

L^2 -CRITICAL NONUNIQUENESS FOR THE 2D NAVIER-STOKES EQUATIONS

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ABSTRACT. In this paper, we consider the 2D incompressible Navier-Stokes equations on the torus. It is well known that for any L^2 divergence-free initial data, there exists a global smooth solution that is unique in the class of $C_t L^2$ weak solutions. We show that such uniqueness would fail in the class $C_t L^p$ if $p < 2$. The non-unique solutions we constructed are almost L^2 -critical in the sense that (i) they are uniformly continuous in L^p for every $p < 2$; (ii) the kinetic energy agrees with any given smooth positive profile except on a set of arbitrarily small measure in time.

1. INTRODUCTION

1.1. Statement of the problem. The 2D Navier-Stokes equations are a fundamental mathematical model of fluid flow written as

$$\begin{cases} \partial_t u - \Delta u + \operatorname{div}(u \otimes u) + \nabla p = 0 \\ \operatorname{div} u = 0 \\ u(0) = u_0 \end{cases} \quad (1.1)$$

posed on a spatial domain $\Omega \subset \mathbb{R}^2$ with a suitable boundary condition, where $u : \Omega \times [0, T] \rightarrow \mathbb{R}^2$ is the unknown velocity with initial data u_0 and $p : \Omega \times [0, T] \rightarrow \mathbb{R}$ is a scalar pressure. We consider the Cauchy problem of (1.1) on a time interval $[0, T]$ for some initial data u_0 and $T > 0$. The existence and uniqueness of smooth solutions have been proved by Leray [Ler33] for $\Omega = \mathbb{R}^2$, and Ladyzhenskaya [Lad58] for bounded domains.

In the paper, we focus on the periodic case $\Omega = \mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ and the solutions with zero spatial mean

$$\int_{\mathbb{T}^2} u(x, t) dx = 0,$$

which is also conserved under the evolution of the equations (1.1).

For any divergence-free initial data $u_0 \in L^2(\mathbb{T}^2)$, a standard Galerkin method leads to the existence of a weak solution $u \in L^\infty([0, T]; L^2(\mathbb{T}^2)) \cap L^2([0, T]; H^1(\mathbb{T}^2))$, and the validity of the energy equality

$$\frac{1}{2} \|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 ds = \frac{1}{2} \|u(0)\|_2^2 \quad \text{for all } t \geq 0, \quad (1.2)$$

follows from the Onsager criticality of the energy-ensrophy space in 2D. A weak-strong uniqueness argument and Ladyzhenskaya's inequality imply the regularity and uniqueness of such Leray-Hopf solutions.

In fact, the uniqueness of the 2D Navier-Stokes equation can be stated in a much stronger way. More precisely, Leray-Hopf solutions are unique in a much larger class than the Leray-Hopf class

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itself. As discussed in [GP02], the result [FLRT00] of Furioli, Lemarié-Rieusset, and Terraneo implies that in 2D any weak solution in $C_t L^2$ is unique without any additional regularity assumptions:

Theorem 1.1 ([FLRT00]). *For any divergence-free $u_0 \in L^2(\mathbb{T}^2)$, there is only one weak solution in the class $C_t L^2$ with initial data u_0 .*

Concerning the sharpness of this existence/uniqueness result, we ask two questions:

- (a) Does the existence still hold if initial data $u_0 \in L^p(\mathbb{T}^2)$ for $p < 2$?
- (b) Are weak solutions in the class $C_t L^p$ unique if $p < 2$?

The aim of this paper is to provide a positive answer to the first question, and a negative answer to the second one. In fact, the main results in this paper will be stated in wider Sobolev spaces, but we choose the usual Lebesgue spaces L^p here as the most elementary and significant. For instance, the space $L^2(\mathbb{T}^2)$ is invariant under the natural scaling of the Navier-Stokes equations $u(x, t) \rightarrow u_\lambda(\lambda x, \lambda^2 t)$, and the square of $L^2(\mathbb{T}^2)$ norm represents the total kinetic energy, which is nonincreasing in time for smooth solutions due to (1.2).

It is easy to see that when $p < 2$, the regularity $u \in C_t L^p$ itself alone is not enough to make sense of the weak formulation. By a weak solution of (1.1), we mean a vector field $u \in L^2(\mathbb{T}^2 \times [0, T])$ satisfying all of the following conditions:

- (1) For a.e. $t \in [0, T]$, $u(\cdot, t)$ is weakly divergence-free;
- (2) For any $\varphi \in \mathcal{D}_T$,

$$\int_{\mathbb{T}^2} u_0(x) \varphi(\cdot, t) dx + \int_0^t \int_{\mathbb{T}^2} u \cdot (\Delta \varphi + u \otimes u : \nabla \varphi - u \cdot \partial_t \varphi) dx dt = 0, \quad (1.3)$$

where $\mathcal{D}_T \subset C^\infty(\mathbb{T}^2 \times \mathbb{R})$ is the space of all divergence-free test functions vanishing on $t \geq T$.

The above definition seems too weak at first glance. However, by [FJR72, Theorem 2.1], the above definition of weak solutions is equivalent to the integral equation

$$u(t) = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \mathcal{P} \operatorname{div}(u \otimes u)(s) ds, \quad (1.4)$$

where $e^{t\Delta}$ is the heat semigroup and $\mathcal{P}$ is the Leray projection onto the divergence-free vector fields. This equation (1.4) is often referred to as the *mild formulation* of the Navier-Stokes equations and has been used to construct unique local solutions, also known as the *mild solutions*, when the initial data is critical or subcritical. We will discuss this in the summary below.

The following theorem is one of the main results of this paper, which shows both the existence and nonuniqueness of weak solutions in $C_t L^p$ for $p < 2$.

Theorem 1.2. *Fix $1 \leq p < 2$ and let $u_0 \in L^p(\mathbb{T}^2)$ be divergence free and $e : [0, T] \rightarrow \mathbb{R}^+$ be strictly positive and smooth.*

Then for any $\varepsilon > 0$, there exist a weak solution u with initial data $u(0) = u_0$ and regularity

$$u \in C([0, T]; L^p(\mathbb{T}^2)) \quad \text{and} \quad u \in C((0, T]; L^{p'}(\mathbb{T}^2)) \quad \text{for all } p' < 2,$$

and a Borel set $\mathcal{G} \subset [0, T]$ with $\mathcal{L}^1([0, T] \setminus \mathcal{G}) < \varepsilon$ such that

$$\|u(t)\|_{L^2(\mathbb{T}^2)}^2 = e(t), \quad \forall t \in \mathcal{G}.$$

Theorem 1.2 contributes to both the existence and nonuniqueness in the class $C_t L^p$ for $1 \leq p < 2$. Thanks to the Lusin-type property of the kinetic energy of the constructed solutions, Theorem 1.2 is sharp in both of the following two aspects.

- The regularity $C_t L^p$ for any $1 \leq p < 2$ is sharp: if $p = 2$, then any $C_t L^2$ weak solution is unique in the class $C_t L^2$ with the same initial data;
- The size of set $\mathcal{G}$ is optimal, namely, $[0, T] \setminus \mathcal{G}$ can not be of measure zero: if $\mathcal{L}^1([0, T] \setminus \mathcal{G}) = 0$ and $\|e\|_{L^\infty}$ were small, then u would have a small $L_t^\infty L^2$ norm, which implies uniqueness [GP02].

1.2. Summary of existence and uniqueness results. Before we introduce the main theorems, let us review the existing existence and uniqueness results for the 2D Navier-Stokes equations.

Since the classical work of Leray and Ladyzhenskaya, there has been a vast amount of literature on extending the existence and uniqueness results to a wider class of solutions.

Existence. Since the global existence of Leray-Hopf solutions [Ler34], the existence theory for 2D Navier-Stokes equations mainly focused on two aspects: using the mild formulation for initial data in scale-invariant spaces or using the vorticity formulation in the case where the initial vorticity is a measure.

Using the mild formulation (1.4) for the velocity, in the celebrated work [KT01] Koch and Tataru showed the existence of a unique solution for small initial data $u_0 \in \text{BMO}^{-1}$ in dimensions $d \geq 2$. In 2D, Gallagher and Planchon [GP02] constructed global unique solutions in scale invariant Besov spaces $u_0 \in B_{p,q}^{-1+2/p}$ for any finite p, q . This was later extended by Germain [Ger05] to $u_0 \in \text{VMO}^{-1}$, where the space VMO^{-1} is the closure of Schwartz functions in BMO^{-1} . We also mention related works [Cal90, GMS01, Maz03] and refer to [BCD11, LR16] for the background in Fourier analysis in line of research.

At the level of vorticity $\omega = \text{curl } u$, the critical scaling corresponds to $\omega_0 \in L^1$, which suggests the consideration of measure-valued vorticity. The first existence result in this direction dates back to [BEP85, GMO88] where the existence of weak solutions was shown when ω_0 is a Radon measure. We refer to [BA94, Cot86, Kat94] and reference therein for this line of research. Heuristically, such results would correspond to the velocity of scaling $u_0 \in B_{p,q}^{-1+2/p}$ for $p \leq 2$ and $q = \infty$.

Uniqueness. For the velocity formulation, the known uniqueness results, which are usually corollaries of the existence results, can be summarized as follows: if $u \in C_t B_{p,q}^{-1+2/p}$ with finite p, q , then u is unique in the class of weak solution with regularity $C_t B_{p,q}^{-1+2/p}$. It should be noted that even though the existence results in [KT01, Ger05] surpass [GP02], the uniqueness in [KT01, Ger05] can not be stated with only one continuity condition in time.

The uniqueness of the vorticity formulation however does not follow from the construction of the solutions. Indeed, in [BEP85, GMO88, BA94, Cot86, Kat94], the uniqueness results all have either smallness or structural assumptions. Building upon [GGL05], this problem has finally been settled in [GG05]: if $\omega_0 \in \mathcal{M}$, then there is only one solution $\omega(t)$ which is continuous in $L^1 \cap L^\infty$ for $t > 0$.

The existence and uniqueness of weak solutions for the 2D Navier-Stokes equations have more or less matured except at the endpoint $B_{\infty,\infty}^{-1}$, which is the largest critical space with respect to the scaling the Navier-Stokes equations, independent of the space dimensions. The current existence results in this space require extra assumptions (see for instance [CG06, CGP11, PZ14] and reference therein). It is also not clear at the moment whether weak solutions in $C_t B_{\infty,\infty}^{-1}$ are unique, though other types of ill-posedness results are available in dimensions $d \geq 3$ [BP08, Ger08, CS10].

Supported by the development of the existence and uniqueness theories so far, we can summarize the existence and uniqueness in a non-rigorous principle:

Existence and uniqueness may hold in the critical and sub-critical regime.

This heuristic principle has also been proven quite successful in view of the recent development of the negative results thanks to the convex integration technique.

1.3. Convex integration technique. Convex integration is a technique originated in isometric embedding problems of geometry dating back to the work of Nash [Nas54] and Kuiper [Kui55]. Its application to fluid dynamics was brought to attention only in 2009 by the pioneering work of De Lellis and Székelyhidi Jr. [DLS09]. Since its inception in [DLS09], this technique has been proven very fruitful in the fluids community. Remarkably, its development over a series of works [DLS09, DLS13, DLS14, BDLIS15, BDLS16, BLJV18] has culminated in the resolution of the Onsager conjecture for the 3D Euler equations by Isett [Ise18]. Its application to other models is a very active research direction, and we refer readers to [Shv11, IV15, BSV19, MS20, Nov20, BBV20, FLS21, BMNV21] and surveys [DLS19, BV21] reference therein for other interesting work on convex integration.

Very recently, in a groundbreaking work [BV19], Buckmaster and Vicol successfully applied an “intermittent” variant of this technique to construct non-unique weak solutions of the 3D Navier-Stokes equations. Following this breakthrough development, there have been multiple results in viscous settings, such as extensions of [BV19] to the hyperdissipative case [LT20, LQ20], non-unique weak solutions with partial regularity in time [BCV22], nonconservative $H^{1/2-}$ solutions of the Euler equations [BMNV21], existence of nontrivial stationary weak solutions of d -dimensional Navier-Stokes equations for $d \geq 4$ [Luo19], nonuniqueness of Leray-Hopf solutions for hypodissipative Navier-Stokes equations [CDLDR18, DR19], Hall-MHD [Dai21], and some power-law flows [BMS21] in various solution classes where the Leray structure theorem does not hold.

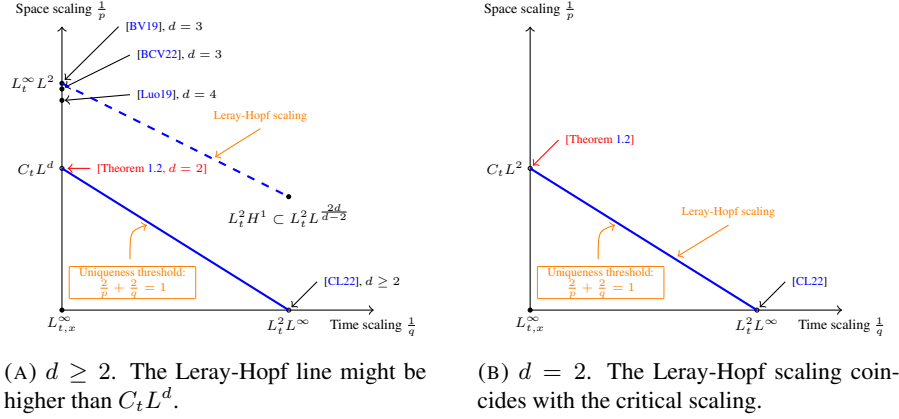
Ultimately, any attempt of using the strategy of [BV19] to extend the nonuniqueness to the 2D Navier-Stokes equations would face substantial difficulty. In a nutshell, this is because the spatially intermittent framework of [BV19] requires the underlined PDE to be at least $C_t L^2$ -supercritical, which, in 2D, is a contradiction to Theorem 1.1. In [CL21, CL22], the authors of the current paper developed a space-time intermittent variant of the framework of [BV19] which allows for more flexibility in the scaling of the nonuniqueness range. Taking advantage of the added temporal intermittency, in [CL22] we constructed nonunique weak solutions to the Navier-Stokes equations in dimensions $d \geq 2$ in $L_t^p L^\infty$ for any $p < 2$, proving the first nonuniqueness result for the 2D Navier-Stokes equations and the sharpness of the classical $L_t^2 L^\infty$ Ladyzhenskaya-Prodi-Serrin criterion.

While the results in [CL22] demonstrate the sharpness of the Ladyzhenskaya-Prodi-Serrin criteria $L_t^p L^q$, $\frac{2}{p} + \frac{d}{q} = 1$, at the $q = \infty$ endpoint, it remains open whether the rest of the cases in these uniqueness criteria are sharp, as shown in Figure 1. In the current paper, we are able to reach the other endpoint $q = d$ in the case of two dimensions $d = 2$.

1.4. Main results. In this subsection, we list additional results, complementary to Theorem 1.2, that follow directly from the construction.

1.4.1. Density of energy near $t = 0$. First, in addition to the Lusin-type property for the kinetic energy, we have the density property (1.5) at $t = 0$, which we state in the case of L^2 initial data for simplicity.

Theorem 1.3. *Let $u_0 \in L^2(\mathbb{T}^2)$ be divergence free and $e : [0, T] \rightarrow \mathbb{R}^+$ be strictly positive, smooth and such that $e(0) = \|u_0\|_2$.*

FIGURE 1. Uniqueness/nonuniqueness in $L_t^p L^q$ scale

There exist a weak solution with initial data $u(0) = u_0$ and regularity

$$u \in C([0, T]; L^{p'}(\mathbb{T}^2)) \quad \text{for all } p' < 2,$$

and a Borel set $\mathcal{G} \subset [0, T]$ such that

$$\|u(t)\|_{L^2}^2 = e(t), \quad \forall t \in \mathcal{G},$$

and the density of $\mathcal{G}$ at $t = 0$ is 1:

$$\lim_{t \rightarrow 0+} t^{-1} \mathcal{L}^1(\mathcal{G} \cap [0, t]) = 1. \quad (1.5)$$

1.4.2. Extension to rough initial data. Our method of constructing weak solutions allows for initial data u_0 to be much rougher than $L^1(\mathbb{T}^2)$. Such a supercritical regularity persists over the whole time interval and is supplemented with an additional smoothing effect, a gain of almost two derivatives.

Theorem 1.4. *For any divergence free $u_0 \in W^{s,1}(\mathbb{T}^2)$ for some $s > -1$ there are infinitely many solutions in the class*

$$u \in C((0, T]; W^{s',1}(\mathbb{T}^2)) \quad \text{for all } s' < 1.$$

Remark 1.5. *A few remarks are in order.*

- (1) *The density estimate (1.5) is a much stronger result than the epsilon result in Theorem 1.2: the kinetic energy of the solution will look more and more like $e(t)$ as $t \rightarrow 0+$.*
- (2) *For every constructed solution, there is an increasing sequence of sets $\mathcal{T}_j \subset [0, 1]$ with $\mathcal{L}^1((\cup_j \mathcal{T}_j)^c) = 0$, such that u continuously maps $\mathcal{T}_j \rightarrow L^2(\mathbb{T}^2)$ for every $j \in \mathbb{N}$. Here the sets $\mathcal{T}_j$ are equipped with the induced Euclidean metric.*
- (3) *Note that $u \in C((0, T]; W^{s,1}(\mathbb{T}^2))$ for all $s < 1$. In addition, since the building blocks are fully intermittent in space, a similar argument implies that the vorticity $\omega = \text{curl } u \in C((0, T]; L^p)$ for all $p < 1$, which means that it is “almost” in L^1 , consistent with the uniqueness at the level of the vorticity formulation.*
- (4) *All solutions constructed in this paper are nowhere regular in time, namely, there does not exist an interval $I \subset [0, T]$ such that $u|_I \in C^\infty(I \times \mathbb{T}^2)$. However, if one drops the control*

on the kinetic energy, one can obtain solutions with a small singular set in time (in the sense of Hausdorff dimension) using the gluing procedure in [CL22, Section 3].

1.4.3. *Smoothness away from the bad set.* Finally, if we drop the control on the energy profile, we can arrange the solution to be close to any given smooth vector field outside of a bad set of an arbitrarily small measure.

Theorem 1.6. *Let u_0 be divergence free, $u_0 \in W^{s,1}(\mathbb{T}^2)$ for some $s > -1$, and $v \in C^\infty(\mathbb{T}^2 \times [0, T])$ be a smooth divergence free vector fields.*

Then for any $\varepsilon > 0$, there exist a weak solution with initial data $u(0) = u_0$, regularity

$$u \in C((0, T]; W^{s',1}(\mathbb{T}^2)) \quad \text{for all } s' < 1,$$

and a Borel set $E \subset [0, T]$ with $\mathcal{L}^1([0, T] \setminus E) < \varepsilon$ such that

$$\|u(t) - v(t)\|_{L^\infty(\mathbb{T}^2)} \leq \varepsilon \quad \text{for all } t \in E. \quad (\dagger)$$

Remark 1.7. *We list a few remarks concerning Theorem 1.6.*

- (1) *The last property $(\dagger)$ is a key feature of the temporal concentration mechanism: the solution is “large” only on a “small” set in time, cf. [CL21, CL22].*
- (2) *In fact, the measuring norm in $(\dagger)$ does not have to be $L^\infty(\mathbb{T}^2)$ —it can be any Sobolev norm.*

1.4.4. *2D Euler equations and the Onsager conjecture.* In general, if a weak solution u of the Euler equations belongs to the dimension independent Onsager’s class $L^3([0, T]; B_{3,c_0}^{1/3})$, then it conserves the kinetic energy [CCFS08]. The $1/3$ -Hölder regularity is known as the Onsager threshold and has been proven to be sharp in dimensions $d \geq 3$ [Ise18] but not yet in 2D.

However, in two dimensions there are additional regularization mechanisms that prevent anomalous dissipation, cf. [CFLS16, CCS20, LMPP21]. For instance, as shown in [CFLS16], weak solutions of the 2D Euler equations obtained as the vanishing viscosity limit of Leray-Hopf solutions of the 2D Navier-Stokes equations conserve energy provided $\omega_0 \in L^p$ for some $p > 1$. We state a version of this result as Theorem A.1, and demonstrate its sharpness in the class of weak solutions of the Navier-Stokes equation in the following theorem, which is a consequence of the main iteration scheme.

Theorem 1.8. *There exists a weak solution of the 2D Euler equations $u \in C([0, T]; W^{s,1}(\mathbb{T}^2))$ for all $s < 1$ with $\text{curl } u(0) \in L^p(\mathbb{T}^2)$ for some $p > 1$ that does not conserve energy, and a family $\{u^\nu\}$ of weak solutions to the 2D Navier-Stokes equations with*

$$\|u^\nu - u\|_{L^\infty(0,T;L^{p'}(\mathbb{T}^2))} \rightarrow 0 \quad \text{as } \nu \rightarrow 0,$$

for all $p' < 2$.

1.5. **Comments on the proof.** Let us briefly explain the main ideas and difficulties of the proof. The main construction is based on a convex integration scheme that consists of adding highly oscillatory and concentrated perturbations to produce a weak solution of the 2D Navier-Stokes equations in the limit and an energy pumping mechanism that allows us to achieve the Lusin-type property for the kinetic energy profile.

One of the biggest difficulties in any convex integration scheme designed for the Navier-Stokes equations is the presence of the Laplacian, which in 2D makes the equations L^2 -critical. This is the very reason that we need to work in a regularity setting below this threshold. The framework of intermittent convex integration developed in [BV19] only works above the regularity level $C_t L^2$ which is not sufficient for 2D Navier-Stokes equations.

The main building block in the convex integration scheme employed in this paper is a family of accelerating jet flows that are oscillatory and intermittent in space as well as in time. These jet flows combine the advantages of the two constructions from [BCV22] and [CL22], and can be fully intermittent both in space and in time, in contrast to the ones in [BCV22] and [CL22]. The main nonuniqueness results are the manifestation of the wild behaviors of such jet flows:

- (1) Accelerating jet flows are at rest for the majority of the time and only move on a union of small time intervals whose length models the strength of temporal intermittency;
- (2) In space, these flows are divergence-free and supported in small moving balls. The extreme spatial intermittency is needed for achieving the $C_t L^p$ regularity for p close to 2.
- (3) On each time interval in the temporal supports, the jet flows first accelerate, travel along the torus $\mathbb{T}^2$ with large speed, and then come to a rest. They are periodic in both space and time.
- (4) The jet flows solve the Navier-Stokes equations with a small $L_t^1 W^{-1,1}$ error. In other words, the “Reynolds stress” of these jet flows is small in $L_{t,x}^1$.

Once the main building blocks are obtained, we employ a space-time convex integration reducing the Reynolds stress to a highly oscillating in time error, which can be balanced by appropriate temporal and acceleration correctors, taking advantage of the space-time intermittency of the building blocks. To obtain nonuniqueness in the class $C_t L^p$ for $p < 2$, we optimize the parameters in the accelerating flows so that they are fully intermittent in space but only slightly intermittent in time. The membership in this class and reducing the stress error at the same time require a very delicate choice of parameters in the accelerating flows. Once the accelerating flows are in the class $C_t L^p$, it is then fairly straightforward to obtain nonuniqueness in $\cap_{p < 2} C_t L^p$ by a limiting argument in the iteration scheme, at least for smooth initial data.

To achieve nonuniqueness for rough initial data, we take advantage of the smoothing property of the fractional heat flow. Choosing the power of Laplacian large enough, we show that any divergence-free initial data $u_0 \in W^{s,1}$ for some $s > -1$ admits a solution (u, R) of the Navier-Stokes-Reynolds system with the stress error $R \in L_{x,t}^1$, which is compatible with our space-time convex integration scheme. Thanks to the smoothing property of the fractional heat flow, we can build a solution of the 2D Navier-Stokes equations on top of (u, R) , approaching the initial time as iterations progress.

Another key ingredient needed for the Lusin-type property is an energy pumping mechanism, done separately from the convex integration step, which essentially exploits the fact that the smallness in $L_t^\infty L^2$ falls in the subcritical regime of the 2D Navier-Stokes equations. Due to the time-intermittent nature of the convex integration scheme, the solution is mostly intact on a large portion of the time axis, what we call the good set $\mathcal{G}$, whose complement is of arbitrarily small measure. During the convex integration step, the perturbation can be made arbitrarily small on the good set $\mathcal{G}$ in any Sobolev space. On the other hand, we are able to correct the solution on the good set $\mathcal{G}$ to achieve desired properties, such as the L^2 -continuity and gaining the prescribed energy level.

Organization. The rest of the paper is organized as follows.

- (1) In Section 2, we prove all the main theorems stated in the introduction assuming the main Proposition 2.2 of the convex integration iteration.

- (2) Section 3 constitutes the first step of the proof of Proposition 2.2—we construct the building blocks, accelerating jets, define the velocity perturbation, and decompose the resulting Reynolds stress.
- (3) Section 4 is devoted to estimating the velocity perturbation and the Reynolds stress error. We specify the oscillation, concentration, and acceleration parameters, and show that the inductive assumption of the iteration Proposition 2.2 are satisfied.
- (4) We show the conservation of energy for vanishing viscosity solutions to the 2D Euler equations with initial vorticity in L^p , $p > 1$ in Appendix A. In Appendices B, C we recall some technical inequalities and properties of the antidivergence operators used in the convex integration scheme.

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2. OUTLINE OF THE PROOF

2.1. Notations. Throughout the paper, we fix our spatial domain $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ which we often identify with a periodic square $[0, 1]^2$.

For any $d \in \mathbb{N}_+ := \mathbb{N} \setminus \{0\}$, $\mathcal{L}^d$ denotes the Lebesgue measure in $\mathbb{R}^d$. For the Lebesgue spaces $L^p(\mathbb{T}^2)$, the Lebesgue norms are denoted by

$$\|f\|_p = \|f\|_{L^p} = \|f\|_{L^p(\mathbb{T}^2)}$$

where we do not distinguish scalar, vector or tensor valued functions. If the function $f : \mathbb{T}^2 \times [0, T] \rightarrow \mathbb{R}$ is time-dependent, we write

$$\|f(t)\|_p = \|f(t)\|_{L^p(\mathbb{T}^2)},$$

to indicate that the spatial norm is taken at each time slice $t \in [0, T]$.

The differentiation operations such as ∇ , Δ , and div are meant for differentiation in space only, while for differentiation in time we use ∂_t .

In general, repeated indexed are summed without mentioning when the meaning is clear in the context. For matrix-valued function $A : \mathbb{T}^2 \rightarrow \mathbb{R}^{2 \times 2}$, the usual (spatial) divergence is defined by

$$\operatorname{div} A = \partial_j A_{ij}.$$

For two vector valued functions $f, g : \mathbb{T}^2 \rightarrow \mathbb{R}^2$, the tensor product $\otimes$ is given by $f \otimes g = f_i g_j : \mathbb{T}^2 \rightarrow \mathbb{R}^{2 \times 2}$ and $\otimes$ denotes the usual traceless tensor product $f \otimes g = f_i g_j - \frac{1}{2} \delta_{ij} f_k g_k$.

For any function f on $\mathbb{T}^2$, we say f is $\sigma^{-1}\mathbb{T}^2$ -periodic if

$$f(x + \sigma^{-1}k) = f(x) \quad \text{for any } k \in \mathbb{Z}^2,$$

and for any $\sigma \in \mathbb{N}$, we use the notation $f(\sigma \cdot)$ for the function $x \mapsto f(\sigma x)$, so $f(\sigma \cdot)$ is at least $\sigma^{-1}\mathbb{T}^2$ -periodic.

We use intensively the notation $X \lesssim Y$ which means $X \leq CY$ for some constant $C > 0$. Sometimes the constant will depend on various parameters and in that case we will write $X \lesssim_{\alpha, \beta, \gamma \dots} Y$ to indicate $X \leq CY$ for some constant $C > 0$ depending on $\alpha, \beta, \gamma \dots$. The notation $\gtrsim$ is similarly defined and $X \sim Y$ means both $X \lesssim Y$ and $X \gtrsim Y$ at the same time.

2.2. The Navier-Stokes-Reynolds system. The proof of our main theorems is based on the construction of a sequence of approximate solutions. These approximate solutions (of (1.1)) satisfy the so-called Navier-Stokes-Reynolds system:

$$\begin{cases} \partial_t u - \Delta u + \operatorname{div}(u \otimes u) + \nabla p = \operatorname{div} R \\ \operatorname{div} u = 0, \end{cases} \quad (2.1)$$

where $R : \mathbb{T}^2 \times [0, T] \rightarrow \mathcal{S}_0^{2 \times 2}$ is a 2×2 symmetric traceless matrix, usually termed as the Reynolds stress in the literature. We refer to the discussion in [BV21] for the motivation of this system and its relation to mathematical turbulence. In a nutshell, the Reynolds stress R measures the approximateness of the solutions to exact solutions of the Navier-Stokes equations.

In what follows, we will say (u, R) is a smooth solution to (2.1) on I if u and R satisfy (2.1) and are both smooth on $\mathbb{T}^2 \times I$ for some interval $I \subset \mathbb{R}$. Note that the pressure is intentionally left out in this formulation since one can take the divergence of (2.1) to obtain the following elliptic equation

$$\Delta p = \operatorname{div} \operatorname{div}(R - u \otimes u),$$

which uniquely determines the pressure when coupled with the usual zero mean condition $\int_{\mathbb{T}^2} p = 0$.

2.3. Universality of the Navier-Stokes-Reynolds flows. We prove the following auxiliary result used in the proof of Theorem 1.2 concerning the existence of (weak) solutions of (2.1) for general rough data.

Throughout the paper, we say (u, R) is a weak solution of (2.1) on $[0, T]$ if

$$\int_{\mathbb{T}^2} u_0(x) \varphi(x, 0) dx = - \int_{\mathbb{T}^2 \times [0, 1]} (u \cdot \Delta \varphi + u \otimes u : \nabla \varphi + u \cdot \partial_t \varphi) dx dt - \int_{\mathbb{T}^2 \times [0, 1]} R : \nabla \varphi dx dt \quad (2.2)$$

holds for all test functions $\varphi \in \mathcal{D}_T$ (smooth, divergence-free, and vanishing on $t \geq T$).

Theorem 2.1. *For any divergence-free $u_0 \in W^{s,p}(\mathbb{T}^2)$, $s > -1$, $1 \leq p \leq \infty$, there exists a weak solution (u, R) of (2.1) on $\mathbb{T}^2 \times [0, T]$ such that (u, R) is smooth on $(0, T]$ and*

$$u \in C([0, T]; W^{s,p}(\mathbb{T}^2)) \quad \text{and} \quad R \in L^1(\mathbb{T}^2 \times [0, T]).$$

In fact, the velocity can be chosen to be $u(t) = e^{-\nu t(-\Delta)^\alpha} u_0$ for any $\alpha \geq 2$ and $\nu > 0$.

Proof. Take $u(t) = e^{-\nu t(-\Delta)^\alpha} u_0$ for some $\alpha \geq 2$, $\nu > 0$, and define

$$R := \mathcal{R}(\partial_t u - \Delta u) + u \otimes u, \quad (2.3)$$

where $\mathcal{R}$ is the antidivergence defined in Appendix C. Note that the smoothness of R on $\mathbb{T}^2 \times (0, T]$ follows from that of u .

To prove the claims, we estimate each term in (2.3). For any $0 < t \leq 1$, by (C.2), L^q boundedness of the Riesz transform for some $q > 1$, and well-known estimates for the fractional heat semigroup, we have

$$\begin{aligned} \|\mathcal{R} \partial_t u(t)\|_{L^1} &= \|\mathcal{R}(-\Delta)^\alpha u(t)\|_{L^1} \lesssim \|(-\Delta)^{\alpha - \frac{1}{2}} u(t)\|_{L^q} \\ &\lesssim t^{-1 + \frac{1}{\alpha}(\frac{1}{q} - \frac{1-s}{2})} \|\nabla|^s u_0\|_{L^1}, \end{aligned}$$

which is integrable by choosing q sufficiently close to 1 when $\alpha > 0$ and $s > -1$; and

$$\|\mathcal{R} \Delta u(t)\|_{L^1} \lesssim \|\nabla|u(t)\|_{L^q} \lesssim t^{\frac{1}{2\alpha}(s-1+2(1-\frac{1}{q}))} \|\nabla|^s u_0\|_{L^1},$$

which is integrable by choosing q sufficiently close to 1 when $\alpha \geq 1$ and $s > -1$; as well as

$$\|u \otimes u(t)\|_{L^1} \lesssim \|u(t)\|_{L^2}^2 \lesssim t^{\frac{s-1}{\alpha}} \| |\nabla|^s u_0 \|_{L^1}^2,$$

which is integrable when $\alpha \geq 2$ and $s > -1$;

□

2.4. The main iteration proposition. We now state the main proposition of the paper.

Proposition 2.2. *There exist a geometric constant $M > 0$ such that for any $1 \leq p < 2$, there exists $r > 1$ depending only on p with the following properties.*

Let (u, R) be a weak solution of (2.1) on $\mathbb{T}^2 \times [0, 1]$ such that (u, R) is smooth on $(0, 1]$. Given any $\delta > 0$ and any closed interval $I \subset (0, 1]$, there exists another weak solution (u_1, R_1) of (2.1) on $\mathbb{T}^2 \times [0, 1]$ and an open set $E \subset I$ such that

(1) *The velocity perturbation $w := u_1 - u$ is smooth on $[0, 1]$,*

$$\text{supp}_t w \subset I,$$

and

$$\mathcal{L}^1([0, t] \cap E) \leq t\delta, \quad \text{for all } 0 \leq t \leq 1. \quad (2.4)$$

(2) *For any $t \notin E$*

$$\|w(t)\|_{W^{\frac{1}{\delta}, \infty}(\mathbb{T}^2)} \leq \delta, \quad (2.5)$$

$$\left| \|u_1(t)\|_{L^2(\mathbb{T}^2)}^2 - \|u(t)\|_{L^2(\mathbb{T}^2)}^2 \right| \leq \delta^2. \quad (2.6)$$

(3) *There hold the estimates,*

$$\|w\|_{L^2(\mathbb{T}^2 \times [0, 1])} \leq M \|R\|_{L^1(\mathbb{T}^2 \times I)}^{\frac{1}{2}}, \quad (2.7)$$

$$\|w\|_{L^\infty(0, 1; L^p(\mathbb{T}^2))} \leq \delta. \quad (2.8)$$

(4) *The new stress error R_1 satisfies*

$$\|R_1\|_{L^1(I; L^r(\mathbb{T}^2))} \leq \delta.$$

In the above proposition, the role of the closed interval $I \subset (0, 1]$ is to progressively get to $t = 0$ along the iterations in the proof. For the current paper, one should think of I as approximations of $(0, 1]$. It also helps us to achieve the exact energy profile of the solution near $t = 0$.

Remark 2.3. *The velocity perturbation in the above proposition enjoys the following additional properties.*

(1) *There exists disjoint open intervals $I_n \subset I$ such that $E \subset \cup_n I_n$ and a small $r > 0$, which goes to zero as $\delta \rightarrow 0$, such that each I_n in $\cup_n I_n$ is of length at most r , and the number of intervals I_n is bounded by $\lesssim r^{-\frac{1}{9}}$.*

(2) *In fact, thanks to Corollary 4.6, estimate (2.8) can be replaced by*

$$\|w\|_{L^\infty(0, 1; W^{s_p, 1}(\mathbb{T}^2))} \leq \delta,$$

for some $0 < s_p < 1$ and $s_p \rightarrow 1$ as $p \rightarrow 2$. This is used to prove (sharp) nonuniqueness in Sobolev spaces $L_t^\infty W^{s, 1}$, $s < 1$, in Theorem 1.4.

2.5. Proof of main nonuniqueness results. We prove Theorems 1.2, 1.3, and 1.4 assuming Proposition 2.2. Without loss of generality we assume $T = 1$ and $\varepsilon < \frac{1}{2}$.

Proof of Theorems 1.2, 1.3, and 1.4. For $n \in \mathbb{N}^+$, we construct a sequence of solutions (u_n, R_n) , positive numbers $\delta_n \rightarrow 0$, monotone decreasing $t_n \rightarrow 0$ as $n \rightarrow \infty$, and open sets $E_n \subset [0, 1]$ such that (u_n, R_n) converges to the desired weak solution u , and the union of E_n gives $[0, 1] \setminus \mathcal{G}$.

The solutions (u_n, R_n) and sets E_n , constructed by induction, will satisfy the following assumptions.

- (a) (u_n, R_n) is a smooth solution of (2.1) on $\mathbb{T}^2 \times (0, 1]$;
- (b) The Reynolds stress is small:

$$\|R_n\|_{L^1(\mathbb{T}^2 \times [0, 1])} \leq \delta_n;$$

- (c) u_n achieves the prescribed the energy level:

$$\begin{aligned} e(t) - \|u_n(t)\|_2^2 &> 0 \quad \text{for all } t \in (0, 1] \setminus \mathcal{B}_n, \\ e(t) - \|u_n(t)\|_2^2 &\leq \frac{\delta_n^2}{100} \quad \text{for all } t \in [t_n, 1] \setminus \mathcal{B}_n, \end{aligned}$$

where we define the bad set $\mathcal{B}_n \subset [0, 1]$ as the union of all E_j , $j \leq n$:

$$\mathcal{B}_n = \bigcup_{1 \leq j \leq n} E_j;$$

- (d) The sets E_n are unions of open intervals and satisfy

$$\mathcal{L}^1([0, t] \cap E_n) \leq t\delta_n, \quad \forall 0 \leq t \leq 1.$$

Here, the parameters p_n and δ_n are given explicitly as

$$p_n = 2 - 2^{-n-1} \quad \text{for } n \in \mathbb{N}^+, \quad \text{and} \quad \delta_n = \varepsilon 2^{-n} \leq 1 \quad \text{if } n \geq 2.$$

The value of $\delta_1 > 0$ and the sequence t_n are chosen separately after we specify the initial solution (u_1, R_1) in Step 1 below.

In addition, the perturbation $w_n := u_n - u_{n-1}$ satisfies for $n \geq 2$:

- (i) Regularity estimates:

$$\begin{aligned} \|w_n\|_{L^2(\mathbb{T}^2 \times [0, 1])} &\leq C_M \delta_{n-1}^{\frac{1}{2}}, \\ \|w_n\|_{L^\infty(0, 1; L^{p_n})} &\leq \delta_{n-1}, \end{aligned}$$

where $C_M > 0$ depends only on the initial solution (u_1, R_1) and the geometric constant M from Proposition 2.2;

- (ii) Support away from the origin:

$$\text{supp } w_n \subset (0, 1];$$

- (iii) Small energy outside of E_n :

$$\|w_n\|_{L^2(\mathbb{T}^2)} \leq \delta_n, \quad t \in [t_n, 1] \setminus E_n.$$

Step 1: Initialization

We choose the initial solution (u_1, R_1) according to the given energy profile and $\varepsilon > 0$ in the main theorems. Take

$$u_1(t) = e^{-\nu t(-\Delta)^2} u_0,$$

for some $\nu > 0$ that will be fixed large enough depending on $e(t)$. Thanks to Theorem 2.1, there exists a Reynolds stress $R_1 \in L^1(\mathbb{T}^2 \times [0, 1])$ such that (u_1, R_1) is a smooth solution to the Navier-Stokes-Reynolds system (2.1) on $(0, 1]$. The associated pressure can be obtained by a routine computation.

Now we define $E_1 = [0, \frac{\varepsilon}{2}]$ if $u_0 \notin L^2(\mathbb{T}^2)$ (recall that $\varepsilon < \frac{1}{2}$), and $E_1 = \emptyset$ when $u_0 \in L^2(\mathbb{T}^2)$. Note that $\|u_1(t)\|_{L^2}$ is bounded on any compact subset of $(0, 1]$, and, in the case $p = 2$, we also have $\|u_1(t)\|_2^2 \leq \|u_0\|_2^2 e^{-\nu t}$. Thus, in either case, we can choose ν large enough so that

$$\|u_1(t)\|_2^2 < e(t), \quad \text{for all } t \in (0, T] \setminus E_1.$$

Let us point out that u_1 is responsible for the difference in final regularity of u between the case $p < 2$ and $p = 2$.

Since $R_1 \in L^1(\mathbb{T}^2 \times [0, 1])$, we can specify the value of $\delta_1 > 1$ as

$$\delta_1 = 1 + 100 \max \left\{ \|R_1\|_{L^1(\mathbb{T}^2 \times [0, 1])}, \sup_{t \in [0, 1] \setminus E_1} [e(t) - \|u_1(t)\|_2^2]^{\frac{1}{2}} \right\}, \quad (2.9)$$

and choose a strictly decreasing sequence of $\{t_n\}_{n \geq 1}$, such that $t_n \rightarrow 0$ and

$$100 \|R_1\|_{L^1(\mathbb{T}^2 \times [0, t_n])} \leq \delta_n, \quad (2.10a)$$

$$t_n^{\frac{1}{2}} \|e\|_{L^\infty(0, 1)} \leq \frac{1}{4} \delta_n. \quad (2.10b)$$

It follows that (u_1, R_1) satisfies all the conditions (a)–(d). We also remark that condition (2.10b) is to get the energy density statement at $t = 0$, so it is only used for the case $p = 2$, as when $p < 2$, we have $0 \in E_1 \subset \mathcal{B}_n$.

In the step 2–4 below, we will construct the solution (u_n, R_n) and the sets E_n for $n \geq 2$ by induction.

Step 2: Energy correction $(u_n, R_n) \mapsto (\bar{u}_n, \bar{R}_n)$

In this step, our main objective is to correct the energy level of (u_n, R_n) to obtain a new solution pair $(\bar{u}_n, \bar{R}_n)$ with a slightly larger $\bar{R}_n$. This is done by adding a small energy corrector $\bar{w}_{n+1}$ to u_n .

In view of inductive assumptions, the correction $\bar{w}_{n+1} := \bar{u}_n - u_n$ needs to improve the energy level of u_n on both $[t_{n+1}, t_n]$ and $[t_n, 1] \setminus \mathcal{B}_n$. On the set $[t_n, 1] \setminus \mathcal{B}_n$, the energy is already quite close to $e(t)$ due to (c), and hence $\bar{w}_{n+1}$ is small in L^2 in this region. On $[t_{n+1}, t_n]$, the correction might be larger than δ_n since it is the first time we correct the energy there, but due to $t_n \rightarrow 0$ as in (2.10b) the correction will be small in L_t^2 norm. These observations ensure the needed convergence of $\bar{w}_{n+1}$ whose estimates are verified in the next step.

Now we turn to the specific construction of $\bar{w}_{n+1}$. Throughout the iteration, we fix a function $\psi \in C_c^\infty(\mathbb{R}^2)$ with $\text{supp } \psi \subset (0, 1)^2$ and rescale it by a large parameter $\mu > 0$, still denoting the resulting function by ψ , so that

$$\|\nabla \psi\|_{L^2(\mathbb{T}^2)} = 1, \quad \|\nabla^n \psi\|_{L^q(\mathbb{T}^2)} \lesssim \mu^{n-1} \mu^{2(\frac{1}{2} - \frac{1}{q})} \quad \text{for all } 1 \leq q \leq \infty. \quad (2.11)$$

We will use ψ as the stream function on $\mathbb{T}^2$ for the energy corrector $\bar{w}_{n+1}$. The exact value of μ depends on many factors and will be specified later.

Now we can introduce a smooth nonnegative energy profile $\rho : [0, 1] \rightarrow \mathbb{R}^+$ such that all of the following holds.

(1) $\rho(t)$ pumps the exact amount of energy outside of the bad set:

$$[\rho(t)]^2 = \left(1 - \frac{\delta_{n+1}^2}{600\|e\|_{L^\infty(0,1)}}\right) [e(t) - \|u_n(t)\|_2^2] \quad \text{if } t \in [t_{n+1}, 1] \setminus \mathcal{B}_n; \quad (2.12)$$

(2) $\rho(t)$ is under control on $[0, 1]$:

$$[\rho(t)]^2 < \begin{cases} e(t) - \|u_n(t)\|_2^2 & \text{if } t \in [t_{n+2}, t_{n+1}] \\ \|e(t) - \|u_n(t)\|_2^2\|_{L^\infty([t_n, 1] \setminus \mathcal{B}_n)} & \text{if } t \in [t_n, 1]; \end{cases} \quad (2.13)$$

(3) $\rho(t)$ is supported away from $t = 0$:

$$\rho(t) = 0, \quad \text{for all } t \in [0, t_{n+2}]. \quad (2.14)$$

Such a profile $\rho(t)$ exists thanks to the fact that $\mathcal{B}_n \subset [t_n, 1]$ and the inductive assumption (c), and we use it together with the stream function ψ to define the energy corrector $\bar{w}_{n+1}$:

$$\bar{w}_{n+1} = \rho(t) \nabla^\perp \psi.$$

We remark that in (2.13) we use t_n to separate the two cases as $\rho(t)$ could be large on $[t_{n+1}, t_n]$ when comparing to δ_n , cf. (2.16) below. The exact choice of $\rho(t)$ is not substantial to our purpose as long as it fulfills (2.12)–(2.14). Immediately, we have

$$\|\bar{w}_{n+1}(t)\|_2 = \rho(t), \quad \text{for all } t \in [0, 1]. \quad (2.15)$$

In addition, by assumption (c) and (2.13), it follows that

$$0 \leq \|\bar{w}_{n+1}(t)\|_2 \leq \frac{1}{10} \delta_n, \quad \text{for all } t \in [t_n, 1]. \quad (2.16)$$

The new solution $(\bar{u}_n, \bar{R}_n)$ will then be defined as

$$\bar{u}_n := u_n + \bar{w}_{n+1},$$

and

$$\bar{R}_n := R_n + \mathcal{R}(-\Delta \bar{w}_{n+1} + \partial_t \bar{w}_{n+1}) + (\bar{w}_{n+1} \otimes u_n + u_n \otimes \bar{w}_{n+1}) + (\bar{w}_{n+1} \otimes \bar{w}_{n+1}). \quad (2.17)$$

Step 3: Estimates of $(\bar{u}_n, \bar{R}_n)$ and $\bar{w}_{n+1}$

In this step we establish the following estimates that will be passed to the next step. The estimates that we need are:

$$\|\bar{R}_n\|_{L^1(\mathbb{T}^2 \times [0, 1])} \leq C \delta_n, \quad (2.18)$$

where $C > 0$ is independent of n , as well as

$$\|\bar{w}_{n+1}\|_{L^2(0, 1; L^2(\mathbb{T}^2))} \leq \frac{1}{2} \delta_n, \quad (2.19a)$$

$$\|\bar{w}_{n+1}\|_{L^\infty(0, 1; L^{p_n}(\mathbb{T}^2))} \leq \frac{1}{2} \delta_n, \quad (2.19b)$$

and

$$e(t) - \|\bar{u}_n\|_2^2 > 0 \quad \text{for all } t \in (0, 1] \setminus \mathcal{B}_n, \quad (2.20a)$$

$$0 < e(t) - \|\bar{u}_n\|_2^2 \leq \frac{1}{500} \delta_{n+1}^2 \quad \text{for all } t \in [t_{n+1}, 1] \setminus \mathcal{B}_n. \quad (2.20b)$$

The easiest one is (2.19b), which follow directly from (2.11) and (2.16) by taking the free parameter μ sufficiently large. For (2.19a), we consider the split

$$\|\bar{w}_{n+1}\|_{L^2(0,1;L^2(\mathbb{T}^2))} \leq \|\bar{w}_{n+1}\|_{L^2(0,t_n;L^2(\mathbb{T}^2))} + \|\bar{w}_{n+1}\|_{L^2(t_n,1;L^2(\mathbb{T}^2))}.$$

The bound for the first term follows from (2.10b), the first part in (2.13), and (2.14):

$$\|\bar{w}_{n+1}\|_{L^2(0,t_n;L^2(\mathbb{T}^2))} \leq t_n^{\frac{1}{2}} \sup_{t \in [0,t_n]} \rho(t) \leq \frac{1}{4} \delta_n,$$

while the second term follows from (2.16):

$$\|\bar{w}_{n+1}\|_{L^2(t_n,1;L^2(\mathbb{T}^2))} \leq \frac{\delta_n}{10}.$$

Next, let us examine the energy level of $\bar{u}_n = u_n + \bar{w}_{n+1}$. Note that due to (2.11), for any $t \in [t_{n+2}, 1]$ we have

$$\begin{aligned} |\langle u_n, \bar{w}_{n+1} \rangle| &= \left| \int_{\mathbb{T}^2} u_n \cdot \bar{w}_{n+1} dx \right| \\ &\leq \|u_n\|_{L^\infty(\mathbb{T}^2 \times [t_{n+2}, 1])} \rho(t) \|\nabla^\perp \psi\|_{L^1(\mathbb{T}^2)} \\ &\leq C(u_n) \rho(t) \mu^{-1}, \end{aligned} \tag{2.21}$$

which tends to 0 uniformly in time as $\mu \rightarrow \infty$. With this, we consider the energy difference,

$$e(t) - \|\bar{u}_n(t)\|_2^2 = e(t) - \|u_n(t)\|_2^2 - \|\bar{w}_{n+1}(t)\|_2^2 - 2\langle u_n, \bar{w}_{n+1} \rangle. \tag{2.22}$$

Next, we will use (2.22) to prove (2.20a) and (2.20b). Since $[t_{n+2}, 1] \setminus \mathcal{B}_n$ is closed in $[0, 1]$, it follows from (c) that $e(t) - \|u_n(t)\|_2^2$ is bigger than some positive constant on that set. Now thanks to (2.14), (2.12), and (2.21) we may choose $\mu > 0$ sufficiently large such that (2.20a) holds:

$$e(t) - \|\bar{u}_n(t)\|_2^2 > 0 \quad \text{for all } t \in (0, 1] \setminus \mathcal{B}_n.$$

For (2.20b), by (2.12) and (2.15), we see that the main terms in (2.22) satisfies

$$\begin{aligned} e(t) - \|u_n(t)\|_2^2 - \|\bar{w}_{n+1}(t)\|_2^2 &= \frac{\delta_{n+1}^2}{600\|e\|_{L^\infty(0,1)}} [e(t) - \|u_n(t)\|_2^2] \\ &< \frac{1}{500} \delta_{n+1}^2 \quad \text{for all } t \in [t_{n+1}, 1] \setminus \mathcal{B}_n. \end{aligned}$$

Thanks to the strict inequality above, we can thus take $\mu > 0$ sufficiently large such that (2.21) and (2.22) yield

$$e(t) - \|\bar{u}_n(t)\|_2^2 \leq \frac{1}{500} \delta_{n+1}^2 \quad \text{for all } t \in [t_{n+1}, 1] \setminus \mathcal{B}_n,$$

thereby establishing (2.20b).

The remaining task is to obtain the estimate of the new stress $\bar{R}_n$ defined in (2.17). First of all, we see that by (2.11) and the Hölder inequality,

$$\begin{aligned} \|u_n \otimes \bar{w}_{n+1}\|_{L^1(\mathbb{T}^2 \times [0,1])} &\leq \|u_n\|_{L^\infty(\mathbb{T}^2 \times [0,1])} \|\bar{w}_{n+1}\|_{L^1(\mathbb{T}^2 \times [0,1])} \\ &\leq C(e, u_n) \mu^{-1}, \\ \|\mathcal{R}(\partial_t \bar{w}_{n+1})\|_{L^1(\mathbb{T}^2 \times [0,1])} &\lesssim \|\partial_t \rho\|_{L^\infty([0,1])} \|\mathcal{R} \nabla^\perp \psi\|_{L^q(\mathbb{T}^2)} \\ &\leq C(e, u_n, q) \mu^{1-\frac{2}{q}} \quad \text{for any } 1 < q < 2, \end{aligned}$$

and

$$\|\bar{w}_{n+1} \overset{\circ}{\otimes} \bar{w}_{n+1}\|_{L^1(\mathbb{T}^2 \times [0,1])} \leq 4\|\bar{w}_{n+1}\|_{L^2(0,1;L^2(\mathbb{T}^2))}^2.$$

Since $\mu > 0$ is a free parameter, choosing suitably $q < 2$ and taking μ sufficiently large once again, we can ensure

$$\|u_n \overset{\circ}{\otimes} \bar{w}_{n+1}\|_{L^1(\mathbb{T}^2 \times [0,1])} + \|\mathcal{R}(\partial_t \bar{w}_{n+1})\|_{L^1(\mathbb{T}^2 \times [0,1])} \leq \delta_n^2. \quad (2.23)$$

On the other hand, (2.19a) implies

$$\|\bar{w}_{n+1} \overset{\circ}{\otimes} \bar{w}_{n+1}\|_{L^1(\mathbb{T}^2 \times [0,1])} \leq \delta_n^2. \quad (2.24)$$

Finally, for the term involving the Laplacian in (2.17), by (C.2) and (2.11) we have

$$\begin{aligned} \|\mathcal{R}(\Delta \bar{w}_{n+1})\|_{L^1(\mathbb{T}^2 \times [0,1])} &= 2\|\nabla \bar{w}_{n+1}\|_{L^1(\mathbb{T}^2 \times [0,1])} \\ &\leq C_\psi \left(\int_0^{t_n} \rho(t) + \sup_{t \in [t_n, 1]} \rho(t) \right) \leq C_\psi \delta_n, \end{aligned} \quad (2.25)$$

where in the last step we have used bounds (2.13). We note that C_ψ is a constant depending only on the profile ψ .

Now we add up (2.23)–(2.25) and, using the inductive assumption (b), obtain the desired (2.18):

$$\|\bar{R}_n\|_{L^1(\mathbb{T}^2 \times [0,1])} \leq \|R_n\|_{L^1(\mathbb{T}^2 \times [0,1])} + 2\delta_n^2 + C_\psi \delta_n \leq C\delta_n,$$

for some constant C independent of n , but dependent on δ_1 (which is larger than 1) and hence dependent on (u_1, R_1) .

Step 4: Error reduction $(\bar{u}_n, \bar{R}_n) \mapsto (u_{n+1}, R_{n+1})$

In this step, we use the main proposition of convex integration to reduce the stress error.

We apply Proposition 2.2 for the solution pair $(\bar{u}_n, \bar{R}_n)$ with $I = [t_{n+1}, 1]$ and parameters

$$\delta = \frac{1}{100} \min \left\{ \delta_{n+1}, \min_{t \in [t_{n+1}, 1] \setminus \mathcal{B}_n} [e(t) - \|\bar{u}_n(t)\|_2^2]^{\frac{1}{2}} \right\} \quad \text{and} \quad p = p_{n+1}.$$

where $\delta > 0$ due to (2.20b) and the fact that $[t_{n+1}, 1] \setminus \mathcal{B}_n$ is closed. Let us denote the obtained solution by (u_{n+1}, R_{n+1}) , the obtained set by E_{n+1} , and the velocity perturbation $u_{n+1} - \bar{u}_n$ by $\tilde{w}_{n+1}$. Note that from the proposition, we have $\text{supp}_t \tilde{w}_{n+1} \subset [t_{n+1}, 1]$ and the bound

$$\|\tilde{w}_{n+1}(t)\|_2 \leq \delta_{n+1}/100 \quad \text{for all } t \notin E_{n+1} \quad (2.26)$$

Now by (2.7), (2.8), and (2.18) we can fix $C_M \geq \delta_1^{\frac{1}{2}}$ throughout the iteration, depending only on the geometric constant M and (u_1, R_1) , such that

$$\begin{aligned} \|\tilde{w}_{n+1}\|_{L^2(\mathbb{T}^2 \times [0,1])} &\leq M\|\bar{R}_n\|_{L^1(\mathbb{T}^2 \times [0,1])}^{\frac{1}{2}} \\ &\leq M \cdot C^{\frac{1}{2}} \delta_n^{\frac{1}{2}} \leq \frac{1}{2} C_M \delta_n^{\frac{1}{2}}, \end{aligned} \quad (2.27a)$$

$$\|\tilde{w}_{n+1}\|_{L^\infty(0,1;L^{p_n}(\mathbb{T}^2))} \leq \frac{1}{2} \delta_{n+1} \leq \frac{1}{2} \delta_n, \quad (2.27b)$$

and

$$\|R_{n+1}\|_{L^1(\mathbb{T}^2 \times [t_{n+1}, 1])} \leq \frac{1}{2} \delta_{n+1}. \quad (2.28)$$

The pair (u_{n+1}, R_{n+1}) solves equation (2.1) on $[0, 1] \times \mathbb{T}^2$ and the total stress error satisfies

$$\|R_{n+1}\|_{L^1(\mathbb{T}^2 \times [0,1])} \leq \|R_{n+1}\|_{L^1(\mathbb{T}^2 \times [0, t_{n+1}])} + \|R_{n+1}\|_{L^1(\mathbb{T}^2 \times [t_{n+1}, 1])},$$

which by (2.10a) and (2.28) reduces to

$$\|R_{n+1}\|_{L^1(\mathbb{T}^2 \times [0,1])} \leq \delta_{n+1}.$$

In addition, due to (2.6), the energy level for any $t \in [t_{n+1}, 1] \setminus E_{n+1}$ satisfies

$$\left| \|u_{n+1}(t)\|_2^2 - \|\bar{u}_n(t)\|_2^2 \right| \leq \frac{\delta_{n+1}^2}{100^2}. \quad (2.29)$$

Then for the inductive assumption (c), we first observe that $e(t) - \|u_{n+1}(t)\|_2^2 > 0$ on $t \in (0, t_{n+1}]$ by the fact that $\text{supp}_t \tilde{w}_{n+1} \subset [t_{n+1}, 1]$ and (2.20a). The same $e(t) - \|u_{n+1}(t)\|_2^2 > 0$ holds also on $t \in [t_{n+1}, 1] \setminus \mathcal{B}_{n+1}$ thanks to the definition of δ and (2.6) in the proposition. These establish the first bound in (c). Similarly, the second bound in follows from (2.29) and (2.20b)

Step 5: Verification of inductive assumptions

From Step 3 and Step 4, we see that (u_{n+1}, R_{n+1}) satisfies the listed items (a)–(d) at level $n+1$. To close the induction argument, we only need to verify items (i), (ii) and (iii) for the total perturbation

$$w_{n+1} := u_{n+1} - u_n = \bar{w}_{n+1} + \tilde{w}_{n+1}.$$

Now, we can conclude that:

- Item (i) follows from (2.19a)–(2.19b) and (2.27a)–(2.27b) by definition of C_M ;
- Item (ii) is a direct consequence of (2.14) and the definition of $\tilde{w}_{n+1}$;
- Item (iii) follows from (2.16) and (2.26).

Step 6: Passing to the limit $n \rightarrow \infty$

Finally, we will conclude that the constructed sequence (u_n, R_n) converges to a weak solution u of (1.1), and u satisfies all the listed properties in the statement of Theorem 1.2.

We define the final weak solution u as the point-wise in time limit of u_n in $L^1(\mathbb{T}^2)$:

$$u(t) := \lim_{n \rightarrow \infty} u_n(t), \quad (2.30)$$

which is well-defined due to the inductive assumption (i). In other words,

$$u(t) = u_1(t) + \sum_{n=2}^{\infty} w_n(t). \quad (2.31)$$

Since we know that $u_1 \in C([0, 1]; L^p(\mathbb{T}^2)) \cap C^\infty(\mathbb{T}^2 \times (0, T])$, to show the claimed regularity of u it suffices to prove that

$$\sum_{n=2}^{\infty} w_n \in C([0, 1]; L^{p'}(\mathbb{T}^2)), \quad \text{for any } p' < 2.$$

Indeed, for each $p' < 2$ we can find $N_{p'} \in \mathbb{N}$ such that $p_n > p'$ provided $n \geq N_{p'}$. Then by the inductive estimate (i),

$$\lim_{N \rightarrow \infty} \sum_{n=N}^{\infty} \|w_n\|_{L^\infty L^{p'}} \leq \lim_{N \rightarrow \infty} \sum_{n=N}^{\infty} \delta_{n-1} = 0.$$

Sobolev space $L_t^\infty W^{s,1}$ regularity in Theorem 1.4 similarly follows from estimates in Remark 2.3.

Next, we show that the weak formulation holds. Indeed, for any $\varphi \in \mathcal{D}_T$, using weak formulation of (2.1) or integrating by parts we obtain

$$\begin{aligned} \int_{\mathbb{T}^2} u_0(x) \varphi(x, 0) dx &= - \int_{\mathbb{T}^2 \times [0, 1]} (u_n \cdot \Delta \varphi + u_n \otimes u_n : \nabla \varphi + u_n \cdot \partial_t \varphi) dx dt \\ &\quad - \int_{\mathbb{T}^2 \times [0, 1]} R : \nabla \varphi dx dt. \end{aligned} \quad (2.32)$$

Inductive assumptions (a)–(d) imply the convergence of all terms to their natural limits in (2.32). So u is a weak solution on $\mathbb{T}^2 \times [0, 1]$.

Step 7: Final conclusion

The good set in Theorems 1.2 and 1.3 is defined as

$$\mathcal{G} = [0, 1] \setminus \bigcup_{j \geq 1} E_j.$$

By inductive assumption (c), the energy achieves the prescribed level on $\mathcal{G}$:

$$\|u(t)\|_{L^2}^2 = e(t), \quad \forall t \in \mathcal{G}.$$

Now by inductive assumption (d) and definition of δ_n ,

$$\mathcal{L}^1([0, 1] \setminus \mathcal{G}) \leq \sum_{n \geq 1} \mathcal{L}^1(E_n) \leq \frac{\varepsilon}{2} + \sum_{n \geq 2} \delta_n = \varepsilon,$$

which implies the Lusin-type property for the kinetic energy in Theorem 1.2.

In the case when $u_0 \in L^2(\mathbb{T}^2)$, the set E_1 is empty by definition. Recall also that $E_n \subset [t_n, 1]$ for some sequence $t_n \rightarrow 0$. Hence, employing the inductive assumption (d) again, we obtain

$$t^{-1} \mathcal{L}^1([0, t] \setminus \mathcal{G}) \leq t^{-1} \sum_{n: t_n < t} \mathcal{L}^1([0, t] \cap E_n) \leq t^{-1} \sum_{n: t_n < t} t \delta_n \rightarrow 0,$$

as $n \rightarrow \infty$, since $t_n \rightarrow 0$. This establishes density property (1.5) in Theorem 1.3.

Finally, we prove the second statement in Remark 1.5 concerning the L^2 -continuity of u . Define the bad set

$$\mathcal{B} = \bigcap_{n \geq 1} \bigcup_{j \geq n} E_j.$$

By continuity of the Lebesgue measure from above,

$$\mathcal{L}^1(\mathcal{B}) = \lim_{n \rightarrow \infty} \mathcal{L}^1\left(\bigcup_{j \geq n} E_j\right) = \lim_{n \rightarrow \infty} \varepsilon 2^{1-n} = 0.$$

In fact, it follows from Remark 2.3 that the Hausdorff dimension of $\mathcal{B}$ is less than or equal to $\frac{1}{9}$. Note that

$$[0, 1] \setminus \mathcal{B} = \bigcup_{n \geq 1} \mathcal{T}_n, \quad \text{where} \quad \mathcal{T}_n = \{0\} \cup [t_n, 1] \setminus \bigcup_{j \geq n} E_j.$$

Clearly $\mathcal{T}_n$ are increasing, and by inductive assumption (iii),

$$\|w_j(t)\|_{L^2(\mathbb{T}^2)} \leq \delta_j, \quad \forall t \in \mathcal{T}_n, \quad j \geq n.$$

Since $\delta_j \rightarrow 0$, we have that u continuously maps $\mathcal{T}_n \rightarrow L^2(\mathbb{T}^2)$ for every $n \in \mathbb{N}$, which establishes the second item in Remark 1.5. □

2.6. Proof of other main results. Here we sketch the minor modifications needed to prove Theorem 1.6 and Theorem 1.8.

Proof of Theorem 1.6. We can use the same iteration as in Subsection 2.5, but without the inductive items (c) and (iii) which are only related to the kinetic energy.

Take the initial solution u_1 to be v and compute the associated Reynolds stress R_1 . The smallness of $u - v$ in any Sobolev norm is then a consequence (2.5), which we can add to the inductive assumptions for (u_n, R_n) . To conclude, we just delete Step 2 of energy correction from the proof and pass to the limit as before. $\square$

Proof of Theorem 1.8. We use a standard approximation argument. First, we note that Proposition 2.2 holds for the Euler equations, and hence we can construct a weak solution of the 2D Euler equations $u \in C([0, T]; W^{s,1}(\mathbb{T}^2))$ for all $s < 1$ with $\text{curl } u(0) \in L^p(\mathbb{T}^2)$ for some $p > 1$ that does not conserve energy.

Now for each value of the viscosity $0 < \nu < 1$ we mollify the Euler solution u at a lengthscale $\sim \nu$ to obtain a smooth vector field v^ν which we use as the initial solution to the Navier-Stokes-Reynolds system $u_1 := v^\nu$ with an appropriate Reynolds stress R_1 . Omitting the energy correction step again, we can construct a solution of the Navier-Stokes equations u^ν with

$$\|u^\nu - v^\nu\|_{L^\infty(0,T;L^{2-\nu}(\mathbb{T}^2))} \leq \nu.$$

Hence the family of Navier-Stokes solutions u^ν approaches u in the desired spaces:

$$\|u^\nu - u\|_{L^\infty(0,T;L^{p'}(\mathbb{T}^2))} \leq \|u^\nu - v^\nu\|_{L^\infty L^{p'}} + \|u^\nu - v^\nu\|_{L^\infty L^{p'}} \rightarrow 0 \quad \text{as } \nu \rightarrow 0,$$

for all $p' < 2$. $\square$

3. PROOF OF MAIN PROPOSITION: VELOCITY PERTURBATION

In this section, we initiate the proof Proposition 2.2. Typical in any convex integration scheme, the goal is to design a suitable velocity perturbation w so that the new solution

$$u_1 := u + w$$

solves the equation (2.1) with a much smaller Reynolds stress R_1 .

The main objective here is to introduce the necessary preparation to define the velocity perturbation and the associated Reynolds stress. The full estimates claimed in Proposition 2.2 will be verified in the next section.

3.1. Building blocks in space. The velocity perturbation will consist of suitable superpositions of a family of vector fields with coefficients depending smoothly on the given stress error R . We recall a crucial lemma from [DS17] which motivates the construction of our building blocks in the convex integration.

Recall that $\mathcal{S}_+^{2 \times 2}$ denotes the set of positive definite symmetric 2×2 matrices and $\mathbf{e}_k := \frac{k}{|k|}$ for any $k \in \mathbb{Z}^2$. For any matrix $A \in \mathbb{R}^{2 \times 2}$, we use the norm $|A| := \sup_{|x|=1} |Ax|$ for definiteness (the choice is insignificant for our purpose).

Lemma 3.1. *Let $B_{\frac{1}{2}}(\text{Id})$ denote the ball of radius $1/2$ centered at identity Id in $\mathcal{S}_+^{2 \times 2}$. There exists a finite set $\Lambda \subset \mathbb{Z}^2$ and smooth functions $\Gamma_k \in C^\infty(B_{\frac{1}{2}}(\text{Id}))$ for any $k \in \Lambda$ such that*

$$R = \sum_{k \in \Lambda} \Gamma_k^2(R) \mathbf{e}_k \otimes \mathbf{e}_k \quad \text{for all } R \in B_{\frac{1}{2}}(\text{Id}).$$

To utilize Lemma 3.1, we need to choose proper cutoffs functions for the stress error R . Let $\chi : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^+$ be a positive smooth function such that χ is monotone increasing with respect to $|x|$ and

$$\chi(x) = \begin{cases} 2 & \text{if } 0 \leq |x| \leq \frac{1}{2} \|R\|_{L^1(\mathbb{T}^2 \times I)} \\ 2|x| & \text{if } |x| \geq \|R\|_{L^1(\mathbb{T}^2 \times I)}. \end{cases} \quad (3.1)$$

Let $\rho \in C^\infty(\mathbb{T}^2 \times [0, 1])$ be

$$\rho = \chi(R).$$

Then we see that

$$\text{Id} - \frac{R}{\rho} \in B_{\frac{1}{2}}(\text{Id}) \quad \text{for any } (x, t) \in \mathbb{T}^2 \times [0, 1],$$

so we can use it as the arguments in the amplitude functions given by Lemma 3.1. In particular, we have the following useful decomposition

$$\rho \text{Id} - R = \sum_{k \in \Lambda} R_k, \quad (3.2)$$

where the “primitive” stresses $R_k : \mathbb{T}^2 \times [0, 1] \rightarrow \mathcal{S}_+^{2 \times 2}$ are given by

$$R_k := \rho \Gamma_k^2 \left(\text{Id} - \frac{R}{\rho} \right) \mathbf{e}_k \otimes \mathbf{e}_k. \quad (3.3)$$

The main building block of the convex integration scheme is a family of periodic accelerating jet flows $\mathbf{W}_k : \mathbb{T}^2 \times [0, 1] \rightarrow \mathbb{R}^2$. This construction builds on previous work of Buckmaster, Colombo, and Vicol [BCV22] and our temporal intermittency framework used in [CL21, CL22].

We first construct a family of autonomous vector fields W_k and then use intermittent temporal phase shifts $\Phi_k : \mathbb{T}^2 \times [0, 1] \rightarrow \mathbb{T}^2$ to obtain the final jets $\mathbf{W}_k := W_k \circ \Phi_k$.

Theorem 3.2 (Stationary jets). *Let $\Lambda \subset \mathbb{Z}^2$ be a finite set. There exists $\mu_0 > 0$ such that for any $\nu, \mu \in \mathbb{R}$ with $\mu_0 < \nu \leq \mu$ the following holds.*

For every $k \in \Lambda$, there exist smooth vector fields $W_k, W_k^{(c)} \in C_0^\infty(\mathbb{T}^2, \mathbb{R}^2)$ with the following properties.

- (1) *Each $W_k + W_k^{(c)}$ is divergence-free on $\mathbb{T}^2$, and there is a stream function $\Psi_k \in C_0^\infty(\mathbb{T}^2)$ such that*

$$W_k + W_k^{(c)} = \nabla^\perp \Psi_k,$$

and for each $k \in \Lambda$

$$\mathbf{e}_k \parallel W_k \quad \text{and} \quad \mathbf{e}_k \perp W_k^{(c)},$$

where $\mathbf{e}_k = \frac{k}{|k|}$.

- (2) *Each W_k has disjoint support*

$$\text{supp } W_k \cap \text{supp } W_{k'} = \emptyset \quad \text{if } k \neq k'.$$

and for any $k \in \Lambda$, the following identities hold

$$\oint_{\mathbb{T}^2} W_k \otimes W_k = \mathbf{e}_k \otimes \mathbf{e}_k, \quad (3.4)$$

$$\operatorname{div}(W_k \otimes W_k) = \mathbf{e}_k \cdot \nabla |W_k|^2 \mathbf{e}_k. \quad (3.5)$$

(3) For any $1 \leq p \leq \infty$, the estimates

$$\begin{aligned} \|W_k\|_{L^p(\mathbb{T}^2)} + \mu^{-1} \|\nabla W_k\|_{L^p(\mathbb{T}^2)} &\lesssim (\nu\mu)^{\frac{1}{2} - \frac{1}{p}}, \\ \|W_k^{(c)}\|_{L^p(\mathbb{T}^2)} + \mu^{-1} \|\nabla W_k^{(c)}\|_{L^p(\mathbb{T}^2)} &\lesssim \nu\mu^{-1} (\nu\mu)^{\frac{1}{2} - \frac{1}{p}}, \end{aligned} \quad (3.6)$$

and

$$\|\Psi_k\|_{L^p(\mathbb{T}^2)} \lesssim \mu^{-1} (\nu\mu)^{\frac{1}{2} - \frac{1}{p}}, \quad (3.7)$$

holds uniformly in ν, μ .

Proof. We choose a collection of distinct points $p_k \in [0, 1]^2$ for $k \in \Lambda$ and a number $\mu_0 > 0$ such that

$$\bigcup_{k \in \Lambda} B_{\frac{2}{\mu_0}}(p_k) \subset [0, 1]^2,$$

where $B_{\frac{2}{\mu_0}}(p_k)$ denotes the ball of radius $\frac{2}{\mu_0}$ and center p_k and

$$\operatorname{dist}(p_i, p_j) > \frac{2}{\mu_0} \quad \text{if } i \neq j. \quad (3.8)$$

The points p_i will be the centers of W_k and W_k^c .

For $k \in \Lambda$, let us introduce unit vectors $\mathbf{e}_k, \mathbf{e}_k^\perp \in \mathbb{R}^2$ by

$$\mathbf{e}_k = \frac{k}{|k|}, \quad \mathbf{e}_k^\perp = \frac{1}{|k|}(-k_2, k_1), \quad (3.9)$$

and their associated coordinates: for any $x \in \mathbb{R}^2$,

$$\begin{aligned} x_k &= (x - p_k) \cdot \mathbf{e}_k, \\ y_k &= (x - p_k) \cdot \mathbf{e}_k^\perp. \end{aligned}$$

Now we choose compactly supported nontrivial $\varphi, \psi \in C_c^\infty((-\frac{1}{\mu_0}, \frac{1}{\mu_0}))$ and define non-periodic potentials $\tilde{\Psi}_k \in C_c^\infty(\mathbb{R}^2)$ and vector fields $\tilde{W}_k, \tilde{W}_k^{(c)} \in C_c^\infty(\mathbb{R}^2)$

$$\begin{aligned} \tilde{\Psi}_k &= c\mu^{-1}(\nu\mu)^{\frac{1}{2}} \varphi(\nu x_k) \psi(\mu y_k), \\ \tilde{W}_k &= -c(\nu\mu)^{\frac{1}{2}} \varphi(\nu x_k) \psi'(\mu y_k) \mathbf{e}_k, \\ \tilde{W}_k^{(c)} &= c\nu\mu^{-1}(\nu\mu)^{\frac{1}{2}} \varphi'(\nu x_k) \psi(\mu y_k) \mathbf{e}_k^\perp, \end{aligned}$$

where $c > 0$ is a normalizing constant such that (3.4) holds. Periodizing, we obtain

$$\Psi_k(x) = \sum_{n \in \mathbb{Z}^2} \tilde{\Psi}_k(x + n), \quad W_k(x) = \sum_{n \in \mathbb{Z}^2} \tilde{W}_k(x + n), \quad W_k^{(c)}(x) = \sum_{n \in \mathbb{Z}^2} \tilde{W}_k^{(c)}(x + n),$$

which will be used as periodic building blocks, as we can now identify them with corresponding functions on $\mathbb{T}^2$:

$$\Psi_k : \mathbb{T}^2 \rightarrow \mathbb{R}, \quad W_k : \mathbb{T}^2 \rightarrow \mathbb{R}^2, \quad W_k^{(c)} : \mathbb{T}^2 \rightarrow \mathbb{R}^2.$$

Note that

$$\begin{aligned}\nabla^\perp \Psi_k &= -\partial_{y_k} \Psi_k \mathbf{e}_k + \partial_{x_k} \Psi_k \mathbf{e}_k^\perp \\ &= W_k + W_k^{(c)},\end{aligned}$$

and the rest of the conclusions follow trivially by direct computations as well. $\square$

Remark 3.3. *It is clear that the derivative of Ψ_k in the direction $\mathbf{e}_k$ is of order ν (rather than μ for the full gradient):*

$$\|(\mathbf{e}_k \cdot \nabla) \Psi_k\|_{L^r(\mathbb{T}^2)} \lesssim \nu \cdot \mu^{-1} (\nu \mu)^{\frac{1}{2} - \frac{1}{r}}. \quad (3.10)$$

This fact will be used in Lemma 4.8 later.

Apparently, W_k or $W_k + W_k^{(c)}$ are not approximate solutions of the Navier-Stokes (or Euler) equations due to (3.5). This error term can not be absorbed by a pressure gradient due to its anisotropic nature. However, this very identity (3.5) allows for letting W_k travel along the geodesics on $\mathbb{T}^2$ so that (3.5) can be balanced by the acceleration of a small corrector.

Letting W_k move along the geodesics creates one addition problem in 2D: without a suitable discretization of the temporal velocity, different W_k might collide with each other and thus create harmful interactions. We will use a temporal concentration mechanism to get around this issue.

3.2. Phase shift by acceleration. Next, we introduce a simple method to avoid the collision of the support sets of different W_k . This is based on our previous construction of temporal concentration in [CL22]. We concentrate each W_k on disjoint time intervals so that they have disjoint supports in $\mathbb{T}^2 \times [0, 1]$. The size of those time intervals then determines the level of temporal concentration.

Now we turn to the specifics. Let us first choose temporal functions g_k and h_k to oscillate the building blocks W_k intermittently in time. Let $G \in C_c^\infty(0, 1)$ be such that

$$\int_0^1 G^2(t) dt = 1, \quad \int_0^1 G(t) dt = 0. \quad (3.11)$$

For any $\kappa \geq 1$, we define $\tilde{g}_k : [0, 1] \rightarrow \mathbb{R}$ as the 1-periodic extension of $\kappa^{1/2} G(\kappa(t - t_k))$, where t_k are chosen so that g_k have disjoint supports for different k . In other words,

$$\tilde{g}_k(t) = \sum_{n \in \mathbb{Z}} \kappa^{1/2} G(n + \kappa(t - t_k)). \quad (3.12)$$

In the convex integration scheme below, we will also oscillate the velocity perturbation at a large frequency $\sigma \in \mathbb{N}$. So we define

$$g_k(t) = \tilde{g}_k(\sigma t). \quad (3.13)$$

Note that g_k concentrates on different small time intervals, and we have

$$\|g_k\|_{W^{n,p}([0,1])} \lesssim (\sigma \kappa)^n \kappa^{\frac{1}{2} - \frac{1}{p}} \quad \text{for all } 1 \leq p \leq \infty, n \in \mathbb{N}. \quad (3.14)$$

Remark 3.4. *With g_k defined, we will essentially use $g_k W_k$ as the “building blocks” for the convex integration. Due to the temporal concentration of g_k , all $g_k W_k$ have disjoint supports on $\mathbb{T}^2 \times [0, 1]$, and hence there will be no interference between each different W_k .*

For the corrector term that we will be using to balance (3.5), define $h_k : [0, 1] \rightarrow \mathbb{R}$ by

$$h_k(t) = \int_0^{\sigma t} (\tilde{g}_k^2(s) - 1) ds.$$

In view of the zero-mean condition for $g_k^2(t) - 1$, these h_k are σ^{-1} -periodic on $[0, 1]$ and we have

$$\|h_k\|_{L^\infty([0,1])} \leq 1 \quad (3.15)$$

uniformly in κ . Moreover, we have the identity

$$\partial_t (\sigma^{-1} h_k) = g_k^2 - 1, \quad (3.16)$$

which will imply the smallness of the temporal oscillation corrector, cf (3.29).

3.3. Accelerating jets. Let us recall the five parameters used for the building blocks.

- spatial concentrations $\mu \geq 1$ and $\nu \geq 1$: these are reciprocals of the longitudinal and lateral dimension of the building blocks W_k .
- Oscillation $\sigma \in \mathbb{N}$: we use it to oscillate the “building blocks” $g_k W_k$ both in space and in time so that the stress error R is canceled weakly.
- Temporal concentration $\kappa \geq 1$: this parameter models the concentration in time of the perturbation (through g_k), which is crucial for controlling the new stress error emanating from the Laplacian.
- Acceleration $\omega \geq 1$: this parameter represents the acceleration of the flow in the building block, we use it to define the phase shift in (3.18).

To make our notation more compact, let us introduce the phase function $\Phi_k : \mathbb{T}^2 \times [0, 1] \rightarrow \mathbb{T}^2$ defined by

$$\Phi_k : (x, t) \mapsto \sigma x + \phi_k(t) \mathbf{e}_k, \quad (3.17)$$

where $\phi_k(t)$ is defined by the relation

$$\phi_k'(t) = \omega g_k(t). \quad (3.18)$$

Note that due to the zero-mean condition in (3.11), such ϕ_k always exists and for definiteness, we fix one of such choices throughout the construction.

By design, Φ_k is $\sigma^{-1}\mathbb{T}^2$ -periodic in space and σ^{-1} -periodic in time. Also, by definition of the phase shift (3.18), we have the important identities

$$\begin{aligned} \nabla (f \circ \Phi_k) &= \sigma \nabla f \circ \Phi_k, \\ \partial_t (f \circ \Phi_k) &= \phi_k'(\mathbf{e}_k \cdot \nabla f) \circ \Phi_k = \sigma^{-1} \omega g_k \mathbf{e}_k \cdot \nabla (f \circ \Phi_k). \end{aligned} \quad (3.19)$$

Now we will let the stationary flows W_k travel along $\mathbb{T}^2$ in time, relating the velocity of the moving support sets to the intermittent oscillator g_k . More precisely, we define

$$\begin{aligned} \mathbf{W}_k(x, t) &:= W_k \circ \Phi_k = W_k(\sigma x + \phi_k(t) \mathbf{e}_k), \\ \mathbf{W}_k^{(c)}(x, t) &:= W_k^{(c)} \circ \Phi_k = W_k^{(c)}(\sigma x + \phi_k(t) \mathbf{e}_k), \\ \Psi_k(x, t) &:= \Psi_k \circ \Phi_k = \Psi_k(\sigma x + \phi_k(t) \mathbf{e}_k). \end{aligned} \quad (3.20)$$

Hence

$$\sigma^{-1} \nabla^\perp \Psi_k = \mathbf{W}_k + \mathbf{W}_k^{(c)}. \quad (3.21)$$

Also, thanks to (3.19) and (3.5), we have the important identities

$$\partial_t |\mathbf{W}_k|^2 \mathbf{e}_k = \sigma^{-1} \omega g_k \operatorname{div}(\mathbf{W}_k \otimes \mathbf{W}_k), \quad (3.22)$$

and

$$\partial_t \Psi_k = \sigma^{-1} \omega g_k (\mathbf{e}_k \cdot \nabla) \Psi_k, \quad (3.23)$$

3.4. The velocity perturbation. We are in the position to define the velocity perturbation. In summary, the perturbation consists of three parts:

$$w = w^{(p)} + w^{(c)} + w^{(t)}, \quad (3.24)$$

where $w^{(p)}$ is the principle part, accounting for the main contribution in the nonlinear term, $w^{(c)}$ is an incompressibility corrector, rectifying the divergence of $w^{(p)}$, and $w^{(t)}$ is a temporal corrector with zero divergence, balancing the errors introduced by (3.5) and (3.17) below.

Next, we choose a cutoff for the velocity perturbation so that it lives strictly within the interval I . Since the stress error might accumulate near the endpoints of I , the cutoff will be sufficiently sharp so that the “leftover” error is negligible. Let $\theta \in C_c^\infty(\mathbb{R})$ be a smooth temporal cutoff function such that $\|\theta\|_{L^\infty} \leq 1$ and

$$\theta(t) = \begin{cases} 1 & \text{if } \text{dist}(t, I^c) \geq \frac{\delta}{8\|R\|_{L^\infty(\mathbb{T}^2 \times I)}} \\ 0 & \text{if } \text{dist}(t, I^c) \leq \frac{\delta}{16\|R\|_{L^\infty(\mathbb{T}^2 \times I)}}. \end{cases} \quad (3.25)$$

In view of (3.3), the amplitude functions of the perturbation are given by

$$a_k = \theta \rho^{1/2} \Gamma_k \left(\text{Id} - \frac{R}{\rho} \right). \quad (3.26)$$

The principle part of the perturbation consists of superpositions of the building blocks $\mathbf{W}_k$ oscillating with period σ^{-1} on $\mathbb{T}^2$ and traveling with a velocity $\phi'_k(t)$.

$$w^{(p)}(x, t) := \sum_{k \in \Lambda} a_k(x, t) g_k(t) \mathbf{W}_k. \quad (3.27)$$

In what follows, we will omit the set of the summation indexes $k \in \Lambda$ so that

$$w^{(p)} = \sum_k a_k g_k \mathbf{W}_k.$$

Note that (3.27) is not divergence-free. To fix this, we introduce a divergence-free corrector

$$w^{(c)}(x, t) := \sum_k a_k(x, t) g_k(t) \mathbf{W}_k^{(c)} + \sigma^{-1} \nabla^\perp a_k(x, t) g_k(t) \Psi_k.$$

Then, thanks to (3.21), we have

$$\begin{aligned} w^{(p)} + w^{(c)} &= \sigma^{-1} \sum_k g_k(t) [a_k \nabla^\perp \Psi_k + \nabla^\perp a_k \Psi_k] \\ &= \sigma^{-1} \sum_k \nabla^\perp [a_k g_k \Psi_k]. \end{aligned} \quad (3.28)$$

Finally, we define a temporal corrector $w^{(t)}$. The goal of this corrector $w^{(t)}$ is two-fold: (i) to balance the high temporal frequency part of the stress error; and (ii) to balance the error in the interaction due to the acceleration in the phase shifts Φ_k .

To introduce these correctors, we first recall the well-known Leray projection on $\mathbb{T}^2$.

Definition 3.5 (Leray projection). *Let $v \in C^\infty(\mathbb{T}^2, \mathbb{R}^2)$ be a smooth vector field. Define the operator $\mathcal{Q}$ as*

$$\mathcal{Q}v := \nabla f + \oint_{\mathbb{T}^2} v,$$

where $f \in C^\infty(\mathbb{T}^2)$ is the unique smooth zero-mean solution of

$$\Delta f = \text{div } v, \quad x \in \mathbb{T}^2.$$

Furthermore, let $\mathcal{P} = \text{Id} - \mathcal{Q}$ be the Leray projection onto divergence-free vector fields with zero mean.

The temporal corrector $w^{(t)}$ is defined as

$$w^{(t)} = w^{(o)} + w^{(a)}, \quad (3.29)$$

where $w^{(o)}$ is the temporal oscillation corrector

$$w^{(o)}(x, t) := -\sigma^{-1} \theta^2 \mathcal{P} \sum_{k \in \Lambda} h_k(t) \operatorname{div} R_k, \quad (3.30)$$

and $w^{(a)}$ is the acceleration corrector

$$w^{(a)}(x, t) := -\omega^{-1} \sigma \mathcal{P} \sum_{k \in \Lambda} a_k(x, t)^2 g_k(t) |\mathbf{W}_k|^2 \mathbf{e}_k. \quad (3.31)$$

Let us say a few words about the corrector $w^{(t)}$. The corrector $w^{(t)}$ will be used to balance part of the new stress error through its temporal derivative $\partial_t w^{(t)}$, but the roles of each part $w^{(o)}$ and $w^{(a)}$ are very different. To see the leading order temporal derivative of $w^{(a)}$, thanks to fact that $\phi'(t) = \omega g(t)$ and due to (3.22), we have the heuristic

$$\partial_t w^{(a)} = -\mathcal{P} \sum_k a_k^2 g_k^2 \operatorname{div} (\mathbf{W}_k \otimes \mathbf{W}_k) + \text{lower order terms},$$

which is needed for a cancellation in the oscillation error. On the other hand, thanks to (3.16),

$$\partial_t w^{(o)} = -\theta^2 \mathcal{P} \left(\sum_{k \in \Lambda} (g_k^2 - 1) \operatorname{div} R_k \right) + \text{lower order terms},$$

which we use to cancel a highly time oscillating remainder of the Reynolds stress, specific to the use of temporal intermittency in our scheme. We compute this in the following lemma.

Lemma 3.6. *For any $(x, t) \in \mathbb{T}^2 \times [0, 1]$, there holds*

$$\sum_{k \in \Lambda} a_k^2 g_k^2 \int_{\mathbb{T}^2} \mathbf{W}_k \otimes \mathbf{W}_k dx = \theta^2 (\rho \operatorname{Id} - R) + \theta^2 \sum_{k \in \Lambda} (g_k^2 - 1) R_k,$$

where R_k is defined by (3.3).

Proof. First we note that by Theorem 3.2, for any t ,

$$\int_{\mathbb{T}^2} \mathbf{W}_k(x, t) \otimes \mathbf{W}_k(x, t) dx = \int_{\mathbb{T}^2} W_k(x) \otimes W_k(x) dx = \mathbf{e}_k \otimes \mathbf{e}_k.$$

So by Lemma 3.1, a direct computation gives

$$\begin{aligned} \sum_{k \in \Lambda} a_k^2 g_k^2 \mathbf{e}_k \otimes \mathbf{e}_k &= \theta^2 \rho \sum_{k \in \Lambda} g_k^2 \Gamma_k^2 \left(\operatorname{Id} - \frac{R}{\rho} \right) \mathbf{e}_k \otimes \mathbf{e}_k \\ &= \theta^2 \sum_{k \in \Lambda} R_k + \theta^2 \sum_{k \in \Lambda} (g_k^2 - 1) R_k \\ &= \theta^2 (\rho \operatorname{Id} - R) + \theta^2 \sum_{k \in \Lambda} (g_k^2 - 1) R_k. \end{aligned}$$

□

3.5. The new Reynolds stress. In this subsection, our goal is to design a suitable stress tensor $R_1 : \mathbb{T}^2 \times [0, 1] \rightarrow \mathcal{S}_0^{2 \times 2}$ such that the pair (u_1, R_1) is a smooth solution of 2.1 for a suitable smooth pressure p_1 .

We will define the new Reynolds stress by

$$R_1 = R_{\text{lin}} + R_{\text{cor}} + R_{\text{osc}},$$

such that

$$\operatorname{div} R_{\text{osc}} = \partial_t w^{(t)} + \operatorname{div}(w^{(p)} \otimes w^{(p)} + R) + \nabla P, \quad (3.32)$$

$$\operatorname{div} R_{\text{cor}} = \operatorname{div} \left((w^{(c)} + w^{(t)}) \otimes w + w^{(p)} \otimes (w^{(c)} + w^{(t)}) \right), \quad (3.33)$$

$$\operatorname{div} R_{\text{lin}} = \partial_t (w^{(p)} + w^{(c)}) - \Delta u + \operatorname{div}(u \otimes w + w \otimes u). \quad (3.34)$$

To this end, we will use the antidivergence operator $\mathcal{R} : C^\infty(\mathbb{T}^2, \mathbb{R}^2) \rightarrow C^\infty(\mathbb{T}^2, \mathcal{S}_0^{2 \times 2})$ defined in Appendix C. True to its name, this antidivergence operator $\mathcal{R}$ satisfies

$$\operatorname{div}(\mathcal{R}v) = v - \oint_{\mathbb{T}^2} v \quad \text{for any } v \in C^\infty(\mathbb{T}^2, \mathbb{R}^2). \quad (3.35)$$

Since (3.32)-(3.34) are time-dependent, it should be understood that the antidivergence $\mathcal{R}$ is applied for each time slice $t \in [0, 1]$, and the resulting stresses are smooth on $\mathbb{T}^2 \times [0, 1]$.

In addition, we will also use a bilinear antidivergence operator $\mathcal{B}$ in Appendix C, which satisfies

$$\operatorname{div}(\mathcal{B}(v, A)) = vA - \oint vA, \quad \text{for } v \in C^\infty(\mathbb{T}^2, \mathbb{R}^2) \text{ and } A \in C_0^\infty(\mathbb{T}^2, \mathbb{R}^{2 \times 2}), \quad (3.36)$$

where we denote $vA = v_j A_{ij}$ instead of the usual Av . In what follows the matrix A is often symmetric, so there will be no confusion in writing vA versus Av .

This bilinear antidivergence $\mathcal{B}$ has the advantage of gaining derivative from the second argument A when it only has high spatial frequencies. This has a flavor of the classical stationary phase and has been a useful tool for many estimates in convex integration [BMS21].

3.6. Computing the oscillation error. Our first step of solving for R_1 is to derive R_{osc} . Let us compute the nonlinear term $\operatorname{div}(w^{(p)} \otimes w^{(p)} + R)$. Due to (3.12) and (3.13) (see Remark 3.4), we have

$$\operatorname{supp} g_k \mathbf{W}_k \cap \operatorname{supp} g_{k'} \mathbf{W}_{k'} = \emptyset \quad \text{if } k \neq k'.$$

It follows that we only have self-interactions of the accelerating jets:

$$\operatorname{div}(w^{(p)} \otimes w^{(p)} + R) = \operatorname{div} \left[\sum_{k \in \Lambda} a_k^2 g_k^2 \mathbf{W}_k \otimes \mathbf{W}_k + R \right].$$

We use Lemma 3.6 to remove the leading order interaction given by the spatial mean of $\mathbf{W}_k \otimes \mathbf{W}_k$ and obtain

$$\begin{aligned} \operatorname{div}(w^{(p)} \otimes w^{(p)} + R) &= \operatorname{div} \sum_k a_k^2 g_k^2 (\mathbf{W}_k \otimes \mathbf{W}_k - \oint \mathbf{W}_k \otimes \mathbf{W}_k) + \nabla(\theta^2 \rho) + (1 - \theta^2) \operatorname{div} R \\ &\quad + \theta^2 \sum_k (g_k^2 - 1) \operatorname{div} R_k. \end{aligned} \quad (3.37)$$

Putting $\partial_t w^{(t)}$ into action, using $\mathcal{P} = \text{Id} - \mathcal{Q}$ and the product rule, we separate the terms into four different groups:

$$\begin{aligned}
& \partial_t w^{(t)} + \text{div}(w^{(p)} \otimes w^{(p)} + R) \\
&= \underbrace{\sum_k g_k^2 \nabla(a_k^2) \cdot (\mathbf{W}_k \otimes \mathbf{W}_k - \oint \mathbf{W}_k \otimes \mathbf{W}_k) + \mathcal{Q} \sum_k a_k^2 g_k^2 \text{div}(\mathbf{W}_k \otimes \mathbf{W}_k)}_{=: E_1} \\
&+ \underbrace{\partial_t w^{(a)} + \mathcal{P} \sum_k a_k^2 g_k^2 \text{div}(\mathbf{W}_k \otimes \mathbf{W}_k)}_{=: E_2} \\
&+ \underbrace{\partial_t w^{(o)} + \theta^2 \sum_k (g_k^2 - 1) \text{div} R_k}_{=: E_3} \\
&+ \underbrace{\nabla(\theta^2 \rho) + (1 - \theta^2) \text{div} R}_{=: E_4}.
\end{aligned} \tag{3.38}$$

Before we move on to the analysis of each term E_i , let us point out the following simple consequence of (3.35) and the definition of $\mathcal{P}$:

$$\mathcal{P}f = \text{div}(Rf) - \nabla(\Delta^{-1} \text{div} f) \quad \text{for any } f \in C^\infty(\mathbb{T}^2, \mathbb{R}^2), \tag{3.39}$$

where we note that f does not need to be mean-free. We will use (3.36) and (3.39) to re-write each E_i as the sum of the divergence of a stress and the gradient of a pressure.

Analysis of E_1 :

Recall that by Definition 3.5,

$$\mathcal{Q}v = \nabla \Delta^{-1} \text{div} v + \oint v, \tag{3.40}$$

and consequently $\oint \mathcal{Q}v = \oint v$, which implies that E_1 has zero spatial mean. Therefore, if we define

$$R_{\text{osc},x} = \sum_{k \in \Lambda} g_k^2 \mathcal{B}(\nabla(a_k^2), \mathbf{W}_k \otimes \mathbf{W}_k - \oint \mathbf{W}_k \otimes \mathbf{W}_k), \tag{3.41}$$

and the pressure

$$p_1 = \Delta^{-1} \text{div} \sum_{k \in \Lambda} a_k^2 g_k^2 \text{div}(\mathbf{W}_k \otimes \mathbf{W}_k),$$

then by (3.36) and (3.40) we have

$$E_1 = \text{div} R_{\text{osc},x} + \nabla p_1, \tag{3.42}$$

where the cancellation of the means follow from the fact that $\oint E_1 = 0$.

Analysis of E_2 :

We use (3.31) and (3.22) to compute

$$\partial_t w^{(a)} = -\mathcal{P} \sum_k a_k^2 g_k^2 \text{div}(\mathbf{W}_k \otimes \mathbf{W}_k) - \omega^{-1} \sigma \mathcal{P} \sum_k \partial_t(a_k^2 g_k) |\mathbf{W}_k|^2 \mathbf{e}_k.$$

which implies that

$$E_2 = -\omega^{-1} \sigma \mathcal{P} \sum_k \partial_t(a_k^2 g_k) |\mathbf{W}_k|^2 \mathbf{e}_k. \tag{3.43}$$

In view of (3.39), we define an approximation stress

$$R_{\text{osc},a} = -\omega^{-1}\sigma \sum_{k \in \Lambda} \mathcal{R} \left(\partial_t (a_k^2 g_k) |\mathbf{W}_k|^2 \mathbf{e}_k \right) \quad (3.44)$$

and a pressure term

$$p_2 = \omega^{-1}\sigma \Delta^{-1} \operatorname{div} \sum_{k \in \Lambda} \partial_t (a_k^2 g_k) |\mathbf{W}_k|^2 \mathbf{e}_k,$$

such that

$$E_2 = \operatorname{div} R_{\text{osc},a} + \nabla p_2.$$

Analysis of E_3 :

For this term, let us first compute the time derivative of $w^{(o)}$ (3.30) using (3.16):

$$\partial_t w^{(o)} = -\theta^2 \mathcal{P} \left(\sum_{k \in \Lambda} (g_k^2 - 1) \operatorname{div} R_k \right) - \sigma^{-1} \theta^2 \mathcal{P} \sum_{k \in \Lambda} h_k \operatorname{div} \partial_t R_k.$$

Thus

$$\begin{aligned} E_3 &= \partial_t w^{(o)} + \theta^2 \sum_k (g_k^2 - 1) \operatorname{div} R_k \\ &= \theta^2 \mathcal{Q} \sum_k (g_k^2 - 1) \operatorname{div} R_k - \sigma^{-1} \theta^2 \mathcal{P} \sum_{k \in \Lambda} h_k \operatorname{div} \partial_t R_k. \end{aligned}$$

Using definition of $\mathcal{Q}$ and (3.39) once again, we define

$$p_3 = \theta^2 \sum_k (g_k^2 - 1) \Delta^{-1} \operatorname{div} \operatorname{div} R_k + \sigma^{-1} \theta^2 \sum_{k \in \Lambda} h_k \Delta^{-1} \operatorname{div} \operatorname{div} \partial_t R_k \quad (3.45)$$

and

$$R_{\text{osc},t} = -\sigma^{-1} \theta^2 \mathcal{R} \sum_{k \in \Lambda} h_k \operatorname{div} \partial_t R_k \quad (3.46)$$

to obtain

$$E_3 = \operatorname{div} R_{\text{osc},t} + \nabla p_3.$$

Analysis of E_4 :

We just define

$$p_4 = \theta^2 \rho, \quad (3.47)$$

and

$$R_{\text{rem}} = (1 - \theta^2) R.$$

Then

$$E_4 = \operatorname{div} R_{\text{rem}} + \nabla p_4.$$

Combining all the terms, we have thus proven:

Lemma 3.7. *Define the oscillation error R_{osc}*

$$R_{\text{osc}} = R_{\text{osc},x} + R_{\text{osc},t} + R_{\text{osc},a} + R_{\text{rem}}.$$

and the pressure $P := -p_1 - p_2 - p_3 - p_4$. Then

$$\partial_t w^{(t)} + \operatorname{div}(w^{(p)} \otimes w^{(p)} + R) + \nabla P = \operatorname{div} R_{\text{osc}}.$$

3.7. Finalizing the new solution. Finally, we define the correction error and the linear error in the usual way:

$$R_{\text{cor}} = \mathcal{R} \operatorname{div} \left((w^{(c)} + w^{(t)}) \otimes w + w^{(p)} \otimes (w^{(c)} + w^{(t)}) \right), \quad (3.48)$$

and

$$R_{\text{lin}} = \mathcal{R} \left(\partial_t (w^{(p)} + w^{(c)}) - \Delta u + \operatorname{div}(u \otimes w) + \operatorname{div}(w \otimes u) \right). \quad (3.49)$$

Now we can conclude the construction of the new solution to the Navier-Stokes-Reynolds system.

Lemma 3.8. *Define the new Reynolds stress by*

$$R_1 = R_{\text{lin}} + R_{\text{cor}} + R_{\text{osc}},$$

and the new pressure by

$$p_1 = p + P.$$

Then (u_1, R_1) is a solution to (2.1),

$$\partial_t u_1 - \Delta u_1 + \operatorname{div}(u_1 \otimes u_1) + \nabla p_1 = \operatorname{div} R_1.$$

Proof. Since (u, R) solves (2.1) with pressure p , a direct computation gives

$$\begin{aligned} & \partial_t u_1 - \Delta u_1 + \operatorname{div}(u_1 \otimes u_1) + \nabla p_1 \\ &= \partial_t u - \Delta u + \operatorname{div}(u \otimes u) + \nabla p \\ & \quad + \partial_t w - \Delta w + \operatorname{div}(u \otimes w) + \operatorname{div}(w \otimes u) + \operatorname{div}(w \otimes w) + \nabla P \\ &= \operatorname{div} R + \partial_t w - \Delta w + \operatorname{div}(u \otimes w) + \operatorname{div}(w \otimes u) + \operatorname{div}(w \otimes w) + \nabla P. \end{aligned}$$

Now by Lemma 3.7 and definitions (3.48), (3.49) we conclude that (u_1, R_1) solves (2.1). $\square$

4. ESTIMATES FOR THE VELOCITY AND THE STRESS ERROR

We will show in this section that the velocity perturbation w and the new Reynolds stress R_1 derived in Section 3.5 verify the claimed properties in Proposition 2.2.

Throughout this section, we denote by C_u a large constant depending on the previous solution (u, R) . The value of C_u addresses a different norm of (u, R) from line to line, but most importantly, it does not depend on any of the parameters $\sigma, \nu, \mu, \kappa, \omega$ introduced in the velocity perturbation.

4.1. Concentration and oscillation: choice of parameters. First of all, we assume that $p > 1$ as the proposition is the strongest when p is close to 2. We now specify all the parameters appearing in the velocity perturbation and the constant $r > 1$ entering Proposition 2.2.

For a small parameter $0 < \gamma < 1/14$ depending only on $p < 2$ as in the Lemma 4.1 below, we choose

- (1) Oscillation $\sigma \in \mathbb{N}$:

$$\sigma = \lceil \lambda^{2\gamma} \rceil.$$

Without loss of generality, we only consider values of λ such that $\sigma = \lambda^{2\gamma}$ in what follows.

- (2) Concentration κ, ν, μ :

$$\begin{aligned} \kappa &= \lambda^{16\gamma} \\ \nu &= \lambda^{1-8\gamma} \\ \mu &= \lambda. \end{aligned}$$

- (3) Acceleration ω :

$$\omega = \lambda.$$

For $r > 1$ to be fixed in the following lemma, define $q = q(r) > 2$ as

$$\frac{1}{r} = \frac{1}{2} + \frac{1}{q}. \quad (4.1)$$

Now we show that there is an admissible choice for $r > 1$ when $\gamma \ll 1$.

Lemma 4.1. *For all $\gamma > 0$ sufficiently small depending only on $1 < p < 2$, there exists $1 < r \leq p$ such that for any $\lambda \geq 2$ with $\lambda^{2\gamma} \in \mathbb{N}$, there hold*

$$\kappa^{\frac{1}{2}}(\nu\mu)^{\frac{1}{2}-\frac{1}{p}} \leq \lambda^{-\gamma} \quad (w^{(p)} \in L^\infty L^p) \quad (4.2)$$

$$\omega^{-1}\sigma\kappa^{\frac{1}{2}}(\nu\mu)^{1-\frac{1}{p}} \leq \lambda^{-\gamma} \quad (w^{(a)} \in L^\infty L^p) \quad (4.3)$$

$$\omega^{-1}\sigma(\nu\mu)^{\frac{3}{2}-\frac{1}{r}} \leq \lambda^{-\gamma} \quad (w^{(a)} \in L^2 L^q) \quad (4.4)$$

$$\nu\mu^{-1}(\nu\mu)^{1-\frac{1}{r}} \leq \lambda^{-\gamma} \quad (w^{(c)} \in L^2 L^q) \quad (4.5)$$

$$\sigma\kappa^{-\frac{1}{2}}\mu(\nu\mu)^{\frac{1}{2}-\frac{1}{r}} \leq \lambda^{-\gamma} \quad \text{Laplacian error for } w^{(p)} \quad (4.6)$$

$$\omega^{-1}\sigma^2\kappa^{-\frac{1}{2}}\mu(\nu\mu)^{1-\frac{1}{r}} \leq \lambda^{-\gamma} \quad \text{Laplacian error for } w^{(a)} \quad (4.7)$$

$$(\kappa^{\frac{1}{2}} + \omega\sigma^{-1}\nu)\mu^{-1}(\nu\mu)^{\frac{1}{2}-\frac{1}{r}} \leq \lambda^{-\gamma} \quad \text{Acceleration error} \quad (4.8)$$

$$\sigma^{-1}(\nu\mu)^{1-\frac{1}{r}} \leq \lambda^{-\gamma} \quad \text{Oscillation error for } w^{(p)} \quad (4.9)$$

$$\omega^{-1}\sigma^2\kappa^{\frac{1}{2}}(\nu\mu)^{1-\frac{1}{r}} \leq \lambda^{-\gamma} \quad \text{Oscillation error for } w^{(a)} \quad (4.10)$$

Proof. First, notice that (4.2) and (4.3) hold automatically for any $\gamma > 0$ small enough. Once γ is chosen according to (4.2) and (4.3), we can find $1 < r \leq p$, close enough to 1, so that the rest of the conditions hold as well.

Indeed, expressing the left hand sides of the conditions as powers of λ , we notice that all the exponents are continuous in r . So the desired $r > 1$ exists because with $r = 1$, the left hand side of each condition (4.4)–(4.10) is less than or equal to $\lambda^{-2\gamma}$. □

4.2. Estimates on velocity perturbation.

Lemma 4.2. *The coefficients a_k are smooth on $\mathbb{T}^2 \times [0, 1]$ and*

$$\|a_k\|_{C^n(\mathbb{T}^2 \times [0, 1])} \leq C_{u,n} \quad \text{for any } n \in \mathbb{N},$$

where $C_{u,n}$ are constants independent of λ . In addition, the bound

$$\|a_k(t)\|_{L^2(\mathbb{T}^2)} \lesssim \theta(t) \left(\int_{\mathbb{T}^2} \rho(x, t) dx \right)^{\frac{1}{2}}$$

holds uniformly for all time $t \in [0, 1]$.

Proof. The smoothness of a_k follows from definition (3.26). Since the implicit constant is allowed to depend on (u, R) , the first bound follows immediately from the definition of a_k .

The second bound follows from the definition of a_k as well:

$$\|a_k(t)\|_{L^2(\mathbb{T}^2)} \lesssim \theta \left(\int_{\mathbb{T}^2} \rho \Gamma_k^2 \left(\text{Id} - \frac{R}{\rho} \right) dx \right)^{\frac{1}{2}} \lesssim \theta \left(\int_{\mathbb{T}^2} \rho dx \right)^{\frac{1}{2}}.$$

□

Proposition 4.3. *The principle part $w^{(p)}$ satisfies*

$$\|w^{(p)}\|_{L^2(\mathbb{T}^2 \times [0,1])} \lesssim \|R\|_{L^1(\mathbb{T}^2 \times I)}^{\frac{1}{2}} + C_u \sigma^{-\frac{1}{2}},$$

and

$$\|w^{(p)}\|_{L^\infty(0,1;L^p(\mathbb{T}^2))} \lesssim C_u \lambda^{-\gamma}.$$

In particular, for sufficiently large λ ,

$$\begin{aligned} \|w^{(p)}\|_{L^2(\mathbb{T}^2 \times [0,1])} &\lesssim \|R\|_{L^1(\mathbb{T}^2 \times I)}^{\frac{1}{2}}, \\ \|w^{(p)}\|_{L^\infty(0,1;L^p(\mathbb{T}^2))} &\leq \frac{\delta}{4}. \end{aligned}$$

Proof. We first show the estimate for $L_{t,x}^2$ and then for $L^\infty L^p$.

$L_{t,x}^2$ estimate:

For each $t \in [0, 1]$, we take L^2 norm in space and use Lemma B.1 to obtain

$$\|w^{(p)}(t)\|_{L^2(\mathbb{T}^2)} \lesssim \sum_k g_k(\|a_k(t)\|_{L^2} \|\mathbf{W}_k(t)\|_{L^2} + \sigma^{-\frac{1}{2}} C_u),$$

where we used the fact that $\mathbf{W}_k$ is $\sigma^{-1}\mathbb{T}^2$ -periodic in space. Recall that $\|\mathbf{W}_k\|_{L_t^\infty L^2} \lesssim 1$. Then using Lemma 4.2 and taking L^2 norm in time gives

$$\|w^{(p)}\|_{L^2(\mathbb{T}^2 \times [0,1])} \lesssim \sum_k \left(\int_0^1 g_k^2 \theta^2(t) \int_{\mathbb{T}^2} \rho(x, t) dx dt \right)^{\frac{1}{2}} + C_u \sigma^{-\frac{1}{2}}. \quad (4.11)$$

Notice that

$$t \mapsto \theta^2(t) \int_{\mathbb{T}^2} \rho(x, t) dx$$

is a smooth map on $[0, 1]$ and $g_k(t)$ is σ^{-1} -periodic. Thus, we may apply Lemma B.1 once again (with $p = 1$) to obtain that

$$\int_0^1 g_k^2 \theta^2(t) \int_{\mathbb{T}^2} \rho(x, t) dx dt \lesssim \|R\|_{L^1(\mathbb{T}^2 \times I)} + C_u \sigma^{-1}, \quad (4.12)$$

where we used the fact that $\int g_k^2 = 1$, $\text{supp } \theta \subset I$, and the bound

$$\int_{\mathbb{T}^2} \rho(x, t) dx \lesssim \|R(t)\|_{L^1(\mathbb{T}^2)} + \|R\|_{L^1(\mathbb{T}^2 \times I)}.$$

Thus, combining (4.11) and (4.12) gives

$$\|w^{(p)}\|_{L^2(\mathbb{T}^2 \times [0,1])} \lesssim \|R\|_{L^1(\mathbb{T}^2 \times I)}^{\frac{1}{2}} + C_u \sigma^{-\frac{1}{2}}.$$

$L_t^\infty L^p$ estimate:

For each $t \in [0, 1]$, we take L^p norm in space and then use Hölder's inequality to obtain

$$\|w^{(p)}(t)\|_{L^p(\mathbb{T}^2)} \lesssim \sum_k \|a_k(t)\|_{L^\infty} |g_k(t)| \|\mathbf{W}_k(t)\|_{L^p}.$$

Now, thanks to Lemma 4.2, (3.14), and Theorem 3.2, taking L^∞ in time, we get

$$\begin{aligned} \|w^{(p)}\|_{L^\infty(0,1;L^p(\mathbb{T}^2))} &\lesssim \sum_k \|a_k\|_{L_{x,t}^\infty} \|g_k\|_{L^\infty} (\nu\mu)^{\frac{1}{2}-\frac{1}{p}} \\ &\leq C_u \kappa^{\frac{1}{2}} (\nu\mu)^{\frac{1}{2}-\frac{1}{p}} \\ &\lesssim C_u \lambda^{-\gamma}, \end{aligned}$$

where we used (4.2) for the last inequality. $\square$

The last estimate of the velocity perturbation concerns the temporal corrector $w^{(t)}$.

Proposition 4.4. *The temporal corrector $w^{(t)} = w^{(o)} + w^{(a)}$ satisfies*

$$\begin{aligned} \|w^{(t)}\|_{L^2(0,1;L^q(\mathbb{T}^2))} &\lesssim C_u \lambda^{-\gamma}, \\ \|w^{(t)}\|_{L^\infty(0,1;L^p(\mathbb{T}^2))} &\lesssim C_u \lambda^{-\gamma}, \\ \|w^{(o)}\|_{L^\infty(0,1;W^{\frac{1}{\delta},\infty}(\mathbb{T}^2))} &\lesssim C_u \lambda^{-\gamma}, \end{aligned}$$

where $q > 2$ is as defined in (4.1).

In particular, for all sufficiently large λ ,

$$\begin{aligned} \|w^{(t)}\|_{L^2(\mathbb{T}^2 \times [0,1])} &\leq \|R\|_{L^1(\mathbb{T}^2 \times I)}, \\ \|w^{(t)}\|_{L^\infty(0,1;L^p(\mathbb{T}^2))} &\leq \frac{\delta}{4}, \\ \|w^{(o)}\|_{L^\infty(0,1;W^{\frac{1}{\delta},\infty}(\mathbb{T}^2))} &\leq \delta. \end{aligned}$$

Proof. Recall that the temporal corrector $w^{(t)}$ consists of the temporal oscillation corrector $w^{(o)}$ (3.30) and the acceleration corrector $w^{(a)}$ (3.31):

$$\begin{aligned} w^{(o)} &= -\sigma^{-1} \theta^2 \mathcal{P} \sum_{k \in \Lambda} h_k \operatorname{div} R_k, \\ w^{(a)} &= -\omega^{-1} \sigma \mathcal{P} \sum_{k \in \Lambda} a_k^2 g_k |\mathbf{W}_k|^2 \mathbf{e}_k. \end{aligned}$$

The estimate of $w^{(o)}$ follows from a cheap Hölder estimate:

$$\|w^{(o)}\|_{L^\infty(0,1;W^{\frac{1}{\sigma},\infty}(\mathbb{T}^2))} \lesssim \sigma^{-1} \sum_{k \in \Lambda} \|h_k\|_{L^\infty([0,1])} \|\operatorname{div} R_k\|_{L^\infty(I;W^{\frac{1}{\delta},\infty}(\mathbb{T}^2))} \lesssim C_u \lambda^{-\gamma}.$$

Now we turn to the $L_t^2 L^q$ and $L_t^\infty L^p$ estimates for the acceleration corrector $w^{(a)}$. For each $t \in [0, 1]$ we have

$$\|w^{(a)}(t)\|_{L^p(\mathbb{T}^2)} \lesssim \omega^{-1} \sigma \sum_{k \in \Lambda} \left\| a_k^2 g_k |\mathbf{W}_k|^2 \mathbf{e}_k \right\|_{L^p(\mathbb{T}^2)},$$

since $\mathcal{P}$ is bounded on $L^p(\mathbb{T}^2)$. Using Hölder's inequality we get

$$\begin{aligned} \|w^{(a)}(t)\|_{L^p(\mathbb{T}^2)} &\lesssim \omega^{-1} \sigma \sum_k |g_k(t)| \|a_k\|_{L_{x,t}^\infty}^2 \|\mathbf{W}_k\|^2 \mathbf{e}_k\|_{L^\infty(0,1;L^p(\mathbb{T}^2))} \\ &\lesssim C_u \omega^{-1} \sigma \sum_k |g_k(t)| \|\mathbf{W}_k\|_{L^\infty(0,1;L^{2p}(\mathbb{T}^2))}^2 \\ &\lesssim C_u \omega^{-1} \sigma (\nu\mu)^{1-\frac{1}{p}} \sum_k |g_k(t)|, \end{aligned}$$

where we used Theorem 3.2 at the last step. Now by (3.14) we have

$$\|w^{(a)}\|_{L^2(0,1;L^q(\mathbb{T}^2))} \lesssim C_u \omega^{-1} \sigma (\nu\mu)^{1-\frac{1}{q}} \lesssim C_u \lambda^{-\gamma},$$

due to (4.4) in Lemma 4.1, and

$$\|w^{(a)}\|_{L^\infty(0,1;L^p(\mathbb{T}^2))} \lesssim C_u \omega^{-1} \sigma (\nu\mu)^{1-\frac{1}{p}} \kappa^{\frac{1}{2}} \lesssim C_u \lambda^{-\gamma},$$

due to (4.3), which are the desired estimates. $\square$

Proposition 4.5. *Then divergence-free corrector $w^{(c)}$ satisfies*

$$\|w^{(c)}\|_{L^2(0,1;L^q(\mathbb{T}^2))} \lesssim C_u \lambda^{-\gamma},$$

and

$$\|w^{(c)}\|_{L^\infty(0,1;L^p(\mathbb{T}^2))} \lesssim C_u \lambda^{-\gamma}.$$

In particular, for sufficiently large λ ,

$$\begin{aligned} \|w^{(c)}\|_{L^2(\mathbb{T}^2 \times [0,1])} &\leq \|R\|_{L^1(\mathbb{T}^2 \times I)}, \\ \|w^{(c)}\|_{L^\infty(0,1;L^p(\mathbb{T}^2))} &\leq \frac{\delta}{4}. \end{aligned}$$

Proof. Using Theorem 3.2, for any $1 \leq p \leq \infty$ and $t \in [0, 1]$, there holds

$$\begin{aligned} \|w^{(c)}(t)\|_{L^p(\mathbb{T}^2)} &\lesssim \sum_k \|a_k(t)\|_{L^\infty} |g_k(t)| \|\mathbf{W}_k^{(c)}\|_{L_t^\infty L^p} + \sigma^{-1} \|\nabla^\perp a_k(t)\|_{L^\infty} |g_k(t)| \|\Psi_k\|_{L_t^\infty L^p} \\ &\lesssim \left[\nu\mu^{-1} (\nu\mu)^{\frac{1}{2}-\frac{1}{p}} + \mu^{-1} (\nu\mu)^{\frac{1}{2}-\frac{1}{p}} \right] \sum_k \|a_k\|_{C_{x,t}^1} |g_k(t)|. \end{aligned}$$

Recall that $\frac{1}{r} = \frac{1}{2} + \frac{1}{q}$. Then, using Lemma 4.2 and (3.14), we have

$$\|w^{(c)}\|_{L^2(0,1;L^q(\mathbb{T}^2))} \lesssim C_u \nu\mu^{-1} (\nu\mu)^{1-\frac{1}{r}} \lesssim C_u \lambda^{-\gamma},$$

due to (4.5), and for the specific $L_t^\infty L^p$ norm,

$$\|w^{(c)}\|_{L^\infty(0,1;L^p(\mathbb{T}^2))} \lesssim C_u \kappa^{\frac{1}{2}} \nu\mu^{-1} (\nu\mu)^{\frac{1}{2}-\frac{1}{p}} \lesssim C_u \lambda^{-\gamma},$$

due to (4.2). $\square$

4.3. Sobolev estimates and the exceptional set E . Finally we show the Sobolev estimates in $L^\infty(0, 1; W^{s_p, 1}(\mathbb{T}^2))$ and specify the exceptional set E with the estimates (2.4)–(2.6) in the main proposition.

The proof of the Sobolev estimate follows closely the estimation we have done so far, therefore we only sketch the outline.

Corollary 4.6. *Let $s_p := 1 - 40\gamma$. Then for all sufficiently large λ , there holds*

$$\|w\|_{L^\infty(0, 1; W^{s_p, 1}(\mathbb{T}^2))} \leq \delta.$$

Proof. From Lemma 4.1 and the previous estimations, it is clear that the estimate of w is majorized by $w^{(p)}$. By a Sobolev interpolation, for any $t \in [0, 1]$,

$$\|w^{(p)}(t)\|_{W^{s_p, 1}(\mathbb{T}^2)} \lesssim \|w^{(p)}(t)\|_{L^1(\mathbb{T}^2)}^{1-s_p} \|w^{(p)}(t)\|_{W^{1, 1}(\mathbb{T}^2)}^{s_p} \leq C_u (\sigma\mu)^{s_p} \kappa^{1/2} (\nu\mu)^{-1/2}.$$

The conclusion would follow if the exponent is negative. Indeed, using the choice of parameters in Section 4.1, we see that

$$(\sigma\mu)^{s_p} \kappa^{1/2} (\nu\mu)^{-1/2} \leq \lambda^{s_p(2\gamma+1)-1+20\gamma},$$

which implies

$$\|w^{(p)}(t)\|_{W^{s_p, 1}(\mathbb{T}^2)} \leq \delta, \quad t \in [0, 1],$$

for all sufficiently large λ . □

For the exceptional set E , recall from the definition of g_k , (3.12) and (3.13), that

$$\text{supp } g_k \subset \bigcup_{n \in \mathbb{Z}, k \in \Lambda} (t_k, t_k + (\kappa\sigma)^{-1}) + n\sigma^{-1}.$$

Note that these open intervals are disjoint by definition. So we just define E as

$$E := [0, 1] \cap \bigcup_{n \in \mathbb{Z}, k \in \Lambda} (t_k, t_k + (\kappa\sigma)^{-1}) + n\sigma^{-1}. \quad (4.13)$$

Then for any $t > 0$, we have

$$\mathcal{L}([0, t] \cap E) \leq \sum_{n, k} (\kappa\sigma)^{-1},$$

where n, k are such that $t_k + n\sigma^{-1} \in [0, t]$. Since $\Lambda \ni k$ is finite and fixed, the total number of pairs of n, k is bounded by $Ct\sigma$, where C is independent of t and κ . It follows that

$$\mathcal{L}([0, t] \cap E) \leq Ct\kappa^{-1}$$

which implies (2.4) provided λ is sufficiently large.

Next, we show (2.5) and (2.6), namely the Sobolev and energy level estimate away from E .

Lemma 4.7. *For any $t \in [0, 1] \setminus E$*

$$\begin{aligned} \|w(t)\|_{W^{\frac{1}{8}, \infty}(\mathbb{T}^2)} &\leq \delta, \\ \left| \|u_1(t)\|_{L^2(\mathbb{T}^2)}^2 - \|u(t)\|_{L^2(\mathbb{T}^2)}^2 \right| &\leq \delta^2, \end{aligned}$$

provided that λ is sufficiently large.

Proof. By the definition of E , we see that

$$w^{(p)} = w^{(c)} = w^{(a)} = 0 \quad \text{for all } t \in [0, 1] \setminus E.$$

So, for all $t \in [0, 1] \setminus E$ we have

$$\begin{aligned} \left| \|u_1(t)\|_{L^2(\mathbb{T}^2)}^2 - \|u(t)\|_{L^2(\mathbb{T}^2)}^2 \right| &\leq 2 \left| \langle w^{(o)}, u \rangle \right| + \|w^{(o)}(t)\|_{L^2(\mathbb{T}^2)}^2, \\ \|w(t)\|_{W^{\frac{1}{8}, \infty}(\mathbb{T}^2)} &\leq \|w^{(o)}(t)\|_{W^{\frac{1}{8}, \infty}(\mathbb{T}^2)}, \end{aligned}$$

and the conclusion follows from Proposition 4.4. $\square$

4.4. Estimates on the new Reynolds stress. The last step of the proof is to estimate R_1 . We will proceed with the decomposition in Lemma 3.8 and show that for any sufficiently large λ , each part of the stress R_1 is less than $\frac{\delta}{4}$.

It is also worth noting that we estimate all the errors on $\mathbb{T}^2 \times [0, 1]$ except the oscillation error, for which we only look at $\mathbb{T}^2 \times I$.

4.4.1. *Linear error.*

Lemma 4.8. *For sufficiently large λ ,*

$$\|R_{\text{lin}}\|_{L^1(0,1;L^r(\mathbb{T}^2))} \leq \frac{\delta}{4}.$$

Proof. We split the linear error into three parts:

$$R_{\text{lin}} = \underbrace{-\mathcal{R}(\Delta w)}_{:=R_{\text{lap}}} + \underbrace{\mathcal{R}(\partial_t(w^{(p)} + w^{(c)}))}_{:=R_{\text{acc}}} + \underbrace{\mathcal{R}(\text{div}(w \otimes u) + \text{div}(u \otimes w))}_{:=R_{\text{dri}}}.$$

The estimate of the Laplacian error R_{lap} relies crucially on the temporal concentration κ , the R_{acc} error uses the stream functions Ψ_k so that we gain a factor of $(\sigma\mu)^{-1}$, and the drifts R_{dri} can be handle by standard estimates.

Estimate of R_{lap} :

We recall that $w = w^{(p)} + w^{(c)} + w^{(t)}$ and, using (C.2), decompose the Laplacian error into two parts:

$$\begin{aligned} \|R_{\text{lap}}\|_{L^1(0,1;L^r(\mathbb{T}^2))} &\leq 2\|\nabla w\|_{L^1(0,1;L^r(\mathbb{T}^2))} \\ &\leq 2\|\nabla(w^{(p)} + w^{(c)})\|_{L^1(0,1;L^r(\mathbb{T}^2))} + 2\|\nabla w^{(t)}\|_{L^1(0,1;L^r(\mathbb{T}^2))}. \end{aligned}$$

Now recall from (3.28) that

$$w^{(p)} + w^{(c)} = \sigma^{-1} \sum_k \nabla^\perp [a_k g_k \Psi_k], \quad (4.14)$$

and hence we can use estimates in Lemma 4.2, (3.14), and Theorem 3.2 to obtain

$$\begin{aligned} \|\nabla(w^{(p)} + w^{(c)})\|_{L^1(0,1;L^r(\mathbb{T}^2))} &= \sigma^{-1} \left\| \sum_k \nabla \nabla^\perp [a_k g_k \Psi_k] \right\|_{L^1(0,1;L^r(\mathbb{T}^2))} \\ &\lesssim \sigma^{-1} \sum_k \|a_k\|_{C_{x,t}^2} \|g_k\|_{L^1} \|\Psi_k\|_{L^\infty(0,1;W^{2,r}(\mathbb{T}^2))} \\ &\lesssim C_u \sigma^{-1} \cdot \kappa^{-\frac{1}{2}} \cdot (\sigma\mu)^2 \mu^{-1} (\nu\mu)^{\frac{1}{2} - \frac{1}{r}}. \end{aligned} \quad (4.15)$$

As for $\nabla w^{(t)}$, the oscillation part (3.30) simply satisfies

$$\|\nabla w^{(o)}\|_{L^\infty(0,1;L^r(\mathbb{T}^2)\times[0,1])} \lesssim \sigma^{-1} \sum_{k \in \Lambda} \|h_k\|_{L^\infty([0,1])} \|\nabla \operatorname{div} R_k\|_{L^\infty(0,1;L^r(\mathbb{T}^2))} \lesssim C_u \lambda^{-\gamma},$$

due to the choice of parameter σ , and the acceleration part (3.31) enjoys the bound

$$\begin{aligned} \|\nabla w^{(a)}\|_{L^1(0,1;L^r(\mathbb{T}^2))} &\lesssim \omega^{-1} \sigma \sum_{k \in \Lambda} \|a_k^2\|_{C_{x,t}^1} \|g_k\|_{L^1} \|\nabla |\mathbf{W}_k|^2 \mathbf{e}_k\|_{L^\infty(0,1;L^r(\mathbb{T}^2))} \\ &\lesssim C_u \omega^{-1} \sigma \cdot \kappa^{-\frac{1}{2}} \cdot (\sigma \mu)(\nu \mu)^{1-\frac{1}{r}} \\ &\lesssim C_u \lambda^{-\gamma}, \end{aligned}$$

where we used Lemma 4.2, (3.14), Theorem 3.2, and Condition (4.7) for the last inequality.

Estimate of R_{acc} :

For the acceleration part of the linear error, taking time derivative of (4.14) and using identity (3.23), we obtain

$$\begin{aligned} \partial_t(w^{(p)} + w^{(c)}) &= \sigma^{-1} \sum_k \nabla^\perp [\partial_t(a_k g_k) \Psi_k] + \sigma^{-1} \sum_k \nabla^\perp [a_k g_k \partial_t \Psi_k] \\ &= \sigma^{-1} \sum_k \nabla^\perp [\partial_t(a_k g_k) \Psi_k] + \sigma^{-2} \omega \sum_k \nabla^\perp [a_k g_k^2 (\mathbf{e}_k \cdot \nabla) \Psi_k]. \end{aligned}$$

Now thanks to the fact that $\mathcal{R} \nabla^\perp$ is a Calderón-Zygmund operator on $\mathbb{T}^2$ (see (B.2)), we can use Lemma 4.2, (3.14), and Theorem 3.2 to estimate the first term:

$$\begin{aligned} \sigma^{-1} \left\| \sum_k \mathcal{R} \nabla^\perp [\partial_t(a_k g_k) \Psi_k] \right\|_{L^1(0,1;L^r)} &\lesssim \sigma^{-1} \sum_k \|a_k\|_{C_{x,t}^1} \|g_k\|_{W^{1,1}} \|\Psi_k\|_{L_t^\infty L^r} \\ &\lesssim C_u \sigma^{-1} (\sigma \kappa) \kappa^{-\frac{1}{2}} \cdot \mu^{-1} (\nu \mu)^{\frac{1}{2}-\frac{1}{r}}. \end{aligned}$$

As for the second term, we recall that the derivative of Ψ_k in the direction $\mathbf{e}_k$ is of order $\sigma \nu$ (rather than $\sigma \mu$ for the full gradient, see (3.10)):

$$\|(\mathbf{e}_k \cdot \nabla) \Psi_k\|_{L_t^\infty L^r} \lesssim \sigma \nu \cdot \mu^{-1} (\nu \mu)^{\frac{1}{2}-\frac{1}{r}}.$$

This estimate together with Lemma 4.2 and (3.14) implies

$$\begin{aligned} \sigma^{-2} \omega \left\| \sum_k \mathcal{R} \nabla^\perp [a_k g_k^2 (\mathbf{e}_k \cdot \nabla) \Psi_k] \right\|_{L^1(0,1;L^r)} &\lesssim \sigma^{-2} \omega \sum_k \|a_k\|_{L_{x,t}^\infty} \|g_k^2\|_{L^1} \|(\mathbf{e}_k \cdot \nabla) \Psi_k\|_{L_t^\infty L^r} \\ &\lesssim C_u \sigma^{-2} \omega \cdot \sigma \nu \mu^{-1} (\nu \mu)^{\frac{1}{2}-\frac{1}{r}}. \end{aligned}$$

Due to (4.8), combining both terms we get

$$\begin{aligned} \|R_{\text{acc}}\|_{L^1(0,1;L^r(\mathbb{T}^2))} &\lesssim C_u \left(\kappa^{\frac{1}{2}} \mu^{-1} (\nu \mu)^{\frac{1}{2}-\frac{1}{r}} + \omega \sigma^{-1} \nu \mu^{-1} (\nu \mu)^{\frac{1}{2}-\frac{1}{r}} \right) \\ &\lesssim C_u \lambda^{-\gamma}. \end{aligned}$$

Estimate of R_{dri} :

Using the L^r boundedness of $\mathcal{R}$,

$$\begin{aligned} \|R_{\text{dri}}\|_{L^1(0,1;L^r(\mathbb{T}^2))} &= \|\mathcal{R}(\operatorname{div}(w \otimes u) + \operatorname{div}(u \otimes w))\|_{L^1(0,1;L^r(\mathbb{T}^2))} \\ &\lesssim \|w \otimes u\|_{L^1(0,1;L^r(\mathbb{T}^2))} \\ &\lesssim \|w\|_{L^1(0,1;L^r(\mathbb{T}^2))} \|u\|_{L_{x,t}^\infty} \\ &\lesssim C_u \lambda^{-\gamma}, \end{aligned}$$

by $L^\infty L^p$ estimates on w in Section 4.2, and the fact that $r \leq p$.

Combining the above estimates, we can conclude that for all sufficiently large λ , there holds

$$\|R_{\text{lin}}\|_{L^1(0,1;L^r(\mathbb{T}^2))} \leq \frac{\delta}{4}.$$

□

4.4.2. Correction error.

Lemma 4.9. *For sufficiently large λ ,*

$$\|R_{\text{cor}}\|_{L^1(0,1;L^r(\mathbb{T}^2))} \leq \frac{\delta}{4}.$$

Proof. Using Hölder's inequality, the fact that $\mathcal{R} \operatorname{div}$ is bounded in L^r , and (4.1), we obtain

$$\begin{aligned} \|R_{\text{cor}}\|_{L^1(0,1;L^r(\mathbb{T}^2))} &\lesssim \|(w^{(c)} + w^{(t)}) \otimes w\|_{L_t^1 L_x^r} + \|w^{(p)} \otimes (w^{(c)} + w^{(t)})\|_{L_t^1 L_x^r} \\ &\lesssim \left(\|w^{(c)}\|_{L_t^2 L_x^q} + \|w^{(t)}\|_{L_t^2 L_x^q} \right) \|w\|_{L_{x,t}^2} \\ &\quad + \|w^{(p)}\|_{L_{x,t}^2} \left(\|w^{(c)}\|_{L_t^2 L_x^q} + \|w^{(t)}\|_{L_t^2 L_x^q} \right). \end{aligned}$$

Recall that $w^{(t)} = w^{(o)} + w^{(a)}$. Then by Propositions 4.3, 4.4, and 4.5 we have

$$\begin{aligned} \|w\|_{L_{x,t}^2} &\leq \|w^{(p)}\|_{L_{x,t}^2} + \|w^{(c)}\|_{L_{x,t}^2} + \|w^{(t)}\|_{L_{x,t}^2} \\ &\lesssim \|R\|_{L^1(\mathbb{T}^2 \times [0,1])}^{\frac{1}{2}}, \end{aligned}$$

as well as

$$\|w^{(c)}\|_{L_t^2 L_x^q} + \|w^{(t)}\|_{L_t^2 L_x^q} \leq C_u \lambda^{-\gamma}.$$

Thus for all sufficiently large λ , we can conclude that

$$\|R_{\text{cor}}\|_{L^1(0,1;L^r(\mathbb{T}^2))} \leq \frac{\delta}{4}.$$

□

4.4.3. *Oscillation error.* As we mentioned before, we estimate the oscillation error on $\mathbb{T}^2 \times I$ since the perturbations are only designed to balance the old stress on $\mathbb{T}^2 \times I$, cf. the remainder R_{rem} below.

Lemma 4.10. *For sufficiently large λ ,*

$$\|R_{\text{osc}}\|_{L^1(I;L^r(\mathbb{T}^2))} \leq \frac{\delta}{4}.$$

Proof. We will use the decomposition from Lemma 3.7

$$R_{\text{osc}} = R_{\text{osc},x} + R_{\text{osc},t} + R_{\text{osc},a} + R_{\text{rem}}$$

where we recall

$$\begin{aligned} R_{\text{osc},x} &= g_k^2 \sum_{k \in \Lambda} \mathcal{B}(\nabla(a_k^2), \mathbf{W}_k \otimes \mathbf{W}_k - \oint \mathbf{W}_k \otimes \mathbf{W}_k), \\ R_{\text{osc},t} &= -\sigma^{-1} \theta^2 \mathcal{R} \sum_{k \in \Lambda} h_k \operatorname{div} \partial_t R_k \\ R_{\text{osc},a} &= -\omega^{-1} \sigma \sum_{k \in \Lambda} \mathcal{R}(\partial_t(a_k^2 g_k) | \mathbf{W}_k|^2 \mathbf{e}_k) \\ R_{\text{rem}} &= (1 - \theta^2) R. \end{aligned}$$

Estimate of $R_{\text{osc},x}$:

To reduce notations, denote $\mathbf{T}_k : \mathbb{T}^2 \times [0, 1] \rightarrow \mathbb{R}^{2 \times 2}$ by

$$\mathbf{T}_k = \mathbf{W}_k \otimes \mathbf{W}_k - \oint \mathbf{W}_k \otimes \mathbf{W}_k,$$

so that

$$R_{\text{osc},x}(x, t) = g_k^2 \sum_{k \in \Lambda} \mathcal{B}(\nabla(a_k^2), \mathbf{T}_k).$$

Using Theorem C.3 and the fact that $\mathbf{T}_k$ has zero spatial mean, we can estimate the L^r norm of $R_{\text{osc},x}$ at each time $t \in [0, 1]$ as follows.

$$\begin{aligned} \|R_{\text{osc},x}(t)\|_{L^r(\mathbb{T}^2)} &= g_k^2 \left\| \sum_k \mathcal{B}(\nabla(a_k^2), \mathbf{T}_k) \right\|_{L^r} \\ &\lesssim g_k^2 \sum_k \|\nabla(a_k^2)\|_{C^1} \|\mathcal{R}(\mathbf{T}_k)\|_{L^r} \\ &\lesssim \sigma^{-1} g_k^2 \sum_k \|\nabla(a_k^2)\|_{C^1} \|\mathbf{T}_k\|_{L^r}, \end{aligned}$$

where the last inequality used the fact that $\mathbf{T}_k$ has zero spatial mean. Thanks to Theorem 3.2, for any $k \in \Lambda$

$$\|\mathbf{T}_k\|_{L_t^\infty L^r} \leq \|\mathbf{W}_k \otimes \mathbf{W}_k\|_{L_t^\infty L^r} \lesssim \|\mathbf{W}_k\|_{L_t^\infty L^{2r}}^2 \lesssim (\nu\mu)^{1-\frac{1}{r}}$$

Therefore, integrating in time and using Lemma 4.2, (3.14), and (4.9), we obtain

$$\begin{aligned} \|R_{\text{osc},x}\|_{L^1(0,1;L^r(\mathbb{T}^2))} &\lesssim C_u \sigma^{-1} (\nu\mu)^{1-\frac{1}{r}} \\ &\lesssim C_u \lambda^{-\gamma}. \end{aligned}$$

Estimate of $R_{\text{osc},t}$:

Using the bound on h_k (3.15), we infer

$$\begin{aligned} \|R_{\text{osc},t}\|_{L^1(0,1;L^r(\mathbb{T}^2))} &\leq \left\| \sigma^{-1} \sum_k h_k \operatorname{div} \partial_t R_k \right\|_{L^1(0,1;L^r(\mathbb{T}^2))} \\ &\lesssim \sigma^{-1} \left\| \sum_k h_k \right\|_{L^1} C_u \\ &\leq C_u \sigma^{-1}. \end{aligned}$$

Estimate of $R_{\text{osc},a}$:

Thanks to Theorem 3.2 ,

$$\begin{aligned}
\|R_{\text{osc},a}\|_{L^1(0,1;L^r(\mathbb{T}^2))} &= \omega^{-1}\sigma \left\| \sum_k \mathcal{R}(\partial_t(a_k^2 g_k) |\mathbf{W}_k|^2 \mathbf{e}_k) \right\|_{L^1(0,1;L^r(\mathbb{T}^2))} \\
&\lesssim \omega^{-1}\sigma \sum_k \|\partial_t(a_k^2 g_k)\|_{L^1(0,1;C^1(\mathbb{T}^2))} \|(|\mathbf{W}_k|^2 \mathbf{e}_k)\|_{L_t^\infty L^r} \\
&= \omega^{-1}\sigma \sum_k \|\partial_t(a_k^2 g_k)\|_{L^1(0,1;C^1(\mathbb{T}^2))} \|\mathbf{W}_k\|_{L_t^\infty L^{2r}}^2 \\
&\lesssim \omega^{-1}\sigma(\nu\mu)^{1-\frac{1}{r}} \sum_k \|\partial_t(a_k^2 g_k)\|_{L^1(0,1;C^1(\mathbb{T}^2))} .
\end{aligned}$$

Using the product rule,

$$\partial_t(a_k^2 g_k) = (\partial_t a_k^2)g_k + a_k^2 g'_k,$$

thanks to Lemma 4.2 and (3.14) we obtain

$$\begin{aligned}
\|\partial_t(a_k^2 g_k)\|_{L^1(0,1;C^1(\mathbb{T}^2))} &\lesssim C_u (\|g_k\|_{L^1} + \|g'_k\|_{L^1}) \\
&\lesssim C_u (\kappa^{-\frac{1}{2}} + \sigma\kappa^{\frac{1}{2}}).
\end{aligned} \tag{4.16}$$

Hence

$$\|R_{\text{osc},a}\|_{L^1(0,1;L^r(\mathbb{T}^2))} \lesssim C_u \omega^{-1}(\nu\mu)^{1-\frac{1}{r}} \cdot \sigma^2 \kappa^{\frac{1}{2}} \lesssim C_u \lambda^{-\gamma},$$

due to (4.10).

Estimate of R_{rem} :

Finally, the estimate of R_{rem} follows from the definition of the cutoff θ as in (3.25):

$$\begin{aligned}
\|R_{\text{rem}}\|_{L^1(I;L^r(\mathbb{T}^2))} &= \|(1-\theta^2)R\|_{L^1(I;L^r(\mathbb{T}^2))} \\
&\leq \int_{I \cap \{\theta \neq 1\}} \|R(t)\|_{L^\infty(\mathbb{T}^2)} dt \\
&\leq |\{t \in I : \theta \neq 1\}| \|R\|_{L^\infty(\mathbb{T}^2 \times I)} \\
&\leq \delta/8.
\end{aligned}$$

Combining this estimate with

$$\|R_{\text{osc},x}\|_{L^1(0,1;L^r(\mathbb{T}^2))} + \|R_{\text{osc},t}\|_{L^1(0,1;L^r(\mathbb{T}^2))} + \|R_{\text{osc},a}\|_{L^1(0,1;L^r(\mathbb{T}^2))} \lesssim C_u \lambda^{-\gamma},$$

we conclude that the desired bound

$$\|R_{\text{osc}}\|_{L^1(I;L^r(\mathbb{T}^2))} \leq \frac{\delta}{4}$$

holds for all λ large enough. □

APPENDIX A. VANISHING VISCOSITY LIMIT IN $L^\infty L^2$

Here, we adopt a proof from [CFLS16] to our settings and show that vanishing viscosity solution of the 2D Euler equations with initial vorticity in L^p , $p > 1$, conserve the kinetic energy.

Theorem A.1. *Let $u \in C([0, T]; W^{1,1}(\mathbb{T}^2))$ be a weak solution of the 2D Euler equations with $\text{curl } u(0) \in L^p(\mathbb{T}^2)$ for some $p > 1$, such that there is a family $\{u^\nu\}$ of weak solutions to the 2D Navier-Stokes equations with*

$$\|u^\nu - u\|_{L^\infty(0,T;L^2(\mathbb{T}^2))} \rightarrow 0 \quad \text{as } \nu \rightarrow 0.$$

Then $u(t)$ conserves the kinetic energy.

Proof. Thanks to the Sobolev embedding $W^{1,1}(\mathbb{T}^2) \subset L^2(\mathbb{T}^2)$, we have that $u \in C([0, T]; L^2(\mathbb{T}^2))$, and hence for any $\varepsilon > 0$, there exists a decomposition $u^\nu = u_1^\nu + u_2^\nu$ such that

$$\|u_1^\nu\|_{L_t^\infty L^2} \leq \varepsilon \quad \text{and} \quad u_2^\nu \in L_{t,x}^\infty,$$

for ν small enough. Therefore, by the previous theorem, for all ν small enough, u^ν is a Leray-Hopf solution of the 2D NSE and is smooth for $t > 0$.

Without loss of generality, assume that $\omega_0 \notin L^2$, as otherwise the result is trivial. Note that

$$\frac{d}{dt} \|\omega^\nu\|_{L^2}^2 = -2\nu \|\nabla \omega^\nu\|_{L^2}^2 \lesssim -\nu \|\omega_0\|_{L^p}^{\frac{2p}{p-2}} \|\omega^\nu\|_{L^2}^{\frac{4}{2-p}}.$$

Therefore

$$\|\omega^\nu(t)\|_{L^2}^2 \lesssim (\nu t)^{\frac{p-2}{p}}.$$

Hence

$$0 \geq \|u^\nu(t)\|_{L^2}^2 - \|u^\nu(0)\|_{L^2}^2 \geq -c(\nu t)^{\frac{2p-2}{p}}.$$

Taking a limit as $\nu \rightarrow 0$, this concludes the proof. $\square$

APPENDIX B. SOME TECHNICAL INEQUALITIES

B.1. Calderon-Zygmund operators on $\mathbb{T}^d$. For $1 < p < \infty$, the classical Calderon-Zygmund estimates hold on $\mathbb{R}^d$:

$$\|\nabla^2 f\|_{L^p(\mathbb{R}^d)} \lesssim \|\Delta f\|_{L^p(\mathbb{R}^d)}.$$

It is standard to transfer the estimates on $\mathbb{R}^d$ to $\mathbb{T}^d$:

$$\|\nabla^2 f\|_{L^p(\mathbb{T}^d)} \lesssim \|\Delta f\|_{L^p(\mathbb{T}^d)}. \quad (\text{B.1})$$

In particular, (B.1) implies the bounds used in the paper:

$$\|\mathcal{R}\nabla f\|_{L^p(\mathbb{T}^2)} \lesssim \|f\|_{L^p(\mathbb{T}^2)} \quad \text{for any } f \in C^\infty(\mathbb{T}^2), \quad (\text{B.2})$$

where $\mathcal{R}$ is the antidivergence operator defined in the next section.

B.2. Improved Hölder's inequality on $\mathbb{T}^d$. We recall the following result due to Modena and Székelyhidi [MS18, Lemma 2.1], which extends the first type of such result [BV19, Lemma 3.7].

Lemma B.1. *Let $d \geq 2$, $r \in [1, \infty]$, and $a, f : \mathbb{T}^d \rightarrow \mathbb{R}$ be smooth functions. Then for every $\sigma \in \mathbb{N}$,*

$$\left| \|af(\sigma \cdot)\|_r - \|a\|_r \|f\|_r \right| \lesssim \sigma^{-\frac{1}{r}} \|a\|_{C^1} \|f\|_r. \quad (\text{B.3})$$

This result is used to control the energy of the perturbations in Section 4. Note that the error term on the right-hand side can be made arbitrarily small by increasing the oscillation σ .

APPENDIX C. ANTIDIVERGENCE OPERATORS

For any $f \in C^\infty(\mathbb{T}^2)$, there exists a $v \in C_0^\infty(\mathbb{T}^2)$ such that

$$\Delta v = f - \oint_{\mathbb{T}^2} f.$$

And we denote this solution v by $\Delta^{-1}f$. Note that if $f \in C_0^\infty(\mathbb{T}^2)$, then by rescaling we have

$$\Delta^{-1}(f(\sigma \cdot)) = \sigma^{-2}v(\sigma \cdot) \quad \text{for } \sigma \in \mathbb{N}.$$

C.1. Tensor-valued antidergence $\mathcal{R}$. We recall the following antidergence operator $\mathcal{R}$ introduced in [DLS13].

Definition C.1. $\mathcal{R} : C^\infty(\mathbb{T}^2, \mathbb{R}^2) \rightarrow C^\infty(\mathbb{T}^2, \mathcal{S}_0^{2 \times 2})$ is defined by

$$(\mathcal{R}v)_{ij} = \mathcal{R}_{ijk}v_k \quad (\text{C.1})$$

where

$$\mathcal{R}_{ijk} = -\Delta^{-1}\partial_k\delta_{ij} + \Delta^{-1}\partial_i\delta_{jk} + \Delta^{-1}\partial_j\delta_{ik}.$$

It is clear that $\mathcal{R}$ is well-defined since $\mathcal{R}_{ijk}$ is symmetric in i, j and taking the trace gives

$$\text{Tr } \mathcal{R}v = -2\Delta^{-1}\partial_k v_k + \Delta^{-1}\partial_k v_k + \Delta^{-1}\partial_k v_k = 0.$$

By a direct computation, one can also show that

$$\text{div}(\mathcal{R}v) = v - \oint_{\mathbb{T}^2} v \quad \text{for any } v \in C^\infty(\mathbb{T}^2, \mathbb{R}^2)$$

and

$$\mathcal{R}\Delta v = \nabla v + \nabla v^T \quad \text{for any divergence-free } v \in C^\infty(\mathbb{T}^2, \mathbb{R}^2). \quad (\text{C.2})$$

The antidergence operator $\mathcal{R}$ is bounded on $L^p(\mathbb{T}^2)$ for any $1 \leq p \leq \infty$ [MS18].

Theorem C.2. Let $1 \leq p \leq \infty$. For any vector field $f \in C_0^\infty(\mathbb{T}^2, \mathbb{R}^2)$, there holds

$$\|\mathcal{R}f\|_{L^p(\mathbb{T}^2)} \lesssim \|f\|_{L^p(\mathbb{T}^2)}.$$

In particular, if $f \in C_0^\infty(\mathbb{T}^2, \mathbb{R}^2)$, then

$$\|\mathcal{R}f(\sigma \cdot)\|_{L^p(\mathbb{T}^2)} \lesssim \sigma^{-1}\|f\|_{L^p(\mathbb{T}^2)} \quad \text{for any } \sigma \in \mathbb{N}.$$

C.2. Bilinear antidergence $\mathcal{B}$. We can also introduce the bilinear version $\mathcal{B} : C^\infty(\mathbb{T}^2, \mathbb{R}^2) \times C^\infty(\mathbb{T}^2, \mathbb{R}^{2 \times 2}) \rightarrow C^\infty(\mathbb{T}^2, \mathcal{S}_0^{2 \times 2})$ of $\mathcal{R}$.

Let

$$(\mathcal{B}(v, A))_{ij} = v_l \mathcal{R}_{ijk} A_{lk} - \mathcal{R}(\partial_i v_l \mathcal{R}_{ijk} A_{lk})$$

or by a slight abuse of notations

$$\mathcal{B}(v, A) = v\mathcal{R}A - \mathcal{R}(\nabla v \mathcal{R}A).$$

This bilinear antidergence $\mathcal{B}$ allows us to gain derivative when the later argument has zero mean and a small period.

Theorem C.3. Let $1 \leq p \leq \infty$. For any $v \in C^\infty(\mathbb{T}^2, \mathbb{R}^2)$ and $A \in C_0^\infty(\mathbb{T}^2, \mathbb{R}^{2 \times 2})$,

$$\text{div}(\mathcal{B}(v, A)) = vA - \oint_{\mathbb{T}^2} vA, \quad (\text{C.3})$$

and

$$\|\mathcal{B}(v, A)\|_{L^p(\mathbb{T}^2)} \lesssim \|v\|_{C^1(\mathbb{T}^2)} \|\mathcal{R}A\|_{L^p(\mathbb{T}^2)}.$$

Proof. A direct computation gives

$$\begin{aligned} \text{div}(\mathcal{B}(v, A)) &= v_l \partial_j \mathcal{R}_{ijk} A_{lk} + \partial_j v_l \mathcal{R}_{ijk} A_{lk} - \text{div } \mathcal{R}(\partial_i v_l \mathcal{R}_{ijk} A_{lk}) \\ &= v_l A_{il} + \oint \partial_i v_l \mathcal{R}_{ijk} A_{lk} \end{aligned}$$

where we have used the fact that A has zero mean and $\mathcal{R}$ is symmetric.

Integrating by parts, we have

$$\oint \partial_i v_l \mathcal{R}_{ijk} A_{lk} = - \oint v_l \partial_i \mathcal{R}_{ijk} A_{lk} = - \oint v_l A_{lj},$$

which implies that

$$\operatorname{div}(\mathcal{B}(v, A)) = vA - \oint vA.$$

The second estimate follows immediately from the definition of $\mathcal{B}$ and Theorem C.2. $\square$

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REFERENCES

- [BA94] Matania Ben-Artzi. Global solutions of two-dimensional Navier-Stokes and Euler equations. *Arch. Rational Mech. Anal.*, 128(4):329–358, 1994.
- [BBV20] Rajendra Beekie, Tristan Buckmaster, and Vlad Vicol. Weak solutions of ideal MHD which do not conserve magnetic helicity. *Ann. PDE*, 6(1):Paper No. 1, 40, 2020.
- [BCD11] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.
- [BCV22] Tristan Buckmaster, Maria Colombo, and Vlad Vicol. Wild solutions of the Navier-Stokes equations whose singular sets in time have Hausdorff dimension strictly less than 1. *J. Eur. Math. Soc. (JEMS)*, 24(9):3333–3378, 2022.
- [BDLS15] Tristan Buckmaster, Camillo De Lellis, Philip Isett, and László Székelyhidi, Jr. Anomalous dissipation for $1/5$ -Hölder Euler flows. *Ann. of Math. (2)*, 182(1):127–172, 2015.
- [BDLS16] Tristan Buckmaster, Camillo De Lellis, and László Székelyhidi, Jr. Dissipative Euler flows with Onsager-critical spatial regularity. *Comm. Pure Appl. Math.*, 69(9):1613–1670, 2016.
- [BEP85] G. Benfatto, R. Esposito, and M. Pulvirenti. Planar Navier-Stokes flow for singular initial data. *Nonlinear Anal.*, 9(6):533–545, 1985.
- [BLJV18] T. Buckmaster, C. De Lellis, L. Székelyhidi Jr., and V. Vicol. Onsager’s conjecture for admissible weak solutions. *Comm. Pure Appl. Math.*, to appear, 2018.
- [BMNV21] Tristan Buckmaster, Nader Masmoudi, Matthew Novack, and Vlad Vicol. Non-conservative $H^{\frac{1}{2}-}$ weak solutions of the incompressible 3d euler equations. *arXiv:2101.09278*, 2021.
- [BMS21] Jan Burczak, Stefano Modena, and László Székelyhidi. Non uniqueness of power-law flows. *Comm. Math. Phys.*, 388(1):199–243, 2021.
- [BP08] Jean Bourgain and Nataša Pavlović. Ill-posedness of the Navier-Stokes equations in a critical space in 3D. *J. Funct. Anal.*, 255(9):2233–2247, 2008.
- [BSV19] Tristan Buckmaster, Steve Shkoller, and Vlad Vicol. Nonuniqueness of weak solutions to the SQG equation. *Comm. Pure Appl. Math.*, 72(9):1809–1874, 2019.
- [BV19] Tristan Buckmaster and Vlad Vicol. Nonuniqueness of weak solutions to the Navier-Stokes equation. *Ann. of Math. (2)*, 189(1):101–144, 2019.
- [BV21] Tristan Buckmaster and Vlad Vicol. Convex integration constructions in hydrodynamics. *Bull. Amer. Math. Soc. (N.S.)*, 58(1):1–44, 2021.
- [Cal90] Calixto P. Calderón. Existence of weak solutions for the Navier-Stokes equations with initial data in L^p . *Trans. Amer. Math. Soc.*, 318(1):179–200, 1990.
- [CCFS08] A. Cheskidov, P. Constantin, S. Friedlander, and R. Shvydkoy. Energy conservation and Onsager’s conjecture for the Euler equations. *Nonlinearity*, 21(6):1233–1252, 2008.
- [CCS20] Gennaro Ciampa, Gianluca Crippa, and Stefano Spirito. Weak solutions obtained by the vortex method for the 2D Euler equations are Lagrangian and conserve the energy. *J. Nonlinear Sci.*, 30(6):2787–2820, 2020.
- [CDLDR18] Maria Colombo, Camillo De Lellis, and Luigi De Rosa. Ill-posedness of Leray solutions for the hypodissipative Navier-Stokes equations. *Comm. Math. Phys.*, 362(2):659–688, 2018.
- [CFLS16] A. Cheskidov, M. C. Lopes Filho, H. J. Nussenzveig Lopes, and R. Shvydkoy. Energy conservation in two-dimensional incompressible ideal fluids. *Comm. Math. Phys.*, 348(1):129–143, 2016.

- [CG06] Jean-Yves Chemin and Isabelle Gallagher. On the global wellposedness of the 3-D Navier-Stokes equations with large initial data. *Ann. Sci. École Norm. Sup. (4)*, 39(4):679–698, 2006.
- [CGP11] Jean-Yves Chemin, Isabelle Gallagher, and Marius Paicu. Global regularity for some classes of large solutions to the Navier-Stokes equations. *Ann. of Math. (2)*, 173(2):983–1012, 2011.
- [CL21] Alexey Cheskidov and Xiaoyutao Luo. Nonuniqueness of weak solutions for the transport equation at critical space regularity. *Ann. PDE*, 7(1):Paper No. 2, 45, 2021.
- [CL22] Alexey Cheskidov and Xiaoyutao Luo. Sharp nonuniqueness for the Navier-Stokes equations. *Invent. Math.*, 229(3):987–1054, 2022.
- [Cot86] Georges-Henri Cottet. Équations de Navier-Stokes dans le plan avec tourbillon initial mesure. *C. R. Acad. Sci. Paris Sér. I Math.*, 303(4):105–108, 1986.
- [CS10] A. Cheskidov and R. Shvydkoy. Ill-posedness of the basic equations of fluid dynamics in Besov spaces. *Proc. Amer. Math. Soc.*, 138(3):1059–1067, 2010.
- [Dai21] Mimi Dai. Nonunique weak solutions in Leray-Hopf class for the three-dimensional Hall-MHD system. *SIAM J. Math. Anal.*, 53(5):5979–6016, 2021.
- [DLS09] C. De Lellis and L. Székelyhidi, Jr. The Euler equations as a differential inclusion. *Ann. of Math. (2)*, 170(3):1417–1436, 2009.
- [DLS13] Camillo De Lellis and László Székelyhidi, Jr. Dissipative continuous Euler flows. *Invent. Math.*, 193(2):377–407, 2013.
- [DLS14] Camillo De Lellis and László Székelyhidi, Jr. Dissipative Euler flows and Onsager’s conjecture. *J. Eur. Math. Soc. (JEMS)*, 16(7):1467–1505, 2014.
- [DLS19] Camillo De Lellis and László Székelyhidi, Jr. On turbulence and geometry: from Nash to Onsager. *Notices Amer. Math. Soc.*, 66(5):677–685, 2019.
- [DR19] Luigi De Rosa. Infinitely many Leray-Hopf solutions for the fractional Navier-Stokes equations. *Comm. Partial Differential Equations*, 44(4):335–365, 2019.
- [DS17] Sara Daneri and László Székelyhidi, Jr. Non-uniqueness and h-principle for Hölder-continuous weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.*, 224(2):471–514, 2017.
- [FJR72] E. B. Fabes, B. F. Jones, and N. M. Riviere. The initial value problem for the Navier-Stokes equations with data in L^p . *Arch. Rational Mech. Anal.*, 45:222–240, 1972.
- [FLRT00] Giulia Furioli, Pierre G. Lemarié-Rieusset, and Elide Terraneo. Unicité dans $L^3(\mathbb{R}^3)$ et d’autres espaces fonctionnels limites pour Navier-Stokes. *Rev. Mat. Iberoamericana*, 16(3):605–667, 2000.
- [FLS21] Daniel Faraco, Sauli Lindberg, and László Székelyhidi, Jr. Bounded solutions of ideal MHD with compact support in space-time. *Arch. Ration. Mech. Anal.*, 239(1):51–93, 2021.
- [Ger05] Pierre Germain. Solutions globales d’énergie infinie de l’équation de Navier-Stokes 2D. *C. R. Math. Acad. Sci. Paris*, 340(7):547–550, 2005.
- [Ger08] Pierre Germain. The second iterate for the Navier-Stokes equation. *J. Funct. Anal.*, 255(9):2248–2264, 2008.
- [GG05] Isabelle Gallagher and Thierry Gallay. Uniqueness for the two-dimensional Navier-Stokes equation with a measure as initial vorticity. *Math. Ann.*, 332(2):287–327, 2005.
- [GGL05] Isabelle Gallagher, Thierry Gallay, and Pierre-Louis Lions. On the uniqueness of the solution of the two-dimensional Navier-Stokes equation with a Dirac mass as initial vorticity. *Math. Nachr.*, 278(14):1665–1672, 2005.
- [GMO88] Yoshikazu Giga, Tetsuro Miyakawa, and Hirofumi Osada. Two-dimensional Navier-Stokes flow with measures as initial vorticity. *Arch. Rational Mech. Anal.*, 104(3):223–250, 1988.
- [GMS01] Y. Giga, S. Matsui, and O. Sawada. Global existence of two-dimensional Navier-Stokes flow with nondecaying initial velocity. *J. Math. Fluid Mech.*, 3(3):302–315, 2001.
- [GP02] Isabelle Gallagher and Fabrice Planchon. On global infinite energy solutions to the Navier-Stokes equations in two dimensions. *Arch. Ration. Mech. Anal.*, 161(4):307–337, 2002.
- [Ise18] Philip Isett. A proof of Onsager’s conjecture. *Ann. of Math. (2)*, 188(3):871–963, 2018.
- [IV15] Philip Isett and Vlad Vicol. Hölder continuous solutions of active scalar equations. *Ann. PDE*, 1(1):Art. 2, 77, 2015.
- [Kat94] Tosio Kato. The Navier-Stokes equation for an incompressible fluid in $\mathbf{R}^2$ with a measure as the initial vorticity. *Differential Integral Equations*, 7(3-4):949–966, 1994.
- [KT01] Herbert Koch and Daniel Tataru. Well-posedness for the Navier-Stokes equations. *Adv. Math.*, 157(1):22–35, 2001.
- [Kui55] Nicolaas H. Kuiper. On C^1 -isometric imbeddings. I, II. *Nederl. Akad. Wetensch. Proc. Ser. A*. **58** = *Indag. Math.*, 17:545–556, 683–689, 1955.

- [Lad58] O. A. Ladyzhenskaja. Solution “in the large” to the boundary-value problem for the Navier-Stokes equations in two space variables. *Dokl. Akad. Nauk SSSR*, 123:427–429, 1958.
- [Ler33] Jean Leray. Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l’hydrodynamique. *Journal de Mathématiques Pures et Appliquées*, 12:1–82, 1933.
- [Ler34] J. Leray. Sur le mouvement d’un liquide visqueux emplissant l’espace. *Acta Math.*, 63(1):193–248, 1934.
- [LMPP21] S. Lanthaler, S. Mishra, and C. Pares-Pulido. On the conservation of energy in two-dimensional incompressible flows. *Nonlinearity*, 34(2):1084–1136, 2021.
- [LQ20] Tianwen Luo and Peng Qu. Non-uniqueness of weak solutions to 2D hypoviscous Navier-Stokes equations. *J. Differential Equations*, 269(4):2896–2919, 2020.
- [LR16] Pierre Gilles Lemarié-Rieusset. *The Navier-Stokes problem in the 21st century*. CRC Press, Boca Raton, FL, 2016.
- [LT20] Tianwen Luo and Edriss S. Titi. Non-uniqueness of weak solutions to hyperviscous Navier-Stokes equations: on sharpness of J.-L. Lions exponent. *Calc. Var. Partial Differential Equations*, 59(3):Paper No. 92, 15, 2020.
- [Luo19] Xiaoyutao Luo. Stationary solutions and nonuniqueness of weak solutions for the Navier-Stokes equations in high dimensions. *Arch. Ration. Mech. Anal.*, 233(2):701–747, 2019.
- [Maz03] Anna L. Mazzucato. Besov-Morrey spaces: function space theory and applications to non-linear PDE. *Trans. Amer. Math. Soc.*, 355(4):1297–1364, 2003.
- [MS18] Stefano Modena and László Székelyhidi, Jr. Non-uniqueness for the transport equation with Sobolev vector fields. *Ann. PDE*, 4(2):Art. 18, 38, 2018.
- [MS20] Stefano Modena and Gabriel Sattig. Convex integration solutions to the transport equation with full dimensional concentration. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 37(5):1075–1108, 2020.
- [Nas54] John Nash. C^1 isometric imbeddings. *Ann. of Math. (2)*, 60:383–396, 1954.
- [Nov20] Matthew Novack. Nonuniqueness of weak solutions to the 3 dimensional quasi-geostrophic equations. *SIAM J. Math. Anal.*, 52(4):3301–3349, 2020.
- [PZ14] Marius Paicu and Zhifei Zhang. Global well-posedness for 3D Navier-Stokes equations with ill-prepared initial data. *J. Inst. Math. Jussieu*, 13(2):395–411, 2014.
- [Shv11] R. Shvydkoy. Convex integration for a class of active scalar equations. *J. Amer. Math. Soc.*, 24(4):1159–1174, 2011.

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