Counting Hamiltonian cycles in planar triangulations

Xiaonan Liu, Zhiyu Wang, and Xingxing Yu *
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332 USA

Abstract

Hakimi, Schmeichel, and Thomassen [*J. Graph Theory, 1979*] conjectured that every 4-connected planar triangulation G on n vertices has at least $2(n-2)(n-4)$ Hamiltonian cycles, with equality if and only if G is a double wheel. In this paper, we show that every 4-connected planar triangulation on n vertices has $\Omega(n^2)$ Hamiltonian cycles. Moreover, we show that if G is a 4-connected planar triangulation on n vertices and the distance between any two vertices of degree 4 in G is at least 3, then G has $2^{\Omega(n^{1/4})}$ Hamiltonian cycles.

1 Introduction

A graph G is called *Hamiltonian* if it contains a *Hamiltonian cycle*, i.e., a cycle that contains every vertex in G. A k-vertex cycle (or a k-*cycle*) C in a connected graph G is said to be *separating* if the graph obtained from G by deleting C is not connected. A separating 3-cycle is also called a *separating triangle*. A graph G is k-*connected* if it has more than k vertices and if it remains connected when fewer than k vertices are removed. The *distance* between two vertices in a graph is the number of edges in a shortest path in the graph connecting them. A *planar triangulation* is an edge-maximal plane graph with at least three vertices, i.e., every face is bounded by a triangle. By Euler's theorem, an n-vertex planar triangulation has exactly $3n-6$ edges.

Whitney [20] showed in 1931 that every planar triangulation without separating triangles is Hamiltonian. In 1956, Tutte [19] extended Whitney's result by showing that every 4-connected planar graph is Hamiltonian. Thomassen [18] further strengthened Tutte's result in 1983 by showing that every 4-connected planar graph is *Hamiltonian connected*, i.e., any two distinct vertices are connected by a Hamiltonian path. These results have been extended to graphs on other surfaces, see e.g., [5, 15–17]. It is possible that results there can be combined with the methods in this paper to obtain similar counting results on Hamiltonian cycles in certain triangulations on other surfaces.

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It is a natural problem to consider the number of cycles in a graph, which also has applications in coding theory according to [2–4]. The problem of determining the number of Hamiltonian cycles in 4-connected planar triangulations was initated by Hakimi, Schmeichel, and Thomassen [10] who showed in 1979 that every 4-connected planar triangulation on n vertices has at least $n/\log_2 n$ Hamiltonian cycles. In the same paper, they conjectured a lower bound which is quadratic in the number of vertices and realized by the double wheel. A *double wheel* is a planar triangulation obtained from a cycle by adding two vertices and all edges from these two vertices to the vertices of the cycle.

Conjecture 1.1 (Hakimi, Schmeichel, and Thomassen [10])**.** *If G is a 4-connected planar triangulation on n vertices, then G has at least $2(n-2)(n-4)$ Hamiltonian cycles, with equality if and only if G is a double wheel.*

It was not until recently that Brinkmann, Souffriau, and Van Cleemput [7] gave the first linear lower bound of $\frac{12}{5}(n-2)$ for n-vertex 4-connected planar triangulations. Subsequently, Brinkmann and Van Cleemput [8] proved linear lower bounds for 4-connected plane graphs and plane graphs with at most one 3-cut and sufficiently many edges. Since then, there has been more progress on this problem. Lo [13] showed that every n-vertex 4-connected planar triangulation with $O(\log n)$ separating 4-cycles has $\Omega((n/\log n)^2)$ Hamiltonian cycles. The first and third author [12] further showed that every n-vertex 4-connected planar triangulation with $O(n/\log_2 n)$ separating 4-cycles has $\Omega(n^2)$ Hamiltonian cycles. Very recently, Lo and Qian [14] showed that every n-vertex 4-connected planar triangulation with $O(n)$ separating 4-cycles has $2^{\Omega(n)}$ Hamiltonian cycles.

Thus Conjecture 1.1 holds for large graphs with $O(n)$ separating 4-cycles. In this paper, we remove the assumption on separating 4-cycles and settle Conjecture 1.1 asymptotically.

Theorem 1.2. *If G is a 4-connected planar triangulation on n vertices, then G has at least cn^2 Hamiltonian cycles, where $c = (12 \times 90 \times 541 \times 301)^{-2}/2$.*

The number of Hamiltonian cycles in a planar triangulation G can be significantly larger if one increases the connectivity or the minimum degree of G. Alahmadi, Aldred, and Thomassen [1] showed that every 5-connected n-vertex planar triangulation has $2^{\Omega(n)}$ Hamiltonian cycles, improving the earlier bound $2^{\Omega(n^{1/4})}$ by Böhme, Harant, and Tkáč [6]. Note that the more recent result of Lo and Qian [14] is stronger, but the technique used in [1] played an important role in [14]. The first and third author [12] weakened the assumption in the Böhme-Harant-Tkáč result by replacing the 5-connectedness condition with "minimum degree at least 5". In this paper, we observe that the relative locations of degree 4 vertices play an essential role for 4-connected planar triangulations to have exponentially many Hamiltonian cycles.

Theorem 1.3. *There exists a constant $c > 0$ such that for any 4-connected planar triangulation G on n vertices in which the distance between any two vertices of degree 4 is at least three, G has at least $2^{cn^{1/4}}$ Hamiltonian cycles.*

In Section 2, we discuss an idea similar to the key idea in [1] for finding an edge set F in a 4-connected planar triangulation G such that removing F from G still gives a 4-connected graph. We collect several results on the number of Hamiltonian paths between two given vertices in planar graphs. We also cite some known results on "Tutte paths" and "Tutte cycles" in planar graphs. Such results will be used to find a Hamiltonian cycle through specific edges in a planar graph.

In Section 3, we prove Theorem 1.2. We first show that every n-vertex 4-connected planar triangulation G has $\Omega(n)$ Hamiltonian cycles through two specified edges in any given triangle. Moreover, if G does not contain two adjacent vertices of degree 4, then G has $\Omega(n^2)$ Hamiltonian cycles. We then use these results and apply induction on n to complete the proof of Theorem 1.2.

In Section 4, we consider 4-connected planar triangulations G in which any two vertices of degree 4 have distance at least three. We show that either G has a large independent set with nice properties, or G has many separating 4-cycles with pairwise disjoint interiors, or G has many "nested" separating 4-cycles. In all cases, we can find the desired number of Hamiltonian cycles in G.

We conclude this section with some terminology and notation. For any positive integer k, let $[k] = \{1, 2, \ldots, k\}$.

Let G and H be graphs. We use $G \cup H$ and $G \cap H$ to denote the union and intersection of G and H, respectively. For any $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S, and let $G - S = G[V(G) \backslash S]$. A set $S \subseteq V(G)$ is a *cut* in G if $G - S$ has more components than G, and if $|S| = k$ then S is a cut of *size* k or k-*cut* for short. For a subgraph T of G, we often write $G - T$ for $G - V(T)$ and write $G[T]$ for $G[V(T)]$. A path (respectively, cycle) is often represented as a sequence (respectively, cyclic sequence) of vertices, with consecutive vertices being adjacent. Given a path P and distinct vertices $x, y \in V(P)$, we use xPy to denote the subpath of P between x and y.

Let G be a graph. For $v \in V(G)$, we use $N_G(v)$ (respectively, $N_G[v]$) to denote the neighborhood (respectively, closed neighborhood) of v, and use $d_G(v)$ to denote $|N_G(v)|$. For distinct vertices u, v of G, we use $d_G(u, v)$ to denote the distance between u and v, and if u and v are adjacent in G, we use uv to denote the edge of G between u and v. (If there is no confusion we omit the reference to G.) If H is a subgraph of G, we write $H \subseteq G$. For any set R consisting of vertices of G and 2-element subsets of $V(G)$, we use $H + R$ (respectively, $H - R$) to denote the graph with vertex set $V(H) \cup (R \cap V(G))$ (respectively, $V(H) \backslash (R \cap V(G))$) and edge set $E(H) \cup (R \backslash V(G))$ (respectively, $E(H) \backslash (R \backslash V(G))$). If $R = \{\{x, y\}\}$ (respectively, $R = \{v\}$), we write $H + xy$ (respectively, $H + v$) instead of $H + R$, and write $H - xy$ (respectively, $H - v$) instead of $H - R$.

Let G be a plane graph. Two elements of $V(G) \cup E(G)$ are *cofacial* if they are incident with a common face of G. The *outer walk* of G consists of vertices and edges of G incident with the infinite face of G. If the outer walk is a cycle in G, we call it *outer cycle* instead. If all vertices of G are incident with its infinite face, then we say that G is an *outer planar* graph. For a cycle C in G, we use $\overline{C}$ to denote the subgraph of G consisting of all vertices and edges of G contained in the closed disc in the plane bounded by C. The *interior* of C is then defined as the subgraph $\overline{C} - C$. For any distinct vertices $u, v \in V(C)$, we use uCv to denote the subpath of C from u to v in clockwise order.

2 Preliminaries

In this section, we state and prove a number of lemmas needed for the proofs of Theorems 1.2 and 1.3.

A *near triangulation* is a plane graph in which all faces except possibly its infinite face are bounded by triangles. The first and third author [12] considered the number of Hamiltonian paths between two given vertices in the outer cycle of a near triangulation.

Lemma 2.1 (Liu and Yu [12])**.** *Let G be a near triangulation with outer cycle $C := uvwxu$ and assume that $G \neq C + vx$ and G has no separating triangles. Then one of the following holds:*

(i) *$G - \{v, x\}$ has at least two Hamiltonian paths between u and w.*

(ii) *$G - \{v, x\}$ is a path between u and w and, hence, $G - \{v, x\}$ is outer planar.*

Lemma 2.2 (Liu and Yu [12])**.** *Let G be a near triangulation with outer cycle $C := uvwxu$ and assume that G has no separating triangles. Then one of the following holds:*

(i) *$G - \{w, x\}$ has at least two Hamiltonian paths between u and v.*

(ii) *$G - \{w, x\}$ is an outer planar near triangulation.*

We need an observation about degree 4 vertices and the number of Hamiltonian paths in a near triangulation.

Lemma 2.3. *Let G be a near triangulation with outer cycle $C := uvwxu$ and assume that $|V(G)| \geq 6$ and G has no separating triangles. Suppose there exist distinct $a, b \in V(C)$ such that $G - (V(C)\backslash\{a, b\})$ has at most one Hamiltonian path between a and b. Then G has two adjacent vertices of degree 4 that are contained in $V(G)\backslash V(C)$.*

Proof. By symmetry, we only need to consider two cases: $\{a, b\} = \{u, w\}$ or $\{a, b\} = \{u, v\}$. If $\{a, b\} = \{u, w\}$ and $G - (V(C)\backslash\{a, b\}) = G - \{v, x\}$ has at most one Hamiltonian path between u and w, then by Lemma 2.1, $G - \{v, x\}$ is a path. Hence, by planarity, all vertices in $V(G)\backslash V(C)$ have degree 4 in G; so the assertion holds as $|V(G)\backslash V(C)| \geq 2$.

Now suppose $\{a, b\} = \{u, v\}$ and there exists at most one Hamiltonian path between u and v in $G - (V(C)\backslash\{a, b\}) = G - \{w, x\}$. Then by Lemma 2.2, $G - \{w, x\}$ is an outer planar near triangulation. Let $D = u_1u_2 \dots u_tu_1$ denote the outer cycle of $G - \{w, x\}$ such that $u_1 = u$ and $u_t = v$. Note that $t \geq 4$ and that u_i is adjacent to w or x for every $i \in [t]$. Let u_s, where $s \in [t]$, be the common neighbor of w and x in $V(D)$. (The existence of u_s is guaranteed by the fact that G is a near triangulation with outer cycle $uvwxu$.) Since G has no separating triangles, $2 \leq s \leq t-1$ and every edge of $(G - \{w, x\}) - E(D)$ is incident with both paths $u_1 \dots u_{s-1}$ and $u_{s+1} \dots u_t$. It follows that $d_G(u_s) = 4$. Moreover, $s \geq 3$ and $d_G(u_{s-1}) = 4$, or $s \leq t-2$ and $d_G(u_{s+1}) = 4$, as $|V(G)\backslash V(C)| \geq 2$ and G is a near triangulation. This completes the proof of the lemma. ∎

By Lemma 2.3, it is natural to expect that 4-connected planar triangulations without too many vertices of degree 4 should have many Hamiltonian cycles. We now prove a technical lemma, which will be used in the proof of Lemma 4.2 to produce two Hamiltonian paths in a near triangulation.

Lemma 2.4. *Let G be a near triangulation with outer cycle C, and let $x_1, w_1, w_2, x_2 \in V(C)$ be distinct and occur on C in clockwise order such that $x_1x_2, w_1w_2 \in E(C)$ and each edge of $G - E(C)$ is incident with both x_1Cw_1 and w_2Cx_2. Let $N_G(x_1) \cap N_G(x_2) = \{r\}$ and $N_G(w_1) \cap N_G(w_2) = \{y\}$, and assume $r \notin \{y, w_1, w_2\}$ and $y \notin \{r, x_1, x_2\}$. Suppose any two degree 3 vertices of G contained in $V(G)\backslash\{x_1, x_2, w_1, w_2\}$ have distance at least three in G. Then $G - \{x_1, x_2, w_1, w_2\}$ has a Hamiltonian path between r and y.*

Proof. Note that $|V(G)| \geq 6$ as $r \notin \{y, w_1, w_2\}$ and $y \notin \{r, x_1, x_2\}$. We apply induction on $|V(G)|$. Without loss of generality, we may assume $r \in V(x_1Cw_1)$. Then $d_G(x_1) = 2$.

Suppose $|V(G)| = 6$. If $ry \in E(G)$ then we are done. So assume $ry \notin E(G)$. Then $y \in V(w_2Cx_2)$, $x_2w_1 \in E(G)$, and $d_G(r) = d_G(y) = 3$. This gives a contradiction since $d_G(r, y) = 2$.

Now assume $|V(G)| > 6$. We have two cases: $y \in V(x_1Cw_1)$ or $y \in V(w_2Cx_2)$.

Case 1. $y \in V(x_1Cw_1)$. Then $d_G(w_1) = 2$.

Consider $G_1 = G - w_1$. Let y' denote the unique vertex in $N_{G_1}(y) \cap N_{G_1}(w_2)$. If $y' \notin \{r, x_2\}$ then, by induction, $G_1 - \{x_1, x_2, y, w_2\}$ has a Hamiltonian path H_1 between r and y'; so $H_1 + y'y$ gives a Hamiltonian path between r and y in $G - \{x_1, x_2, w_1, w_2\}$. Hence, we may assume $y' \in \{r, x_2\}$.

If $y' = r$ then $ry, rw_2 \in E(G)$ and $d_G(y) = 3$. Since $|V(G)| \geq 7$, $|V(w_2Cx_2)| \geq 3$. Now, y and a degree 3 vertex of G contained in $V(w_2Cx_2)\backslash\{x_2, w_2\}$ are distance 2 apart in G, a contradiction. So $y' = x_2$. Then $x_2y, x_2w_2 \in E(G)$. Hence, since each edge of $G - E(C)$ is incident with both x_1Cw_1 and x_2Cw_2, $V(rCy) \subseteq N_G(x_2)$, and all vertices in $V(rCy - y)$ have degree 3 in G. Since $|V(G)| \geq 7$ and $x_2w_2 \in E(G)$, $rCy - y$ contains two adjacent vertices of degree 3 in G, a contradiction.

Case 2. $y \in V(w_2Cx_2)$. Then $d_G(w_2) = 2$.

Consider $G_2 = G - w_2$. Let y' denote the unique vertex in $N_{G_2}(y) \cap N_{G_2}(w_1)$. Similar to Case 1, if $y' \notin \{r, x_2\}$ then, by induction, $G_2 - \{x_1, x_2, y, w_1\}$ has a Hamiltonian path H_2 between r and y'. Hence, $H_2 + y'y$ is a Hamiltonian path between r and y in $G - \{x_1, x_2, w_1, w_2\}$.

If $y' = r$, then let x_2' be the neighbor of x_2 on w_2Cx_2; now $rx_2' \cup yCx_2'$ is a Hamiltonian path between r and y in $G - \{x_1, x_2, w_1, w_2\}$. If $y' = x_2$, then $x_2w_1, x_2y \in E(G)$. It follows that $d_G(r) = d_G(y) = 3$, which gives a contradiction since $d_G(r, y) = 2$. ∎

Next, we discuss results related to the counting idea from [1]. Let S be an independent set in a 4-connected planar triangulation G and $F \subseteq E(G)$ consist of $|S|$ edges incident with S. Alahmadi *et al.* [1] observed that $G - F$ is not 4-connected only if some vertex in S is contained in a separating 4-cycle, or some vertex in S is adjacent to three vertices in a separating 4-cycle, or two vertices in S are contained in a separating 5-cycle, or three vertices in S occur in some diamond-6-cycle. A *diamond*-6-*cycle* is a graph isomorphic to the graph shown on the left in Figure 1, in which the vertices of degree 3 are called *crucial* vertices. (A *diamond*-4-*cycle* is a graph isomorphic to the graph shown on the right in Figure 1, where the two degree 3 vertices not adjacent to the degree 2 vertex are its *crucial* vertices.) We say that S *saturates* a 4-cycle or 5-cycle C in G if $|S \cap V(C)| = 2$, and S *saturates* a diamond-6-cycle D in G if S contains three crucial vertices of D.

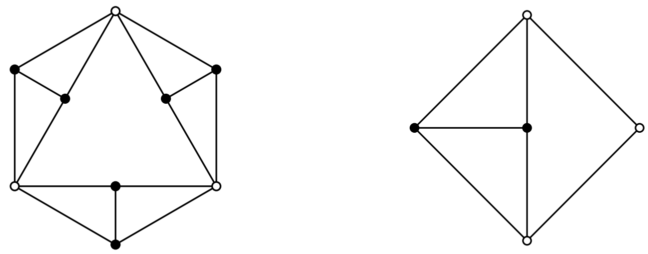

Figure 1: diamond-6-cycle (left); diamond-4-cycle (right); solid vertices represent the crucial vertices.

For a 5-connected planar triangulation G, Alahmadi *et al.* [1] showed that there exists an independent set S consisting of $\Omega(n)$ vertices of degree at most 6 in G, such that $G - F$ is 4-connected for each set F consisting of $|S|$ edges of G that are incident with S. Then it follows from a simple calculation that G has $2^{\Omega(n)}$ Hamiltonian cycles. Such large independent sets need not exist in 4-connected planar triangulations because of the existence of vertices of degree 4 or separating 4-cycles.

Next, we prove two lemmas that will help us deal with vertices of degree 4 and separating 4-cycles. Let G be a plane graph. Suppose u is a vertex of degree at most 6 in G. Define the *link* of u in G, denoted by A_u, as

$$A_u = \begin{cases} E(G[N(u)]), & \text{if } d(u) = 4, \\ \{e \in E(G) : e \text{ is incident with } u \text{ and } G - e \text{ is 4-connected}\}, & \text{if } d(u) \in \{5, 6\}. \end{cases}$$

Lemma 2.5. *Let G be a 4-connected planar triangulation. Suppose S is an independent set of vertices of degree at most 6 in G such that, for any $u \in S$ with $d(u) \in \{5, 6\}$, no degree 4 neighbors of u are adjacent in G. Then the following statements hold:*

(i) *For $u \in S$ with $d(u) \in \{5, 6\}$, $\{v \in N(u) : uv \notin A_u\}$ is independent in G and, hence, $|A_u| \geq \lceil d(u)/2 \rceil$.*

(ii) *If S saturates no 4-cycle in G, then, for any distinct $u_1, u_2 \in S$, $E(G[N[u_1]]) \cap E(G[N[u_2]]) = \emptyset$.*

Proof. Suppose $u \in S$ and $d(u) \in \{5, 6\}$, and suppose there exist two edges $e_1 = uv_1, e_2 = uv_2 \in E(G) \backslash A_u$ with $v_1v_2 \in E(G)$. Let $v_0, v_3 \in N(u) \backslash \{v_1, v_2\}$ be the neighbors of v_1, v_2 in $G[N(u)]$, respectively. Since $G - e_1$ is not 4-connected, there exists a vertex $z \in V(G)$ such that $\{z, v_0, v_2\}$ is a 3-cut in $G - e_1$. Since G is a planar triangulation, we have $zv_0, zv_2 \in E(G)$. Since $G - e_2$ is not 4-connected, we see from planarity that $\{z, v_1, v_3\}$ is a 3-cut in $G - e_2$. Thus, $zv_1, zv_3 \in E(G)$ as G is a planar triangulation. Since G has no separating triangles, we have $d(v_1) = d(v_2) = 4$, a contradiction. Thus, (i) holds.

For (ii), suppose S saturates no 4-cycle in G, and let $u_1, u_2 \in S$ be distinct. Suppose there exists $e \in E(G[N[u_1]]) \cap E(G[N[u_2]])$. Since S is an independent set, it follows that u_1, u_2, and the two vertices incident with e form a 4-cycle in G, contradicting the assumption that S saturates no 4-cycle in G. Hence $E(G[N[u_1]]) \cap E(G[N[u_2]]) = \emptyset$. ∎

The following lemma is derived by using an idea similar to one in [1].

Lemma 2.6. *Let G be a 4-connected planar triangulation and S be an independent set of vertices of degree at most 6 in G. Suppose that $A_u \neq \emptyset$ for all $u \in S$, that S saturates no 4-cycle, or 5-cycle, or diamond-6-cycle in G, and that no degree 4 vertex of G in S has a neighbor of degree 4 in G. Let $F \subseteq \bigcup_{u\in S} A_u$ with $|F \cap A_u| \leq 1$ for all $u \in S$. Then $G - F$ is 4-connected.*

Proof. Suppose there exists some $F \subseteq \bigcup_{u\in S} A_u$ such that $|F \cap A_u| \leq 1$ for all $u \in S$, and $G - F$ is not 4-connected. Let K be a minimum cut of $G - F$; so $|K| \leq 3$. Let G_1, G_2 be subgraphs of $G - F$ such that $G - F = G_1 \cup G_2$, $V(G_1 \cap G_2) = K, E(G_1 \cap G_2) = \emptyset$, and $V(G_i) \neq K$ for $i = 1, 2$. Let F' be the set of the edges of G between $G_1 - K$ and $G_2 - K$. Then $F' \subseteq F$ and $F' \neq \emptyset$ (as $G - K$ is connected).

Observation 1. *Since G is a 4-connected planar triangulation, for each $e \in F'$, the two vertices incident with e have exactly two common neighbors, which must be contained in K.*

Observation 2. *For any two edges $e_1, e_2 \in F'$, there do not exist distinct vertices $u, v \in K$ such that all vertices incident with e_1 or e_2 are contained in $N_G(u) \cap N_G(v)$. For, otherwise, $G[N_G[u] \cup N_G[v]]$ contains a 4-cycle with two vertices in S, contradicting the assumption that S saturates no 4-cycle in G.*

By Observation 1, $|K| \geq 2$. By Observations 1 and 2, $|F'| \leq \binom{|K|}{2}$. Hence, $1 \leq |F'| \leq 3$. Moreover, $|K| = 3$ as, otherwise, $|K| = 2$ and $|F'| \leq \binom{2}{2} = 1$, contradicting the assumption that G is 4-connected. By the definition of A_u, if $e \in A_u \cap F'$ and $d_G(u) = 4$, then $u \in K$; if $e \in A_u \cap F'$ and $d_G(u) \in \{5, 6\}$, then e is incident with u and $u \notin K$.

Suppose $|F'| = 1$ and let $e \in F'$ with $e \in A_u$ for some $u \in S$. If $d_G(u) = 5$ or 6, then u is incident with e and $G - e$ is 4-connected by the definition of A_u, contradicting the fact that K is a 3-cut of $G - F' = G - e$. Thus $d_G(u) = 4$ and $u \in K$. Let $e = w_1w_2$ and $K = \{u, v, w\}$ such that $w_1 \in V(G_1)\backslash V(G_2)$, $w_2 \in V(G_2)\backslash V(G_1)$, and $N_G(w_1) \cap N_G(w_2) = \{u, v\}$. Again since G is a planar triangulation and K is a 3-cut in $G - e$, we have $wu, wv \in E(G)$. Hence $C_1 = uw_1vwu$ and $C_2 = uw_2vwu$ are 4-cycles in G. Let $x \in N_G(u)\backslash\{w, w_1, w_2\}$. Then $G[N_G(u)] = xw_2w_1wx$ or $G[N_G(u)] = xw_1w_2wx$. In the former case, $V(G_1)\backslash K = \{w_1\}$ as, otherwise, $\{w_1, w, v\}$ would be a 3-cut in G; so w_1 and u are two adjacent vertices of degree 4 in G, a contradiction. In the latter case, $V(G_2)\backslash K = \{w_2\}$ as, otherwise, $\{w_2, w, v\}$ would be a 3-cut in G; so w_2 and u are two adjacent vertices of degree 4 in G, a contradiction.

If $|F'| = 2$ and let $F' = \{e_1, e_2\}$, then by Observations 1 and 2, each vertex in K is adjacent to both vertices incident with some edge in F', and exactly one vertex of K is adjacent to all vertices incident with e_1 or e_2. Hence, some 5-cycle in the subgraph of G induced by K and the vertices incident with F' contains two vertices from S, contradicting the assumption that S saturates no 5-cycle in G.

Hence, $|F'| = 3$, and let $e_1, e_2, e_3 \in F'$ where $e_i \in A_{u_i}$ and $u_i \in S$ for $i = 1, 2, 3$. Since S is independent and saturates no 4-cycle or 5-cycle, F' is a matching in G. If two vertices in $\{u_1, u_2, u_3\}$ have degree 4 in G, then these two vertices are contained in K and in a 4-cycle in G, a contradiction. If exactly one vertex in $\{u_1, u_2, u_3\}$, say u_1, has degree 4 in G, then $u_1 \in K$ and u_1 must be adjacent to a vertex in $\{u_2, u_3\}$, contradicting the assumption that S is independent. So u_1, u_2, and u_3 all have degree 5 or 6 in G. But then by Observations 1 and 2, we see that the subgraph of G induced by K and the vertices of G incident with

F' contains a diamond-6-cycle in which u_1, u_2, u_3 are three crucial vertices, contradicting the assumption that S saturates no diamond-6-cycle. ∎

We also need the following two lemmas from Lo [13] and Alahmadi *et al.* [1], that will help us to find an independent set saturating no 4-cycle, or 5-cycle, or diamond-6-cycle.

Lemma 2.7 (Lo [13])**.** *Let G be a 4-connected planar triangulation and let S be an independent set of vertices of degree at most 6 in G, such that S saturates no 4-cycle in G. Then there exists a subset $S' \subseteq S$ of size at least $|S|/541$ such that S' saturates no 5-cycle in G.*

Lemma 2.8 (Alahmadi, Aldred, and Thomassen [1]; Lo [13])**.** *Let G be a 4-connected planar triangulation and let S be an independent set of vertices of degree at most 6 in G, such that S saturates no 4-cycle in G. Then there exists a subset $S' \subseteq S$ of size at least $|S|/301$ such that S' saturates no diamond-6-cycle in G.*

We need another result from Lo [13], which shows that any 4-connected planar triangulation either has a large independent set saturating no 4-cycle, or contains two vertices with many common neighbors.

Lemma 2.9 (Lo [13])**.** *Let G be a 4-connected planar triangulation. Let S be an independent set of vertices of degree at most 6 in G, and S' be a maximal subset of S such that S' saturates no 4-cycle in G. Then there exist distinct vertices $v, x \in V(G)$ such that $|(N(v) \cap N(x)) \cap S| \geq |S|/(9|S'|)$.*

The following result can be easily deduced from the previous three lemmas.

Lemma 2.10. *Let G be a 4-connected planar triangulation on n vertices. Let I be an independent set of vertices of degree at most 6 in G. For any positive integer t, one of the following statements holds:*

(i) *There exist distinct vertices $v, x \in V(G)$ such that $|N(v) \cap N(x) \cap I| \geq t$.*

(ii) *There is a subset $S \subseteq I$, such that $|S| > |I|/(t \times 9 \times 541 \times 301)$ and S saturates no 4-cycle, or 5-cycle, or diamond-6-cycle in G.*

Proof. Let S_1 be a maximal subset of I such that S_1 saturates no 4-cycle in G. If $|S_1| \leq |I|/(t \times 9)$, then by Lemma 2.9 there are distinct vertices v, x in G such that $|N(v) \cap N(x) \cap I| \geq |I|/(9|S_1|) \geq |I|/(9|I|/(t \times 9)) = t$; so (i) holds.

Now suppose $|S_1| > |I|/(t \times 9)$. By Lemmas 2.7 and 2.8, there exists $S \subseteq S_1$ such that S saturates no 4-cycle, or 5-cycle, or diamond-6-cycle in G, and

$$|S| \geq |S_1|/(541 \times 301) > |I|/(t \times 9 \times 541 \times 301);$$

thus (ii) holds. ∎

We conclude this section by stating two results on Tutte paths and Tutte cycles. Let G be a graph and $H \subseteq G$. An *H-bridge* of G is a subgraph of G induced by either an edge in $E(G) \backslash E(H)$ with both incident vertices in $V(H)$, or all edges in $G - H$ with at least one incident vertex in a single component of $G - H$. For an H-bridge B of G, the vertices in $V(B \cap H)$ are the *attachments* of B on H. A path (or cycle) P in a graph G is called a *Tutte path* (or *Tutte cycle*) if every P-bridge of G has at most three attachments on P. If, in addition, every P-bridge of G containing an edge of some subgraph C of G has at most two attachments on P, then P is called a *C-Tutte path* (or *C-Tutte cycle*) in G. Thomassen [18] proved the following result on Tutte paths in 2-connected planar graphs.

Lemma 2.11 (Thomassen [18])**.** *Let G be a 2-connected plane graph and C be its outer cycle, and let $x \in V(C)$, $y \in V(G)\backslash\{x\}$, and $e \in E(C)$. Then G has a C-Tutte path P between x and y such that $e \in E(P)$.*

Note that Lemma 2.11 implies that every 4-connected planar graph is Hamiltonian connected and has a Hamiltonian cycle through two given edges that are cofacial.

A *circuit graph* is an ordered pair (G, C) consisting of a 2-connected plane graph G and a facial cycle C of G such that, for any 2-cut U of G, each component of $G - U$ contains a vertex of C. Jackson and Yu [11] showed that every circuit graph (G, C) has a C-Tutte cycle through a given edge of C and two other vertices.

Lemma 2.12 (Jackson and Yu [11])**.** *Let (G, C) be a circuit graph, and let r, z be vertices of G and $e \in E(C)$. Then G contains a C-Tutte cycle H such that $e \in E(H)$ and $r, z \in V(H)$.*

3 Quadratic bound

We start with a technical lemma for finding distinct Hamiltonian cycles. Recall the link A_u for a vertex u of degree at most 6 in a plane graph.

Lemma 3.1. *Let G be a 4-connected plane graph and $e \in E(G)$. Suppose u is a vertex of degree at most 6 in G such that $G[N(u)]$ is a cycle and $e \notin E(G[N[u]])$. Moreover, assume that if $d(u) \in \{5, 6\}$ then $\{v \in N(u) : uv \notin A_u\}$ is an independent set in G, and that if $d(u) = 4$ then there exist two nonadjacent neighbors of u each having degree at least 5 in G. Then the following statements hold.*

(i) *G has a Hamiltonian cycle through e as well as an edge in A_u.*

(ii) *For any $y \in V(G)\backslash\{u\}$ cofacial with e but not incident with e, $G-y$ has a Hamiltonian cycle through e and an edge in A_u not incident with y.*

Proof. Let $y \in V(G)\backslash\{u\}$ be cofacial with e but not incident with e. Consider a drawing of G in which y is contained in the infinite face of $G - y$. Let C denote the facial cycle of G containing e and y, and C' denote the outer cycle of $G - y$. Then $e \in E(C')$ as y is cofacial with e and not incident with e. Since G is 4-connected, both (G, C) and $(G - y, C')$ are circuit graphs.

Case 1. $d_G(u) \in \{5, 6\}$ and $d_G(u) - |A_u| \leq 1$.

By Lemma 2.12, G has a C-Tutte cycle D through e, and $G-y$ has a C'-Tutte cycle D_1 through e. Since G is 4-connected, D is a Hamiltonian cycle in G, and D_1 is a Hamiltonian cycle in $G - y$. Since $d_G(u) - |A_u| \leq 1$, both D and D_1 contain some edge in A_u. Thus, (i) and (ii) hold.

Case 2. $d_G(u) \in \{5,6\}$ and $d_G(u) - |A_u| = 2$.

Then let $r \in N_G(u)$ such that $ur \notin A_u$, and let $G' := G - ur$.

Let C_1 be a facial cycle of G' containing e. Since G is 4-connected, (G', C_1) is a circuit graph. By Lemma 2.12, G' has a C_1-Tutte cycle D_1 through e, r, and u, which is a Hamiltonian cycle in G containing e. Since $d_G(u) - |A_u| = 2$ and $ur \notin A_u$, D_1 must contain an edge in A_u. Hence, (i) holds.

Let C_2 be the outer cycle of $G' - y$. Since G is 4-connected, $(G' - y, C_2)$ is a circuit graph. By Lemma 2.12, $G' - y$ has a C_2-Tutte cycle D_2 through e and every vertex in $\{r, u\} \backslash \{y\}$. Now D_2 is a Hamiltonian cycle in $G - y$ containing e as G is 4-connected. Moreover, D_2 contains an edge in A_u as $d_G(u) - |A_u| = 2$ and $ur \notin A_u$. Hence, (ii) holds.

Case 3. $d_G(u) \in \{5,6\}$ and $d_G(u) - |A_u| \geq 3$.

Since $\{v \in N_G(u) : uv \notin A_u\}$ is an independent set in G, $|A_u| \geq \lceil d_G(u)/2 \rceil = 3$ (as $d_G(u) \in \{5,6\}$). Hence, $d_G(u) = 6$ and $|A_u| = 3$ (as $d_G(u) - |A_u| \geq 3$). Let $r_1, r_2 \in N_G(u)$ such that $ur_1, ur_2 \notin A_u$, and let $G' = G - \{ur_1, ur_2\}$. Note that $r_1 r_2 \notin E(G)$.

Let C_1 be a facial cycle of G' containing e. Then (G', C_1) is a circuit graph as G is 4-connected and $d_G(u) = 6$. It follows from Lemma 2.12 that G' has a C_1-Tutte cycle D_1 through e, r_1, and r_2. If D_1 is a Hamiltonian cycle in G, then (i) holds, since D_1 contains an edge of A_u (because $d_G(u) - |A_u| = 3$ and $ur_1, ur_2 \notin A_u$). So suppose $V(G) \backslash V(D_1) \neq \emptyset$.

Then there exists a D_1-bridge of G', say B, such that $V(B) \backslash V(D_1) \neq \emptyset$ and $V(B \cap D_1) \leq 3$. Observe that $u \in V(B) \backslash V(D_1)$ otherwise, $V(B \cap D_1)$ is a 3-cut in G, a contradiction. Since $G[N_G(u)]$ is a cycle and $r_1 r_2 \notin E(G)$, $V(B) \cap V(r_1 D_1 r_2 - \{r_1, r_2\}) \neq \emptyset$ and $V(B) \cap V(r_2 D_1 r_1 - \{r_1, r_2\}) \neq \emptyset$. Thus, since $|V(B) \cap V(D_1)| \leq 3$, we may assume $V(B) \cap V(r_2 D_1 r_1 - \{r_1, r_2\}) = \{z\}$. Now since $d_G(u) = 6$, $\{u, z\}$ is a 2-cut in G, or $\{u\} \cup (V(B \cap D_1) \backslash \{z\})$ is a 3-cut in G, a contradiction.

Let C_2 be the outer cycle of $G' - y$. Since G is 4-connected and $d_G(u) = 6$, $(G' - y, C_2)$ is a circuit graph. By Lemma 2.12, $G' - y$ has a C_2-Tutte cycle D_2 through e and every vertex in $\{r_1, r_2\} \backslash \{y\}$. Similarly, we can show that D_2 is a Hamiltonian cycle in $G - y$ containing e as G is 4-connected and D_2 is a C_2-Tutte cycle. (In particular, note that if B is a D_2-bridge and $V(B) \backslash V(D_2)$ contains a neighbor of y, then $|V(B) \cap V(D_2)| \leq 2$.) Moreover, D_2 contains an edge in A_u as $|A_u| = 3$ and $ur_1, ur_2 \notin A_u$. Hence, (ii) holds.

Case 4. $d_G(u) = 4$.

Let $G[N_G(u)] = x_1 x_2 x_3 x_4 x_1$. By our assumption on u, two nonadjacent neighbors of u must each have degree at least 5 in G. Without loss of generality, assume that $d_G(x_2) \geq 5$ and $d_G(x_4) \geq 5$.

We claim that $(G-u)+x_1x_3$ or $(G-u)+x_2x_4$ is 4-connected. For, suppose $(G-u)+x_1x_3$ is not 4-connected, and let S be a 3-cut in $(G-u)+x_1x_3$. Then $\{x_1, x_3\} \subseteq S$, and $S \cup \{u\}$ is a 4-cut in G separating x_2 and x_4. Suppose G_1, G_2 are the components of $G - (S \cup \{u\})$ containing x_2, x_4, respectively. Since $d_G(x_{2i}) \geq 5$ for $i = 1,2$, $|V(G_i)| \geq 2$ for $i = 1,2$. Let $w_i \in V(G_i) \backslash \{x_{2i}\}$ for $i = 1,2$. Since G is 4-connected, there exist a path Q_i' from w_i to x_1 in $(G - (S \backslash \{x_1\})) - x_{2i}$ and a path Q_i'' from w_i to x_3 in $(G - (S \backslash \{x_3\})) - x_{2i}$. Observe that $V(Q_i' \cup Q_i'') \subseteq (V(G_i) \backslash \{x_{2i}\}) \cup \{x_1, x_3\}$. Hence, $G - \{u, x_2, x_4\}$ has two internally disjoint paths between x_1 and x_3. This implies that $(G - u) + x_2 x_4$ is 4-connected.

So without loss of generality assume that $G^* = (G - u) + x_1 x_3$ is 4-connected and that the edge $x_1 x_3$ is inside the face of $G - u$ bounded by $x_1 x_2 x_3 x_4 x_1$. Let G' be the plane graph obtained from G^* by inserting two vertices r and z into the faces of G^* bounded by $x_1x_2x_3x_1$ and $x_1x_3x_4x_1$, respectively, and then adding edges rx_i for $i = 1, 2, 3$ and zx_i for $i = 1, 3, 4$.

Since G^* is 4-connected, (G', C) is a circuit graph. By Lemma 2.12, G' has a C-Tutte cycle D' containing e, r, and z, which is a Hamiltonian cycle in G' as G^* is 4-connected. It is easy to check that D' can be modified at r and z to give a Hamiltonian cycle in G containing e and an edge in A_u; so (i) holds.

To prove (ii), we apply Lemma 2.12 to the circuit graph $(G' - y, C_1)$, where C_1 is the outer cycle of $G' - y$ containing e. Then $G' - y$ has a C_1-Tutte cycle D_1 through e, r, and z. (Note that if $uy \in E(G)$, then $\{r, z\} \cap V(C_1) \neq \emptyset$.) Since G^* is 4-connected, D_1 is a Hamiltonian cycle in $G' - y$. It is straightforward to check that D_1 can be modified to give a Hamiltonian cycle in $G - y$ containing e and an edge in A_u not incident with y. ∎

We now prove that in a 4-connected planar triangulation on n vertices, any two cofacial edges are contained in $\Omega(n)$ Hamiltonian cycles.

Lemma 3.2. *Let n be an integer with $n \geq 4$, G be a 4-connected planar triangulation on n vertices, T be a facial triangle in G, and $e_1, e_2 \in E(T)$. Then G contains at least c_1n Hamiltonian cycles through e_1 and e_2, where $c_1 = (12 \times 63 \times 541 \times 301)^{-1}$.*

Proof. We apply induction on n. Since G is a 4-connected plane graph and e_1, e_2 are cofacial in G, it follows from Lemma 2.11 that G has a Hamiltonian cycle through e_1 and e_2. So the assertion holds when $n \leq 1/c_1$. Now assume $n > 1/c_1$ and the assertion holds for 4-connected planar triangulations on fewer than n vertices.

Consider a drawing of G in which T is its outer cycle. Let $y \in V(T)$ be incident with both e_1 and e_2, and let e_3 be the edge in $E(T)\backslash\{e_1, e_2\}$.

We may assume that if there exist two adjacent vertices u_1, u_2 in G with $d_G(u_1) = d_G(u_2) = 4$, then $u_1u_2 = e_3$ or $y \in \{u_1, u_2\}$. For, suppose there exist $u_1, u_2 \in V(G)\backslash\{y\}$ such that $d_G(u_1) = d_G(u_2) = 4$ and $u_1u_2 \neq e_3$. We contract the edge u_1u_2 to obtain a planar triangulation G^* on $n - 1$ vertices. (We retain the edges e_1 and e_2.) Note that G^* is 4-connected (as $n > 1/c_1 > 6$) and T is a triangle in G^*. So by induction, G^* has $c_1(n - 1)$ Hamiltonian cycles through e_1 and e_2. Observe that all such cycles in G^* can be modified to give $c_1(n - 1)$ distinct Hamiltonian cycles in G through the edges e_1, e_2, and u_1u_2. Therefore, it suffices to show that G has a Hamiltonian cycle through e_1 and e_2 but not u_1u_2, as $c_1(n - 1) + 1 \geq c_1n$. So let C_1 denote the outer cycle of $G_1 := (G - y) - u_1u_2$. Observe that $e_3 \in E(C_1)$ and (G_1, C_1) is a circuit graph as G is 4-connected and planar. By Lemma 2.12, G_1 contains a C_1-Tutte cycle H_1 through e_3, u_1, and u_2. Moreover, H_1 is a Hamiltonian cycle in G_1 (since G is 4-connected). Therefore, $(H_1 - e_3) + \{y, e_1, e_2\}$ is a Hamiltonian cycle in G through e_1, e_2 and avoiding u_1u_2.

Since G has minimum degree at least 4 and $|E(G)| = 3n - 6$ by Euler's formula, we have

$$\begin{aligned}
2(3n-6) = 2|E(G)| &= \sum_{\{v \in V(G):4 \leq d(v) \leq 6\}} d(v) + \sum_{\{v \in V(G):d(v) \geq 7\}} d(v) \\
&\geq 4|\{v \in V(G) : d(v) \leq 6\}| + 7(n - |\{v \in V(G) : d(v) \leq 6\}|).
\end{aligned}$$

It follows that $|\{v \in V(G) : d(v) \leq 6\}| \geq n/3 + 4$. By the Four Color Theorem, there exists an independent set I of vertices of degree at most 6 in G with $I \cap V(T) = \emptyset$ and $|I| \geq (n/3 + 4 - 3)/4 \geq n/12$. By Lemma 2.10 (with $t = 7$), either there exist distinct $v, x \in V(G)$ such that $|N(v) \cap N(x) \cap I| \geq 7$, or there is a subset $S \subseteq I$ such that $|S| > c_1 n$ and S saturates no 4-cycle, or 5-cycle, or diamond-6-cycle in G. Moreover, $S \cap V(T) = \emptyset$ as $I \cap V(T) = \emptyset$.

Case 1. There exist distinct $v, x \in V(G)$ such that $|N(v) \cap N(x) \cap I| \geq 7$.

Recall that any two adjacent degree 4 vertices of G cannot be contained in $V(G)\backslash V(T)$. Since $|N(v) \cap N(x) \cap I| \geq 7$, G has at least two separating 4-cycles D_1 and D_2, such that $|V(\overline{D_i})| \geq 6$ for $i = 1, 2$, and $\overline{D_1} - D_1$ and $\overline{D_2} - D_2$ are disjoint. Without loss of generality, we may assume $|V(\overline{D_1} - D_1)| \leq n/2$. By our assumptions on G and applying Lemma 2.3, we see that $\overline{D_1} - (V(D_1)\backslash\{a, b\})$ has at least two Hamiltonian paths between a and b for any distinct $a, b \in V(D_1)$.

Let G_1^* be obtained from G by contracting $\overline{D_1} - D_1$ to a new vertex v_1. Observe that G_1^* is a 4-connected planar triangulation with outer cycle T. It follows by induction that G_1^* has at least $c_1(n - |V(\overline{D_1} - D_1)| + 1) \geq c_1 n/2$ Hamiltonian cycles through e_1 and e_2. For each such Hamiltonian cycle in G_1^*, say H^*, let $a_1, b_1 \in N_{G_1^*}(v_1)$ such that $a_1 v_1 b_1 \subseteq H^*$. We can then form a Hamiltonian cycle in G through e_1 and e_2 by taking the union of $H^* - v_1$ and a Hamiltonian path between a_1 and b_1 in $\overline{D_1} - (V(D_1)\backslash\{a_1, b_1\})$. Thus G has at least $2(c_1 n/2) = c_1 n$ Hamiltonian cycles through e_1 and e_2.

Case 2. There is an independent set S of vertices of degree at most 6 in G such that $|S| > c_1 n$, $S \cap V(T) = \emptyset$, and S saturates no 4-cycle, or 5-cycle, or diamond-6-cycle in G.

If there exist distinct $u_1, u_2 \in S$ such that $|N_G(u_i) \cap V(T)| \geq 2$ for $i \in [2]$, then u_1, u_2 are contained in a 4-cycle or a 5-cycle in G, a contradiction. Hence, at most one vertex in S, say x, is adjacent to two vertices in $V(T)$. Let $S' = S$ if x does not exist, and $S' = S\backslash\{x\}$ if x exists. Hence, $|S'| \geq |S| - 1$ and for all $u \in S'$, $|N_G(u) \cap V(T)| \leq 1$.

Next we show that S' satisfies the conditions of Lemma 2.5 and Lemma 2.6. First suppose $d_G(y) > 4$. If two degree 4 vertices are adjacent in G, they must be the two vertices in $V(T)\backslash\{y\}$. Hence, for any $u \in S'$, since $|N_G(u) \cap V(T)| \leq 1$, if $d_G(u) = 4$ then u is not adjacent to a degree 4 vertex in G, and if $d_G(u) \in \{5, 6\}$ then no degree 4 neighbors of u are adjacent in G. Now assume that $d_G(y) = 4$. Notice that for any $v \in N_G(y)$, $|N_G(v) \cap V(T)| = 2$, and thus, $N_G(y) \cap S' = \emptyset$. Hence, for any $u \in S'$, if $d_G(u) = 4$ then u is adjacent to no degree 4 vertex in G, and if $d_G(u) \in \{5, 6\}$ then no degree 4 neighbors of u are adjacent in G. Therefore, S' satisfies the conditions of Lemma 2.5 and Lemma 2.6.

Let $k := |S'|$ and $S' = \{u_1, u_2, \ldots, u_k\}$. Recall the definition of A_{u_i} for $i \in [k]$, and let $A_i := A_{u_i}\backslash\{e \in E(G) : e \text{ is incident with } y\}$. By Lemma 2.5, $E(G[N_G[u_i]]) \cap E(G[N_G[u_j]]) = \emptyset$ for $i \neq j$, and if $d_G(u_i) \in \{5, 6\}$ then $\{v \in N_G(u_i) : vu_i \notin A_{u_i}\}$ is independent in G; so $|A_{u_i}| \geq 3$ for all $u_i \in S'$. Hence, $|A_i| \geq 2$ for all $i \in [k]$. Note that $e_3 \notin E(G[N_G[u_i]])$, as $S' \cap V(T) = \emptyset$ and $|N_G(u_i) \cap V(T)| \leq 1$. We now find $k + 1 > c_1 n$ Hamiltonian cycles $H_1, \ldots, H_{k+1}$ in G, as follows.

Let $F_0 = \emptyset$ and $X_1 := G - F_0 = G$. Note that $e_3 \notin E(G[N_G[u_1]])$ and by Lemma 2.5, u_1 satisfies the conditions of Lemma 3.1 (with u_1, e_3, X_1 as u, e, G in Lemma 3.1, respectively). Since y is cofacial with e_3 but not incident with e_3, it follows from (ii) of Lemma 3.1 that $X_1 - y = G - y$ has a Hamiltonian cycle D_1 through e_3 and an edge $f_1 \in A_1$. Hence,

$H_1 = (D_1 - e_3) + \{y, e_1, e_2\}$ is a Hamiltonian cycle in G through e_1, e_2, and $f_1 \in A_1$. Set $F_1 = \{f_1\}$.

Suppose for some $j \in [k+1]$ ($j \geq 2$) we have found an edge set $F_{j-1} = \{f_1, \ldots, f_{j-1}\}$ where $f_i \in A_i$ for each $i \in [j-1]$, and a Hamiltonian cycle H_l in $X_l := G - F_{l-1}$ for each $l \in [j-1]$, such that $\{e_1, e_2, f_l\} \subseteq E(H_l)$ and $F_{l-1} \cap E(H_l) = \emptyset$. Consider the graph $X_j := G - F_{j-1}$. By Lemma 2.6, X_j is 4-connected. When $j = k+1$, $X_{k+1} := G - F_k$ is 4-connected; so by Lemma 2.11, X_{k+1} has a Hamiltonian cycle H_{k+1} through e_1 and e_2. We stop this process and output the desired $H_1, \ldots, H_{k+1}$. Now suppose $j \leq k$. Note that $G[N_G[u_j]]$ is a subgraph of X_j (as $E(G[N_G[u_j]]) \cap A_{u_l} = \emptyset$ for any $l \in [j-1]$), and that $e_3 \in E(X_j) \backslash E(G[N_G[u_j]])$. We now show that u_j satisfies the conditions of Lemma 3.1 (with u_j, e_3, X_j as u, e, G in Lemma 3.1, respectively). Since $u_j \in S'$, $X_j - f$ is 4-connected for any $f \in A_{u_j}$ (by Lemma 2.6), and the link of u_j in X_j is A_{u_j} (as $G[N_G[u_j]] \subseteq X_i \subseteq G$). Hence, if $d_{X_j}(u_j) = d_G(u_j) \in \{5, 6\}$ then by Lemma 2.5, $\{v \in N_{x_j}(u_j) : vu_j \notin A_{u_j}\} = \{v \in N_G(u_j) : vu_j \notin A_{u_j}\}$ is independent in G (hence, in X_j); if $d_{X_j}(u_j) = d_G(u_j) = 4$ then all neighbors of u_j each have degree at least 5 in X_j, as $A_{u_j} = E(G[N_G(u_j)])$ and $X_j - f$ is 4-connected for any $f \in A_{u_j}$. Therefore, by (ii) of Lemma 3.1, $X_j - y$ has a Hamiltonian cycle D_j through e_3 and some edge $f_j \in A_j$. Now $H_j = (D_j - e_3) + \{y, e_1, e_2\}$ is a Hamiltonian cycle in G such that $\{e_1, e_2, f_j\} \subseteq E(H_j)$. Note that $F_{j-1} \cap E(H_j) = \emptyset$ as $D_j \subseteq X_j$. Set $F_j = F_{j-1} \cup \{f_j\}$.

Therefore, G has at least $k + 1 = |S'| + 1 > c_1 n$ Hamiltonian cycles through e_1, e_2. ∎

Proof of Theorem 1.2. Let $c_2 := (12 \times 90 \times 541 \times 301)^{-1}$ and $c = c_2^2/2$. We show that every 4-connected planar triangulation on n vertices has at least cn^2 Hamiltonian cycles. It is easy to check that the assertion holds when $n \leq 1/\sqrt{c} = \sqrt{2}/c_2$ as every 4-connected planar graph is Hamiltonian by Tutte's theorem (or by Lemma 2.11). Hence we may assume that $n > \sqrt{2}/c_2$ and that the assertion holds for 4-connected planar triangulations on fewer than n vertices.

Case 1. G contains two adjacent vertices of degree 4.

Let $u_1, u_2 \in V(G)$ such that $u_1u_2 \in E(G)$ and $d_G(u_1) = d_G(u_2) = 4$. Let G^* be the graph obtained from G by contracting the edge u_1u_2 to a new vertex u^*. By induction, G^* has at least $c(n-1)^2$ Hamiltonian cycles from which we obtain at least $c(n-1)^2$ Hamiltonian cycles in G through the edge u_1u_2.

Let x_1, x_2, x_3, x_4 be the vertices that occur on $G[N_{G^*}(u^*)]$ in the clockwise order such that $N_G(u_1) \cap N_G(u_2) = \{x_2, x_4\}$. Note that $u_1u_2x_2u_1, u_1u_2x_4u_1$ are two triangles in G. By Lemma 3.2, G has at least $c_1 n$ Hamiltonian cycles through u_1x_{2i} and u_2x_{2i} for each $i \in [2]$. Observe that if H is a Hamiltonian cycle in G through u_1x_{2i} and u_2x_{2i}, then H is a Hamiltonian cycle in $G - u_1u_2$. Therefore, G contains at least $2c_1 n$ Hamiltonian cycles all avoiding the edge u_1u_2. Hence, there exist at least $c(n-1)^2 + 2c_1 n \geq cn^2$ Hamiltonian cycles in G.

Case 2. No two vertices of degree 4 in G are adjacent.

Recall that G contains an independent set I of vertices of degree at most 6 with $|I| \geq n/12$. By Lemma 2.10 (with $t = 10$), either there exist distinct vertices $v, x \in V(G)$ such that $|N(v) \cap N(x) \cap I| \geq 10$, or G contains $S \subseteq I$ such that $|S| \geq c_2 n$ and S saturates no 4-cycle, or 5-cycle, or diamond-6-cycle in G.

Suppose the former case holds. Since no two vertices of degree 4 in G are adjacent, we can find a separating 4-cycle D such that $1 < |V(\overline{D} - D)| \leq n/4$. We contract $\overline{D} - D$ to a vertex and denote this new graph by G_0. Note that G_0 is a 4-connected planar triangulation with $3n/4 \leq |V(G_0)| = n - |V(\overline{D} - D)| + 1 \leq n - 1$; so G_0 has at least $c(3n/4)^2$ Hamiltonian cycles by induction. Therefore, G has at least $2c(3n/4)^2 \geq cn^2$ Hamiltonian cycles, as by Lemma 2.3, $\overline{D} - (V(D) \backslash \{a, b\})$ has at least two Hamiltonian paths between a and b for any distinct $a, b \in V(D)$.

Now assume that there exists an independent set S of vertices of degree at most 6 in G with $|S| \geq c_2 n$ such that S saturates no 4-cycle, or 5-cycle, or diamond-6-cycle in G. By our assumptions on G, S satisfies the conditions in Lemma 2.5 and Lemma 2.6. Let $S = \{u_1, u_2, \ldots, u_k\}$. Recall the definition of A_{u_i}, the link of u_i for each $i \in [k]$.

Let $F = \{f_1, f_2, \ldots, f_k\}$, where $f_i \in A_{u_i}$ for $i \in [k]$. Let $F_j = \{f_1, \ldots, f_j\}$ for each $j \in [k]$ and let $F_0 = \emptyset$.

We claim that for each integer $j \in [k]$, there exists a collection of Hamiltonian cycles in G, say $\mathcal{C}_j$, such that $|\mathcal{C}_j| = k - j + 1$ and every cycle in $\mathcal{C}_j$ contains f_j but no edge from F_{j-1}. For each $j \in [k]$, let $X_j := G - F_{j-1}$. By Lemma 2.6, X_j is 4-connected for each j. If $j = k$, it follows from Lemma 2.11 that X_k has a Hamiltonian cycle $H^{(k)}_{k+1}$ through the edge f_k. Moreover, $F_{k-1} \cap E(H^{(k)}_{k+1}) = \emptyset$ as $H^{(k)}_{k+1} \subseteq X_k$. Let $\mathcal{C}_k = \{H^{(k)}_{k+1}\}$.

Now assume $j < k$. Let $F^{(j)}_j = \emptyset$ and $Y^{(j)}_{j+1} := X_j - F^{(j)}_j = X_j$. Note that $f_j \in X_j$ as $X_j = G - F_{j-1}$. Note that u_{j+1} satisfies the conditions in Lemma 3.1 (with u_{j+1}, f_j, X_j as u, e, G in Lemma 3.1, respectively), since $u_{j+1} \in S$, S satisfies the conditions in Lemmas 2.5 and 2.6, and $F_{j-1} \subseteq \cup_{i=1}^{j-1} A_{u_i}$ if $j \geq 1$. Then by (i) of Lemma 3.1, X_j has a Hamiltonian cycle $H^{(j)}_{j+1}$ containing f_j and some edge $f^{(j)}_{j+1} \in A_{u_{j+1}}$. Set $F^{(j)}_{j+1} = \{f^{(j)}_{j+1}\}$. For $j + 2 \leq l \leq k + 1$, suppose we have found an edge set $F^{(j)}_{l-1} = \{f^{(j)}_{j+1}, \ldots, f^{(j)}_{l-1}\}$, where $f^{(j)}_t \in A_{u_t}$ for $j + 1 \leq t \leq l - 1$, such that, for each $j + 1 \leq t \leq l - 1$, $Y^{(j)}_t := X_j - F^{(j)}_{t-1}$ is 4-connected and has a Hamiltonian cycle $H^{(j)}_t$ through f_j and $f^{(j)}_t$. Consider $Y^{(j)}_l := X_j - F^{(j)}_{l-1}$. Then $Y^{(j)}_l$ is 4-connected by Lemma 2.6. If $l = k + 1$, then, by Lemma 2.11, $Y^{(j)}_{k+1} := X_j - F^{(j)}_k$ has a Hamiltonian cycle $H^{(j)}_{k+1}$ through f_j and $F^{(j)}_k \cap H^{(j)}_{k+1} = \emptyset$. We stop the process and output the desired $\mathcal{C}_j = \{H^{(j)}_{j+1}, H^{(j)}_{j+2}, \ldots, H^{(j)}_{k+1}\}$. Now assume that $l < k + 1$. Then $G[N_G[u_l]]$ is a subgraph of $Y^{(j)}_l$, and u_l in $Y^{(j)}_l$ satisfies the conditions in Lemma 3.1. Since $f_j \in E(Y^{(j)}_l) \backslash E(G[N_G[u_l]])$, we apply (i) of Lemma 3.1 to find a Hamiltonian cycle $H^{(j)}_l$ in $Y^{(j)}_l$ through f_j and an edge $f^{(j)}_l$ in A_{u_l}. Set $F^{(j)}_l = F^{(j)}_{l-1} \cup \{f^{(j)}_l\}$.

Hence, by the above claim, the number of Hamiltonian cycles in G is at least $\sum_{j=1}^{k} |\mathcal{C}_j| \geq \sum_{j=1}^{k} (k + 1 - j) = k(k + 1)/2 > c_2^2 n^2/2 = cn^2$. ∎

4 Restricting degree 4 vertices

In this section, we prove Theorem 1.3. We first show several lemmas.

Lemma 4.1. *Let G be a 4-connected planar triangulation. Let S be an independent set in G saturating no 4-cycle or 5-cycle in G. Let $u, u' \in S$ be distinct and let D_u and $D_{u'}$ be 4-cycles* *containing u and u', respectively. Then $|V(D_u) \cap V(D_{u'})| \leq 2$, $V(D_u) \cap V(D_{u'}) \cap S = \emptyset$, and if $V(D_u) \cap V(D_{u'})$ consists of two vertices, say a and b, then $ab \in E(D_u) \cap E(D_{u'})$.*

Proof. Note that $u' \notin V(D_u)$ since $\{u, u'\}$ saturates no 4-cycle in G. Similarly, $u \notin V(D_{u'})$. So $V(D_u) \cap V(D_{u'}) \cap S = \emptyset$. Moreover, $|V(D_u) \cap V(D_{u'})| \leq 2$ since otherwise u, u' are contained in a 4-cycle in $G[D_u + u']$, contradicting the assumption that S saturates no 4-cycle in G.

Now suppose $V(D_u) \cap V(D_{u'}) = \{a, b\}$ with $a \neq b$. If $ab \in E(D_u) \backslash E(D_{u'})$ then $G[D_u + u']$ has a 5-cycle containing u and u', contradicting the assumption that S saturates no 5-cycle in G. So, $ab \notin E(D_u) \backslash E(D_{u'})$. Similarly, $ab \notin E(D_{u'}) \backslash E(D_u)$. If $ab \notin E(D_u) \cup E(D_{u'})$, then u, u' are contained in a 4-cycle in G, a contradiction. Thus, $ab \in E(D_u) \cap E(D_{u'})$. ∎

Recall that for a cycle D in G, $\overline{D}$ is the subgraph of G consisting of all vertices and edges of G contained in the closed disc bounded by D. In the proof of Theorem 1.3, we will need to consider the subgraphs of a planar triangulation that lie between two separating 4-cycles and use the following result on Hamiltonian paths in those subgraphs.

Lemma 4.2. *Let G be a 4-connected planar triangulation in which the distance between any two vertices of degree 4 is at least three. Let S be an independent set in G such that S saturates no 4-cycle or 5-cycle in G. Let $u, u' \in S$ be distinct, and $D_u, D_{u'}$ be separating 4-cycles in G containing u and u', respectively. Suppose $\overline{D_{u'}} \subseteq \overline{D_u}$, and $D_{u'}$ is a maximal separating 4-cycle containing u' in G, i.e., $\overline{D_{u'}}$ is not contained in $\overline{D}$ for any other separating 4-cycle $D \neq D_{u'}$ with $u' \in V(D)$. Let H denote the graph obtained from $\overline{D_u}$ by contracting $\overline{D_{u'}} - D_{u'}$ to a new vertex z so that H is a near triangulation with outer cycle D_u. Then one of the following holds:*

(i) *For any distinct $a, b \in V(D_u)$, $H - (V(D_u) \backslash \{a, b\})$ has at least two Hamiltonian paths between a and b.*

(ii) *There exist distinct $a, b \in V(D_u)$ such that $H - (V(D_u) \backslash \{a, b\})$ has a unique Hamiltonian path, say P, between a and b; but for any distinct $c, d \in V(D_u)$ with $\{c, d\} \neq \{a, b\}$, $H - (V(D_u) \backslash \{c, d\})$ has at least two Hamiltonian paths between c and d and avoiding an edge of P incident with z.*

Proof. Let $D_u = uvwxu$ and $D_{u'} = u'v'w'x'u'$. Without loss of generality, assume that u, v, w, x occur on D_u in clockwise order, and u', v', w', x' occur on $D_{u'}$ in clockwise order.

By Lemma 4.1, we have $|V(D_u) \cap V(D_{u'})| \leq 1$ or $|E(D_u) \cap E(D_{u'})| = 1$. Thus, $|V(H)| \geq 7$, and for any distinct $a, b \in V(D_u)$ with $ab \notin E(D_u)$, $H - (V(D_u) \backslash \{a, b\})$ is not a path. So by Lemma 2.1, we have

Claim 1. *For any distinct $a, b \in V(D_u)$ with $ab \notin E(D_u)$, $H - (V(D_u) \backslash \{a, b\})$ has at least two Hamiltonian paths between a and b.*

Claim 2. *We may assume $V(D_u) \cap V(D_{u'}) \neq \emptyset$.*

Proof. For, suppose $V(D_u) \cap V(D_{u'}) = \emptyset$. Then z is not incident with the infinite face of $H - D_u$. So for any distinct $a, b \in V(D_u)$ with $ab \in E(D_u)$, $H - (V(D_u) \backslash \{a, b\})$ cannot be an outer planar graph. Thus, by Lemma 2.2, $H - (V(D_u) \backslash \{a, b\})$ has at least two Hamiltonian paths between a and b. So (i) holds by Claim 1. ∎

Claim 3. *We may further assume that* $|V(D_u) \cap V(D_{u'})| = 2$.

Proof. For, suppose $V(D_u) \cap V(D_{u'})$ consists of exactly one vertex, say y. Then $y \in \{v, w, x\} \cap \{v', w', x'\}$. We show that (i) holds. By Claim 1, it suffices to consider distinct $a, b \in V(D_u)$ with $ab \in E(D_u)$. By Lemma 2.2, it suffices to show that $H - (V(D_u)\backslash\{a, b\})$ is not outer planar.

Let $a, b \in V(D_u)$ with $ab \in E(D_u)$. If $y \in \{a, b\}$, then z is not incident with the infinite face of $H - (V(D_u)\backslash\{a, b\})$; so $H - (V(D_u)\backslash\{a, b\})$ is not outer planar. Hence we may assume that $y \notin \{a, b\}$. Let $D_u = yy_1y_2y_3y$ and assume that y, y_1, y_2, y_3 occur on D_u in clockwise order. Then $\{a, b\} = \{y_1, y_2\}$ or $\{a, b\} = \{y_2, y_3\}$.

First, assume that $y \in \{v', x'\}$. We consider $y = v'$, as the other case $y = x'$ is symmetric. If $H - \{y, y_3\}$ is outer planar, then u' is adjacent to the vertices y, y_3 in $V(D_u)$. Since $yy_3 \in E(D_u)$, $G[D_u + u']$ has a 5-cycle containing u and u', contradicting the assumption that S saturates no 5-cycle. Hence, $H - (V(D_u)\backslash\{y_1, y_2\}) = H - \{y, y_3\}$ is not outer planar. It remains to consider $H - (V(D_u)\backslash\{y_2, y_3\}) = H - \{y, y_1\}$. Suppose $H - \{y, y_1\}$ is outer planar. Then w' and x' are incident with the infinite face of $H - \{y, y_1\}$ and $w'y_1 \in E(G)$. We claim that $x'y_1 \in E(G)$; otherwise $x'y \in E(G)$, implying that $x'w'yx'$ or $x'u'yx'$ is a separating triangle in G, a contradiction. But then $D = u'yy_1x'u'$ is a separating 4-cycle in G containing u', and $\overline{D}$ properly contains $\overline{D_{u'}}$, contradicting the maximality of $D_{u'}$.

Suppose $y = w'$. For $\{a, b\} = \{y_1, y_2\}$ or $\{a, b\} = \{y_2, y_3\}$, if $H - (V(D_u)\backslash\{a, b\})$ is not outer planar, then $u'y_3 \in E(G)$ or $u'y_1 \in E(G)$. So $D = u'y_3w'x'u'$ or $D = u'y_1w'v'u'$ is a separating 4-cycle in G such that $\overline{D}$ properly contains $\overline{D_{u'}}$. Thus, $H - (V(D_u)\backslash\{a, b\})$ cannot be outer planar for $\{a, b\} = \{y_1, y_2\}$ or $\{a, b\} = \{y_2, y_3\}$. ∎

By Claim 3, $|E(D_u) \cap E(D_{u'})| = 1$; so $V(D_u) \cap V(D_{u'}) = \{v, w\}$ or $V(D_u) \cap V(D_{u'}) = \{w, x\}$. By the symmetry among the edges in D_u and between the two orientations of D_u, we may further assume $V(D_u) \cap V(D_{u'}) = \{v, w\}$.

Claim 4. *For* $\{a, b\} \subseteq V(D_u)$ *with* $ab \in E(D_u)$*, if* $\{a, b\} \neq \{u, x\}$*, then* $H - (V(D_u)\backslash\{a, b\})$ *has at least two Hamiltonian paths between* a *and* b.

Proof. For $\{a, b\} = \{v, w\}$, since z is not incident with the infinite face of $H - \{x, u\}$, $H - (V(D_u)\backslash\{a, b\}) = H - \{x, u\}$ is not outer planar and has at least two Hamiltonian paths between a and b by Lemma 2.2.

For $\{a, b\} = \{u, v\}$ or $\{a, b\} = \{w, x\}$, $H - (V(D_u)\backslash\{a, b\})$ cannot be outer planar. Otherwise, one can check that $\{u, u'\}$ is contained in a 4-cycle or 5-cycle in G. Hence, by Lemma 2.2, there exist at least two Hamiltonian paths between a and b in $H - (V(D_u)\backslash\{a, b\})$ when $\{a, b\} = \{u, v\}$ or $\{a, b\} = \{w, x\}$. ∎

By Claim 4, we may assume that $H - \{v, w\}$ has a unique Hamiltonian path P between u and x, as otherwise (i) holds. It follows from Lemma 2.2 that $H - \{v, w\}$ is an outer planar near triangulation. Let y' denote the vertex in $V(D_{u'})\backslash\{u', v, w\}$; so $y' = x'$ or $y' = v'$. Note that $V(uPz) \subseteq N_H(v)$, that $V(xPz) \subseteq N_H(w)$, and that P contains $u'zy'$. Let r denote the unique vertex in $N_H(u) \cap N_H(x)$. Observe that $\{v, w\} = \{v', w'\}$ or $\{v, w\} = \{w', x'\}$. Recall the definition of diamond-4-cycle in Figure 1.

Claim 5. *There exists a vertex* $y \in V(P)\backslash\{u, x, z\}$ *such that* $yu' \in E(P)$ *and* $D' := G[D_{u'} + y]$ *is a diamond-4-cycle with* u' *and* y *as crucial vertices. Moreover,* $r \notin \{y, y'\}$*.*

Proof. First, suppose $\{v, w\} = \{v', w'\}$, i.e. $v = v'$ and $w = w'$. Then $y' = x'$. Hence, there exists a vertex y in $V(uPu')\backslash\{u, u'\}$ such that $yu' \in E(P)$ and $yv \in E(G)$. If $yx' \notin E(G)$, then u' has a neighbor z' in $V(xPx')$ since $H - \{v, w\}$ is an outer planar near triangulation; now $u'v'w'z'u'$ is a separating 4-cycle in G containing u' (as $z'w = z'w' \in E(G)$), contradicting the maximality of $D_{u'}$. Therefore, $yx' \in E(G)$ and $G[D_{u'} + y]$ is a diamond-4-cycle with crucial vertices u' and y. Moreover, $r \notin \{y, x'\} = \{y, y'\}$; otherwise, $uvu'yu$ (when $r = y$) or $uvu'x'u$ (when $r = y'$) is a 4-cycle saturated by S, a contradiction.

Now assume that $\{v, w\} = \{w', x'\}$, i.e., $v = w'$ and $w = x'$. Then $y' = v'$. Observe that $u'x \notin E(G)$, otherwise $uvwu'xu$ is a 5-cycle in G saturated by S, a contradiction. Hence, there exists $y \in V(u'Px)\backslash\{u', x\}$ such that $yu' \in E(P)$ and $yw \in E(G)$. Now $yv' \in E(G)$ by the maximality of $D_{u'}$. Therefore, $G[D_{u'} + y]$ is a diamond-4-cycle in G in which u', y are crucial vertices. If $r = y' = v'$ then $uvwu'y'u$ is a 5-cycle in G containing $\{u, u'\}$, and if $r = y$ then $uyu'wxu$ is a 5-cycle in G containing $\{u, u'\}$. This contradicts the assumption that S saturates no 5-cycle in G, completing the proof of Claim 5. ∎

We need another claim, in order to show that for any $\{c, d\} \neq \{u, x\}$, $H-(V(D_u)\backslash\{c, d\})$ has at least two Hamiltonian paths between c and d and not containing $u'zy'$. Let $H' := H - (V(D_u) \cap V(D_{u'}) \cup \{z\}) = H - \{v, w, z\}$. Then H' is an outer planar near triangulation and $H' \subseteq G$.

Claim 6. $r \in V(H')\backslash\{y, u', y'\}$ *and* $H' - \{u, x, u', y'\}$ *has a Hamiltonian path* P_1 *between* r *and* y*.*

Proof. Since $r \in N_G(u)$ and S is independent, $r \neq u'$. By Claim 5, $r \notin \{y, y'\}$ and $y \notin \{u, x, y'\}$. Thus, $r \notin \{y, y', u'\}$.

Let C denote the outer cycle of H'. Then $ux, u'y' \in E(C)$. We may assume $y' = x'$; the other case is similar.

Since G contains no separating triangle, each edge in $H' - E(C)$ is incident with both uPu' and xPy'. Since $V(uPu') \subseteq N_G(v)$ and $V(xPy') \subseteq N_G(w)$, every degree 4 vertex of G in $V(H')\backslash\{u, x, u', y'\}$ has degree 3 in H'. Hence, by assumption of the lemma, the distance between any two degree 3 vertices of H', contained in $V(H')\backslash\{u, x, u', y'\}$, is at least three in H'. Applying Lemma 2.4 to H', we see that $H' - \{u, x, u', y'\}$ has a Hamiltonian path P_1 between r and y. ∎

Let $Q_1 := P_1 \cup yu'y'z$ and $Q_2 := P_1 \cup yy'u'z$. Then Q_1 and Q_2 are two distinct Hamiltonian paths between r and z in $H-V(D_u)$, and neither contains $u'zy'$. We now show that (ii) holds with $\{a, b\} = \{u, x\}$. Let $c, d \in V(D_u)$ be distinct such that $\{c, d\} \neq \{u, x\}$. Observe that one vertex in $\{c, d\}$ is a neighbor of r and the other is a neighbor of z. We may assume $c \in N_H(r)$ and $d \in N_H(z)$. Then $cr \cup Q_1 \cup zd, cr \cup Q_2 \cup zd$ are two distinct Hamiltonian paths in $H - (V(D_u)\backslash\{c, d\})$ between c and d and not containing $u'zy'$. ∎

We also need the following result, which is given implicitly in the proof of Theorem 1.3 in [12].

Lemma 4.3 (Liu and Yu [12]). *Let G be a 4-connected planar triangulation. Assume that G contains a collection of separating 4-cycles, say $\mathcal{D} = \{D_1, D_2, \dots, D_{t+1}\}$, such that $\overline{D_1} \supseteq \overline{D_2} \supseteq \cdots \supseteq \overline{D_{t+1}}$. For $j \in [t]$, let G_j be the graph obtained from $\overline{D_j}$ by contracting $\overline{D_{j+1}} - D_{j+1}$ to a new vertex, denoted by z_{j+1}. Suppose the conclusion of Lemma 4.2 holds for G_j and z_{j+1} (as H and z, respectively, in Lemma 4.2). Then G has at least $2^{\sqrt{t}}$ Hamiltonian cycles.*

Proof of Theorem 1.3. Note that, for any two distinct vertices x, y of degree 4 in G, we have $N_G(x) \cap N_G(y) = \emptyset$ as $d_G(x, y) \geq 3$. Hence, the number of vertices of degree 4 in G is at most $n/5$. Thus, since $|E(G)| = 3n - 6$ and $\delta(G) \geq 4$, there exist at least $n/5$ vertices of degree 5 or 6 in G. Then by the Four Color Theorem, there is an independent set I such that every vertex in I has degree 5 or 6 in G and $|I| \geq (n/5)/4 = n/20$. We may assume that

(1) G has an independent set $S \subseteq I$ of size $\Omega(n^{3/4})$ such that S saturates no 4-cycle, or 5-cycle, or diamond-6-cycle in G.

For, otherwise, by Lemma 2.10, there exist distinct $v, x \in V(G)$ such that $|N_G(v) \cap N_G(x) \cap I| \geq c_0 n^{1/4}$ for some constant $c_0 > 0$. Since any two vertices of degree 4 in G have distance at least three, $G[N_G[v] \cup N_G[x]]$ contains separating 4-cycles $C_1, \dots, C_k$ in G, where $k \geq c_0 n^{1/4} - 1$, such that $|V(\overline{C_i})| \geq 6$ for each $i \in [k]$, and $\overline{C_i} - C_i$, $\overline{C_j} - C_j$ are disjoint whenever $1 \leq i \neq j \leq k$. Let G^* be the graph obtained from G by contracting $\overline{C_i} - C_i$ to a new vertex v_i, for $i \in [k]$. Then G^* is a 4-connected planar triangulation and, hence, has a Hamiltonian cycle, say H.

Let $a_i, b_i \in N_{G^*}(v_i)$ such that $a_i v_i b_i \subseteq H$ for $i \in [k]$. Since $|V(\overline{C_i})| \geq 6$ and no vertices of degree 4 in G are adjacent, it follows from Lemma 2.3 that $\overline{C_i} - (V(C_i) \backslash \{a_i, b_i\})$ has at least two Hamiltonian paths between a_i and b_i. We can form a Hamiltonian cycle in G by taking the union of $H - \{v_i : i \in [k]\}$ and one Hamiltonian path between a_i and b_i in $\overline{C_i} - (V(C_i) \backslash \{a_i, b_i\})$ for each $i \in [k]$. Thus, G has at least $2^k \geq 2^{c_0 n^{1/4} - 1}$ Hamiltonian cycles and we are done. This completes the proof of (1).

For each $u \in S$, recall the link A_u defined in Section 2. We may assume that

(2) there exists $S_1 \subseteq S$ such that $|S_1| \geq |S|/2$ and, for each $u \in S_1$, $d_G(u) - |A_u| \geq 2$ and u is contained in a separating 4-cycle D in G with $|V(\overline{D})| \geq 6$.

Suppose we have $S_2 \subseteq S$ with $|S_2| \geq |S|/2$ such that $d_G(u) - |A_u| \leq 1$ for all $u \in S_2$. Hence, for any $u \in S_2$, $|A_u| \geq 4$ if $d_G(u) = 5$; and $|A_u| \geq 5$ if $d_G(u) = 6$. Let F be any subset of $E(G)$ with $|F| = |S_2|$ and $|F \cap A_u| = 1$ for each $u \in S_2$. By Lemma 2.6, $G - F$ is 4-connected; so $G - F$ has a Hamiltonian cycle by Tutte's theorem (or by Lemma 2.11). Let $\mathcal{C}$ be a collection of Hamiltonian cycles in G by taking precisely one Hamiltonian cycle in $G - F$ for each choice of F. Let a_1 and a_2 denote the number of vertices in S_2 of degree 5 and 6 in G, respectively. There are at least $4^{a_1} 5^{a_2}$ choices of the edge set $F \subseteq E(G)$. Each Hamiltonian cycle of G in $\mathcal{C}$ is chosen at most $(5-2)^{a_1}(6-2)^{a_2} = 3^{a_1} 4^{a_2}$ times. Thus $|\mathcal{C}| \geq (4/3)^{a_1} (5/4)^{a_2} \geq (5/4)^{a_1 + a_2} = (5/4)^{|S_2|} \geq (5/4)^{\Omega(n^{3/4})}$.

Hence, we may assume that there exists $S_1 \subseteq S$ such that $|S_1| \geq |S|/2$ and $d_G(u) - |A_u| \geq 2$ for all $u \in S_1$. For each $u \in S_1$, since $d_G(u) - |A_u| \geq 2$, there exist at least two edges

e_1 and e_2 incident with u such that $G - e_i$ is not 4-connected for $i \in [2]$. Since u has at most one neighbor of degree 4 in G (by assumption), there exists $i \in [2]$ such that a 3-cut of $G - e_i$ and u induce a separating 4-cycle D_u in G with $|V(\overline{D_u})| \geq 6$. This completes the proof of (2).

For each $u \in S_1$, we choose a maximal separating 4-cycle D_u containing u. Note that $|V(\overline{D_u})| \geq 6$. Let $\mathcal{D} = \{D_u : u \in S_1\}$. Since S_1 saturates no 4-cycle, $D_u \neq D_{u'}$ for any distinct $u, u' \in S_1$ and $|\mathcal{D}| = |S_1| \geq |S|/2$. By Lemma 4.1, for any distinct $D_1, D_2 \in \mathcal{D}$, either $\overline{D_1} - D_1$ and $\overline{D_2} - D_2$ are disjoint, or $\overline{D_1}$ contains $\overline{D_2}$ or vice versa. We may assume that

(3) there exist $D_1, D_2, \dots, D_{t+1} \in \mathcal{D}$, where $t = \Omega(n^{1/2})$, such that $\overline{D_1} \supseteq \overline{D_2} \supseteq \cdots \supseteq \overline{D_{t+1}}$.

For, otherwise, since $|\mathcal{D}| = |S_1| \geq |S|/2 = \Omega(n^{3/4})$, there exist separating 4-cycles $D'_1, \dots, D'_k \in \mathcal{D}$, where $k = \Omega(n^{1/4})$, such that $|V(\overline{D'_i})| \geq 6$ for $i \in [k]$, and $\overline{D'_i} - D'_i$, $\overline{D'_j} - D'_j$ are disjoint for $1 \leq i \neq j \leq k$. Hence, G has at least 2^k Hamiltonian cycles, as shown in the first paragraph in the proof of (1). This completes the proof of (3).

For each $j \in [t]$, let G_j denote the graph obtained from $\overline{D_j}$ by contracting $\overline{D_{j+1}} - D_{j+1}$ to a new vertex z_{j+1}. Note that G_j is a near triangulation with outer cycle D_j and that G_j contains the 4-cycle D_{j+1}.

By Lemma 4.1 and the definition of $\mathcal{D}$, we see that D_{j+1}, D_j, G_j, and G (as $D_{u'}, D_u, H, G$, respectively, in Lemma 4.2) for $j \in [t]$, satisfy the conditions in Lemma 4.2. Hence, by Lemma 4.2 and Lemma 4.3, G has at least $2^{\sqrt{t}} = 2^{\Omega(n^{1/4})}$ Hamiltonian cycles. ∎

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