changes in the base point $x = x_o$, or to time evolution of q according to the focusing NLS equation. Indeed, in the context of the integrability of NLS on the torus, the Dirichlet eigenvalues correspond to angle variables and are used to coordinatize the isospectral level sets. As we discuss next, the set of Dirichlet eigenvalues is discrete, consists of movable and immovable points, and the number of movable Dirichlet eigenvalues is tied to the genus of the corresponding Riemann surface (see [34,41]).

The monodromy matrix $M(z)$ in (2.10) was defined in terms of the fundamental matrix solution $\Phi(x; z)$ normalized as $\Phi(0; z) \equiv \mathbf{1}$. The monodromy matrix $M(z; x_o)$ associated with a "shifted" solution $\tilde{\Phi}(x; x_o, z)$ normalized as $\tilde{\Phi}(x_o; x_o, z) \equiv \mathbf{1}$, with $x_0 \in \mathbb{R}$, is given by

$$M(z; x_o) = \Phi(x_o; z) M(z) \Phi^{-1}(x_o; z). \tag{2.20}$$

Let $\Sigma_{\mathrm{Dir}}(L; x_o)$ be the corresponding set of Dirichlet eigenvalues.

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

Definition 2.12 *(Movable and immovable Dirichlet eigenvalues).* Let $z \in \mathbb{C}$ be a Dirichlet eigenvalue associated to the monodromy matrix $M(z; x_o)$ with a given base point $x = x_o$, i.e., $z \in \Sigma_{\mathrm{Dir}}(L; x_o)$. Following [34], we say that z is an *immovable* Dirichlet eigenvalue if $z \in \Sigma_{\mathrm{Dir}}(L; x)$ for all $x \in \mathbb{R}$. Otherwise, we say $z \in \mathbb{C}$ is a *movable* Dirichlet eigenvalue.

Remark 2.13. If q is an N-band potential of the non-self-adjoint Dirac operator (1.2), then the number of movable Dirichlet eigenvalues is $N - 1$ (see [34,41]).

An immediate consequence of (2.16) and the symmetries of $M(z)$, $\Delta(z)$, $c(z)$ and $s(z)$ is the following lemma, which will be useful later (see also [34,74]):

Lemma 2.14. *If $z \in \mathbb{R}$ and $|\Delta(z)| = 1$, then $c(z) = s(z) = 0$, so that z is an immovable Dirichlet eigenvalue. Conversely, if $s(z) = 0$ with $z \in \mathrm{i}\mathbb{R}$, then $|\Delta(z)| \geq 1$.*

Lemma 2.15. *Let $z_\pm \in \Sigma_\pm(L)$. If $\partial_z \Delta(z_\pm) \neq 0$, then the corresponding eigenspace has dimension one.*

Proof. Suppose that there exist two linearly independent periodic (or antiperiodic) eigenfunctions. Consider the normalized fundamental matrix solution $\Phi(x; z)$ of (1.1), namely, $L\Phi(x; z) = z\Phi(x; z)$ with $\Phi(0; z) \equiv \mathbf{1}$. Differentiating with respect to z and using variation of parameters one gets

$$\Delta_z = \tfrac{1}{2}\,\mathrm{tr}\left(-\mathrm{i}\Phi(l; z) \int_0^l \Phi^{-1}(x; z)\sigma_3\Phi(x; z)\,\mathrm{d}x \right). \tag{2.21}$$

By Floquet's theorem $\Phi(l; z_\pm) = \pm\mathbf{1}$, respectively. Then (2.21) yields $\partial_z \Delta(z_\pm) = 0$. □

The following lemma is a direct consequence of Lemmas 2.14, and 2.15:

Lemma 2.16. *If $z_\pm \in \Sigma_\pm(L) \cap \mathbb{R}$, then the geometric multiplicity is two and $\partial_z \Delta(z^\pm) = 0$, respectively.*

2.3. Limits $m \to 0$ and $m \to 1$; $z = 0$

The two distinguished limits $m \to 0^+$ and $m \to 1^-$ of the two-parameter family of elliptic potentials (1.3) provide convenient limits of the results of this work. Interestingly, both of these limits yield exactly solvable models. Here it will be convenient to keep track of the dependence on m explicitly.

Since $\mathrm{dn}(x; 0) \equiv 1$, when $m = 0$ the potential (1.3) reduces to a constant background, i.e., $q \equiv A$ with period $l = 2K(0) = \pi$. Thus, (1.1) becomes a linear system of ODEs with constant coefficients, for which one easily obtains a fundamental matrix solution

$$\Phi(x; z, m = 0) = \mathrm{e}^{-\mathrm{i}(z\sigma_3 - A\sigma_2)x}\,. \tag{2.22}$$

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

Hence the monodromy matrix is

$$M(z,m=0) = \cos\left(\sqrt{z^2+A^2}\pi\right)\mathbf{1} - \frac{\mathrm{i}\sin\left(\sqrt{z^2+A^2}\pi\right)}{\sqrt{z^2+A^2}}\left(z\sigma_3 - A\sigma_2\right), \tag{2.23}$$

implying $\Sigma(L;A,0) = \mathbb{R}\cup[-\mathrm{i}A,\mathrm{i}A]$. Further, $z=\pm\mathrm{i}A$ are the only simple periodic eigenvalues; all other periodic (resp. antiperiodic) eigenvalues are double points. Hence, for any $A\neq 0$, $q\equiv A$ is a 1-band (i.e., genus-0) potential of (1.1). Moreover, the associated solution of the focusing NLS equation [i.e., (1.5) with $s=1$] is simply $q(x,t) = A\,\mathrm{e}^{2\mathrm{i}A^2t}$.

On the other hand, the limit $m\to 1^-$ is singular, since $K(m)$, and therefore the period $l=2K(m)$ of the potential (1.3), diverges in this limit. Indeed, $\mathrm{dn}(x;1)\equiv\operatorname{sech}x$, so letting $m=1$ results in the eigenvalue problem (1.1) with potential $q\equiv A\operatorname{sech}x$. This case is also exactly solvable, and was first studied by Satsuma and Yajima [86]. The point spectrum is comprised of a set of discrete eigenvalues located along the imaginary z-axis. Moreover, for $A\in\mathbb{N}$ the potential is reflectionless, and the point spectrum is given by $z_n = \mathrm{i}(n-1/2)$ for $n=1,\dots,A$. That is, when $A\in\mathbb{N}$, $q\equiv A\operatorname{sech}x$ corresponds to a pure bound-state A-soliton solution of the focusing NLS equation [86]. When $A=1$, the solution of the NLS equation (1.5) is simply $q(x,t)=\mathrm{e}^{\mathrm{i}t}\operatorname{sech}x$. When $A>1$, the solutions are much more complicated [69,86]. Indeed, the potential $A\operatorname{sech}x$ was used to study the semiclassical limit of the focusing NLS equation in the pure soliton regime [57].

Lastly, we discuss the origin $z=0$ of the spectral plane. When $z=0$, the ZS system (1.6a) admits closed-form solutions (see Appendix A.3). These solutions then allow one to obtain the following lemma, which is proved in Appendix A.3:

Lemma 2.17. *Consider (1.2) with potential (1.3) and $m\in[0,1)$. If $A\in\mathbb{N}$ is even or odd, then $z=0$ is a periodic or antiperiodic eigenvalue, respectively, with geometric multiplicity two in each case.*

3. Transformation to Hill's equation

In this section we introduce a transformation of (1.1) that will be instrumental in proving Theorem 1.2, and we consider the effect of this transformation on the Lax spectrum.

First we transform (1.1) to Hill's equation with a complex-valued potential via the unitary linear transformation

$$\phi\mapsto v = \Lambda\phi\,, \qquad \Lambda := \frac{1}{\sqrt{2}}\begin{pmatrix}1 & \mathrm{i}\\ 1 & -\mathrm{i}\end{pmatrix}, \tag{3.1}$$

where $v := v(x;z^2) = (v^+,v^-)^{\mathrm{T}}$. Differentiation of (1.1) and use of (3.1) show that, if q in (1.2) is a real-valued differentiable potential, then (3.1) maps (1.1) into the diagonal system

$$Hv := (-\partial_x^2 + Q^2 - \mathrm{i}Q_x\sigma_1)v = z^2 v\,. \tag{3.2}$$

Or, in component form,

$$v_{xx}^\pm + (\pm\mathrm{i}q_x + z^2 + q^2)v^\pm = 0\,. \tag{3.3}$$

Equation (3.3) is Hill's equation with the complex (Riccati) potential $V^\pm := \mp\mathrm{i}q_x - q^2$. Thus, (3.3) amounts to the pair of eigenvalue problems

$$H^\pm v^\pm = \lambda v^\pm\,, \quad \lambda := z^2\,, \tag{3.4}$$

where

$$H^\pm := -\partial_x^2 + V^\pm(x)\,. \tag{3.5}$$

Remark 3.1. If the potential q in (1.2) is real and even, then $V^\pm(-x) = \overline{V^\pm(x)}$, i.e., $V^\pm$ is PT-symmetric.

Next, similarly to (2.1), we introduce the corresponding *Floquet* BCs for $H^\pm$:

$$\mathrm{BC}_\nu(H^\pm) := \{v^\pm : v^\pm(l;\lambda) = \mathrm{e}^{\mathrm{i}\nu l}\, v^\pm(0;\lambda)\,, \quad v_x^\pm(l;\lambda) = \mathrm{e}^{\mathrm{i}\nu l} v_x^\pm(0;\lambda)\,, \quad \nu \in \mathbb{R}\}\,. \tag{3.6}$$

Definition 3.2 *(Floquet eigenvalues of Hill's operator).* Let the operators $H^\pm : H^2([0,l]) \to L^2([0,l])$ be defined by (3.5). Let $\mathrm{dom}(H^\pm) := \{v^\pm \in H^2([0,l]) : v^\pm \in \mathrm{BC}_\nu(H^\pm)\}$. The set of Floquet eigenvalues of $H^\pm$ is given by

$$\Sigma_\nu(H^\pm) := \{\lambda \in \mathbb{C} : \exists v^\pm \not\equiv 0 \in \mathrm{dom}(H^\pm) \text{ s.t. } H^\pm v^\pm = \lambda v^\pm\}\,. \tag{3.7}$$

In particular, $\nu = 2n\pi/l$, $n \in \mathbb{Z}$, identifies periodic eigenfunctions, while $\nu = (2n-1)\pi/l$, $n \in \mathbb{Z}$, identifies antiperiodic eigenfunctions. We will call the corresponding eigenvalues periodic and antiperiodic, respectively, and we will denote the set of periodic and antiperiodic eigenvalues by $\Sigma_\pm(H^\pm)$, respectively.

(H^2 denotes the space of square-integrable functions with square-integrable first and second derivatives.) It is well-known that $\Sigma_\nu(H^\pm)$ is discrete and countably infinite [15,25,28,72].

Lemma 3.3. *If the potential q in (1.2) is real and even, then $\Sigma_\nu(H^+) = \Sigma_{-\nu}(H^-)$, the dimension of the corresponding eigenspaces are equal, and each of $\Sigma_\nu(H^\pm)$ is symmetric about the real λ-axis.*

Proof. Let $\lambda \in \Sigma_\nu(H^+)$ with eigenfunction $v^+(x;\lambda)$. Since q is even, it is easy to check $\tilde v := v^+(-x;\lambda)$ satisfies $H^-\tilde v = \lambda\tilde v$. Moreover,

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

$$\tilde{v}(l;\lambda) = v^+(-l;\lambda) = \mathrm{e}^{-\mathrm{i}\nu l}\tilde{v}(0;\lambda)\,, \tag{3.8a}$$

$$\tilde{v}_x(l;\lambda) = -v_x^+(-l;\lambda) = \mathrm{e}^{-\mathrm{i}\nu l}\tilde{v}_x(0;\lambda)\,. \tag{3.8b}$$

Hence, $\lambda \in \Sigma_{-\nu}(H^-)$. Conversely, if $\lambda \in \Sigma_{-\nu}(H^-)$ with eigenfunction $v^-(x;\lambda)$, a completely symmetric argument shows that $\lambda \in \Sigma_\nu(H^+)$. Finally, since the map $v(x;\lambda) \mapsto v(-x;\lambda)$ is a (unitary) isomorphism, the dimension of the corresponding eigenspaces are the same.

Next, we prove the symmetry. Assume that $\lambda \in \Sigma_\nu(H^\pm)$ with corresponding eigenfunction $v^\pm(x;\lambda)$, respectively. Then it is easy to check that $\tilde{v}^\pm := \overline{v(-x;\lambda)^\pm}$ satisfies $H^\pm\tilde{v}^\pm = \overline{\lambda}\tilde{v}^\pm$. Moreover,

$$\tilde{v}^\pm(l;\lambda) = \overline{v^\pm(-l;\lambda)} = \mathrm{e}^{\mathrm{i}\nu l}\tilde{v}^\pm(0;\lambda)\,, \tag{3.9a}$$

$$\tilde{v}_x^\pm(l;\lambda) = -\overline{v_x^\pm(-l;\lambda)} = \mathrm{e}^{\mathrm{i}\nu l}\tilde{v}_x^\pm(0;\lambda)\,. \tag{3.9b}$$

Thus, $\overline{\lambda} \in \Sigma_\nu(H^\pm)$ with eigenfunction $\tilde{v}^\pm(x;\lambda)$, respectively. □

Remark 3.4. It is easy to see that Lemma 3.3 implies $\Sigma_\pm(H^+) = \Sigma_\pm(H^-)$, respectively.

Next, since the Lax spectrum $\Sigma(H^\pm) = \cup_{\nu\in[0,2\pi/l)}\Sigma_\nu(H^\pm)$, we have the following key equivalence:

Lemma 3.5. *If the potential q in (1.2) is real and even, then the unitary map (3.1) implies:*

$$\Sigma(H^+) = \Sigma(H^-) = \{\lambda = z^2 : z \in \Sigma(L)\}\,. \tag{3.10}$$

That is, the Lax spectrum of these three operators is related through the relation $\lambda = z^2$. In particular,

$$z \in \Sigma_+(L) \;\Leftrightarrow\; \lambda = z^2 \in \Sigma_+(H^\pm)\,, \quad z \in \Sigma_-(L) \;\Leftrightarrow\; \lambda = z^2 \in \Sigma_-(H^\pm)\,. \tag{3.11}$$

Finally, for $z \neq 0$, the geometric multiplicity of an eigenvalue $z \in \Sigma_+(L)$ equals that of $\lambda = z^2 \in \Sigma_+(H^\pm)$, and similarly for $z \in \Sigma_-(L)$ and $\lambda = z^2 \in \Sigma_-(H^\pm)$.

Proof. If $z \in \Sigma(L)$, the transformation (3.1) implies that $v^\pm(x;\lambda)$ are both bounded solutions of Hill's ODE (3.4), respectively, implying $\lambda \in \Sigma(H^\pm)$. Conversely, if $v^+(x;\lambda)$ is a bounded solution of (3.4) with the plus sign, it follows that $\tilde{v} := v^+(-x,\lambda)$ is a bounded solution of (3.4) with the minus sign. Further, $\phi_1 = (v^+ + v^-)/\sqrt{2}$, and $\phi_2 = \mathrm{i}(v^- - v^+)/\sqrt{2}$ are both bounded, and the map (3.1) then implies that $\phi(x;z) = (\phi_1,\phi_2)^{\mathrm{T}}$ solves (1.1), implying $z \in \Sigma(L)$. A similar argument follows if one starts with $v^-(x;\lambda)$ bounded. Thus, (3.10) follows. Equation (3.11) follows directly from Lemma 3.3.

It remains to show that, for $z \neq 0$, the dimension of the corresponding eigenspaces are equal. The argument follows [26] where the self-adjoint case was studied. To this end, let

$E_\pm(L,z)$ denote the eigenspace associated with an eigenvalue $z \in \Sigma_\pm(L)$, and similarly for $L^2 := L \circ L$ and $H^\pm$. First, note that $\phi \mapsto \mathrm{i}\sigma_2\phi$ is a (unitary) isomorphism between the eigenspaces $E_\pm(L,z)$ and $E_\pm(L,-z)$. Thus, applying the operator twice, for $z \neq 0$ one easily gets

$$\mathrm{dim} E_\pm(L^2, \lambda = z^2) = 2\mathrm{dim} E_\pm(L,z)\,. \tag{3.12}$$

Next, note that L^2 is (unitary) equivalent to the diagonal system (3.2), i.e., $H = \frac{1}{2}\Lambda L^2 \Lambda^{-1}$ Moreover,

$$H = \begin{pmatrix} H^+ & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & H^- \end{pmatrix}, \tag{3.13}$$

and so

$$E_\pm(H,\lambda) = (E_\pm(H^+,\lambda) \oplus 0) \oplus (0 \oplus E_\pm(H^-,\lambda))\,. \tag{3.14}$$

Hence, by (3.12)–(3.14) and Lemma 3.3 it follows $\mathrm{dim} E_\pm(L,z) = \mathrm{dim} E_\pm(H^\pm,\lambda)$, respectively. □

Remark 3.6. By Lemma 3.5, the spectrum of the Dirac operator L in (1.2) with real and even potential is associated to that of the spectrum of the Hill operators $H^\pm$ in (3.5). Importantly, note that the final statement of Lemma 3.5 does not hold at $z = 0$; that is, the geometric multiplicity of the periodic (or antiperiodic) eigenvalue $z = 0$ of the Dirac and Hill operators need not be equal (see Appendix A.3).

All of the above results hold for generic real and even potentials. Moving forward, we restrict our attention to the Jacobi elliptic potential (1.3). By Lemma 3.5 we fix $y := v^-(x;\lambda)$ without loss of generality (dependence on A and m is omitted for brevity). Then Hill's equation $H^-v^- = \lambda v^-$ is given by

$$y_{xx} + (\mathrm{i}Am\,\mathrm{sn}(x;m)\,\mathrm{cn}(x;m) + \lambda + A^2\,\mathrm{dn}^2(x;m))y = 0\,. \tag{3.15}$$

Remark 3.7. Since $\mathrm{dn}^2(x;m) \equiv 1 - m\,\mathrm{sn}^2(x;m)$, (3.15) can be viewed as an imaginary deformation of the celebrated Lamé equation [4,31,48,72], $y_{xx} + (\lambda + V(x))\,y = 0$ up to a shift of the eigenvalue λ. The Lamé equation has the remarkable property that solutions can coexist if and only if $A^2 = n(n+1)$ where n is an integer [4,31,48,72]. Recall that solutions coexist if two linearly independent periodic (or respectively antiperiodic) solutions exist for a given λ. In the case of Hill's equation with a real potential this amounts to a "closed gap" in the spectrum (corresponding to finite gap potentials).

4. Transformation to a trigonometric ODE

In this section we introduce a second transformation of (1.1). By part (ii) of Lemma 2.10 moving forward we only need to consider the periodic and antiperiodic eigenfunctions.

4.1. Second-order ODE with trigonometric coefficients

Consider the following change of independent variable:

$$x \mapsto t := 2\mathrm{am}(x; m)\,, \tag{4.1}$$

where $\mathrm{am}(x; m)$ is the Jacobi amplitude [14,43]. Equation (4.1) establishes a conformal map between the strip $|\,\mathrm{Im}\, x\,| < K(1-m)$ and the complex t-plane cut along the rays $(2j+1)\pi \pm 2\mathrm{i}\tau r$, $\tau \geq 1$, $j \in \mathbb{Z}$, where $r = \ln[(2-m)/m]/2$ [43,79]. We then arrive at our second reformulation of the Dirac eigenvalue problem:

$$4(1 - m\sin^2 \tfrac{t}{2})y_{tt} - (m\sin t)y_t + (\lambda + A^2(1 - m\sin^2 \tfrac{t}{2}) + \tfrac{\mathrm{i}}{2}Am\sin t)y = 0\,. \tag{4.2}$$

(The independent variable t introduced above should not be confused with the time variable of the NLS equation (1.5).)

Remark 4.1. Equation (4.2) can be written as the eigenvalue problem

$$By = \lambda y\,, \tag{4.3}$$

where the operator $B : H^2([0, 2\pi]) \to L^2([0, 2\pi])$ is defined by

$$B := -4(1 - m\sin^2 \tfrac{t}{2})\partial_t^2 + (m\sin t)\partial_t - (A^2(1 - m\sin^2 \tfrac{t}{2}) + \tfrac{\mathrm{i}}{2}Am\sin t))\,. \tag{4.4}$$

The coefficients are now 2π-periodic and as before $\Sigma_\pm(B)$ will denote the periodic and antiperiodic eigenvalues of the operator B, respectively (see Definition 3.2).

This leads to the following result which connects the periodic/antiperiodic eigenvalues of Hill's equation (3.15) to the periodic/antiperiodic eigenvalues of the trigonometric equation (4.2).

Lemma 4.2. *Let B be the trigonometric operator (4.4). Then $\lambda \in \Sigma_\pm(B)$ if and only if $\lambda \in \Sigma_\pm(H^-)$.*

Proof. By (4.2) one gets $By = \lambda y$ if and only if $H^-\tilde{y} = \lambda\tilde{y}$, with $\tilde{y}(x; \lambda) = y(t; \lambda)$ and $t = 2\mathrm{am}(x; m)$ as per (4.1). Next, note that $\mathrm{am}(x; m)$ is monotonic increasing for $x \in (0, 2K)$, $\mathrm{am}(x + 2K; m) = \mathrm{am}(x; m) + \pi$, and $\mathrm{am}(0; m) = 0$. Hence, the map (4.1)

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

is a bijection between $x \in [0, 2K]$ and $t \in [0, 2\pi]$. Moreover, $\tilde{y}(0;\lambda) = \pm\tilde{y}(2K;\lambda)$ if and only if $y(0;\lambda) = \pm y(2\pi;\lambda)$. Similarly, $\tilde{y}_x(0;\lambda) = \pm\tilde{y}_x(2K;\lambda)$ if and only if $y_t(0;\lambda) = \pm y_t(2\pi;\lambda)$. Thus, $2K$-periodic (resp. antiperiodic) solutions of (3.15) map to 2π-periodic (resp. antiperiodic) solutions of (4.2), and vice versa. □

Remark 4.3. The trigonometric ODE (4.2) can be viewed as a complex deformation of Ince's equation (see Chapter 7 of [72] for more details). Namely, one can write (4.2) as

$$(1 + a\cos t)y_{tt} + (b\sin t)y_t + (h + d\cos t + \mathrm{i}e\sin t)y = 0\,, \tag{4.5}$$

where $a = m/(2-m)$, $b = -a/2$, $h = \lambda/(4-2m) + A^2/4$, $d = A^2a/4$, $e = Aa/4$. To the best of our knowledge this is the first example of a non-self-adjoint version of Ince's equation arising from applications.

4.2. Fourier series expansion and three-term recurrence relation

Recall that any Floquet solution $y(t;\lambda)$ of (4.2) bounded for all $t \in \mathbb{R}$ has the form $y(t;\lambda) = \mathrm{e}^{\mathrm{i}\nu t}f(t;\lambda)$ where $f(t+2\pi;\lambda) = f(t;\lambda)$ and $\nu \in \mathbb{R}$ (cf. Section 2.1). Moreover, since $f(t;\lambda)$ is 2π-periodic, we can express it in terms of a Fourier series on $L^2(\mathbb{S}^1)$, where $\mathbb{S}^1 := \mathbb{R}/\mathbb{Z}$ is the unit circle. By direct calculation, let $y(t;\lambda)$ be a Floquet solution of (4.2) given by

$$y(t;\lambda) = \mathrm{e}^{\mathrm{i}\nu t}\sum_{n\in\mathbb{Z}} c_n\mathrm{e}^{\mathrm{i}nt}\,. \tag{4.6}$$

Then the coefficients $\{c_n\}_{n\in\mathbb{Z}}$ are given by the following three-term recurrence relation:

$$\alpha_n c_{n-1} + (\beta_n - \lambda)c_n + \gamma_n c_{n+1} = 0\,, \qquad n \in \mathbb{Z}\,, \tag{4.7}$$

where

$$\alpha_n = -\tfrac{1}{4}m\,[A - (2n+2\nu-2)][A + (2n+2\nu-1)]\,, \tag{4.8a}$$

$$\beta_n = (1 - \tfrac{1}{2}m)[(2n+2\nu)^2 - A^2]\,, \tag{4.8b}$$

$$\gamma_n = -\tfrac{1}{4}m\,[A - (2n+2\nu+2)][A + (2n+2\nu+1)]\,. \tag{4.8c}$$

Remark 4.4. In turn, the recurrence relation (4.7) can be written as the eigenvalue problem

$$B_\nu c = \lambda c\,, \tag{4.9}$$

where $c = \{c_n\}_{n\in\mathbb{Z}}$

$$B_\nu := \begin{pmatrix} \ddots & \ddots & \ddots & & \\ & \alpha_n & \beta_n & \gamma_n & \\ & & \ddots & \ddots & \ddots \end{pmatrix} . \tag{4.10}$$

Note: $\nu \in \mathbb{Z}$ corresponds to periodic, and $\nu \in \mathbb{Z} + \frac{1}{2}$ to antiperiodic eigenfunctions of (4.3).

Next, define the space $\ell^{2,p}(\mathbb{Z}) := \{c \in \ell^2(\mathbb{Z}) : \sum_{n\in\mathbb{Z}} |n|^p |c_n|^2 < \infty\}$. The requirement that $c \in \ell^{2,4}(\mathbb{Z})$ ensures $By \in L^2([0, 2\pi])$. The reason why this is the case is that B is a second-order differential operator, which implies that the Fourier coefficients of By will grow n^2 faster as $|n| \to \infty$ than those of y.

Definition 4.5 *(Eigenvalues of the tridiagonal operator).* Let the operator $B_\nu : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be defined by (4.10). The set of eigenvalues is given by

$$\Sigma(B_\nu) := \{\lambda \in \mathbb{C} : \exists c \not\equiv 0 \in \ell^{2,4}(\mathbb{Z}) \text{ s.t. } B_\nu c = \lambda c\} . \tag{4.11}$$

We have the following important result:

Lemma 4.6. *If $\nu \in \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$, then $\Sigma_\pm(B) = \Sigma(B_\nu)$, and the dimension of the corresponding eigenspaces are equal, respectively.*

Proof. By standard results in Fourier analysis [82] one defines the bijective linear map

$$U : \ell^2(\mathbb{Z}) \to L^2(\mathbb{S}^1) , \qquad (Uc)(t) = \sum_{n\in\mathbb{Z}} c_n \mathrm{e}^{\mathrm{i}nt} , \tag{4.12}$$

and the multiplication operator

$$M_\nu : L^2([0, 2\pi]) \to L^2([0, 2\pi]) , \qquad (M_\nu w)(t) = \mathrm{e}^{\mathrm{i}\nu t} w(t) . \tag{4.13}$$

By construction $B_\nu = (M_\nu U)^{-1} B M_\nu U$ in the standard basis and U, M_ν are unitary. Also, $y = M_\nu U c$ (see (4.6)). Hence, it follows $\Sigma_\pm(B) = \Sigma(B_\nu)$ and the dimensions of the corresponding eigenspaces are equal. □

Remark 4.7. The Floquet exponent ν can be shifted by any integer amount without loss of generality, since doing so simply corresponds to a shift in the numbering of the Fourier coefficients in (4.6). So, for example, $\nu \mapsto \nu + s$ simply corresponds to $(\alpha_n, \beta_n, \gamma_n) \mapsto (\alpha_{n+s}, \beta_{n+s}, \gamma_{n+s})$ for all $n \in \mathbb{Z}$.

4.3. Reducible tridiagonal operators and ascending and descending Fourier series

We show that the tridiagonal operator B_ν is reducible. Recall that a tridiagonal operator is reducible if there exists a zero element along the subdiagonal, or superdiagonal [46].

Lemma 4.8. *If $A \in \mathbb{N}$ and $\nu \in \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$, then B_ν is reducible.*

Proof. There are two cases to consider: (i) $\nu \in \mathbb{Z}$, corresponding to periodic eigenvalues, and (ii) $\nu \in \mathbb{Z} + \frac{1}{2}$, corresponding to antiperiodic eigenvalues. In either case, however, when $A \in \mathbb{N}$ one has

$$\alpha_n = 0 \iff n = \tfrac{A}{2} + 1 - \nu \quad \vee \quad n = -\tfrac{A}{2} + \tfrac{1}{2} - \nu\,, \tag{4.14a}$$

$$\gamma_n = 0 \iff n = \tfrac{A}{2} - 1 - \nu \quad \vee \quad n = -\tfrac{A}{2} - \tfrac{1}{2} - \nu\,. \tag{4.14b}$$

In both cases, one can find two values of n that make α_n and γ_n zero, respectively, but only one of them is an integer, depending on whether A is even or odd. Note also that $\beta_n = 0$ for $n = -\nu \pm A/2$, but the corresponding value of n is integer only if A is even and $\nu \in \mathbb{Z}$ or A is odd and $\nu \in \mathbb{Z} + \frac{1}{2}$. (The equalities in (4.14) hold for all $\nu \in \mathbb{R}$, but only when $\nu \in \mathbb{Z}$ or $\nu \in \mathbb{Z} + \frac{1}{2}$ do they yield integer values of n.) □

We emphasize that, when $A \notin \mathbb{N}$, a similar statement (namely, that B_ν is reducible) can be made for different values of ν. The precise values of ν can be immediately obtained from the definition of the coefficients α_n, β_n and γ_n in (4.8). On the other hand, the particular significance of integer and half-integer values of ν is that they are associated with periodic and antiperiodic eigenvalues, which are the endpoints of the spectral bands. In Section 5.3 we will also see how the periodic and antiperiodic eigenvalues are related to the solution of a connection problem for a particular Heun ODE.

Consider the tridiagonal operator $B_\nu : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ in (4.10). Let $\ell^2_+ = \ell^2(\mathbb{N}_o)$ ($\mathbb{N}_o := \mathbb{N} \cup \{0\}$) and $\ell^2_- = \ell^2(\mathbb{Z} \setminus \mathbb{N}_o)$, so that $\ell^2(\mathbb{Z}) = \ell^2_- \oplus \ell^2_+$, and denote by $P_\pm$ orthogonal projectors from $\ell^2(\mathbb{Z})$ onto $\ell^2_\pm$ respectively. Finally, introduce the block decomposition

$$B_\nu = \begin{pmatrix} B_- & A_- \\ A_+ & B_+ \end{pmatrix}, \tag{4.15}$$

where the semi-infinite tridiagonal operators $B_\pm$ are defined as

$$B_- := \begin{pmatrix} \ddots & \ddots & \ddots & \\ & \alpha_{-2} & \beta_{-2} & \gamma_{-2} \\ & & \alpha_{-1} & \beta_{-1} \end{pmatrix}, \qquad B_+ := \begin{pmatrix} \beta_0 & \gamma_0 & & \\ \alpha_1 & \beta_1 & \gamma_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}, \tag{4.16}$$

and $A_\pm$ only have one nontrivial entry each, equal to α_0 and γ_{-1} respectively, in their upper right corner and lower left corner, respectively. If $A \in \mathbb{N}$ and $\nu = (1-A)/2$ (corresponding to the case of periodic eigenvalues when A is odd and antiperiodic eigenvalues when A is even), it is easy to see that $\alpha_0 = \gamma_{-1} = 0$ and therefore $A_\pm \equiv 0$, which implies that $B_\nu = B_- \oplus B_+$ and $\ell^2_\pm$ are invariant subspaces of B_ν. The above considerations imply the following:

Lemma 4.9. *If $A \in \mathbb{N}$ and $\nu = (1-A)/2$, then $\Sigma(B_\nu) = \Sigma(B_-) \cup \Sigma(B_+)$, where $B_\pm$ are given by (4.16).*

The case $A \in \mathbb{N}$ and $\nu = A/2$ is similar, but more complicated. In this case, it is necessary to also introduce a second block decomposition of B_ν in addition to (4.15), namely:

$$B_\nu = \begin{pmatrix} \tilde{B}_- & \tilde{A}_- \\ \tilde{A}_+ & \tilde{B}_+ \end{pmatrix}, \tag{4.17}$$

where

$$\tilde{B}_- := \begin{pmatrix} \ddots & \ddots & \ddots & \\ & \alpha_{-1} & \beta_{-1} & \gamma_{-1} \\ & & \alpha_0 & \beta_0 \end{pmatrix}, \qquad \tilde{B}_+ := \begin{pmatrix} \beta_1 & \gamma_1 & & \\ \alpha_2 & \beta_2 & \gamma_2 & \\ & \ddots & \ddots & \ddots \end{pmatrix}, \tag{4.18}$$

and $\tilde{A}_\pm$ only have one nontrivial entry each, equal to α_1 and γ_0 respectively, in their upper right corner and lower left corner, respectively. If $A \in \mathbb{N}$ and $\nu = A/2$ (corresponding to the case of periodic eigenvalues when A is even and antiperiodic eigenvalues when A is odd), it is easy to see that $\gamma_{-1} = \beta_0 = \alpha_1 = 0$ and therefore $A_- = \tilde{A}_+ \equiv 0$. On the other hand, A_+ and $\tilde{A}_-$ are not identically zero. Thus, B_ν cannot be split into a direct sum of two semi-infinite tridiagonal operators. Nevertheless, an analog of Lemma 4.9 still holds.

Lemma 4.10. *If $A \in \mathbb{N}$ and $\nu = A/2$, then $\Sigma(B_\nu) = \Sigma(B_-) \cup \Sigma(B_+) = \Sigma(\tilde{B}_-) \cup \Sigma(\tilde{B}_+)$, where $B_\pm$ and $\tilde{B}_\pm$ are given by (4.16) and (4.18), respectively.*

Proof. We first show that $\Sigma(B_\nu) \subset \Sigma(B_-) \cup \Sigma(B_+)$. Recall that $A_- = 0$ but $A_+ \neq 0$. Let λ and c be an eigenpair of B_ν, and let $c_\pm = P_\pm c$, so that $c = (c_-, c_+)^{\mathrm{T}}$. If $c_- \neq 0$, we have $B_- c_- = \lambda c_-$, and therefore $\lambda \in \Sigma(B_-)$. Otherwise, $c_- = 0$ implies $c_+ \neq 0$ and $c = (0, c_+)^{\mathrm{T}}$, and $B_+ c_+ = \lambda c_+$, i.e., $\lambda \in \Sigma(B_+)$.

We show that $\Sigma(B_-) \cup \Sigma(B_+) \subset \Sigma(B_\nu)$. Suppose that λ and $c_+ \neq 0$ are an eigenpair of B_+, and let $c = (0, c_+)^{\mathrm{T}}$. Then $B_\nu c = \lambda c$, implying $\lambda \in \Sigma(B_\nu)$. Finally, suppose that $\lambda \in \Sigma(B_-) \setminus \Sigma(B_+)$, with associated eigenvector $c_- \neq 0$. In this case, let $c = (c_-, p)^{\mathrm{T}}$. We choose p such that $p = -(B_+ - \lambda)^{-1} A_+ c_-$. One can show (similarly to Lemma 6.7) that it is always possible to do so since B_+ is closed with compact resolvent. Therefore, the operator $(B_+ - \lambda)^{-1}$ exists and is bounded, and $\lambda \notin \Sigma(B_+)$ implies that λ is in the resolvent set of B_+. But then we have $B_\nu c = \lambda c$, which implies $\lambda \in \Sigma(B_\nu)$.

The proof that $\Sigma(B_\nu) = \Sigma(\tilde{B}_-) \cup \Sigma(\tilde{B}_+)$ is entirely analogous, but we report it because it is useful later. If λ and c are an eigenpair of B_ν, let $\tilde{c}_\pm = \tilde{P}_\pm c$, with $\tilde{P}_\pm$ defined similarly as $P_\pm$. If $\tilde{c}_+ \neq 0$, we have $\tilde{B}_+ \tilde{c}_+ = \lambda \tilde{c}_+$ and therefore $\lambda \in \Sigma(\tilde{B}_+)$, since $\tilde{A}_+ = 0$. Otherwise, similar arguments as before show that $\tilde{B}_- \tilde{c}_- = \lambda \tilde{c}_-$ and therefore $\lambda \in \Sigma(\tilde{B}_-)$. We therefore have $\Sigma(B_\nu) \subset \Sigma(\tilde{B}_-) \cup \Sigma(\tilde{B}_+)$. Finally, to show that $\Sigma(\tilde{B}_-) \cup \Sigma(\tilde{B}_+) \subset \Sigma(B_\nu)$, we first observe that if λ and $\tilde{c}_-$ are an eigenpair of $\tilde{B}_-$, and $c = (\tilde{c}_-, 0)^{\mathrm{T}}$, one has $B_\nu c = \lambda c$ and therefore $\lambda \in \Sigma(B_\nu)$. Conversely, if $\lambda \in \Sigma(\tilde{B}_+) \setminus \Sigma(\tilde{B}_-)$, with eigenvector $\tilde{c}_+$, it is always possible to choose p such that $p = -(\tilde{B}_- - \lambda)^{-1} \tilde{A}_- \tilde{c}_+$ (again, cf. Lemma 6.7), and therefore $c = (p, \tilde{c}_+)^{\mathrm{T}}$ satisfies $B_\nu c = \lambda c$, implying $\lambda \in \Sigma(B_\nu)$. □

Remark 4.11. If $A \in \mathbb{N}$ and $\nu = A/2$, then B_+ and $\tilde{B}_-$ can be decomposed as

$$B_+ = \begin{pmatrix} 0 & \gamma_0 \\ 0 & \tilde{B}_+ \end{pmatrix}, \qquad \tilde{B}_- = \begin{pmatrix} B_- & 0 \\ \alpha_0 & 0 \end{pmatrix}. \tag{4.19}$$

Corollary 4.12. *If $A \in \mathbb{N}$ and $\nu = A/2$, then $\Sigma(B_\nu) = \Sigma(B_-) \cup \Sigma(\tilde{B}_+) \cup \{0\}$.*

Importantly, the proofs of Lemmas 4.9 and 4.10 also imply the following:

Theorem 4.13. *If $A \in \mathbb{N}$ and $\lambda \in \mathbb{C} \setminus \{0\}$ is a periodic or antiperiodic eigenvalue of the trigonometric operator (4.4), then there exists an associated eigenfunction generated by either an ascending or descending Fourier series.*

Proof. The proof is trivial when $\nu = (1 - A)/2$, since in this case $B_\nu = B_- \oplus B_+$. On the other hand, the case $\nu = A/2$ requires more care. The proof of Lemma 4.10 shows that, if λ and c_+ are an eigenpair of B_+, then $c = (0, c_+)^{\mathrm{T}}$ is a corresponding eigenvector of B_ν. Next, if λ and $\tilde{c}_-$ are an eigenpair of $\tilde{B}_-$, then $c = (\tilde{c}_-, 0)^{\mathrm{T}}$ is a corresponding eigenvector of B_ν. Finally, note $\Sigma(\tilde{B}_-) = \Sigma(B_-) \cup \{0\}$. □

Corollary 4.14. *If $A \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ is a periodic or antiperiodic eigenvalue with geometric multiplicity two, then a first eigenfunction can be written in terms of an ascending Fourier series, while a second linearly independent eigenfunction is given by a descending Fourier series.*

5. Transformation to a Heun ODE

We now introduce a final change of independent variable that maps the trigonometric ODE (4.2) into Heun's equation. All the results of Sections 5.1 and 5.2 below will hold for integer as well as non-integer values of A except where expressly indicated. This further

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

reformulation allows us to interpret the Dirac problem (1.1) as a connection problem for Heun's ODE.

5.1. Transformation from the trigonometric ODE to Heun's equation

Recall that Heun's equation is a second-order linear ODE with four regular singular points [31,50,85]. We first rewrite (4.2) using Euler's formula. Then we perform the following change of independent variable:

$$t \mapsto \zeta := \mathrm{e}^{\mathrm{i}t}\,. \tag{5.1}$$

We then obtain a third reformulation of our spectral problem, since the transformation (5.1) maps the trigonometric ODE (4.2) (and therefore (1.1) with elliptic potential (1.3)) into the following Heun ODE:

$$\zeta^2 F(\zeta;m) y_{\zeta\zeta} + \zeta G(\zeta;m) y_\zeta + H(\zeta;\lambda,A,m) y = 0\,, \tag{5.2}$$

where

$$F(\zeta;m) := -m\zeta^2 + (2m-4)\zeta - m\,, \tag{5.3a}$$

$$G(\zeta;m) := -\tfrac{3}{2}m\zeta^2 + (2m-4)\zeta - \tfrac{1}{2}m\,, \tag{5.3b}$$

$$H(\zeta;\lambda,A,m) := \tfrac{1}{4}A(A+1)m\,\zeta^2 + \big(\lambda + A^2(1-\tfrac{m}{2})\big)\,\zeta + \tfrac{1}{4}A(A-1)m\,. \tag{5.3c}$$

Note that the trigonometric ODE (4.2) does not explicitly contain the Floquet exponent ν. The role of ν for Heun's ODE will be played by the Frobenius exponents discussed below.

Equation (5.2) has three regular singular points in the finite complex plane plus a regular singular point at infinity. Specifically, in the finite complex plane one has a regular singular point at $\zeta = 0$ and two additional regular singular points where $F(\zeta;m) = 0$, i.e., when

$$\zeta^2 - 2(1-\tfrac{2}{m})\,\zeta + 1 = 0\,, \tag{5.4}$$

which is satisfied for

$$\zeta_{1,2} = \frac{m-2 \pm 2\sqrt{1-m}}{m}\,. \tag{5.5}$$

Note that $\zeta_{1,2} < 0$ for all $m \in (0,1)$, and $\zeta_2 = 1/\zeta_1$. Without loss of generality, we take $|\zeta_1| < 1 < |\zeta_2|$. Summarizing, the four real regular singular points are at $0, \zeta_1, \zeta_2, \infty$, with $\zeta_2 \in (-\infty,-1)$ and $\zeta_1 \in (-1,0)$.

Table 1
Frobenius exponents corresponding to the Heun ODE (5.2).

	$\zeta = 0$	$\zeta = \zeta_1$	$\zeta = \zeta_2$	$\zeta = \infty$
ρ_1	$\rho_1^o = A/2$	$\rho_1^1 = 0$	$\rho_1^2 = 0$	$\rho_1^\infty = A/2$
ρ_2	$\rho_2^o = -(A-1)/2$	$\rho_2^1 = 1/2$	$\rho_2^2 = -1/2$	$\rho_2^\infty = -(A+1)/2$

Remark 5.1. One can equivalently map the first-order ZS system (1.6a) into a first-order Heun system with the same four singular points using the same change of independent variable (5.1) [cf. Appendix A.4]:

$$\zeta w_\zeta = -\left[\frac{A}{2}\sigma_3 + \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{2\lambda\zeta}{4\zeta + m(\zeta-1)^2} & \frac{(\zeta^2-1)m}{2(4\zeta+m(\zeta-1)^2)} \end{pmatrix}\right] w\,, \tag{5.6}$$

where $w(\zeta;\lambda) = (w_1, w_2)^{\mathrm{T}}$.

5.2. Frobenius analysis of Heun's ODE

Next we apply the method of Frobenius to (5.2) at the regular singular points $\zeta = 0$ and $\zeta = \infty$. Then we construct half-infinite tridiagonal operators whose eigenvalues coincide with those of the tridiagonal operators discussed in Section 4.3. By direct calculation, one can easily check that the Frobenius exponents of (5.2) are as in Table 1. The Frobenius exponents $\rho_{1,2}$ at $\zeta = 0$ and $\zeta = \infty$ are obtained by looking for solutions of (5.2) in the form

$$y_o(\zeta;\lambda) = \zeta^\rho \sum_{n=0}^{\infty} c_n\,\zeta^{\,n}\,, \tag{5.7a}$$

and

$$y_\infty(\zeta;\lambda) = \zeta^\rho \sum_{n=0}^{\infty} c_n\,\zeta^{-n}\,, \tag{5.7b}$$

respectively, with $c_0 \neq 0$ in each case. Note, when A is even, ρ_1^o and ρ_1^∞ are integer while ρ_2^o and ρ_2^∞ are half-integer, and vice versa when A is odd. Note also that $\rho_2^o - \rho_1^o = 1/2 - A$ and $\rho_1^\infty - \rho_2^\infty = 1/2 + A$, so when $A \in \mathbb{N}$, these differences are never integer, and no exceptional cases (i.e., resonances) arise.

Next we study the three-term recurrence relations at $\zeta = 0$ and $\zeta = \infty$, since they are key to proving the reality of the λ eigenvalues. We begin by plugging (5.7a) and (5.7b) into (5.2). The coefficients of the Frobenius series (5.7a) at $\zeta = 0$ solve the following three-term recurrence relations. For $\rho = \rho_1^o = A/2$:

$$-\lambda c_0 + \tfrac{m}{2}(2A+1)\,c_1 = 0\,, \qquad n = 0\,, \tag{5.8a}$$

$$P_n c_{n-1} + (R_n - \lambda) c_n + S_n c_{n+1} = 0\,, \qquad n \geq 1\,, \tag{5.8b}$$

where

$$P_n = \tfrac{m}{2}(n-1)(2A+2n-1)\,, \quad R_n = (1-\tfrac{m}{2})((A+2n)^2 - A^2)\,,$$
$$S_n = \tfrac{m}{2}(n+1)(2A+2n+1)\,. \tag{5.8c}$$

For $\rho = \rho_2^o = -(A-1)/2$:

$$\left[(\tfrac{m}{2}-1)(2A-1) - \lambda\right] c_0 - \tfrac{m}{2}(2A-3)\, c_1 = 0\,, \qquad n = 0\,, \tag{5.9a}$$
$$\widetilde{P}_n c_{n-1} + (\widetilde{R}_n - \lambda) c_n + \widetilde{S}_n c_{n+1} = 0\,, \qquad n \geq 1\,, \tag{5.9b}$$

where

$$\widetilde{P}_n = -\tfrac{m}{2} n(2A-2n+1)\,, \quad \widetilde{R}_n = (1-\tfrac{m}{2})((2n+1-A)^2 - A^2)\,,$$
$$\widetilde{S}_n = -\tfrac{m}{2}(n+1)(2A-2n-3)\,. \tag{5.9c}$$

Similarly, the coefficients of the Frobenius series (5.7b) at $\zeta = \infty$ are given by the following three-term recurrence relations. For $\rho = \rho_1^\infty = A/2$:

$$-\lambda c_0 - \tfrac{m}{2}(2A-1) c_1 = 0\,, \qquad n = 0\,, \tag{5.10a}$$
$$X_n c_{n-1} + (Y_n - \lambda) c_n + Z_n c_{n+1} = 0\,, \qquad n \geq 1\,, \tag{5.10b}$$

where

$$X_n = -\tfrac{m}{2}(n-1)(2A-2n+1)\,, \quad Y_n = (1-\tfrac{m}{2})((2n-A)^2 - A^2)\,,$$
$$Z_n = -\tfrac{m}{2}(n+1)(2A-2n-1)\,. \tag{5.10c}$$

For $\rho = \rho_2^\infty = -(A+1)/2$:

$$\left[(1-\tfrac{m}{2})(2A+1) - \lambda\right] c_0 + \tfrac{m}{2}(2A+3)\, c_1 = 0\,, \qquad n = 0\,, \tag{5.11a}$$
$$\widetilde{X}_n c_{n-1} + (\widetilde{Y}_n - \lambda) c_n + \widetilde{Z}_n c_{n+1} = 0\,, \qquad n \geq 1\,, \tag{5.11b}$$

where

$$\widetilde{X}_n = \tfrac{m}{2} n(2A+2n-1)\,, \quad \widetilde{Y}_n = (1-\tfrac{m}{2})((2n+1+A)^2 - A^2)\,,$$
$$\widetilde{Z}_n = \tfrac{m}{2}(n+1)(2A+2n+3)\,. \tag{5.11c}$$

Remark 5.2. The three-term recurrence relations at $\zeta = 0$ can be written as the eigenvalue problems

$$T_o^{\pm} c = \lambda c\,, \tag{5.12}$$

where $T_o^{\pm} : \ell^{2,4}(\mathbb{N}_o) \subset \ell^2(\mathbb{N}_o) \to \ell^2(\mathbb{N}_o)$, and

$$T_o^- := \begin{pmatrix} R_0 & S_0 & & \\ P_1 & R_1 & S_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}, \qquad T_o^+ := \begin{pmatrix} \widetilde{R}_0 & \widetilde{S}_0 & & \\ \widetilde{P}_1 & \widetilde{R}_1 & \widetilde{S}_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}. \tag{5.13}$$

Similarly, the three-term recurrence relations at $\zeta = \infty$ can be written as the eigenvalue problems

$$T_\infty^{\pm} c = \lambda c\,, \tag{5.14}$$

where $T_\infty^{\pm} : \ell^{2,4}(\mathbb{N}_o) \subset \ell^2(\mathbb{N}_o) \to \ell^2(N_o)$, and

$$T_\infty^- := \begin{pmatrix} Y_0 & Z_0 & & \\ X_1 & Y_1 & Z_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}, \qquad T_\infty^+ := \begin{pmatrix} \widetilde{Y}_0 & \widetilde{Z}_0 & & \\ \widetilde{X}_1 & \widetilde{Y}_1 & \widetilde{Z}_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}. \tag{5.15}$$

5.3. Relation between Fourier series and the connection problem for Heun's ODE

Recall that: (i) If $\lambda \in \mathbb{C}$ is a periodic or antiperiodic eigenvalue of (4.3), one has $\nu \in \mathbb{Z}$ or $\nu \in \mathbb{Z} + \frac{1}{2}$, respectively. (ii) The Floquet exponents can be shifted by an arbitrary integer amount by shifting the indices of the Fourier coefficients (cf. Remark 4.7). (iii) By Theorem 4.13, each periodic or antiperiodic eigenvalue has an associated ascending or descending Fourier series when $A \in \mathbb{N}$. (iv) The transformation $\zeta = \mathrm{e}^{\mathrm{i}t}$ maps the Frobenius series (5.7) to ascending or descending Fourier series (4.6), and vice versa. (v) Finally, when $A \in \mathbb{N}$, the values of the Frobenius exponents for the expansions at $\zeta = 0$ and at $\zeta = \infty$ are either integer or half-integer.

Moreover, the Floquet exponents $\nu = (1 - A)/2$ and $\nu = A/2$ in Lemmas 4.9 and 4.10 coincide exactly with the Frobenius exponents ρ_2^o and ρ_1^o at $\zeta = 0$, respectively. The Frobenius exponents ρ_2^∞ and ρ_1^∞ at $\zeta = \infty$ are also equivalent to the above Floquet exponents upon a shift of indices. As a result, the recurrence relations (5.8), (5.9), (5.10) and (5.11) associated to the Frobenius series (5.7) of Heun's ODE (5.2) are equivalent to those associated to the Fourier series solutions of the trigonometric ODE (4.2). More precisely:

Lemma 5.3. *If $A \in \mathbb{N}$ and $\lambda \in \Sigma(B_\nu)$ is either a periodic or antiperiodic eigenvalue, (i.e., ν integer or half-integer, respectively) then the following identities map the recurrence relations generated by the Frobenius series solution of (5.2) at $\zeta = 0$ and $\zeta = \infty$ to the ascending and descending recurrence relations generated by the Fourier series solution of (4.2), respectively. Namely:*

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

(i) *For* $\nu = \rho^o_{1,2}$ *one has, respectively:*

$$(\alpha_n,\, \beta_n,\, \gamma_n) = (P_n,\, R_n,\, S_n)\,, \quad n \geq 0\,, \tag{5.16a}$$

$$(\alpha_n,\, \beta_n,\, \gamma_n) = (\widetilde{P}_n,\, \widetilde{R}_n,\, \widetilde{S}_n)\,, \quad n \geq 0\,. \tag{5.16b}$$

(ii) *For* $\nu = \rho^\infty_{1,2}$ *one has, respectively:*

$$(\alpha_{-n},\, \beta_{-n},\, \gamma_{-n}) = (Z_n,\, Y_n,\, X_n)\,, \quad n \geq 0\,, \tag{5.17a}$$

$$(\alpha_{-n-1},\, \beta_{-n-1},\, \gamma_{-n-1}) = (\widetilde{Z}_n,\, \widetilde{Y}_n,\, \widetilde{X}_n), \qquad n \geq 0\,, \tag{5.17b}$$

Proof. When $\nu = \rho^o_{1,2}$, the result follows immediately by direct comparison. Likewise when $\nu = \rho^\infty_1$. Finally, when $\nu = \rho^\infty_2$ we can simply shift $\nu \mapsto \nu + 1$, which sends $n \mapsto -n - 1$. □

Corollary 5.4. *If* A *is odd, then the eigenvalues of* T^+_o *and* T^+_∞ *correspond to the periodic eigenvalues of the Dirac operator* (1.2) *and* T^-_o *and* T^-_∞ *to the antiperiodic ones, via the map* $\lambda = z^2$. *Conversely, if* A *is even, then the eigenvalues of* T^-_o *and* T^-_∞ *correspond to the periodic eigenvalues of the Dirac operator and those of* T^+_o *and* T^+_∞ *to the antiperiodic ones.*

Remark 5.5. We emphasize that, when $A \in \mathbb{N}$, Lemma 5.3 only holds for periodic or antiperiodic solutions of (4.2) (i.e., ν integer or half-integer). On the other hand, even when $A \notin \mathbb{N}$, a similar conclusion holds for certain Floquet solutions of (4.2). Namely, even for generic values of A, one can establish a one-to-one correspondence between certain Floquet exponents and ascending or descending Floquet eigenfunctions of (4.2), and in turn with Frobenius series solutions of (5.2).

So far we have analyzed the properties of solutions corresponding to periodic and antiperiodic eigenvalues of the problem. We now turn to the question of identifying these eigenvalues. Doing so yields the desired characterization of the Lax spectrum of (1.2).

Remark 5.6. A periodic/antiperiodic eigenfunction of (1.1) with potential (1.3) corresponds to a Fourier series solution (4.6) of the trigonometric ODE (4.2) that is convergent for $t \in \mathbb{R}$. The transformation (5.1) given by $\zeta = \mathrm{e}^{\mathrm{i}t}$, which maps the real t-axis onto the unit circle $|\zeta| = 1$ (cf. Fig. 2), maps these solutions into a Laurent series representation for the solutions of Heun's ODE (5.2). The question of identifying which solutions of Heun's ODE define periodic/antiperiodic eigenfunctions of (1.1) is discussed next.

Lemma 5.7. *Let* T_o *be either one of the operators* $T^\pm_o$ *defined in Remark* 5.2 *and let* $y_o(\zeta) = \zeta^\rho w_o(\zeta)$ *be a corresponding Frobenius series solution of Heun's equation at* $\zeta = 0$. *Then:*

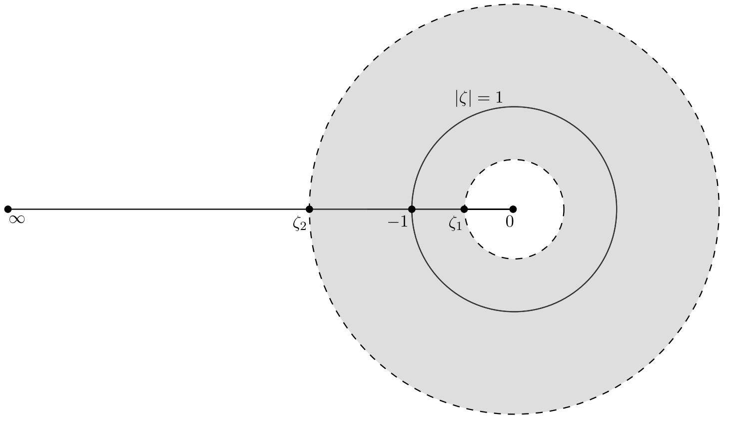

Fig. 2. The singular points $\zeta = 0, \zeta_1, \zeta_2$, and ∞ and the region $|\zeta| = 1$ in the complex ζ plane.

(i) *λ is an eigenvalue of T_o if and only if $w_o(\zeta)$ is analytic in the disk $|\zeta| < |\zeta_2|$; i.e., if $y_o(\zeta)$ is analytic up to a branch cut when the Frobenius exponent ρ_o is not integer.*

Similarly, let T_∞ be either one of the operators $T_\infty^\pm$ defined in Remark 5.2 and let $y_\infty(\zeta) = \zeta^\rho w_\infty(\zeta)$ be a corresponding Frobenius series solution of Heun's equation at $\zeta = \infty$. Then:

(ii) *λ is an eigenvalue of T_∞ if and only if $w_\infty(\zeta)$ is analytic in the exterior disk $|\zeta| > |\zeta_1|$; i.e., if $y_\infty(\zeta)$ is analytic up to a branch cut when the Frobenius exponent ρ_∞ is not integer.*

Proof. We consider T_o first. The radius of convergence of the Frobenius series representing $y(\zeta)$ in a neighborhood of $\zeta = 0$ is at least $|\zeta_1|$. Moreover, $\lambda \in \mathbb{C}$ is an eigenvalue of T_o if and only if the corresponding eigenvector $c \in \ell^{2,4}(\mathbb{N}_o)$ (see Remark 5.2). Since the entries of c coincide with the coefficients of the Frobenius power series representing $y(\zeta)$, we conclude that $\lambda \in \mathbb{C}$ is an eigenvalue of T_o if and only if the radius of convergence of this series is at least one. In this case $y(\zeta)$ is analytic in the disk $|\zeta| < |\zeta_2|$ (up to a possible branch cut), since there are no singular points of the Heun's equation in the annulus $|\zeta_1| < |\zeta| < |\zeta_2|$. The proof for T_∞ follows along the same lines. □

Remark 5.8. The above results relate the existence of eigenvalues to the connection problem for Heun's equation (5.2). For simplicity, consider the case of periodic eigenvalues. Assume $A \in \mathbb{N}$. The Frobenius analysis of Section 5.2 yields two linearly independent solutions of Heun's ODE near each of the four singular points. Let $y^o_{1,2}(\zeta;\lambda)$ be the Frobenius series with base point $\zeta = 0$ and $y^1_{1,2}(\zeta;\lambda)$ those with base point $\zeta = \zeta_1$. Both $y^o_{1,2}(\zeta;\lambda)$ and $y^1_{1,2}(\zeta;\lambda)$ form a basis for the solutions of Heun's ODE (5.2) in their respective domains of convergence. Since these domains overlap, in the intersection region one can express one set of solutions in terms of the other, i.e., $(y^1_1, y^1_2) = (y^o_1, y^o_2)\, C$, with a constant non-singular connection matrix C. The Frobenius exponents at $\zeta = \zeta_1$ are 0 and $\frac{1}{2}$, and, when $A \in \mathbb{N}$, one of the Frobenius exponents at $\zeta = 0$ is integer and the other is half-integer. Therefore, the values of λ for which the analytic solution at $\zeta = 0$ converges up to $|\zeta| = |\zeta_2|$ are precisely those values for which the Frobenius series with integer exponent at $\zeta = 0$ is exactly proportional to that with integer exponent at $\zeta = \zeta_1$. Similar arguments hold for the solutions near $\zeta = \zeta_2$ and $\zeta = \infty$. In other words, when λ is a periodic eigenvalue, the analytic solutions at $\zeta = 0$ and $\zeta = \zeta_1$ or those at $\zeta = \zeta_2$ and $\zeta = \infty$ must be proportional. This is the manifestation of an eigenvalue in terms of the connection problem for the Heun's equation (5.2). If both pairs of analytic solutions are proportional to each other, λ is a double eigenvalue, otherwise λ is a simple eigenvalue. (In Section 6 we will also see that all positive eigenvalues have multiplicity two and all negative eigenvalues have multiplicity one.) Similar results hold for the antiperiodic eigenvalues once the square root branch cut resulting from the half-integer Frobenius exponent is taken into account.

We also mention that there is an alternative but in a sense equivalent way to look at the problem, which is to study the convergence of the Frobenius series solutions (5.7) using Perron's rule [80]. This connection is briefly discussed in Appendix A.5.

6. Real eigenvalues of the operators $T_o^\pm$ and $T_\infty^\pm$

Thus far we have shown that the periodic and antiperiodic eigenvalues of (1.2) with Jacobi elliptic potential (1.3) and amplitude $A \in \mathbb{N}$ can be obtained from the eigenvalues of certain unbounded tridiagonal operators, namely, $T_o^\pm$ and $T_\infty^\pm$ defined in Section 5.2. We now prove that all eigenvalues of these operators are real. We do so in two steps: First, in Section 6.1, we show that finite truncations of these operators have purely real eigenvalues. Then, in Section 6.2, we use semicontinuity to show these operators have purely real eigenvalues.

6.1. Real eigenvalues of the truncated operators $T_{o,N}^\pm$ and $T_{\infty,N}^\pm$

Here we show that finite truncations of the operators $T_o^\pm$, $T_\infty^\pm$ have purely real eigenvalues. We form the truncations by considering only the first $N-1$ terms of the corresponding three-term recurrence relations. To this end let $T_{o,N}^\pm$ and $T_{\infty,N}^\pm$ be the $N \times N$ truncations of $T_o^\pm$ and $T_\infty^\pm$, respectively.

Lemma 6.1. *If $A \in \mathbb{N}$ and $m \in (0,1)$, then for any $N > 0$ the matrices $T_{o,N}^-$ and $T_{\infty,N}^\pm$ have purely real eigenvalues.*

The result is a consequence of the fact that $P_{n+1}S_n \geq 0$, $X_{n+1}Z_n \geq 0$, and $\widetilde{X}_{n+1}\widetilde{Z}_n > 0$, $n \geq 0$, which makes it possible to symmetrize $T_{o,N}^-$ and $T_{\infty,N}^\pm$ via a similarity transformation (see [42,46]). The result does not apply to $T_{o,N}^+$, since there exists an $n > 0$ such that $\widetilde{P}_{n+1}\widetilde{S}_n < 0$, and, as a result, some of the entries of the resulting symmetrized matrix would be complex. Thus, another approach is needed to show the eigenvalues of $T^+_{o,N}$ are all real. To this end we introduce the following definition [46]:

Definition 6.2 *(Irreducibly diagonally dominant).* An $N \times N$ tridiagonal matrix is irreducibly diagonally dominant if (i) it is irreducible; (ii) it is diagonally dominant, i.e., $|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|$, for all $i \in \{0, \ldots, N-1\}$; and (iii) there exists an $i \in \{0, \ldots, N-1\}$ such that $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$. Here a_{ij} denotes the entry in the i-th row and j-th column of the matrix.

Theorem 6.3. *(Veselic, [99] p. 171) Let $T^+_{o,N}$ be an $N \times N$ tridiagonal matrix which is irreducibly diagonally dominant and such that $\operatorname{sign}(\widetilde{P}_n\widetilde{S}_{n-1}) = \operatorname{sign}(\widetilde{R}_n\widetilde{R}_{n-1})$ for $n = 1, \ldots, N-1$. Then $T^+_{o,N}$ has N real simple eigenvalues.*

Next we show that $T^+_{o,N}$ satisfies the hypotheses of Theorem 6.3 and thus has only real eigenvalues.

Lemma 6.4. *If $A \in \mathbb{N}$ and $m \in (0, 1)$, then for any $N > 0$ all eigenvalues of $T^+_{o,N}$ are real and distinct.*

Proof. First, $A \in \mathbb{N}$ implies $\widetilde{P}_n\widetilde{S}_{n-1} \neq 0$ for $n \geq 1$. Thus, $T^+_{o,N}$ is irreducible. Next, $\widetilde{R}_n < 0$ when $n \leq \lfloor A - \frac{1}{2} \rfloor$. Similarly, $\widetilde{R}_{n-1} < 0$ when $n \leq \lfloor A + \frac{1}{2} \rfloor$. (Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.) Thus, $\operatorname{sign}(\widetilde{R}_n\widetilde{R}_{n-1}) < 0$ if and only if $A = n$. Likewise, $\widetilde{P}_n < 0$ when $n \leq \lfloor A + \frac{1}{2} \rfloor$, and $\widetilde{S}_{n-1} < 0$ when $n \leq \lfloor A - \frac{1}{2} \rfloor$. Thus, $\operatorname{sign}(\widetilde{P}_n\widetilde{S}_{n-1}) < 0$ if and only if $A = n$. Hence,

$$\operatorname{sign}(\widetilde{P}_n\widetilde{S}_{n-1}) = \operatorname{sign}(\widetilde{R}_n\widetilde{R}_{n-1}), \qquad n \geq 1. \tag{6.1}$$

Finally, consider the transpose $(T^+_{o,N})^{\mathrm{T}}$. Note (6.1) remains valid. For $n = 0$ one easily gets $|\widetilde{R}_0| > |\widetilde{P}_1|$. Moreover, for $n \geq 1$, one has $|\widetilde{R}_n| = (1 - \frac{m}{2})(2n + 1)|2n + 1 - 2A|$ and $|\widetilde{P}_{n+1}| + |\widetilde{S}_{n-1}| = \frac{m}{2}(2n + 1)|2n + 1 - 2A|$. Thus, $|\widetilde{R}_n| > |\widetilde{P}_{n+1}| + |\widetilde{S}_{n-1}|$ for $n \geq 1$. Hence $(T^+_{o,N})^{\mathrm{T}}$ is an $N \times N$ irreducibly diagonally dominant tridiagonal matrix and satisfies (6.1). The result follows from Theorem 6.3. □

Theorem 6.5. *If $A \in \mathbb{N}$ and $m \in (0, 1)$, then for any $N > 0$ all eigenvalues of the tridiagonal matrices $T^\pm_{o,N}$ and $T^\pm_{\infty,N}$ are real and have geometric multiplicity one. Moreover, all eigenvalues of $T^+_{o,N}$ and $T^+_{\infty,N}$, and all nonzero eigenvalues of $T^-_{o,N}$ and $T^-_{\infty,N}$ are simple.*

Proof. For $T^-_{o,N}$ and $T^\pm_{\infty,N}$, the reality of all eigenvalues was proved in Lemma 6.1, and for $T^+_{o,N}$ it was proved in Lemma 6.4. Moreover, Lemma 6.4 also proved that the eigenvalues of $T^+_{o,N}$ are simple.

Let λ be an eigenvalue of $T^-_{o,N}$ and $c = (c_0, \ldots, c_{N-1})^{\mathrm{T}}$ the corresponding eigenvector. Assume $c_0 = 0$. Then it follows from the three-term recurrence relation (5.8) that $c_n = 0$

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

for $n \geq 1$. (Note that S_0 is nonzero.) Since c is an eigenvector this is a contradiction. Hence the first component of the eigenvector is necessarily nonzero. Next, let c and $\tilde{c}$ be two eigenvectors corresponding to the same eigenvalue of $T^-_{o,N}$. Consider the linear combination $b = \alpha c + \tilde{\alpha}\tilde{c}$. Then there exists $(\alpha, \tilde{\alpha}) \neq 0$ such that $b_0 = 0$. By the first part of the argument $b \equiv 0$. Hence the eigenvectors c and $\tilde{c}$ are linearly dependent. The proofs for $T^+_{o,N}$ and $T^{\pm}_{\infty,N}$ are identical. □

6.2. Generalized convergence and reality of periodic and antiperiodic eigenvalues

In Section 6.1 we showed that, for $A \in \mathbb{N}$, the $N \times N$ truncations of the tridiagonal operators have only real eigenvalues. It remains to show that the tridiagonal operators $T^{\pm}_o$ and $T^{\pm}_\infty$ also have only real eigenvalues. This result will follow from the fact that the eigenvalues of the tridiagonal operators possess certain continuity properties as the truncation parameter N tends to infinity. Some of the proofs in this section follow from Volkmer [101]. For brevity we only present the details of the analysis for T^+_o.

Lemma 6.6. *Consider the operator T^+_o. There exists $\theta \in (0,1)$ and $n_* \in \mathbb{N}$ such that*

$$2\max(\widetilde{P}_n^{\,2} + \widetilde{S}_n^{\,2},\ \widetilde{P}_{n+1}^{\,2} + \widetilde{S}_{n-1}^{\,2}) \leq \theta^2 \widetilde{R}_n^{\,2}, \qquad n \geq n_*\,. \tag{6.2}$$

The same estimate holds for the operators T^-_o and $T^{\pm}_\infty$.

Proof. It follows from the definition of $\widetilde{P}_n$, $\widetilde{R}_n$, and $\widetilde{S}_n$ in (5.9c) that

$$\widetilde{P}_n^2 + \widetilde{S}_n^2 = 2m^2 n^4 (1 + o(1))\,, \tag{6.3a}$$

$$\widetilde{P}_{n+1}^2 + \widetilde{S}_{n-1}^2 = 2m^2 n^4 (1 + o(1))\,, \tag{6.3b}$$

$$\widetilde{R}_n^2 = (4 - 2m)^2 n^4 (1 + o(1))\,, \tag{6.3c}$$

as $n \to \infty$. Hence, let $\theta = m$. For n sufficiently large one gets $4m^2n^4 \leq \theta^2 \widetilde{R}_n^2 = m^2(4-2m)^2 n^4$. The result holds for $m \in (0,1)$. It is easy to check that the same estimate holds also for the operators T^-_o, $T^{\pm}_\infty$. □

Next we decompose $T^{\pm}_o$, $T^{\pm}_\infty$ into their diagonal and off-diagonal parts. Namely, if T is any one of the operators $T^{\pm}_o$, $T^{\pm}_\infty$, we write

$$T := T_D + T_O\,, \tag{6.4}$$

where T_D is the diagonal, and T_O the off-diagonal. This decomposition is instrumental in proving the following:

Lemma 6.7. *The operators $T^{\pm}_o$ and $T^{\pm}_\infty$ are closed with compact resolvent.*

Proof. The proof follows closely that of the analogous result in [101]. We provide details of the proof for the operator T_o^+. First, by replacing $\widetilde{R}_n$ by $\widetilde{R}_n + \omega$ with sufficiently large ω, we may assume, without loss of generality, that $\widetilde{R}_n > 0$ and that (6.2) holds for all $n \geq 0$. Then

$$\|T_O c\| \leq \theta \|T_D c\| \quad \forall c \in \ell^{2,4}(\mathbb{N}_o)\,. \tag{6.5}$$

Since $0 < \widetilde{R}_n \to \infty$ it follows T_D^{-1} exists and is a compact operator. Moreover, by (6.5) it follows that $\|T_O T_D^{-1}\| \leq \theta < 1$. Hence $T^{-1} = T_D^{-1}(I + T_O T_D^{-1})^{-1}$ is a compact operator (see [58] p. 196). Therefore, T_o^+ is a closed operator with compact resolvent. The proofs for T_o^-, $T_\infty^\pm$ are identical. □

The proof of the next lemma is identical to that of Theorem 6.5.

Lemma 6.8. *All eigenvalues of the operators $T_o^\pm$ and $T_\infty^\pm$ have geometric multiplicity one.*

Next we begin to address reality of the eigenvalues. By Lemma 6.7 it follows $\Sigma(T_o^\pm)$ and $\Sigma(T_\infty^\pm)$ are comprised of a set of discrete eigenvalues with finite multiplicities. Recall that, for $z \in \Sigma(L)$, we have $|\operatorname{Im} z| < \|q\|_\infty$. Moreover, we also have $|\operatorname{Im} z|\,|\operatorname{Re} z| \leq \frac{1}{2}\|q_x\|_\infty$ for any $z \in \Sigma(L)$ (see [12]). Hence, by the correspondence between the Dirac and Hill equations (see Section 3) we have

$$\operatorname{Re}\lambda \geq -\|q\|_\infty^2\,, \qquad |\operatorname{Im}\lambda| \leq \|q_x\|_\infty\,. \tag{6.6}$$

Thus there exists a curve $\mathscr{C}$ such that the region in the complex λ-plane bounded by $\mathscr{C}$ contains finitely many periodic (resp. antiperiodic) eigenvalues of Hill's equation with complex elliptic potential (3.15) counting multiplicity. This suggests to apply the concept of generalized convergence of closed linear operators (see Appendix A.6 for a discussion of generalized convergence). In particular, we will use the following result:

Theorem 6.9. *(Kato, [58] p. 206) Let T, $T_n \in \mathfrak{C}(\mathscr{X},\mathscr{Y})$, $n = 1, 2, \ldots$ the space of closed operators between Banach spaces. If T^{-1} exists and belongs to $\mathfrak{B}(\mathscr{X},\mathscr{Y})$, the space of bounded operators, then $T_n \to T$ in the generalized sense if and only if T_n^{-1} exists and is bounded for sufficiently large n and $\|T_n^{-1} - T^{-1}\| \to 0$.*

Theorem 6.9 implies the semicontinuity of a finite system of eigenvalues counted according to multiplicity ([58] p. 213). To this end, we introduce a sequence of tridiagonal operators:

$$T_n := T_D + P_n T_O\,, \tag{6.7}$$

where P_n is the orthogonal projection of $\ell^2(\mathbb{N}_o)$ onto $\operatorname{span}\{e_0, e_1, \ldots, e_{n-1}\}$, with $\{e_i\}_{n\in\mathbb{N}_o}$ being the canonical basis. Thus, for example, T_n is determined by, say, T_o^+

with the off-diagonal entries $\widetilde{P}_j$, $\widetilde{S}_j$ replaced by zeros for $j \geq n$. Clearly, $\Sigma(T_N) = \Sigma(T^+_{o,N}) \cup \{\widetilde{R}_n\}_{n\geq N}$ for any $N \in \mathbb{N}$. The following result is obtained from Theorem 2 in [101], the difference being the additional zero column for T^-_o and T^-_∞. Once the first column and row are deleted, the proof is identical. We therefore omit the proof for brevity.

Lemma 6.10. *Let T be any one of the operators $T^\pm_o$, $T^\pm_\infty$. If T_n is defined by (6.7), with T_D and T_O defined by (6.4), then $T_n \to T$ in the generalized sense (see Theorem 6.9 above).*

Using convergence in the generalized sense, we are now ready to show that $T^\pm_o$ and $T^\pm_\infty$ have real eigenvalues only:

Lemma 6.11. *If $\lambda_n \in \Sigma(T^\pm_o)$ or $\lambda_n \in \Sigma(T^\pm_\infty)$, then $\lambda_n \in \mathbb{R}$.*

Proof. Let T be any one of the operators $T^\pm_o$ or $T^\pm_\infty$. Count eigenvalues according to their multiplicity. Fix $n \in \mathbb{N}$, and let $\lambda_n \in \Sigma(T)$. Let $\epsilon > 0$ and $C_\epsilon := \{\lambda \in \mathbb{C} : |\lambda - \lambda_n| = \epsilon\}$. Since $\|T^{-1}_n - T^{-1}\| \to 0$ as $n \to \infty$, we know for each $\delta > 0$ there exists $N \in \mathbb{N}$ such that $\|T^{-1}_N - T^{-1}\| < \delta$. By semicontinuity of a finite system of eigenvalues (see [58] p. 212), we can choose $\delta > 0$ such that C_ϵ contains an eigenvalue of T_N. Call this eigenvalue λ_N. Since ϵ is arbitrary, and λ_N is real for any N it follows $\lambda_n \in \mathbb{R}$. □

Summarizing, we have shown that the periodic (resp. antiperiodic) eigenvalue problems for (1.2) with Jacobi elliptic potential (1.3) can be mapped to eigenvalue problems for four tridiagonal operators obtained from a Frobenius analysis of the Heun equation (5.2). Moreover, all eigenvalues of the tridiagonal operators are real with geometric multiplicity one. Putting everything together, we are now ready to prove the first part of Theorem 1.2:

Theorem 6.12. *Consider (1.2) with potential (1.3) and $m \in (0, 1)$. If $A \in \mathbb{N}$, then $\Sigma(L; A, m) \subset \mathbb{R} \cup (-\mathrm{i}A, \mathrm{i}A)$, and q is a finite-band potential.*

Proof. Let $z \in \Sigma_\pm(L)$. Recall that we have established a direct correspondence between the periodic (resp. antiperiodic) eigenvalues of the tridiagonal operators $T^\pm_o$, $T^\pm_\infty$ and the periodic (resp. antiperiodic) eigenvalues of the Dirac operator (1.2) with elliptic potential (1.3). Also, $\Sigma(T^\pm_o) \cup \Sigma(T^\pm_\infty) \subset \mathbb{R}$. Hence, by Lemma 3.5, and since $\lambda = z^2$, it follows that $\Sigma(L; A, m) \subset \mathbb{R} \cup (-\mathrm{i}A, \mathrm{i}A)$ (see also Lemma 2.10). Thus, the periodic (resp. antiperiodic) eigenvalues of the Dirac operator (1.2) with elliptic potential (1.3) are real or purely imaginary. Then, by symmetry (see Lemmas 2.9 and 2.10), the entire Lax spectrum is only real and purely imaginary. Finally, that q is finite-band for all $A \in \mathbb{N}$ and $m \in (0, 1)$ follows from Lemma 2.10. □

Lemma 6.13. *If $\lambda \in \Sigma(T^-_o) \cup \Sigma(T^+_\infty)$, then $\lambda \geq 0$.*

Proof. All entries of the tridiagonal operator T_∞^+ are positive. Consider the truncation $T_{\infty,N}^+$. Without loss of generality take the transpose. A simple calculation shows that $(T_{\infty,N}^+)^{\mathrm{T}}$ is strictly diagonally dominant. Hence, by the Gershgorin circle theorem all eigenvalues of $(T_{\infty,N}^+)^{\mathrm{T}}$ are strictly positive. By semicontinuity, in the limit $N \to \infty$ it follows that $\Sigma(T_\infty^+) \subset [0,\infty)$.

Next, note that the first column of T_o^- is comprised of all zeros. Thus, $\Sigma(T_o^-) = \Sigma(\tilde{T}_o^-) \cup \{0\}$, where $\tilde{T}_o^-$ is defined by T_o^- with the first row and the first column removed. Moreover, $\tilde{T}_o^-$ has strictly positive entries, and the transpose is diagonally dominant. Arguing as in the previous case gives the result. □

7. Lax spectrum for non-integer values of A

All of the results in this work up to the Fourier series expansion and the three-term recurrence relation in Section 4.2 hold independently of whether or not A is integer. The same holds for the Frobenius analysis in Section 5.2. On the other hand, the reducibility of the tridiagonal operator B_ν with integer and half-integer Floquet exponents ν in Section 4.3 only holds when $A \in \mathbb{N}$ (because it is only in that case that zeros appear in the upper and lower diagonal entries). Similarly, the Frobenius exponents at $\zeta = 0$ and $\zeta = \infty$ in Section 5.2 are integer or half-integer only when $A \in \mathbb{N}$. We next show that these are not just technical difficulties, but instead reflect a fundamental difference in the properties of the Lax spectrum of (1.2) when $A \notin \mathbb{N}$.

Lemma 7.1. *If $A \notin \mathbb{N}$, and $m \in (0,1)$, then all periodic and antiperiodic eigenvalues of (1.2) with Jacobi elliptic potential (1.3) have geometric multiplicity one.*

Proof. The proof proceeds by contradiction. For simplicity, we focus on the periodic eigenvalues. Suppose that for $A \notin \mathbb{N}$ and $\nu \in \mathbb{Z}$ there exist two linearly independent eigenfunctions. Then the transformation $\zeta = \mathrm{e}^{\mathrm{i}t}$ yields two linearly independent solutions of Heun's ODE (5.2) on $|\zeta| = 1$. Let us denote these solutions as $\hat{y}_1(\zeta;\lambda)$ and $\hat{y}_2(\zeta;\lambda)$. Note all points on $|\zeta| = 1$ are ordinary points for Heun's ODE and, therefore, both $\hat{y}_1(\zeta;\lambda)$ and $\hat{y}_2(\zeta;\lambda)$ are analytic and single-valued in the annulus $|\zeta_1| < |\zeta| < |\zeta_2|$ (cf. Fig. 2). Moreover, recall that the Frobenius exponents at $\zeta = \zeta_1$ are $\rho_1^1 = 0$ and $\rho_2^1 = 1/2$. Let $y_1^1(\zeta;\lambda)$ and $y_2^1(\zeta;\lambda)$ denote the corresponding solutions. Since $\hat{y}_1(\zeta;\lambda)$ and $\hat{y}_2(\zeta;\lambda)$ are linearly independent solutions, we have $y_1^1(\zeta;\lambda) = c_1\hat{y}_1(\zeta;\lambda) + c_2\hat{y}_2(\zeta;\lambda)$ for some constants c_1 and c_2. Then $y_1^1(\zeta;\lambda)$ is analytic and single valued in the region $0 < |\zeta| < |\zeta_2|$

On the other hand, $y_1^1(\zeta;\lambda)$ is a linear combination of the Frobenius solutions $\tilde{y}_1(\zeta;\lambda)$ and $\tilde{y}_2(\zeta;\lambda)$ defined at the singular point $\zeta = 0$, with Frobenius exponents $\rho_1^o = A/2$ and $\rho_2^o = (1-A)/2$ respectively, neither of which is an integer. Thus, no single-valued solution can exist around $\zeta = 0$. Therefore, there cannot be two linearly independent periodic eigenfunctions. Similar considerations apply for the antiperiodic eigenvalues. □

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

Note Lemma 7.1 does not hold for $m = 0$, as in the limit $m \to 0$ Heun's equation (5.2) degenerates into a Cauchy-Euler equation (with two regular singular points at $\zeta = 0$ and $\zeta = \infty$). Still, together with Lemma 2.16, Lemma 7.1 implies:

Corollary 7.2. *If $A \notin \mathbb{N}$, and $m \in (0,1)$, then $\Sigma_\pm(L) \cap \mathbb{R} = \emptyset$.*

In turn, since both $\Sigma_\pm(L)$ are infinite (see [25]), and since the periodic and antiperiodic eigenvalues are the endpoints of the spectral bands, Corollary 7.2 directly implies:

Corollary 7.3. *If $A \notin \mathbb{N}$, and $m \in (0,1)$, then $\Sigma(L)$ with Jacobi elliptic potential (1.3) has an infinite number of spines along the real z-axis.*

We conclude that when $A \notin \mathbb{N}$, the potential q in (1.3) is not finite-band according to Definition 2.5, which proves the only if part of Theorem 1.2, namely that $A \in \mathbb{N}$ is not only sufficient, but also necessary in order for q in (1.3) to be finite-band, as well as Theorem 1.4.

8. Further characterization of the spectrum and determination of the genus

It remains to prove the last part of Theorem 1.2, namely the determination of the genus. To this end, we need a more precise characterization of the Lax spectrum for $A \in \mathbb{N}$, which will also yield the proof of the remaining parts of Theorem 1.3. We turn to this task in this section.

8.1. Multiplicity of imaginary eigenvalues

Theorem 8.1. *If $z \in (-\mathrm{i}A, \mathrm{i}A) \setminus \{0\}$ is a periodic or an antiperiodic eigenvalue of (1.2) with potential (1.3) with $A \in \mathbb{N}$, and $m \in (0,1)$, then it has geometric multiplicity one.*

Proof. By Lemma 3.5 it follows $z \in \Sigma_\pm(L)$ if and only if $\lambda = z^2 \in \Sigma_\pm(H^-)$, respectively. Moreover, for $z \neq 0$ the geometric multiplicity of the periodic (resp. antiperiodic) eigenvalues is the same. Next, by the results of Section 5, each periodic (resp. antiperiodic) eigenfunction of H^- is associated with an eigenvector of $T_o^\pm$ or $T_\infty^\pm$, and

$$\Sigma(T_o^\pm) \cup \Sigma(T_\infty^\pm) = \{\lambda = z^2 : z \in \Sigma_\pm(L)\}, \tag{8.1}$$

with T_o^+, T_∞^+ yielding periodic eigenvalues and T_o^-, T_∞^- antiperiodic eigenvalues when A is odd, and vice versa when A is even (cf. Corollary 5.4). By Lemma 6.8, each eigenvalue of $T_o^\pm$, $T_\infty^\pm$ has geometric multiplicity one. Therefore, a periodic (resp. antiperiodic) eigenvalue $z \in \mathbb{C}$ of L can have geometric multiplicity two if and only if $\lambda = z^2$ is simultaneously an eigenvalue of both T_o^+ and T_∞^+ or simultaneously an eigenvalue of both T_o^- and T_∞^-. On the other hand, Lemma 6.13 showed that the eigenvalues of T_o^- and T_∞^+ are non-negative. Hence, by the relation $\lambda = z^2$ all periodic (resp. antiperiodic) eigenvalues $z \in (-\mathrm{i}A, \mathrm{i}A) \setminus \{0\}$ of (1.2) with potential (1.3) have geometric multiplicity one. □

Corollary 8.2. *For all $m \in (0,1)$, if $z \in (-\mathrm{i}A, \mathrm{i}A) \setminus \{0\}$ is a periodic or an antiperiodic eigenvalue of (1.2) with potential (1.3), then $s(z) \neq 0$.*

Proof. Recall that $s(z)$ is defined by (2.15). If $s(z) = 0$, the monodromy matrix $M(z)$ would be diagonal, but this would imply the existence of two periodic (resp. antiperiodic) eigenfunctions, which would contradict Theorem 8.1. □

Note that the above results do not hold for $m = 0$ (a constant background potential), since in that case all periodic and antiperiodic eigenvalues except $z = \pm\mathrm{i}A$ have geometric multiplicity two.

8.2. Dirichlet eigenvalues and behavior of the Floquet discriminant near the origin

In this subsection we prove some technical but important results that will be used later in the proof of Theorem 1.3.

As in Section 2.3, here it will be convenient to explicitly keep track of the dependence on m by writing the potential, fundamental matrix solution, and monodromy matrix respectively as $q(x;m)$, $\Phi(x;z,m)$ and $M(z;m)$. We begin by recalling some relevant information. We will use the structure of the monodromy matrix $M(z;m) = \Phi(2K(m);z,m)$ introduced in (2.15). Also recall that, when $m = 0$ (in which case $q(x,0) \equiv A$), $M(z,0)$ is given by (2.23). (Recall that $l = 2K(m)$ is the (real) period of $\mathrm{dn}(x;m)$, and $2K(0) = \pi$.) Thus, (2.15) implies

$$\Delta(z;0) = \cos\left(\sqrt{z^2 + A^2}\pi\right), \quad s(z;0) = -\frac{A}{\sqrt{z^2 + A^2}}\sin\left(\sqrt{z^2 + A^2}\pi\right). \tag{8.2}$$

Recall from Section 2.2 that $\Delta(z;m)$ and $s(z;m)$ are even functions of z while $c(z;m)$ is an odd function of z. Let $\Delta_j(m)$, $-\mathrm{i}c_j(m)$ and $s_j(m)$ denote, respectively, the coefficients of z^{2j}, z^{2j+1} and z^{2j} in the Taylor series of $\Delta(z;m)$, $c(z;m)$ and $s(z;m)$ around $z = 0$. Combining (2.23) and (A.12b), we obtain the following expansions near $z = 0$:

$$\Delta(z;m) = (-1)^A + \Delta_1(m)z^2 + O(z^4)\,, \tag{8.3a}$$

$$c(z;m) = -\mathrm{i}c_0(m)z - \mathrm{i}c_1(m)z^3 + O(z^5)\,, \tag{8.3b}$$

$$s(z;m) = s_1(m)z^2 + O(z^4)\,. \tag{8.3c}$$

We want to study in detail the behavior of $\Delta(z;m)$ near $z = 0$. We begin by looking at the dynamics of (closed) gaps as a function of A at $m = 0$, to show how the number of bands grows as A increases. According to (8.2), the periodic and antiperiodic eigenvalues are, respectively,

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

$$z_n = \pm\sqrt{4n^2 - A^2}\,, \qquad z_n = \pm\sqrt{(2n+1)^2 - A^2}\,, \qquad n \in \mathbb{Z}\,. \tag{8.4}$$

It follows that $z = 0$ is a periodic or antiperiodic eigenvalue when $A \in \mathbb{Z}$. Direct calculations show that

$$\Delta_{zz}(z;0) = -\frac{\pi^2 z^2}{z^2 + A^2}\cos(\sqrt{z^2 + A^2}\pi) - \frac{\pi A^2}{(z^2 + A^2)^{\frac{3}{2}}}\sin(\sqrt{z^2 + A^2}\pi), \tag{8.5}$$

so that

$$\Delta_{zz}(0;0) = -\frac{\pi \sin(A\pi)}{A}. \tag{8.6}$$

Observe that, as a function of A, $\Delta_{zz}(0;0)$ changes sign as A passes through an integer value. For example, if A passes through an even value $n \in \mathbb{N}$, the sign of $\Delta_{zz}(0;0)$ changes from "+" to "−", corresponding to the transition of a pair of critical points of $\Delta(z;0)$ from $\mathbb{R}$ to $[-\mathrm{i}A, \mathrm{i}A]$. Correspondingly, a pair of zero level curves of $\operatorname{Im}\Delta(z) = 0$ intersecting $\mathbb{R}$ transversally will pass through $z = 0$ and intersect $[-\mathrm{i}A, \mathrm{i}A]$ forming an extra closed gap on $[-\mathrm{i}A, \mathrm{i}A]$. This is the mechanism of increase of the number of gaps on $[-\mathrm{i}A, \mathrm{i}A]$. Note that (8.5) implies that $\Delta_z(z;0)$ has a third order zero at $z = 0$ when $A \in \mathbb{Z}$. Next we show that this mechanism works for any $m \in (0,1)$. This will be accomplished through several intermediate steps.

Lemma 8.3. *For fixed $A \in \mathbb{N}$ we have $\Delta_{zz}(0;0) = 0$, and $(-1)^A\Delta_{zz}(0;m)$ is a strictly monotonically decreasing function of m for $m \in [0,1)$.*

Proof. The first statement follows from (8.5). The rest of the proof is devoted to show that

$$(-1)^A\Delta_{zz}(0;m) < 0\,, \tag{8.7}$$

when $m \in (0,1)$. Substitution of (8.3) into (2.16) yields

$$\tfrac{1}{2}\Delta_{zz}(0;m) = \Delta_1(m) = \tfrac{1}{2}(-1)^{A+1}c_0^2(m)\,. \tag{8.8}$$

Note that $(-1)^A\Delta_{zz}(0;m) \le 0$ since $M(z;m)$ is real on $z \in [-\mathrm{i}A, \mathrm{i}A]$. Thus, it remains to show that $c_0(m) \ne 0$ for $m \in (0,1)$. In fact, we will show below that $c_0(m)$ is monotonically increasing on $m \in [0,1)$. That, combined with $c_0(0) = 0$ (see (8.8)), will complete the proof.

Recall that $\Phi = \Phi(x;z,m)$ is the solution of the ZS system (1.6a) normalized as $\Phi(0;z,m) \equiv \mathbf{1}$. Differentiating (1.6a) with respect to z we get the system

$$\Phi_{xz} = (-\mathrm{i}z\sigma_3 + \mathrm{i}q\sigma_2)\Phi_z - \mathrm{i}\sigma_3\Phi\,. \tag{8.9}$$

Considering system (8.9) as a non-homogeneous ZS system [i.e., treating the term $-\mathrm{i}\sigma_3\Phi$ as a "forcing"] and integrating, we obtain the solution

$$\Phi_z(x;z,m) = -\mathrm{i}\Phi(x;z,m)\int_0^x \Phi^{-1}(\xi;z,m)\sigma_3\Phi(\xi;z,m)\,\mathrm{d}\xi\,, \qquad z\in\mathbb{C}\,. \tag{8.10}$$

Also recall that the (real) period of q in (1.3) is $l = 2K(m)$ and that $M(z;m) = \Phi(l;z,m)$. By Lemma 2.17, $A\in\mathbb{N}$ implies $\Phi(l;0,m)\equiv(-1)^A\mathbf{1}$. At $z=0$, we therefore have

$$\begin{aligned} M_z(0;m) &= -\mathrm{i}(-1)^A\int_0^l \Phi^{-1}(\xi)\sigma_3\Phi(\xi)\,\mathrm{d}\xi \\ &= -\mathrm{i}(-1)^A\int_0^l \begin{pmatrix} u_1v_2+u_2v_1 & 2v_1v_2 \\ -2u_1u_2 & -u_1v_2-u_2v_1\end{pmatrix}\mathrm{d}\xi\,, \end{aligned} \tag{8.11}$$

where we introduced the notation

$$\Phi(x;z,m) = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2\end{pmatrix}, \tag{8.12}$$

which we will use extensively below. On the other hand, in light of (2.15) we have

$$M_z = \Delta_z\mathbf{1} + c_z\sigma_3 - \mathrm{i}s_z\sigma_2\,, \qquad z\in\mathbb{C}\,, \tag{8.13}$$

which implies

$$\begin{aligned} c_z(0;m) &= -\mathrm{i}(-1)^A\int_0^l (u_1v_2+u_2v_1)\,\mathrm{d}x\,, \\ s_z(0;m) &= 2\mathrm{i}(-1)^A\int_0^l u_1u_2\,\mathrm{d}x = 2\mathrm{i}(-1)^A\int_0^l v_1v_2\,\mathrm{d}x\,. \end{aligned} \tag{8.14}$$

Comparing this with the expansion (8.3) we then have

$$c_0(m) = \mathrm{i}c_z(0;m) = (-1)^A\int_0^l (u_1v_2+u_2v_1)\,\mathrm{d}x\,. \tag{8.15}$$

At $z=0$, according to Section A.3, we also have

$$\Phi(x;0,m) = \cos(A\mathrm{am}x)\,\mathbf{1} + \sin(A\mathrm{am}x)\,\mathrm{i}\sigma_2\,, \tag{8.16}$$

where $\mathrm{am}x = \mathrm{am}(x;m)$, so that

$$\Phi^{-1}(x;0,m)\sigma_3\Phi(x;0,m) = \cos(2A\mathrm{am}x)\sigma_3 + \sin(2A\mathrm{am}x)\sigma_1\,. \tag{8.17}$$

We obtain

$$c_0(m) = (-1)^A \int_0^{2K(m)} \cos(2A\mathrm{am}x)\,\mathrm{d}x = (-1)^A \int_0^{\pi} \frac{\cos 2Ay\,\mathrm{d}y}{\sqrt{1-m\sin^2 y}}\,, \tag{8.18}$$

where we used $y = \mathrm{am}x$, $\mathrm{d}y = \sqrt{1-m\sin^2 y}\,\mathrm{d}x$. Now, from [14], 806.01, for $m \in (0,1)$ we have

$$(-1)^A \int_0^{\pi} \frac{\cos 2Ay\,\mathrm{d}y}{\sqrt{1-m\sin^2 y}} = \pi \sum_{j=A}^{\infty} \frac{[(2j-1)!]^2 m^j}{4^{2j-1}(j-A)!(j+A)![(j-1)!]^2} > 0 \tag{8.19}$$

since all the coefficients of the convergent Taylor series are positive. □

Corollary 8.4. *One has $|\Delta(z;m)| > 1$ in a deleted neighborhood of $z = 0$ on $(-\mathrm{i}A, \mathrm{i}A)$ for any $A \in \mathbb{N}$ and $m \in (0,1)$. Moreover, $z = 0$ is a simple critical point of $\Delta(z;m)$ and $\Delta(0;m) = (-1)^A$.*

Remark 8.5. Corollary 8.4 shows that no critical points of $\Delta(z;m)$ can move from $\mathbb{R}$ to $\mathrm{i}\mathbb{R}$ when we vary $m \in (0,1)$ with a fixed $A \in \mathbb{N}$. Similarly to the case $m = 0$, the change of genus in the case $m > 0$, happens when we vary A (see also Fig. 4).

Next, recall that the monodromy matrix $M(z;x_o)$ normalized at a base point x_o is given by (2.20). Using (2.20), we prove the following lemma regarding Dirichlet eigenvalues.

Lemma 8.6. *Let $A \in \mathbb{N}$ and $m \in (0,1)$. If an open gap γ on $(-\mathrm{i}A, \mathrm{i}A)$ contains a zero of $s(z)$, then the associated Dirichlet eigenvalue is movable.*

Proof. Equations (2.15) and (2.20) and direct calculation show that

$$M(z;x_0)=\begin{pmatrix} \Delta+c(u_1v_2+u_2v_1)+s(u_1u_2+v_1v_2) & -s(u_1^2+v_1^2)-2cu_1v_1 \\ s(u_2^2+v_2^2)+2cu_2v_2 & \Delta-c(u_1v_2+u_2v_1)-s(u_1u_2+v_1v_2) \end{pmatrix}, \tag{8.20}$$

where $c = c(z)$ and $s = s(z)$ were defined in (2.15) and the functions u_1, u_2, v_1 and v_2, defined as in (8.12), are evaluated at $x = x_0$.

Consider first an open gap $\gamma \subset \mathrm{i}\mathbb{R}$ that does not contain $z = 0$. From (2.16) it follows that $c(z) \neq 0$ on γ. Note $u_1(x_0)v_1(x_0) \neq 0$ for small $x_0 > 0$ [because $u_1(0) = 1$ and $v_1(0) = 0$ and u_1 and v_1 are analytic in x as solutions of (1.6a) with the potential (1.3)].

Therefore, it follows from (8.20) that $M_{12}(z;x_0)=0$ implies $s(z)\neq 0$. But $M_{12}(z;0)=s(z)$. Thus, each Dirichlet eigenvalue in such a gap is movable.

Consider now a gap $\gamma_0\subset \mathrm{i}\mathbb{R}$ containing $z=0$, i.e., the *central gap*. By Corollary 8.4, such gap exists for any $m\in(0,1)$, and by (8.2), it does not exist when $m=0$. We consider $m\in(0,1)$. Then by Lemma 8.3 (see (8.8)), $c(z)$ has a simple zero at $z=0$ and, by (8.3), $s(z)$ has at least a double zero at $z=0$. Thus, the condition $M_{12}(z;x_0)=0$ near $z=0$ becomes

$$-s_1z^2(u_1^2(x_0;z)+v_1^2(x_0;z))+2\mathrm{i}c_0zu_1(x_0;z)v_1(x_0;z)=R(x_0;z)\,, \tag{8.21}$$

where $R(x_0;z)\in\mathbb{R}$ when $z\in\mathrm{i}\mathbb{R}$ and $R(x_0;z)=O(z^3)$ uniformly in small real x_0. By Lemma 8.3 (see (8.8)) we have $c_0\neq 0$. If $s_1\neq 0$, (8.21) shows that $M_{12}(z;x_0)$ has one fixed zero at $z=0$ whereas the location of the second zero depends on x_0 and is given by

$$z=\frac{2\mathrm{i}c_0u_1(x_0;z)v_1(x_0;z)-\frac{R(x_0;z)}{z}}{s_1(u_1^2(x_0;z)+v_1^2(x_0;z))}=\frac{2\mathrm{i}c_0u_1(x_0;0)v_1(x_0;0)+O(z)}{s_1(u_1^2(x_0;0)+v_1^2(x_0;0)+O(z))}\in\mathrm{i}\mathbb{R}\,, \tag{8.22}$$

which is a point inside the central gap on $(-\mathrm{i}A,\mathrm{i}A)$. Indeed, the requirement $\det M(z;x_0)\equiv 1$ and (8.20) imply that a Dirichlet eigenvalue can not be in the interior of any band located on $(-\mathrm{i}A,\mathrm{i}A)\setminus\{0\}$.

Equations (8.21) and (8.22) show that a zero of $M_{12}(z;x_0)$ in the gap $\gamma_0\subset(-\mathrm{i}A,\mathrm{i}A)\setminus\{0\}$ is always fixed at $z=0$, and therefore corresponds to an immovable Dirichlet eigenvalue, whereas a second zero is located at a point changing with x_0, and is therefore a movable Dirichlet eigenvalue. Indeed, the point $z=z(x_0)$ defined by (8.22) attains $z(0)=0$ and $z(x_0)\neq 0$ at least for small $x_0>0$ since $v_1(0)=0$ and $v_1(x_0)\neq 0$ in a deleted neighborhood of zero.

Finally, if $s_1=\cdots=s_{k-1}=0$ and $s_k\neq 0$, with $k>1$, the leading-order portion of each term in the 1,2 entry of (8.21) yields instead

$$-s_kz^{2k}(u_1^2+v_1^2)+2\mathrm{i}c_0zu_1v_1=R\,, \tag{8.23}$$

where again $R=O(z^3)$ is real-valued for $z\in\mathrm{i}\mathbb{R}$ and where for brevity we dropped the arguments. Repeating the same arguments as for (8.21), we see that at least one of the roots in (8.23) is purely imaginary. □

Remark 8.7. It follows from (8.21) and (8.23) that $s_1(m)\neq 0$ if and only if there is exactly one movable Dirichlet eigenvalue in a vicinity of $z=0$ for small $x_0\in\mathbb{R}$.

8.3. Proof of the remaining statements of Theorem 1.3

Lemma 2.14 proves items 4 and 5 of Theorem 1.3. Theorem 1.2 together with Lemma 2.2 and the symmetries (2.17) implies item 3. Thus, it remains to prove items 2 and 6 only, namely:

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

Theorem 8.8. *Consider* (1.2) *with Jacobi elliptic potential* (1.3)*. For all* $m \in (0,1)$*, if* $A \in \mathbb{N}$ *then:*

1. *For any* $m \in (0,1)$*, there are exactly* $2A$ *symmetric bands of* $\Sigma(L; A, m)$ *on* $(-\mathrm{i}A, \mathrm{i}A)$ *separated by* $2A-1$ *symmetric open gaps. The central gap (i.e., the gap surrounding the origin) contains an eigenvalue at* $z = 0$*. This eigenvalue is periodic when* A *is even and antiperiodic when* A *is odd.*
2. *Each of the open* $2A-1$ *gaps on* $(-\mathrm{i}A, \mathrm{i}A)$ *contains exactly one movable Dirichlet eigenvalue. Thus, all of the* $2A-1$ *movable Dirichlet eigenvalues of the finite-band solution with genus* $2A-1$ *are located in the gaps of the interval* $(-\mathrm{i}A, \mathrm{i}A)$*.*

Proof. The idea of the proof is based on continuous deformation of the elliptic parameter m, starting from $m = 0$ and going into $m \in (0,1)$. The proof is based on the following three main steps, each of which will be discussed more fully below:

1. *Analysis of the spectrum for* $m = 0$*.* When $m = 0$, $\mathrm{dn}(x, 0) \equiv 1$, and the ZS system (1.1) has a simple solution $\Phi(x; z, m)$. The monodromy matrix $M(z; m)$ based on $\Phi(x; z, m)$ was given explicitly in (2.23) for $m = 0$, and the Lax spectrum is $\Sigma(L) = \mathbb{R} \cup [-\mathrm{i}A, \mathrm{i}A]$ in this case. In particular, the vertical segment $[-\mathrm{i}A, \mathrm{i}A]$ is a single band that contains $2A-1$ double periodic/antiperiodic eigenvalues, which we consider as being closed gaps. Each of these closed gaps contains a Dirichlet eigenvalue (a zero of $s(z; m)$, see (8.2)), which for $m = 0$ is immovable according to Lemma 2.14 (see also (8.20)).

2. *Analysis of the spectrum for small nonzero values of* m*.* Corollary 8.4 states that for all $m \in (0,1)$ the double eigenvalue at $z = 0$ is embedded in the central gap $\gamma_0 \subset (-\mathrm{i}A, \mathrm{i}A)$. Moreover, Corollary 8.4 and Lemma 8.6 show that there is at least one movable Dirichlet eigenvalue on γ_0. Next, we show that under a small deformation $m > 0$ all the remaining closed gaps on $(-\mathrm{i}A, \mathrm{i}A)$ must open, creating $2A$ bands and $2A-1$ gaps on $(-\mathrm{i}A, \mathrm{i}A)$, with each gap containing exactly one movable Dirichlet eigenvalue. Our proof of this statement is based on the fact that any periodic/antiperiodic eigenvalue on $(-\mathrm{i}A, \mathrm{i}A) \setminus \{0\}$ has geometric multiplicity one (see Theorem 8.1), whereas a double eigenvalue at a closed gap would have geometric multiplicity two. By a continuity argument, each gap on $(-\mathrm{i}A, \mathrm{i}A) \setminus \{0\}$ has exactly one movable Dirichlet eigenvalue. Thus, items 2,6 are proved for small $m > 0$.

3. *Control of the spectrum for arbitrary values of* $m \in (0,1)$*.* We finally prove that the number $2A$ of separate bands on $(-\mathrm{i}A, \mathrm{i}A)$, as well as the fact that each gap on $(-\mathrm{i}A, \mathrm{i}A)$ contains exactly one movable Dirichlet eigenvalue, cannot change when m varies on $(0,1)$.

In what follows, we prove all the statements in items 1–3 above.

1. From (8.2) it follows that, for $m = 0$, we have $\Sigma(L) = \mathbb{R} \cup [-\mathrm{i}A, \mathrm{i}A]$ with

$$z_n^2 = n^2 - A^2, \quad n = 0, \pm 1, \ldots, \pm A \tag{8.24}$$

being (interlaced) periodic or antiperiodic eigenvalues on $[-\mathrm{i}A, \mathrm{i}A]$. Note that $\Delta_z(z; 0)$ has a simple zero at each z_n for $n \neq 0, \pm A$. Therefore, each $z_n \neq 0, \pm \mathrm{i}A$ identifies a

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

closed gap. On the other hand, since $\Delta_z(\pm \mathrm{i}A; 0) \neq 0$, the endpoints $\pm\mathrm{i}A$ of the spectrum are simple periodic eigenvalues. [By Lemma 2.10, when $m > 0$ it follows $\pm\mathrm{i}A \notin \Sigma(L)$.] Finally, note that $\Delta_z(z; 0)$ has a double zero at $z = 0$, which will be relevant in the discussion of item 3 below.

Recall that the Dirichlet eigenvalues are the zeros of $s(z; 0)$. By (8.2), each closed gap on $[-\mathrm{i}A, \mathrm{i}A] \setminus \{0\}$ contains exactly one Dirichlet eigenvalue μ_n, $n = \pm 1, \ldots, \pm(A - 1)$, which is a simple zero of $s(z; 0)$. Note also that there are exactly $2A - 1$ closed gaps on $(-\mathrm{i}A, \mathrm{i}A)$, and at each such gap with the exception of $z = 0$ there is exactly one zero level curve Γ_n of $\operatorname{Im} \Delta$ orthogonally crossing $\mathrm{i}\mathbb{R}$. Moreover, by Lemma 8.3, there are eight zero level curves of $\operatorname{Im} \Delta$ passing through $z = 0$, including the real and imaginary axes (e.g., see Fig. 4, upper right panel). Note that, by Lemma 8.6, a Dirichlet eigenvalue μ_n becomes movable if the closed gap opens up as m is deformed away from $m = 0$.

2. Recall that the monodromy matrix $M(z; m)$ is entire in z and A and analytic in $m \in [0, 1)$ (cf. Lemma 2.10). Also recall that $s(z; m)$ is real-valued on $\mathrm{i}\mathbb{R}$. Finally, recall that by Lemma 2.14 zeros of $s(z; m)$ cannot lie in the interior of a band. Let μ_n be the zeros of $s(z; 0)$ for $z \in \mathrm{i}\mathbb{R}$, i.e., the Dirichlet eigenvalues along the imaginary axis when $m = 0$. Since the zeros of $s(z; m)$ are isolated, for sufficiently small $m > 0$, each Dirichlet eigenvalue μ_n must remain on $(-\mathrm{i}A, \mathrm{i}A)$ by continuity. Thus, for sufficiently small values of m, all the gaps on $(-\mathrm{i}A, \mathrm{i}A) \setminus \{0\}$ (independently of whether they are open or closed) must survive the small m deformation, with exactly one Dirichlet eigenvalue in each gap.

Importantly, the above arguments imply that, for small $m \in (0, 1)$, all the gaps on $(-\mathrm{i}A, \mathrm{i}A)\setminus\{0\}$ must be open. Indeed, the assumption that for small $m \in (0, 1)$ there exists a closed gap at $z_* \in (-\mathrm{i}A, \mathrm{i}A) \setminus \{0\}$ leads to a contradiction, because by Corollary 8.2, $s(z; m) \neq 0$ at the endpoints of each band. There are $2A-2$ such open gaps. By continuity, each of them contains a zero of $s(z; m)$ and therefore a movable Dirichlet eigenvalue by Lemma 8.6. Moreover, by continuity, $s(z; m)$ must have opposite signs at the endpoints of any gap in $(-\mathrm{i}A, \mathrm{i}A) \setminus \{0\}$.

Next, recall that by Theorem 6.12, $\Sigma(L) \subset \mathbb{R} \cup (-\mathrm{i}A, \mathrm{i}A)$ for all $m \in (0, 1)$. It follows from Corollary 8.4 that, for all $m \in (0, 1)$, the (double) Floquet eigenvalue $z = 0$ is immersed in a gap $\gamma_o \subset (-\mathrm{i}A, \mathrm{i}A)$. Then, by Lemma 8.6, for small $m > 0$ there are exactly $2A - 1$ open gaps on $(-\mathrm{i}A, \mathrm{i}A)$, with each gap containing a movable Dirichlet eigenvalue. Therefore there are $2A$ (disjoint) bands on $(-\mathrm{i}A, \mathrm{i}A)$.

Finally, differentiation of $s(z; m)$ in (8.2) yields

$$s_z(z; 0) = \frac{zA}{z^2 + A^2} \left[\frac{\sin\left(\sqrt{z^2 + A^2}\pi\right)}{\sqrt{z^2 + A^2}} - \cos\left(\sqrt{z^2 + A^2}\pi\right) \right], \tag{8.25}$$

which shows that $s_z(z; 0)$ has a simple zero at the origin. Therefore, $s_1(0) = s_{zz}(0; 0) \neq 0$ and so, by Remark 8.7, there is a unique movable Dirichlet eigenvalue in a vicinity of $z = 0$ and it is situated on γ_o. Hence there is exactly one movable Dirichlet eigenvalue in each gap implying that the genus of the corresponding Riemann surface in $2A - 1$. Thus, items 2 and 6 are proved for small $m > 0$.

3. It remains to prove that the number of bands on $(-\mathrm{i}A, \mathrm{i}A)$ and the number of movable Dirichlet eigenvalues (which were established for small $m \in (0,1)$ in item 2 above) do not change as m varies in $(0,1)$. Let us consider the deformation of the collection of bands (with genus $2A-1$) established for small $m \in (0,1)$. A possible change of the genus can be caused only by one of the following four possibilities: (a) a collapse of a band into a point; (b) a splitting of a band into two or more separate bands; (c) a splitting of a gap into two or more separate gaps; (d) a collapse of an open gap into a closed one.

We next prove that none of these possibilities can occur. Indeed, regarding (a), the collapse of a band into a point would contradict the analyticity of $\Delta(z;m)$, since it would imply that the same value of z is simultaneously a periodic and antiperiodic eigenvalue. (Note that each band along $(-\mathrm{i}A, \mathrm{i}A)$ must necessarily start at a periodic eigenvalue and end at an antiperiodic one or vice versa, since otherwise there would necessarily be a critical point z_o inside the band. But a critical point z_o inside the band would imply the existence of a second band emanating transversally from the imaginary axis, contradicting Theorem 6.12.) Similarly, regarding (b), the splitting of a band would require a critical point of $\Delta(z;m)$ at some z_o inside the band. But, again, a critical point at z_o would mean that there is a zero-level curve of $\operatorname{Im}\Delta$ crossing $\mathrm{i}\mathbb{R}$ at z_0, which in turn would contradict Theorem 6.12.

For the same reasons we have $\Delta_z(z_*;m) \neq 0$ at any non-periodic and non-antiperiodic Floquet eigenvalue z_*, separating a band and a gap on $(-\mathrm{i}A, \mathrm{i}A)$. Indeed, the contrary would lead to Δ_z having an even-order zero at z_*. But that would imply at least two pairs of zero-level curves emanating from $\mathrm{i}\mathbb{R}$ at z_* and, thus, again would contradict Theorem 6.12.

We now turn our attention to the gaps, and specifically to the possibility (c) listed above. The splitting of a gap into two or more separate gaps would imply that $\Delta(z;m)$ has a local minimum z_0 on the gap at some $m \in (0,1)$ and simultaneously $|\Delta(z_0;m)| \leq 1$. That, again, would contradict Theorem 6.12.

Thus, it remains to exclude possibility (d), namely the collapse of a gap. By Lemma 8.6, the central gap γ_o containing $z=0$ stays open for any $m \in (0,1)$. Also, for small $m \in (0,1)$, it was shown in the proof of item 2 above that the signs of $s(z;m)$ at the endpoints of any gap $\gamma \subset (-\mathrm{i}A, \mathrm{i}A) \setminus \{0\}$ are opposite. These signs cannot change in the course of a deformation with respect to $m \in (0,1)$, by Corollary 8.2. Thus, each gap that was open for small $m \in (0,1)$ must contain a zero of $s(z;m)$ and, therefore, as it was proven in item 2 above, must stay open for all $m \in (0,1)$. So, the genus $2A-1$ is preserved for all $m \in (0,1)$. Moreover, each gap contains a zero of $s(z;m)$ and thus, according to Lemma 8.6, a movable Dirichlet eigenvalue.

So, we proved that each gap on $(-\mathrm{i}A, \mathrm{i}A)$ contains exactly one movable Dirichlet eigenvalue. It is well known [34,40,74] that the number of movable Dirichlet eigenvalues is equal to the genus $2A-1$. This, completes the proof of Theorem 8.8 for all $m \in (0,1)$. □

Remark 8.9. It follows from Remark 8.7 that $s_1(m) \neq 0$ for all $m \in (0,1)$.

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

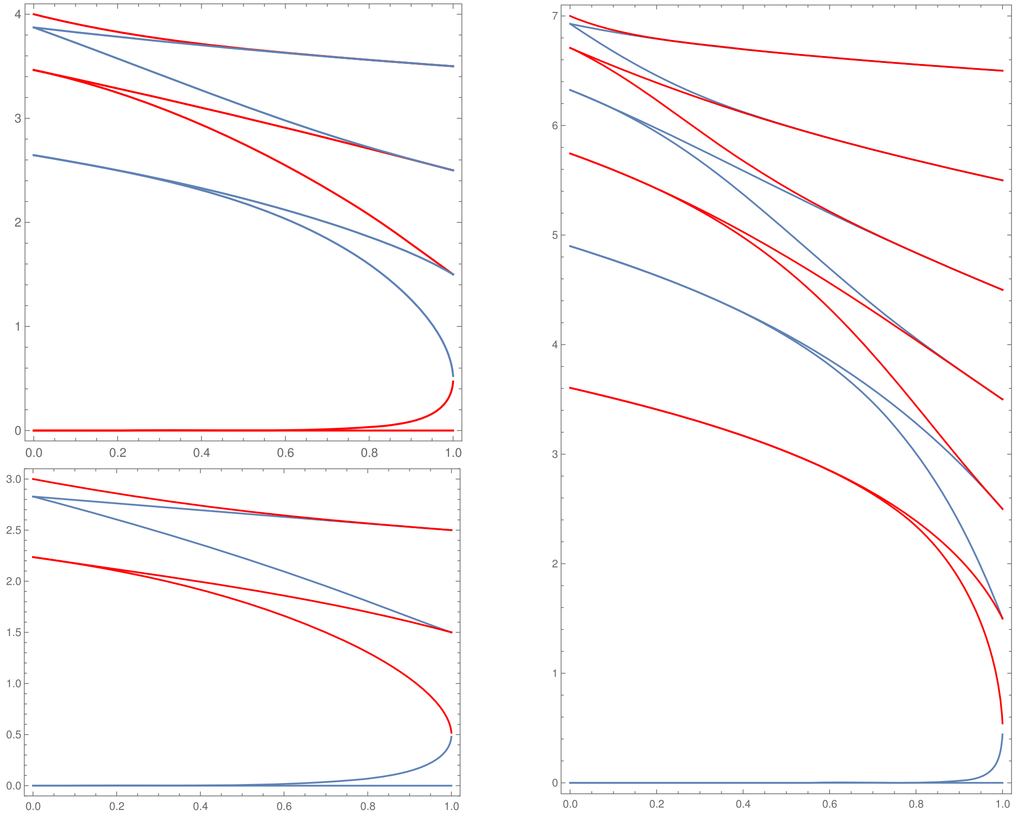

Fig. 3. Periodic (red) and antiperiodic (blue) eigenvalues (vertical axis) of the spectrum as a function of the elliptic parameter m (horizontal axis) for a few integer values of A. Bottom left: $A = 3$. Top left: $A = 4$. Right: $A = 7$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

9. Dynamics of the spectrum as a function of A and m

We further illustrate the results of this work by presenting some concrete plots of the spectrum. We begin with the case of $A \in \mathbb{N}$. Fig. 3 shows the periodic (red) and antiperiodic (blue) eigenvalues along the imaginary z-axis (vertical axis in the plot) as a function of the elliptic parameter m (horizontal axis) for a few integer values of A, namely: $A = 3$ (bottom left), $A = 4$ (top left) and $A = 7$ (right). Note how all gaps are closed when $m = 0$ and how they open immediately as soon as $m > 0$ and remain open for all $m \in (0, 1)$. In the singular limit $m \to 1^-$, the band widths tend to zero, and the periodic and antiperiodic eigenvalues “collide” into the point spectrum of the operator L on the line.

Next, Fig. 4 shows the Lax spectrum (blue curves) in the complex z-plane for several non-integer values of A, illustrating the formation of new bands and gaps as function of A. Note that the range of values for the real and imaginary parts of z allows one to see only a small portion of the Lax spectrum. For example, not visible outside the

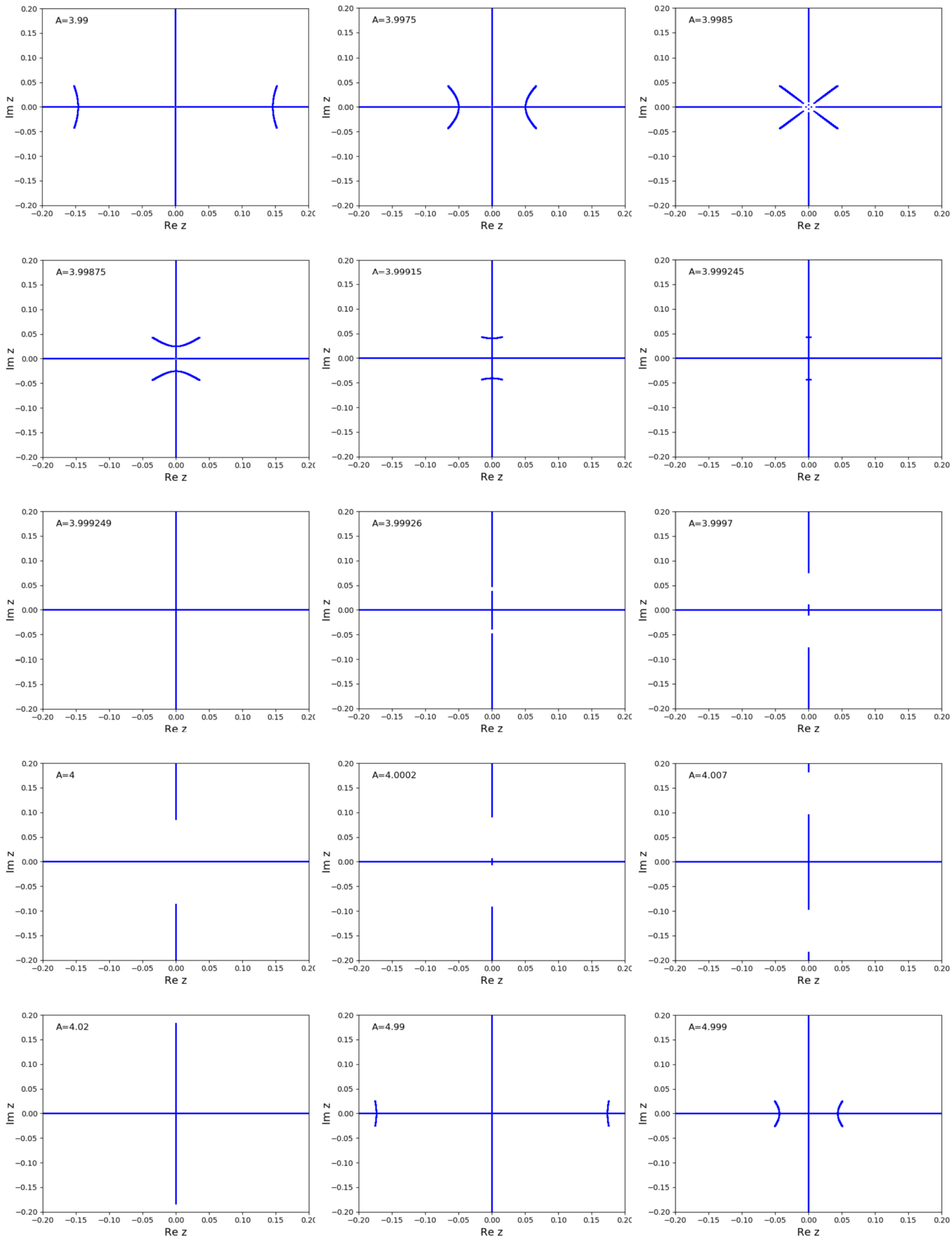

Fig. 4. The Lax spectrum $\Sigma(L)$ [computed numerically using Hill's method (see [21])] with potential $q \equiv A\,\mathrm{dn}(x;m)$, $m = 0.9$, and increasing values of A, illustrating the formation of new bands and gaps for non-integer values of A.

plot window are various bands and gaps along the imaginary axis (cf. Fig. 3) as well as the infinite number of spines growing off the real axis when $A \notin \mathbb{N}$. However, selecting a larger portion of the complex z-plane would have made it more difficult to see the dynamics of the bands and gaps near the origin. Starting from the smallest value of A in the set ($A = 3.99$, top left panel), one can see how, as A increases, a spine is pulled towards the origin, which it reaches at approximately $A = 3.9985$ (top right panel). As A increases further, the spine moves along the imaginary axis, simultaneously shrinking to zero at approximately $A = 3.999249$ (left plot in the third row). (Note that, even though the band is effectively gone at this value of A, the corresponding potential is still not finite-band due to the infinitely many spines that are still present outside the plot window.) As A increases further, the band edges of the previous spine bifurcate along the imaginary axis, giving rise to a new gap. Finally, at $A = 4$ (left plot in the fourth row), the lower edge of this new gap reaches the origin. This is also exactly the value of A at which the infinitely many spines shrink to zero. As A increases further, the band centered at $z = 0$ reappears, a new spine gets sucked towards the origin, and the cycle repeats.

In summary, every time A increases by one unit, two more spines from the real axis gets pulled into the imaginary axis, and a two new Scwarz symmetric gaps open on the imaginary axis. When A hits the next integer value, the lower band edges in the upper-half plane and the corresponding one in the lower-half plane reach the point $z = 0$. Simultaneously, all remaining spines emanating from the real z-axis shrink to zero, giving rise to a finite-band potential. As A keeps increasing, the spines grow back, and the process repeats.

It is also interesting to briefly describe the dynamics of the zero-level curves of $\operatorname{Im}\Delta(z;m)$ near $z = 0$. Thanks to (8.5) and (8.6), we know that $\Delta_{zz}(0;0) = 0$ if and only if $A \in \mathbb{Z}$. One can also see that $\Delta_{zzzz}(0;0) \neq 0$ when $A \in \mathbb{Z}$. So, when $A \in \mathbb{Z}$, there are exactly eight zero-level curves of $\operatorname{Im}\Delta$ emanating from $z = 0$. As it follows from Corollary 8.4, under a small $m > 0$ deformation from $m = 0$, a pair of these level curves will move up along the imaginary z-axis, while the other pair will symmetrically move down along $\mathrm{i}\mathbb{R}^-$. It also follows from (8.2) that $s(z;0)$ has a second-order zero at $z = 0$; that is, $s_1(0) \neq 0$, which will remain in place under a small $m > 0$ deformation according to (2.16), (2.17) and Corollary 8.4. This is another way to show that $s_1(m) \neq 0$ for small $m > 0$.

10. Discussion and concluding remarks

The results of this work provide an extension to the non-self-adjoint operator (1.2) of the classical works of Ince [47–49]. The results of this work also provide: (i) an example of Hill's equation with a complex, PT-symmetric potential (and a corresponding complex deformation of Ince's equation) whose spectrum is purely real, which is especially relevant, since the study of quantum mechanics with non-Hermitian, PT-symmetric potential continues to attract considerable interest (e.g. see [5,9,30] and references therein),

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

(ii) an example of an exactly solvable connection problem for Heun's ODE, and (iii) for the first time a perturbation approach to study the determination of the genus, and the movable Dirichlet eigenvalues was presented.

We point out that the fact that the elliptic potential (1.3) is finite-band for any $A \in \mathbb{N}$ can also be obtained as a consequence of the results of [40], where the potential $q(x) = n\,(\zeta(x) - \zeta(x-\omega_2) - \zeta(\omega_2))$ was studied (where $\zeta(x)$ is Weierstrass' zeta function and ω_2 one of the lattice generators [79]) and was shown to be finite-band when $n \in \mathbb{N}$ using the criteria introduced there (see Appendix A.7 for details). In Appendix A.7 we also discuss other elliptic potentials satisfying the criteria laid out in [40]. On the other hand, no discussion of the spectrum (i.e., location of the periodic/antiperiodic eigenvalues and of the spectral bands) was present in [40].

It is also the case that the elliptic potential (1.3) is associated with the so-called Trebich-Verdier potentials [92] for Hill's equation (which are known to be algebro-geometric finite-band, see [88]) if and only if $A \in \mathbb{N}$, as we show in Appendix A.8. To the best of our knowledge, this connection had not been previously made in the literature.

The family of elliptic potentials (1.3) is especially important from an applicative point of view, since (as was discussed in Section 1) it interpolates between the plane wave potential $q(x) \equiv A$ when $m = 0$ and the Satsuma-Yajima (i.e., sech) potential $q(x) \equiv A\,\mathrm{sech}\,x$ when $m = 1$, which, when $A \in \mathbb{N}$, gives rise to the celebrated A-soliton bound-state solution of the focusing NLS equation [(1.5) with $s = 1$].

The potential $q(x) \equiv A\,\mathrm{sech}\,x$ has also been used in relation to the semiclassical limit of the focusing NLS equation. This is because, by letting $A = 1/\epsilon$ and performing a simple rescaling $x \mapsto \epsilon x$ and $t \mapsto \epsilon t$ of the spatial temporal variables, (1.5) is mapped into the semiclassical focusing NLS equation

$$\mathrm{i}\epsilon q_t + \epsilon^2 q_{xx} + 2|q|^2 q = 0\,, \tag{10.1}$$

with the rescaled initial data $q(x,0) \equiv \mathrm{dn}(x;m)$. The dynamics of solutions of (10.1) has been studied extensively in the literature (e.g., see [10,13,17,19,29,55,57,70,96]). In particular, it is known that, for a rather broad class of single-lobe initial conditions (including $q(x) \equiv \mathrm{sech}\,x$), the dynamics gives rise to a focusing singularity (gradient catastrophe) that is regularized by the formation of high-intensity peaks regularly arranged in the pattern of genus-2 solutions of the NLS equation. The caustic (i.e., breaking) curve along which the genus-2 region breaks off from the genus-0 region (characterized by a slowly modulated plane wave, in which the solution does not exhibit short-scale oscillations) has also been characterized, and it is conjectured that additional breaking curves exist, giving rise to regions of higher genus.

All of the above-cited works studied localized potentials on the line. However, similar behavior was observed for (10.1) with periodic potentials in [11], where a formal asymptotic characterization of the spectrum of the Zakharov-Shabat system (1.6a) in the semiclassical scaling was obtained using WKB methods, and in particular it was shown that the genus is $O(1/\epsilon)$ as $\epsilon \to 0^+$. Some of the numerical results of [11] about the localization of the spectrum were rigorously proved in [12]. The results of the present work provide some rigorous evidence, for the dn potential (1.3), in support of the formal results of [11] about the genus of the potential as a function of ϵ.

We emphasize that, even though we limited our attention to the focusing NLS equation for simplicity, all the equations of the infinite NLS hierarchy (including the modified KdV equation, higher-order NLS equation, sine-Gordon equation, etc.) share the same Zakharov-Shabat scattering problem (1.6a). Therefore, the results of this work provide a two-parameter family of finite-band potentials for all the equations in the focusing NLS hierarchy.

The results of this work open up a number of interesting avenues for further study. In particular, an obvious question is whether these potentials are stable under perturbations. The stability of genus-1 solutions of the focusing NLS equation was recently studied in [22] by taking advantage of the machinery associated with the Lax representation. A natural question is therefore whether similar results can also be used for the higher-genus potentials when $A > 1$ or whether different methods are necessary.

Another interesting question is whether more general elliptic finite-band potentials related to (1.3) exist. Recall that, for the focusing NLS equation on the line, the potential $q(x) = A \operatorname{sech} x \, \mathrm{e}^{-\mathrm{i}a \log(\cosh x)}$ (which reduces to $q(x) = A \operatorname{sech} x$ when $a = 0$) was shown in [95,96] to be amenable to exact analytical treatment. It is then natural to ask whether exactly solvable periodic potentials also exist related to $q(x) = A \operatorname{dn}(x; m)$ but with an extra non-trivial periodic phase.

Yet another question is related to the time evolution of the potential (1.3) according to the focusing NLS equation. When $A = 1$, time evolution is trivial, and the corresponding solution of the NLS equation is simply $q(x,t) = \operatorname{dn}(x; m)\, \mathrm{e}^{\mathrm{i}(2-m)t}$. That is not the case when $A > 1$, however. For the Dirac operator (1.2) on the line with reflectionless potentials, sufficient conditions were obtained in [68] guaranteeing that, if the discrete spectrum is purely imaginary, the corresponding solution of the focusing NLS equation is periodic in time. The natural question is then whether a similar result is also true for the elliptic potential (1.3), namely, whether such potentials generates a time-periodic solution of the focusing NLS equation when $A \in \mathbb{N}$.

The semiclassical limit of certain classes of periodic potentials (including the potential $\operatorname{dn}(x; m)$) generates a so-called breather gas for the focusing NLS equation (e.g., see [11,97]), which is to be understood as the thermodynamic limit of a finite-band solution of the focusing NLS equation where the genus $G \to \infty$ and simultaneously all bands but one shrink in size exponentially fast in G (see [94] for details). It was proposed that such gases be called periodic breather gases. Periodic gases have the important feature that, together with their spectral data (i.e., independent of the phase variables) such as the density of states, one also can obtain some information on a “realization” of the gas, namely, on the semiclassical evolution of the given periodic potential. Thus, progress in studying the $(2A-1)$-band solutions of the focusing NLS equation (with $A \in \mathbb{N}$) generated by the potential $q(x) = A \operatorname{dn}(x; m)$, and especially its large A limit, is of definite interest. In fact, finite-band solutions to integrable systems (such as the KdV and NLS equations) generated by elliptic potentials, based on the work of Krichever [64], were studied in the literature. We will not go into the details of those results here, but it should be clear that any details about the family of finite-band solutions of the focusing NLS equation that homotopically "connect" the known behavior of the plane wave and the multi-soliton solutions will be very interesting to obtain and analyze.

Acknowledgments

This work was partially supported by the U.S. National Science Foundation under grant numbersDMS-2009487 (GB), and DMS-2009647 (AT), and the National Natural Science Foundation of China under grant number 12101590 (XDL). Part of this work was supported by the U.S. National Science Foundation under grant number 1440140 while JO was in residence at the Mathematical Sciences Research Institute (MSRI) in Berkeley, California, during the Fall 2021 semester program "Universality and Integrability in Random Matrix Theory and Interacting Particle Systems".

Appendix A

A.1. Notation and function spaces

The Pauli spin matrices, used throughout this work, are defined as

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1}$$

Moreover, $L^\infty(\mathbb{R};\mathbb{C}^2)$ is the space of essentially bounded Lebesgue measurable two-component vector functions with the essential supremum norm. Given the interval $I_{x_o} = [x_o, x_o + l]$ we define the inner product between two-component Lebesgue measurable functions ϕ and ψ as

$$\langle \phi, \psi \rangle := \int_{x_o}^{x_o+l} \left(\phi_1 \overline{\psi}_1 + \phi_2 \overline{\psi}_2\right) \mathrm{d}x\,. \tag{A.2}$$

Then $L^2(I_{x_o};\mathbb{C}^2)$ denotes the set of two-component Lebesgue measurable vector functions that are square integrable, i.e., $\|\phi\|_{L^2(I_{x_o};\mathbb{C}^2)} := \langle \phi, \phi \rangle^{1/2} < \infty$. Similarly, we define the inner product of two scalar Lebesgue measurable functions f and g as

$$\langle f, g \rangle := \int_{x_o}^{x_o+l} f\overline{g}\,\mathrm{d}x\,. \tag{A.3}$$

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

Then $L^2(I_{x_o};\mathbb{C})$ denotes the set of scalar Lebesgue measurable functions that are square integrable, i.e., $\|f\|_{L^2(I_{x_o};\mathbb{C})} := \langle f, f\rangle^{1/2} < \infty$. Finally, we define the inner product of two infinite sequences $c = \{c_n\}_{n\in\mathbb{Z}}$ and $d = \{d_n\}_{n\in\mathbb{Z}}$ as

$$\langle c, d\rangle := \sum_{n\in\mathbb{Z}} c_n \overline{d}_n \,. \tag{A.4}$$

Then $\ell^2(\mathbb{Z})$ denotes the set of square-summable sequences, i.e., $\|c\|_{\ell^2(\mathbb{Z})} := \langle c, c\rangle^{1/2} < \infty$. Finally, the space of continuous functions on the real axis is denoted $C(\mathbb{R})$, and $\mathbb{N}_o := \mathbb{N} \cup \{0\}$.

A.2. Proof of two lemmas

Proof of Lemma 2.10. To prove part (i) we begin by writing (1.1) as the coupled system of linear differential equations (1.6a). By Floquet theory $z \in \Sigma(L)$ if and only if $\phi = \mathrm{e}^{\mathrm{i}\nu x}\psi$, where $\psi = (\psi_1, \psi_2)^T$ with $\psi(x+l; z) = \psi(x; z)$, and $\nu \in [0, 2\pi/l)$. Plugging this expression into (1.6a) yields the modified system

$$\mathrm{i}\psi_{1,x} - \mathrm{i}q\psi_2 = (z + \nu)\psi_1 \,, \qquad \mathrm{i}\psi_{2,x} + \mathrm{i}q\psi_1 = (-z + \nu)\psi_2 \,. \tag{A.5}$$

Multiplying the first of these equations by $\overline{\psi}_1$ and taking the complex conjugate yields two equations, which we integrate over a full period, arriving at the expressions

$$\begin{aligned} \mathrm{i}\langle q\psi_2, \psi_1\rangle &= -\mathrm{i}\langle \psi_1, \psi_{1,x}\rangle - (z + \nu)\|\psi_1\|^2_{L^2([0,l])} \,, \\ \mathrm{i}\langle \psi_1, q\psi_2\rangle &= \mathrm{i}\langle \psi_1, \psi_{1,x}\rangle + (\overline{z} + \nu)\|\psi_1\|^2_{L^2([0,l])} \,, \end{aligned}$$

where boundary terms vanish since $\psi_1(x+l; z) = \psi_1(x; z)$. Adding these two expressions then one gets

$$-\operatorname{Im} z\|\psi_1\|^2_{L^2([0,l])} = \operatorname{Re}\langle q\psi_2, \psi_1\rangle \,. \tag{A.6}$$

Similarly, the second equation of (A.5) yields

$$\begin{aligned} \mathrm{i}\langle \psi_1, q\psi_2\rangle &= \mathrm{i}\langle \psi_2, \psi_{2,x}\rangle + (-z + \nu)\|\psi_2\|^2_{L^2([0,l])} \,, \\ \mathrm{i}\langle q\psi_2, \psi_1\rangle &= -\mathrm{i}\langle \psi_2, \psi_{2,x}\rangle + (\overline{z} - \nu)\|\psi_2\|^2_{L^2([0,l])} \,, \end{aligned}$$

as well as

$$-\operatorname{Im} z\|\psi_2\|^2_{L^2([0,l])} = \operatorname{Re}\langle q\psi_2, \psi_1\rangle \,. \tag{A.7}$$

Equating (A.6) and (A.7) we conclude that if $|\operatorname{Im} z| > 0$, then

$$\|\psi_1\|_{L^2([0,l])} = \|\psi_2\|_{L^2([0,l])} \,. \tag{A.8}$$

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

Next, note that $|\langle q\psi_2, \psi_1\rangle| \le \langle |q\psi_2|, |\psi_1|\rangle$. Also, since q is not constant, there exists $(a,b) \subset (0,l)$ such that $|q(x)| < \|q\|_\infty$ for $x \in (a,b)$. Thus, for $|\operatorname{Im} z| > 0$ it follows from (A.8) and the Hölder inequality that

$$0 < |\operatorname{Im} z| \|\psi_1\|^2_{L^2([0,l])} = |\operatorname{Re}\langle q\psi_2, \psi_1\rangle| < \|q\|_\infty \langle |\psi_2|, |\psi_1|\rangle \\ \le \|q\|_\infty \|\psi_1\|^2_{L^2([0,l])}\,.$$

Hence $|\operatorname{Im} z| < \|q\|_\infty$ for $z \in \Sigma(L)$. The proof of (ii) which can be found in [12] follows from Lemma 2.9 and since $\Delta(z)$ is an analytic function of z. □

Lemma A.1. *Consider the Dirac operator (1.2). If the potential $q \equiv A\,\mathrm{dn}(x;m)$, then the monodromy matrix $M(z;m)$ is an analytic function of m for any $m \in [0,1)$.*

Proof. The result follows from two key facts. One is that $\mathrm{dn}(x;m)$ is an analytic function of m for all $|m| \le 1$ [103]. The second is the fact that solutions of ODEs with analytic dependence on variables and parameters are analytic (see [18] pp. 23–32 and [50] p. 72). □

A.3. Solution of the ZS system at $z = 0$

We have seen that, for q real, the eigenvalue problem (1.1) can be reduced to second-order scalar ODEs (3.3). Consider (3.3) with $\lambda = 0$, namely $v^\pm_{xx} + (\pm \mathrm{i} q_x + q^2)v^\pm = 0$. Using the ansatz $v^\pm = \mathrm{e}^{\pm f}$, one gets $\pm f_{xx} + (f_x)^2 \pm \mathrm{i} q_x + q^2 = 0$. Then, letting $g = f_x$ yields $\pm g_x + g^2 = \mp \mathrm{i} q_x - q^2$ with a solution given by $g = \mp \mathrm{i} q$. In particular, it follows $g^2 = -q^2$. Hence, we have derived the following solution to the ODEs (3.3) for $\lambda = 0$, namely,

$$v^\pm(x;0) = \mathrm{e}^{\mp \mathrm{i}\int_0^x q(s)\,\mathrm{d}s}\,. \tag{A.9}$$

Next, using the invertible change of variables (3.1), one gets the following solution to the eigenvalue problem (1.1) when $z = 0$:

$$\phi(x;0) = \left(\cos\left(\int_0^x q(s)\,\mathrm{d}s\right), -\sin\left(\int_0^x q(s)\,\mathrm{d}s\right)\right)^{\mathrm{T}}. \tag{A.10a}$$

Moreover, using Rofe-Beketov's formula [83], one obtains a second linearly independent solution as:

$$\widetilde{\phi}(x;0) = \left(\sin\left(\int_0^x q(s)\,\mathrm{d}s\right), \cos\left(\int_0^x q(s)\,\mathrm{d}s\right)\right)^{\mathrm{T}}. \tag{A.10b}$$

Thus, the Floquet discriminant (2.5) for eigenvalue problem (1.1) at $z = 0$ is

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

$$\Delta(0) = \cos\Big(\int_0^l q(s)\,\mathrm{d}s\Big)\,, \tag{A.11}$$

where l is the period of the potential. We can now prove Lemma 2.17.

Proof of Lemma 2.17. Using well-known properties of the Jacobi elliptic functions (see [43,79]), when $q(x) = A\,\mathrm{dn}(x;m)$, (A.10a), (A.10b) and (A.11) yield, respectively

$$\phi(x;0,A,m) = \Big(\cos(A\,\mathrm{am}(x;m)), -\sin(A\,\mathrm{am}(x;m))\Big)^{\mathrm{T}}, \tag{A.12a}$$

$$\widetilde{\phi}(x;0,A,m) = \Big(\sin(A\,\mathrm{am}(x;m)), \cos(A\,\mathrm{am}(x;m))\Big)^{\mathrm{T}}, \tag{A.12b}$$

$$\Delta(0;A,m) = \cos(A\pi)\,. \tag{A.12c}$$

In particular, $\phi(0;0,A,m) = (1,0)^{\mathrm{T}}$, and $\phi(2K;0,A,m) = (\cos(A\pi),\sin(A\pi))^{\mathrm{T}}$. Therefore, we have that $\phi(x+2K;0,A,m) = \phi(x;0,A,m)$ if and only if $A \in 2\mathbb{Z}$, and $\phi(x+2K;0,A,m) = -\phi(x;0,A,m)$ if and only if $A \in 2\mathbb{Z}+1$, with similar relations for $\widetilde{\phi}$. Thus, when A is an even integer, $z = 0 \in \Sigma_+(L)$, whereas when A is an odd integer, $z = 0 \in \Sigma_-(L)$. Finally, the above calculations also show that, for $q \equiv A\,\mathrm{dn}(x;m)$ with $A \in \mathbb{Z}$ the eigenvalue $z = 0$ has geometric multiplicity two. □

A.4. Transformation of the ZS system into a Heun system

If the potential q is real, then the change of dependent variable $\phi = \frac{1}{2}\,\mathrm{diag}(1,-\mathrm{i})(\sigma_3 + \sigma_1)\,v$, maps the ZS system (1.6a) into the equivalent system

$$v_x = -\mathrm{i}(z\sigma_1 + q\sigma_3)v\,. \tag{A.13}$$

Then the transformation $t = 2\mathrm{am}(x;m)$ maps (A.13) to the trigonometric first-order system

$$v_t = -\tfrac{\mathrm{i}}{2}\left(A\sigma_3 + \frac{z\sigma_1}{\sqrt{1 - m\sin^2\frac{t}{2}}}\right)v\,, \tag{A.14}$$

which is equivalent to (4.2). Finally, the transformation $\zeta = \mathrm{e}^{\mathrm{i}t}$ maps (A.14) to

$$\zeta v_\zeta = -\tfrac{1}{2}\left(A\sigma_3 + \frac{z\sigma_1}{\sqrt{1 - \frac{m}{2}\big(1 - \frac{1}{2}(\zeta + \zeta^{-1})\big)}}\right)v\,, \tag{A.15}$$

and the transformation

$$v = \Xi w\,, \quad \Xi = \mathrm{diag}\left(1, \frac{1}{z}\sqrt{1 - \frac{m}{2}(1 - \frac{1}{2}(\zeta + \zeta^{-1}))}\right)\,, \tag{A.16}$$

maps (A.15) to the Heun system (5.6) where $\lambda = z^2$. The Heun system (5.6) has four (regular) singular points, located at $\zeta = 0, \zeta_{1,2}, \infty$, where $\zeta_{1,2}$ are zeros of the denominator in (5.6). The Frobenius exponents at the singularities can be derived directly from (5.6).

A.5. Augmented convergence and Perron's rule

In general the Frobenius series (5.7a) with base point $\zeta = 0$ only converges for $|\zeta| < |\zeta_1|$ and the series (5.7b) with base point $\zeta = \infty$ only converges for $|\zeta| > |\zeta_2|$. Therefore, neither expansion is convergent on $|\zeta| = 1$ in general. However, there exist certain values of λ for which one or both of the Frobenius series have a larger (i.e., augmented) radius of convergence. These are precisely the periodic/antiperiodic eigenvalues of the problem, and Perron's rule provides a constructive way to identify them (see also [4,31,48,50,85] for further details).

We begin by noting that, by dividing all coefficients by n^2, all four three-term recurrence relations (5.8), (5.9), (5.10) and (5.11) can be rewritten as

$$e_0 c_0 + f_0 c_1 = 0\,, \qquad n = 0\,, \tag{A.17a}$$

$$d_n c_{n-1} + e_n c_n + f_n c_{n+1} = 0\,, \qquad n = 1, 2, \ldots \tag{A.17b}$$

with $f_n \neq 0$, and $d_n \to d$, $e_n \to e$, and $f_n \to f$ as $n \to \infty$. Perron's rule [31,80] states that, if $\xi_\pm$ are the roots of the quadratic equation

$$f\xi^2 + e\xi + d = 0\,, \tag{A.18}$$

with $|\xi_-| < |\xi_+|$, then $\lim_{n\to\infty} c_{n+1}/c_n = \xi_+$, unless the coefficients d_n, e_n, f_n satisfy the infinite continued fraction equation

$$\frac{e_o}{f_o} = \cfrac{d_1}{e_1 - \cfrac{d_2 f_1}{e_2 - \cfrac{d_3 f_2}{e_3 - \cdots}}}\,, \tag{A.19}$$

in which case $\lim_{n\to\infty} c_{n+1}/c_n = \xi_-$. That is, Perron's rule implies that, generically, the radius of convergence of the series $\Sigma_{n=0}^{\infty} c_n \zeta^n$ is $1/|\xi_+|$. However, if and only if (A.19) holds, the radius of convergence is $1/|\xi_-|$, and therefore larger. In our case, the roots $\xi_\pm$ of (A.18) are exactly the singular points $\zeta_{1,2}$ of Heun's ODE (5.2). Then, since e_n depends on λ, (A.19) is a condition that determines the exceptional values of λ that guarantee augmented convergence. Indeed, (A.19) is equivalent to requiring that λ is an eigenvalue of $T_o^\pm$ (resp. $T_\infty^\pm$).

A.6. Generalized convergence of closed operators

Here we briefly discuss the generalized convergence of closed operators, (see [58] p. 197 for a detailed discussion). Consider $\mathfrak{C}(\mathscr{X},\mathscr{Y})$ the space of closed linear operators between Banach spaces. If T, $S \in \mathfrak{C}(\mathscr{X},\mathscr{Y})$, their graphs $G(T)$, $G(S)$ are closed linear manifolds on the product space $\mathscr{X} \times \mathscr{Y}$. Set $\hat{\delta}(T,S) = \hat{\delta}(G(T), G(S))$, i.e., the *gap* between T and S. (See [58] p. 197 for the definition of $\hat{\delta}(T,S)$.) Similarly, we can define the *distance* $\hat{d}(T,S)$ between T and S as equal to $\hat{d}(G(T), G(S))$. (See [58] p. 198 for the definition of $\hat{d}(T,S)$.) With this distance function $\mathfrak{C}(\mathscr{X},\mathscr{Y})$ becomes a metric space.

In this space the convergence of a sequence $T_n \to T \in \mathfrak{C}(\mathscr{X},\mathscr{Y})$ is defined by $\hat{(}T_n,T) \to 0$. Since $\hat{\delta}(T,S) \le \hat{d}(T,S) \le 2\hat{\delta}(T,S)$ ([58] p. 198) this is true if and only if $\hat{\delta}(T_n,T) \to 0$. In this case we say $T_n \to T$ *in the generalized sense*. This notion of generalized convergence for closed operators is a generalization of convergence *in norm*. Importantly, the convergence of closed operators in the generalized sense implies the continuity of a finite system of eigenvalues ([58] p. 213).

A.7. Gesztesy-Weikard criterion for finite-band potentials

According to Theorem 1.2 from [40], an elliptic potential $Q(x)$ of the Dirac operator (1.2) is finite-band if and only if its fundamental matrix solution $\Phi(x;z)$ is meromorphic in x for all $z \in \mathbb{C}$.

Theorem A.2. *Consider (1.2), then $q \equiv A\,\mathrm{dn}(x;m)$ with $m \in (0,1)$ is finite-band if and only if $A \in \mathbb{Z}$.*

Proof. The (simple) poles of $\mathrm{dn}(x;m)$ within the fundamental period $2jK + 4n\mathrm{i}K'$ are at $x = \mathrm{i}K'$ and $x = 3\mathrm{i}K'$ where $K' := K(1-m)$. By the Schwarz symmetry, it is sufficient to consider only the pole at $\mathrm{i}K'$. The residue at $\mathrm{i}K'$ is $-\mathrm{i}$ ([43], 8.151) and the local Laurent expansion is odd. Let $\Phi(u) := \Phi(x - \mathrm{i}K'; z)$ and note $\mathrm{dn}(x + \mathrm{i}K'; m) = -\mathrm{i}\,\mathrm{cn}(u;m)/\,\mathrm{sn}(u;m)$ ([14], p. 20). Substitution into (A.13) gives

$$u\Phi_u(u) = [-\mathrm{i}zu\sigma_3 + (A + u^2 p(u))\sigma_2]\Phi(u) =: B(u)\Phi(u), \tag{A.20}$$

where $p(u)$ is analytic near $u = 0$ and even, and is meromorphic near $u = 0$ for all $z \in \mathbb{C}$. The leading order term of $B(u)$ is $A\sigma_2$ with eigenvalues $\lambda = \pm A$. Thus, meromorphic $\Phi(u)$ requires $A \in \mathbb{Z}$.

To show that $A \in \mathbb{Z}$ is also a sufficient condition we need to show that $\Phi(u)$ does not contain logarithms, i.e., regular singular point $u = 0$ is non-resonant. To do so we need to shift the spectrum of the leading term of $B(u)$ to a single point, for example, $-A$. Without loss of generality, we can assume $A > 0$. Since

$$\tfrac{1}{2}(\mathbf{1} - \mathrm{i}\sigma_1)\sigma_2(\mathbf{1} + \mathrm{i}\sigma_1) = \sigma_3, \qquad \tfrac{1}{2}(\mathbf{1} - \mathrm{i}\sigma_1)\sigma_3(\mathbf{1} + \mathrm{i}\sigma_1) = -\sigma_2, \tag{A.21}$$

we first diagonalize the leading term $A\sigma_2$ by the transformation $\Phi = (\mathbf{1} + \mathrm{i}\sigma_1)\tilde{\Phi}$. Then (A.20) becomes

$$u\tilde{\Phi}_u = [\mathrm{i}zu\sigma_2 + (A + u^2 p(u))\sigma_3]\tilde{\Phi}. \tag{A.22}$$

Then the shearing transformation $\tilde{\Phi} = \mathrm{diag}(u, 1)\Psi$ reduces (A.22) to

$$u\Psi_u = \left[\begin{pmatrix} A-1 & z \\ 0 & -A \end{pmatrix} + u^2 \begin{pmatrix} p(u) & 0 \\ -z & -p(u) \end{pmatrix}\right]\Psi. \tag{A.23}$$

After diagonalizing the leading term, we obtain

$$u\tilde{\Psi}_u = \left[\begin{pmatrix} A-1 & 0 \\ 0 & -A \end{pmatrix} + u^2 \begin{pmatrix} \tilde{p}(u) & r(u) \\ -z & -\tilde{p}(u) \end{pmatrix}\right]\tilde{\Psi}, \tag{A.24}$$

where $\tilde{p}(u), r(u)$ are even and analytic at $u = 0$ functions. Thus, the coefficient of (A.24) is an analytic and even matrix function.

If $A = 1$, one more shearing transformation would produce leading order term $-\mathbf{1}$, which is non-resonant (no non trivial Jordan block) and, thus, the result would follow. If $A > 1$, we apply shearing transformations with the matrix $\mathrm{diag}(u^2, 1)$ with consequent diagonalizations that will shift the $(1,1)$ entry of the leading term by -2 and preserve the analyticity and evenness of the coefficient. When the difference of the eigenvalues of the leading term becomes one, we repeat the last step of the case $A = 1$. □

Corollary A.3. *For the Dirac operator (1.2), $q \equiv A\,\mathrm{cn}(x; m)$ with $m \in (0, 1)$ and $A > 0$ is finite-band if and only if $A = \sqrt{m}n$ with $n \in \mathbb{Z}$, while $q \equiv A\,\mathrm{sn}(x; m)$ is not finite-band for any $A > 0$.*

Proof. The function $\mathrm{cn}(x; m)$ has the same locations of simple poles as $\mathrm{dn}(x; m)$. Given that the residues of the poles $2jK + \mathrm{i}K'$ for $j \in \mathbb{Z}$ of $\mathrm{cn}(x; m)$ are $(-1)^{j-1}\mathrm{i}/\sqrt{m}$, it is clear that the choice of A given above leads to integer Frobenius exponents. To prove the non-resonance conditions, we notice that in Theorem A.2 we used only the fact that $\mathrm{dn}(x; m)$ has an odd Laurent expansion at the pole. Since this is also true for ([43], 8.151) the proof is complete. Similar arguments show that $A\,\mathrm{sn}(x; m)$ is never finite-band. □

A.8. Connection between Heun's equation and Treibich-Verdier potentials

It was shown in [88] that the Heun ODE in standard form:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\zeta^2} + \left(\frac{\gamma}{\zeta} + \frac{\delta}{\zeta - 1} + \frac{\epsilon}{\zeta - a}\right)\frac{\mathrm{d}y}{\mathrm{d}\zeta} + \frac{\alpha\beta\zeta - \xi}{\zeta(\zeta-1)(\zeta-a)}y = 0 \tag{A.25}$$

is associated with the so-called Treibich-Verdier potentials (defined below) for Hill's equation [92], where $\alpha, \beta, \gamma, \delta, \epsilon, \xi, a$ (with each of them $\neq 0, 1$) are complex parameters linked by the relation $\gamma+\delta+\epsilon=\alpha+\beta+1$. Specifically, the Heun equation (A.25) can be transformed into

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i)-E\right) f(x)=0 \tag{A.26}$$

via the transformation $f(x)=y\zeta^{-l_1/2}(\zeta-1)^{-l_2/2}(\zeta-a)^{-l_3/2}$, where $\wp(x)$ is the Weierstrass $\wp$-function with periods $\{2\omega_1,2\omega_3\}$, where $\omega_1/\omega_3\notin\mathbb{R}$ and

$$\omega_0=0\,,\quad \omega_2=-\omega_1-\omega_3\,,\quad e_i=\wp(\omega_i)\,,\quad z=\frac{\wp(x)-e_1}{e_2-e_1}\,,\quad a=\frac{e_3-e_1}{e_2-e_1}\,,$$

$E=(e_2-e_1)[-4\xi+(-(\alpha-\beta)^2+2\gamma^2+6\gamma\epsilon+2\epsilon^2-4\gamma-4\epsilon-\delta^2+2\delta+1)/3+(-(\alpha-\beta)^2+2\gamma^2+6\gamma\delta+2\delta^2-4\gamma-4\delta-\epsilon^2+2\epsilon+1)a/3]$, and the coefficients l_i in (A.26) are connected with the parameters in (A.25) as follows:

$$l_0=\beta-\alpha-\tfrac{1}{2}\,,\quad l_1=-\gamma+\tfrac{1}{2}\,,\quad l_2=-\delta+\tfrac{1}{2}\,,\quad l_3=-\epsilon+\tfrac{1}{2}\,. \tag{A.27}$$

It is known that the potential $\sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i)$ is a (finite-band) Treibich-Verdier potential if and only if $l_i\in\mathbb{Z}$, $i=0,1,2,3$ [92]. Note that the periods $\{2\omega_1,2\omega_3\}$ of $\wp(x)$ are not uniquely determined.

In this subsection we show the special case of the Heun equation (5.2) considered in this work corresponds to a Treibich-Verdier potential if and only if $A\in\mathbb{Z}$. To show this, we employ the conformal mapping $\tilde\zeta:=\zeta/\zeta_1$ Under this transformation, and recalling the relation $\zeta_2=1/\zeta_1$, the Heun equation (5.2) is mapped into

$$\frac{\mathrm{d}^2y}{\mathrm{d}\tilde\zeta^2}+\frac{\frac{3}{2}\tilde\zeta^2-\big(\frac{2m-4}{m\zeta_1}\big)\tilde\zeta+\frac{1}{2\zeta_1^2}}{\tilde\zeta(\tilde\zeta-1)(\tilde\zeta-1/\zeta_1^2)}\frac{\mathrm{d}y}{\mathrm{d}\tilde\zeta}-\frac{\frac{1}{4}A(A+1)\tilde\zeta^2+\big(\frac{\lambda+A^2(1-m/2)}{m\zeta_1}\big)\tilde\zeta+\frac{1}{4\zeta_1^2}A(A-1)}{\tilde\zeta^2(\tilde\zeta-1)(\tilde\zeta-1/\zeta_1^2)}y=0\,. \tag{A.28}$$

The four regular singularities $\{0,\zeta_1,\zeta_2,\infty\}$ of (5.2) are mapped into $\{0,1,1/\zeta_1^2,\infty\}$. Moreover, applying the change of dependent variable $y(\zeta)=\zeta^\rho\tilde y(\zeta)$ to (A.25)) yields

$$\tilde y_{\zeta\zeta}+P(\zeta)\tilde y_\zeta+Q(\zeta)\tilde y=0\,, \tag{A.29}$$

where

$$P(\zeta)=\frac{\gamma+2\rho}{\zeta}+\frac{\delta}{\zeta-1}+\frac{\epsilon}{\zeta-a}\,,$$
$$Q(\zeta)=\frac{\rho(\rho-1+\gamma)}{\zeta^2}+\frac{\delta\rho}{\zeta(\zeta-1)}+\frac{\epsilon\rho}{\zeta(\zeta-a)}+\frac{\alpha\beta\zeta-\xi}{\zeta(\zeta-1)(\zeta-a)}\,.$$

Note that (A.28)) and (A.29) are of the same form with $a=1/\zeta_1^2$ and $\zeta_1=[m-2+2\sqrt{1-m}]/m$. Reducing to a common denominator for $P(\zeta)$ and comparing the

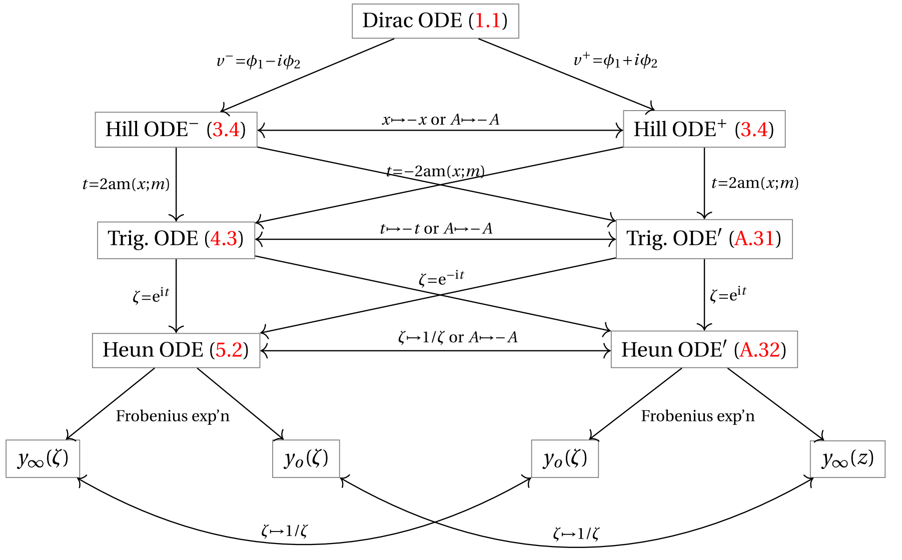

Fig. A.5. Relations between the various ODEs and solutions discussed throughout this work.

corresponding coefficients with (A.28) leads to $\gamma + 2\rho = 1/2$, $\delta = 1/2$ and $\epsilon = 1/2$, which implies that $l_2 = l_3 = 0$. Repeating the same procedure for $Q(\zeta)$, we find that: (i) $\rho = (A-1)/2$ or $\rho = -A/2$, and; (ii) $-A(A+1)/4 = -\rho(\rho - 1/2) + \alpha\beta$.

Now we discuss the two possible cases for ρ: If $\rho = (A-1)/2$, then $\gamma = 3/2 - A$ and $\alpha\beta = 1/2 - A$. Combining $\alpha + \beta = \gamma + \delta + \epsilon - 1 = \gamma$, one obtains $l_1 = A - 1$ and $l_0 = -1 - A$ or A. Alternatively, if $\rho = -A/2$, then $\gamma = 1/2 + A$, $\alpha\beta = 0$, and $\alpha + \beta = 1/2 + A$. It turns out that $l_1 = -A$ and $l_0 = -1 - A$ or A. Either way, we therefore have that $l_0, l_1 \in \mathbb{Z}$ if and only if $A \in \mathbb{Z}$.

A.9. Transformations $A \mapsto -A$ and $\zeta \mapsto 1/\zeta$

The maps $A \mapsto -A$ and $\zeta \mapsto 1/\zeta$ allow one to establish a connection between several related objects. Specifically, using the change of independent variable (4.1), Hill's equation $H^+v^+ = \lambda v^+$ is mapped into the following second-order ODE with trigonometric coefficients

$$4(1 - m\sin^2\tfrac{t}{2})y_{tt} - (m\sin t)y_t + (\lambda + A^2(1 - m\sin^2\tfrac{t}{2}) - \tfrac{\mathrm{i}}{2}Am\sin t)y = 0\,. \qquad \text{(A.30)}$$

Next, applying the transformation $\zeta = \mathrm{e}^{\mathrm{i}t}$ to (A.30) yields another Heun ODE, namely,

$$\zeta^2 F(\zeta; m)y_{\zeta\zeta} + \zeta G(\zeta; m)y_\zeta + \tilde{H}(\zeta; \lambda, A, m)y = 0\,, \qquad \text{(A.31)}$$

where $F(\zeta; m)$ and $G(\zeta; m)$ are still given by (5.3a) and (5.3b), respectively, and with $\tilde{H} := H(\zeta; \lambda, -A, m)$. Note that the four regular singular points of (5.2)) and (A.31) are

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

the same. The full chain of transformations and correspondences is summarized in the commutative diagram in Fig. A.5.

Uncited references

[23] [44] [61] [89] [90]

References

[1] M.J. Ablowitz, H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia, 1981.
[2] M.J. Ablowitz, B. Prinari, A.D. Trubatch, Discrete and Continuous Nonlinear Schrödinger Systems, Cambridge University Press, Cambridge, 2004.
[3] G.P. Agrawal, Nonlinear Fiber Optics, Academic Press, New York, 2007.
[4] F.M. Arscott, Periodic Differential Equations and Introduction to Mathieu, Lamé and Allied Functions, Pergamon Press, Oxford, 1964.
[5] F. Bagarello, J.-P. Gazeau, F.H. Szafraniec, M. Znojil (Eds.), Non-Selfadjoint Operators in Quantum Physics: Mathematical Aspects, Wiley, 2015.
[6] R. Beals, R. Coifman, Scattering and inverse scattering for first order systems, Commun. Pure Appl. Math. 37 (1984) 39–90.
[7] R. Beals, P. Deift, C. Tomei, Direct and Inverse Scattering on the Line, American Mathematical Society, 1988.
[8] E.D. Belekolos, A.I. Bobenko, V.Z. Enol'skii, A.R. Its, V.B. Matveev, Algebro-Geometric Approach to Nonlinear Integrable Equations, Springer-Verlag, Berlin, 1994.
[9] C.M. Bender, Introduction to PT-symmetric quantum theory, Contemp. Phys. 46 (2005) 277–292.
[10] M. Bertola, A. Tovbis, Universality for the focusing nonlinear Schrödinger equation at the gradient catastrophe point: rational breathers and poles of the tritronquée solution to Painlevé I, Commun. Pure Appl. Math. 66 (2013) 678–752.
[11] G. Biondini, J. Oregero, Semiclassical dynamics in self-focusing nonlinear media with periodic initial conditions, Stud. Appl. Math. 145 (2020) 325–356.
[12] G. Biondini, J. Oregero, A. Tovbis, On the spectrum of the periodic focusing Zakharov-Shabat operator, J. Spectr. Theory 12 (2022) 939–992.
[13] J.C. Bronski, J.N. Kutz, Numerical simulation of the semiclassical limit of the focusing nonlinear Schrodinger equation, Phys. Lett. A 254 (1999) 335–336.
[14] P.F. Byrd, M.D. Friedman, Handbook of Elliptic Integrals for Engineers and Scientists, Springer-Verlag, Berlin, 1971.
[15] B.M. Brown, M.S.P. Eastham, K.M. Schmidt, Periodic Differential Operators, Springer, New York, 2013.
[16] J.D. Carter, H. Segur, Instabilities in the two-dimensional cubic nonlinear Schrödinger equation, Phys. Rev. E 68 (2003) 045601.
[17] S.R. Clarke, P.D. Miller, Semi-classical limit for the focusing nonlinear Schrödinger equation: sensitivity to analytic properties of the initial data, Proc. R. Soc. Lond. A 458 (2002) 135–156.
[18] E.A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, Krieger, 1995.
[19] C. Conti, A. Fratalocchi, M. Peccianti, G. Ruocco, S. Trillo, Observation of a gradient catastrophe generating solitons, Phys. Rev. Lett. 102 (2009) 083902.
[20] E. Date, S. Tanaka, Periodic multi-soliton solutions of Korteweg-de Vries equation and Toda lattice, Prog. Theor. Phys. Suppl. 59 (1976) 107–125.
[21] B. Deconinck, J. Kutz, Computing spectra of linear operators using the Floquet-Fourier-Hill method, J. Comput. Phys. 219 (2006) 296–321.
[22] B. Deconinck, B.L. Segal, The stability spectrum for elliptic solutions to the focusing NLS equation, Physica D 346 (2017) 1–19.
[23] P.A. Deift, Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach, AMS, Providence, 1998.
[24] P. Deift, E. Trubowitz, Inverse scattering on the line, Commun. Pure Appl. Math. 32 (1979) 121–252.
[25] P. Djakov, B. Mityagin, Instability zones of periodic 1-dimensional Schrödinger and Dirac operators, Russ. Math. Surv. 61 (2006) 663–766.

[26] P. Djakov, B. Mityagin, Multiplicities of the eigenvalues of periodic Dirac operators, J. Differ. Equ. 210 (2005) 178–216.
[27] B. Dubrovin, Periodic problems for the Korteweg–de Vries equation in the class of finite band potentials, Funct. Anal. Appl. 9 (3) (1975) 215–223.
[28] M.S.P. Eastham, The Spectral Theory of Periodic Differential Equations, Scottish Academic Press, London, 1973.
[29] G.A. El, E.G. Khamis, A. Tovbis, Dam break problem for the focusing nonlinear Schrödinger equation and the generation of rogue waves, Nonlinearity 29 (2016) 2798.
[30] R. El-Ganainy, K.G. Makris, M. Khajavikhan, Z.H. Musslimani, S. Rotter, D.N. Christodoulides, Non-Hermitian physics and PT symmetry, Nat. Phys. 14 (2018) 11–19.
[31] A. Erdélyi (Ed.), Bateman Manuscript Project. Vol. III - Higher Transcendental Functions, McGraw-Hill, New York, 1955.
[32] L.D. Faddeev, L.A. Takhtajan, Hamiltonian Methods in the Theory of Solitons, Springer, Berlin, 1987.
[33] G. Floquet, Sur les équations différentielles linéaires à coefficients périodiques, Ann. Sci. Éc. Norm. Supér. 12 (1883) 47–88.
[34] M.G. Forest, J.E. Lee, Geometry and modulation theory for the periodic nonlinear Schrödinger equation, in: IMA Volumes in Mathematics and Its Applications, Springer, New York, 1986, pp. 35–69.
[35] A. Fratalocchi, C. Conti, G. Ruocco, S. Trillo, Free-energy transition in a gas of noninteracting nonlinear wave particles, Phys. Rev. Lett. 101 (2008) 044101.
[36] C.S. Gardner, J.M. Greene, M.D. Kruskal, R.M. Miura, Method for solving the Korteweg-deVries equation, Phys. Rev. Lett. 19 (1967) 1095.
[37] F. Gesztesy, H. Holden, Soliton Equations and Their Algebro–Geometric Solutions, vol. I, Cambridge University Press, Cambridge, 2003.
[38] F. Gesztesy, P. Deift, C. Galvez, P. Perry, W. Schlag (Eds.), Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon's 60th Birthday, AMS, 2007.
[39] F. Gesztesy, R. Weikard, Picard potentials and Hill's equation on a torus, Acta Math. 176 (1996) 73–107.
[40] F. Gesztesy, R. Weikard, A characterization of all elliptic algebro-geometric solutions of the AKNS hierarchy, Acta Math. 181 (1998) 63–108.
[41] F. Gesztesy, R. Weikard, Elliptic algebro-geometric solutions of the KdV and AKNS hierarchy: an analytic approach, Bull. Am. Math. Soc. 35 (1998) 271–317.
[42] G.H. Golub, G. Meurant, Matrices, Moments, and Quadrature with Applications, Princeton University Press, 2010.
[43] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products Seventh Ed., Academic Press, Burlington, 2007.
[44] B. Grébert, T. Kappeler, Estimates on periodic and Dirichlet eigenvalues for the Zakharov-Shabat system, Asymptot. Anal. 25 (2001) 201–237.
[45] A. Haese-Hill, A. Hallnäs, P. Veselov, On the spectra of real and complex Lamé operators, SIGMA 13 (2017) 049.
[46] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, New York, 1985.
[47] E.L. Ince, A linear differential equation with periodic coefficients, Proc. Lond. Math. Soc. 2 (1925) 56–74.
[48] E.L. Ince, V.–The periodic Lamé functions, Proc. R. Soc. Edinb. 60 (1940) 47–63.
[49] E.L. Ince, VII.–Further investigations into the periodic Lamé functions, Proc. R. Soc. Edinb. 60 (1940) 83–99.
[50] E.L. Ince, Ordinary Differential Equations, Dover, New York, 1956.
[51] A.R. Its, V.P. Kotlyarov, Explicit formulas for the solutions of a nonlinear Schroedinger equation, Dokl. Akad. Nauk Ukr. 10 (1976) 965–968, English translation: arXiv:1401.4445 [nlin.si].
[52] A.R. Its, V.B. Matveev, Hill's operator with finitely many gaps, Funct. Anal. Appl. 9 (1975) 69–70.
[53] A.R. Its, V.B. Matveev, Schrödinger operators with finite-gap spectrum and soliton solutions of the Korteweg-de Vries equation, Theor. Math. Phys. 23 (1975) 51–67.
[54] C. Itzykson, J.-B. Zuber, Quantum Field Theory, McGraw-Hill, 1980.
[55] R. Jenkins, K.D.T.-R. McLaughlin, Semiclassical limit of focusing NLS for a family of square barrier initial data, Commun. Pure Appl. Math. 67 (2013) 246–320.
[56] A.M. Kamchatnov, Nonlinear Periodic Waves and Their Modulations, World Scientific, 2000.
[57] S. Kamvissis, K.D.T-R. McLaughlin, P.D. Miller, Semiclassical Soliton Ensembles for the Focusing Nonlinear Schrödinger Equation, Princeton University Press, Princeton, 2003.

[58] T. Kato, Perturbation Theory for Linear Operators, Springer, Berlin, 1995.
[59] M. Klaus, J.K. Shaw, On the eigenvalues of Zakharov–Shabat systems, SIAM J. Math. Anal. 34 (4) (2003) 759–773.
[60] M. Klaus, J.K. Shaw, Purely imaginary eigenvalues of Zakharov-Shabat systems, Phys. Rev. E 65 (2002) 036607.
[61] V.P. Kotlyarov, Periodic problem for the Schrödinger nonlinear equation, in: Voprosy Matematicheskoi Fiziki i Funkcionalnogo Analiza, vol. 1, Naukova Dumka, Kiev, 1976, pp. 121–131, English translation: arXiv:1401.4445 [nlin.si].
[62] I. Krichever, Potentials with zero reflection coefficient on the background of the finite-gap potentials, Funct. Anal. Appl. 9 (1975) 77–78.
[63] I. Krichever, An algebraic-geometric construction of the Zakharov-Shabat equations and their periodic solutions, Dokl. Akad. Nauk SSSR 227 (1976) 291–294.
[64] I. Krichever, Elliptic solutions of the Kadomtsev-Petviashili equation and many-body problems, Funct. Anal. Appl. 14 (1980) 45.
[65] G. Lamé, Sur les surfaces isothermes dan les corps homogénes en équilibre de température, J. Math. Pures Appl. 2 (1837) 147–188.
[66] P.D. Lax, Integrals of nonlinear equations of evolution and solitary waves, Commun. Pure Appl. Math. 21 (1968) 467–490.
[67] P.D. Lax, Periodic solutions of the KdV equation, Commun. Pure Appl. Math. 28 (1975) 141–188.
[68] S. Li, G. Biondini, Soliton interactions and degenerate soliton complexes in focusing media with non-zero background, Eur. Phys. J. Plus 133 (2018) 400.
[69] S. Li, G. Biondini, C. Schiebold, On the degenerate soliton solutions of the focusing nonlinear Schrödinger equation, J. Math. Phys. 58 (2017) 033507.
[70] L. Lee, G. Lyng, I. Vankova, The Gaussian semiclassical soliton ensemble and numerical methods for the focusing nonlinear Schrödinger equation, Physica D 241 (2012) 1767–1781.
[71] Y.-C. Ma, M.J. Ablowitz, The periodic cubic NLS equation, Stud. Appl. Math. 63 (1981) 113–158.
[72] W. Magnus, S. Winkler, Hill's Equation, Dover, New York, 1966.
[73] H.P. McKean, E. Trubowitz, Hill's operator and hyperelliptic function theory in the presence of infinitely many branch points, Commun. Pure Appl. Math. 29 (1976) 143–226.
[74] D.W. McLaughlin, E.A. Overman II, Whiskered tori for integrable PDE's: chaotic behavior in near integrable PDEs, in: J.P. Keller, D.W. McLaughlin, G.P. Papanicolaou (Eds.), Surv. Appl. Math., vol. I, Plenum, 1995.
[75] K.D.T.-R. McLaughlin, P.V. Nabelek, A Riemann-Hilbert problem approach to infinite gap Hill's operators and the Korteweg–de Vries equation, Int. Math. Res. Not. 2021 (2019) 1288–1352.
[76] A. Messiah, Quantum Mechanics, North-Holland, 1961.
[77] S.P. Novikov, The periodic problem for the Korteweg–de Vries equation, Funct. Anal. Appl. 8 (1974) 236–246.
[78] S. Novikov, S.V. Manakov, L.P. Pitaevskii, V.E. Zakharov, Theory of Solitons: The Inverse Scattering Method, Monographs in Contemporary Mathematics, Plenum, New York, 1984.
[79] F.W. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark, NIST Handbook of Mathematical Functions, Cambridge University Press, New York, 2010.
[80] O. Perron, Die Lehre von den Kettenbrüchen, Teubner Verlag, Leipzig, 1913.
[81] L.P. Pitaevskii, S. Stringari, Bose-Einstein Condensation, Clarendon Press, Oxford, 2003.
[82] M. Reed, B. Simon, Method of Mathematical Physics I: Functional Analysis, revised and enlarged ed., Academic Press, San Diego, 1980.
[83] F.S. Rofe-Beketov, Spectrum perturbations, the Kneser-type constants and the effective masses of zones-type potentials, in: Constructive Theory of Functions, Sofia, 1984, 1984, pp. 757–766.
[84] F.S. Rofe-Beketov, A.M. Khol'kin, Spectral analysis of differential operators: interplay between spectral and oscillatory properties, in: World Scientific Monograph Series in Mathematics, New Jersey, 2005.
[85] A. Ronveaux (Ed.), Heun's Differential Equation, Oxford University Press, New York, 1995.
[86] J. Satsuma, N. Yajima, Initial value problems of one-dimensional self-modulation of nonlinear waves in dispersive media, Prog. Theor. Phys. Suppl. 55 (1974) 284–306.
[87] O.A. Smirnov, The Dirac operator with elliptic potential, Sb. Math. 186 (1995) 1213–1221.
[88] K. Takemura, Finite-gap potential, Heun's differential equation and WKB analysis, RIMS Kôkyûroku Bessatsu 5 (2008) 61–74.
[89] G. Teschl, Jacobi Operators and Completely Integrable Nonlinear Lattices, AMS, Providence, 2000.
[90] V. Tkachenko, Non-selfadjoint periodic Dirac operators with finite band spectra, in: Integr. Eq. Oper. Theory, Birkhäuser, Basel, 2000, pp. 325–348.

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

[91] E.R. Tracy, H.H. Chen, Nonlinear self-modulation: an exactly solvable model, Phys. Rev. A 37 (1998) 815–839.
[92] A. Treibich, J.-L. Verdier, Revetements exceptionnels et sommes de 4 nombres triangulaires, Duke Math. J. 68 (1992) 217–236.
[93] T. Trogdon, B. Deconinck, A Riemann-Hilbert problem for the finite-genus solutions of the KdV equation and its numerical solution, Physica D 251 (2013) 1–18.
[94] G.A. El, A. Tovbis, Spectral theory of soliton and breather gases for the focusing nonlinear Schrödinger equation, Phys. Rev. E 101 (2020) 052207.
[95] A. Tovbis, S. Venakides, The eigenvalue problem for the focusing nonlinear Schrödinger equation: new solvable cases, Physica D 146 (2000) 150–164.
[96] A. Tovbis, S. Venakides, X. Zhou, On semiclassical (zero dispersion limit) solutions of the focusing nonlinear Schrödinger equation, Commun. Pure Appl. Math. 57 (2004) 877–985.
[97] A. Tovbis, F. Wang, Recent developments in spectral theory of the focusing NLS soliton and breather gases: the thermodynamic limit of average densities, fluxes and certain meromorphic differentials; periodic gases, arXiv:2203.03566, 2022.
[98] E. Trubowitz, The inverse problem for periodic potentials, Commun. Pure Appl. Math. 30 (1977) 321–337.
[99] K. Veselić, On real eigenvalues of real tridiagonal matrices, Linear Algebra Appl. 27 (1979) 167–171.
[100] H. Volkmer, Coexistence of periodic solutions of Ince's equation, Analysis 23 (2003) 97–105.
[101] H. Volkmer, Approximation of eigenvalues of some differential equations by zeros of orthogonal polynomials, J. Comput. Appl. Math. 213 (2008) 488–500.
[102] H. Volkmer, Eigenvalue problems for Lamé's differential equation, SIGMA 14 (2018) 131.
[103] P. Walker, The analyticity of Jacobian functions with respect to the parameter k, Proc. R. Soc. A 459 (2003) 2569–2574.
[104] S. Weinberg, The Quantum Theory of Fields, Cambridge University Press, Cambridge, 1995.
[105] V.E. Zakharov, A.B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, Sov. Phys. JETP 34 (1972) 62–69.
[106] V.E. Zakharov, A.B. Shabat, Interaction between solitons in a stable medium, Sov. Phys. JETP 37 (1973) 1627–1639.

Please cite this article in press as: G. Biondini et al., Elliptic finite-band potentials of a non-self-adjoint Dirac operator, Adv. Math. (2023), https://doi.org/10.1016/j.aim.2023.109188

Sponsor names

Do not correct this page. Please mark corrections to sponsor names and grant numbers in the main text.

National Science Foundation, *country*=United States, *grants*=DMS-2009487, DMS-2009647

National Natural Science Foundation of China, *country*=China, *grants*=12101590

National Science Foundation, *country*=United States, *grants*=1440140