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# GEOMETRIC EVOLUTION OF BILAYERS UNDER THE DEGENERATE FUNCTIONALIZED CAHN–HILLIARD EQUATION*

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**Abstract.** Using a multiscale analysis, we derive a sharp interface limit for the dynamics of bilayer structures of the functionalized Cahn–Hilliard equation with a cutoff diffusion mobility $M(u)$ that is degenerate for $u \leq 0$ and a continuously differentiable double-well potential $W(u)$. We show that the bilayer interface does not move in the $t = O(1)$ time scale. The interface motion occurs in the $t = O(\varepsilon^{-1})$ time scale and is determined by porous medium diffusion processes in both phases with no jumps on the interface. In the longer $O(\varepsilon^{-2})$ time scale, the interface motion is a complex combination of porous medium diffusion processes in both phases and the property of mass conservation.

**Key words.** bilayers, functionalized Cahn–Hilliard energy, degenerate mobility, asymptotic analysis, geometric evolution

**MSC codes.** 35B40, 35Q74, 35Q92

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## 1. Introduction.

Natural and synthetic amphiphilic materials, such as phospholipids, detergents, and block copolymers, play an essential role in biological and medical sciences [1, 18, 26, 29], and in engineering of new materials and devices [25]. These amphiphiles contain both hydrophobic and hydrophilic segments. When dispersed in aqueous solutions at a concentration higher than the critical micelle concentration, they self-assemble into aggregates with various structures and sizes, such as bilayer vesicles, micelles (spheres and tubes), and multilamellar compounds [17, 19]. The system must have defined boundaries that separate it from the environment [26, 28]. It is, therefore, of mathematical interest to understand the geometric evolution of amphiphilic structures during the self-assembly. The functionalized Cahn–Hilliard (FCH) equations,

$$u_t = \nabla \cdot (M(u)\nabla \mu), \tag{1.1}$$

$$\mu = (-\varepsilon^2\Delta + W''(u) - \varepsilon^2\eta_2)(-\varepsilon^2\Delta u + W'(u)), \tag{1.2}$$

was introduced to study phase-separated mixtures with an amphiphilic structure [16]. This equation is usually subject to periodic or zero-flux boundary conditions on $\partial\Omega$, where $\Omega$ is an open subset of $\mathbb{R}^n$. The initial data should be taken as

$$u(x,0) = \Phi(x), \tag{1.3}$$

$$\mu(x,0) = \Psi(x) := (-\varepsilon^2\Delta + W''(\Phi) - \varepsilon^2\eta_2)(-\varepsilon^2\Delta\Phi + W'(\Phi)) \tag{1.4}$$

in $\Omega$, where $\Phi \in H^4(\Omega)$.

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The function $\mu$ in equation (1.2) is the chemical energy which is defined by the variational derivative of the FCH free energy

$$\mathcal{F}(u) = \varepsilon^{-2} \int_{\Omega} \frac{1}{2}(-\varepsilon^2 \Delta u + W'(u))^2 - \varepsilon^2 \eta_2 \left( \frac{\varepsilon^2}{2} |\nabla u|^2 + W(u) \right) dx, \tag{1.5}$$

and $M(u)$ in denotes the diffusion mobility. $W(u)$ represents the double-well potential with two unequal depth local minima $b_- < b_+$ for which $W(b_-) > W(b_+)$, and $W'$ has exactly three zeros $b_- < b_0 < b_+$. The phase $u = b_-$ with the higher self-energy is the majority phase, while the phase $u = b_+$ is the minority phase. In this paper we take

$$W(u) = u^2(u-1)(u-2), \tag{1.6}$$

which has two local minima $b_- = 0 < b_+$. In this paper, we choose the diffusion mobility $M(u)$ to be

$$M(u) = \begin{cases} 0, & u < 0, \\ u, & u \geq 0, \end{cases} \tag{1.7}$$

which is degenerate at $u = 0$. This corresponds to an attempt to model the elimination of exchange of amphiphilic molecules between disjoint morphologies; see [30] for experimental descriptions.

The FCH free energy (1.5) is a phenomenological model describing the free energy of amphiphilic mixtures that supports codimension-one bilayer interfaces separating two identical phases $u = b_-$ by a thin region of another phase $u = b_+$ [4, 10, 11, 13, 14, 15, 20]. Using asymptotic analysis, we will derive sharp interface models of the system with respect to different scales of time $t$. In this paper we use a periodic boundary condition on $\partial\Omega$ for (1.1)–(1.2). It was established in [9] that the FCH equations (1.1)–(1.2) with $W(u)$ and $M(u)$ defined by (1.6) and (1.7), along with periodic boundary condition on $\partial\Omega$, have a nonnegative weak solution that is not identically zero in $\Omega$. So in this paper we make the following assumptions. Specifically, we assume the initial data are compatible with these assumptions.

*Assumption* 1.1. There exists a solution $u \geq 0$ to (1.1)–(1.2) that is smooth enough to carry out the formal calculation.

*Assumption* 1.2. There is a smooth, codimensional-one initial interface $\Gamma_0 \subset \mathbb{R}^n$ that splits $\Omega$ into the interior $\Omega_+$ and the exterior $\Omega_-$, which is parametrized by

$$\Gamma_0 = \{\phi_0(s) : s = (s_1, \ldots, s_{n-1}) \in Q_0 \subset \mathbb{R}^{n-1}\}, \tag{1.8}$$

where $\phi_0 : Q_0 \mapsto \mathbb{R}^n$ is a smooth function.

In section 3, we define the outer expansion

$$u(x,t) = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots. \tag{1.9}$$

Throughout this paper we assume $u \approx u_0 = b_- = 0$ in the bulk phase $\Omega_\pm$. Since $u \geq 0$, we assume the following.

*Assumption* 1.3. $u_1(x,t) \geq 0$. Specifically we require the initial data to also satisfy this assumption.

In subsection 6.2, we show that in the $t_2 = \varepsilon^2 t$ time scale, $u_1(x, t_2) = 0$, and since $u \geq 0$, we assume the following.

*Assumption* 1.4. In the $t_2 = \varepsilon^2 t$ time scale, $u_2(x, t_2) \geq 0$.

It is important to point out that the stability of codimension-one solutions is an important and nontrivial problem, even for the relatively simpler case when the diffusion mobility $M(u)$ is constant [3, 13, 20, 27]. To the best of our knowledge there is no result about the stability of solutions when $M(u)$ depends on $u$. In this paper we concentrate on formal asymptotic analysis, assuming that the parameters in our equations are in a range where meandering or pearling instabilities do not occur.

We will describe the motion of the bilayer interfaces $\Gamma(t)$, which is parametrized by

$$\Gamma(t) = \{\phi(s,t) : s = (s_1, \dots, s_{n-1}) \in Q(t) \subset \mathbb{R}^{n-1}\}, \tag{1.10}$$

as time $t$ changes. The parametrization $\phi(s,t)$ is chosen so that $s_i$ corresponds to the arc length along the $i$th coordinate curve, and the coordinate curves are curvature lines. Then the tangent vectors

$$\mathbf{T}^i = \frac{\partial \phi}{\partial s_i}, \quad i = 1, \dots, n-1, \tag{1.11}$$

form an orthonormal basis for the tangent space to $\Gamma$ at the point $\phi(s,t)$. Let $\mathbf{n}(s,t)$ be the outer normal vector of $\Gamma$ pointing toward $\Omega_-$; then we have the relations

$$\frac{\partial \mathbf{T}^i}{\partial s_i} = -k_i \mathbf{n}, \quad \frac{\partial \mathbf{n}}{\partial s_i} = k_i \mathbf{T}^i, \quad i = 1, \dots, n-1, \tag{1.12}$$

where $k_i = k_i(s)$ $(i = 1, \dots, n-1)$ are principal curvatures of $\Gamma$ at $\phi(s,t)$. Our main result is the following statement about the quasi-equilibrium evolutions.

MAIN RESULT 1.5. *With the choice of $W(u)$ and $M(u)$ in* (1.6) *and* (1.7)*, the nontrivial interface motion under the FCH equations* (1.1)–(1.2) *with the initial data* (1.3)–(1.4) *starts to occur in the time scale $t_1 = \varepsilon t$, or $t = O(\varepsilon^{-1})$. The normal velocity $V_\mathbf{n}$ of the interface in the $t_1 = \varepsilon t$ time scale is determined by the following free boundary problem:*

$$\frac{\partial \mu_1}{\partial t_1} = \nabla \cdot (\mu_1 \nabla \mu_1) \quad \text{in } \Omega \backslash \Gamma, \tag{1.13}$$

$$[\![\partial_\mathbf{n} \mu_1]\!] = 0 \quad \text{on } \Gamma, \tag{1.14}$$

$$\mu_1 \text{ is periodic on } \partial\Omega, \tag{1.15}$$

$$\mu_1(x,0) = \Psi_1(x) \quad \text{in } \Omega, \tag{1.16}$$

$$V_\mathbf{n} = \mu_1 \kappa_0 \quad \text{on } \Gamma, \tag{1.17}$$

*where $\kappa_0 = k_1 + \cdots + k_{n-1}$ is the mean curvature, and $\Psi_1(x)$ is a given function defined by* (3.4).

*In the $t_2 = \varepsilon^2 t$ time scale, i.e., $t = O(\varepsilon^{-2})$, the normal velocity $V_\mathbf{n}$ of the interface is determined by the following free boundary problem:*

$$\frac{\partial \mu_2}{\partial t_2} = \nabla \cdot (\mu_2 \nabla \mu_2) \quad \textit{in } \Omega \backslash \Gamma, \tag{1.18}$$

$$[\![\partial_{\mathbf{n}} \mu_2]\!] = 0 \quad \textit{on } \Gamma, \tag{1.19}$$

$$\mu_2 \textit{ is periodic on } \partial\Omega, \tag{1.20}$$

$$\mu_2(x,0) = \Psi_2(x) \quad \textit{in } \Omega, \tag{1.21}$$

$$V_{\mathbf{n}} = \left(\mu_2 + \frac{m_2}{m_1}\eta_2\right)\kappa_0 + \frac{m_2}{m_1}\left(\Delta_s \kappa_0 - \frac{\kappa_0^3}{2} - \kappa_0\kappa_1\right) \quad \textit{on } \Gamma, \tag{1.22}$$

*where* $\kappa_1 = -(k_1^2 + \cdots + k_{n-1}^2)$, $\Psi_2(x)$ *is a given function defined by* (3.4), *and* $m_1 = \int_{-\infty}^{\infty} U dz$, $m_2 = \int_{-\infty}^{\infty} U_z^2 dz$. *Here,* $U(z)$ *is the homoclinic solution to the codimensional-one bilayer equation*

$$-U''(z) + W'(U(z)) = 0,\ U(0) = 1,\ \lim_{z\to\pm\infty} U(z) = 0. \tag{1.23}$$

By Assumption 1.3 and equality (3.8), $\mu_1 = 16u_1 \geq 0$. Hence the porous medium equations (1.13)–(1.6) are well-posed, and the mean-curvature flow (1.17) is also well-posed. By Assumption 1.4 and equality (6.22), $\mu_2(x,t_2) = 16u_2(x,t_2) \geq 0$. So the porous medium equations (1.18)–(1.21) are also well-posed.

*Remark* 1.6. Our main results show a few interesting things.

1. When the double-well potential $W$ is smooth, the degeneracy in the diffusion mobility is not enough to completely cut off the diffusion in the bulk phase. This is similar to what happens in the degenerate Cahn–Hilliard (CH) equations [6, 5, 7, 8, 22, 23]. However, rather than a quasi-stationary porous medium diffusion in the degenerate CH equation, we have a porous medium diffusion in the bulk phase.
2. Similar to the FCH equation with constant mobility [10, 11], the $t_1$-dynamics of the degenerate FCH is still a quenched mean curvature flow with a variable coefficient. However, the coefficient is determined by the porous medium diffusion. In comparison, for the FCH equation with constant mobility, the coefficient in the quenched mean curvature flow is a spatial constant that varies with time.
3. Similarly to the FCH equation with constant mobility, the $t_2$-dynamics is still a Willmore-type flow, but with some coefficient determined by a porous medium diffusion. For the constant mobility case, the corresponding coefficient in the Willmore-type flow is again a spatial constant.
4. Based on these observations, we conjecture that in order to completely suppress diffusion in the bulk phase, we should consider a combination of degenerate mobility and a nonsmooth potential, as suggested in [12], or maybe some completely different approaches.

## 2. The whiskered coordinates and inner expansion.

**2.1. The whiskered coordinates.** By the implicit function theorem, for each $\bar{x}$ on $\Gamma$, there exists a neighborhood $\mathcal{N}_{\bar{x}} \subset \Gamma$ of $\bar{x}$ such that the map $x \mapsto (s,r)$ defined by

$$x = \phi(s,t) + r\mathbf{n}(s,t) \tag{2.1}$$

is locally and smoothly invertible for each fixed time $t$. The thickness of the bilayer interfaces is of order $\varepsilon$, so we rescale the normal local coordinate $r(x)$ by $z = r/\varepsilon$. Lemma 2.1 below summarizes the properties of the local coordinate system [2].

LEMMA 2.1. *Let $\Gamma = \Gamma(t)$ be a smooth interface of the form (1.10) with curvatures $\{k_i\}_{i=1}^{n-1}$ uniformly of order 1. The normal velocity of $\Gamma$ at $\phi(s,t)$ is $V_{\mathbf{n}} = -\frac{\partial r}{\partial t}(s,t)$, which is positive when $\Gamma$ moves in the direction of $\mathbf{n}$, and the tangential coordinates $(r,s)$ satisfy the formulae*

$$\nabla_x s_i = \frac{1}{1+rk_i}\mathbf{T}^i, \quad \Delta_x s_i = -\frac{r}{(1+rk_i)^3}\frac{\partial k_i}{\partial s_i}, \quad i = 1,\dots,n-1, \tag{2.2}$$

*and*

$$\nabla_x r = \mathbf{n}, \quad \Delta_x r = \sum_{j=1}^{n-1}\frac{k_j}{1+rk_j}. \tag{2.3}$$

*In the scaled local coordinates $(z,s)$, the Cartesian-Laplacian $\Delta_x$ can be represented in terms of the Laplace–Beltrami operator $\Delta_s$ and the curvatures*

$$\Delta_x = \varepsilon^{-2}\partial_{zz} + \varepsilon^{-1}\kappa_0\partial_z + z\kappa_1\partial_z + \Delta_s + \varepsilon\Delta_1 + O(\varepsilon^2), \tag{2.4}$$

*where $\kappa_0 = k_1 + \cdots + k_{n-1}$ is the mean curvature, $\kappa_1 = -(k_1^2 + \cdots + k_{n-1}^2)$, $\kappa_2 = k_1^3 + \cdots + k_{n-1}^3$, and*

$$\Delta_1 = -z\sum_{j=1}^{n-1}\frac{\partial k_j}{\partial s_j}\frac{\partial}{\partial s_j} + z^2\kappa_2\partial_z - 2z\kappa_0\Delta_s. \tag{2.5}$$

*The Jacobian matrix $\mathbf{J}$ of the transformation $x \mapsto (z,s)$ has the form*

$$\mathbf{J} = \left((1+\varepsilon z k_1)\mathbf{T}^1, \dots, (1+\varepsilon z k_{n-1})\mathbf{T}^{n-1}, \varepsilon\mathbf{n}\right), \tag{2.6}$$

*and the Jacobian $J = |det\mathbf{J}|$ satisfies*

$$J(s,z) = \varepsilon\prod_{i=1}^{n-1}(1+\varepsilon z k_i) = \varepsilon + \varepsilon^2 z\kappa_0 + O(\varepsilon^3). \tag{2.7}$$

For a given constant $l > 0$, we define the so-called "whiskers" at every point on the interface $\Gamma$ along the normal direction as [10]:

$$w(s) = \left\{\phi(s) + z\mathbf{n}(s) : z \in \left[-\frac{l}{\varepsilon}, \frac{l}{\varepsilon}\right]\right\}. \tag{2.8}$$

The interface $\Gamma$ is called *far from intersection* if there exists $l > 0$ such that none of the whiskers of length $l$ intersect each other or $\partial\Omega$.

DEFINITION 2.1. *Let $\Gamma$ be far from self-intersection and let*

$$\Gamma_l = \bigcup_{s\in Q} w(s) \tag{2.9}$$

*be the subset of $\Omega$ consisting of all points $x \in \Omega$ with $dist(x,\Gamma) \le l$. A function $f \in L^1(\Omega)$ is called* localized *on $\Gamma$ if there exist positive constants $M$ and $\alpha$, not depending on $\varepsilon$, such that*

$$|f(x(z,s))| \le Me^{-\alpha|z|} \tag{2.10}$$

*for all $x \in \Gamma_l$*

Lemma 2.2 below shows the relation between the integral with respect to the global variable $x$ and the integral with respect to the scaled local variables $(z, s)$ [10].

LEMMA 2.2. *If $\Gamma$ is far from self-intersection and $f$ is localized on $\Gamma$, then the following integral formula holds:*

$$\int_\Omega f(x)dx = \int_Q \int_{-l/\varepsilon}^{l/\varepsilon} f(x(z,s))J(z,s)dzds + O(e^{-\nu(l/\varepsilon)}), \tag{2.11}$$

*where $J$ is the Jacobian associated with the immersion $\phi : Q = Q(t) \mapsto \Gamma(t) \subset \mathbb{R}^n$.*

Let $U = U(z)$ be the homoclinic solution to the codimensional-one bilayer equation

$$-U''(z) + W'(U(z)) = 0 \tag{2.12}$$

representing the standard transition layer profile. We assume that the initial data $\Phi$ of $u$ is close to a bilayer interface, that is, for some interface $\Gamma$, $\Phi$ is close to the $\Gamma$-extension $U_\Gamma$ of $U$. Lemma 2.3 below establishes the existence of the maximum of the homoclinic solution $U$ and some properties of the associated linearization [10].

LEMMA 2.3. *Let $U$ be the solution of equation* (2.12) *which is homoclinic to $b_-$, and even in $z$, that is, $U(z) = U(-z)$, then $U$ attains its maximum value $U_M$ at $z = 0$, where $U_M$ is the unique zero of $W$ in $(b_-, b_+)$. Moreover, there exists $\nu > 0$ such that the linearization $\mathcal{L}$ on $H^2(\mathbb{R})$ defined by*

$$\mathcal{L} = -\partial_{zz} + W''(U) \tag{2.13}$$

*has the spectrum satisfying*

$$\sigma(\mathcal{L}) \subset \{\lambda_0, \lambda_1 = 0\} \cup [\nu, \infty), \tag{2.14}$$

*where $\lambda_0 < 0$ is the ground-state eigenvalue. The corresponding eigenfunctions of $\lambda_0$ and $\lambda_1$ are $\Psi_0 \geq 0$ and $\Psi_1 = U_z$, respectively. Also, $\mathcal{L}$ satisfies the following identities:*

$$\mathcal{L}\left(\frac{z}{2}U_z\right) = -U_{zz}, \tag{2.15}$$

$$\mathcal{L}(U_{zz}) = -W'''(U)U_z^2. \tag{2.16}$$

*Finally, there exist even functions $\varphi_1, \varphi_2 \in L^\infty(\mathbb{R})$ that satisfy*

$$\mathcal{L}\varphi_1 = 1, \quad \mathcal{L}\varphi_2 = \varphi_1, \tag{2.17}$$

*and $\varphi_j$'s are orthogonal to the kernel of $\mathcal{L}$ which is spanned by $U_z$.*

With the choice of $W(u) = u^2(u-1)(u-2)$, the maximum value of $U$ is $U_M = 1$ attained only at $z = 0$, and $U$ is homoclinic to 0, that is, $\lim_{z\to\pm\infty} U(z) = 0$.

**2.2. Inner expansion.** At a time scale $\tau$, we have the inner spatial expansions

$$u(x,t) = \tilde{u}(s,z,\tau) = \tilde{u}_0 + \varepsilon\tilde{u}_1 + \varepsilon^2\tilde{u}_2 + \cdots, \tag{2.18}$$

$$\mu(x,t) = \tilde{\mu}(s,z,\tau) = \tilde{\mu}_0 + \varepsilon\tilde{\mu}_1 + \varepsilon^2\tilde{\mu}_2 + \cdots. \tag{2.19}$$

Using (2.4) we get

$$\Delta_x u = \varepsilon^{-2}\tilde{u}_{0zz} + \varepsilon^{-1}(\tilde{u}_{1zz} + \kappa_0\tilde{u}_{0z}) + (\tilde{u}_{2zz} + \kappa_0\tilde{u}_{1z} + \kappa_1 z\tilde{u}_{0z} + \Delta_s\tilde{u}_0) + \varepsilon(\tilde{u}_{3zz} + \kappa_0\tilde{u}_{2z} + \kappa_1 z\tilde{u}_{1z} + \Delta_s\tilde{u}_1 + \Delta_1\tilde{u}_0) + O(\varepsilon^2). \tag{2.20}$$

From (1.2), we write $\mu = PA$, where $P = -\varepsilon^2\Delta + W''(u) - \varepsilon^2\eta_2$ and $A = -\varepsilon^2\Delta u + W'(u)$. Expanding $P$ and $A$ in local coordinates we get

$$\begin{aligned} P = {}& [-\partial_{zz} + W''(\tilde{u}_0)] + \varepsilon[-\kappa_0\partial_z + W'''(\tilde{u}_0)\tilde{u}_1] \\ & + \varepsilon^2\left[-z\kappa_1\partial_z - \Delta_s - \eta_2 + W'''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W^{(4)}(\tilde{u}_0)\tilde{u}_1^2\right] \\ & + \varepsilon^3[-\Delta_1 + \tilde{u}_1\tilde{u}_2 + W'''(\tilde{u}_0)\tilde{u}_3 + W^{(4)}(\tilde{u}_0)\tilde{u}_1\tilde{u}_2] + O(\varepsilon^4), \end{aligned} \tag{2.21}$$

$$\begin{aligned} A = {}& [-\tilde{u}_{0zz} + W'(\tilde{u}_0)] + \varepsilon[-\tilde{u}_{1zz} - \kappa_0\tilde{u}_{0z} + W''(\tilde{u}_0)\tilde{u}_1] + \varepsilon^2\Big[-\tilde{u}_{2zz} - \kappa_0\tilde{u}_{1z} - z\kappa_1\tilde{u}_{0z} \\ & - \Delta_s\tilde{u}_0 + W''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W^{(4)}(\tilde{u}_0)\tilde{u}_1^2\Big] + \varepsilon^3\Big[-\tilde{u}_{3zz} - \kappa_0\tilde{u}_{2z} - z\kappa_1\tilde{u}_{1z} - \Delta_s\tilde{u}_1 \\ & - \Delta_1\tilde{u}_0 + W''(\tilde{u}_0)\tilde{u}_3 + W'''(\tilde{u}_0)\tilde{u}_1\tilde{u}_2 + \frac{1}{6}W^{(4)}(\tilde{u}_0\tilde{u}_1^3)\Big] + O(\varepsilon^4). \end{aligned} \tag{2.22}$$

Grouping the orders of $\mu$ we get

$$\tilde{\mu}_0 = [-\partial_{zz} + W''(\tilde{u}_0)][-\tilde{u}_{0zz} + W'(\tilde{u}_0)], \tag{2.23}$$

$$\begin{aligned} \tilde{\mu}_1 = {}& [-\partial_{zz} + W''(\tilde{u}_0)][-\tilde{u}_{1zz} - \kappa_0\tilde{u}_{0z} + W''(\tilde{u}_0)\tilde{u}_1] \\ & + [-\kappa_0\partial_z + W'''(\tilde{u}_0)\tilde{u}_1][-\tilde{u}_{0zz} + W'(\tilde{u}_0)], \end{aligned} \tag{2.24}$$

$$\begin{aligned} \tilde{\mu}_2 = {}& [-\partial_{zz} + W''(\tilde{u}_0)]\Big[-\tilde{u}_{2zz} - \kappa_0\tilde{u}_{1z} - z\kappa_1\tilde{u}_{0z} - \Delta_s\tilde{u}_0 + W''(\tilde{u}_0)\tilde{u}_2 \\ & + \frac{1}{2}W^{(4)}(\tilde{u}_0)\tilde{u}_1^2\Big] + [-\kappa_0\partial_z + W'''(\tilde{u}_0)\tilde{u}_1][-\tilde{u}_{1zz} - \kappa_0\tilde{u}_{0z} + W''(\tilde{u}_0)\tilde{u}_0] \\ & + \left[-z\kappa_1\partial_z - \Delta_s - \eta_2 + W'''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W^{(4)}(\tilde{u}_0)\tilde{u}_1^2\right][-\tilde{u}_{0zz} + W'(\tilde{u}_0)], \end{aligned} \tag{2.25}$$

$$\begin{aligned} \tilde{\mu}_3 = {}& [-\partial_{zz} + W''(\tilde{u}_0)]\Big[-\tilde{u}_{3zz} - \kappa_0\tilde{u}_{2z} - z\kappa_1\tilde{u}_{1z} - \Delta_s\tilde{u}_1 - \Delta_1\tilde{u}_0 + W''(\tilde{u}_0)\tilde{u}_3 \\ & + W'''(\tilde{u}_0)\tilde{u}_1\tilde{u}_2 + \frac{1}{6}W^{(4)}(\tilde{u}_0)\tilde{u}_1^3\Big] + [-\kappa_0\partial_z + W'''(\tilde{u}_0)\tilde{u}_1]\Big[-\tilde{u}_{2zz} - \kappa_0\tilde{u}_{1z} \\ & - z\kappa_1\tilde{u}_{0z} - \Delta_s\tilde{u}_0 + W''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W^{(4)}(\tilde{u}_0)\tilde{u}_1^2\Big] + \Big[-z\kappa_1\partial_z - \Delta_s - \eta_2 \\ & + W'''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W^{(4)}(\tilde{u}_0)\tilde{u}_1^2\Big][-\tilde{u}_{1zz} - \kappa_0\tilde{u}_{0z} + W''(\tilde{u}_0)\tilde{u}_1] \\ & + [-\Delta_1 + W'''(\tilde{u}_0)\tilde{u}_3 + W^{(4)}(\tilde{u}_0)\tilde{u}_1\tilde{u}_2][-\tilde{u}_{0zz} + W'(\tilde{u}_0)]. \end{aligned} \tag{2.26}$$

Also,

$$\Delta_x\mu = \varepsilon^{-2}\tilde{\mu}_{0zz} + \varepsilon^{-1}(\tilde{\mu}_{1zz} + \kappa_0\tilde{\mu}_{0z}) + (\tilde{\mu}_{2zz} + \kappa_0\tilde{\mu}_{1z} + z\kappa_1\tilde{\mu}_{0z} + \Delta_s\tilde{\mu}_0) + \varepsilon(\tilde{\mu}_{3zz} + \kappa_0\tilde{\mu}_{2z} + z\kappa_1\tilde{\mu}_{1z} + \Delta_s\tilde{\mu}_1 + \Delta_1\tilde{\mu}_0) + O(\varepsilon^2). \tag{2.27}$$

Since we are interested in the quasi-equilibrium long-time behavior of the system, we will assume that the leading order transition profile $\tilde{u}_0$ reaches its equilibrium state, which is the homoclinic solution $U(z)$ of (2.12) satisfying $0 < U(z) \leq 1, U(0) = 1$, and $\lim_{z\to\pm\infty} U(z) = 0$.

Since $W'(0) = 0$ and $W''(0) > 0$, $U$ exponentially approaches 0 as $z \to \pm\infty$. Assuming that $\tilde{u}_0(z) \sim e^{-|z|/\beta}$ as $z \to \pm\infty$, then by taking $\eta = \beta ln(\frac{1}{\varepsilon})$ we have

$$\tilde{u}_0(z) \leq \begin{cases} O(1), & |z| < \eta, \\ O(\varepsilon), & |z| \geq \eta, \\ O(\varepsilon^2), & |z| \geq \eta^2, \\ O(\varepsilon^3), & |z| \geq \eta^3, \\ O(\varepsilon^4), & |z| \geq \eta^4. \end{cases} \tag{2.28}$$

So it's reasonable to split $(-\infty, \infty)$ into subsets

$$\{z : |z| < \eta\}, \{z : \eta \leq |z| < 2\eta\}, \{z : 2\eta \leq |z| < 3\eta\}, \{z : 3\eta \leq |z| < 4\eta\}, \{z : |z| \geq 4\eta\}.$$

Letting $\chi_0, \chi_1, \chi_2, \chi_3, \chi_4$ be the characteristic functions of these sets, we have the following expansion of $\tilde{u}_0$:

$$\tilde{u}_0 = \tilde{u}_0\chi_0 + \varepsilon\tilde{u}_0\chi_1\varepsilon^{-1} + \varepsilon^2\tilde{u}_0\chi_2\varepsilon^{-2} + \varepsilon^3\tilde{u}_0\chi_3\varepsilon^{-3} + \varepsilon^4\tilde{u}_0\chi_4\varepsilon^{-4}. \tag{2.29}$$

The first four terms on the right-hand side are, respectively, of orders $1, \varepsilon, \varepsilon^2, \varepsilon^3$, and the last one is a residual term of order $\varepsilon^4$ and higher.

Since $\tilde{u}_{0z}$ decays exponentially to 0 as $z \to \pm\infty$ at the same rate as $\tilde{u}_0$, we have a similar expansion for $\tilde{u}_{0z}$:

$$\tilde{u}_{0z} = \tilde{u}_{0z}\chi_0 + \varepsilon\tilde{u}_{0z}\chi_1\varepsilon^{-1} + \varepsilon^2\tilde{u}_{0z}\chi_2\varepsilon^{-2} + \varepsilon^3\tilde{u}_{0z}\chi_3\varepsilon^{-3} + \varepsilon^4\tilde{u}_{0z}\chi_4\varepsilon^{-4}. \tag{2.30}$$

By the choice (1.7) of $M(u)$ we obtain

$$\nabla \cdot (M(u)\nabla\mu) = \begin{cases} 0, & u < 0, \\ u\Delta\mu + \nabla u \cdot \nabla\mu, & u \geq 0. \end{cases} \tag{2.31}$$

By Assumption 1.1, $u \geq 0$, so we have

$$\nabla \cdot (M(u)\nabla\mu) = u\Delta\mu + \nabla u \cdot \nabla\mu. \tag{2.32}$$

Using Lemma 2.1 and (2.29), (2.30), we expand

$$\nabla \cdot (M(u)\nabla\mu) = \varepsilon^{-2}P_{-2} + \varepsilon^{-1}P_{-1} + P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + O(\varepsilon^3), \tag{2.33}$$

where

$$P_{-2} = \tilde{u}_0\tilde{\mu}_{0zz}\chi_0 + \tilde{\mu}_{0z}\tilde{u}_{0z}\chi_0, \tag{2.34}$$

$$\begin{aligned} P_{-1} &= \tilde{u}_0\chi_0(\tilde{\mu}_{1zz} + \kappa_0\tilde{\mu}_{0z}) + \tilde{\mu}_{0zz}(\tilde{u}_0\chi_1\varepsilon^{-1} + \tilde{u}_1) \\ &\quad + \tilde{\mu}_{0z}(\tilde{u}_{0z}\chi_1\varepsilon^{-1} + \tilde{u}_{1z}) + \tilde{\mu}_{1z}\tilde{u}_{0z}\chi_0, \end{aligned} \tag{2.35}$$

$$\begin{aligned} P_0 &= \tilde{u}_0\chi_0(\tilde{\mu}_{2zz} + \kappa_0\tilde{\mu}_{1z} + z\kappa_1\tilde{\mu}_{0z} + \Delta_s\tilde{\mu}_0) + (\tilde{u}_0\chi_1\varepsilon^{-1} + \tilde{u}_1)(\tilde{\mu}_{1zz} + \kappa_0\tilde{\mu}_{0z}) \\ &\quad + (\tilde{u}_0\chi_2\varepsilon^{-2} + \tilde{u}_2)\tilde{\mu}_{0zz} + \tilde{\mu}_{0z}(\tilde{u}_{0z}\chi_2\varepsilon^{-2} + \tilde{u}_{2z}) + \tilde{\mu}_{1z}(\tilde{u}_0\chi_1\varepsilon^{-1} + \tilde{u}_{1z}) \\ &\quad + \tilde{\mu}_{2z}\tilde{u}_{0z}\chi_0 + \sum_{j=1}^{n-1}\tilde{u}_{0s_j}\tilde{\mu}_{0s_j}, \end{aligned} \tag{2.36}$$

$$\begin{aligned} P_1 &= \tilde{u}_0\chi_0(\tilde{\mu}_{3zz} + \kappa_0\tilde{\mu}_{2z} + z\kappa_1\tilde{\mu}_{1z} + \Delta_s\tilde{\mu}_1 + \Delta_1\tilde{\mu}_0) \\ &\quad + (\tilde{u}_0\chi_1\varepsilon^{-1} + \tilde{u}_1)(\tilde{\mu}_{2zz} + \kappa_0\tilde{\mu}_{1z} + z\kappa_1\tilde{\mu}_{0z} + \Delta_s\tilde{\mu}_0) \\ &\quad + (\tilde{u}_0\chi_2\varepsilon^{-2} + \tilde{u}_2)(\tilde{\mu}_{1zz} + \kappa_0\tilde{\mu}_{0z}) + (\tilde{u}_0\chi_3\varepsilon^{-3} + \tilde{u}_3)\tilde{\mu}_{0zz} \\ &\quad + \tilde{\mu}_{0z}(\tilde{u}_{0z}\chi_3\varepsilon^{-3} + \tilde{u}_{3z}) + \tilde{\mu}_{1z}(\tilde{u}_{0z}\chi_2\varepsilon^{-2} + \tilde{u}_{2z}) + \tilde{\mu}_{2z}(\tilde{u}_{0z}\chi_1\varepsilon^{-1} + \tilde{u}_{1z}) \\ &\quad + \tilde{\mu}_{3z}\tilde{u}_{0z}\chi_0 - 2z\sum_{j=1}^{n-1}\tilde{u}_{0s_j}\tilde{\mu}_{0s_j}\kappa_j + \sum_{j=1}^{n-1}\tilde{u}_{0s_j}\tilde{\mu}_{1s_j} + \sum_{j=1}^{n-1}\tilde{u}_{1s_j}\tilde{\mu}_{0s_j}. \end{aligned} \tag{2.37}$$

## 3. Outer expansion.

Away from $\Gamma_l$, we have the outer expansion

$$u(x,t) = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots, \tag{3.1}$$

$$\mu(x,t) = \mu_0 + \varepsilon \mu_1 + \varepsilon^2 \mu_2 + \varepsilon^3 \mu_3 + \cdots, \tag{3.2}$$

where $u \approx u_0 \equiv 0$ in the bulk phase $\Omega_\pm$. We also expand the initial data

$$\Phi(x) = \Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2 + \varepsilon^3\Phi_3 + \cdots, \tag{3.3}$$

$$\Psi(x) = \Psi_0 + \varepsilon\Psi_1 + \varepsilon^2\Psi_2 + \varepsilon^3\Psi_3 + \cdots. \tag{3.4}$$

Using Taylor series to expand $W'(u)$ and $W''(u)$ about $u_0$, we have

$$W'(u) = W'(u_0) + \varepsilon u_1 W''(u_0) + \varepsilon^2\left[u_2 W''(u_0) + \frac{1}{2}u_1^2 W'''(u_0)\right] + O(\varepsilon^3), \tag{3.5}$$

$$W''(u) = W''(u_0) + \varepsilon u_1 W'''(u_0) + \varepsilon^2\left[u_2 W'''(u_0) + \frac{1}{2}u_1^2 W^{(4)}(u_0)\right] + O(\varepsilon^3). \tag{3.6}$$

Then from (1.2) we obtain

$$\mu_0 = W''(u_0)W'(u_0) = 0, \tag{3.7}$$

$$\mu_1 = [W'''(u_0)W'(u_0) + W''(u_0)^2]u_1 = 16u_1. \tag{3.8}$$

Hence

$$\nabla\cdot(M(u)\nabla\mu) = \varepsilon^2\nabla\cdot(u_1\nabla\mu_1) + O(\varepsilon^3). \tag{3.9}$$

The following match condition connects the inner and outer expansions:

$$\mu_0^{\pm} = \lim_{z\to\pm\infty} \tilde{\mu}_0, \tag{3.10}$$

$$\mu_1^{\pm} + z\partial_{\mathbf{n}}\mu_0^{\pm} = \tilde{\mu}_1 + o(1) \text{ as } z \to \pm\infty, \tag{3.11}$$

$$\mu_2^{\pm} + z\partial_{\mathbf{n}}\mu_1^{\pm} + \frac{1}{2}z^2\partial_{\mathbf{n}}^2\mu_0^{\pm} = \tilde{\mu}_2 + o(1) \text{ as } z \to \pm\infty, \tag{3.12}$$

$$\mu_3^{\pm} + z\partial_{\mathbf{n}}\mu_2^{\pm} + \frac{1}{2}z^2\partial_{\mathbf{n}}^2\mu_1^{\pm} + \frac{1}{6}z^3\partial_{\mathbf{n}}^3\mu_0^{\pm} = \tilde{\mu}_3 + o(1) \text{ as } z \to \pm\infty, \tag{3.13}$$

and

$$u_0^{\pm} = \lim_{z\to\pm\infty} \tilde{u}_0, \tag{3.14}$$

$$u_1^{\pm} + z\partial_{\mathbf{n}}u_0^{\pm} = \tilde{u}_1 + o(1) \text{ as } z \to \pm\infty, \tag{3.15}$$

$$u_2^{\pm} + z\partial_{\mathbf{n}}u_1^{\pm} + \frac{1}{2}z^2\partial_{\mathbf{n}}^2u_0^{\pm} = \tilde{u}_2 + o(1) \text{ as } z \to \pm\infty. \tag{3.16}$$

**4. Time scale $t = O(1)$.** In this time scale, the outer solution satisfies $\partial_t u_0 = 0$ and $\partial_t u_1 = 0$. For the inner solution, we have

$$u_t = \tilde{u}_t + \nabla_s\tilde{u} \cdot \frac{\partial s}{\partial t} + \varepsilon^{-1}\frac{\partial r}{\partial t}\tilde{u}_z = \varepsilon^{-1}\chi_0\tilde{u}_{0z}\frac{\partial r}{\partial t} + O(1). \tag{4.1}$$

Expanding the normal distance $r = r_0 + \varepsilon r_1 + O(\varepsilon^2)$ and matching (2.33) and (4.1), also recalling that $\tilde{u}_0 = U(z)$ and $\tilde{\mu}_0 \equiv 0$, the $\varepsilon^{-1}$ terms give

$$\chi_0 U_z\frac{\partial r_0}{\partial t} = P_{-1} = \chi_0\frac{\partial}{\partial z}(U\tilde{\mu}_{1z}). \tag{4.2}$$

Hence

$$\frac{\partial}{\partial z}\left(U\left(\frac{\partial r_0}{\partial t} - \tilde{\mu}_{1z}\right)\right) = 0 \quad \text{for } z \in (-\eta, \eta). \tag{4.3}$$

So there is $a_1$ independent of $z \in (-\eta, \eta)$ such that $U\left(\frac{\partial r_0}{\partial t} - \tilde{\mu}_{1z}\right) = a_1$. Since $U \to O(\varepsilon)$ as $z \to \pm\eta$ and $\frac{\partial r_0}{\partial t} - \tilde{\mu}_{1z} = a_1U^{-1}$, the only way for $\frac{\partial r_0}{\partial t} - \tilde{\mu}_{1z}$ to remain $O(1)$ is that $a_1 = 0$ for $z \in (-\eta, \eta)$. So

$$\frac{\partial r_0}{\partial t} = \tilde{\mu}_{1z} \quad \text{for } z \in (-\eta, \eta). \tag{4.4}$$

To find $\frac{\partial r_0}{\partial t}$, let $\mathcal{L} := -\partial_{zz} + W''(U)$. Since $\tilde{u}_0 = U(z)$ and $-U_{zz} + W'(U) = 0$, (2.24) implies

$$\tilde{\mu}_1 = -\kappa_0\mathcal{L}(U_z) + \mathcal{L}^2(\tilde{u}_1) = -\kappa_0\frac{d}{dz}(-U_{zz} + W'(U)) + \mathcal{L}^2(\tilde{u}_1) = \mathcal{L}^2(\tilde{u}_1). \tag{4.5}$$

By Lemma 2.3, equation (4.5) has a solution $\tilde{u}_1 \in L^2(\mathbb{R})$ if and only if $\tilde{\mu}_1 \perp U_z$. Using integration by parts and the solvability condition, we get

$$\begin{aligned}\int_{-\eta}^{\eta} U\tilde{\mu}_{1z}dz &= \int_{-\infty}^{\infty} U\tilde{\mu}_{1z}dz - \int_{|z|>\eta} U\tilde{\mu}_{1z}dz \\ &= -\int_{-\infty}^{\infty} U_z\tilde{\mu}_1 dz + O(\varepsilon) = O(\varepsilon).\end{aligned} \tag{4.6}$$

Combining (4.4) and (4.6) we get $\frac{\partial r_0}{\partial t}\int_{-\eta}^{\eta} U dz = O(\varepsilon)$, and since the left-hand side is of order 1, then $\frac{\partial r_0}{\partial t}\int_{-\eta}^{\eta} U dz = 0$. Thus $\frac{\partial r_0}{\partial t} = 0$, and hence the interface $\Gamma(t)$ doesn't move to the leading order in this time scale.

By (4.4), $\tilde{\mu}_{1z} = \frac{\partial r_0}{\partial t} = 0$ for $z \in (-\eta, \eta)$. This is also consistent with the behavior of $\tilde{\mu}_1$ as $z \to \pm\infty$, which is $\lim_{z\to\pm\infty} \tilde{\mu}_{1z} = \partial_{\mathbf{n}}\mu_0^{\pm} = 0$ by the match condition (3.11). So we expect the equilibrium state of $\tilde{\mu}_1$ to be independent of $z$ in the whole transition layer. Thus there exists $\tilde{B}_1(s,t)$ independent of $z$ such that $\tilde{\mu}_1(z,s,t) = \tilde{B}_1(s,t)$, which is determined by matching the inner expansion $\tilde{\mu}_1$ with the outer expansion $\mu_1$ using

$$\lim_{r\to 0^{\pm}} \mu_1(\phi(s,t) + r\mathbf{n}) = \lim_{z\to\pm\infty} \tilde{\mu}_1(z,s,t) = \tilde{B}_1(s,t).$$

Recalling the function $\varphi_2$ in Lemma 2.3, we find the solution to equation (4.5):

$$\tilde{u}_1 = \tilde{B}_1(s,t)\varphi_2(z), \tag{4.7}$$

where we assume that $\tilde{u}_1 \perp Ker(\mathcal{L})$ on each whisker $w(s)$.

## 5. Time scale $t_1 = \varepsilon t$.

### 5.1. Outer expansion.

Since $\partial_t = \varepsilon\partial_{t_1}$, we have

$$u_t = \varepsilon u_{0t_1} + \varepsilon^2 u_{1t_1} + \varepsilon^3 u_{2t_1} + O(\varepsilon^4). \tag{5.1}$$

Matching (3.9) and (5.1), the $\varepsilon^2$ terms give

$$\frac{\partial u_1}{\partial t_1} = \nabla\cdot(u_1\nabla\mu_1). \tag{5.2}$$

Combining (3.8) and (5.2), it turns out that $\mu_1$ satisfies a porous medium equation

$$\frac{\partial \mu_1}{\partial t_1} = \nabla\cdot(\mu_1\nabla\mu_1) \quad \text{in } \Omega_{\pm}. \tag{5.3}$$

### 5.2. Inner expansion.

We have $u_t = \varepsilon(\tilde{u}_{t_1} + \nabla_s\tilde{u}\cdot\frac{\partial s}{\partial t_1}) + \frac{\partial r}{\partial t_1}\tilde{u}_z$. Combining with (2.27) we have

$$u_t = \frac{\partial r}{\partial t_1}\tilde{u}_{0z}\chi_0 + O(\varepsilon). \tag{5.4}$$

Expanding the normal distance $r = r_0 + \varepsilon r_1 + O(\varepsilon^2)$, and matching (2.33) and (4.1), the order 1 terms give

$$\frac{\partial r_0}{\partial t_1}U_z\chi_0 = P_0 = \chi_0\frac{\partial}{\partial z}(U\tilde{\mu}_{2z}). \tag{5.5}$$

Hence

$$\frac{\partial}{\partial z}\left(U\left(\frac{\partial r_0}{\partial t_1} - \tilde{\mu}_{2z}\right)\right) = 0 \quad \text{for } z\in(-\eta,\eta). \tag{5.6}$$

Using an argument similar to that in section 6, we get

$$\frac{\partial r_0}{\partial t_1} = \tilde{\mu}_{2z} \quad \text{for } z\in(-\eta,\eta). \tag{5.7}$$

Hence $\tilde{\mu}_2 = \frac{\partial r_0}{\partial t_1} z + C_2(s,t_1)$ for $z \in (-\eta, \eta)$. It is reasonable to assume that $\tilde{\mu}_2$ is linear in terms of $z$ in the whole transition layer, i.e.,

$$\tilde{\mu}_2 = \frac{\partial r_0}{\partial t_1} z + C_2(s,t_1) \quad \text{for } z \in (-\infty, \infty), \tag{5.8}$$

provided we have already waited long enough for the whole transition profile to equilibrate.

To find $\frac{\partial r_0}{\partial t_1}$, we recall the form of $\tilde{\mu}_2$ in (2.25). From (4.7) we deduce that $\mathcal{L}\tilde{u}_1 = \tilde{B}_1 \varphi_1$, where $\varphi_1$ is the function in Lemma 2.3. Then we can simplify (2.25) to get

$$\begin{aligned} \tilde{\mu}_2 = {} & \mathcal{L}(-\tilde{u}_{2zzz} - \kappa_0 \tilde{u}_{1z} - z\kappa_1 U_z + W''(U)\tilde{u}_2 + \frac{1}{2} W'''(U)\tilde{u}_1^2) \\ & + (-\kappa_0 \partial_z + W'''(U)\tilde{B}_1 \varphi_2)(\tilde{B}_1 \varphi_1 - \kappa_0 U_z). \end{aligned} \tag{5.9}$$

The solvability condition for (5.9) gives

$$\int_{-\infty}^{\infty} [\tilde{\mu}_2 - (-\kappa_0 \partial_z + W'''(U)\tilde{B}_1 \varphi_2)(\tilde{B}_1 \varphi_1 - \kappa_0 U_z)] U_z dz = 0. \tag{5.10}$$

Since $U, \varphi_1, \varphi_2$ are even functions and $\tilde{B}_1$ is independent of $z$, we simplify the left-hand side of (5.10) to get

$$\int_{-\infty}^{\infty} [\tilde{\mu}_2 + \tilde{B}_1 \kappa_0 ({\varphi_1}' + W'''(U)\varphi_2 U_z)] U_z dz = 0. \tag{5.11}$$

Using Lemma 2.3, we rewrite the last term of (5.11) as

$$\begin{aligned} \int_{-\infty}^{\infty} W'''(U) U_z^2 \varphi_2 dz &= -\int_{-\infty}^{\infty} \varphi_2 \mathcal{L}(U_{zz}) dz = -\langle \varphi_2, \mathcal{L}(U_{zz}) \rangle \\ &= -\langle \mathcal{L}\varphi_2, U_{zz} \rangle = -\langle \varphi_1, U_{zz} \rangle. \end{aligned} \tag{5.12}$$

And since $\mathcal{L}(\frac{z}{2} U_z) = -U_{zz}$, we get

$$\begin{aligned} \int_{-\infty}^{\infty} W'''(U) U_z^2 \varphi_2 dz &= -\langle \varphi_1, U_{zz} \rangle = \left\langle \varphi_1, \mathcal{L}\left(\frac{z}{2} U_z\right) \right\rangle \\ &= \left\langle \frac{z}{2} U_z, \mathcal{L}\varphi_1 \right\rangle = \left\langle \frac{z}{2} U_z, 1 \right\rangle \\ &= \int_{-\infty}^{\infty} \frac{z}{2} U_z dz = -\int_{-\infty}^{\infty} \frac{U}{2} dz. \end{aligned} \tag{5.13}$$

Using (5.13), the second term of (5.11) is simplified to

$$\tilde{B}_1 \kappa_0 \int_{-\infty}^{\infty} {\varphi_1}' U_z dz = -\tilde{B}_1 \kappa_0 \int_{-\infty}^{\infty} \varphi_1 U_{zz} dz = -\frac{\tilde{B}_1 \kappa_0}{2} \int_{-\infty}^{\infty} U dz. \tag{5.14}$$

By (5.8), the first term of (5.11) is

$$\int_{-\infty}^{\infty} \tilde{\mu}_2 U_z dz = -\int_{-\infty}^{\infty} \tilde{\mu}_{2z} U dz = -\frac{\partial r_0}{\partial t_1} \int_{-\infty}^{\infty} U dz. \tag{5.15}$$

Substituting (5.13), (5.14), and (5.15) into (5.11) we get

$$\left( -\frac{\partial r_0}{\partial t_1} - \tilde{B}_1 \kappa_0 \right) \int_{-\infty}^{\infty} U dz = 0. \tag{5.16}$$

Since $\int_{-\infty}^{\infty} U dz \neq 0$, we get

$$\frac{\partial r_0}{\partial t_1} = -\tilde{B}_1 \kappa_0. \tag{5.17}$$

Finally, since $\tilde{\mu}_{2z} = \frac{\partial r_0}{\partial t_1} = -\tilde{B}_1\kappa_0$ is independent of $z$, then $\lim_{z\to\infty} \tilde{\mu}_{2z} = \lim_{z\to-\infty} \tilde{\mu}_{2z}$, and together with the match condition (3.12) we get

$$\partial_{\mathbf{n}}\mu_1^+ = \partial_{\mathbf{n}}\mu_1^-, \text{ i.e., } [\![\partial_{\mathbf{n}}\mu_1]\!] = 0. \tag{5.18}$$

We summarize the $t_1 = \varepsilon t$ evolution in the following model:

$$\begin{aligned} \frac{\partial \mu_1}{\partial t_1} &= \nabla\cdot(\mu_1\nabla\mu_1) \quad \text{in } \Omega\backslash\Gamma, \\ [\![\partial_{\mathbf{n}}\mu_1]\!] &= 0 \quad \text{on } \Gamma, \\ \mu_1 &\text{ is periodic on } \partial\Omega, \\ \mu_1(x,0) &= \Psi_1(x) \quad \text{in } \Omega, \end{aligned}$$

and the leading order of the normal velocity is

$$V_{\mathbf{n}} = -\frac{\partial r_0}{\partial t_1} = \tilde{B}_1\kappa_0,$$

where $\tilde{B}_1(s,t_1) = \lim_{r\to 0}\mu_1(\phi(s,t_1)+r\mathbf{n})$. Hence $\tilde{B}_1(s,t_1) = \mu_1(\phi(s,t_1))$ on $\Gamma$ if $\mu_1$ is continuous across $\Gamma$. By Assumption 1.3 and equality (3.8), $\mu_1(x,t_1) = 16u_1(x,t_1) \geq 0$, so the above porous medium equation is well-posed.

## 6. Time scale $t_2 = \varepsilon^2 t$.

**6.1. Outer expansion.** Since $\partial_t = \varepsilon^2\partial_{t_2}$ and $u_0 \equiv 0$, we have $u_t = \varepsilon^3 u_{1t_2} + O(\varepsilon^4)$. Matching with (3.9), the $\varepsilon^2$ terms give

$$\nabla\cdot(u_1\nabla\mu_1) = 0 \quad \text{in } \Omega_\pm.$$

Combining with (5.18) and recalling that $\mu_1 = 16u_1$, we obtain that $\mu_1$ satisfies a quasi-stationary porous medium equation

$$\begin{aligned} \nabla\cdot(\mu_1\nabla\mu_1) &= 0 \quad \text{in } \Omega\backslash\Gamma, \\ [\![\partial_{\mathbf{n}}\mu_1]\!] &= 0 \quad \text{on } \Gamma, \\ \mu_1 &\text{ is periodic on } \partial\Omega. \end{aligned} \tag{6.1}$$

Using integration by parts with periodic boundary condition, equation (6.1) implies that $\Delta(\mu_1^2) = 0$ in $\Omega$. Since $\mu_1 = 16u_1 \geq 0$, and with periodic boundary condition, it follows from the maximum principle that $\mu_1$ is a spatial constant $\mu_1(x,t_2) = B_1(t_2)$ for all $x \in \Omega$. Consequently, $\partial_{\mathbf{n}}\mu_1 = 0$ on $\Gamma$, and by the continuity of the inner and outer expansions of $\mu$, we have $\tilde{\mu}_1 = \tilde{B}_1(s,t_2) = B_1(t_2)$. Hence $\tilde{u}_1 = B_1(t_2)\varphi_2(z)$.

**6.2. Inner expansion.** We have

$$u_t = \varepsilon^2\left(\tilde{u}_{t_2} + \nabla_s\tilde{u}\cdot\frac{\partial s}{\partial t_2}\right) + \varepsilon\frac{\partial r}{\partial t_2}\tilde{u}_z = \varepsilon\frac{\partial r}{\partial t_2}\tilde{u}_{0z}\chi_0 + O(\varepsilon^2). \tag{6.2}$$

Matching with (2.33), the order 1 terms give

$$P_0 = \chi_0\frac{\partial}{\partial z}(U\tilde{\mu}_{2z}) = 0. \tag{6.3}$$

Hence

$$\frac{\partial}{\partial z}(U\tilde{\mu}_{2z}) = 0 \quad \text{for } z \in (-\eta, \eta). \tag{6.4}$$

So there is $a_2$ independent of $z \in (-\eta, \eta)$ such that $U\tilde{\mu}_{2z} = a_2$. Since $U \to O(\varepsilon)$ as $z \to \pm\eta$ and $\tilde{\mu}_{2z} = a_2 U^{-1}$, the only way for $\tilde{\mu}_{2z}$ to remain $O(1)$ is that $a_2 = 0$, and thus $\tilde{\mu}_{2z} = 0$ for $z \in (-\eta, \eta)$. This is also consistent with the behavior of $\tilde{\mu}_2$ as $z \to \pm\infty$, which is $\lim_{z\to\pm\infty} \tilde{\mu}_{2z} = \partial_{\mathbf{n}}\mu_1^{\pm} = 0$ by the match condition (3.12). So we expect the equilibrium state of $\tilde{\mu}_2$ to be independent of $z$ in the whole transition layer. Thus there exists $\tilde{B}_2(s, t_2)$ such that $\tilde{\mu}_2(z, s, t_2) = \tilde{B}_2(s, t_2)$, which is determined by matching the inner expansion $\tilde{\mu}_2$ with the outer expansion $\mu_2$ using

$$\lim_{r\to 0^{\pm}} \mu_2(\phi(s, t_2) + r\mathbf{n}) = \lim_{z\to\pm\infty} \tilde{\mu}_2(z, s, t_2) = \tilde{B}_2(s, t_2).$$

Then using Lemma 2.3 and (2.25), with $\tilde{u}_1 = B_1\varphi_2$, we get

$$\begin{aligned}\tilde{B}_2 &= \mathcal{L}\left(\mathcal{L}\tilde{u}_2 - \kappa_0 B_1 \varphi_2{}' - z\kappa_1 U_z + \frac{1}{2}W^{(4)}(U)B_1^2\varphi_2^2\right)\\ &\quad + [-\kappa_0\partial_z + W'''(U)B_1\varphi_2][B_1\mathcal{L}\varphi_2 - \kappa_0 U_z]\\ &= \mathcal{L}^2(\tilde{u}_2) - \kappa_0 B_1\mathcal{L}(\varphi_2{}') + z\kappa_1 U_{zz} + \frac{1}{2}W^{(4)}(U)B_1^2\mathcal{L}(\varphi_2^2) - \kappa_0 B_1\varphi_1{}'\\ &\quad + \kappa_0^2 U_{zz} + W'''(U)B_1^2\varphi_1\varphi_2 - \kappa_0 W'''(U)B_1\varphi_2 U_z.\end{aligned} \tag{6.5}$$

By the solvability condition, to solve for $\tilde{u}_2$, we need

$$\Big[\tilde{B}_2 + \kappa_0 B_1\mathcal{L}(\varphi_2{}') - z\kappa_1 U_{zz} - \frac{1}{2}W^{(4)}(U)B_1^2\mathcal{L}(\varphi_2^2) + \kappa_0 B_1\varphi_1{}' \\ -\kappa_0^2 U_{zz} - W'''(U)B_1^2\varphi_1\varphi_2 + \kappa_0 W'''(U)B_1\varphi_2 U_z\Big] \perp U_z.$$

Simplifying the solvability condition integral, we get

$$B_1\kappa_0 \int_{-\infty}^{\infty} (\varphi_1{}' + W'''(U)\varphi_2 U_z)U_z dz = 0. \tag{6.6}$$

Since $(\varphi_1{}' + W'''(U)\varphi_2 U_z)U_z$ is even, we assume that $\int_{-\infty}^{\infty}(\varphi_1{}' + W'''(U)\varphi_2 U_z)U_z dz \neq 0$. Thus $B_1 \equiv 0$, and hence

$$\tilde{u}_1 \equiv 0,\ \mu_1 \equiv 0,\ \text{and } u_1 = \frac{1}{16}\mu_1 \equiv 0, \tag{6.7}$$

and (6.5) is simplified to

$$\tilde{B}_2 = \mathcal{L}\tilde{u}_2 + (2\kappa_1 + \kappa_0^2)U_{zz}. \tag{6.8}$$

And since $U_{zz}$ and $\tilde{B}_2$ are orthogonal to $Ker(\mathcal{L})$, we can solve for $\tilde{u}_2$:

$$\tilde{u}_2 = \tilde{B}_2\varphi_2 - (2\kappa_1 + \kappa_0^2)\psi_2, \tag{6.9}$$

where $\psi_j \perp U_z$ are even functions that satisfy $\mathcal{L}\psi_1 = U_{zz}$ and $\mathcal{L}\psi_2 = \psi_1$.

Expanding the normal distance $r = r_0 + \varepsilon r_1 + O(\varepsilon^2)$, and matching (2.33) with (6.2), the $\varepsilon$ terms give

$$\frac{\partial r_0}{\partial t_2} U_z \chi_0 = P_1 = \chi_0 \frac{\partial}{\partial z}(U\tilde{\mu}_{3z}). \tag{6.10}$$

Hence

$$\frac{\partial}{\partial z}\left(U\left(\frac{\partial r_0}{\partial t_2}-\tilde{\mu}_{3z}\right)\right) \quad \text{for } z\in(-\eta,\eta). \tag{6.11}$$

Using an argument similar to that in section 6, we get

$$\frac{\partial r_0}{\partial t_2}=\tilde{\mu}_{3z} \quad \text{for } z\in(-\eta,\eta). \tag{6.12}$$

Hence $\tilde{\mu}_3=\frac{\partial r_0}{\partial t_2}z+C_3(s,t_2)$ for $z\in(-\eta,\eta)$. Again, it is reasonable to assume that $\tilde{\mu}_3$ is linear in terms of $z$ in the whole transition layer, i.e.,

$$\tilde{\mu}_3=\frac{\partial r_0}{\partial t_2}z+C_3(s,t_2) \quad \text{for } z\in(-\infty,\infty), \tag{6.13}$$

provided we have already waited long enough for the whole transition profile to equilibrate.

To find $\frac{\partial r_0}{\partial t_2}$, we recall the form of $\tilde{\mu}_3$ in (2.26), and since $\tilde{u}_0=U,\tilde{u}_1\equiv 0$, we simplify (2.26) to get

$$\begin{aligned}\tilde{\mu}_3&=\mathcal{L}(\mathcal{L}\tilde{u}_3-\kappa_0\tilde{u}_{2z}-z^2\kappa_2U_z)-\kappa_0\partial_z(\mathcal{L}\tilde{u}_2-z\kappa_1U_z)\\&\quad+(-z\kappa_1\partial_z+W'''(U)\tilde{u}_2-\Delta_s-\eta_2)(-\kappa_0U_z).\end{aligned} \tag{6.14}$$

Using (6.9), we rewrite (6.14) as

$$\mathcal{L}(\mathcal{L}\tilde{u}_3-\kappa_0\tilde{u}_{2z}-z^2\kappa_2U_z)=R_2, \tag{6.15}$$

where

$$\begin{aligned}R_2&=\tilde{\mu}_3+\kappa_0\tilde{B}_2({\varphi_1}'+W'''(U)\varphi_2U_z)-(2\kappa_1+\kappa_0^2)\kappa_0({\psi_1}'+W'''(U)\psi_2U_z)\\&\quad-\kappa_0\kappa_1(U_z+2zU_{zz})-(\Delta_s+\eta_2)\kappa_0U_z.\end{aligned}$$

By the solvability condition, to solve for $\tilde{u}_3$, we require $R_2\perp U_z$. We examine the terms of the integral $\int_{-\infty}^{\infty}R_2U_z dz$ one by one. From (6.13), using integration by parts we get

$$\int_{-\infty}^{\infty}\tilde{\mu}_3U_z dz=-\int_{-\infty}^{\infty}\tilde{\mu}_{3z}U dz=-\frac{\partial r_0}{\partial t_2}\int_{-\infty}^{\infty}U dz. \tag{6.16}$$

Using Lemma 2.3 and integration by parts, we get

$$\begin{aligned}\int_{-\infty}^{\infty}({\varphi_1}'+W'''(U)\varphi_2U_z)U_z dz&=-\int_{-\infty}^{\infty}\varphi_1U_{zz}dz-\langle\mathcal{L}U_{zz},\varphi_2\rangle\\&=-\langle U_{zz},\varphi_1\rangle-\langle U_{zz},\mathcal{L}\varphi_2\rangle\\&=-2\langle U_{zz},\varphi_1\rangle=\langle\mathcal{L}(zU_z),\varphi_1\rangle\\&=\langle zU_z,\mathcal{L}\varphi_1\rangle=\int_{-\infty}^{\infty}zU_z dz\\&=-\int_{-\infty}^{\infty}U dz.\end{aligned} \tag{6.17}$$

Similarly, we have

$$\begin{aligned}\int_{-\infty}^{\infty}(\psi_1{}' + W'''(U)\psi_2 U_z)U_z dz &= \langle zU_z, \mathcal{L}\psi_1\rangle = \langle zU_z, U_{zz}\rangle \\ &= \int_{-\infty}^{\infty} zU_zU_{zz}dz = \frac{1}{2}\int_{-\infty}^{\infty} zd(U_z^2) \\ &= -\frac{1}{2}\int_{-\infty}^{\infty} U_z^2 dz. \end{aligned} \tag{6.18}$$

Using integration by parts, it is easy to see that

$$\int_{-\infty}^{\infty}(U_z + 2zU_{zz})U_z dz = 0. \tag{6.19}$$

Substituting (6.16), (6.17), (6.18), and (6.19) into the solvability condition integral $\int_{-\infty}^{\infty} R_2 U_z dz$, we get

$$\begin{aligned}-\frac{\partial r_0}{\partial t_2}\int_{-\infty}^{\infty} U dz - \kappa_0 \tilde{B}_2 \int_{-\infty}^{\infty} U dz + (2\kappa_1 + \kappa_0^2)\frac{\kappa_0}{2}\int_{-\infty}^{\infty} U_z^2 dz \\ -(\Delta_s + \eta_2)\kappa_0 \int_{-\infty}^{\infty} U_z^2 dz = 0.\end{aligned} \tag{6.20}$$

Hence the leading order of the normal velocity is

$$V_{\mathbf{n}} = \left(\tilde{B}_2 + \frac{m_2}{m_1}\eta_2\right)\kappa_0 + \frac{m_2}{m_1}\left(\Delta_s\kappa_0 - \frac{\kappa_0^3}{2} - \kappa_0\kappa_1\right), \tag{6.21}$$

where we have introduced the constants

$$m_1 = \int_{-\infty}^{\infty} U dz \quad \text{and} \quad m_2 = \int_{-\infty}^{\infty} U_z^2 dz.$$

To determine $\tilde{B}_2 = \tilde{\mu}_2$, we need to find $\mu_2$ and then determine $\tilde{\mu}_2$ by matching with $\mu_2$ using

$$\lim_{r\to 0^{\pm}} \mu_2(\phi(s,t_2) + r\mathbf{n}) = \lim_{z\to\pm\infty} \tilde{\mu}_2(z,s,t_2) = \tilde{B}_2(s,t_2).$$

To find $\mu_2$, we need the $\varepsilon^2$ terms in the outer expansions on both sides of (1.2). Since $u_0, u_1, \mu_0$, and $\mu_1$ are both 0, we simplify (1.2) to get

$$\mu_2 = W''(u_0)^2 u_2 = 16u_2. \tag{6.22}$$

Substituting $u = \varepsilon^2 u_2 + O(\varepsilon^3)$ and $\mu = \varepsilon^2\mu_2 + O(\varepsilon^3)$ into (1.1) then simplifying, we get

$$\varepsilon^4\frac{\partial u_2}{\partial t_2} + O(\varepsilon^5) = \varepsilon^4\nabla\cdot(u_2\nabla\mu_2) + O(\varepsilon^5). \tag{6.23}$$

Hence the $\varepsilon^4$ terms give

$$\frac{\partial u_2}{\partial t_2} = \nabla\cdot(u_2\nabla\mu_2). \tag{6.24}$$

Combining (6.22) and (6.24), it turns out that $\mu_2$ satisfies a porous medium equation

$$\frac{\partial \mu_2}{\partial t_2} = \nabla \cdot (\mu_2 \nabla \mu_2) \quad \text{in } \Omega\backslash\Gamma. \tag{6.25}$$

By (6.13), $\lim_{z\to-\infty} \tilde{\mu}_{3z} = \lim_{z\to\infty} \tilde{\mu}_{3z}$, together with the match condition (3.13), we get

$$\partial_{\mathbf{n}} \mu_2^+ = \partial_{\mathbf{n}} \mu_2^-, \text{ i.e., } [\![\partial_{\mathbf{n}} \mu_2]\!] = 0. \tag{6.26}$$

We summarize the $t_2 = \varepsilon^2 t$ evolution in the following model:

$$\begin{aligned} \frac{\partial \mu_2}{\partial t_2} &= \nabla \cdot (\mu_2 \nabla \mu_2) \quad \text{in } \Omega\backslash\Gamma, \\ [\![\partial_{\mathbf{n}} \mu_2]\!] &= 0 \quad \text{on } \Gamma, \\ \mu_2 & \text{ is periodic on } \partial\Omega, \\ \mu_2(x,0) &= \Psi_2(x) \quad \text{in } \Omega, \end{aligned}$$

and the leading order of the normal velocity is

$$V_{\mathbf{n}} = \left(\tilde{B}_2 + \frac{m_2}{m_1}\eta_2\right)\kappa_0 + \frac{m_2}{m_1}\left(\Delta_s \kappa_0 - \frac{\kappa_0^3}{2} - \kappa_0\kappa_1\right),$$

where $\tilde{B}_2(s,t_2) = \lim_{r\to 0} \mu_2(\phi(s,t_2) + r\mathbf{n})$. Hence $\tilde{B}_2(s,t_2) = \mu_2(\phi(s,t_2))$ on $\Gamma$ if $\mu_2$ is continuous across $\Gamma$. By Assumption 1.4 and (6.22), $\mu_2(x,t_2) = 16u_2(x,t_2) \geq 0$, so the above porous medium equation is well-posed.

**6.3. The mass constraint.** Even though we cannot get an explicit formula for $\tilde{B}_2(s,t_2)$, we may extract a bit more information about it by looking into the conservation of mass. The total mass is

$$M = \int_\Omega u(x,t)dx, \tag{6.27}$$

which is fixed by the initial data. In the outer region $\Omega\backslash\Gamma_l$, we have the expansion

$$u = u_0 + \varepsilon u_1 + O(\varepsilon^2), \tag{6.28}$$

where $u_0 \equiv 0$ and $u_1 \equiv 0$. In the inner region $\Gamma_l$, the inner expansion is

$$\tilde{u} = \tilde{u}_0 + \varepsilon \tilde{u}_1 + O(\varepsilon^2), \tag{6.29}$$

where $\tilde{u}_0 = U(z)$ and $\tilde{u}_1 = B_1(t_2)\varphi_2(z) \equiv 0$. We insert these expansions into (6.27) to get

$$M = \int_{\Gamma_l} U(z)dx + O(\varepsilon^2). \tag{6.30}$$

Assuming that $|\Gamma| = O(1)$, changing to whiskered coordinates, and using the local integral formula (2.11) and the Jacobian expansion (2.7), we get

$$\int_{\Gamma_l} U(z)dx = \int_Q \int_{-l/\varepsilon}^{l/\varepsilon} U(z)J(s,z)dzds = \varepsilon \int_Q \int_{-l/\varepsilon}^{l/\varepsilon} U(z)dzds. \tag{6.31}$$

Substituting (6.31) into (6.30) we get

$$M = \varepsilon \int_Q \int_{-l/\varepsilon}^{l/\varepsilon} U(z)dzds + O(\varepsilon^2). \tag{6.32}$$

We expand $M = \varepsilon M_1 + \varepsilon^2 M_2 + O(\varepsilon^3)$ and the surface area $|\Gamma| = \gamma_0 + \varepsilon\gamma_1 + O(\varepsilon^2)$. Since $U(z) \geq O(\varepsilon)$ with $|z|$ large enough, then

$$\int_{-\infty}^{\infty} U(z)dz = \int_{-l/\varepsilon}^{l/\varepsilon} U(z)dz + O(\varepsilon). \tag{6.33}$$

Hence

$$\int_Q \int_{-l/\varepsilon}^{l/\varepsilon} U(z)dzdx = |\Gamma| \int_{-\infty}^{\infty} U(z)dz + O(\varepsilon^2) = m_1|\Gamma| + O(\varepsilon^2), \tag{6.34}$$

where $m_1 = \int_{-\infty}^{\infty} U(z)dz$. Then we obtain $M_1 = m_1\gamma_0$, so $\gamma_0 = M_1/m_1$, and hence $d\gamma_0/dt_2 = 0$. On the other hand, when subject to a normal velocity $V_{\mathbf{n}}$, measured in time unit $t_2$, the interfacial surface area grows at the rate

$$\frac{d|\Gamma|}{dt_2} = \int_\Gamma \kappa_0(s)V_{\mathbf{n}}(s)ds. \tag{6.35}$$

Combining with (6.21), the interfaces $\Gamma(t)$ have the leading order growth

$$\frac{d}{dt_2}\gamma_0(t_2) = \int_\Gamma \left(\tilde{B}_2 + \frac{m_2}{m_1}\eta_2\right)\kappa_0^2 + \frac{m_2}{m_1}\left(\Delta_s\kappa_0 - \frac{\kappa_0^3}{2} - \kappa_0\kappa_1\right)\kappa_0 ds. \tag{6.36}$$

Since $d\gamma_0/dt_2 = 0$, $\tilde{B}_2(s, t_2)$ satisfies the identity

$$\int_\Gamma \left(\tilde{B}_2 + \frac{m_2}{m_1}\eta_2\right)\kappa_0^2 - \frac{m_2}{m_1}\left(|\nabla_s\kappa_0|^2 + \frac{\kappa_0^4}{2} + \kappa_0^2\kappa_1\right)ds = 0. \tag{6.37}$$

**7. Discussion.** We formally derive the sharp interface models for different time scales for the FCH equation with the cutoff diffusion mobility $M(u)$ that is degenerate for $u \leq 0$. Even with a degenerate mobility, we still get porous medium equations in the regions away from the interface, and that influences the evolution of the interface. The interface motion occurs in the $t = O(\varepsilon^{-1})$ time scale and is determined by porous medium diffusion processes in both phases with no jumps on the interface. In the longer $O(\varepsilon^{-2})$ time scale, the interface motion is a complex combination of porous medium diffusion processes in both phases and the property of mass conservation. The FCH equation with a degenerate mobility is more complicated than the one with a constant mobility, and is unlikely to obtain simple models. We have shown that the two phases of a degenerate FCH model communicate through a porous medium process in both phases, not by a common mean field on the interface like constant mobility ones (for example, see [10]). With a degenerate diffusion mobility, the long-range communication of the two phases becomes weaker through the diffusion process, but it is still connected. This means just the degenerate mobility itself is not enough to cut off the long-range communication. We need some other mechanism, such as some singularity in the double-well potential $W(u)$ as studied in [12], for the communication to be cut off completely.

In the process of self-assembly of vesicles, the existence of intermolecular interactions between vesicles makes the system difficult to reach equilibrium. Indeed, there are several types of noncovalent interactions, such as hydrophobic, electrostatic, hydrogen bonding, and van der Waals interactions [21, 24]. However, the structures eventually become stable once the system reaches equilibrium. To understand and explain this process mathematically, other strategies will be explored in our future studies.

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