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A better bound on the size of rainbow matchings

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Aharoni and Howard conjectured that, for positive integers n, k, t with $n \geq kt$, if $F_1, \dots, F_t \subseteq \binom{[n]}{k}$ such that $|F_i| > \max\left\{\binom{n}{k} - \binom{n-t+1}{k}, \binom{kt-1}{k}\right\}$ for $i \in [t]$ then there exist $e_i \in F_i$ for $i \in [t]$ such that $e_1, \dots, e_t$ are pairwise disjoint. Huang, Loh, and Sudakov proved this conjecture for $t < n/(3k^2)$. In this paper, we show that this conjecture holds for $t < n/(2k)$ and t sufficiently large.

1. Introduction

For a positive integer n, let $[n]$ denote the set $\{1, \dots, n\}$. For a nonnegative integer k and set S with at least k elements, let $\binom{S}{k} = \{e \subseteq S \ : \ |e| = k\}$. Let $k \geq 2$ be an integer.

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A *k-uniform hypergraph* or *k-graph* is a pair $H = (V, E)$, where $V = V(H)$ is a finite set of *vertices* and $E = E(H) \subseteq \binom{V}{k}$ is the set of *edges*. We use $e(H)$ to denote the number of edges in H. For any $S \subseteq V(H)$, let $H[S]$ denote the subgraph of H with $V(H[S]) = S$ and $E(H[S]) = \{e \in E(H) : e \subseteq S\}$, and let $H - S := H[V(H) \setminus S]$.

A *matching* in a hypergraph H is a subset of $E(H)$ consisting of pairwise disjoint edges. The maximal size of a matching in a hypergraph H is denoted by $\nu(H)$. A classical problem in extremal set theory is to determine $\max e(H)$ with $\nu(H)$ fixed. Erdős [6] in 1965 made the following conjecture: For positive integers k, n, t with $n \geq kt$, every k-graph H on n vertices with $\nu(H) < t$ satisfies $e(H) \leq \max\left\{\binom{n}{k} - \binom{n-t+1}{k}, \binom{kt-1}{k}\right\}$. This bound is tight for the complete k-graph on $kt - 1$ vertices and for the k-graph on n vertices in which every edge intersects a fixed set of $t - 1$ vertices. There have been recent activities on this conjecture, see [2,3,7–9,12,18]. In particular, Frankl [8] proved that if $n \geq (2t - 1)k - (t - 1)$ and $\nu(H) < t$ then $e(H) \leq \binom{n}{k} - \binom{n-t+1}{k}$, with further improvement by Frankl and Kupavskii [10].

There are also attempts to extend the above conjecture of Erdős to a family of hypergraphs. Let $\mathcal{F} = \{F_1, \dots, F_t\}$ be a family of hypergraphs. A set of pairwise disjoint edges, one from each F_i, is called a *rainbow matching* for $\mathcal{F}$. In this case, we also say that $\mathcal{F}$ or $\{F_1, \dots, F_t\}$ *admits* a rainbow matching. Aharoni and Howard [1] made the following conjecture, also see Huang, Loh, and Sudakov [12].

Conjecture 1.1. *Let $\mathcal{F} = \{F_1, \dots, F_t\}$ be a family of subsets of $\binom{[n]}{k}$. If*

$$e(F_i) > \max\left\{\binom{n}{k} - \binom{n-t+1}{k}, \binom{kt-1}{k}\right\}$$

for all $1 \leq i \leq t$, then $\mathcal{F}$ admits a rainbow matching.

Huang, Loh, and Sudakov [12] proved that Conjecture 1.1 holds for $n > 3k^2t$.

Theorem 1.2 *(Huang, Loh, and Sudakov). Let n, k, t be three positive integers such that $n > 3k^2t$. Let $\mathcal{F} = \{F_1, \dots, F_t\}$ be a family of subsets of $\binom{[n]}{k}$. If*

$$e(F_i) > \binom{n}{k} - \binom{n-t+1}{k}$$

for all $1 \leq i \leq t$, then $\mathcal{F}$ admits a rainbow matching.

Recently, Frankl and Kupavskii [11] proved that Conjecture 1.1 holds when $n \geq 12kt\log(e^2t)$, providing an almost linear bound. In this paper, we show in Theorem 1.3 that Conjecture 1.1 holds when $n > 2kt$ and t is sufficiently large. More recently, Keevash, Lifshitz, Long and Minzer [15] independently proved a more general version of Theorem 1.3 with $n = \Omega(kt)$ using sharp threshold techniques developed in [14].

Theorem 1.3. *For a given integer $k \geq 3$, there exists $t_0 = t_0(k)$ such that the following holds. Let $t > t_0$ and $n > 2kt$ be two positive integers, and $\mathcal{F} = \{F_1, \ldots, F_t\}$ be a family of subsets of $\binom{[n]}{k}$. If*

$$e(F_i) > \binom{n}{k} - \binom{n-t+1}{k}$$

for all $1 \leq i \leq t$, then $\mathcal{F}$ admits a rainbow matching.

Note that the lower bound on $e(F_i)$ is best possible. Indeed, for $i \in [t]$ let F_i be the k-graph on $[n]$ consisting of all edges intersecting $[t-1]$. Then for $i \in [t]$, $e(F_i) = \binom{n}{k} - \binom{n-t+1}{k}$ and $\nu(F_i) = t-1$. Hence, $\{F_1, \ldots, F_t\}$ does not admit any rainbow matching.

This example naturally corresponds to a special class of $(k+1)$-graphs $\mathcal{F}_t(k, n)$. This is defined in Section 2, where we reduce the problem for finding one such rainbow matching to a problem about finding "near" perfect matchings in a larger class of $(k+1)$-graphs, denoted by $\mathcal{F}^t(k, n)$. This will allow us to apply various techniques used previously to find large matchings in uniform hypergraphs.

We show in Section 3 that Theorem 1.3 holds when $\mathcal{F}^t(k, n)$ is close to $\mathcal{F}_t(k, n)$, in the sense that most edges of $\mathcal{F}_t(k, n)$ are also edges of $\mathcal{F}^t(k, n)$. To deal with the case $\mathcal{F}^t(k, n)$ is not close to $\mathcal{F}_t(k, n)$, we follow the approach in [5] and [20]. First, we find a small absorbing matching M_1 in $\mathcal{F}^t(k, n)$ which is done in Section 4. (However, the existence of this absorbing matching does not require that $\mathcal{F}^t(k, n)$ be not close to $\mathcal{F}_t(k, n)$.) Then we take random samples of subgraphs of $\mathcal{F}^t(k, n) - V(M_1)$ so that they satisfy various properties, in particular they all have fractional perfect matchings, see Section 5. In Section 6, we use fractional perfect matchings in those random samples to perform a second round of randomization to find a spanning subgraph H' of $\mathcal{F}^t(k, n) - V(M_1)$. We then apply a result of Pippenger to find a matching in H' covering all but a small constant fraction of the vertices, and use the matching M_1 to find the desired matching in $\mathcal{F}^t(k, n)$ covering all but fewer than k vertices.

2. Notation and reduction

To prove Theorem 1.3, we convert this rainbow matching problem on k-graphs to a matching problem for a special class of $(k+1)$-graphs. Let Q, V be two disjoint sets. A $(k+1)$-graph H with vertex $Q \cup V$ is called $(1, k)$*-partite* with *partition classes* Q, V if, for each edge $e \in E(H)$, $|e \cap Q| = 1$ and $|e \cap V| = k$. A $(1, k)$-partite $(k+1)$-graph H with partition classes Q, V is *balanced* if $|V| = k|Q|$. We say that $S \subseteq V(H)$ is *balanced* if $|S \cap V| = k|S \cap Q|$.

Let $F_1, \ldots, F_t$ be a family of subsets of $\binom{[n]}{k}$ and $X := \{x_1, \ldots, x_t\}$ be a set of t vertices. We use $\mathcal{F}^t(k, n)$ to denote the $(1, k)$-partite $(k+1)$-graph with partition classes $X, [n]$ and edge set

$$E(\mathcal{F}^t(k,n)) = \bigcup_{i=1}^{t} \{\{x_i\} \cup e \ : \ e \in F_i\}.$$

If $F_1 = \cdots = F_t = H_k(t,n)$, where $H_k(t,n)$ denotes the k-graph with vertex set $[n]$ and edge set $\binom{[n]}{k} \setminus \binom{[n]\setminus[t]}{k}$, then we denote $\mathcal{F}^t(k,n)$ by $\mathcal{F}_t(k,n)$.

Observation 1. $\{F_1,\ldots,F_t\}$ admits a rainbow matching if, and only if, $\mathcal{F}^t(k,n)$ has a matching of size t.

Hence, to prove Theorem 1.3, it suffices to show that $\mathcal{F}^t(k,n)$ has a matching of size t. For convenience, we further reduce this problem to a near perfect matching problem. Write $n-kt = km+r$, where $0 \le r \le k-1$. Let $F_1,\ldots,F_t \subseteq \binom{[n]}{k}$, and let $F_i = \binom{[n]}{k}$ for $i = t+1,\ldots,t+m$. Let $Q = \{x_1,\ldots,x_{m+t}\}$ and let $\mathcal{H}^t(k,n)$ be the $(1,k)$-partite $(k+1)$-graph with partition classes $Q, [n]$ and edge set

$$E(\mathcal{H}^t(k,n)) = \bigcup_{i=1}^{m+t} \{\{x_i\} \cup e \ : \ e \in F_i\}.$$

When $F_1 = \cdots = F_t = H_k(t,n)$, we denote $\mathcal{H}^t(k,n)$ by $\mathcal{H}_t(k,n)$. Note that $\nu(\mathcal{H}_t(k,n)) = m+t = (n-r)/k$, i.e., $\mathcal{H}_t(k,n)$ has a matching covering all but less than k vertices (and such a matching is said to be *near perfect*).

Lemma 2.1. *$\mathcal{F}^t(k,n)$ has a matching of size t if, and only if, $\mathcal{H}^t(k,n)$ has a matching of size $m+t = \lfloor n/k \rfloor$.*

Proof. First, suppose that $\mathcal{F}^t(k,n)$ has a matching M_1 of size t. Since $n-kt = km+r \ge km$, $[n] \setminus V(M_1)$ contains m pairwise disjoint k-sets, say $e_1,\ldots,e_m$. Let $M_2 = \{e_i \cup \{x_{i+t}\} \ : \ i \in [m]\}$. Then $M_1 \cup M_2$ is a matching of size $m+t$ in $\mathcal{H}^t(k,n)$.

Now assume that $\mathcal{H}^t(k,n)$ has a matching M of size $m+t$. Note that each edge in M contains exactly one vertex in $\{x_1,\ldots,x_{m+t}\}$. Thus, the t edges in M intersecting $\{x_1,\ldots,x_t\}$ form a matching in $\mathcal{F}^t(k,n)$ of size t. $\square$

For the proof of Theorem 1.3, we need additional concepts and notation. Given two hypergraphs H_1, H_2 with $V(H_1) = V(H_2)$, let $c(H_1,H_2)$ be the minimum of $|E(H_1)\setminus E(H')|$ taken over all isomorphic copies H' of H_2 with $V(H') = V(H_2)$. For a real number $\varepsilon > 0$, we say that H_2 is *ε-close* to H_1 if $V(H_1) = V(H_2)$ and $c(H_1,H_2) \le \varepsilon|E(H_1)|$. The following is obvious.

Observation 2. If $\mathcal{F}^t(k,n)$ is ε-close to $\mathcal{F}_t(k,n)$, then $\mathcal{H}^t(k,n)$ is ε-close to $\mathcal{H}_t(k,n)$.

As mentioned in Section 1, our proof of Theorem 1.3 will be divided into two parts, according to whether or not $\mathcal{F}^t(k,n)$ is ε-close to $\mathcal{F}_t(n,k)$. If $\mathcal{F}^t(k,n)$ is close to $\mathcal{F}_t(n,k)$, we will apply greedy argument to construct a matching of size t. If $\mathcal{F}^t(k,n)$ is not close to $\mathcal{F}_t(n,k)$, then $\mathcal{H}^t(k,n)$ is not close to $\mathcal{H}_t(n,k)$, and we will show that $\mathcal{H}^t(k,n)$ has a spanning subgraph with properties that enable us to find a large matching M_2 and to use absorbing matching M_1 to enlarge M_2 to a near perfect matching.

3. The extremal case: $\mathcal{F}^t(k,n)$ is ε-close to $\mathcal{F}_t(k,n)$

We say that a claim holds when $0 < a \ll b$ if there exists a function $f : (0,1) \to (0,1)$ such that the claim holds whenever $0 < a < f(b)$. We will not specify this implicit function f.

In this section, we prove Theorem 1.3 for the case when $\mathcal{F}^t(k,n)$ is ε-close to the extremal configuration $\mathcal{F}_t(k,n)$ where $0 < \varepsilon \ll \zeta < 1$ and $\varepsilon \ll \frac{1}{k}$. Note that ζ will be determined when we consider the non-extremal case where $\mathcal{F}^t(k,n)$ is not ε-close to $\mathcal{F}_t(k,n)$.

Let H be a $(k+1)$-graph and $v \in V(H)$. We define the neighborhood $N_H(v)$ of v in H to be the set $\{S \in \binom{V(H)}{k} : S \cup \{v\} \in E(H)\}$. Let H be a $(k+1)$-graph with the same vertex set as $\mathcal{F}_t(k,n)$. Given real number α with $0 < \alpha < 1$, a vertex v in H is called *α-good* with respect to $\mathcal{F}_t(k,n)$ if

$$\left|N_{\mathcal{F}_t(k,n)}(v) \setminus N_H(v)\right| \le \alpha n^k$$

and, otherwise, v is called *α-bad*. Clearly, if H is ε-close to $\mathcal{F}_t(k,n)$, then the number of α-bad vertices in H is at most $(k+1)\varepsilon n/\alpha$.

Lemma 3.1. *Let ζ, α be real numbers and n, k, t be positive integers such that $0 < \alpha \ll \zeta < 1$, $\alpha \ll 1/k$, $n \ge 24k^3$ and $n/(6k^2) \le t < (1-\zeta)n/k$. Let H be a $(1,k)$-partite $(k+1)$-graph with $V(H) = V(\mathcal{F}_t(k,n))$. If every vertex of H is α-good with respect to $\mathcal{F}_t(k,n)$, then H has a matching of size t.*

Proof. Let $X := \{x_1, x_2, ..., x_t\}$, $W := [t]$, and $U := [n] \setminus [t]$, such that $X, [n]$ are the partition classes of H. Let M be a maximum matching in H such that $|e \cap X| = |e \cap W| = 1$ for all $e \in E(H)$. Let $X' = X \setminus V(M)$, $W' = W \setminus V(M)$, and $U' = U \setminus V(M)$.

We claim that $|M| \ge n/(12k^2)$. For, otherwise, assume $|M| < n/(12k^2)$. Consider any vertex $x \in X'$. Since x is α-good, we have

$$\left|\left(W \times \binom{U}{k-1}\right) \setminus N_H(x)\right| \le \alpha n^k.$$

Note that, since $n/6k^2 \le t < (1-\zeta)n/k$ and $\alpha < \zeta^{k-1}(6k^2 2^k (k-1)!)^{-1}$,

$$\left|W' \times \binom{U'}{k-1}\right| = (|W| - |M|)\binom{n-kt}{k-1} > \frac{n}{12k^2}\binom{\zeta n}{k-1} > \frac{n}{12k^2}\frac{(\zeta n/2)^{k-1}}{(k-1)!} > \alpha n^k.$$

Thus there exists $f \in N_H(x) \cap \left(W' \times \binom{U'}{k-1}\right)$. Now $M' = M \cup \{\{x\} \cup f\}$ is a matching of size $|M|+1$ in H, and $|f \cap X| = |f \cap W| = 1$ for all $f \in M'$. Hence, M' contradicts the choice of M, completing the proof of Claim.

Let $S = \{u_1, \dots, u_{k+1}\} \subseteq V(H) \setminus V(M)$, where $u_1 \in X'$, $u_{k+1} \in W'$ and $u_i \in U'$ for $i \in [k] \setminus \{1\}$. Let $\{e_1, \dots, e_k\}$ be an arbitrary k-subset of M, and let $e_i := \{v_{i,1}, v_{i,2}, \dots, v_{i,k+1}\}$ with $v_{i,1} \in X$, $v_{i,k+1} \in W$, and $v_{i,j} \in U$ for $i \in [k]$ and $j \in [k] \setminus \{1\}$. For $j \in [k+1]$, let $f_j := \{u_j, v_{1,j+1}, v_{2,j+2}, \dots, v_{k,j+k}\}$ with addition in the subscripts modulo $k+1$ (except we write $k+1$ instead of 0). Note that $f_1, \dots, f_{k+1}$ are pairwise disjoint.

If $f_j \in E(H)$ for all $j \in [k+1]$ then $M' := (M \cup \{f_1, \dots, f_{k+1}\}) \setminus \{e_1, \dots, e_k\}$ is a matching in H such that $|M'| = |M|+1 > |M|$ and $|f \cap X| = |f \cap W| = 1$ for all $f \in M'$, contradicting the choice of M. Hence, $f_j \notin E(H)$ for some $j \in [k+1]$.

Note that there are $\binom{|M|}{k} k!$ choices of $(e_1, \dots, e_k) \subseteq M^k$ and that for any two different such choices the corresponding f_j's are distinct. Hence,

$$\begin{aligned}
&|\{e \in E(\mathcal{F}_t(k,n)) \setminus E(H) : |e \cap \{u_i : i \in [k+1]\}| = 1\}| \\
&\geq |M|(|M|-1) \cdots (|M|-k+1) \\
&> \left(n/(12k^2) - k\right)^k \\
&> \left(n/(24k^2)\right)^k \quad \text{(since } n \geq 24k^3) \\
&> (k+1)\alpha n^k \quad \text{(since } \alpha < ((k+1)24^k k^{2k}))^{-1}.
\end{aligned}$$

This implies that there exists $i \in [k+1]$ such that $|N_{\mathcal{F}_t(k,n)}(u_i) \setminus N_H(u_i)| > \alpha n^k$, contradicting the fact that all u_i are α-good with respect to $\mathcal{F}_t(k,n)$ $\square$

We can now prove Theorem 1.3 when $\mathcal{F}^t(k,n)$ is ε-close to $\mathcal{F}_t(k,n)$.

Lemma 3.2. *Let $k \geq 3$, $t \geq 1$ be integers, and let ε, ζ be real numbers such that $0 < \varepsilon \ll \zeta < 1$ and $\varepsilon \ll 1/k$, $48k^2 \leq t < (1-\zeta)(1-k(k+1)\sqrt{\varepsilon})n/k$. Let $(F_1, \dots, F_t)$ be a family of subsets of $\binom{[n]}{k}$ such that $e(F_i) > \binom{n}{k} - \binom{n-t+1}{k}$ for $i \in [t]$, and let $\mathcal{F}^t(k,n)$ denote the corresponding $(1,k)$-partite $(k+1)$-graph. Suppose $\mathcal{F}^t(k,n)$ is ε-close to $\mathcal{F}_t(k,n)$. Then $\mathcal{F}^t(k,n)$ has a matching of size t.*

Proof. We may assume $n \leq 3k^2 t$ as otherwise the assertion follows from Theorem 1.2. Let B denote the set of $\sqrt{\varepsilon}$-bad vertices in $\mathcal{F}^t(k,n)$. Since $\mathcal{F}^t(k,n)$ is ε-close to $\mathcal{F}_t(k,n)$, $|B| \leq (k+1)\sqrt{\varepsilon}n$. Let $X, [n]$ be the partition classes of $\mathcal{F}_t(k,n)$, and let $X := \{x_1, x_2, ..., x_t\}$, $W := [t]$, and $U := [n] \setminus [t]$. Note that each edge of $\mathcal{F}_t(k,n)$ intersects W.

Let $b := \max\{|B \cap X|, |B \cap W|\}$; so $b \leq (k+1)\sqrt{\varepsilon}n$. We choose $X_1 \subseteq X, W_1 \subseteq W$ such that $B \cap X \subseteq X_1$, $B \cap W \subseteq W_1$, and $|X_1| = |W_1| = b$. Let $\mathcal{F}_1 = \mathcal{F}^t(k,n)[X_1 \cup W_1 \cup U]$. For every $x \in X_1$, we have

$$|N_{\mathcal{F}_1}(x)| \geq |N_{\mathcal{F}}(x)| - \left(\binom{n}{k} - \binom{n-(t-b)}{k}\right) > \binom{n-(t-b)}{k} - \binom{n-(t-1)}{k}.$$

Since $n-(t-b) > n/2 \geq 3k^2(k+1)\sqrt{\varepsilon}n > 3k^2 b$, it follows from Theorem 1.2 that the family $\{N_{\mathcal{F}_1}(x) : x \in X_1\}$ admits a rainbow matching. Thus, by Observation 1, $\mathcal{F}_1$ has a matching, say M, of size b. Clearly, M covers $B \cap X$.

Let $\mathcal{F}_2 := \mathcal{F}^t(k,n)[(X\setminus X_1)\cup([n]\setminus(V(M)\cup B)]$, and let $a := |B\setminus V(M)|$. By the choice of W_1 and X_1, we have $B\cap(W\setminus W_1) = \emptyset$. Note that $\mathcal{F}_2$ may be viewed as the $(1,k)$-partite $(k+1)$-graph $\mathcal{F}_2 = \mathcal{F}_{t-b}(k, n-kb-a)$, with partition classes $X\setminus X_1, [n]\setminus(V(M)\cup B)$, corresponding to the family $(F_i[(X\setminus X_1)\cup([n]\setminus(V(M)\cup B)] : i \in X\setminus X_1)$. Put $n' = n-kb-a$ and $t' = t-b$. We wish to apply Lemma 3.1 to $\mathcal{F}_2$.

Note that $n' = n-kb-a \geq n-k|B| \geq n-k(k+1)\sqrt{\varepsilon}n \geq n/2 \geq 24k^3$. Moreover, since $b \leq (k+1)\sqrt{\varepsilon}n \leq n/(6k^2) \leq t/2$, we have $n'/(6k^2) \leq n/(6k^2) \leq t/2 < t-b = t'$. Also, $t' \leq t < (1-\zeta)(n-k(k+1)\sqrt{\varepsilon}n)/k \leq (1-\zeta)(n-k|B|)/k \leq (1-\zeta)(n-kb-a)/k = (1-\zeta)n'/k$.

For every $x \in V(\mathcal{F}_2)$, since x is $\sqrt{\varepsilon}$-good with respect to $\mathcal{F}_t(k,n)$,

$$\begin{aligned}
|N_{\mathcal{F}_{t'}(k,n')}(x)\setminus N_{\mathcal{F}_2}(x)| &\leq |N_{\mathcal{F}_t(k,n)}(x)\setminus N_{\mathcal{F}}(x)| \\
&\leq \sqrt{\varepsilon}n^k \\
&< 2^k\sqrt{\varepsilon}(n-kb-a)^k \quad (\text{since } kb+a \leq (k+1)^2\sqrt{\varepsilon}n < n/2) \\
&= 2^k\sqrt{\varepsilon}(n')^k.
\end{aligned}$$

Thus every vertex x of $\mathcal{F}_2$ is $2^k\sqrt{\varepsilon}$-good with respect to $\mathcal{F}_{t'}(k,n')$. By Lemma 3.1, $\mathcal{F}_2$ has a matching M' of size $t' = t-b$. Hence $M\cup M'$ is a matching in $\mathcal{F}$ of size t. □

4. Absorbing lemma

The purpose of this section is to prove the existence of a small matching M in $\mathcal{H}^t(k,n)$ (defined before Observation 2) such that for any small balanced set S, $\mathcal{H}^t(k,n)[V(M)\cup S]$ has a perfect matching. We need to use Chernoff bounds here and in the next section. Let $Bi(n,p)$ denote a binomial random variable with parameters n and p. The following well-known concentration inequalities, i.e. Chernoff bounds, can be found in Appendix A in [4], or Theorem 2.8, inequalities (2.9) and (2.11) in [13].

Lemma 4.1 *(Chernoff inequality for small deviation). If $X = \sum_{i=1}^n X_i$, each random variable X_i has Bernoulli distribution with expectation p_i, and $\alpha \leq 3/2$, then*

$$\mathbb{P}(|X-\mathbb{E}X| \geq \alpha\mathbb{E}X) \leq 2e^{-\frac{\alpha^2}{3}\mathbb{E}X}.$$

In particular, when $X \sim Bi(n,p)$ and $\lambda < \frac{3}{2}np$, then

$$\mathbb{P}(|X-np| \geq \lambda) \leq e^{-\Omega(\lambda^2/(np))}.$$

We can now prove an absorbing lemma for $H = \mathcal{H}^t(k,n)$.

Lemma 4.2. *Let $k \geq 3$ be an integer, $\zeta > 0$ be a real number, and $t \geq t_1(k, \zeta)$ be a sufficiently large integer. Let H be a $(1,k)$-partite $(k+1)$-graph with partition classes $\{x_1, \ldots, x_{\lfloor n/k \rfloor}\}, [n]$ such that $d_H(x_i) > \binom{n}{k} - \binom{n-t+1}{k}$ for $i \in [t]$ and $d_H(x_i) = \binom{n}{k}$ for $i = t+1, \ldots, \lfloor n/k \rfloor$. Suppose $n/3k^2 \leq t \leq (1-\zeta)n/k$. Then for any c with $0 < c \ll \zeta < 1$, $c \ll \frac{1}{k}$, there exists a matching M in H such that $|M| \leq 2kcn$ and, for any balanced subset $S \subseteq V(H)$ with $|S| \leq (k+1)c^{1.5}n/2$, $H[V(M) \cup S]$ has a perfect matching.*

Proof. For balanced $R \in \binom{V(H)}{k+1}$ and balanced $Q \in \binom{V(H)}{k(k+1)}$, we say that Q is R-*absorbing* if $\nu(H[Q \cup R]) = k+1$ and Q is the vertex set of a matching in H. Let $\mathcal{L}(R)$ denote the collection of all R-absorbing sets in H.

Claim 1. For each balanced $(k+1)$-set $R \subseteq V(H)$, the number of R-absorbing sets in H is at least $\zeta^k \binom{n}{k}^{k+1} (6k^2 2^k k^2!)^{-1}$.

Let $R = \{x, u_1, \ldots, u_k\}$ be fixed with $x \in X$ and $u_i \in [n]$ for $i \in [k]$. Note that the number of edges in H containing x and intersecting $\{u_1, \ldots, u_k\}$ is at most $k\binom{n}{k-2}$, and $d_{\mathcal{H}}(x) > \binom{n}{k} - \binom{n-t+1}{k}$. So the number of edges $\{x, v_1, \ldots, v_k\}$ in H such that $v_i \in [n]$ for $i \in [k]$ and $\{v_1, \ldots, v_k\} \cap \{u_1, \ldots, u_k\} = \emptyset$ is at least

$$\binom{n}{k} - \binom{n-t+1}{k} - k\binom{n}{k-2} \geq \frac{1}{6k^2}\binom{n}{k},$$

since $3k^2 t \geq n \geq kt$.

Fix a choice of an edge $\{x, v_1, \ldots, v_k\}$ in H such that $v_i \in [n]$ for $i \in [k]$ and $\{v_1, \ldots, v_k\} \cap \{u_1, \ldots, u_k\} = \emptyset$, and let $W_0 = \{v_1, \ldots, v_k\}$. For each $j \in [k]$ and each pair u_j, v_j, we choose a k-set U_j such that U_j is disjoint from $W_{j-1} \cup R$ and both $U_j \cup \{u_j\}$ and $U_j \cup \{v_j\}$ are edges in H, and let $W_j := U_j \cup W_{j-1}$. Thus, if W_k is defined then W_k is an absorbing $k(k+1)$-set for R.

Note that in each step $j \in [k]$ there are $k+1+jk$ vertices in $W_{j-1} \cup R$. Thus, the number of edges in H containing u_j (respectively, v_j) and at least one other vertex in $W_{j-1} \cup R$ is at most $(k+1+jk)\binom{n}{k-2}\lfloor n/k \rfloor < (k+1)n\binom{n}{k-2}$. Note that by definition of $x_{t+1}, x_{t+2}, \ldots, x_{\lfloor n/k \rfloor}$, there are at least $\binom{n-2}{k-1}(\lfloor n/k \rfloor - t) \geq \binom{n-2}{k-1}\zeta n/k$ sets U_j such that both $U_j \cup \{u_j\}$ and $U_j \cup \{v_j\}$ are edges in $\mathcal{H}$ for large n. Hence, for each $j \in [k]$, there are at least $\binom{n-2}{k-1}\zeta n/k - (k+1)n\binom{n}{k-2} \geq \zeta n\binom{n-1}{k-1}/(2k)$ such choices for U_j (as n is sufficiently large). Thus, in total we obtain $\frac{1}{6k^2}\binom{n}{k}(\zeta n\binom{n-1}{k-1}/(2k))^k$ absorbing, ordered $k(k+1)$-sets for R, with multiplicity at most $(k^2)!$; so

$$\mathcal{L}(R) \geq \frac{\frac{1}{6k^2}\binom{n}{k}(\zeta n\binom{n-1}{k-1}/(2k))^k}{(k^2)!} \geq \frac{\zeta^k\binom{n}{k}^{k+1}}{6k^2 2^k (k^2)!}.$$

This completes the proof of Claim 1.

Now, let c be a fixed constant with $0 < c < \zeta^{2k}(12k^2 2^k (k!)^k)^{-2}$, and choose a family $\mathcal{G}$ of balanced $k(k+1)$-sets by selecting each of the $\binom{\lfloor n/k \rfloor}{k}\binom{n}{k^2}$ balanced sets of size $k(k+1)$ with probability

$$p := \frac{cn}{\binom{\lfloor n/k \rfloor}{k}\binom{n}{k^2}}.$$

It follows from Lemma 4.1 that, with probability $1 - o(1)$, the family $\mathcal{G}$ satisfies the following properties:

$$|\mathcal{G}| \leq 2cn \tag{1}$$

and

$$|\mathcal{L}(R) \cap \mathcal{G}| \geq p|\mathcal{L}(R)|/2 \geq \frac{c\zeta^k n}{12k^2 2^k (k!)^k} \geq c^{1.5} n \tag{2}$$

for all balanced $(k+1)$-sets R. Furthermore, we can bound the expected number of intersecting pairs of $k(k+1)$-sets from above by

$$\binom{\lfloor n/k \rfloor}{k}\binom{n}{k^2} k(k+1)\left(\binom{\lfloor n/k \rfloor - 1}{k-1}\binom{n}{k^2} + \binom{\lfloor n/k \rfloor}{k}\binom{n-1}{k^2-1}\right) p^2 \leq c^{1.9} n.$$

Thus, using Markov's inequality, we derive that with probability at least 1/2

$$\mathcal{G} \text{ contains at most } c^{1.9} n \text{ intersecting pairs of } k(k+1)\text{-sets.} \tag{3}$$

Hence, there exists a family $\mathcal{G}$ satisfying (1), (2) and (3). Delete one $k(k+1)$-set from each intersecting pair in such a family $\mathcal{G}$. Further removing all non-absorbing $k(k+1)$-sets, we obtain a subfamily $\mathcal{G}'$ consisting of pairwise disjoint balanced, absorbing $k(k+1)$-sets, which satisfies

$$|\mathcal{L}(R) \cap \mathcal{G}'| \geq \frac{1}{2} c^{1.5} n,$$

for all balanced $(k+1)$-sets R.

Since $\mathcal{G}'$ consists only of absorbing $k(k+1)$-sets, $H[V(\mathcal{G}')]$ has a perfect matching M, of size at most $2kcn$ by (1). For a balanced set $S \subseteq V(H)$ of size $|S| \leq (k+1)c^{1.5}n/2$, S can be partitioned into at most $c^{1.5}n/2$ balanced $(k+1)$-sets. For each balanced $(k+1)$-set R, since $|\mathcal{L}(R) \cap \mathcal{G}'| \geq \frac{1}{2}c^{1.5}n$, we can successively choose a distinct absorbing $k(k+1)$-set for R in $\mathcal{G}'$. Hence, $\mathcal{H}[V(M) \cup S]$ has a perfect matching. $\square$

5. Fractional perfect matchings

When $\mathcal{F}^t(k,n)$ is not ε-close to $\mathcal{F}_t(k,n)$, we use fractional perfect matchings in random subgraphs of $\mathcal{H}^t(k,n)$.

Let H be a hypergraph. A *fractional* matching in H is a function $h : E(H) \to [0,1]$ such that $\sum_{e \ni x} h(e) \leq 1$ for all $x \in V(H)$. Let $\nu_f(H) := \max_h \sum_{e \in E(H)} h(e)$, where maximum is taken over all fractional matchings h of H. A fractional matching in a k-uniform hypergraph with n vertices is *perfect* if its size is n/k.

First, we need a concept of dense graphs used in the hypergraph container result of Balogh, Morris, and Samotij [5] and independently Sexton and Thomassen [20]. Let H be a hypergraph, $\lambda > 0$ be a real number, and $\mathcal{A}$ be a family of subsets of $V(H)$. We say that H is $(\mathcal{A}, \lambda)$-*dense* if $e(H[A]) \geq \lambda e(H)$ for every $A \in \mathcal{A}$.

Lemma 5.1. *Let n, k, t be positive integers and ε be a constant such that $n \leq 3k^2t$, $0 < \varepsilon \ll 1$, and $n \geq 40k^2/\varepsilon$. Let $a_0 = \varepsilon/(8k), a_1 = \varepsilon/(24k^2), a_2 = \varepsilon/(8k^2)$, and $a_3 < \varepsilon/(2^k \cdot k! \cdot 30k)$. Let H be a $(1,k)$-partite $(k+1)$-graph with vertex partition classes $X, [n]$ and with $|X| = t$. Suppose $d_H(x) \geq \binom{n}{k} - \binom{n-t+1}{k} - a_3n^k$ for any $x \in X$. If H is not ε-close to $\mathcal{F}_t(k,n)$, then H is $(\mathcal{A}, a_0)$-dense, where $\mathcal{A} = \{A \subseteq V(H) : |A \cap X| \geq (t/n - a_1)n,\ |A \cap [n]| \geq (1 - t/n - a_2)n\}$.*

Proof. We prove this by way of contradiction. Suppose that there exists $A \subseteq V(H)$ such that $|A \cap X| \geq (t/n - a_1)n$, $|A \cap [n]| \geq (1 - t/n - a_2)n$, and $e(H[A]) \leq a_0e(H)$. Without loss of generality, we may choose A such that $|A \cap X| = (t/n - a_1)n$ and $|A \cap [n]| = (1 - t/n - a_2)n$. Let $U \subseteq [n]$ such that $A \cap [n] \subseteq U$ and $|U| = n - t$. Let $A_1 = A \cap X$, $A_2 = X \setminus A$, $B_1 = A \cap [n]$, and $B_2 = U \setminus A$.

Let H_0 denote the isomorphic copy of H by naming vertices such that $X = \{x_1, ..., x_t\}$ and $U = [n] \setminus [t]$. We derive a contradiction by showing that $|E(\mathcal{F}_t(k,n)) \setminus E(H_0)| < \varepsilon e(\mathcal{F}_t(k,n))$. Note that, since $n \leq 3k^2t$,

$$e(\mathcal{F}_t(k,n)) = t\left(\binom{n}{k} - \binom{n-t}{k}\right) \geq t\left(\binom{n}{k} - \binom{n - n/3k^2}{k}\right) \geq t\binom{n}{k}/(3k).$$

Moreover,

$$e(\mathcal{F}_t(k,n)) \geq t\binom{n}{k}/(3k) = \frac{tn}{3k^2}\binom{n-1}{k-1},$$

and since $n > 2k$,

$$e(\mathcal{F}_t(k,n)) \geq t\binom{n}{k}/(3k) > \frac{tn^k}{2^k \cdot k! \cdot 3k}.$$

Consider $x \in A_1$. Let $E_{H_0}(B_1, x)$ denote the set of edges of H_0 contained entirely in $B_1 \cup \{x\}$ in H_0. The number of edges in H_0 containing x that also exist in $\mathcal{F}_t(k,n)$ is the number of edges in H_0 containing x and intersecting $[t]$. Hence,

$$\begin{aligned} &|\{e : x \in e, e \in E(H_0), e \cap [t] \neq \emptyset\}| \\ &\geq d_{H_0}(x) - |\{e : x \in e, e \in E(H_0 - [t]), e \cap B_2 \neq \emptyset\}| - |E_{H_0}(B_1, x)| \\ &\geq \left(\binom{n}{k} - \binom{n-t+1}{k} - a_3n^k\right) - a_2n\binom{n-t}{k-1} - |E_{H_0}(B_1, x)|. \end{aligned}$$

Then we have

$$\begin{aligned}
&\sum_{x\in A_1} |\{e : x \in e, e \in E(\mathcal{F}_t(k,n)) \setminus E(H_0)\}| \\
&\leq \sum_{x\in A_1} \left(\binom{n}{k} - \binom{n-t}{k} - |\{e : x \in e, e \in E(H_0), e \cap [t] \neq \emptyset\}|\right) \\
&\leq \sum_{x\in A_1} \left[\left(\binom{n}{k} - \binom{n-t}{k}\right)\right. \\
&\quad \left. - \left(\binom{n}{k} - \binom{n-t+1}{k} - a_3 n^k - a_2 n\binom{n-t}{k-1} - E_{H_0}(B_1,x)\right)\right] \\
&\leq \sum_{x\in A_1} \left[\binom{n-t+1}{k} - \binom{n-t}{k} + a_3 n^k + a_2 n\binom{n-t}{k-1} + E_{H_0}(B_1,x)\right] \\
&= |A_1|\binom{n-t}{k-1} + a_3 t n^k + a_2 t n\binom{n-t}{k-1} + \sum_{x\in A_1} E_{H_0}(B_1,x) \\
&\leq (3k^2/n)\cdot e(\mathcal{F}_t(k,n)) + (2^k\cdot k!\cdot 3ka_3)\cdot e(\mathcal{F}_t(k,n)) + (3k^2 a_2)\cdot e(\mathcal{F}_t(k,n)) + e(H_0[A]) \\
&< a_0 e(H_0) + \left(3k^2/n + 2^k\cdot k!\cdot 3ka_3 + 3k^2 a_2\right)\cdot e(\mathcal{F}_t(k,n))
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\sum_{x\in A_2} |\{e : x \in e, e \in E(\mathcal{F}_t(k,n)) \setminus E(H_0)\}| \\
&\leq |A_2|\left(\binom{n}{k} - \binom{n-t}{k}\right) \\
&\leq (a_1 n/t)\cdot e(\mathcal{F}_t(k,n)) \\
&\leq (3k^2 a_1)\cdot e(\mathcal{F}_t(k,n))
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&|E(\mathcal{F}_t(k,n)) \setminus E(H_0)| \\
&= \sum_{x\in A_1} |\{e : x \in e, e \in E(\mathcal{F}_t(k,n)) \setminus E(H_0)\}| \\
&\quad + \sum_{x\in A_2} |\{e : x \in e, e \in E(\mathcal{F}_t(k,n)) \setminus E(H_0)\}| \\
&< a_0 e(H_0) + \left(3k^2/n + 2^k\cdot k!\cdot 3ka_3 + 3k^2 a_2\right)\cdot e(\mathcal{F}_t(k,n)) + 3k^2 a_1\cdot e(\mathcal{F}_t(k,n)) \\
&\leq a_0 t\binom{n}{k} + \left(3k^2/n + 2^k\cdot k!\cdot 3ka_3 + 3k^2 a_2 + 3k^2 a_1\right)\cdot e(\mathcal{F}_t(k,n)) \\
&\leq \left(3ka_0 + 3k^2/n + 2^k\cdot k!\cdot 3ka_3 + 3k^2 a_2 + 3k^2 a_1\right)\cdot e(\mathcal{F}_t(k,n))
\end{aligned}$$

$$\leq \varepsilon \cdot e(\mathcal{F}_t(k,n)),$$

a contradiction since H is not ε-close to $\mathcal{F}_t(k,n)$. $\square$

We also need a result of Lu, Yu, and Yuan [16], which is a stability result on matchings in "stable" graphs. For subsets $e = \{u_1, ..., u_k\}, f = \{v_1, ..., v_k\} \subseteq [n]$ with $u_i < u_{i+1}$ and $v_i < v_{i+1}$ for $i \in [k-1]$, we write $e \leq f$ if $u_i \leq v_i$ for all $i \in [k]$. A hypergraph H with $V(H) = [n]$ and $E(H) \subseteq \binom{[n]}{k}$ is said to be *stable* with respect to the ordering of vertices if for $e, f \in \binom{[n]}{k}$ with $e \leq f$, $e \in E(H)$ implies $f \in E(H)$. The following is Lemma 4.2 in [16].

Lemma 5.2 *(Lu, Yu, and Yuan). Let k be a positive integer and let b and η be constants, such that $0 < b < 1/(2k)$ and $0 < \eta \leq (1 + 18(k-1)!/b)^{-2}$. Let n, m be positive integers such that n is sufficiently large and $bn \leq m \leq n/(2k)$. Let H be a k-graph with vertex set $[n]$. Suppose H is stable and $e(H) > \binom{n}{k} - \binom{n-m}{k} - \eta n^k$. If H is not $\sqrt{\eta}$-close to $H_k(m,n)$, then $\nu(H) > m$.*

We now state and prove the main result of this section.

Lemma 5.3. *Let n, k, t be positive integers such that $n \equiv 0 \pmod k$ and let c', c, ε be constants such that $0 \leq c' \ll c \ll \varepsilon \ll 1/k$. Suppose that t is sufficiently large and $n/(3k^2) \leq t \leq (\frac{1}{2} - c')n/k$. Let H be a balanced $(1,k)$-partite $(k+1)$-graph with partition classes $X, [n]$, and let $X' \subseteq X$ with $|X'| = t$. Suppose $d_H(x) \geq \binom{n}{k} - \binom{n-t+1}{k} - \sqrt{c}n^k$ for $x \in X'$, and $d_H(x) = \binom{n}{k}$ for $x \in X \setminus X'$, and assume that for any independent set S in H, $|S \cap X| \leq (t/n - \varepsilon)n$ or $|S \cap [n]| \leq (1 - t/n - \varepsilon)n$. Then H has a fractional perfect matching.*

Proof. We use linear programming duality between vertex cover and matchings. Let $\omega : V(H) \to [0,1]$ such that $\sum_{v \in e} \omega(v) \geq 1$ for all $e \in E(H)$, and, subject to this, $\omega(H) := \sum_{v \in V(H)} \omega(v)$ is minimum. (Thus, ω is a minimum fractional vertex cover of H.) Without loss of generality, we may assume that $\omega(x_1) \leq \cdots \leq \omega(x_{n/k})$, where $X = \{x_1, \ldots, x_k\}$, and $\omega(1) \leq \omega(2) \cdots \leq \omega(n)$. Let $CL(H)$ be a graph with vertex set $V(H)$ and edge set

$$E(CL(H)) = \left\{ e \in \binom{V(H)}{k+1} \ : \ |e \cap X| = 1 \text{ and } \sum_{x \in e} \omega(x) \geq 1 \right\}.$$

Note that H is a subgraph of $CL(H)$ and ω is also a vertex cover of $CL(H)$. Thus ω is also a minimum vertex cover of $CL(H)$.

By Linear Programming Duality Theory, we have $\nu_f(H) = w(H) = w(CL(H)) = \nu_f(CL(H))$. Thus it suffices to show that $CL(H)$ has a fractional perfect matching. Indeed, we will prove that $\nu(CL(H)) = n/k$, i.e., $CL(H)$ has a perfect matching.

By the definition of $E(CL(H))$ and since $\omega(x_1) \leq \cdots \leq \omega(x_n)$, we have

$$N_{CL(H)}(x_1) \subseteq N_{CL(H)}(x_2) \subseteq \cdots \subseteq N_{CL(H)}(x_{n/k}). \tag{4}$$

Hence, $N_{CL(H)}(x_i) = \binom{[n]}{k}$ for $i \in [n/k] \setminus [t]$. It is also easy to see that $N_{CL(H)}(x_i)$ is stable for all $i \in [n/k]$.

Let $t' = t - c'n/k$. One can see that $t' \leq n/(2k)$. Let η be a constant satisfying $c^{1/4} \ll \eta \leq \min\{(1 + 54k^2(k-1)!)^{-1}, \varepsilon(k(k+1))^{-2}\}$. We distinguish two cases.

Case 1. $N_H(x_1)$ is not η-close to $H_k(t, n)$.

Let $\mathcal{M}$ be a matching of size $c'n/k$ in $N_{CL(H)}(x_1)$ (Note one can find it greedily). Write $\mathcal{N} = N_H(x_1) - \{e \in N_H(x_1) \mid e \cap V(\mathcal{M}) \neq \emptyset\}$. We observe that

$$\begin{aligned} e(\mathcal{N}) \geq d_H(x_1) - c'n^k &\geq \binom{n}{k} - \binom{n-t+1}{k} - 2\sqrt{c}n^k \\ &= \binom{n}{k} - \binom{n-t}{k} - \binom{n-t}{k-1} - 2\sqrt{c}n^k. \end{aligned}$$

By Lemma 5.2 with $m = t'$ and $b = 1/(3k^2)$, $\mathcal{N}$ has a matching $\mathcal{M}'$ of size t'. So $M_1 := \mathcal{M} \cup \mathcal{M}'$ is a matching of size t in $N_H(x_1)$. Let $M_1 = \{e_1, \ldots, e_t\}$. By (4), $M_1 \subseteq N_{CL(H)}(x_i)$ for $i \in [n/k]$. Thus $M_2 = \{e_i \cup \{x_i\} \ : \ i \in [t]\}$ is a matching in $CL(H)$.

Partition $[n] \setminus V(M_2)$ into $n/k - t$ pairwise disjoint k-sets, say $f_1, \ldots, f_{n/k-t}$. Then by (4), $M_2' = \{f_i \cup \{x_{i+t}\} \ : \ i \in [n/k - t]\}$ is a matching in $CL(H) \setminus V(M_2)$. Hence $M_2 \cup M_2'$ is a perfect matching in $CL(\mathcal{H})$.

Case 2. $N_H(x_1)$ is η-close to $H_k(t, n)$.

Then by (4) $N_{CL(H)}(x_i)$ is η-close to $H_k(t, n)$ for all $i \in [t]$. Without loss of generality, we may assume the set $[t]$ in $V(CL(H))$ corresponds to the set $[t]$ in $V(H_k(t, n))$.

Let B denote the set of $\sqrt{\eta}$-bad vertices in $V(CL(H))$ with respect to $\mathcal{H}_t(k, n)$ and let $b = |B|$. Since $N_{CL(H)}(x_i)$ is η-close to $H_k(t, n)$ for all $i \in [t]$, we have $B \subseteq [n]$ and $b \leq (k+1)\sqrt{\eta}n$. Consider $H' = CL(H) - (\{x_{t+1}, \ldots, x_{n/k}\} \cup [t])$. Note that $kb \leq k(k+1)\sqrt{\eta}n < \varepsilon n$; so $b < \varepsilon n/k$.

Since by assumption for any independent set S in H', $|S \cap X| \leq (t/n - \varepsilon)n$ or $|S \cap [n]| \leq (1 - t/n - \varepsilon)n$, we can greedily find pairwise disjoint edges $f_1, \ldots, f_b$ in H' such that $x_{t-i+1} \in f_i$ in H'. Write $M_{21} = \{f_1, \ldots, f_b\}$.

Let $n' = |[n] \setminus (V(M_{21}) \cup B)|$. Since $V(M_{21}) \cap [t] = \emptyset$, we may assume $[t - b] \subseteq [t] \setminus B$. So there is a natural correspondence between $V(H_k(t - b, n'))$ and $V(N_{CL(H)-(V(M_{21}) \cup B)}(\{x_i\}))$, i.e. $[t - b]$ in $V(H_k(t - b, n'))$ corresponds to $[t - b]$ in $V(N_{CL(H)-(V(M_{21}) \cup B)}(\{x_i\}))$.

Let $H'' = CL(H) - (\{x_{t+1}, \ldots, x_{n/k}\} \cup V(M_{21}) \cup B)$. Note that for each vertex $v \in [n] \setminus (V(M_{21}) \cup B)$, we have

$$\begin{aligned}
&|N_{\mathcal{F}_{t-b}(k,n')}(v) \setminus N_{H''}(\{v\})| \\
\leq &(t-b) \cdot |N_{H_k(t,n)}(v) \setminus N_{CL(H)}(\{v, x_1\})| \\
< &(n/(2k)) \cdot \sqrt{\eta} n^{k-1} \\
< &\eta^{1/3}(n')^k.
\end{aligned}$$

Thus, all vertices in $V(H'')$ are $\eta^{1/3}$-good with respect to $\mathcal{F}_{t-b}(k,n')$. Hence by Lemma 3.1, H'' has a matching M_{22} of size $t-b$.

Partition $[n] \setminus V(M_{21} \cup M_{22})$ into $n/k - t$ disjoint k-sets, say $g_1, \ldots, g_{n/k-t}$. Let $M_{23} = \{g_i \cup \{x_{i+t}\} \ : \ i \in [n/k] \setminus [t]\}$. Then $M_{21} \cup M_{22} \cup M_{23}$ is a perfect matching in $CL(H)$. □

6. Random rounding

In this section, we complete the proof of Theorem 1.3. For convenience, we do not round certain numbers to integers as this does not affect calculations.

First, we need another result of Lu, Yu, and Yuan [17] on the independence number of a subgraph of a k-graph induced by a random subset of vertices, which is a generalization of Lemma 4.3 in [17] where it was shown for $(1,3)$-partite graphs. The same proof for Lemma 4.3 in [17] works here as well, except we use Lemma 5.1 in place of Lemma 4.1 in [17].

Lemma 6.1 *(Lu, Yu, and Yuan). Let $l, \varepsilon', \alpha_1, \alpha_2$ be positive reals, let $\alpha > 0$ with $\alpha \ll \min\{\alpha_1, \alpha_2\}$, let k, n be positive integers, and let H be a $(1,k)$-partite $(k+1)$-graph with partition classes Q, P such that $|Q| = t$, $|P| = n$, $n/(3k^2) \leq t \leq n/k$, $e(H) \geq ln^{k+1}$, and $e(H[F]) \geq \varepsilon' e(H)$ for all $F \subseteq V(H)$ with $|F \cap P| \geq \alpha_1 n$ and $|F \cap Q| \geq \alpha_2 n$. Let $R \subseteq V(H)$ be obtained by taking each vertex of H uniformly at random with probability $n^{-0.9}$. Then, with probability at least $1 - n^{O(1)} e^{-\Omega(n^{0.1})}$, every independent set J in $H[R]$ satisfies $|J \cap P| \leq (\alpha_1 + \alpha + o(1))n^{0.1}$ or $|J \cap Q| \leq (\alpha_2 + \alpha + o(1))n^{0.1}$.*

Next, we also need Janson's inequality to provide an exponential upper bound for the lower tail of a sum of dependent zero-one random variable. (See Theorem 8.7.2 in [4])

Lemma 6.2 *(Janson). Let Γ be a finite set and $p_i \in [0,1]$ be a real for $i \in \Gamma$. Let Γ_p be a random subset of Γ such that the elements are chosen independently with $\mathbb{P}[i \in \Gamma_p] = p_i$ for $i \in \Gamma$. Let S be a family of subsets of Γ. For every $A \in S$, let $I_A = 1$ if $A \subseteq \Gamma_p$ and 0 otherwise. Define $X = \sum_{A \in S} I_A$, $\lambda = \mathbb{E}[X]$, $\Delta = \frac{1}{2} \sum_{A \neq B} \sum_{A \cap B \neq \emptyset} \mathbb{E}[I_A I_B]$ and $\bar{\Delta} = \lambda + 2\Delta$. Then, for $0 \leq t \leq \lambda$, we have*

$$\mathbb{P}[X \leq \lambda - t] \leq \exp(-\frac{t^2}{2\bar{\Delta}}).$$

Now, we use Chernoff bound and Janson's inequality to prove a result on several properties of certain random subgraphs.

Lemma 6.3. *Let n, k be integers such that $k \geq 3$ and n is sufficiently large, let H be a $(1,k)$-partite $(k+1)$-graph with partition classes A, B and $k|A| = |B| = n$, let A_1, A_2 be a partition of A with $|A_1| \geq n/(3k^2)$ and $|A_2| \geq n/(3k^2)$, and let $A_3 \subseteq A$ and $A_4 \subseteq B$ with $|A_i| = n^{0.99}$ for $i = 3, 4$. Take $n^{1.1}$ independent copies of R and denote them by R^i, $1 \leq i \leq n^{1.1}$, where R is chosen from $V(H)$ by taking each vertex uniformly at random with probability $n^{-0.9}$ and then deleting $O(n^{0.06})$ vertices arbitrarily so that $|R| \in (k+1)\mathbb{Z}$ and $k|R \cap A| = |R \cap B|$. For each $S \subseteq V(H)$, let $Y_S := |\{i : \ S \subseteq R^i\}|$. Then, with probability at least $1 - o(1)$, all of the following statements hold:*

- (*i*) $Y_{\{v\}} = (1 \pm n^{-0.01})n^{0.2}$ *for all* $v \in V(H)$.
- (*ii*) $Y_{\{u,v\}} \leq 2$ *for all* $\{u, v\} \subseteq V(H)$.
- (*iii*) $Y_e \leq 1$ *for all* $e \in E(H)$.
- (*iv*) *For all* $i = 1, \ldots, n^{1.1}$, *we have* $|R_i \cap A| = (1/k \pm o(n^{-0.04}))n^{0.1}$ *and* $|R_i \cap B| = (1 \pm o(n^{-0.04}))n^{0.1}$,
- (*v*) *Suppose* $n/k^3 \leq m \leq n/k$ *and* ρ *is a constant with* $0 < \rho < 1$ *such that* $d_H(v) \geq \binom{n}{k} - \binom{n-m}{k} - \rho n^k$ *for all* $v \in A$. *Then for* $1 \leq i \leq n^{1.1}$ *and* $v \in R_i \cap A$, *we have*
$$d_{R_i}(v) > \binom{|R_i \cap B|}{k} - \binom{|R_i \cap B| - mn^{-0.9}}{k} - 3\rho|R_i \cap B|^k,$$
- (*vi*) $|R_i \cap A_j| = |A_j|n^{-0.9} \pm n^{0.06}$ *for* $1 \leq i \leq n^{1.1}$ *and* $j \in [4]$.

Proof. For $1 \leq i \leq n^{1.1}$ and $j \in [4]$, $\mathbb{E}[|R_i \cap A|] = n^{0.1}/k$, $\mathbb{E}[|R_i \cap B|] = n^{0.1}$ and $\mathbb{E}[|R_i \cap A_j|] = n^{-0.9}|A_j|$. Recall the assumptions $|A_1| \geq n/(3k^2)$, $|A_2| \geq n/(3k^2)$, and $|A_3| = |A_4| = n^{0.99}$. By Lemma 4.1, we have

$$\mathbb{P}\left(\left||R_i \cap A| - n^{0.1}/k\right| \geq n^{0.06}\right) \leq e^{-\Omega(n^{0.02})},$$
$$\mathbb{P}\left(\left||R_i \cap B| - n^{0.1}\right| \geq n^{0.06}\right) \leq e^{-\Omega(n^{0.02})}, \text{ and}$$
$$\mathbb{P}\left(\left||R_i \cap A_j| - |A_j|n^{-0.9}\right| \geq n^{0.06}\right) \leq e^{-\Omega(n^{0.02})}.$$

Hence, with probability at least $1 - O(n^{1.1})e^{-\Omega(n^{0.02})}$, (*iv*) and (*vi*) hold.

For every $v \in V(H)$, $\mathbb{E}[Y_{\{v\}}] = n^{1.1} \cdot n^{-0.9} = n^{0.2}$. By Lemma 4.1,

$$\mathbb{P}\left(\left||Y_{\{v\}}| - n^{0.2}\right| \geq n^{0.19}\right) \leq e^{-\Omega(n^{0.18})}$$

Hence, with probability at least $1 - O(n)e^{-\Omega(n^{0.18})}$, (*i*) holds.

Let $Z_{p,q} = \left|S \in \binom{V(H)}{p} : Y_S \geq q\right|$. Then

$$\mathbb{E}\left[Z_{p,q}\right] \leq \binom{n + n/k}{p}\binom{n^{1.1}}{q}(n^{-0.9})^{pq} \leq 2^{p+1}n^{p+1.1q-0.9pq}.$$

So $\mathbb{E}[Z_{2,3}] \leq 8n^{-0.1}$ and $\mathbb{E}[Z_{k,2}] \leq 2^{k+1}n^{2.2-0.8k} \leq 2^{k+1}n^{-0.2}$ for $k \geq 3$. Hence by Markov's inequality, (*ii*) and (*iii*) hold with probability at least $1 - o(1)$.

Finally, we show (v). For all $v \in A$, since $d_H(v) \geq \binom{n}{k} - \binom{n-m}{k} - \rho n^k$, we see that, for $1 \leq i \leq n^{1.1}$ and $v \in R_i \cap A$,

$$\begin{aligned}\mathbb{E}\left[d_{R_i}(v)\right] &> \binom{n}{k} n^{-0.9k} - \binom{n-m}{k} n^{-0.9k} - \rho n^k n^{-0.9k} \\ &> \binom{n^{0.1}}{k} - \binom{n^{0.1} - mn^{-0.9}}{k} - \rho n^{0.1k}.\end{aligned}$$

By (iv), with probability at least $1 - O(n^{1.1})e^{-\Omega(n^{0.02})}$, for all $i = 1, \ldots, n^{1.1}$, we have $|R_i \cap B| = (1 + o(n^{-0.04}))n^{0.1}$. Thus for all $v \in A \cap R_i$,

$$\mathbb{E}\left[d_{R_i}(v)\right] > \binom{|R_i \cap B|}{k} - \binom{|R_i \cap B| - mn^{-0.9}}{k} - 2\rho |R_i \cap B|^k.$$

We wish to apply Lemma 6.2 with $\Gamma = B$, $\Gamma_p = R_i$, and $S \subseteq \binom{B}{k}$. We define

$$\Delta = \frac{1}{2} \sum_{b_1, b_2 \subseteq B, b_1 \neq b_2, b_1 \cap b_2 \neq \emptyset} \mathbb{E}[I_{b_1} I_{b_2}] \leq \frac{1}{2} |R_i \cap B|^{2k-1}.$$

By Lemma 6.2,

$$\begin{aligned}&\mathbb{P}\left(d_{R_i}(v) \leq \binom{|R_i \cap B|}{k} - \binom{|R_i \cap B| - mn^{-0.9}}{k} - 3\rho |R_i \cap B|^k\right) \\ \leq &\mathbb{P}\left(d_{R_i}(v) \leq \mathbb{E}[d_{R_i}(v)] - \rho |R_i \cap B|^k\right) \\ \leq &\exp\left(-\frac{\rho^2 |R_i \cap B|^{2k}}{2\binom{|R_i \cap B|}{k} + 2|R_i \cap B|^{2k-1}}\right) \\ \leq &\exp(-\Omega(n^{0.1})).\end{aligned}$$

Therefore, with probability at least $1 - O(n^{1.1})e^{-\Omega(n^{0.1})}$, (v) holds.

By applying union bound, (i) – (vi) all hold with probability $1 - o(1)$. $\square$

Now we use random subgraphs and fractional matchings to perform a second round of randomization to find a sparse subgraph in a hypergraph that is not ε-close to $\mathcal{H}_t(k, n)$.

Lemma 6.4. *Let $k \geq 3$ be an integer, $0 < c \ll \rho \ll \varepsilon \ll 1$ be reals, $n \in k\mathbb{Z}$ sufficiently large, and let t be an integer with $n/(3k^2) \leq t \leq (\frac{1}{2} - c)n/k$. Let H is a $(1, k)$-partite $(k+1)$-graph with partition classes A, B such that $k|A| = |B| = n$. Let A_1 and A_2 be a partition of A such that $|A_1| = t$ and $|A_2| = n/k - t$. Suppose that $d_H(x) > \binom{n}{k} - \binom{n-t+1}{k} - \rho n^k$ for all $x \in A_1$ and $d_H(x) = \binom{n}{k}$ for all $x \in A_2$. If H is not ε-close to $\mathcal{H}_t(k, n)$, then there exists a spanning subgraph H' of H such that the following conditions hold:*

(1) *For all $x \in V(H')$, with at most $n^{0.99}$ exceptions, $d_{H'}(x) = (1 \pm n^{-0.01})n^{0.2}$.*
(2) *For all $x \in V(H')$, $d_{H'}(x) < 2n^{0.2}$.*
(3) *For any two distinct $x, y \in V(H')$, $d_{H'}(\{x, y\}) < n^{0.19}$.*

Proof. Let $A_3 \subseteq A$ and $A_4 \subseteq B$ with $|A_i| = n^{0.99}$ for $i = 3, 4$. Let $R_1, \ldots, R_{n^{1.1}}$ be defined as in Lemma 6.3. By Lemma 6.3 (iv), we have, with probability $1 - o(1)$, for all $i = 1, \ldots, n^{1.1}$,

$$|R_i \cap A| = (1/k + o(n^{-0.04}))n^{0.1} \text{ and } |R_i \cap B| = (1 + o(n^{-0.04}))n^{0.1}.$$

By Lemma 6.3 (vi), with probability $1 - o(1)$, we have

$$|R_i \cap A_1| = (t/n + o(n^{-0.04}))n^{0.1} \text{ and } |R_i \cap A_2| = (1/k - t/n + o(n^{-0.04}))n^{0.1}.$$

By Lemma 6.3 (v), with probability $1 - o(1)$, we have for $1 \leq i \leq n^{1.1}$ and $x \in A \cap R^i$,

$$d_{R_i}(x) > \binom{|R_i \cap B|}{k} - \binom{|R_i \cap B| - (t-1)n^{-0.9}}{k} - 3\rho|R_i \cap B|^k;$$

By (iv) and (vi) of Lemma 6.3, we may choose $I_i \subseteq R_i \cap (A_3 \cup A_4)$ such that $R_i \setminus I_i$ is balanced and $|R'_i| = (1 - o(1))|R_i|$, where $R'_i = R_i \setminus I_i$ for $i = 1, \ldots, n^{1.1}$. Let $H_1 = H[A_1 \cup B]$. Then, since $t \geq n/(3k^2)$, $e(H_1) \geq n^{k+1}/(6k^2 \cdot k!)$.

Since H is not ε-close to $\mathcal{H}_t(k, n)$, H_1 is not ε-close to $\mathcal{F}_t(k, n)$ (by Observation 2 in Section 2). Let $a_0 = \varepsilon/(8k), a_1 = \varepsilon/(24k^2), a_2 = \varepsilon/(8k^2)$, and $a_3 < \varepsilon(2^k \cdot k! \cdot 30k)^{-1}$. By applying Lemma 5.1 to H_1, a_0, a_1, a_2, a_3, we see that H_1 is $(\mathcal{F}, a_0)$-dense, where

$$\mathcal{F} = \{U \subseteq V(H) : |U \cap A_1| \geq (t/n - a_1)n,\ |U \cap B| \geq (1 - t/n - a_2)n\}.$$

Now we apply Lemma 6.1 to H_1 with $l = (3k^3k!)^{-1}$, $\alpha_1 = t/n - a_1$, $\alpha_2 = 1 - t/n - a_2$, and $\varepsilon' = a_0$. Therefore, with probability at least $1 - n^{O(1)}e^{-\Omega(n^{0.1})}$, for any independent set S of R'_i, $|S \cap R'_i \cap A_1| \leq (t/n - a_1 + o(1))n^{0.1}$ or $|S \cap R'_i \cap B| \leq (1 - t/n - a_2 + o(1))n^{0.1}$. By definition, for $x \in R'_i \cap A_2$, $d_{R'_i}(x) = \binom{|R'_i|}{k}$.

By applying Lemma 5.3 to each $H[R'_i]$, we see that each $H[R'_i]$ contains a fractional perfect matching ω_i. Let $H^* = \cup_{i=1}^{n^{1.1}} R'_i$. We select a generalized binomial subgraph H' of H^* by letting $V(H') = V(H)$ and independently choosing edge e from $E(H^*)$, with probability $\omega_{i_e}(e)$ if $e \subseteq R'_{i_e}$. (By Lemma 6.3 (iii), for each $e \in E(H^*)$, i_e is uniquely defined.)

Note that since w_i is a fractional perfect matching of $H[R'_i]$ for $1 \leq i \leq n^{1.1}$, $\sum_{e \ni v} w_i(e) = 1$ for $v \in R'_i$. By Lemma 6.3 (i) and by Lemma 4.1, with probability $1 - o(1)$, $d_{H'}(v) = (1 \pm n^{-0.01})n^{0.2}$ for each vertex $v \in V(H) - (\cup_{i=1}^{n^{1.1}} I_i) \subseteq V(H) - (A_3 \cup A_4)$ and $d_{H'}(v) \leq (1 \pm n^{-0.01})n^{0.2} < 2n^{0.2}$ for each vertex $v \in \cup_{i=1}^{n^{1.1}} I_i$. By Lemma 6.3 (ii), with probability $1 - o(1)$, $d_{H'}(\{x, y\}) < n^{0.19}$ for any $\{x, y\} \in \binom{V(H)}{2}$. Therefore, there exists a hypergraph H' as desired. $\square$

To prove Theorem 1.3, we also need the following result which was attributed to Pippenger [19] (see Theorem 4.7.1 in [4]). An *edge cover* in a hypergraph H is a set of edges whose union is $V(H)$.

Theorem 6.5 *(Pippenger). For every integer $k \geq 2$ and real $r \geq 1$ and $a > 0$, there are $\gamma = \gamma(k, r, a) > 0$ and $d_0 = d_0(k, r, a)$ such that for every n and $D \geq d_0$ the following holds: Every k-uniform hypergraph $H = (V, E)$ on a set V of n vertices in which all vertices have positive degrees and which satisfies the following conditions:*

(1) *For all vertices $x \in V$ but at most γn of them, $d_H(x) = (1 \pm \gamma)D$;*
(2) *For all $x \in V$, $d_H(x) < rD$;*
(3) *For any two distinct $x, y \in V$, $d_H(\{x, y\}) < \gamma D$;*

contains an edge cover of at most $(1 + a)(n/k)$ edges.

Proof of Theorem 1.3. By Theorem 1.2, we may assume that $2kt < n \leq 3k^2t$. Let $0 < \varepsilon \ll \frac{1}{k} < 1$ be sufficiently small and $t > t_0(k)$ be sufficiently large. By Observation 1, it suffices to show $\mathcal{F}^t(k, n)$ has a matching of size t. Applying Lemma 3.2 to $\mathcal{F}^t(k, n)$ with $\zeta = 1/3$, we may assume that $\mathcal{F}^t(k, n)$ is not ε-close to $\mathcal{F}_t(k, n)$. That is, $\mathcal{H}^t(k, n)$ is not ε-close to $\mathcal{H}_t(k, n)$ by Observation 2.

Now we apply Lemma 4.2 to $\mathcal{H}^t(k, n)$ with $\zeta = 1/3$. Thus there exists some constant $0 < c \ll \varepsilon$ such that $n - kcn \geq 2kt$ and $\mathcal{H}^t(k, n)$ contains an absorbing matching M_1 with $m_1 := |M_1| \leq cn$ and for any balanced subset S of vertices with $|S| \leq (k+1)c^{1.5}n/(4k)$, $\mathcal{H}^t(k, n)[V(M_1) \cup S]$ has a perfect matching. Let $H := \mathcal{H}^t(k, n) - V(M_1)$ and $n_1 := n - km_1$.

Next, we see that H is not $(\varepsilon/2)$-close to $\mathcal{H}_t(k, n - km_1)$. For, suppose otherwise. Then

$$
\begin{aligned}
&|E(\mathcal{H}_t(k, n)) \setminus E(\mathcal{H}^t(k, n))| \\
&\leq |E(\mathcal{H}_t(k, n - km_1)) - E(H)| + |e \in E(\mathcal{H}_t(k, n)) : e \cap V(M_1) \neq \emptyset| \\
&\leq (\varepsilon/2)|E(\mathcal{H}_t(k, n - km_1))| + (k+1)cn \cdot n^k \\
&\leq \varepsilon|E(\mathcal{H}_t(k, n))|.
\end{aligned}
$$

This is a contradiction as $\mathcal{H}^t(k, n)$ is not ε-close to $\mathcal{H}_t(k, n)$.

Since $n_1 \geq n - kcn \geq 2kt$, by Lemma 6.4 H has a spanning subgraph H_1 such that

(1) For all vertices $x \in V(H_1)$ but at most $n_1^{0.99}$ of them, $d_{H_1}(x) = (1 \pm n_1^{-0.01})n_1^{0.2}$;
(2) For all $x \in V(H_1)$, $d_{H_1}(x) < 2n_1^{0.2}$;
(3) For any two distinct $x, y \in V(H_1)$, $d_{H_1}(\{x, y\}) < n_1^{0.19}$.

Hence by applying Lemma 6.5 to H_1 with $0 < a \ll c^{1.5}$, H_1 contains an edge cover of at most $(1 + a)((n_1/k + n_1)/(k + 1))$ edges. Thus, at most $a(n_1/k + n_1)$ vertices are each covered by more than one edge in the cover. Hence, after removing at most $a(n_1/k + n_1)$ edges from the edge cover, we obtain a matching M_2 covering all but at most $(k+1)a(n_1/k + n_1) \leq 3kan_1 \leq 3kan$ vertices.

Now we may choose a balanced subset S of $V(H) \setminus V(M_2)$ such that $|V(H) \setminus (V(M_2) \cup S)| \leq k$. Since $|S| \leq 3kan \leq (k+1)c^{1.5}n$, $\mathcal{H}^t(k,n)[V(M_1) \cup S]$ has a perfect matching, say M_3. Thus, $M_2 \cup M_3$ is matching of $H^t(k,n)$ covering all but at most k vertices, and, hence, has size $\lfloor n/k \rfloor$. Therefore, by Lemma 2.1, $\mathcal{F}^t(k,n)$ has a matching of size t. $\square$

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