



# EXPLICIT RIP MATRICES: AN UPDATE

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**Abstract.** Leveraging recent advances in additive combinatorics, we exhibit explicit matrices satisfying the Restricted Isometry Property with better parameters. Namely, for  $\varepsilon = 3.26 \cdot 10^{-7}$ , large  $k$  and  $k^{2-\varepsilon} \leq N \leq k^{2+\varepsilon}$ , we construct  $n \times N$  RIP matrices of order  $k$  with  $k = \Omega(n^{1/2+\varepsilon/4})$ .

## 1. Introduction

Suppose  $1 \leq k \leq n \leq N$  and  $0 < \delta < 1$ . A ‘signal’  $\mathbf{x} = (x_j)_{j=1}^N$  is said to be  $k$ -sparse if  $\mathbf{x}$  has at most  $k$  nonzero coordinates. An  $n \times N$  matrix  $\Phi$  is said to satisfy the Restricted Isometry Property (RIP) of order  $k$  with constant  $\delta$  if for all  $k$ -sparse vectors  $\mathbf{x}$  we have

$$(1.1) \quad (1 - \delta)\|\mathbf{x}\|_2^2 \leq \|\Phi\mathbf{x}\|_2^2 \leq (1 + \delta)\|\mathbf{x}\|_2^2.$$

While most authors work with real signals and matrices, in this paper we work with complex matrices for convenience. Given a complex matrix  $\Phi$  satisfying (1.1), the  $2n \times 2N$  real matrix  $\Phi'$ , formed by replacing each element  $a + ib$  of  $\Phi$  by the  $2 \times 2$  matrix  $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ , also satisfies (1.1) with the same parameters  $k, \delta$ .

We know from Candès, Romberg and Tao that matrices satisfying RIP have application to sparse signal recovery (see [7–9]). Given  $n, N, \delta$ , we wish to find  $n \times N$  RIP matrices of order  $k$  with constant  $\delta$ , and with  $k$  as large

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as possible. If the entries of  $\Phi$  are independent Bernoulli random variables with values  $\pm 1/\sqrt{n}$ , then with high probability,  $\Phi$  will have the required properties for  $k$  of order close to  $\delta n$ ; in different language, this was first proved by Kashin [13].

It is an open problem to find good *explicit* constructions of RIP matrices; see Tao's Weblog [17] for a discussion of the problem. All existent explicit constructions of RIP matrices are based on number theory. Prior to the work of Bourgain, Dilworth, Ford, Konyagin and Kutzarova [3], there were many constructions, e.g. Kashin [12], DeVore [10] and Nelson and Temlyakov [15], producing matrices with  $\delta$  small and order

$$(1.2) \quad k \approx \delta \frac{\sqrt{n} \log n}{\log N}.$$

The  $\sqrt{n}$  barrier was broken by the aforementioned authors in [3]:

**THEOREM A** [3]. *There are effective constants  $\varepsilon > 0$ ,  $\varepsilon' > 0$  and explicit numbers  $k_0, c > 0$  such that for any positive integers  $k \geq k_0$  and  $k^{2-\varepsilon} \leq N \leq k^{2+\varepsilon}$ , there is an explicit  $n \times N$  RIP matrix of order  $k$  with  $k \geq cn^{1/2+\varepsilon/4}$  and constant  $\delta = k^{-\varepsilon'}$ .*

As reported in [4], the construction in [3] produces a value  $\varepsilon \approx 2 \cdot 10^{-22}$ . An improved construction was presented in [4], giving Theorem A with  $\varepsilon = 3.6 \cdot 10^{-15}$ . The values of  $\varepsilon$  depend on two constants in additive combinatorics, which have since been improved. Incorporating these improvements into the argument in [4], we will deduce the following.

**THEOREM 1.** *Let  $\varepsilon = 3.26 \cdot 10^{-7}$ . There are  $\varepsilon' > 0$  and effective numbers  $k_0, c > 0$  such that for any positive integers  $k \geq k_0$  and  $k^{2-\varepsilon} \leq N \leq k^{2+\varepsilon}$ , there is an explicit  $n \times N$  RIP matrix of order  $k$  with  $k \geq cn^{1/2+\varepsilon/4}$  and constant  $\delta = k^{-\varepsilon'}$ .*

As of this writing, the constructions in [3] and [4] remain the only explicit constructions of RIP matrices which exceed the  $\sqrt{n}$  barrier for  $k$ .

The proof of Theorem 1 depends on two key results in additive combinatorics. For subsets  $A, B$  of an additive finite group  $G$ , we write

$$A \pm B = \{a \pm b : a \in A, b \in B\},$$

$$E(A, B) = \#\{(a_1, a_2, b_1, b_2) : a_1 + b_1 = a_2 + b_2; a_1, a_2 \in A; b_1, b_2 \in B\}.$$

Also set  $x \cdot B = \{xb : b \in B\}$ . Here we will mainly work with the group of residues modulo a prime  $p$ .

**PROPOSITION 2.** *For some  $c_0$ , the following holds. Assume  $A, B$  are subsets of residue classes modulo  $p$ , with  $0 \notin B$  and  $|A| \geq |B|$ . Then*

$$(1.3) \quad \sum_{b \in B} E(A, b \cdot A) = O((\min(p/|A|, |B|)^{-c_0} |A|^3 |B|)).$$

This theorem, without an explicit  $c_0$ , was proved by Bourgain [2]. The first explicit version of Proposition 2, with  $c_0 = 1/10430$ , is given in Bourgain and Glibuchuk [6], and this is the value used in the papers [3,4]. Murphy and Petridis [14, Lemma 13] made a great improvement, showing that Proposition 2 holds with  $c_0 = 1/3$ . It is conceivable that  $c_0$  may be taken to be any number less than 1. Taking  $A = B$  we see that  $c_0$  cannot be taken larger than 1.

We also need a version of the Balog–Szemerédi–Gowers lemma, originally proved by Balog and Szemerédi [1] and later improved by Gowers [11]. The version we use is a later improvement due to Schoen [16].

**PROPOSITION 3.** *For some positive  $c_1, c_2, c_3$  and  $c_4$ , the following holds. If  $E(A, A) = |A|^3/K$ , then there exist  $A', B' \subseteq A$  with  $|A'|, |B'| \geq c_2 \frac{|A|}{K^{c_4}}$  and  $|A' - B'| \leq c_3 K^{c_1} |A'|^{1/2} |B'|^{1/2}$ .*

The constants  $c_2, c_3$  are relatively unimportant. The best result to date is due to Schoen [16], who showed that any  $c_1 > 7/2$  and  $c_4 > 3/4$  is admissible. It is conjectured that  $c_1 = 1$  is admissible. The papers [3,4] used Proposition 3 with the weaker values  $c_1 = 9$  and  $c_4 = 1$ , this deducible from Bourgain and Garaev [5, Lemma 2.2].

## 2. Construction of the matrix

Our construction is identical to that in [4]. We fix an even integer  $m \geq 100$  and let  $p$  be a large prime. For  $x \in \mathbb{Z}$ , let  $e_p(x) = e^{2\pi i x/p}$ . Let

$$(2.1) \quad \mathbf{u}_{a,b} = \frac{1}{\sqrt{p}} (e_p(ax^2 + bx))_{1 \leq x \leq p}.$$

We take

$$(2.2) \quad \alpha = \frac{1}{2m}, \quad \mathcal{A} = \{1, 2, \dots, \lfloor p^\alpha \rfloor\}.$$

To define the set  $\mathcal{B}$ , we take

$$\beta = \frac{1}{2.01m}, \quad r = \left\lfloor \frac{\beta \log p}{\log 2} \right\rfloor, \quad M = \lfloor 2^{2.01m-1} \rfloor,$$

and let

$$(2.3) \quad \mathcal{B} = \left\{ \sum_{j=1}^r x_j (2M)^{j-1} : x_1, \dots, x_r \in \{0, \dots, M-1\} \right\}.$$

We interpret  $\mathcal{A}$ ,  $\mathcal{B}$  as sets of residue classes modulo  $p$ . We notice that all elements of  $\mathcal{B}$  are at most  $p/2$ , and  $|\mathcal{A}||\mathcal{B}|$  lies between two constant multiples of  $p^{1+\alpha-\beta} = p^{1+1/(402m)}$ .

Given large  $k$  and  $k^{2-\varepsilon} \leq N \leq k^{2+\varepsilon}$ , let  $p$  be a prime in the interval  $[k^{2-\varepsilon}, 2k^{2-\varepsilon}]$  (such  $p$  exists by Bertrand's postulate). Let  $\Phi_p$  be a  $p \times (|\mathcal{A}| \cdot |\mathcal{B}|)$  matrix formed by the column vectors  $\mathbf{u}_{a,b}$  for  $a \in \mathcal{A}$ ,  $b \in \mathcal{B}$  (the columns may appear in any order). We also have

$$(2.4) \quad \text{if } \varepsilon \leq \frac{1}{403m}, \quad \text{then } N \leq p^{\frac{2+\varepsilon}{2-\varepsilon}} \leq |\mathcal{A}||\mathcal{B}|.$$

Take  $\Phi$  to be the matrix formed by the first  $N$  columns of  $\Phi_p$ . Let  $n = p$ . Our task is to show that  $\Phi$  satisfies the RIP condition with  $\delta = p^{-\varepsilon'}$  for some constant  $\varepsilon' > 0$ , and of order  $k$ .

### 3. Main tools

LEMMA 4. Assume that  $c_0 \leq 1$  and that Proposition 2 holds. Fix an even integer  $m \geq 100$ , and define  $\alpha$ ,  $\mathcal{A}$ ,  $\mathcal{B}$  by (2.2) and (2.3). Suppose that  $p$  is sufficiently large in terms of  $m$ . Assume also that for some constant  $c_5 > 0$  and constant  $0 < \gamma \leq \frac{1}{4m}$ ,  $\mathcal{B}$  satisfies

$$(3.1) \quad \forall S \subseteq \mathcal{B} \text{ with } |S| \geq p^{0.49}, \quad E(S, S) \leq c_5 p^{-\gamma} |S|^3.$$

Define the vectors  $\mathbf{u}_{a,b}$  by (2.1). Then for any disjoint sets  $\Omega_1, \Omega_2 \subset \mathcal{A} \times \mathcal{B}$  such that  $|\Omega_1| \leq \sqrt{p}$ ,  $|\Omega_2| \leq \sqrt{p}$ , the inequality

$$\left| \sum_{(a_1, b_1) \in \Omega_1} \sum_{(a_2, b_2) \in \Omega_2} \langle \mathbf{u}_{a_1, b_1}, \mathbf{u}_{a_2, b_2} \rangle \right| = O(p^{1/2-\varepsilon_1} (\log p)^2)$$

holds, where

$$(3.2) \quad \varepsilon_1 = \frac{\frac{c_0 \gamma}{8} - \frac{47\alpha - 23\gamma}{2m}}{1 + 93/m + c_0/2}.$$

The constant implied by the  $O$ -symbol depends only on  $c_0, \gamma$  and  $m$ .

Lemma 4 follows by combining Lemmas 2 and 4 from [4]; the assumption of Proposition 2 is inadvertently omitted in the statement of [4, Lemma 4].

Using Lemma 4, we shall show the following.

THEOREM 5. Assume the hypotheses of Lemma 4, let  $\varepsilon = 2\varepsilon_1 - 2\varepsilon_1^2$  and assume that  $\varepsilon \leq \frac{1}{403m}$ . There is  $\varepsilon' > 0$  such that for sufficiently large  $k$  and  $k^{2-\varepsilon} \leq N \leq k^{2+\varepsilon}$ , there is an explicit  $n \times N$  RIP matrix of order  $k$  with  $n = O(k^{2-\varepsilon})$  and constant  $\delta = k^{-\varepsilon'}$ .

To prove Theorem 5, we first recall another additive combinatorics result from [4].

LEMMA 6 [4, Theorem 2, Corollary 2]. *Let  $M$  be a positive integer. For the set  $\mathcal{B} \subset \mathbb{F}_p$  defined in (2.3) and for any subsets  $A, B \subset \mathcal{B}$ , we have  $|A - B| \geq |A|^\tau |B|^\tau$ , where  $\tau$  is the unique positive solution of*

$$\left(\frac{1}{M}\right)^{2\tau} + \left(\frac{M-1}{M}\right)^\tau = 1.$$

From [4] we have the easy bounds

$$(3.3) \quad \frac{\log 2}{\log M} \left(1 - \frac{1}{\log M}\right) \leq 2\tau - 1 \leq \frac{\log 2}{\log M}.$$

COROLLARY 7. *Take  $\mathcal{B}$  as in (2.3) and assume Proposition 3. Then (3.1) holds with*

$$\gamma = \frac{0.49(2\tau - 1)}{c_1 + c_4(2\tau - 1)}.$$

PROOF. Just like the proof of [4, Lemma 3], except that we incorporate Proposition 3. Suppose that  $S \subseteq \mathcal{B}$  with  $|S| \geq p^{0.49}$  and  $E(S, S) = |S|^3/K$ . By Proposition 3, there are sets  $T_1, T_2 \subset S$  such that  $|T_1|, |T_2| \geq c_2 \frac{|S|}{K^{c_4}}$  and  $|T_1 - T_2| \leq c_3 K^{c_1} |T_1|^{1/2} |T_2|^{1/2}$ . By Lemma 6,

$$c_3 K^{c_1} |T_1|^{1/2} |T_2|^{1/2} \geq |T_1 - T_2| \geq |T_1|^\tau |T_2|^\tau,$$

and hence

$$c_3 K^{c_1} \geq (|T_1| \cdot |T_2|)^{\tau-1/2} \geq \left(\frac{c_2 p^{0.49}}{K^{c_4}}\right)^{2\tau-1}.$$

It follows that  $K \geq (1/c_5)p^{-\gamma}$  for an appropriate constant  $c_5 > 0$ .  $\square$

Finally, we need a tool from [3] which states that in (1.1) we need only consider vectors  $\mathbf{x}$  whose components are 0 or 1 (so-called *flat* vectors).

LEMMA 8 [3, Lemma 1]. *Let  $k \geq 2^{10}$  and  $s$  be a positive integer. Assume that for all  $i \neq j$  we have  $\langle \mathbf{u}_i, \mathbf{u}_j \rangle \leq 1/k$ . Also, assume that for some  $\delta \geq 0$  and any disjoint  $J_1, J_2 \subset \{1, \dots, N\}$  with  $|J_1| \leq k, |J_2| \leq k$  we have*

$$\left| \left\langle \sum_{j \in J_1} \mathbf{u}_j, \sum_{j \in J_2} \mathbf{u}_j \right\rangle \right| \leq \delta k.$$

*Then  $\Phi$  satisfies the RIP property of order  $2sk$  with constant  $44s\sqrt{\delta} \log k$ .*

Now we show how to deduce Theorem 5. By Lemma 4 and standard bounds for Gauss sums,  $\Phi$  satisfies the conditions of Lemma 8 with  $k = \lfloor \sqrt{p} \rfloor$  and  $\delta = O(p^{-\varepsilon_1} \log^2 p)$ . Let  $\varepsilon_0 < \varepsilon_1/2$  and take  $s = \lfloor p^{\varepsilon_0} \rfloor$ . By Lemma 8,  $\Phi$  satisfies RIP with order  $\geq p^{1/2+\varepsilon_0}$  and constant  $O(p^{-\varepsilon_1/2+\varepsilon_0} (\log p)^3)$ . If  $\varepsilon_0$  is sufficiently close to  $\varepsilon_1/2$ , Theorem 5 follows with

$$\varepsilon = 2 - \frac{2}{1 + 2\varepsilon_0} = \frac{4\varepsilon_0}{1 + 2\varepsilon_0} > 2\varepsilon_1 - 2\varepsilon_1^2.$$

To prove Theorem 1, we take the construction in Section 2. We have (3.1) by Corollary 7. Also take

$$\eta = 10^{-100}, \quad c_0 = \frac{1}{3}, \quad c_1 = 7/2 + \eta, \quad c_4 = 3/4 + \eta, \quad m = 7586.$$

These values were optimized with a computer search. By Corollary 7 and (3.3), we have  $\gamma \geq 9.182 \cdot 10^{-6}$ . It is readily verified that  $\gamma \leq \frac{1}{4m}$ ,  $\varepsilon_1 > 1.631 \cdot 10^{-7}$  and  $\varepsilon = 2\varepsilon_1 - 2\varepsilon_1^2$  satisfies  $3.26 \cdot 10^{-7} \leq \varepsilon \leq \frac{1}{403m}$ . Theorem 1 now follows.

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