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A Note on Exact Minimum Degree Threshold for Fractional Perfect Matchings

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Abstract

Rödl, Ruciński, and Szemerédi determined the minimum $(k-1)$-degree threshold for the existence of fractional perfect matchings in k-uniform hypergrahs, and Kühn, Osthus, and Townsend extended this result by asymptotically determining the d-degree threshold for the range $k-1 > d \geq k/2$. In this note, we prove the following exact degree threshold: let k, d be positive integers with $k \geq 4$ and $k-1 > d \geq k/2$, and let n be any integer with $n \geq 2k(k-1)+1$. Then any n-vertex k-uniform hypergraph with minimum d-degree $\delta_d(H) > \binom{n-d}{k-d} - \binom{n-d-(\lceil n/k \rceil -1)}{k-d}$ contains a fractional perfect matching. This lower bound on the minimum d-degree is best possible. We also determine the minimum d-degree threshold for the existence of fractional matchings of size s, where $0 < s \leq n/k$ (when $k/2 \leq d \leq k-1$), or with s large enough and $s \leq n/k$ (when $2k/5 < d < k/2$).

Keywords Matching · Fractional matching · Perfect matching

1 Introduction

For a positive integer k, let $[k] := \{1, \ldots, k\}$. For a set S, let $\binom{S}{k} := \{T \subseteq S : |T| = k\}$. A *hypergraph* H consists of a vertex set $V(H)$ and an edge set $E(H)$ whose members are subsets of $V(H)$, and H is said to be *k-uniform* if $E(H) \subseteq \binom{V(H)}{k}$. A k-uniform hypergraph is also called a *k-graph*. A *matching* in a hypergraph H is a set of pairwise disjoint edges of H, and a matching in H is *perfect* if the union of all edges in the matching is $V(H)$. We use $\nu(H)$ to denote the largest size of a matching in H. A *maximum matching* in H is a matching in H of size $\nu(H)$.

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There has been much activity on degree thresholds for matchings of certain size in uniform hypergraphs. Let H be a hypergraph. For $S \subseteq V(H)$, let $N_H(S) = \{T \subseteq V(H) \setminus S : T \cup S \in E(H)\}$ and let $d_H(S) := |N_H(S)|$. For any integer $d \geq 0$, let $\delta_d(H) = \min\left\{d_H(S) : S \in \binom{V(H)}{d}\right\}$, which is the *minimum d-degree* of H. Note that $\delta_0(H) = e(H)$, the number of edges in H. For integers n, k, d, s satisfying $0 \leq d \leq k-1$ and $0 < s \leq n/k$, let $m_d^s(k,n)$ denote the minimum integer m such that every k-graph H on n vertices with $\delta_d(H) \geq m$ has a matching of size s.

Rödl, Rucinśki, and Szemerédi [8] determined $m_{k-1}^{n/k}(k,n)$ for all integers $k \geq 3$ and $n \in k\mathbb{Z}$ sufficiently large. Given positive integers k, d with $k \geq 4$ and $k-2 \geq d \geq k/2$, Treglown and Zhao [9, 10] showed that $m_d^{n/k}(k,n) \sim \frac{1}{2}\binom{n-d}{k-d}$.

One approach to finding a large matching in a k-graph is to first find a large fractional matching in the k-graph, and then convert that fractional matching to a matching. This approach has been used quite often, for example, in [1, 3, 6]. A *fractional matching* in a k-graph H is a function $f : E(H) \to [0,1]$ such that, for each $v \in V(H)$, $\sum_{\{e \in E(H): v \in e\}} f(e) \leq 1$. The *size* of f is $\sum_{e \in E(H)} f(e)$, and f is a *fractional perfect matching* if it has size $|V(H)|/k$. We use $\nu'(H)$ to denote the maximum size of a fractional matching in H. For integers n, k, d and positive rational number s satisfying $0 \leq d \leq k-1$ and $s \leq n/k$, let $f_d^s(k,n)$ denote the minimum integer m such that every k-graph H on n vertices with $\delta_d(H) \geq m$ has a fractional matching of size s.

Alon et al. [1] provided a connection between the parameters $m_d^s(k,n)$ and $f_d^s(k,n)$. Let k, d be integers such that $1 \leq d \leq k-1$ and let n be a sufficiently large integer. If there exists $c^* > 0$ such that $f_d^{n/k}(k,n) \sim c^*\binom{n-d}{k-d}$, then $m_d^{n/k}(k,n) \sim \max\{c^*, 1/2\}\binom{n-d}{k-d}$. [For integer-valued functions $h_1(n), h_2(n)$, we write $h_1(n) \sim h_2(n)$ if $\lim_{n\to\infty} h_1(n)/h_2(n) = 1$]. In the same paper, they show a way to convert a large fractional matching to a matching using an absorbing technique and a two-round randomization technique; while Kühn, Osthus, and Townsend [6] used the weak regularity lemma for hypergraphs to show $m_d^{an} \sim (1-(1-a)^{k-d})\binom{n-d}{k-d}$, where $0 \leq a < \min\{(k-d)/2, (1-\varepsilon)n/k\}$ and $\varepsilon > 0$ is a constant.

Rödl, Ruciński, and Szemerédi [7] proved that $f_{k-1}^{n/k}(k,n) = \lceil n/k \rceil$, which is much smaller than $m_{k-1}^{n/k}(k,n)$ when $n \in k\mathbb{Z}$ (which is approximately $n/2$). Kühn, Osthus, and Townsend [6] determined $f_d^s(k,n)$ asymptotically when $s \leq n/(2(k-2))$ or $d \geq k/2$.

Alon et al. [1] conjectured that for all $1 \leq d \leq k-1$, $f_d^{n/k}(k,n) \sim (1-(1-1/k)^{k-d})\binom{n-d}{k-d}$, and proved it for $k \geq 3$ and $k-4 \leq d \leq k-1$. In this note, we determine the exact value of $f_d^{n/k}(k,n)$ for certain ranges of d, using a result of Frankl [4] and a result of Frankl and Kupavskii [5]. This is a special case of the following result.

Theorem 1.1 *Let n, k, d be three positive integers such that $2k/5 \leq d \leq k-1$, and let s be a rational number such that $0 < s \leq n/k$. If*

(i) $k/2 \le d \le k-1$ *and* $n \ge 2k(k-1)+1$, or
(ii) $2k/5 < d < k/2$ *and* $n \ge \max\{(5(k-d)-2)s_0/3+d, k(7k-9)/5+1\}$, *where* s_0 *is a sufficiently large constant.*

then $f_d^s(k,n) = \binom{n-d}{k-d} - \binom{n-d-(\lceil s\rceil-1)}{k-d} + 1$.

In Sect. 2, we prove a technical result, Lemma 2.5, about fractional matchings. In Sect. 3, we give a short proof of Theorem 1.1 by applying Lemma 2.5, a result of Frankl (Lemma 2.2), and a result of Frankl and Kupavskii (Lemma 2.4). We will also discuss other related work on asymptotic and exact bounds for $f_d^s(k,n)$ in Sect. 4.

2 Fractional Matchings

One of the ideas in our proof is to use the strong duality between the size of a largest fractional matching in a hypergraph and the size of a smallest fractional vertex cover of that hypergraph. This idea has been already explored before, e.g., see [1, 6]. Let H be a hypergraph. A *fractional vertex cover* of H is a function $\omega : V(H) \to [0,1]$, such that for each $e \in E(H)$ we have $\sum_{\{v\,:\,v\in e\}} \omega(v) \ge 1$. The *size* of ω is $\sum_{v\in V(H)} \omega(v)$. We use $\mu(H)$ to denote the minimum size of a fractional vertex cover in H. Note that $\nu'(H) = \mu(H)$ for any hypergraph H, as they are optimal solutions of two dual linear programs. In our proof of Theorem 1.1, we will use this fact to transform the fractional matching problem on H to one on another hypergraph H'.

First, observe that $\binom{n-d}{k-d} - \binom{n-d-(\lceil s\rceil-1)}{k-d} + 1$ is a lower bound for $f_d^s(k,n)$. For convenience, we state it below as a lemma. The construction involved in the proof is standard, e.g., see equations (3) and (4) in [1].

Lemma 2.1 *Let* k, d *be integers such that* $k \ge 2$ *and* $0 \le d \le k-1$. *Then, for any integer* n *with* $n \ge k$ *and any rational number* s *with* $0 < s \le n/k$, $f_d^s(k,n) \ge \binom{n-d}{k-d} - \binom{n-d-(\lceil s\rceil-1)}{k-d} + 1$.

Proof Let $H_k(n,s)$ be the k-graph with vertex set $[n]$ and edge set consisting of all k-element subsets of $[n]$ which have non-empty intersection with the subset $[\lceil s\rceil - 1]$.

First, suppose $0 < s \le 1$. Then, by definition, $H_k(n,s)$ has no edge and, thus, has no fractional matching of any positive size. Therefore, in this case, $f_d^s(k,n) \ge 1 = \binom{n-d}{k-d} - \binom{n-d-(\lceil s\rceil-1)}{k-d} + 1$.

Hence, we may assume $s > 1$. Then

$$\delta_d(H_k(n,s)) = \binom{n-d}{k-d} - \binom{n-d-(\lceil s\rceil-1)}{k-d}.$$

Let $\omega : [n] \to [0,1]$ such that $\omega(x) = 1$ for all $x \in [\lceil s\rceil - 1]$ and $\omega(x) = 0$ for all

$x \in [n] \setminus [\lceil s \rceil - 1]$. Clearly, ω is a fractional vertex cover of $H_k(n,s)$. So $\nu'(H_k(n,s)) = \mu(H_k(n,s)) \leq \lceil s \rceil - 1$, and the assertion of the lemma holds. □

We also need two results concerning a famous conjecture of Erdős [2] on the matching number of a k-graph; both have a requirement on the number of vertices. The first result is due to Frankl (Theorem 1.1 in [4]).

Lemma 2.2 (Frankl) *Let k, s be integers with $k \geq 2$ and $s \geq 1$. Then, for any integer n with $n \geq (2k-1)s + k$, $m_0^s(k,n) = \binom{n}{k} - \binom{n-s+1}{k} + 1$.*

The second result is a small variation of the following result of Frankl and Kupavskii (Theorem 1 in [5]).

Lemma 2.3 (Frankl and Kupavskii) *Let k be an integer with $k \geq 2$. There exists an absolute constant $s_0 \geq 1$ such that, for any integer $s \geq s_0$ and any integer $n \geq (5k/3 - 2/3)s$, $m_0^s(k,n) = \binom{n}{k} - \binom{n-s+1}{k} + 1$.*

Lemma 2.4 (*Frankl and Kupavskii*) *Let k be an integer with $k \geq 2$. There exists an absolute constant $s_0 \geq 1$ such that, for any integer $s \geq 1$ and any integer $n \geq (5k/3 - 2/3)\max\{s, s_0\}$, $m_0^s(k,n) = \binom{n}{k} - \binom{n-s+1}{k} + 1$.*

Proof If $s \geq s_0$ then the assertion follows from Lemma 2.3. Now $s < s_0$. Since $n \geq (5k/3 - 2/3)\max\{s, s_0\}$, $n + (s_0 - s) \geq n \geq (5k/3 - 2/3)s_0$. Thus by Lemma 2.3, $m_0^{s_0}(k, n + (s_0 - s)) = \binom{n+s_0-s}{k} - \binom{n+(s_0-s)-s_0+1}{k} + 1$.

Now let H be an arbitrary k-graph with n vertices and $e(H) \geq \binom{n}{k} - \binom{n-s+1}{k} + 1$. Let Q be a set of $s_0 - s$ vertices such that $Q \cap V(H) = \emptyset$. Let H' be the k-graph with vertex set $V(H) \cup Q$ and edge set

$$E(H') = E(H) \cup \{e \in \binom{Q \cup V(H)}{k} \ : \ e \cap Q \neq \emptyset\}.$$

Then $e(H') \geq \binom{n+s_0-s}{k} - \binom{n+(s_0-s)-s_0+1}{k} + 1$. Since $m_0^{s_0}(k, n+(s_0-s)) = \binom{n+s_0-s}{k} - \binom{n+(s_0-s)-s_0+1}{k} + 1$, H' contains a matching M' of size s_0. Then $M = \{e \in M' \ : \ e \cap Q = \emptyset\}$ is a matching of size s in H. Thus $m_0^s(k,n) = \binom{n}{k} - \binom{n-s+1}{k} + 1$. □

We now state and prove the main result of this section, which essentially says that $f_d^s(k,n) \leq f_0^s(k-d, n-d)$. Our proof follows the method used by Alon et al. in [1]. Recall that for a hypergraph H and $S \subseteq V(H)$, $N_H(S) = \{T \subseteq V(H) \setminus S : S \cup T \in E(H)\}$. We also view $N_H(S)$ as a hypergraph with vertex set $V(H) \setminus S$ and edge set $N_H(S)$.

Lemma 2.5 *Let k, d be integers with $k \geq 2$ and $1 \leq d \leq k-1$, and let n be a positive integer and s be a rational constant with $0 < s \leq n/k$. Let H be a k-graph on n vertices such that, for every set $S \subseteq V(H)$ with $|S| = d$, the $(k-d)$-graph $N_H(S)$ has a fractional matching of size at least s. Then H has a fractional matching of size at least s.*

Proof Let ω be a fractional vertex cover of H with size $\mu(H)$, and write $V(H) = \{v_1, \ldots, v_n\}$ such that

(1) $\omega(v_1) \geq \omega(v_2) \geq \cdots \geq \omega(v_n)$.

Let H_ω be the k-graph with vertex set $V(H)$ and edge set

$$E(H_\omega) = \left\{ e \ : \ e \in \binom{V(H)}{k} \text{ and } \sum_{v \in e} \omega(v) \geq 1 \right\}.$$

Then ω is also a fractional vertex cover of H_ω; so $\mu(H_\omega) \leq \mu(H)$. Since every edge of H is also an edge of H_ω, we have $\nu'(H_\omega) \geq \nu'(H)$. Hence, $\nu'(H_\omega) = \mu(H_\omega) \leq \mu(H) = \nu'(H) \leq \nu'(H_\omega)$. Thus, we have

(2) $\nu'(H) = \nu'(H_\omega)$.

Let $S = \{v_{n-d+1}, \ldots, v_n\}$. Then, $|S| = d$. Let $w_0 := \frac{1}{d} \sum_{v \in S} \omega(v)$, and define $\omega' : V(H_\omega) \to [0, 1]$ such that

$$\omega'(v) = \begin{cases} \omega(v), & \text{if } v \in V(H_\omega) \setminus S; \\ w_0, & \text{if } v \in S. \end{cases}$$

We may assume that $w_0 < 1/k$. For, otherwise, $\nu'(H) = \mu(H) = \sum_{v \in V(H)} \omega(v) \geq n\omega_0 \geq n/k \geq s$; so the assertion of the lemma holds.

Let $\omega'' : V(H) \to \mathbb{R}^+ \cup \{0\}$ be a function such that

$$\omega''(v) = \frac{\omega'(v) - w_0}{1 - kw_0} \quad \text{for all } v \in V(H).$$

Then $\omega''(v) = 0$ for $v \in S$. Note that $N_{H_\omega}(S)$ is a $(k-d)$-graph with vertex set $V(H_\omega) \setminus S$ (which has $n - d$ vertices). For any edge $e \in N_{H_\omega}(S)$, since ω is also a vertex cover of H_ω and $e \cup S \in E(H_\omega)$, we have $\sum_{v \in e \cup S} \omega(v) \geq 1$. Recall that $\omega(v) = \omega'(v)$ for any $v \in V(H) - S$ and $\omega'(x) = 0$ for any $x \in S$. So we have

$$\begin{aligned} \sum_{v \in e} \omega''(v) &= \sum_{v \in e} \frac{\omega'(v) - w_0}{1 - kw_0} \\ &= \frac{\sum_{v \in e} \omega'(v) - kw_0}{1 - kw_0} \\ &= \frac{\left(\sum_{v \in e \cup S} \omega(v) \right) - kw_0}{1 - kw_0} \geq 1. \end{aligned}$$

Thus, the function ω'' restricted to $V(H_\omega) \setminus S$ is a fractional vertex cover of $N_{H_\omega}(S)$. Then by hypothesis and Strong Duality Theorem, we have

$$\sum_{v\in V(H_\omega)\setminus S}\omega''(v) \geq \mu(N_{H_\omega}(S)) = \nu'(N_{H_\omega}(S)) \geq s.$$

Recall that ω is a minimum vertex cover of H_ω. Note that $\nu'(H_\omega) \leq n/k$; so

$$k\omega_0 \sum_{v\in V(H_\omega)} \omega(v) \leq k\omega_0(n/k) = n\omega_0.$$

Hence, we have

$$\begin{aligned} s &\leq \sum_{v\in V(H_\omega)\setminus S}\omega''(v) = \sum_{v\in V(H_\omega)}\omega''(v) = \frac{\sum_{v\in V(H_\omega)}\omega(v) - nw_0}{1-kw_0} \\ &\leq \sum_{v\in V(H_\omega)}\omega(v) = \nu'(H_\omega). \end{aligned}$$

Thus by (2), H has a fractional matching of size at least s. □

3 Proof of Theorem 1.1.

First, we give a proof of Theorem 1.1. Let k, d be integers with $k \geq 3$ and $2k/5 < d \leq k-1$. If $d \geq k/2$ let $s_0 = 1$, and if $2k/5 < d \leq k/2$ let $s_0 \geq 1$ be given as in Lemma 2.4. Recall that Rödl, Ruciński, and Szemerédi (see Corollary 3.1 in [7]) proved that $f_{k-1}^{n/k}(k,n) = \lceil n/k \rceil$. So we may assume that $k-d \geq 2$. By Lemma 2.5, $f_d^s(k,n) \leq f_0^s(k-d, n-d)$.

Since $d < k$, $\lceil s \rceil \leq (n+k-1)/k < (n-d)/(k-d)$; so

$$f_d^s(k,n) \leq f_0^s(k-d, n-d) \leq m_0^{\lceil s \rceil}(k-d, n-d).$$

Therefore, in view of Lemma 2.1, it suffices to show that $m_0^{\lceil s \rceil}(k-d, n-d) \leq \binom{n-d}{k-d} - \binom{(n-d)-\lceil s \rceil+1}{k-d} + 1$ for all s with $1 \leq s \leq n/k$ (in which case $\lceil s \rceil \leq (n-d)/(k-d)$).

We apply Lemma 2.2 (when $d \geq k/2$) and Lemma 2.4 (when $2k/5 < d < k/2$) on a $(k-d)$-graph of order $n-d$. Thus, we need to verify that, for every s with $0 < s \leq n/k, f(d) := (n-d) - [(2(k-d)-1)\lceil s \rceil + (k-d)] \geq 0$ when $d \geq k/2$, and $g(d) := (n-d) - (5(k-d)/3 - 2/3)\max\{\lceil s \rceil, s_0\} \geq 0$ when $2k/5 < d < k/2$. Note that the first derivatives $f'(d) = 2\lceil s \rceil > 0$ and $g'(d) = 5\max\{\lceil s \rceil, s_0\}/3 - 1 > 0$ when $s > 0$.

Suppose $d \geq k/2$ and $n \geq 2k(k-1)+1$. Then

$$f(d) \geq f(k/2) = n-k-(k-1)\lceil s \rceil \geq n-k-(k-1)(n+k-1)/k \geq 0.$$

as $s \leq n/k$ and $n \geq 2k(k-1)+1$.

Now suppose $2k/5 < d < k/2$ and $n \geq \max\{k(7k-9)/5+1, (5(k-d)-2)s_0/3 + d\}$. We have $d \geq (2k+1)/5$. Hence,

$$g(d) \geq g((2k+1)/5) = n - (2k+1)/5 - (k-1)\lceil s \rceil \geq n - (2k+1)/5 - (k-1)(n+k-1)/k \geq 0$$

as $n \geq k(7k-9)/5 + 1$. On the other hand, $g(d) \geq (n-d) - (5(k-d)/3 - 2/3)s_0 \geq 0$ as $n \geq (5(k-d)-2)s_0/3 + d\}$. So $g(d) \geq 0$. □

4 Concluding Remarks

Rödl, Ruciński, and Szemerédi [7] determined $f^s_{k-1}(k,n)$ for $0 < s \leq n/k$. For the entire range $1 \leq d \leq k-2$, Kühn, Osthus, and Townsend [6] proved the following asymptotic result.

Theorem 4.1 (*Kuhn, Osthus, and Townsend*) *Let k, d be integers with $k \geq 3$ and $1 \leq d \leq k-2$, and let $0 \leq a \leq \min\{1/(2(k-d)), 1/k\}$. Then, for positive integers n,*

$$f^{an}_d(k,n) \sim \left(1 - (1-a)^{k-d}\right)\binom{n-d}{k-d}.$$

Thus, $f^s_d(k,n)$ is asymptotically determined when $1 \leq d \leq k-2$ and $s \leq n/(2(k-d))$, and when $d \geq k/2$ and $s \in (0, n/k]$. Theorem 1.1 determines $f^s_d(k,n)$ exactly when $d > 2k/5$ and n, s large enough.

For matchings, Kühn, Osthus, and Townsend [6] proposed the following conjecture.

Conjecture 4.2 (*Kühn, Osthus, and Townsend*) *For all $\varepsilon > 0$ and all integers n, k, d, s with $1 \leq d \leq k-1$ and $1 \leq s \leq (1-\varepsilon)n/k$,*

$$m^s_d(k,n) \sim \left(1 - (1-s/n)^{k-d}\right)\binom{n-d}{k-d}.$$

Kühn, Osthus, and Townsend [6] proved that Conjecture 4.2 holds for $k/2 \leq d \leq k-1$. Han [3] showed that this conjecture holds for $0.42k < d < k/2$. Alon et al. [1] showed for any two constants α, α' with $0 < \alpha'^{1/r} \ll \alpha < 1/k$, where r is a sufficiently large integer, there exists n_0 such that for all $n \geq n_0$, $m^{(1-\alpha)n/k}_d(k,n) \leq f^{(1/k-\alpha+\alpha')n}_d(k,n)$. By Lemma 2.1, we have $\binom{n-d}{k-d} - \binom{n-d-(n/k-\alpha n)}{k-d} \leq m^{(1/k-\alpha)n}_d(k,n)$. Recall that Alon et al. [1] proved $m^{n/k}_d(k,n) \sim \max\{c^*, 1/2\}\binom{n-d}{k-d}$, where $f^{n/k}_d(k,n) \sim c^*\binom{n-d}{k-d}$. Note that for $k \geq 3$ and $2k/5 \leq d \leq k-1$, $1-(1-1/k)^{k-d} < 1/2$. As a consequence of Theorem 1.1 and another result [1] (see Theorem 1.1), we can derive the following result.

Corollary 4.3 *Let k, d be integers such that $k \geq 2$ and $d > 2k/5$. For any constant α with $0 < \alpha < 1/k$, there exists n_0 such that for any $n \geq n_0$,*

$$m_d^{(1/k-\alpha)n}(k,n) \sim \binom{n-d}{k-d}\left(1-(1-1/k+\alpha)^{k-d}\right),$$

and

$$m_d^{n/k}(k,n) \sim \frac{1}{2}\binom{n-d}{k-d}$$

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Declarations

Conflict of interest The authors have not disclosed any competing interests.

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