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On the Cahn–Hilliard equation with no-flux and strong anchoring conditions

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Abstract. The Cahn–Hilliard equation is a common model to describe phase separation processes of a mixture of two components. We study the Cahn–Hilliard equation coupled with the homogeneous strong anchoring condition (i.e., homogeneous Dirichlet condition) on the relative concentration u of the two phases. Moreover, we adopt no-flux boundary condition to keep conservation of mass. With a specific quartic form of the double-well potential, we prove the existence and uniqueness of the weak solution to this model by interpreting the problem as a gradient flow of the Cahn–Hilliard free energy. Utilizing the minimizing movement scheme and time discretization method, we show that the approximation solutions converge to the weak solution of the Cahn–Hilliard equation. Finally, we prove that the weak solution satisfies an energy dissipation inequality.

Mathematics Subject Classification. 35B36, 35B30, 35K35, 35K61, 35Q92.

1. Introduction

The Cahn–Hilliard equation

$$\partial_t u = \Delta(-\kappa\Delta u + W'(u)), \quad x \in \Omega, t \in (0, \infty), \tag{1.1}$$

was proposed in 1958 to model phase separation that occurs in binary alloys [12,13]. Here, Ω is a bounded domain in $\mathbb{R}^d$, $d = 1, 2, 3, \ldots$. $u(x, t)$ is the relative concentration of the two phases, which evolves in time and space, $W(u)$ is a double-well potential with two equal minima at $u^- < u^+$ corresponding to the two pure phases, and $\kappa > 0$ is a parameter whose square root $\sqrt{\kappa}$ is proportional to the thickness of the transition region between the two phases. The Cahn–Hilliard equation appears in modeling many other phenomena, including the dynamics of two populations [17], the biomathematical modeling of a bacterial film [40], phase separations in polymers [54], the growth of tumor tissues [60], and certain thin film problems [49,57]. It has been found to help

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describe various phenomena ranging from nanoscale precipitation [54,63] to the clumping of galaxies in the universe [52]. This equation also has applications in image processing, where it is used for image inpainting and segmentation[9,61]. Furthermore, it plays an important role in phase field methods [48], where it is employed to model behavior of conserved order variables.

To model the two-phase system, different forms of double-well potential $W(u)$ have been used. A common choice is the smooth double-well potential, for example,

$$W(u) = \gamma(u - u^+)^2(u - u^-)^2, \quad \gamma > 0, \tag{1.2}$$

since it is convenient for theoretical analysis and numerical simulations. In this paper we will focus on double-well potentials of the form (1.2). Other choices for the double-well potential $W(u)$ include double-barrier potentials and logarithmic potentials, see, e.g., [11,20] for a comparison of these potentials.

The Cahn–Hilliard equation (1.1) is a fourth-order parabolic equation. To obtain a well-posed problem, we need to complement (1.1) with an initial condition

$$u(x, 0) = \Phi(x) \quad \text{in } \Omega, \tag{1.3}$$

and boundary conditions. A common choice for the boundary conditions is the homogeneous Neumann conditions (see, e.g., [4–6,11,28,34,42,50,58]),

$$\partial_{\mathbf{n}} u = 0, \tag{1.4}$$

$$\partial_{\mathbf{n}}(-\kappa\Delta u + W'(u)) = 0, \tag{1.5}$$

where $\mathbf{n}$ is the exterior unit vector normal to the boundary $\partial\Omega$. Since the flux is postulated to be [24]

$$\mathbf{J} = -\nabla(-\kappa\Delta u + W'(u)), \tag{1.6}$$

the condition (1.5) is also called the no-flux boundary condition, which guarantees the conservation of mass. The Neumann condition (1.4) for u is for mathematical convenience, but a consequence is that near the boundary $\partial\Omega$, the level surfaces of u are required to be perpendicular to $\partial\Omega$ [21]. Another common choice for the boundary conditions is the periodic boundary condition, which also conserves mass and has been widely used particularly in computational studies (cf. e.g., [14–16,18,20,31,37–39,62]). Under either Neumann boundary condition (1.4)–(1.5) or the periodic boundary condition, it is easy to show that the Cahn–Hilliard energy functional

$$E(u) = \int_\Omega \frac{\kappa}{2}|\nabla u|^2 + W(u)\, dx. \tag{1.7}$$

is decreasing in time.

In materials science, it is common that we control the assembly of complex structures through templated substrates or boundaries. The structure formed by self-assembly is strongly affected by the boundary surface pattern and interactions between monomers and the surface. Therefore, we can fine tune the self-assembled structures by altering either the surface pattern or the interactions (cf. [43,59,64] and references therein). Mathematically this means that we want u to match a prescribed function ϕ on $\partial\Omega$. There are different ways to measure how well u matches ϕ on $\partial\Omega$. The strongest match is a pointwise Dirichlet boundary condition

$$u = \phi \quad \text{on } \partial\Omega, \tag{1.8}$$

and we call it the strong anchoring condition. We may also use weaker anchoring conditions. For instance, we may prescribe a tolerance for the $L^2(\partial\Omega)$ norm $\|u-\phi\|_{L^2(\partial\Omega)}$. Since for any given $t > 0$ $u(\cdot,t)$ needs to be at least $H^1(\Omega)$ in space, the trace theorem requires that $u|_{\partial\Omega}(\cdot,t)$ to be in $H^{1/2}(\partial\Omega)$. So for the strong anchoring condition to make sense, we need at least $\phi \in H^{1/2}(\partial\Omega)$. In contrast, the aforementioned L^2–weak anchoring condition is meaningful as long as $\phi \in L^2(\partial\Omega)$. The strong anchoring condition enjoys the advantage of being simple and relatively easy to enforce numerically, while the weak anchoring conditions have broader applications due to the less stringent restrictions on ϕ.

There have been only a few studies that investigated Cahn–Hilliard type problems with Dirichlet boundary conditions for u. In some instances, the equation is complemented with a homogeneous Dirichlet boundary condition for $-\kappa\Delta u + W'(u)$, which can be used to model the propagation of a solidification front into an ambient medium which is at rest relative to the front [10,27,33]. Another work is [7], in which Bates and Han studied an integro-differential extension of the Cahn–Hilliard equation with Dirichlet boundary condition for u on a bounded domain. In addition, people have studied the Dirichlet boundary value problems for the biharmonic equation [26], the Schrödinger equation [25], and the regularized long wave equation [47].

We will explore the strong anchoring condition (1.8) coupled with the no-flux boundary condition (1.5). The latter is a natural requirement since in template modulated pattern formations, the system boundary is usually impermeable, i.e., no mass flows across the boundary. We want to emphasize that our setting is different from tumor growth problems (for example, [33]), in which there is no mass conservation since the tumor is growing (or shrinking). In the literature there are very few studies about problems in our setting. The only paper we found is [44], in which Li, Jeong, Shin, and Kim presented a conservative numerical method for the Cahn–Hilliard equation with Dirichlet boundary condition on u and Neumann boundary condition (1.5) in complex domains. The purpose of our work is to lay the theoretical foundation by establishing the wellposedness of the Cahn–Hilliard equation (1.1) with the no-flux condition (1.5) and the strong anchoring condition (1.8).

Due to nonlinearity, the solution u, if exists, depends not only on the boundary value ϕ, but on a delicate relation between ϕ and the double well potential W. Indeed, in [21] we have proved that with a Dirichlet boundary condition $u = \phi$ on $\partial\Omega$ but without mass conservation, the properties of the minimizers for the Cahn–Hilliard energy functional (1.7) are determined by the symmetry of W. To be more precise, in the symmetric case when W is chosen as (1.2) and ϕ takes the well-mixed homogeneous value

$$\phi = \frac{u^+ + u^-}{2}, \tag{1.9}$$

there is a bifurcation phenomena as the value of κ varies. If ϕ is uniformly above or uniformly below the homogeneous value, the symmetry breaks and the bifurcation does not exist. This motivated us to start our study with the homogeneous boundary value (1.9), and leave the nonhomogeneous situation for future studies.

1.1. Main result

To simplify notations, we choose the smooth quartic double-well potential

$$W(u) = \frac{1}{4}(u^2 - 1)^2, \tag{1.10}$$

which has two minima at $u^{\pm} = \pm 1$. In this setting, the homogeneous boundary value becomes $\phi = 0$ and the corresponding strong anchoring condition for u becomes

$$u = 0 \text{ on } \partial\Omega. \tag{1.11}$$

We also set $\kappa = 1$ since it does not play any role in the analysis in this paper. Therefore the system we study is

$$\partial_t u = \Delta(-\Delta u + W'(u)) \quad (x,t) \in \Omega_T := \Omega \times (0,T), \tag{1.12}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.13}$$

$$\partial_{\mathbf{n}}(-\Delta u + W'(u)) = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.14}$$

where $T \in (0,\infty)$, and Ω is a bounded domain in $\mathbb{R}^d$ with a C^2 boundary $\partial\Omega$. We concentrate on $d = 2, 3$. Moreover, we assume the initial data to be

$$u(x,0) = \Phi(x) \quad \text{in } \Omega, \tag{1.15}$$

where $\Phi \in H_0^1(\Omega)$ is a given function, with initial total mass

$$\mathcal{M} := \int_\Omega \Phi(x)\, dx. \tag{1.16}$$

Since we set $\kappa = 1$, the Cahn–Hilliard energy functional is now defined by

$$E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + W(u)\, dx. \tag{1.17}$$

Recall that $H^1(\Omega)/\mathbb{R}$ is the quotient space of equivalence classes of H^1 functions in which two functions are equivalent if they vary by only a spatial constant.

Definition 1.1. A function $u \in L^2(0,T;H_0^1(\Omega))$ is said to be a weak solution to the Cahn–Hilliard system (1.12) with strong anchoring condition (1.13), no-flux boundary condition (1.14), and initial value (1.15) with prescribed total mass (1.16), if there exists a corresponding chemical potential function $\mu \in L^2(0,T;H^1(\Omega)/\mathbb{R})$ uniquely determined by u such that the following conditions hold:

(i) For any $\xi \in L^2(0,T;H^1(\Omega))$ with $\partial_t\xi \in L^2(\Omega_T)$ and $\xi(T) = 0$, the following integral equality holds:

$$\int_0^T \int_\Omega (u-\Phi)\partial_t\xi \, dxdt = \int_0^T \int_\Omega \nabla\mu \cdot \nabla\xi \, dxdt. \tag{1.18}$$

(ii) For any $\eta \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(\Omega_T)$ with $\int_\Omega \eta(x,t)\, dx = 0$ for all $t \in [0,T]$, the following integral equality holds:

$$\int_0^T \int_\Omega \mu\eta \, dxdt = \int_0^T \int_\Omega \nabla u \cdot \nabla\eta + W'(u)\eta \, dxdt. \tag{1.19}$$

(iii) $u(x,0) = \Phi(x)$ for a.e. $x \in \Omega$.
(iv) $\int_\Omega u(x,t)\, dx = \mathcal{M}$ for a.e. $t \in [0,T]$.

Theorem 1.2. *Let $T \in (0,\infty)$ and $\Phi \in H_0^1(\Omega)$ with $\int_\Omega \Phi\, dx = \mathcal{M}$ and $E(\Phi) < \infty$. There exists a unique function $u \in L^\infty(0,T;H_0^1(\Omega)) \cap C^{0,\beta}([0,T];L^4(\Omega))$, where $\beta = 1/8$ if $d = 2$ and $\beta = 1/16$ if $d = 3$, that is a weak solution to the Cahn–Hilliard equation (1.12)–(1.16) in the sense defined in Definition 1.1. In addition, u and its corresponding chemical potential function $\mu \in L^2(0,T;H^1(\Omega)/\mathbb{R})$ satisfy the following energy inequality:*

$$E(u(\cdot,t)) + \frac{1}{2}\int_0^t ||\nabla\mu(s)||_{L^2(\Omega)} ds \le E(\Phi) \quad \text{for any } t \in [0,T]. \tag{1.20}$$

For convenience we may pick a unique representative of μ by requiring $\int_\Omega \mu(x,t)\, dx = 0$ for all $t > 0$.

Remark 1.3. The requirement for the test function η in (1.19) to have zero average is due to the conservation of mass in the calculation of the variational derivative of the free energy E.

Remark 1.4. The solution u is global in time since $T > 0$ is arbitrary.

Remark 1.5. We need the specific form (1.10) of W to prove the uniqueness of the weak solution. For the nonhomogeneous boundary condition $u = \phi$, we can make a shift $\tilde{u} := u - \phi$ and write the following equations for $\tilde{u}$:

$$\begin{aligned} \tilde{u}_t &= \Delta(-\Delta\tilde{u} + \tilde{W}'(\tilde{u})), \\ \tilde{u} &= 0 \quad \text{on } \partial\Omega, \\ \partial_{\mathbf{n}}(-\Delta\tilde{u} + \tilde{W}'(\tilde{u})) &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Here $\tilde{W}(\tilde{u}) := W(\tilde{u} + \phi) - \tilde{u}\Delta\phi$. Following the same lines, we can still prove the existence of a weak solution $\tilde{u}$. However, since the new potential $\tilde{W}$ takes a different form, the uniqueness argument breaks down. We will explore the uniqueness of the nonhomogeneous problem in future studies.

The relation between u and μ is as follows. Suppose there are sufficiently regular functions u and μ that satisfy (1.19). Then

$$\int_\Omega \mu\eta \, dx = \int_\Omega (-\Delta u + W'(u))\eta \, dx \tag{1.21}$$

for any $\eta \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $\int_\Omega \eta \, dx = 0$. Since we require $\int_\Omega \eta \, dx = 0$, we can only conclude that

$$\mu = -\Delta u + W'(u) + g(t), \tag{1.22}$$

where $g(t)$ is an arbitrary spatial constant that may vary in time. That is, $\mu(\cdot, t) \in H^{-1}(\Omega)/\mathbb{R}$. The requirement $\int_\Omega \mu \; dx = 0$ is for us to remove the uncertainty induced by the arbitrary special constant $g(t)$. To satisfy $\int_\Omega \mu \; dx = 0$, $g(t)$ needs to be

$$g(t) = \frac{1}{|\Omega|} \int_\Omega (\Delta u - W'(u)) \, dx = \frac{1}{|\Omega|} \left(\int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} \, dS - \int_\Omega W'(u) \; dx \right) \tag{1.23}$$

for all $t \in [0, T]$.

The Cahn–Hilliard equation (1.12) is usually written in the form of a system of two equations:

$$\partial_t u = \Delta \omega, \quad (x, t) \in \Omega_T, \tag{1.24}$$

$$\omega = -\Delta u + W'(u), \quad (x, t) \in \Omega_T. \tag{1.25}$$

This formulation is fine if we choose the periodic boundary condition or the Neumann boundary condition (1.4) for u. Indeed, the precise form of the Cahn-Hilliard equation should be written as

$$u_t = \Delta \frac{\delta E}{\delta u}, \tag{1.26}$$

where $\frac{\delta E}{\delta u}$ is the variational derivative of E. Under the periodic or Neumann condition (1.4) and without mass conservation, we do have

$$\frac{\delta E}{\delta u} = -\Delta u + W'(u) \quad \text{ a.e. in } \Omega.$$

However, with homogeneous Dirichlet condition $u = 0$ on $\partial\Omega$ and mass conservation, all we can get is $\frac{\delta E}{\delta u} = -\Delta u + W'(u) + g(t)$, where $g(t)$ is an arbitrary spatial constant, see Sect. 2.1 for details. In other words, $\frac{\delta E}{\delta u}$ is an equivalence class, which can be written in terms of elements in a quotient space,

$$\frac{\delta E}{\delta u} = -\Delta u + W'(u) \text{ in the quotient space } H^{-1}(\Omega)/\mathbb{R}.$$

Our μ defined by (1.22) with $g(t)$ given by (1.23) is a particular representative of $\frac{\delta E}{\delta u}$ in the quotient space. Compared with ω defined by (1.25), we see that

$$\omega = \mu - g(t), \tag{1.27}$$

where g is defined by (1.23). This spatial constant g does not play any role in the original Cahn–Hilliard equation (1.12) since $\nabla\omega = \nabla\mu$.

1.2. The minimizing movement scheme

Our proof of Theorem 1.2 utilizes the gradient flow structure and the minimizing movement scheme for the Cahn–Hilliard equation (1.12). We will give a brief introduction about the minimizing movement method.

1.2.1. Introduction of the minimizing movement method. The concept of *minimizing movement* involves the recursive minimization

(a) $\nu_\tau^0 := \tilde{\nu} \in \mathcal{S}$ is given,
(b) ν_τ^n is a minimizer for $\mathcal{F}(\cdot, \nu_\tau^{n-1}, \tau)$ for any $n = 1, 2, \ldots$,

for a given functional $\mathcal{F} : \mathcal{S} \times \mathcal{S} \times (0,1) \to [-\infty, \infty]$ on a topological space $(\mathcal{S}, \sigma)$. The purpose of a minimizing movement scheme is to study the limit of $\{\nu_\tau\}_{0<\tau<1}$ as $\tau \searrow 0$. The parameter $\tau \in (0,1)$ plays the role of discrete time step size. If a sequence $\{\nu_\tau^n\}_{n=1}^\infty$ satisfies the recursion (a) and (b), we call the corresponding piecewise constant interpolation

$$\begin{aligned} &\nu_\tau(0) := \tilde{\nu}, \\ &\nu_\tau(t) := \nu_\tau^n \quad \text{for all } t \in ((n-1)\tau, n\tau], n = 1, 2, \ldots, \end{aligned}$$

a *discrete solution*. This method plays an important role in the theory of existence of solution of differential equations since it provides a time discrete approximation solution for the differential equation while not requiring the initial condition $\tilde{\nu}$ to be smooth. Moreover, the interpolation method guarantees the compactness of the family of discrete solutions [35,36]. Therefore, the minimizing movement method has been used in many works to study the existence of solution for some classes of PDEs (cf. e.g., [1,2,32,46,51,53,56,65]).

1.2.2. Comparison with the Galerkin approximation method. Another common way to handle the Cahn–Hilliard equation (1.24)–(1.25) with homogeneous Neumann boundary conditions $\partial_{\mathbf{n}} u = \partial_{\mathbf{n}} \omega = 0$ (or periodic boundary conditions) is to use the Galerkin approximation. The idea is to define

$$u^N(x,t) = \sum_{j=1}^N c_j^N(t)\phi_j(x), \quad \omega^N(x,t) = \sum_{j=1}^N d_j^N(t)\phi_j(x), \quad N = 1, 2, 3, \ldots, \tag{1.28}$$

where $\phi_j (j = 1, 2, \ldots)$ are eigenfunctions of the eigenvalue problem $-\Delta u = \lambda u$ in Ω subject to homogeneous Neumann boundary condition (resp. periodic boundary condition), which form a complete orthonormal basis for $L^2(\Omega)$. We look for a pair of functions (u^N, ω^N) that solves the following system of ODEs for $\{c_j^N\}_{j=1}^N$

$$\int_\Omega \partial_t u^N \phi_j \, dx = -\int_\Omega \nabla \omega^N \cdot \nabla \phi_j \, dx, \tag{1.29}$$

$$\int_\Omega \omega^N \phi_j \, dx = \int_\Omega \left(\nabla u^N \cdot \nabla \phi_j + W'(u^N)\phi_j \right) dx, \tag{1.30}$$

$$u^N(x,0) = \sum_{j=1}^N \left(\int_\Omega \Phi \phi_j \, dx \right) \phi_j(x). \tag{1.31}$$

The next step is to prove that the sequence $\{u^N\}_{N=1}^\infty$ converges to some function u (in some suitable sense, up to a subsequence) which is a weak solution to (1.24)–(1.25) in some suitable sense. This method is efficient and has been used in many studies (cf. e.g., [3,8,19,22,23,29]).

However, the Galerkin approximation method has some limitations when it comes to mixed boundary conditions like (1.13)–(1.14). First, when defining u^N and ω^N in (1.28), we need to use different bases for u^N and ω^N corresponding to different boundary conditions for u and ω. This would make the calculation more complicated. Furthermore, the Galerkin approximation does not capture the feature that allows an extra spatial constant in the chemical potential as in (1.22). In the equation (1.24)–(1.25) u and ω are always paired, and the Galerkin approxiation method only works with one specific choice ω of the chemical potential, not with the general setting where a spatial constant in the chemical potential, like (1.22), is allowed. The minimizing movement method, on the other hand, can handle this problem elegantly and in the most precise way using functional analysis knowledge. By utilizing the gradient flow structure and minimizing movement method, we obtain the framework that helps us deal with general cases where a constant in the chemical potential is allowed.

2. Preliminaries

In this section we introduce some preliminaries that will be used in the rest of this paper.

(A1) For any real number $\mathcal{M}$, we define

$$X := \{u \in H^1(\Omega) : \int_\Omega u\, dx = 0\},$$
$$X_0 := \{u \in H^1_0(\Omega) : \int_\Omega u\, dx = 0\},$$
$$X_\mathcal{M} := \{u \in H^1_0(\Omega) : \int_\Omega u\, dx = \mathcal{M}\},$$

then X and X_0 are Hilbert spaces with respect to the inner product

$$(u, v)_{H^1(\Omega)} := \int_\Omega uv + \nabla u \cdot \nabla v\, dx, \quad u, v \in X. \tag{2.1}$$

However, for any $u \in X$, by Poincaré's inequality,

$$||\nabla u||_{L^2(\Omega)} \le ||u||_{H^1(\Omega)} \le C||\nabla u||_{L^2(\Omega)}. \tag{2.2}$$

Hence, in X, the inner product (2.1) is equivalent to the following inner product

$$(u, v)_X := \int_\Omega \nabla u \cdot \nabla v\, dx, \quad u, v \in X, \tag{2.3}$$

and X and X_0 are also Hilbert spaces with respect to the inner product (2.3).

(A2) For any $f \in H^1(\Omega)'$ with $\int_\Omega f\, dx = 0$, where $H^1(\Omega)'$ is the dual space of $H^1(\Omega)$, the Neumann problem

$$-\Delta v = f \quad \text{in } \Omega, \tag{2.4}$$
$$\partial_{\mathbf{n}} v = 0 \quad \text{on } \partial\Omega, \tag{2.5}$$

$$\int_\Omega v\,dx = 0, \tag{2.6}$$

has a unique weak solution $v_f \in H^1(\Omega)$. We denote v_f as $(-\Delta)^{-1}f$. Moreover, if $f \in L^2(\Omega)$, then by regularity theory (see [30], §6.3), $v_f \in H^2(\Omega)$, that is,

$$-\Delta((-\Delta)^{-1}f) = -\Delta v_f = f \quad \text{a.e. in } \Omega. \tag{2.7}$$

(A3) Let X_0' and X' be the dual spaces of X_0 and X, respectively. By the Riesz Representation Theorem (see [30], §D.3), for any $f \in X_0'$, there exists a unique function $w_f \in H^1(\Omega)$ such that

$$\langle f, \xi\rangle_{(X_0', X_0)} = \int_\Omega \nabla w_f \cdot \nabla \xi\,dx, \tag{2.8}$$

for any $\xi \in X_0$. If $f \in H^1(\Omega)'$ with $\int_\Omega f\,dx = 0$, then the function w_f is the same as v_f defined in (A2). Thus, we also denote w_f as $(-\Delta)^{-1}f$. Then we define an inner product in X_0' by

$$(f,g)_{X_0'} := \int_\Omega \nabla(-\Delta)^{-1}f \cdot \nabla(-\Delta)^{-1}g\,dx, \quad f,g \in X_0'. \tag{2.9}$$

We also define its induced norm $||u||_{X_0'} := (u,u)_{X_0'}^{1/2}$. Since $X_0 \subset X \subset H^1(\Omega) \subset X' \subset X_0'$, we can also use this inner product and its induced norm for functions in X_0, X and X'.

2.1. Variational derivative of the Cahn–Hilliard energy functional in a quotient space

Since the total mass $\int_\Omega u\,dx$ is conserved, we need to consider the Cahn–Hilliard energy functional

$$E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + W(u)\,dx \tag{2.10}$$

in the admissible set $X_{\mathcal{M}}$. Provided that u is sufficiently regular, for any perturbation $\eta \in X_0 \cap L^\infty(\Omega)$, the variational derivative $\delta E/\delta u$ is defined through

$$\begin{aligned}\left\langle \frac{\delta E}{\delta u}, \eta \right\rangle_{(X_0', X_0)} &= \frac{d}{ds}\bigg|_{s=0} E(u+s\eta) \\ &= \int_\Omega (-\Delta u + W'(u))\eta\,dx. \end{aligned} \tag{2.11}$$

Since $\int_\Omega \eta\,dx = 0$, $\delta E/\delta u$ is not necessarily equal to $-\Delta u + W'(u)$. Instead, we just need

$$\frac{\delta E}{\delta u} = -\Delta u + W'(u) + g, \tag{2.12}$$

where g is a spatial constant that is independent on x. In other words, we can say that

$$\frac{\delta E}{\delta u} = -\Delta u + W'(u) \quad \text{in the quotient space } H^{-1}(\Omega)/\mathbb{R}.$$

2.2. The gradient flow structure for the Cahn–Hilliard equation (1.12)–(1.14)

Provided that u is sufficiently regular, for any $\eta \in X_0 \cap L^\infty(\Omega)$, using integration by parts, (1.12), (1.14) and (A2), we have

$$\begin{aligned}(\partial_t u, \eta)_{X_0'} &= \int_\Omega \nabla(-\Delta)^{-1}\partial_t u \cdot \nabla(-\Delta)^{-1}\eta\, dx \\ &= -\int_\Omega \nabla(-\Delta u + W'(u)) \cdot \nabla(-\Delta)^{-1}\eta\, dx \\ &= \int_\Omega (-\Delta u + W'(u))\Delta((-\Delta)^{-1}\eta)\, dx \\ &= -\int_\Omega (-\Delta u + W'(u))\eta\, dx \\ &= -\left\langle \frac{\delta E}{\delta u}, \eta \right\rangle_{(X_0', X_0)}. \end{aligned} \tag{2.13}$$

So the equation (1.12)–(1.14) is a gradient flow of the Cahn–Hilliard free energy E:

$$(\partial_t u, \eta)_{X_0'} = -\left\langle \frac{\delta E}{\delta u}, \eta \right\rangle_{(X_0', X_0)} \quad \text{for all } \eta \in X_0 \cap L^\infty(\Omega). \tag{2.14}$$

3. Implicit time discretization and preliminary estimates

In this section we introduce the implicit time discretization for the Cahn–Hilliard equation (1.12)–(1.16) and some estimates which are necessary to prove the main result in Theorem 1.2.

3.1. Implicit time discretization

To prepare for the proof of Theorem 1.2, we derive an implicit time discretization of the equation (1.12)–(1.16). Then we will prove that the corresponding time-discrete solution converges to a function u, which is a weak solution to the equation (1.12)–(1.16) in some suitable sense.

Let $N \in \mathbb{N}$ be arbitrary and let $\tau := T/N$ denote the time step size. Without loss of generality, we assume that $\tau < 1$. We define $\phi^n (n = 0, 1, \ldots, N)$ recursively by the following construction:

- $\phi^0 := \Phi$ (the initial data).
- If ϕ^n is already constructed, we choose ϕ^{n+1} to be a minimizer of the functional

$$J_n(\zeta) = \frac{1}{2\tau}||\zeta - \phi^n||^2_{X_0'} + E(\zeta) \tag{3.1}$$

  over the set $X_{\mathcal{M}}$.

The existence of such a minimizer is guaranteed by the following lemma.

Lemma 3.1. *The functional J_n has a global minimizer $\bar{\zeta} \in X_{\mathcal{M}}$, that is, for any $\zeta \in X_{\mathcal{M}}$,*

$$J_n(\bar{\zeta}) \le J_n(\zeta).$$

Proof. Since $E \geq 0$, J_n is bounded below by 0. Thus, $m := \inf_{X_{\mathcal{M}}} J_n$ exists, and we can find a minimizing sequence $(\zeta_k) \subset X_{\mathcal{M}} \subset H_0^1(\Omega)$ such that

$$\lim_{k\to\infty} J_n(\zeta_k) = m, \quad \text{and} \quad J_n(\zeta_k) \leq m+1 \text{ for all } k = 1,2,3,\ldots \tag{3.2}$$

Since $W(\zeta_k) \geq 0$, we have

$$\begin{aligned} m+1 \geq J_n(\zeta_k) &= \frac{1}{2\tau}||\zeta_k - \phi^n||^2_{X_0'} + \int_\Omega \frac{1}{2}|\nabla \zeta_k|^2 + W(\zeta_k)\,dx \\ &\geq \int_\Omega \frac{1}{2}|\nabla\zeta_k|^2\,dx, \end{aligned} \tag{3.3}$$

which implies that

$$||\zeta_k||_{H_0^1(\Omega)} \leq C. \tag{3.4}$$

Hence $\{\zeta_k\}$ is bounded in $H_0^1(\Omega) \subset\subset L^2(\Omega)$. So there exists $\bar\zeta \in H_0^1(\Omega)$ and a subsequence of $\{\zeta_k\}$ (still denoted as $\{\zeta_k\}$) such that $\zeta_k \rightharpoonup \bar\zeta$ weakly in $H_0^1(\Omega)$, $\zeta_k \to \bar\zeta$ strongly in $L^2(\Omega)$, and $\zeta_k \to \bar\zeta$ a.e. in Ω. Hence

$$\left|\int_\Omega \bar\zeta\,dx - \mathcal{M}\right| = \left|\int_\Omega (\bar\zeta - \zeta_k)\,dx\right| \leq \int_\Omega |\bar\zeta - \zeta_k|\,dx \leq C||\bar\zeta - \zeta_k||_{L^2(\Omega)} \to 0$$

as $k \to \infty$. This implies that $\int_\Omega \bar\zeta\,dx = \mathcal{M}$, hence $\bar\zeta \in X_{\mathcal{M}}$.

We rewrite the double well potential W as $W(\zeta) = W_1(\zeta) + W_2(\zeta)$, where $W_1(\zeta) = (\zeta^4+1)/4$ convex and $W_2(\zeta) = -\zeta^2/2$. We have

$$\int_\Omega W_2(\zeta_k) = -\frac{1}{2}\int_\Omega |\zeta_k|^2 \to -\frac{1}{2}\int_\Omega |\bar\zeta|^2 = \int_\Omega W_2(\bar\zeta) \tag{3.5}$$

as $k \to \infty$. Since all other terms of J_n can easily be handled by the weakly lower semicontinuity of convex functionals, we obtain that

$$J_n(\bar\zeta) \leq \liminf_{k\to\infty} J(\zeta_k) = m, \tag{3.6}$$

which implies that $J_n(\bar\zeta) = m$. □

Since ϕ^{n+1} is a minimizer of J_n, for any $\eta \in X_0 \cap L^\infty(\Omega)$, we have

$$\left.\frac{d}{ds}\right|_{s=0} J_n(\phi^{n+1} + s\eta) = 0, \tag{3.7}$$

that is,

$$\left(\frac{\phi^{n+1} - \phi^n}{\tau}, \eta\right)_{X_0'} + \int_\Omega \nabla\phi^{n+1}\cdot\nabla\eta + W'(\phi^{n+1})\eta\,dx = 0 \tag{3.8}$$

for any $\eta \in X_0 \cap L^\infty(\Omega)$. The equation (3.8) can be interpreted as an implicit time discretization of the corresponding gradient flow equation (2.14).

Since $\phi^n, \phi^{n+1} \in X_{\mathcal{M}} \subset H^1(\Omega)'$, we have $\frac{\phi^{n+1}-\phi^n}{\tau} \in H^1(\Omega)'$ and $\int_\Omega \frac{\phi^{n+1}-\phi^n}{\tau}\,dx = 0$. Hence, by (A2), the equation

$$-\Delta\psi^{n+1} = -\frac{\phi^{n+1} - \phi^n}{\tau} \quad \text{in } \Omega, \tag{3.9}$$

$$\partial_{\mathbf{n}}\psi^{n+1} = 0 \quad \text{on } \partial\Omega, \tag{3.10}$$

$$\int_\Omega \psi^{n+1}\,dx = 0, \tag{3.11}$$

has a unique solution $\psi^{n+1} = (-\Delta)^{-1}\left(-\frac{\phi^{n+1}-\phi^n}{\tau}\right)$.

For any $\eta \in X_0 \cap L^\infty(\Omega)$, using integration by parts, we have

$$\begin{aligned}\left(\frac{\phi^{n+1}-\phi^n}{\tau},\eta\right)_{X_0'} &= -\int_\Omega \nabla(-\Delta)^{-1}\left(\frac{\phi^{n+1}-\phi^n}{\tau}\right)\cdot\nabla(-\Delta)^{-1}\eta\,dx\\ &= -\int_\Omega \nabla\psi^{n+1}\cdot\nabla(-\Delta)^{-1}\eta\,dx\\ &= \int_\Omega \psi^{n+1}\Delta((-\Delta)^{-1}\eta)\,dx\\ &= -\int_\Omega \psi^{n+1}\eta\,dx. \end{aligned}\tag{3.12}$$

Combining with (3.8) we get

$$\int_\Omega \psi^{n+1}\eta\,dx = \int_\Omega \nabla\phi^{n+1}\cdot\nabla\eta + W'(\phi^{n+1})\eta\,dx \tag{3.13}$$

for any $\eta \in X_0 \cap L^\infty(\Omega)$. So $(\phi^{n+1},\phi^n,\psi^{n+1})$ is a solution of

$$\int_\Omega\left(-\frac{\phi^{n+1}-\phi^n}{\tau}\right)\xi\,dx = \int_\Omega \nabla\psi^{n+1}\cdot\nabla\xi\,dx \quad \text{for all } \xi\in H^1(\Omega), \tag{3.14}$$

$$\int_\Omega \psi^{n+1}\eta\,dx = \int_\Omega \nabla\phi^{n+1}\cdot\nabla\eta + W'(\phi^{n+1})\eta\,dx \quad \text{for all } \eta\in X_0\cap L^\infty(\Omega), \tag{3.15}$$

$$\int_\Omega \psi^{n+1}\,dx = 0, \tag{3.16}$$

which is an implicit time descretization of the equation (1.12)–(1.14).

Let (ϕ_N,ψ_N) denote the piecewise constant extension of the approximation solution $(\phi^n,\psi^n)_{n\in\{1,\ldots,N\}}$ on the interval $[0,T]$, that is, for any $n\in\{1,\ldots,N\}, x\in\Omega$ and $t\in((n-1)\tau,n\tau]$, we set

$$\phi_N(x,0) := \Phi(x), \tag{3.17}$$

$$(\phi_N,\psi_N)(x,t) := (\phi^n,\psi^n)(x,n\tau). \tag{3.18}$$

By (3.16), $\int_\Omega \psi_N(x,t)\,dx = 0$ for all $t\in[0,T]$ and all $N=1,2,\ldots$ Similarly, we let $(\bar\phi_N,\bar\psi_N)$ denote the piecewise linear extension, that is, for any $n\in\{1,\ldots,N\}, x\in\Omega$ and $t=(1-\alpha)(n-1)\tau+\alpha n\tau$ (where $0<\alpha\le 1$), we set

$$\bar\phi_N(x,0) := \Phi(x), \tag{3.19}$$

$$(\bar\phi_N,\bar\psi_N)(x,t) := (1-\alpha)(\phi^{n-1},\psi^{n-1})(x,(n-1)\tau) + \alpha(\phi^n,\psi^n)(x,n\tau). \tag{3.20}$$

3.2. Uniform bounds on the extensions

In this subsection we establish uniform bounds for the above extensions. Note that since X is a Hilbert space, $L^2(0,T;X)$ is also a Hilbert space [45].

Lemma 3.2. *There exists a constant* $C > 0$ *independent on* N, n, τ *such that*

$$||\phi_N||_{L^\infty(0,T;H_0^1(\Omega))} \le C, \tag{3.21}$$

$$||\bar{\phi}_N||_{C([0,T];H_0^1(\Omega))} \le C, \tag{3.22}$$

$$||\psi_N||_{L^2(0,T;X)} \le C. \tag{3.23}$$

Proof. For each $n \in \{0, \dots, N-1\}$, since ϕ^{n+1} is a minimizer of J_n, then

$$E(\phi^{n+1}) \le J_n(\phi^{n+1}) \le J_n(\phi^n) = E(\phi^n). \tag{3.24}$$

It follows inductively that

$$E(\phi^{n+1}) \le E(\phi^0) = E(\Phi) \tag{3.25}$$

for any $n \in \{0, \dots, N-1\}$. Thus

$$\int_\Omega \frac{1}{2}|\nabla\phi^{n+1}|^2 \le E(\phi^{n+1}) \le E(\Phi). \tag{3.26}$$

Since $\phi^{n+1} \in X_{\mathcal{M}} \subset H_0^1(\Omega)$, (3.26) implies that

$$||\phi^{n+1}||_{H_0^1(\Omega)} \le C, \tag{3.27}$$

where $C > 0$ doesn't depend on t and N. Recalling the definitions of ϕ_N and $\bar{\phi}_N$, we obtain (3.21) and (3.22).

Now pick an arbitrary number $n \in \{1, \dots, N\}$ and let $t := n\tau$. For any $s \in (t-\tau, \tau]$, we have $\phi_N(s) = \phi_N(t) = \phi^n(n\tau)$ and $\psi_N(s) = \psi_N(t) = \psi^n(n\tau)$. Thus, by the definitions of ψ_N and $||\cdot||_{X_0'}$, we get

$$\begin{aligned} &E(\phi_N(t)) + \frac{1}{2}\int_{t-\tau}^t ||\nabla\psi_N(s)||^2_{L^2(\Omega)} ds \\ &= E(\phi_N(t)) + \frac{1}{2}\int_{t-\tau}^t \frac{1}{\tau^2}||\phi_N(s) - \phi_N(s-\tau)||^2_{X_0'} ds \\ &= E(\phi_N(t)) + \frac{1}{2}\int_{t-\tau}^t \frac{1}{\tau^2}||\phi_N(t) - \phi_N(t-\tau)||^2_{X_0'} ds \\ &= E(\phi_N(t)) + \frac{1}{2\tau}||\phi_N(t) - \phi_N(t-\tau)||^2_{X_0'} \\ &\le E(\phi_N(t-\tau)). \end{aligned} \tag{3.28}$$

It follows inductively that

$$E(\phi_N(t)) + \frac{1}{2}\int_0^t ||\nabla\psi_N(s)||^2_{L^2(\Omega)} ds \le E(\phi_1(t)) \le E(\phi^0) = E(\Phi). \tag{3.29}$$

Since $E \ge 0$, we obtain that

$$\int_0^t ||\psi_N(s)||^2_X \, dx = \int_0^t ||\nabla\psi_N(s)||^2_{L^2(\Omega)} ds \le C. \tag{3.30}$$

So (3.23) is established. □

3.3. Hölder estimates for the piecewise linear extension

Now we show the Hölder continuity in time for the piecewise linear extension.

Lemma 3.3. *There exists a constant $C > 0$ independent on N such that for any $t_1, t_2 \in [0, T]$,*

$$||\bar{\phi}_N(t_1) - \bar{\phi}_N(t_2)||_{L^2(\Omega)} \le C|t_1 - t_2|^{1/4}. \tag{3.31}$$

Proof. Let $t_1, t_2 \in [0, T]$ be arbitrary. Without loss of generality, we assume that $t_1 < t_2$. Since $\bar{\phi}_N$ is piecewise linear in t, it is weakly differentiable with respect to t. So we can rewrite (3.14) as

$$\int_\Omega \partial_t \bar{\phi}_N(t)\xi \, dx = -\int_\Omega \nabla\psi_N(t) \cdot \nabla\xi \, dx \tag{3.32}$$

for all $\xi \in H^1(\Omega)$ and all $t \in [0, T]$. Choose $\xi = \bar{\phi}_N(t_2) - \bar{\phi}_N(t_1)$, integrating with respect to t from t_1 to t_2 and using the bounds in Lemma 3.2 we get

$$\begin{aligned} ||\bar{\phi}_N(t_2) - \bar{\phi}_N(t_1)||^2_{L^2(\Omega)} &= \left| \int_{t_1}^{t_2} \int_\Omega \nabla\psi_N(t) \cdot \nabla(\bar{\phi}_N(t_2) - \bar{\phi}_N(t_1)) \, dxdt \right| \\ &\le C||\bar{\phi}_N||_{L^\infty(0,T;H^1(\Omega))}||\psi_N||_{L^2(0,T;X)}|t_1 - t_2|^{1/2} \\ &\le C|t_1 - t_2|^{1/2}. \end{aligned} \tag{3.33}$$

So Lemma 3.3 is established. □

Lemma 3.4. *There exists a constant $C > 0$ independent on N such that for any $t_1, t_2 \in [0, T]$,*

$$||\bar{\phi}_N(t_1) - \bar{\phi}_N(t_2)||_{L^4(\Omega)} \le C|t_1 - t_2|^\beta, \tag{3.34}$$

where $\beta = 1/8$ if $d = 2$ and $\beta = 1/16$ if $d = 3$.

Proof. Inequality (3.34) is slightly more complicated than the Sobolev embedding theorem. In fact we need to use an interpolation inequality such as Ladyzhenskaya's inequality (see the Appendix): for any bounded Lipschitz domain Ω in $\mathbb{R}^d$ ($d = 2$ or 3), there exists a constant $C > 0$ depending only on Ω such that if $f \in H^1_0(\Omega)$, then

$$\begin{aligned} ||f||_{L^4(\Omega)} &\le C||f||^{1/2}_{L^2(\Omega)}||\nabla f||^{1/2}_{L^2(\Omega)} \quad \text{if } d = 2, \\ ||f||_{L^4(\Omega)} &\le C||f||^{1/4}_{L^2(\Omega)}||\nabla f||^{3/4}_{L^2(\Omega)} \quad \text{if } d = 3. \end{aligned}$$

Then (3.34) is a consequence of these two inequalities and Lemmas 3.2 & 3.3. □

4. The existence and uniqueness of the weak solution to the Cahn–Hilliard equation (1.12)–(1.16)

In this section we prove the existence and uniqueness of the weak solution to the equation (1.12)–(1.16) in the sense of Theorem 1.2.

4.1. Convergence of the approximation solutions

In this subsection we prove the convergence of the time-discrete solution.

Lemma 4.1. *There exist functions*

$$u \in L^\infty(0,T;H_0^1(\Omega)) \cap C^{0,\beta}([0,T];L^4(\Omega)),$$
$$\mu \in L^2(0,T;X),$$

where $\beta = 1/8$ *if* $d = 2$ *and* $\beta = 1/16$ *if* $d = 3$*, such that*

$$\phi_N \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;H_0^1(\Omega)), \tag{4.1}$$
$$\bar{\phi}_N \to u \quad \text{in } C^{0,\gamma}([0,T];L^4(\Omega)) \quad \text{for any } \gamma \in (0,\beta], \tag{4.2}$$
$$\phi_N \to u \quad \text{in } L^\infty(0,T;L^4(\Omega)), \tag{4.3}$$
$$\phi_N \to u \quad \text{a.e. in } \Omega_T, \tag{4.4}$$
$$\psi_N \rightharpoonup \mu \quad \text{in } L^2(0,T;X), \tag{4.5}$$

up to a subsequence as $N \to \infty$*.*

Proof. The bounds in Lemma 3.2 imply that there exist functions

$$u \in L^\infty(0,T;H_0^1(\Omega)),$$
$$\mu \in L^2(0,T;X),$$

such that, after extraction of a subsequence,

$$\phi_N \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;H_0^1(\Omega)),$$
$$\psi_N \rightharpoonup \mu \quad \text{in } L^2(0,T;X),$$

as $N \to \infty$.

Since $d = 2$ or 3, by (3.22), $\{\bar{\phi}_N\}$ is bounded uniformly in $C([0,T];L^4(\Omega))$. By Lemma 3.4, $\{\bar{\phi}_N\}$ is equicontinuous. So applying Arzelà–Ascoli theorem for Banach-valued functions (see Lemma 1 in [55]), we can extract of a subsequence of $\{\bar{\phi}_N\}$ (still denoted as $\{\bar{\phi}_N\}$) such that

$$\bar{\phi}_N \to u \quad \text{in } C([0,T];L^4(\Omega)) \quad \text{as } N \to \infty. \tag{4.6}$$

Using Lemma 3.4, one can prove that $u \in C^{0,\beta}([0,T];L^4(\Omega))$, for the value of β indicated. For any $\gamma \in (0,\beta)$, we obtain by interpolation that

$$||\cdot||_{C^{0,\gamma}([0,T];L^4(\Omega))} \le C||\cdot||^{\gamma/\beta}_{C^{0,\beta}([0,T];L^4(\Omega))}||\cdot||^{1-\gamma/\beta}_{C([0,T];L^4(\Omega))}.$$

Hence it implies that

$$\bar{\phi}_N \to u \quad \text{in } C^{0,\gamma}([0,T];L^4(\Omega)) \quad \text{as } N \to \infty.$$

This proves (4.2).

For any $t \in [0,T]$, we can find $n \in \{1,\dots,N\}$ and $\alpha \in (0,1]$ such that $t = (1-\alpha)(n-1)\tau + \alpha n\tau$. Then by the definitions of ϕ_N and $\bar{\phi}_N$, using Lemma 3.4 we have

$$\begin{aligned}||\bar{\phi}_N(t)-\phi_N(t)||_{L^4(\Omega)} &= ||\alpha\phi^n(n\tau)+(1-\alpha)\phi^{n-1}((n-1)\tau)+\alpha\phi^n(n\tau)||_{L^4(\Omega)}\\ &= (1-\alpha)||\phi^n(n\tau)-\phi^{n-1}((n-1)\tau)||_{L^4(\Omega)}\\ &\le C\tau^{\beta}, \end{aligned} \tag{4.7}$$

for the value of β indicated. Since $\tau = T/N$, taking limits as $N \to \infty$ in (4.7) we get

$$||\bar{\phi}_N(t)-\phi_N(t)||_{L^4(\Omega)} \to 0 \quad \text{as } N \to \infty. \tag{4.8}$$

Together with (4.2) we obtain (4.3). This also implies that $\phi_N \to u$ in $L^4(\Omega_T)$. Therefore we can extract a subsequence that converges almost everywhere in Ω_T. This proves (4.4). □

4.2. The existence of a weak solution

In this subsection we prove that the function u in Lemma 4.1 is a weak solution to the Eq. (1.12)–(1.16) in the sense of Theorem 1.2.

Pick any $\xi \in L^2(0,T;H^1(\Omega))$ with $\partial_t\xi \in L^2(\Omega_T)$ and $\xi(T)=0$. Integrating both sides of the Eq. (3.32) with respect to t from 0 to T we get

$$\int_0^T\int_\Omega(\bar{\phi}_N-\Phi)\partial_t\xi\, dxdt = \int_0^T\int_\Omega \nabla\psi_N\cdot\nabla\xi\, dxdt. \tag{4.9}$$

Using the convergence properties from Lemma 4.1 and taking limits as $N\to\infty$ in (4.9), we obtain (1.18).

By the definition of ϕ_N, we have

$$\phi_N(x,0) = \Phi(x) \quad \text{for all } x\in\Omega, \tag{4.10}$$

$$\int_\Omega \phi_N(x,t)\, dx = \mathcal{M} \quad \text{for all } t\in[0,T]. \tag{4.11}$$

Since Ω is bounded, from (4.3) we get $\phi_N \to u$ in $L^\infty(0,T;L^1(\Omega))$, which implies that

$$\int_\Omega u(x,t)\, dx = \lim_{N\to\infty}\int_\Omega \phi_N(x,t)\, dx = \mathcal{M} \tag{4.12}$$

for all $t\in[0,T]$. Moreover, we can extract a subsequence of $\{\phi_N(x,0)\}$ (still denoted as $\{\phi_N(x,0)\}$) such that $\phi_N(x,0)\to u(x,0)$ for a.e. $x\in\Omega$. Thus $u(x,0)=\Phi(x)$ for a.e. $x\in\Omega$.

Now we prove the relation (1.19) of u and μ. Pick an arbitrary function $\eta \in L^2(0,T;X_0)\cap L^\infty(\Omega_T)$. By (3.15) and the definition of ϕ_N, we have

$$\int_0^T\int_\Omega \psi_N\eta\, dxdt = \int_0^T\int_\Omega \nabla\phi_N\cdot\nabla\eta + W'(\phi_N)\eta\, dxdt. \tag{4.13}$$

By (4.3) we have $\phi_N\to u$ in $L^\infty(0,T;L^3(\Omega))$, which implies that $\phi_N\to u$ in $L^3(\Omega_T)$. Hence,

$$\int_0^T\int_\Omega \phi_N^3\, dxdt \to \int_0^T\int_\Omega u^3\, dxdt \quad \text{as } N\to\infty. \tag{4.14}$$

Recalling that $W'(\phi_N)=\phi_N^3-\phi_N$, using (4.14) and the convergence properties in Lemma 4.1, by taking limits as $N\to\infty$ in (4.13) we obtain (1.19).

4.3. The uniqueness of the weak solution

Before we prove that the weak solution is unique, we prove the following lemma:

Lemma 4.2. *Let v be a function defined on Ω with $\int_\Omega v\,dx = 0$. For any two sequences $\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty$ with $a_n, b_n > 0$ for all $n = 1, 2, \ldots,$ we define the cutoff function $\mathcal{P}_n$ to be*

$$\mathcal{P}_n(s) := \begin{cases} s, \ -a_n \le s \le b_n, \\ -a_n, \ s < -a_n, \\ b_n, \ s > b_n. \end{cases} \tag{4.15}$$

Then there exist two sequences $\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty$ such that $\int_\Omega \mathcal{P}_n(v)\,dx = 0$ for all $n = 1, 2, \ldots,$ and $\lim_{n\to\infty} \mathcal{P}_n(v(x)) = v(x)$ for a.e. $x \in \Omega$.

Proof. The case where v is bounded a.e. in Ω is trivial. We will prove Lemma 4.2 for the case $\operatorname{ess\,sup}_\Omega v = +\infty$. The case $\operatorname{ess\,inf}_\Omega v = -\infty$ is similar.

Assume that $\operatorname{ess\,sup}_\Omega v = +\infty$. For each $x \in \Omega$, we define

$$v_+(x) = \max\{v(x), 0\} \quad \text{and} \quad v_-(x) = \max\{-v(x), 0\}. \tag{4.16}$$

Then $v = v_+ - v_-$, and since $\int_\Omega v\,dx = 0$, we have

$$\int_\Omega v_+\,dx = \int_\Omega v_-\,dx. \tag{4.17}$$

For each $\lambda > 0$, we define

$$\mathcal{S}_\lambda(s) := \begin{cases} s, \ s \le \lambda, \\ \lambda, \ s > \lambda. \end{cases} \tag{4.18}$$

Since $\operatorname{ess\,sup}_\Omega v = +\infty$, we have

$$\int_\Omega \mathcal{S}_n(v_+)\,dx < \int_\Omega v_+\,dx \tag{4.19}$$

for any $n = 1, 2, \ldots,$ and we can choose n_0 large enough so that $\int_\Omega \mathcal{S}_{n_0}(v_+)\,dx > 0$. Since $\int_\Omega \mathcal{S}_\lambda(v_+)\,dx$ is increasing with respect to λ, then $\int_\Omega \mathcal{S}_n(v_+)\,dx > 0$ for all $n \ge n_0$. Moreover,

$$\lim_{n\to\infty} \int_\Omega \mathcal{S}_n(v_+)\,dx = \int_\Omega v_+\,dx. \tag{4.20}$$

On the other hand,

$$\lim_{\lambda\to 0^+} \int_\Omega \mathcal{S}_\lambda(v_-)\,dx = 0, \tag{4.21}$$

$$\lim_{\lambda\to\infty} \int_\Omega \mathcal{S}_\lambda(v_-)\,dx = \int_\Omega v_-\,dx = \int_\Omega v_+\,dx. \tag{4.22}$$

Thus, for each $n \ge n_0$, we have

$$\lim_{\lambda\to 0^+} \int_\Omega \mathcal{S}_\lambda(v_-)\,dx < \int_\Omega \mathcal{S}_n(v_+)\,dx < \lim_{\lambda\to\infty} \int_\Omega \mathcal{S}_\lambda(v_-)\,dx. \tag{4.23}$$

Since $\int_\Omega \mathcal{S}_\lambda(v_-)\,dx$ is continuous with respect to λ, there exists $0<\lambda_n<\infty$ with

$$\int_\Omega \mathcal{S}_{\lambda_n}(v_-)\,dx = \int_\Omega \mathcal{S}_n(v_+)\,dx. \tag{4.24}$$

Since v_+ and v_- are nonnegative functions, we see that $\{\lambda_n\}_{n=n_0}^\infty$ is an increasing sequence. Then there are two cases:

Case 1: $\lim_{n\to\infty}\lambda_n=+\infty$. In this case, it is obvious that $-\lim_{n\to\infty}\lambda_n \le v(x) \le \lim_{n\to\infty} n$ for all $x\in\Omega$, and so the conclusion follows.

Case 2: $\lim_{n\to\infty}\lambda_n=\Lambda<+\infty$. We will prove that $\operatorname{ess\,sup}_\Omega v_- = \Lambda$.

If $\operatorname{ess\,sup}_\Omega v_- < \Lambda$, then there exists $n_1>n_0$ such that $\operatorname{ess\,sup}_\Omega v_- < \lambda_{n_1}$. Thus,

$$\int_\Omega \mathcal{S}_{\lambda_{n_1}}(v_-)\,dx = \int_\Omega v_-\,dx, \tag{4.25}$$

which is a contradiction since

$$\int_\Omega \mathcal{S}_{\lambda_n}(v_-)\,dx = \int_\Omega \mathcal{S}_n(v_+)\,dx < \int_\Omega v_+\,dx = \int_\Omega v_-\,dx \tag{4.26}$$

for all $n\ge n_0$.

If $\Lambda < \operatorname{ess\,sup}_\Omega v_-$, then $|\{x\in\Omega : \Lambda < v_- \le \operatorname{ess\,sup}_\Omega v_-\}|>0$. Hence,

$$\begin{aligned}\int_\Omega \mathcal{S}_\Lambda(v_-)\,dx &= \int_{\{v_-\le\Lambda\}} v_-\,dx + \int_{\{v_->\Lambda\}} \Lambda\,dx\\ &< \int_{\{v_-\le\Lambda\}} v_-\,dx + \int_{\{v_->\Lambda\}} v_-\,dx = \int_\Omega v_-\,dx.\end{aligned} \tag{4.27}$$

Note that $\Lambda=\sup\{\lambda_n : n\ge n_0\}$, and since $\int_\Omega \mathcal{S}_\lambda(v_-)\,dx$ is increasing with respect to λ, we have $\int_\Omega \mathcal{S}_{\lambda_n}(v_-)\,dx \le \int_\Omega \mathcal{S}_\Lambda(v_-)\,dx$ for all $n\ge n_0$. Thus,

$$\lim_{n\to\infty}\int_\Omega \mathcal{S}_{\lambda_n}(v_-)\,dx \le \int_\Omega \mathcal{S}_\Lambda(v_-)\,dx < \int_\Omega v_-\,dx, \tag{4.28}$$

which is a contradiction since

$$\lim_{n\to\infty}\int_\Omega \mathcal{S}_{\lambda_n}(v_-)\,dx = \lim_{n\to\infty}\int_\Omega \mathcal{S}_n(v_+)\,dx = \int_\Omega v_+\,dx = \int_\Omega v_-\,dx. \tag{4.29}$$

So we have $\operatorname{ess\,sup}_\Omega v_- = \Lambda$, that is, $\operatorname{ess\,inf}_\Omega v = -\Lambda$.

We relabel $\{\lambda_n\}_{n=n_0}^\infty$ and $\{n\}_{n=n_0}^\infty$ as $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$, respectively. By the above argument, we have $-\lim_{n\to\infty} a_n \le v(x) \le \lim_{n\to\infty} b_n$ for a.e. $x\in\Omega$, which implies that $\lim_{n\to\infty}\mathcal{P}_n(v(x)) = v(x)$ for a.e. $x\in\Omega$, where $\mathcal{P}_n$ is defined by (4.15). Moreover, for each $n=1,2,\ldots$, since $\mathcal{P}_n(v)=\mathcal{S}_{b_n}(v_+)-\mathcal{S}_{a_n}(v_-)$, by (4.24), we have $\int_\Omega \mathcal{P}_n(v)\,dx = 0$. This completes the proof of Lemma 4.2. □

Now we prove the uniqueness of the weak solution of the Eq. (1.12)–(1.16). Assume that there are weak solutions $u_1,u_2\in L^\infty(0,T;H_0^1(\Omega))\cap C^{0,\beta}([0,T];L^4(\Omega))$, where $\beta=1/8$ if $d=2$ and $\beta=1/16$ if $d=3$, in the sense of Theorem 1.2 with corresponding functions $\mu_1,\mu_2\in L^2(0,T;H^1(\Omega))$. We define

$$\bar{u} := u_1-u_2 \quad \text{and} \quad \bar{\mu} := \mu_1-\mu_2. \tag{4.30}$$

By (1.18), we have

$$\int_0^T \int_\Omega \bar{u}\partial_t \xi \, dxdt = \int_0^T \int_\Omega \nabla \bar{\mu} \cdot \nabla \xi \, dxdt \tag{4.31}$$

for any $\xi \in L^2(0,T;H^1(\Omega))$ with $\partial_t \xi \in L^2(\Omega_T)$ and $\xi(T) = 0$. For any $t_0 \in [0,T]$ and any $\zeta \in L^2(0,T;H^1(\Omega))$, we define

$$\xi(x,t) := \begin{cases} \int_t^{t_0} \zeta(x,s)ds, \ t \le t_0, \\ 0, \ t > t_0, \end{cases} \tag{4.32}$$

for all $x \in \Omega$. Since ξ is an admissible test function for (4.31), we have

$$\begin{aligned} -\int_0^{t_0} \int_\Omega \bar{u}\zeta \, dxdt &= \int_0^{t_0} \int_\Omega \nabla \bar{\mu} \cdot \nabla \left(\int_t^{t_0} \zeta ds \right) dxdt \\ &= \int_0^{t_0} \int_\Omega \nabla \left(\int_0^t \bar{\mu} ds \right) \cdot \nabla \zeta \, dxdt \end{aligned} \tag{4.33}$$

Since $t_0 \in [0,T]$ is arbitrary, this implies

$$-\int_\Omega \bar{u}\zeta \, dx = \int_\Omega \nabla \left(\int_0^t \bar{\mu} ds \right) \cdot \nabla \zeta \, dx \tag{4.34}$$

for a.e. $t \in [0,T]$, which implies

$$(-\Delta)^{-1}\bar{u} = -\int_0^t \bar{\mu} ds + c \quad \text{and} \quad \partial_t(-\Delta)^{-1}\bar{u} = -\bar{\mu} \tag{4.35}$$

for any $t \in [0,T]$, with some constant $c \in \mathbb{R}$. Choosing $\zeta = \bar{\mu}$ yields

$$\begin{aligned} -\int_0^{t_0} \int_\Omega \bar{u}\bar{\mu} \, dxdt &= \int_0^{t_0} \int_\Omega \nabla(-\Delta)^{-1}\bar{u} \cdot \nabla \partial_t (-\Delta)^{-1}\bar{u} \, dxdt \\ &= \frac{1}{2} \int_0^{t_0} \int_\Omega \frac{d}{dt} \left(\nabla(-\Delta)^{-1}\bar{u} \cdot \nabla(-\Delta)^{-1}\bar{u} \right) dxdt \\ &= \frac{1}{2} \int_\Omega \nabla(-\Delta)^{-1}\bar{u}(t_0) \cdot \nabla(-\Delta)^{-1}\bar{u}(t_0) \, dx \\ &= \frac{1}{2} ||\bar{u}(t_0)||^2_{X_0'}. \end{aligned} \tag{4.36}$$

From the weak formulation (1.19) we have

$$\int_0^T \int_\Omega \bar{\mu}\eta \, dxdt = \int_0^T \int_\Omega \nabla \bar{u} \cdot \nabla \eta + (W'(u_1) - W'(u_2))\eta \, dxdt \tag{4.37}$$

for any $\eta \in L^2(0,T;X) \cap L^\infty(\Omega_T)$.

Now we consider the cutoff function $\mathcal{P}_n$ defined by (4.15). For each $t \in [0,t_0]$, since $\int_\Omega \bar{u}(x,t)\, dx = 0$, by Lemma 4.2, there exist two sequences $\{a_n(t)\}_{n=1}^\infty, \{b_n(t)\}_{n=1}^\infty$ with $\int_\Omega \mathcal{P}_n(\bar{u}(x,t))dx = 0$ for all $n = 1,2,\ldots,$ and $\lim_{n\to\infty} \mathcal{P}_n(\bar{u}(x,t)) = \bar{u}(x,t)$ for a.e. $x \in \Omega$. Hence $\eta = \chi_{[0,t_0]}\mathcal{P}_n(\bar{u})$ is an admissible test function for (4.37). Recall that we write $W(u) = W_1(u) + W_2(u)$, where $W_1(u) = (u^4+1)/4$ and $W_2(u) = -u^2/2$. Since $W_1'(u) = u^3$ is an increasing function, then $(W_1'(u_1) - W_1'(u_2))\mathcal{P}_n(\bar{u}) \geq 0$ a.e. in Ω_T. Plugging this estimate into (4.37) with $\eta = \chi_{[0,t_0]}\mathcal{P}_n(\bar{u})$, we have

$$\int_0^{t_0}\int_\Omega \bar{\mu}\mathcal{P}_n(\bar{u})\,dxdt \geq \int_0^{t_0}\int_\Omega \nabla\bar{u}\cdot\nabla\mathcal{P}_n(\bar{u}) + (W_2'(u_1) - W_2'(u_2))\mathcal{P}_n(\bar{u})\,dxdt. \tag{4.38}$$

Taking the limits as $n\to\infty$ we obtain

$$\begin{aligned}\int_0^{t_0}\int_\Omega \bar{\mu}\bar{u}\,dxdt &\geq ||\nabla\bar{u}||^2_{L^2(\Omega_{t_0})} + \int_0^{t_0}\int_\Omega (W_2'(u_1) - W_2'(u_2))\bar{u}\,dxdt\\ &\geq ||\nabla\bar{u}||^2_{L^2(\Omega_{t_0})} - ||\bar{u}||^2_{L^2(\Omega_{t_0})}.\end{aligned} \tag{4.39}$$

Combining with (4.36) we get

$$\frac{1}{2}||\bar{u}(t_0)||^2_{X_0'} + ||\nabla\bar{u}||^2_{L^2(\Omega_{t_0})} \leq ||\bar{u}||^2_{L^2(\Omega_{t_0})}. \tag{4.40}$$

Using integration by parts, Hölder's inequality and Young's inequality we get

$$\begin{aligned}||\bar{u}||^2_{L^2(\Omega)} &= \int_\Omega \nabla(-\Delta)^{-1}\bar{u}\cdot\nabla\bar{u}\,dx\\ &\leq ||\bar{u}||_{X_0'}||\nabla\bar{u}||_{L^2(\Omega)}\\ &\leq \frac{1}{4}||\bar{u}||^2_{X_0'} + ||\nabla\bar{u}||^2_{L^2(\Omega)}.\end{aligned} \tag{4.41}$$

Combining (4.40) and (4.41) we obtain

$$||\bar{u}(t_0)||^2_{X_0'} \leq \frac{1}{2}\int_0^{t_0} ||\bar{u}(t)||^2_{X_0'}dt. \tag{4.42}$$

Then Gronwall's inequality implies that $||\bar{u}(t_0)||^2_{X_0'} = 0$. Since $t_0 \in [0,T]$ is arbitrary, we have

$$||\bar{u}(t)||^2_{X_0'} = 0 \quad \text{for all } t\in[0,T],$$

which implies that $\bar{u} = 0$ for all $t\in[0,T]$ and a.e. $x\in\Omega$. Thus $u_1 = u_2$ for all $t\in[0,T]$ and a.e. $x\in\Omega$.

4.4. The energy inequality

The last part of the proof is to prove the energy inequality (1.20).

Let $t\in[0,T]$ be arbitrary. From (4.3) we get $\phi_N \to u$ in $L^\infty(0,T;L^2(\Omega))$, which implies that $\phi_N(t)\to u(t)$ in $L^2(\Omega)$ and a.e. in Ω. Recall that $W(\psi_N) = W_1(\psi_N) + W_2(\psi_N)$, where $W_1(\psi_N) = (\psi_N^4+1)/4$ convex and $W_2(\psi_N) = -\psi_N^2/2$. Since the remaining term of E is convex and ϕ_N, ψ_N are piecewise constant in t, we obtain from (3.29) that

$$\begin{aligned}&E(u(t)) + \frac{1}{2}\int_0^t ||\nabla\mu(s)||_{L^2(\Omega)}ds\\ &\quad\leq \liminf_{N\to\infty}\left(E(\phi_N(t)) + \frac{1}{2}\int_0^t ||\nabla\psi_N(s)||_{L^2(\Omega)}ds\right) \leq E(\Phi).\end{aligned} \tag{4.43}$$

So (1.20) is established. This completes the proof of Theorem 1.2.

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Appendix: Ladyzhenskaya's inequality

The Ladyzhenskaya's inequality was introduced by O. A. Ladyzhenskaya [41] to prove the existence and uniqueness of long-time solutions to the Navier–Stokes equations in $\mathbb{R}^2$ when the initial data is sufficiently smooth. This inequality is a member of a class of inqualities known as *interpolation inequalities*.

Lemma 4.3. (Ladyzhenskaya's inequality) *Let Ω be a Lipschitz domain in $\mathbb{R}^d$ ($d = 2$ or 3) and $f \in H_0^1(\Omega)$. Then there exists a constant $C > 0$ depending only on Ω such that*

$$||f||_{L^4(\Omega)} \le C||f||_{L^2(\Omega)}^{1/2}||\nabla f||_{L^2(\Omega)}^{1/2} \quad \text{if } d = 2,$$
$$||f||_{L^4(\Omega)} \le C||f||_{L^2(\Omega)}^{1/4}||\nabla f||_{L^2(\Omega)}^{3/4} \quad \text{if } d = 3.$$

References

[1] Alessio, F., Wilfrid, G., Türkay, Y.: A variational method for a class of parabolic PDEs. Annali della Scuola Normale Superiore di Pisa - Classe di Scienze **5**(10), 207–252 (2011)

[2] Ambrosio, L., Gigli, N., Savaré, G.: Gradient flows in metric spaces and the Wasserstein spaces of probability measures. Lectures in Mathematics, ETH Zürich, Birkhäuser (2005)

[3] Arendt, W., Chalendar, I., Eymard, R.: Galerkin approximation of linear problems in Banach and Hilbert spaces. IMA Journal of Numerical Analysis, Oxford University Press (OUP), (2020). arXiv:0226.4895v3f [hal]

[4] Baňas, L., Nürnberg, R.: Adaptive finite element methods for Cahn-Hilliard equations. J. Comput. Applied Math. **218**(1), 2–11 (2008)

[5] Baňas, L., Nürnberg, R.: A posteriori estimates for the Cahn-Hilliard equation with obstacle free energy. ESAIM: Math. Modelling Numer. Anal. **43**(5), 1003–1026 (2009)

[6] Badalassi, V.E., Ceniceros, H.D., Banerjee, S.: Computation of multiphase systems with phase field models. J. Comput. Phys. **190**(2), 371–397 (2003)

[7] Bates, P., Han, J.: The Dirichlet boundary problem for a nonlocal Cahn-Hilliard equation. J. Math. Anal. Appl. **311**(1), 289–312 (2005)

[8] Beard, R.W., Saridis, G.N., Wen, J.T.: Galerkin approximations of the generalized Hamilton-Jacobi-Bellman equation. Automatica **33**(12), 2159–2177 (1997)

[9] Bertozzi, A.L., Esedoglu, S., Gillette, A.: Inpainting of binary images using the Cahn-Hilliard equation. IEEE Trans. Image Process. **16**(1), 285–291 (2007)

[10] Bronsard, L., Hilhorst, D.: On the slow dynamics for the Cahn-Hilliard equation in one space dimension. Proc. Royal Soc. A **439**(1907), 669–682 (1992)

[11] Cahn, J.W., Elliott, C.M., Novick-Cohen, A.: The Cahn-Hilliard equation with a concentration-dependent mobility: Motion by minus the Laplacian of the mean curvature. European J. Appl. Math. **7**, 287–301 (1996)

[12] Cahn, J.W., Hilliard, J.E.: Free energy of a nonuniform system. I. Interfacial free energy. J. Chem. Phys. **28**, 258–267 (1958)

[13] Cahn, J.W., Hilliard, J.E.: Free energy of a nonuniform system. II. Thermodynamic basis. J. Chem. Phys. **30**, 1121–1135 (1959)

[14] Ceniceros, H.D., Roma, A.M.: A nonstiff, adaptive mesh refinement-based method for the Cahn-Hilliard equation. J. Comput. Phys. **225**(2), 1849–1862 (2007)

[15] Choksi, R., Peletier, M.A., Williams, J.F.: On the phase diagram for microphase separation of diblock copolymers: An approach via a nonlocal Cahn-Hilliard functional. SIAM J. Applied Math. **69**(6), 1712–1738 (2009)

[16] Choksi, R., Sternberg, P.: Periodic phase separation: the periodic Cahn–Hilliard and isoperimetric problems. Interfaces Free Bound. **8**(3), 371–392 (2006)

[17] Cohen, D., Murray, J.M.: A generalized diffusion model for growth and dispersion in a population. J. Math. Biol. **12**, 237–248 (1981)

[18] Dai, S., Du, Q.: Coarsening mechanism for systems governed by the Cahn–Hilliard equation with degenerate diffusion mobility. Multiscale Model. Simul. **12**(4), 1870–1889 (2014)

[19] Dai, S., Du, Q.: Weak solutions for the Cahn–Hilliard equation with degenerate mobility. Arch. Ration. Mech. Anal. **219**(3), 1161–1184 (2016)

[20] Dai, S., Du, Q.: Weak solutions for the Cahn–Hilliard equation with phase-dependent diffusion mobility. Arch. Ration. Mech. Anal. **219**(3), 1161–1184 (2016)

[21] Dai, S., Li, B., Luong, T.: Minimizers for the Cahn–Hilliard energy under strong anchoring conditions. SIAM J. Appl. Math. **80**(5), 2299–2317 (2020)

[22] Dai, S., Liu, Q., Luong, T., Promislow, K.: On nonnegative solutions for the functionalized Cahn–Hilliard equation with degenerate mobility. Results Appl. Math. **12**, 100195 (2021)

[23] Dai, S., Liu, Q., Promislow, K.: Weak solutions for the Functionalized Cahn–Hilliard equation with degenerate mobility. Appl. Anal. (2019). https://doi.org/10.1080/00036811.2019.1585536

[24] DeGroot, S.R., Mazur, P.: Non-Equilibrium Thermodynamics. Dover Publications Inc., Mineola, N. Y. (1984)

[25] Dehghan, M.: Finite difference procedures for solving a problem arising in modeling and design of certain optoelectronic devices. Math. Comput. Simul. **71**, 16–30 (2006)

[26] Dehghan, M., Mohebbi, A.: Multigrid solution of high order discretisation for three-dimensional biharmonic equation with Dirichlet boundary conditions of second kind. Appl. Math. Comput. **180**, 575–593 (2006)

[27] Du, Q., Nicolaides, R.: Numerical analysis of a continuum model of phase transition. SIAM J. Numer. Anal. **28**(5), 1310–1322 (1991)

[28] Elliott, C., Garcke, H.: On the Cahn–Hilliard equation with degenerate mobility. SIAM J. Math. Anal. **27**(2), 404–423 (1996)

[29] Elliott, C., Garcke, H.: On the Cahn–Hilliard equation with degenerate mobility. SIAM J. Math. Anal. **27**(2), 404–423 (1996)

[30] Evans, L.C.: Partial Differential Equations, 2nd edn. Amer. Math. Soc., Providence (2010)

[31] Feng, W.M., Yu, P., Hu, S.Y., Liu, Z.K., Du, Q., Chen, L.Q.: Spectral implementation of an adaptive moving mesh method for phase-field equations. J. Comput. Phys. **220**(1), 498–510 (2006)

[32] Fleißner, F.C.: A minimizing movement approach to a class of scalar reaction-diffusion equations. ESAIM Control Optim. Calc. Var. **27**(18) (2021)

[33] Garcke, H., Lam, K.F.: Analysis of a Cahn–Hilliard system with non-zero Dirichlet conditions modeling tumor growth with chemotaxis. Disc. Cont. Dyn. Syst. **37**(8), 4277–4308 (2017)

[34] Garcke, H., Nestler, B., Stoth, B.: A multiphase field concept: numerical simulations of moving phase boundaries and multiple junctions. SIAM J. Appl. Math. **60**(1), 295–315 (1999)

[35] De Giorgi, E.: New Problems on Minimizing Movements. In: Boundary Value Problems for PDEs and Applications, pp. 81–98. Masson (1993)

[36] De Giorgi, E., Mariono, A., Tosques, M.: Problems of evolution in metric spaces and maximal decreasing curve. Atti. Accad. Naz. Lincei Rend. Cl. Sci. Mat. Natur. **68**(8), 180–187 (1980)

[37] Gómez, H., Calo, V., Bazilevs, Y., Hughes, T.: Isogeometric analysis of the Cahn–Hilliard phase-field model. Comput. Meth. Appl. Mech. Eng. **197**(49–50), 4333–4352 (2008)

[38] He, L.: Error estimation of a class of stable spectral approximation to the Cahn–Hilliard equation. J. Sci. Comput. **41**(3), 461–482 (2009)

[39] He, Y., Liu, Y., Tang, T.: On large time-stepping methods for the Cahn–Hilliard equation. Appl. Numer. Math. **57**(5–7), 616–628 (2007)

[40] Klapper, I., Dockery, J.: Role of cohesion in the material description of biofilms. Phys. Rev. E **74**, 0319021–0319028 (2006)

[41] Ladyzhenskaya, O.A.: Global solvability of a boundary value roblem for the Navier–Stokes equations in the case of two spatial variables. Doklady USSR **123**, 427–429 (1958)

[42] Lamorgese, A.G., Mauri, R.: Diffuse-interface modeling of phase segregation in liquid mixtures. Intern. J. Multiphase Flow **34**(10), 987–995 (2008)

[43] Li, M., Ober, C.K.: Block copolymer patterns and templates. Mater. Today **9**(9), 30–39 (2006)

[44] Li, Y., Jeong, D., Shin, J., Kim, J.: A conservative numerical method for the Cahn-Hilliard equation with Dirichlet boundary conditions in complex domains. Comput. Math. Appl. **65**(1), 102–115 (2013)

[45] Lions, J.L., Magenes, E.: Non-Homogeneous Boundary Value Problems and Applications, vol. I. Springer, Berlin (1972)

[46] Lisini, S.: Nonlinear diffusion equations with variable coefficients as gradient flows in Wasserstein spaces. ESAIM Control Optim. Calc. Var. **15**(3), 712–740 (2008)

[47] Liu, Y., Liu, W.K.: Rheology of red blood cell aggregation by computer simulation. J. Comput. Phys. **220**, 139–154 (2006)

[48] Moelans, N., Blanpain, B., Wollants, P.: An introduction to phase-field modeling of microstructure evolution. Comput. Coupling Phase Diagr. Thermochem. **32**, 268–294 (2008)

[49] Oron, A., Davis, S.H., Bankoff, S.G.: Long-scale evolution of thin liquid films. Rev. Mod. Phys. **69**, 931–980 (1997)

[50] Pego, R.L.: Front migration in the nonlinear Cahn–Hilliard equation. Proc. Royal Soc. Lond. A **442**, 261–278 (1989)

[51] Rossi, R., Savaré, G.: Gradient flows of non convex functionals in hilbert spaces and applications. ESAIM Control Optim. Calc. Var. **12**(3), 564–614 (2006)

[52] Tremaine, S.: On the origin of irregular structure in Saturn's rings. Astron. J. **125**, 894 (2003)

[53] Santambrogio, F.: Euclidean, metric, and Wasserstein gradient flows: an overview. Bull. Math. Sci. **7**, 87–154 (2017)

[54] Saxena, R., Caneba, G.T.: Studies of spinodal decomposition in a ternary polymer-solvent-nonsolvent systems. Polym. Eng. Sci. **42**, 1019–1031 (2002)

[55] Simon, J.: Compact sets in the space $L^p(0,T;B)$. Ann. Mat. **146**, 65–96 (1986)

[56] Stefanelli, U.: A new minimizing-movements scheme for curves of maximal slope. arXiv:2103.00846. March (2021)

[57] Thiele, U., Knobloch, E.: Thin liquid films on a slightly inclined heated plate. Physica D **190**, 213–248 (2004)

[58] Wells, G.N., Kuhl, E., Garikipati, K.: A discontinuous Galerkin method for the Cahn–Hilliard equation. J. Comput. Phys. **218**(2), 860–877 (2006)

[59] Wheeler, B., Hammadi, R., Ma, X.: Self-assembled 3d nanoporous biomimetic material embedded with green synthesized gold nanoparticles for high-performance non-enzymatic glucose sensor (submitted)

[60] Wise, S., Lowengrub, J., Frieboes, H., Cristini, V.: Three-dimensional multi-species nonlinear tumor growth-I: model and numerical method. J. Theor. Biol. **253**, 524–543 (2008)

[61] Yang, W., Huang, Z., Zhu, W.: Image segmentation using the Cahn–Hilliard equation. J. Sci. Comput. **79**, 1057–1077 (2019)

[62] Ye, X.: The Fourier collocation method for the Cahn–Hilliard equation. Comput. Math. Appl. **44**(1–2), 213–229 (2002)

[63] Zhou, B., Powell, A.: Phase field simulation of early stage structure formation during immersion precipitation of polymeric membranes in 2D and 3D. J. Membr. Sci. **268**, 150–164 (2006)

[64] Zhu, Y., Aissou, K., Andelman, D., Man, X.: Orienting cylinder-forming block copolymer thin films: The combined effect of substrate corrugation and its surface energy. Macromolecules **52**(3), 1241–1248 (2019)

[65] Zinsl, J., Matthes, D.: Discrete approximation of the minimizing movement scheme for evolution equations of Wasserstein gradient flow type with nonlinear mobility. arXiv:1609.06907, September (2016)

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